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On the K-theory of Crossed Product C^*-algebras by Actions of $\mathbb{Z}^n$

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Abstract

We investigate the K-theory of crossed product C^*-algebras by $\mathbb{Z}^n$-actions with emphasis on the case $n = 2$. Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, we define a homomorphism $d_*(\alpha)$ between certain subquotients of $K_*(A)$, which we call the *obstruction homomorphism* associated with $(A, \alpha, \mathbb{Z}^2)$, and which has the property that the K-theory of the crossed product $A \rtimes_\alpha \mathbb{Z}^2$ is determined by $(K_*(A), K_*(\alpha), d_*(\alpha))$ up to group extension problems. A concrete description of this obstruction homomorphism is given as well. Furthermore, we show the existence of C^*-dynamical systems $(A, \alpha, \mathbb{Z}^2)$ whose associated obstruction homomorphisms are non-trivial. Certain C^*-dynamical systems involving Kirchberg algebras turn out to have that property, which we show by using a result of Izumi and Matui [17]. On the other hand, we find C^*-dynamical systems arising from pointwise inner actions whose associated obstruction homomorphisms do not vanish. As an illustrative example, we consider a natural pointwise inner $\mathbb{Z}^2$-action on the group C^*-algebra of the discrete Heisenberg group. Moreover, we show the existence of C^*-dynamical systems $(A, \alpha, \mathbb{Z}^2)$ which are universal for K-theoretical obstructions associated with pointwise inner $\mathbb{Z}^2$-actions.

In the second part of this thesis, we investigate a certain spectral sequence converging to the K-theory of $A \rtimes_\alpha \mathbb{Z}^n$. This spectral sequence is an instance of Kasparov's construction in [20] and was already used in [2] and [43]. We identify the E_1-term with a certain Koszul complex over the integral group ring over $\mathbb{Z}^n$, and use this to recover Kasparov's result that the E_2-term coincides with the group cohomology of $\mathbb{Z}^n$ with coefficients in $K_*(A)$. Moreover, we give a partial description of the higher boundary maps of this spectral sequence in terms of the natural subactions of α. If $n = 2$, we identify the second level differential d_2 with the obstruction homomorphism $d_*(\alpha)$. Using this, and provided that the $\mathbb{Z}^n$-action α induces the trivial action on K-theory, we describe the second level differential in terms of the obstruction homomorphisms associated with the natural $\mathbb{Z}^2$-subactions.

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Contents

Introduction

The study of group actions on C^*-algebras and their associated crossed products plays an important rôle in operator algebra theory. Beside the fact that many prominent C^*-algebras arise as crossed products, their relevance is due to various connections to other mathematical fields, such as representation theory for locally compact groups or topological dynamical systems.

One of the most important invariants for crossed product C^*-algebras is topological K-theory. However, given a C^*-dynamical system, it is in general very difficult to determine the K-theoretical properties of its associated crossed product C^*-algebra. One approach to the computation of the K-theory for reduced crossed product C^*-algebras, which is accessible via topological methods, is proposed by the famous Baum-Connes Conjecture. This conjecture is known to hold for a strikingly large class of groups, of which amenable groups form a prominent subclass [16].

In this thesis we are concerned with the special case of crossed products by $\mathbb{Z}^n$-actions. We investigate their K-theory with emphasis on the case $n = 2$. Since $\mathbb{Z}^n$-actions are given by n commuting automorphisms, crossed product C^*-algebras by $\mathbb{Z}^n$-actions arise in various situations. A rich source of examples is provided by number theory, see for instance the work of Cuntz and Li on C^*-algebras associated with rings of integers in number fields [11], which served as a motivational background for this thesis.

For $\mathbb{Z}$-actions, the celebrated *Pimsner-Voiculescu sequence* [36] is a very powerful tool to compute the K-theory of the corresponding crossed products. Given a C^*-dynamical system $(A, \alpha, \mathbb{Z})$, this six-term exact sequence relates the K-theory of the crossed product $A \rtimes_\alpha \mathbb{Z}$ with the K-theory of A and the induced action of α on K-theory. One important consequence of the Pimsner-Voiculescu sequence is that $K_*(A \rtimes_\alpha \mathbb{Z})$ fits into a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(K_*(\alpha) - \mathrm{id}) \arrow[r] & K_*(A \rtimes_\alpha \mathbb{Z}) \arrow[r] & \ker(K_{*+1}(\alpha) - \mathrm{id}) \arrow[r] & 0. \end{tikzcd}$$

In particular, $K_*(A \rtimes_\alpha \mathbb{Z})$ is determined by the pair $(K_*(A), K_*(\alpha))$ up to a group extension problem.

In order to compute the K-theory of a crossed product $A \rtimes_\alpha \mathbb{Z}^n$, the naïve approach would certainly be to write it as an n-fold crossed product by $\mathbb{Z}$. The hope would be that, as in the case of a single automorphism, the information stored in $K_*(A)$ and $K_*(\alpha)$ is sufficient in order to successfully apply the Pimsner-Voiculescu sequence n times. Closely related to this, we can ask the following question.

Main question. Let $(A, \alpha, \mathbb{Z}^n)$ be a C^*-dynamical system. Is $K_*(A \rtimes_\alpha \mathbb{Z}^n)$ determined by $(K_*(A), K_*(\alpha))$ up to group extension problems?

Although this question has an affirmative answer in the case of a single automorphism, we show that the answer can be negative for $n \geq 2$. In fact, it turns out that there are various counterexamples involving $\mathbb{Z}^2$-actions. This especially shows that $\mathbb{Z}^2$-actions already exhibit a great amount of the difficulties and subtleties that one faces when passing from the known case of a single automorphism to the case of $\mathbb{Z}^n$-actions. Opposed to the general case of $\mathbb{Z}^n$-action, the special case of $\mathbb{Z}^2$-actions is still quite accessible via elementary methods.

The first part of this thesis is devoted to an extensive study of the K-theory for crossed products by $\mathbb{Z}^2$-actions. Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, we define a group homomorphism $d_*(\alpha)$ between certain subquotients of $K_*(A)$ which has the property that $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ is determined by $(K_*(A), K_*(\alpha), d_*(\alpha))$ up to group extension problems. We call $d_*(\alpha)$ the *obstruction homomorphism* associated with $(A, \alpha, \mathbb{Z}^2)$ since it obstructs $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ to be solely determined by $(K_*(A), K_*(\alpha))$ up to group extension problems. Hence, in order to find examples of C^*-dynamical systems $(A, \alpha, \mathbb{Z}^2)$ giving a negative answer to the main question, we have to look for actions with non-trivial obstruction homomorphisms.

Roughly speaking, the examples of $\mathbb{Z}^2$-actions that we present in this thesis can be divided into two classes. The first class is derived from Izumi and Matui's result in [17]. In the spirit of Kirchberg-Phillips classification [21], [35], they use KK-theoretical invariants to classify certain outer $\mathbb{Z}^2$-actions on Kirchberg algebras. We use their result to show the existence of a vast class of $\mathbb{Z}^2$-actions on UCT Kirchberg algebras which induce trivial actions on K-theory and give rise to non-trivial obstruction homomorphisms.

As a second class, we consider pointwise inner $\mathbb{Z}^2$-action. Contrary to the naïve expectation, we find counterexamples to the main question even within this class of actions. This is even more remarkable since unitarily implemented $\mathbb{Z}^2$-actions on A, i.e. actions arising from group representations into the unitary group of A, all give rise to isomorphic crossed products. An instructive example is given by a natural pointwise inner $\mathbb{Z}^2$-action on the group C^*-algebra of the discrete Heisenberg group; this C^*-algebra was extensively studied in [1]. The induced C^*-dynamical system is of its own interest since it is universal in the sense that it admits an equivariant $*$-homomorphism into every other C^*-dynamical system arising from a pointwise inner $\mathbb{Z}^2$-action. Furthermore, we construct counterexamples to the main question which come from certain pointwise inner $\mathbb{Z}^2$-actions on amalgamated free product C^*-algebras. Among these, we find C^*-dynamical systems involving pointwise inner $\mathbb{Z}^2$-actions whose associated obstruction homomorphisms are universal in a suitable sense.

In the second part of this thesis, we are dealing with crossed products by $\mathbb{Z}^n$-actions for arbitrary n. We show that the K-theory of a crossed product $A \rtimes_\alpha \mathbb{Z}^n$ only depends on the homotopy class of the $\mathbb{N}^n$-action induced by α. This extends the known result that $K_*(A \rtimes_\alpha \mathbb{Z}^n)$ is determined by the (automorphic) homotopy class of α. Consequently, if a C^*-dynamical systems $(A, \alpha, \mathbb{Z}^n)$ gives rise to a negative answer to the main question, then α, when considered as an $\mathbb{N}^n$-action, can not be homotopic to the trivial $\mathbb{N}^n$-action on A. In order to investigate the K-theory of a crossed product $A \rtimes_\alpha \mathbb{Z}^n$, we consider a certain spectral sequence going back to a very general result by Kasparov [20, 6.10]. In *loc. cit.*, Kasparov proves the existence of a spectral sequence converging to $K_*(A \rtimes_\alpha G)$ whenever A is σ-unital and G is a countable, discrete, torsion-free subgroup of an amenable, connected, locally compact group. Moreover, he shows that the E_2-term of this spectral sequence equals the group homology of G with coefficients in $K_*(A)$. For $G = \mathbb{Z}$, his result recovers the Pimsner-Voiculescu sequence.

In the case of a C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$, there is a more direct approach to Kasparov's spectral sequence, which is presented in [2]. It uses a natural finite cofiltration of the mapping torus $\mathcal{M}_\alpha(A)$. This C^*-algebra has the same K-theory as the crossed product $A \rtimes_\alpha \mathbb{Z}^n$, which follows from the fact that the Baum-Connes Conjecture with coefficients holds for $\mathbb{Z}^n$. One can then apply a standard technique which associates a spectral sequence converging to the K-theory of the cofiltered C^*-algebra. This construction is closely connected to Schochet's spectral sequence [44] associated with a filtration by closed ideals. In [43], Savignen and Bellissard elaborate on an idea of Pimsner to give a useful description of the E_1-term and its associated differential d_1 in terms of what they call the *Pimsner-Voiculescu complex*. This alternative description reveals a striking similarity with the Pimsner-Voiculescu sequence. We remark that this spectral sequence was also used by Forrest and Hunton in [15], where they determine the K-theory for crossed products $C(X) \rtimes_\alpha \mathbb{Z}^n$ with X denoting the Cantor set.

When working with spectral sequences, it is crucial to understand the occurring differentials. However, this is often very complicated, and it seems that the spectral sequence for the K-theory of crossed products by $\mathbb{Z}^n$ does not constitute an exception. This work is therefore also supposed to provide a starting point for a more systematic investigation of these differentials. The main result of the second part of this thesis is a partial description of the k-th level differential d_k in terms of the natural $\mathbb{Z}^k$-subactions of α. As a corollary, we obtain an elementary proof of Savignen and Bellissard's description of (E_1, d_1). Furthermore, we identify (E_1, d_1) with a certain Koszul complex over the integral group ring of $\mathbb{Z}^n$. If the acting group is $\mathbb{Z}^2$, we compare the methods used in the first part of this thesis with the spectral sequence approach. The crucial observation is that the second level differential d_2 coincides with the associated obstruction homomorphism $d_*(\alpha)$. In combination with the result on the higher boundary maps, this yields some additional information on the second level differentials associated with $\mathbb{Z}^n$-actions.

This thesis is organized as follows. The first chapter serves as a reminder of crossed product C^*-algebras. Moreover, we give a proof (not using the Baum-Connes Conjecture) for the well-known fact that the crossed product $A \rtimes_\alpha \mathbb{Z}^n$ and the mapping torus $\mathcal{M}_\alpha(A)$ are naturally isomorphic in K-theory.

In Chapter 2, we give Loring's [26] and Exel's [14] definition of a *Bott element* associated with two almost commuting unitaries, which lies in the K_0-group of the underlying C^*-algebra. These elements play a crucial rôle when it comes to investigating the boundary map of the Pimsner-Voiculescu sequence and the obstruction homomorphism associated with a $\mathbb{Z}^2$-action. We also show some basic properties of these Bott elements.

In Chapter 3, we recall the Pimsner-Voiculescu exact sequence. Given a C^*-dynamical system $(A, \alpha, \mathbb{Z})$, we describe preimages under the boundary map $\rho_* : K_*(A \rtimes_\alpha \mathbb{Z}) \to K_{*+1}(A)$. The lifts for ρ_1 are well-known and easily obtained by using the partial isometry picture for ρ_1. They are given by what we call *Bott elements* associated with a commuting pair consisting of a projection and a unitary. Both the projection and the unitary lie in a matrix amplification of $A \rtimes_\alpha \mathbb{Z}$. Finding lifts for ρ_0 is more difficult. In fact, we have to use an alternative description for the K_1-group of a unital C^*-algebra, which comes as a special case of a deep result by Dadarlat [12]. In this way, we get lifts in terms of Bott elements associated with almost commuting unitaries in matrix amplifications of $A \rtimes_\alpha \mathbb{Z}$.

In Chapter 4, we study the K-theory of crossed products by $\mathbb{Z}^2$-actions. Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$ with generating automorphisms α_1 and α_2, we use the naturality of the Pimsner-Voiculescu sequence for $(A, \alpha_1, \mathbb{Z})$ applied to α_2. By this, we obtain a morphism of short exact sequences, and the Snake Lemma then yields the associated obstruction homomorphism $d_*(\alpha)$. We show that the K-theory of $A \rtimes_\alpha \mathbb{Z}^2$ is determined by $(K_*(A), K_*(\alpha), d_*(\alpha))$ up to group extension problems. In the case that A is unital, we give a concrete description of $d_*(\alpha)$ in terms of Bott elements in $K_*(A)$. We also describe $d_*(\alpha)$ by using the mapping torus $\mathcal{M}_\alpha(A)$.

In Chapter 5, we obtain the main results of the first part of this thesis. We give examples of C^*-dynamical systems $(A, \alpha, \mathbb{Z}^2)$ whose associated obstruction homomorphisms do not vanish. First, we recall Izumi and Matui's classification result of locally KK-trivial outer $\mathbb{Z}^2$-actions [17]. We then use their result and show the existence of a large class of outer $\mathbb{Z}^2$-actions on Kirchberg algebras with the property that the associated obstruction homomorphisms are non-zero. The second section is concerned with the group C^*-algebra of the discrete Heisenberg group $C^*(H_3)$. We consider a natural pointwise inner $\mathbb{Z}^2$-action α on $C^*(H_3)$, and show that it gives rise to a non-trivial obstruction homomorphism. We also compute the K-theory of the resulting crossed product. The C^*-dynamical system $(C^*(H_3), \alpha, \mathbb{Z}^2)$ turns out to be universal for systems $(B, \beta, \mathbb{Z}^2)$ with β acting by inner automorphisms. Using this, we show that $d_*(\beta)$ is non-trivial if the equivariant map $C^*(H_3) \to B$ induces an injective homomorphism between the respective K_0-groups. In the third section, we present a general procedure to get pointwise inner $\mathbb{Z}^2$-actions with non-trivial obstruction homomorphisms. All occurring C^*-algebras are amalgamated free products of two C^*-algebras over $C(\mathbb{T})$. One of the algebras already carries a pointwise inner $\mathbb{Z}^2$-action, and the other one contributes a non-trivial element in the image of the obstruction homomorphism. As important special cases, we construct C^*-dynamical systems with pointwise inner $\mathbb{Z}^2$-actions which are universal for obstructions coming from pointwise inner $\mathbb{Z}^2$-actions. Moreover, we compute the K-theory for the crossed products associated with these universal systems.

The second part of this thesis is concerned with the general case of crossed products by $\mathbb{Z}^n$-actions. In Chapter 6, we show that two $\mathbb{Z}^n$-actions give rise to crossed products with isomorphic K-theory whenever the actions are homotopic as $\mathbb{N}^n$-actions. This extends the known result that the K-theory of a crossed product by $\mathbb{Z}^n$ only depends on the automorphic homotopy class of the action. The proof uses dilation results for semigroup crossed product C^*-algebras by Laca [23] and by Pask, Raeburn, and Yeend [33].

The last chapter is devoted to a spectral sequence $(E_k, d_k)_{k\geq 1}$ converging to the K-theory of $A \rtimes_\alpha \mathbb{Z}^n$. We first recall the general construction of a spectral sequence associated with a finite cofiltration of C^*-algebras. Afterwards, we obtain the desired spectral sequence by applying this machinery to a natural cofiltration of the corresponding mapping torus $\mathcal{M}_\alpha(A)$. This section's main result is a technical lemma giving a partial description of the k-th level differential d_k in terms of the natural $\mathbb{Z}^k$-subactions of α. In the special case that $k = 1$, this yields an elementary proof for the fact that (E_1, d_1) and the Pimsner-Voiculescu complex are isomorphic. We identify the Pimsner-Voiculescu complex with a certain Koszul complex over the integral group ring of $\mathbb{Z}^n$ with coefficients in $K_*(A)$. This in turn recovers Kasparov's result that the E_2-term coincides with the group cohomology of $\mathbb{Z}^n$ with coefficients in $K_*(A)$. Moreover, if the acting group is $\mathbb{Z}^2$, then we show that the second level differential d_2 coincides with the associated obstruction homomorphism $d_*(\alpha)$. If a $\mathbb{Z}^n$-action α acts trivially on K-theory, this result together with the main technical lemma admits a complete description of d_2 in terms of the obstruction homomorphisms of the natural $\mathbb{Z}^2$-subactions.

I should like to thank Chris Phillips for pointing out the existence of certain C^*-dynamical systems $(A, \alpha, \mathbb{Z}^2)$ with α acting trivially on K-theory and $K_*(A \rtimes_\alpha \mathbb{Z}^2) \not\cong K_*(A \otimes C(\mathbb{T}^2))$ to me.

Chapter 1

Preliminaries

The reader of this thesis is supposed to be familiar with the general theory of C^*-algebras including the definition and basic properties of K-theory. It is also useful to have some background in KK-theory and its connection to K-theory, which particularly includes the Universal Coefficient Theorem (UCT) and the Künneth Theorem [41]. As possible references for K-theory of C^*-algebras, we suggest [4] and [39]. We remark that the sign convention for the boundary maps in K-theory used in *loc. cit.* differs from the one we are using in this work. We shall often regard K-theory as $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. Unless specified otherwise, all graded group homomorphism are supposed to be grading-preserving. For a C^*-algebra A and $n \geq 1$, we write $\mathcal{U}_n(A)$ for the set of unitary elements in $M_n(A)$ and $\mathcal{P}_n(A)$ for the set of projections in $M_n(A)$. Moreover, $\mathcal{T}$ always denotes the Toeplitz algebra, and we write $z := \mathrm{id}_{C(\mathbb{T})} \in C(\mathbb{T})$, and $z_1, z_2 \in C(\mathbb{T}^2)$ for the canonical unitaries $z \otimes 1$ and $1 \otimes z \in C(\mathbb{T}^2)$, respectively.

The first section serves as a reminder of crossed product C^*-algebras. We recall some important results, like Takai duality [46] and Connes' Thom isomorphism [8], which are relevant for what follows. Most of what we present can be found in [50], which we recommend for further information on crossed products.

The second section is concerned with the K-theoretical isomorphism between the mapping torus $\mathcal{M}_\alpha(A)$ and the crossed product $A \rtimes_\alpha \mathbb{Z}^n$. For a single acting automorphism, this was already shown by Connes in [8], where he gives an alternative proof for the Pimsner-Voiculescu sequence. There are also works by Blackadar [5] and Paschke [32] which cover the special case of $\mathbb{Z}$-actions.

Although the aforementioned isomorphism follows from the Baum-Connes Conjecture with coefficients for $\mathbb{Z}^n$, we give an alternative, well-known proof here. In a first step, we use a general result by Olesen and Pedersen [31] and Takai duality in order to establish an isomorphism

$$A \rtimes_\alpha \mathbb{Z}^n \rtimes_{\tilde{\hat{\alpha}}} \mathbb{R}^n \xrightarrow{\cong} \mathcal{M}_\alpha(A) \otimes \mathcal{K}(L(\mathbb{Z}^n)), \tag{1.1}$$

where the action of $\mathbb{R}^n$ is naturally induced by the dual $\mathbb{T}^n$-action. The desired isomorphism

$$\Psi_\alpha : K_*(A \rtimes_\alpha \mathbb{Z}^n) \xrightarrow{\cong} K_{*+n}(\mathcal{M}_\alpha(A)) \tag{1.2}$$

then follows by applying Connes' Thom isomorphism n times.

1.1 A reminder of crossed product C^*-algebras

Let G be a locally compact Hausdorff group and let A be a C^*-algebra. A group homomorphism $\alpha : G \to \mathrm{Aut}(A)$, $g \mapsto \alpha_g$, which is continuous with respect to the point-norm topology on Aut(A) is called a *G-action* on A. A C^*-algebra A equipped with a G-action is called a *G-algebra*, and the induced triple (A, α, G) is called a *C^*-dynamical system*. The trivial G-action on a C^*-algebra A is denoted by tr. We always write $\{e_1, \ldots, e_n\}$ for the canonical $\mathbb{Z}$-basis of $\mathbb{Z}^n$, and define $\alpha_i := \alpha_{e_i}$ for $i = 1, \ldots, n$.

Given a C^*-dynamical system (A, α, G), the crossed product $A \rtimes_\alpha G$ is constructed as a certain completion of the $\mathbb{C}$-vectorspace

$$C_c(G, A) := \{f : G \to A \text{ continuous} \quad : \quad \mathrm{supp}(f) \subseteq G \text{ compact}\}.$$

We write $\int : C_c(G) \to \mathbb{C}$ for the normalized *Haar integral*. It is characterized as the unique normalized positive linear functional which is left-invariant, i.e.

$$\int\limits_G f(gx)\mathrm{d}x = \int\limits_G f(x)\mathrm{d}x \quad \text{for all} \quad g \in G,\ f \in C_c(G).$$

Uniqueness of the Haar integral implies that for every $g \in G$, there is a number $\Delta(g) \in \mathbb{R}$ satisfying

$$\int\limits_G f(x)\mathrm{d}x = \Delta(g) \int\limits_G f(xg)\mathrm{d}x \quad \text{for all} \quad f \in C_c(\mathbb{R}, A).$$

In fact, $\Delta : G \to (0, \infty)$ is a continuous group homomorphism, called the *modular homomorphism* of G. If $\Delta = 1$, then G is called *unimodular*. In other words, a group is unimodular if the Haar-integral is left- and right-invariant. Both discrete and abelian groups, which are subject of our considerations, are unimodular.

We equip $C_c(G, A)$ with a $*$-algebra structure by defining the convolution product and the involution as follows

$$(f * g)(s) := \int\limits_G f(t)\alpha_t(g(t^{-1}s))\,\mathrm{d}t \quad \text{and} \quad f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*).$$

For the definition of the convolution product we have used *vector-valued integration*, which we shortly recall for the reader's convenience. Given a Hilbert space H and a weakly continuous function $f : G \to \mathcal{B}(H)$ with compact support, there is a unique operator $\int\limits_G f(x)dx \in \mathcal{B}(H)$ satisfying

$$\left\langle \left(\int\limits_G f(x)\,\mathrm{d}x \right) \xi, \eta \right\rangle = \int\limits_G \langle f(x)\xi, \eta\rangle\,\mathrm{d}x \quad \text{for all} \quad \xi, \eta \in H.$$

For our purposes, we think of A being represented faithfully on a Hilbert space H, and so we are dealing with norm-continuous functions $f : G \to \mathcal{B}(H)$ with compact support. It can be shown that $\int\limits_G f(x)dx \in A$, and that

$$a\left(\int\limits_G f(x)\,\mathrm{d}x\right)b = \int\limits_G af(x)b\,\mathrm{d}x \quad \text{for} \quad a,b\in\mathcal{M}(A), \tag{1.3}$$

where $\mathcal{M}(A)$ denotes the multiplier algebra of A.

When dealing with crossed products, it is often necessary to consider strictly continuous $*$-homomorphisms between multiplier algebras. The following result characterizes $*$-homomorphisms between C^*-algebras admitting a strictly continuous extension between the respective multiplier algebras, see also [18, 1.1.15].

Proposition 1.1.1. *Let A and B be C^*-algebras, and let $\varphi : A \to \mathcal{M}(B)$ be a $*$-homomorphism. Then the following conditions are equivalent:*

- *i) There is a projection $p\in\mathcal{M}(B)$ with $\overline{\varphi(A)B} = pB$.*
- *ii) There is a strictly continuous $*$-homomorphism $\underline{\varphi} : \mathcal{M}(A)\to\mathcal{M}(B)$ extending φ.*

If $\varphi : A\to\mathcal{M}(B)$ admits a strictly continuous extension $\underline{\varphi} : \mathcal{M}(A)\to\mathcal{M}(B)$, then this extension is unique and satisfies

$$\underline{\varphi}(m)b = \lim_\lambda \varphi(u_\lambda m)b \quad \text{for} \quad m\in\mathcal{M}(B),\ b\in B,$$

and for any approximate unit (u_λ) for A. If $\overline{\varphi(A)B} = B$, then we call φ a *non-degenerate* $*$-homomorphisms.

A *covariant homomorphism* or *covariant pair* of (A,α,G) is a pair (ϕ, U) consisting of a $*$-homomorphism $\phi : A\to\mathcal{M}(D)$ and a strictly continuous homormophism $U : G\to\mathcal{U}(\mathcal{M}(D))$ into the unitary group of the multiplier algebra $\mathcal{M}(D)$ of some C^*-algebra D satisfying the covariance condition

$$\phi(\alpha_g(a)) = U_g\phi(a)U_g^* \quad \text{for all} \quad a\in A,\ g\in G.$$

A covariant homomorphism (ϕ, U) is called *non-degenerate*, if ϕ is non-degenerate. Every covariant pair gives rise to a $*$-homomorphism

$$\phi\times U : C_c(G,A)\longrightarrow\mathcal{M}(D), \quad \phi\times U(f) := \int\limits_G \phi(f(s))U_s\,\mathrm{d}s,$$

called the *integrated form* of (ϕ, U). To see that this expression is well-defined, note that the integrand $s\mapsto\phi(f(s))U_s\in\mathcal{M}(D)$ is compactly supported and continuous with respect to the strict topology. This yields that the integrand is weakly continuous if $\mathcal{M}(A)$ is represented faithfully on a Hilbert space. We can therefore use vector-valued integration, and by (1.3), we obtain a well-defined operator $\phi\times U(f)\in\mathcal{M}(A)\subseteq\mathcal{B}(H)$. If (ϕ, U) is non-degenerate, then the integrated form $\phi\times U$ is non-degenerate as well.

Definition 1.1.2. Let (A, α, G) be a C^*-dynamical system. The *(full) crossed product* $A \rtimes_\alpha G$ is the completion of $C_c(G, A)$ with respect to the maximal norm

$$\|f\| := \sup \{\|\phi \times U(f)\| \ : \ (\pi, U) \text{ covariant homomorphism}\}.$$

It can be shown that the maximal norm is also obtained by restricting to non-degenerate covariant homomorphisms.

There is a canonical non-degenerate covariant homomorphism (i_A, i_G) into $\mathcal{M}(A \rtimes_\alpha G)$ given by

$$\begin{aligned} (i_A(a)f)(s) &= af(s) & (fi_A(a))(s) &= f(s)\alpha_s(a), \\ (i_G(t)f)(s) &= \alpha_t(f(t^{-1}s)) & (fi_G(t))(s) &= \Delta(t^{-1})\alpha_{s^{-1}}(f(st^{-1})). \end{aligned}$$

Both i_A and i_G are injective, and given a covariant pair (ϕ, U), it holds that

$$\underline{\pi \times U} \circ i_A = \phi \quad \text{and} \quad \underline{\pi \times U} \circ i_G = U.$$

This can be used to show the following universal property of crossed products due to Raeburn [37], see also [50, 2.61].

Theorem 1.1.3. *Let (A, α, G) be a C^*-dynamical system and let (j_A, j_G) be a covariant pair into $\mathcal{M}(B)$ such that the following holds:*

i) Given a non-degenerate covariant homomorphism (ϕ, U) into $\mathcal{M}(D)$, there is a non-degenerate $*$-homomorphism $\psi : B \to \mathcal{M}(D)$ such that $\underline{\psi} \circ j_A = \phi$ and $\underline{\psi} \circ j_G = U$.

ii) $B = \overline{\operatorname{span}} \{j_A(a)j_G(g) \ : \ a \in A \text{ and } g \in G\}$.

Then there is an isomorphism $\varphi : B \to A \rtimes_\alpha G$ with $\underline{\varphi} \circ j_A = i_A$ and $\underline{\varphi} \circ j_G = i_G$.

If G is discrete, then A embeds into $C_c(G, A) \subseteq A \rtimes_\alpha G$ via $a \mapsto \delta_e \cdot a$. Moreover, if A is unital, then $A \rtimes_\alpha G$ can be characterized as the universal unital C^*-algebra containing a copy of A and unitaries u_g, $g \in G$, subject to the relations

$$\alpha_g(a) = u_g a u_g^* \quad \text{for all} \quad g \in G, \ a \in A.$$

Every equivariant $*$-homomorphism $\varphi : (A, \alpha, G) \to (B, \beta, G)$ gives rise to a $*$-homomorphism between the crossed products $\varphi \rtimes G : A \rtimes_\alpha G \to B \rtimes_\beta G$ satisfying

$$\varphi \rtimes G(f)(s) = \varphi(f(s)) \quad \text{for} \quad f \in C_c(G, A) \text{ and } s \in G.$$

If the acting groups is given by $\mathbb{Z}$, we usually stick to the notation $\check{\varphi} := \varphi \rtimes \mathbb{Z}$.

It is easy to see that $\_ \rtimes G$ defines a functor from the category of G-algebras with morphisms being equivariant $*$-homomorphisms into the category of C^*-algebras. This functor is exact in the following sense. Given an α-invariant ideal $I \subseteq A$, let $(A/I, \alpha^I, G)$ denote the induced C^*-dynamical system. Then there is a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I \rtimes_\alpha G \arrow[r] & A \rtimes_\alpha G \arrow[r] & A/I \rtimes_{\alpha^I} G \arrow[r] & 0, \end{tikzcd}$$

where both maps are naturally induced. By passing from a C^*-dynamical system (A, α, G) to the induced system of the unitization $(A^\sim, \alpha^\sim, G)$, we therefore get an extension

$$\begin{tikzcd} 0 \arrow[r] & A \rtimes_\alpha G \arrow[r] & A^\sim \rtimes_{\alpha^\sim} G \arrow[r] & \mathbb{C} \rtimes_{\mathrm{id}} G \arrow[r] & 0. \end{tikzcd}$$

Two G-actions α and β on a C^*-algebra A are called *homotopic* if there is a G-action γ on $C([0,1], A)$ with $\mathrm{ev}_0 \circ \gamma_g = \alpha_g$ and $\mathrm{ev}_1 \circ \gamma_g = \beta_g$ for $g \in G$.

Two C^*-dynamical systems (A, α, G) and (B, β, G) are called *conjugate* if there is an equivariant isomorphism $\varphi : (A, \alpha, G) \to (B, \alpha, G)$, i.e. if the associated G-algebras are isomorphic. In this situation, $\varphi \rtimes G$ is an isomorphism between the corresponding crossed products. A strictly continuous map $u : G \to \mathcal{U}(\mathcal{M}(A))$ satisfying $u_{gh} = u_g \alpha_g(u_h)$ for all $g, h \in G$ is called an *α-cocycle.* Every α-cocycle u gives rise to a G-action α^u given by $\alpha^u_g := \mathrm{Ad}(u_g) \circ \alpha_g$. Two C^*-dynamical systems (A, α, G) and (B, β, G) are called *cocycle conjugate* if there is an α-cocycle u such that the perturbed system (A, α^u, G) is conjugate to (B, β, G). Cocycle conjugate systems give rise to isomorphic crossed products, see [50, 2.68]. A cocycle for the trivial G-action is just a continuous group homomorphism $G \to \mathcal{U}(\mathcal{M}(A))$, which in turn induces a C^*-dynamical system $(A, \mathrm{Ad}(u), G)$. Such a system is called *unitarily implemented.*

For every locally compact abelian group G, the *Pontryagin dual* $\widehat{G}$ is given by

$$\widehat{G} := \{\chi \ : \ \chi : G \to \mathbb{T} \ \text{ continuous group homomomorphism}\},$$

which again is an abelian group. Furthermore, we can equip $\widehat{G}$ with the topology induced by the open-compact topology on $C(G, \mathbb{T})$. In this way, $\widehat{G}$ is a locally compact abelian group.

Examples 1.1.4. 1. $\widehat{\mathbb{Z}^n} \cong \mathbb{T}^n \quad \text{via} \quad \chi \mapsto (\chi(e_1), \ldots, \chi(e_n)).$

2. $\mathbb{R}^n \cong \widehat{\mathbb{R}^n} \quad \text{via} \quad s \mapsto \left[t \mapsto \prod_{j=1}^{n} \exp(2\pi i s_j t_j) \right].$

3. Finite cyclic groups are self-dual, i.e. $\widehat{\mathbb{Z}_n} \cong \mathbb{Z}_n$.

4. If G is compact, then $\widehat{G}$ is discrete.

5. If G is discrete, then $\widehat{G}$ is compact.

Let us recall the famous Pontryagin duality, see also [42, Section 1.7].

Theorem 1.1.5 (Pontryagin duality). *For any locally compact abelian group G, the canonical homomorphism*

$$G \longrightarrow \widehat{\widehat{G}}, \quad g \mapsto [\chi \mapsto \chi(g)],$$

is an isomorphism of topological groups.

Definition + Proposition 1.1.6. Let (A, α, G) be a C^*-dynamical system. There is a canonical $\widehat{G}$-action $\hat{\alpha}$ on $A \rtimes_\alpha G$, called the *dual action*, which is given by

$$\hat{\alpha}_\chi(f)(s) = \chi(s)f(s) \quad \text{for} \quad \chi \in \widehat{G},\ f \in C_c(G, A),\ \text{and } s \in G.$$

Next we state the deep Takai duality theorem, which can be considered as an analogue of the Pontryagin duality for crossed product C^*-algebras.

Theorem 1.1.7 (Takai duality, [46])**.** *Let G be a locally compact abelian group. For every C^*-dynamical system (A, α, G), there is an equivariant $*$-isomorphism*

$$\phi_\alpha : (A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \widehat{G}, \hat{\hat{\alpha}}, \widehat{\widehat{G}}) \xrightarrow{\cong} (A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \rho, G),$$

where ρ is induced by the right regular representation and G is identified with $\widehat{\widehat{G}}$ using Pontryagin duality. The isomorphism ϕ_α is natural in the sense that if $\varphi : (A, \alpha, G) \to (B, \beta, G)$ is an equivariant $*$-homomorphism, then the following diagram commutes

$$\begin{tikzcd}
A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \widehat{G} \arrow[r, "\varphi \rtimes G \rtimes \widehat{G}"] \arrow[d, "\phi_\alpha"] & B \rtimes_\beta G \rtimes_{\hat{\beta}} \widehat{G} \arrow[d, "\phi_\beta"] \\
A \otimes \mathcal{K}(L^2(G)) \arrow[r, "\varphi \otimes \mathrm{id}"] & B \otimes \mathcal{K}(L^2(G))
\end{tikzcd}$$

Proof. We only give the construction of ϕ_α without proving that it is an isomorphism. Using the definition of ϕ_α, it is then not hard to verify its equivariance and naturality property. For a complete proof we refer to [34] and [50].

If $(C_0(G), \mathrm{lt}, G)$ denotes the C^*-dynamical system given by left translation of G, then the Stone-von Neumann theorem [50, 4.24] induces an isomorphism

$$C_0(G) \rtimes_{\mathrm{lt}} G \otimes A \cong \mathcal{K}(L^2(G)) \otimes A.$$

There is another isomorphism

$$\widetilde{\phi}_\alpha : A \rtimes_\alpha G \rtimes_{\hat{\alpha}} \widehat{G} \xrightarrow{\cong} C_0(G, A) \rtimes_{\mathrm{lt} \otimes \alpha} G \cong C_0(G) \rtimes_{\mathrm{lt}} G \otimes A,$$

given as follows. For $F \in C_c(\widehat{G} \times G, A)$, the function $\widetilde{\phi}_\alpha(F) \in C_c(G, C_0(G, A))$ is defined as

$$\widetilde{\phi}_\alpha(F)(s, r) = \int\limits_{\widehat{G}} \alpha_r^{-1}(F(\gamma, s))\gamma(s^{-1}r)\, \mathrm{d}\,\mu(\gamma),$$

where μ denotes the Haar measure on $\widehat{G}$. Now, ϕ_α is given as the composition of $\widetilde{\phi}_\alpha$ with the above isomorphism. □

Another milestone in the theory of crossed products is Connes' Thom isomorphism for the K-theory of crossed products by $\mathbb{R}$.

Theorem 1.1.8 (Connes' Thom isomorphism, [8])**.** *For every C^*-dynamical system $(A, \alpha, \mathbb{R})$, there is an isomorphism $K_*(A \rtimes_\alpha \mathbb{R}) \cong K_{*+1}(A)$ which is natural with respect to equivariant $*$-homomorphisms.*

Proof. We only sketch a proof due to Rieffel [38]. Let τ denote the $\mathbb{R}$-action on $C := C_0(\mathbb{R} \cup \{\infty\})$ given by

$$\tau_s(f)(t) := \begin{cases} f(t-s) & , \ t \neq \infty, \\ f(\infty) & , \ t = \infty, \end{cases}$$

and consider the exact sequence

$$0 \longrightarrow SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \longrightarrow C \otimes A \rtimes_{\tau\otimes\alpha} \mathbb{R} \longrightarrow A \rtimes_\alpha \mathbb{R} \longrightarrow 0 \tag{1.4}$$

induced by evaluation at ∞. Let $e \in \mathcal{K}(L^2(\mathbb{R}))$ denote a minimal projection, and let $\bar{e} \in C_0(\mathbb{R}) \rtimes_\tau \mathbb{R}$ be its image under the isomorphism given by the Stone-von Neumann Theorem. One can show that

$$\psi : A \otimes \mathcal{K}(L^2(\mathbb{R})) \xrightarrow{\cong} SA \rtimes_{\tau\otimes\alpha} \mathbb{R}, \quad \psi(a \otimes e)(s,t) = \alpha_t(a)\bar{e}(s,t),$$

is a $*$-isomorphism, and that $C \otimes A \rtimes_{\tau\otimes\alpha} \mathbb{R}$ has trivial K-theory. Connes' Thom isomorphism is now given as the composition of the index map

$$\rho_* : K_*(A \rtimes_\alpha \mathbb{R}) \xrightarrow{\cong} K_{*+1}(SA \rtimes_{\tau\otimes\alpha} \mathbb{R})$$

associated with the six-term exact sequence corresponding to (1.4) and the isomorphism $K_{*+1}(SA \rtimes_{\tau\otimes\alpha} \mathbb{R}) \cong K_{*+1}(A)$ induced by

$$\eta_\alpha : A \longrightarrow A \otimes \mathcal{K}(L^2(\mathbb{R})) \xrightarrow{\psi} SA \rtimes_{\tau\otimes\alpha} \mathbb{R}.$$

Let $\varphi : (A, \alpha, \mathbb{R}) \to (B, \beta, \mathbb{R})$ be an equivariant $*$-homomorphism. Naturality of Connes' Thom isomorphism follows from the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\eta_\alpha"] \arrow[d, "\varphi\rtimes\mathbb{R}"'] & SA \rtimes_{\tau\otimes\alpha} \mathbb{R} \arrow[d, "S\varphi\rtimes\mathbb{R}"] \\
B \arrow[r, "\eta_\beta"] & SB \rtimes_{\tau\otimes\beta} \mathbb{R}
\end{tikzcd}$$

together with naturality of K-theory. □

1.2 Crossed products and mapping tori

We begin with the definition of the mapping torus associated with a C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$.

Definition 1.2.1. Let $(A, \alpha, \mathbb{Z}^n)$ be a C^*-dynamical system. The *mapping torus* is the C^*-algebra given by

$$\mathcal{M}_\alpha(A) := \{f \in C(\mathbb{R}^n, A) \ : \ f(x+z) = \alpha_z(f(x)),\ x \in \mathbb{R}^n,\ z \in \mathbb{Z}^n\}.$$

It is obvious from the definition that the restriction to $[0,1]^n \subseteq \mathbb{R}^n$ gives rise to an isomorphism between the mapping torus $\mathcal{M}_\alpha(A)$ and the C^*-algebra

$$\left\{ f \in C([0,1]^n, A) \ : \ \begin{matrix} f(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) = \\ \alpha_i(f(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)) \end{matrix},\ t_1, \ldots, t_n \in [0,1] \right\}.$$

In the following, we use this identification without further mentioning.

An equivariant $*$-homomorphism $\varphi : (A, \alpha, \mathbb{Z}^n) \to (B, \beta, \mathbb{Z}^n)$ gives rise to a $*$-homomorphism

$$\widetilde{\varphi} : \mathcal{M}_\alpha(A) \longrightarrow \mathcal{M}_\alpha(B), \quad \widetilde{\varphi}(f)(t) := \varphi(f(t)).$$

The map $\widetilde{\varphi}$ is surjective (injective) if φ is surjective (injective). Moreover, the assignment $\varphi \mapsto \widetilde{\varphi}$ is functorial in the obvious sense.

Proposition 1.2.2. *Let $(A, \alpha, \mathbb{Z}^n)$ be a C^*-dynamical system, H a Hilbert space, and let ρ be a unitarily implemented $\mathbb{Z}^n$-action on the C^*-algebra of compact operators $\mathcal{K}(H)$.*

Then there is a $*$-isomorphism $\mathcal{M}_{\alpha\otimes\rho}(A \otimes \mathcal{K}(H)) \cong \mathcal{M}_\alpha(A) \otimes \mathcal{K}(H)$, which is natural in the sense that given an equivariant $*$-homomorphism $\varphi : (A, \alpha, \mathbb{Z}^n) \to (B, \beta, \mathbb{Z}^n)$, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\alpha\otimes\rho}(A \otimes \mathcal{K}(H)) \arrow[r, "\cong"] \arrow[d, "\widetilde{\varphi\otimes\mathrm{id}_\mathcal{K}}"'] & \mathcal{M}_\alpha(A) \otimes \mathcal{K}(H) \arrow[d, "\widetilde{\varphi}\otimes\mathrm{id}_\mathcal{K}"] \\
\mathcal{M}_{\beta\otimes\rho}(B \otimes \mathcal{K}(H)) \arrow[r, "\cong"] & \mathcal{M}_\beta(B) \otimes \mathcal{K}(H)
\end{tikzcd}$$

Proof. Let $u_1, \ldots, u_n \in \mathcal{U}(H)$ be commuting unitaries implementing $\rho_1, \ldots, \rho_n$, respectively. Using Borel functional calculus, we get norm-continuous homotopies $u_{t,k} \in \mathcal{U}(H)$ connecting u_k with 1 and satisfying $u_{t,k}u_j = u_j u_{t,k}$ for all $j \neq k$, $t \in [0,1]$. By setting $\eta_{t,k} := \mathrm{Ad}(u_{t,k})$, we obtain a well-defined $*$-isomorphism

$$\psi : \mathcal{M}_{\alpha\otimes\rho}(A \otimes \mathcal{K}(H)) \xrightarrow{\cong} \mathcal{M}_{\alpha\otimes\mathrm{tr}}(A \otimes \mathcal{K}(H)),$$
$$\psi(f)(t) = (\mathrm{id} \otimes (\eta_{t_1,1} \circ \rho_1^{-1} \ldots \eta_{t_n,n} \circ \rho_n^{-1}))(f(t)).$$

Consider the natural isomorphism

$$\mathcal{M}_{\alpha\otimes\mathrm{tr}}(A \otimes \mathcal{K}(H)) \cong \mathcal{M}_\alpha(A) \otimes \mathcal{K}(H),$$

whose composition with ψ now yields the desired $*$-isomorphism

$$\mathcal{M}_{\alpha\otimes\rho}(A\otimes\mathcal{K}(H))\cong\mathcal{M}_{\alpha}(A)\otimes\mathcal{K}(H).$$

Checking the naturality condition is straightforward. □

Consider the following extension of topological groups

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}^n \arrow[r] & \mathbb{R}^n \arrow[r, "p"] & \mathbb{T}^n \arrow[r] & 0. \end{tikzcd}$$

Given a C^*-dynamical system $(B,\beta,\mathbb{T}^n)$, we get an induced $\mathbb{R}^n$-action $\bar{\beta}$ on B by setting $\bar{\beta}_t := \beta_{p(t)}$ for every $t\in\mathbb{R}^n$. Since $\bar{\beta}_z=\mathrm{id}_B$ for all $z\in\mathbb{Z}^n$, there is a canonical $C(\mathbb{T}^n)$-algebra structure on $B\rtimes_{\bar{\beta}}\mathbb{R}^n$ induced by the natural inclusion

$$C(\mathbb{T}^n)\cong C^*(\mathbb{Z}^n)\longrightarrow\mathcal{Z}(\mathcal{M}(B\rtimes_{\bar{\beta}}\mathbb{R}^n)).$$

The reader unfamiliar with $C_0(X)$-algebras is referred to [6] and [50]. The following result reveals the connection between the two crossed product C^*-algebras $B\rtimes_{\bar{\beta}}\mathbb{R}^n$ and $B\rtimes_{\beta}\mathbb{T}^n$.

Lemma 1.2.3. *Let $\phi: C_c(\mathbb{R}^n,B)\to C(\mathbb{T}^n,B)$ be the $*$-homomorphism given by*

$$\phi(f)(p(x))=\sum_{z\in\mathbb{Z}^n}f(x+z).$$

Then there is a unique extension of ϕ to a surjective $*$-homomorphism

$$\phi: B\rtimes_{\bar{\beta}}\mathbb{R}^n\longrightarrow B\rtimes_{\beta}\mathbb{T}^n.$$

Proof. Straightforward calculations show that ϕ respects the involutions and the convolution products. Hence, ϕ is in fact a $*$-homomorphism. If (π,U) is a covariant pair for $(B,\beta,\mathbb{T}^n)$, then $(\pi,U\circ p)$ is a covariant pair for $(B,\bar{\beta},\mathbb{R}^n)$, which moreover satisfies

$$\pi\rtimes(U\circ p)(f)=\pi\rtimes U(\phi(f))\quad\text{for}\quad f\in C_c(\mathbb{R}^n,B).$$

By the definition of the maximal norm, $\|\phi(f)\|\leq\|f\|$ for any $f\in C_c(\mathbb{R}^n,B)$, and hence, ϕ extends uniquely to a $*$-homomorphism $\phi: B\rtimes_{\bar{\beta}}\mathbb{R}^n\to B\rtimes_{\beta}\mathbb{T}^n$. Elementary considerations reveal that for every $g\in C(\mathbb{T}^n)$ there is a function $f\in C_c(\mathbb{R}^n)$ with compact support satisfying

$$g(p(x))=\sum_{n\in\mathbb{Z}^n}f(x+n)\quad\text{for}\quad x\in\mathbb{R}^n.$$

We conclude that the image of $\phi: B\rtimes_{\bar{\beta}}\mathbb{R}^n\to B\rtimes_{\beta}\mathbb{T}^n$ contains the algebraic tensor product $C(\mathbb{T}^n)\odot B$, and thus ϕ is surjective. □

The natural surjection $[0,1]^n \to \mathbb{T}^n$ induces a $C(\mathbb{T}^n)$-algebra structure on $\mathcal{M}_\alpha(A)$ with all fibres being isomorphic to A. The canonical surjection onto the fibre at $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{T}^n$ corresponds to $\mathrm{ev}_t : \mathcal{M}_\alpha(A) \to A$, where $t = (t_1, \ldots, t_n) \in [0,1]^n$ satisfies $\lambda_j = \exp(2\pi i t_j)$ for $j = 1, \ldots, n$. It is evident that $\mathcal{M}_\alpha(A)$ is even a continuous C^*-bundle over $\mathbb{T}^n$ in this way.

Let us recall the following fact about $C(X)$-linear $*$-homomorphisms.

Proposition 1.2.4. *Let A and B be $C(X)$-algebras with X being a compact Hausdorff space. Assume that $\varphi : A \to B$ is a $C(X)$-linear $*$-homomorphism, and let $\varphi_x : A(x) \to B(x)$ denote the associated $*$-homomorphism between the fibres at $x \in X$. Then the following statements hold true:*

i) If all $\varphi_x : A(x) \to B(x)$ are injective, then φ is injective.

ii) If B is a continuous bundle over X and if all φ_x are surjective, then φ is surjective as well.

Proof. Given $a \in A$ and $x \in X$, we write $a(x) \in A(x)$ for the image under the respective quotient map, and similarly for $b \in B$. If all fibre homomorphisms are injective, then

$$\|\varphi(a)\| = \max_{x \in X} \|\varphi(a(x))\| = \max_{x \in X} \|a(x)\| = \|a\|$$

by [6, 2.8], and thus φ is injective.

Concerning *ii)*, let $\varepsilon > 0$ and take an arbitrary element $b \in B$. For every $x \in X$, there is an $a \in A$ satisfying $\varphi(a)(x) = b(x)$. Since X is compact and since B is a continuous bundle over X, we find elements $a_1, \ldots, a_k \in A$ and a finite open covering $U_1, \ldots, U_k$ of X satisfying

$$\|\varphi(a_j)(x) - b(x)\| < \varepsilon \quad \text{for all} \quad x \in U_j \text{ and } j = 1, \ldots, n.$$

Choose a partition of unity $f_1, \ldots, f_k$ subordinate to the open covering $U_1, \ldots, U_k$ and define $a := \sum_{j=0}^{k} f_k a_k \in A$. Using that φ is $C(X)$-linear, we get

$$\begin{aligned} \|\varphi(a)(x) - b(x)\| &= \|\sum_{j=0}^{k} f_j(x)(\varphi(a_j)(x) - b(x))\| \\ &\leq \sum_{j=0}^{k} f_j(x) \|\varphi(a_j)(x) - b(x)\| \\ &< \varepsilon \end{aligned}$$

for all $x \in X$, since $f_j(x) = 0$ whenever $x \notin U_j$. Another application of [6, 2.8] yields $\|\varphi(a) - b\| < \varepsilon$, and hence the proof is complete by noting that the image of φ is closed. □

The next result is an application of [31, 2.4], and remedies the lack of injectivity of $\phi : B \rtimes_{\bar{\beta}} \mathbb{R}^n \to B \rtimes_\beta \mathbb{T}^n$.

Theorem 1.2.5. *Let $(B, \beta, \mathbb{T}^n)$ be a C^*-dynamical system and denote by $\bar{\beta}$ the induced $\mathbb{R}^n$-action on B. There exists a $*$-isomorphism*

$$\psi_\beta : B \rtimes_{\bar{\beta}} \mathbb{R}^n \xrightarrow{\cong} \mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n)$$

which is natural in the sense that if $\varphi : (B, \beta, \mathbb{T}^n) \to (C, \gamma, \mathbb{T}^n)$ is an equivariant $*$-homomorphism, then the following diagram commutes

$$\begin{tikzcd}
B \rtimes_{\bar{\beta}} \mathbb{R}^n \arrow[r, "\psi_\beta"] \arrow[d, "\varphi \rtimes \mathbb{R}^n"'] & \mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n) \arrow[d, "\widetilde{\varphi \rtimes \mathbb{T}^n}"] \\
C \rtimes_{\bar{\gamma}} \mathbb{R}^n \arrow[r, "\psi_\gamma"] & \mathcal{M}_{\hat{\gamma}}(C \rtimes_\gamma \mathbb{T}^n)
\end{tikzcd}$$

Proof. Using the identification $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$ mentioned in the first section, the dual action $\hat{\bar{\beta}}$ on $B \rtimes_{\bar{\beta}} \mathbb{R}^n$ is given by

$$\hat{\bar{\beta}}_t(f)(s) = \left(\prod_{j=1}^n \exp(2\pi i s_j t_j) \right) f(s) \quad \text{for} \quad f \in C_c(\mathbb{R}^n, B) \text{ and } s, t \in \mathbb{R}^n.$$

Let $\phi : B \rtimes_{\bar{\beta}} \mathbb{R}^n \to B \rtimes_\beta \mathbb{T}^n$ be the surjective $*$-homomorphism from Lemma 1.2.3. For every element $x \in B \rtimes_\beta \mathbb{T}^n$, we define a continuous map

$$\psi_\beta(x) : \mathbb{R}^n \longrightarrow B \rtimes_\beta \mathbb{T}^n, \quad \psi_\beta(x)(t) := \phi(\hat{\bar{\beta}}_t(x)).$$

Moreover, if $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{Z}^n$, then

$$\begin{aligned}
\phi(\hat{\bar{\beta}}_{y+e_j}(f))(p(x)) &= \sum_{z \in \mathbb{Z}^n} \hat{\bar{\beta}}_{y+e_j}(f)(x+z) \\
&= \exp(2\pi i x_j) \sum_{z \in \mathbb{Z}^n} \left(\textstyle\prod_{k=1}^n \exp(2\pi i y_k(x_k + z_k)) \right) f(x+z) \\
&= \exp(2\pi i x_j) \phi(\hat{\bar{\beta}}_y(f))(p(x)) \\
&= \hat{\beta}_j(\phi(\hat{\bar{\beta}}_y)(f))(p(x))
\end{aligned}$$

for every $f \in C_c(\mathbb{R}^n, B)$. Altogether, we have defined a $*$-homomorphism

$$\psi_\beta : B \rtimes_{\bar{\beta}} \mathbb{R}^n \longrightarrow \mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n)$$

in this way. Considering $B \rtimes_{\bar{\beta}} \mathbb{T}^n$ and $\mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n)$ as $C(\mathbb{T}^n)$-algebras, a straightforward computation reveals that ψ_β is even $C(\mathbb{T}^n)$-linear.

Let $x \in B \rtimes_\beta \mathbb{T}^n$ be an arbitrary element, and find $\bar{x} \in B \rtimes_{\bar{\beta}} \mathbb{R}^n$ with $\phi(\bar{x}) = x$. Then

$$\psi_\beta(\hat{\bar{\beta}}_{-t}(\bar{x}))(t) = \phi(\hat{\bar{\beta}}_t(\hat{\bar{\beta}}_{-t}(\bar{x}))) = \phi(\bar{x}) = x,$$

showing that $\mathrm{ev}_t \circ \psi_\beta$ is surjective for each $t \in \mathbb{R}^n$. Hence, all fibre homomorphisms $(\psi_\beta)_\lambda$ are surjective. Since $\mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n)$ is a continuous bundle over $\mathbb{T}^n$, we are in the situation of Proposition 1.2.4 *ii)* and deduce that ψ_β is a surjective $C(\mathbb{T}^n)$-linear $*$-homomorphism.

Now suppose that ψ_β is not injective. For $x \in \ker(\psi_\beta)$ and $r, s \in \mathbb{R}^n$, we have that

$$\psi_\beta(\hat{\hat{\beta}}_r(x))(s) = \phi(\hat{\hat{\beta}}_s(\hat{\hat{\beta}}_r(x))) = \phi(\hat{\hat{\beta}}_{r+s}(x)) = \psi_\beta(x)(r+s) = 0,$$

showing that $\ker(\psi_\beta) \subseteq B \rtimes_{\bar{\beta}} \mathbb{R}^n$ is a non-trivial $\hat{\hat{\beta}}$-invariant ideal. Consider the unique strictly continuous extension

$$\underline{\psi_\beta} : \mathcal{M}(B \rtimes_{\bar{\beta}} \mathbb{R}^n) \longrightarrow \mathcal{M}(\mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n)),$$

which exists since ψ_β is surjective. The proof of [34, 7.9.6] reveals that the $*$-homomorphism

$$B \xrightarrow{\iota_B} \mathcal{M}(B \rtimes_{\bar{\beta}} \mathbb{R}^n) \xrightarrow{\underline{\psi_\beta}} \mathcal{M}(\mathcal{M}_{\hat{\beta}}(B \rtimes_\beta \mathbb{T}^n))$$

is not injective as well. If $b \in \ker(\underline{\psi_\beta} \circ \iota_B)$, then for every $f \in C_c(\mathbb{R}^n)$ it holds that

$$0 = \underline{\psi_\beta}(f)\underline{\psi_\beta}(\iota_B(b)) = \psi_\beta(f\iota_B(b)).$$

However, a function $f \in C_c(\mathbb{R}^n)$ with $f(0) = 1$ and $\mathrm{supp}(f) \subseteq [0,1]^n$ yields the contradiction

$$0 = \psi_\beta(f\iota_B(b))(1)(0) = b \sum_{z \in \mathbb{Z}^n} \hat{\hat{\beta}}_0(f)(z) = b.$$

We conclude that ψ_β is injective. $\square$

We now have all ingredients to establish the K-theoretical isomorphism between the crossed product and the mapping torus associated with a given C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$.

Theorem 1.2.6. *For every C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$, there is an isomorphism*

$$A \rtimes_\alpha \mathbb{Z}^n \rtimes_{\bar{\alpha}} \mathbb{R}^n \cong \mathcal{M}_\alpha(A) \otimes \mathcal{K}(L^2(\mathbb{Z}^n)).$$

In particular, we get an isomorphism

$$\Psi_\alpha : K_*(A \rtimes_\alpha \mathbb{Z}^n) \xrightarrow{\cong} K_{*+n}(\mathcal{M}_\alpha(A))$$

by applying Connes' Thom isomorphism n times. The isomorphism Ψ_α is natural in the following sense. If $\varphi : (A, \alpha, \mathbb{Z}^n) \to (B, \beta, \mathbb{Z}^n)$ is an equivariant $*$-homomorphism, then the following diagram commutes

$$\begin{tikzcd}
K_*(A \rtimes_\alpha \mathbb{Z}^n) \arrow[r, "\Psi_\alpha"] \arrow[d, "K_*(\varphi \rtimes \mathbb{Z}^n)"'] & K_{*+n}(\mathcal{M}_\alpha(A)) \arrow[d, "K_{*+n}(\widetilde{\varphi})"] \\
K_*(B \rtimes_\beta \mathbb{Z}^n) \arrow[r, "\Psi_\beta"] & K_{*+n}(\mathcal{M}_\beta(B))
\end{tikzcd}$$

Proof. By applying Theorem 1.2.5 to the equivariant $*$-homomorphism

$$\varphi \rtimes \mathbb{Z}^n : (A \rtimes_\alpha \mathbb{Z}^n, \hat{\alpha}, \mathbb{T}^n) \longrightarrow (B \rtimes_\beta \mathbb{Z}^n, \hat{\beta}, \mathbb{T}^n),$$

we obtain a commutative diagram

$$\begin{tikzcd}
A \rtimes_\alpha \mathbb{Z}^n \rtimes_{\bar{\hat{\alpha}}} \mathbb{R}^n \arrow[r, "\psi_{\hat{\alpha}}"] \arrow[d, "\varphi\rtimes\mathbb{Z}^n\rtimes\mathbb{R}^n"'] & \mathcal{M}_{\hat{\hat{\alpha}}}(A \rtimes_\alpha \mathbb{Z}^n \rtimes_{\hat{\alpha}} \mathbb{T}^n) \arrow[d, "\varphi\rtimes\widetilde{\mathbb{Z}^n\rtimes}\mathbb{T}^n"] \\
B \rtimes_\beta \mathbb{Z}^n \rtimes_{\bar{\hat{\beta}}} \mathbb{R}^n \arrow[r, "\psi_{\hat{\beta}}"] & \mathcal{M}_{\hat{\hat{\beta}}}(B \rtimes_\beta \mathbb{Z}^n \rtimes_{\hat{\beta}} \mathbb{T}^n)
\end{tikzcd}$$

Since Takai duality respects equivariant $*$-homomorphisms, we also get the following commutative diagram

$$\begin{tikzcd}
A \rtimes_\alpha \mathbb{Z}^n \rtimes_{\bar{\hat{\alpha}}} \mathbb{R}^n \arrow[r, "\cong"] \arrow[d, "\varphi\rtimes\mathbb{Z}^n\rtimes\mathbb{R}^n"'] & \mathcal{M}_{\alpha\otimes\rho}(A \otimes \mathcal{K}(L^n(\mathbb{Z}^n))) \arrow[r, "\cong"] \arrow[d, "\widetilde{\varphi\otimes\mathrm{id}}"] & \mathcal{M}_\alpha(A) \otimes \mathcal{K}(L^n(\mathbb{Z}^n)) \arrow[d, "\widetilde{\varphi}\otimes\mathrm{id}"] \\
B \rtimes_\beta \mathbb{Z}^n \rtimes_{\bar{\hat{\beta}}} \mathbb{R}^n \arrow[r, "\cong"] & \mathcal{M}_{\beta\otimes\rho}(B \otimes \mathcal{K}(L^n(\mathbb{Z}^n))) \arrow[r, "\cong"] & \mathcal{M}_\beta(B) \otimes \mathcal{K}(L^n(\mathbb{Z}^n))
\end{tikzcd}$$

Here, ρ denotes the right regular representation, and the right hand diagram is induced by the isomorphism from Proposition 1.2.2. Naturality of Connes' Thom isomorphism completes the proof. □

Chapter 2

Bott elements associated with almost commuting unitaries

Generalizing the classical Bott element in $K_0(C_0(\mathbb{R}^2))$, Loring [26] has presented a concrete method of associating a K_0-element to a pair of unitaries $u, v \in A$ with sufficiently small commutator $[u, v]$. Indeed, if $A = C(\mathbb{T}^2)$, then Loring's element associated with the first and second coordinate function coincides with the Bott element $\mathfrak{b} \in K_0(C_0((0,1)^2)) \subseteq K_0(C(\mathbb{T}^2))$. In [14], Exel generalized this definition to pairs of unitaries whose commutator has norm less than 2. We recall both definitions and show some useful properties of these so-called Bott elements.

Consider the following extension of C^*-algebras

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \otimes C(\mathbb{T}) \arrow[r] & \mathcal{T} \otimes C(\mathbb{T}) \arrow[r] & C(\mathbb{T}) \otimes C(\mathbb{T}) \arrow[r] & 0. \end{tikzcd}$$

Up to a sign, the Bott element $\mathfrak{b} \in K_0(C(\mathbb{T}^2))$ is characterized by the property that its image under the index map $\rho_0 : K_0(C(\mathbb{T}^2)) \to K_1(C(\mathbb{T}))$ is a generator for $K_1(C(\mathbb{T}))$. We fix the convention that $\rho_0(\mathfrak{b}) = [z]$.

For $\varepsilon > 0$, define the *soft torus* A_ε [14] as

$$A_\varepsilon := C^*(u_\varepsilon,\ v_\varepsilon \text{ unitaries} \quad : \quad \|[u_\varepsilon, v_\varepsilon]\| \leq \varepsilon).$$

It is obvious from the definition that $A_0 = C(\mathbb{T}^2)$, and that for $\varepsilon \geq 2$ the soft torus A_ε coincides with the full group C^*-algebra of the free group in two generators. There is a canonical surjective $*$-homomorphism

$$\varphi_\varepsilon : A_\varepsilon \longrightarrow C(\mathbb{T}^2) \quad \text{with} \quad \varphi(u_\varepsilon) := z_1,\ \varphi(v_\varepsilon) := z_2.$$

By [14, 2.4], $K_*(\varphi_\varepsilon)$ is an isomorphism whenever $\varepsilon < 2$, and in this case we define

$$\mathfrak{b}_\varepsilon := K_0(\varphi_\varepsilon)^{-1}(\mathfrak{b}) \in K_0(A_\varepsilon).$$

Let B be a unital C^*-algebra, and let $u, v \in B$ be unitaries satisfying $\|[u, v]\| \leq \varepsilon < 2$. The universal property of the soft torus A_ε yields a unique

$*$-homomorphism $\varphi : A_\varepsilon \to B$ with $\varphi(u_\varepsilon) = u$ and $\varphi(v_\varepsilon) = v$. Define the *Bott element* associated with u and v as

$$\kappa(u,v) := K_0(\varphi)(\mathfrak{b}_\varepsilon) \in K_0(B).$$

Note that $\kappa(u,v)$ is independent of ε as long as $\|[u,v]\| \leq \varepsilon$. By definition, $\kappa(z_1,z_2) = \mathfrak{b} \in K_0(C(\mathbb{T}^2))$.

For small tolerance $\varepsilon > 0$, the Bott element $\kappa(u,v)$ is given (up to a sign) by the following description due to Loring [26]. Consider the real-valued functions $f,g,h \in C(\mathbb{T})$ defined as

$$f(e^{2\pi it}) = \begin{cases} 1-2t & , \quad \text{if } 0 \leq t \leq 1/2, \\ -1+2t & , \quad \text{if } 1/2 \leq t \leq 1, \end{cases}$$

$$g(e^{2\pi it}) = \begin{cases} (f(\exp(2\pi it)) - f(\exp(2\pi it))^2)^{1/2} & , \quad \text{if } 0 \leq t \leq 1/2, \\ 0 & , \quad \text{if } 1/2 \leq t \leq 1, \end{cases}$$

$$h(e^{2\pi it}) = \begin{cases} 0 & , \quad \text{if } 0 \leq t \leq 1/2, \\ (f(\exp(2\pi it)) - f(\exp(2\pi it))^2)^{1/2} & , \quad \text{if } 1/2 \leq t \leq 1, \end{cases}$$

and set

$$e(u,v) := \begin{pmatrix} f(v) & g(v)+h(v)u \\ g(v)+u^*h(v) & 1-f(v) \end{pmatrix} \in M_2(B).$$

Observe that $e(u,v)$ is self-adjoint for any choice of unitaries u,v. Moreover, direct calculations show that $e(u,v)$ is a projection whenever u and v commute. Loring observed in [26, 3.5] that there is a universal constant $\delta > 0$ such that whenever $\|[u,v]\| < \delta$, then the spectrum of $e(u,v)$ does not contain $\frac{1}{2}$. In this case, $\chi_{[1/2,\infty)}(e(u,v)) \in M_2(B)$ is a projection, and Loring's Bott element is given as

$$[\chi_{[1/2,\infty)}(e(u,v))] - [1] \in K_0(B).$$

It follows directly from the definition of the Bott elements that for any unital $*$-homomorphism $\varphi : A \to B$ and for any unitaries $u,v \in A$ with $\|[u,v]\| < 2$, we have that

$$K_0(\varphi)(\kappa(u,v)) = \kappa(\varphi(u),\varphi(v)).$$

The Bott elements also have the following properties.

Proposition 2.1. *Let $u,v,u_1,v_1,\dots,u_n,v_n \in B$ be unitary elements in a unital C^*-algebra B. Then the following statements hold true:*

i) If $u_t \in B$ is a homotopy of unitaries with $\|[u_t,v]\| < 2$ for all $t \in [0,1]$, then $\kappa(u_0,v) = \kappa(u_1,v)$.

ii) If $\|u_i - v_i\| < 2$ for $i = 1,\dots,n$, then

$$\kappa(\mathrm{diag}(u_1,\dots,u_n), \mathrm{diag}(v_1,\dots,v_n)) = \sum_{i=1}^{n} \kappa(u_i,v_i).$$

iii) If $\sum\limits_{i=1}^{n} \|[u, v_i]\| < 2$*, then* $\kappa(u, v_1 v_2 \dots v_n) = \sum\limits_{i=1}^{n} \kappa(u, v_i)$*.*

iv) If $\|[u, v]\| < 2$*, then* $\kappa(u, v) = -\kappa(u, v^*) = -\kappa(v, u)$*.*

Proof. For the proof of *i)*, first note that by compactness of $[0,1]$, there is an $\varepsilon < 2$ with $\|[u_t, v]\| \leq \varepsilon$ for all $t \in [0,1]$. For $i = 0, 1$, define a $*$-homomorphism $\varphi_i : A_\varepsilon \to B$ via $\varphi(u_\varepsilon) = u_i$ and $\varphi(v_\varepsilon) = v$. Moreover, let $\psi : A_\varepsilon \to C([0,1], B)$ be given by $\psi(u_\varepsilon)(t) = u_t$ and $\psi(v_\varepsilon)(t) = v$ for all $t \in [0,1]$. Then there is a commutative diagram

$$\begin{tikzcd}
A_\varepsilon \arrow[r, "\psi"] \arrow[dr, "\varphi_i"'] & C([0,1], B) \arrow[d, "\mathrm{ev}_i"] \\
 & B
\end{tikzcd}$$

for $i = 0, 1$. The claim of *i)* now follows from the naturality property of the Bott elements and the fact that $K_*(\mathrm{ev}_0) = K_*(\mathrm{ev}_1)$.

Concerning *ii)*, find an $\varepsilon < 2$ such that $\|u_i - v_i\| \leq \varepsilon$ for $i = 1, \dots, n$. By the universal property of the soft torus A_ε, we find $*$-homomorphisms $\varphi_i : A_\varepsilon \to B$ with $\varphi_i(u_\varepsilon) = u_i$ and $\varphi_i(v_\varepsilon) = v_i$. Let $\varphi : A_\varepsilon \to M_n(B)$ denote the unital $*$-homomorphism $\varphi := \mathrm{diag}(\varphi_1, \dots, \varphi_n)$. Additivity of K-theory implies

$$\begin{aligned} \kappa(\mathrm{diag}(u_1, \dots, u_n), \mathrm{diag}(v_1, \dots, v_n)) &= K_0(\varphi)(\mathfrak{b}_\varepsilon) \quad = \sum_{i=1}^{n} K_0(\varphi_i)(\mathfrak{b}_\varepsilon) \\ &= \textstyle\sum_{i=1}^{n} \kappa(u_i, v_i). \end{aligned}$$

We only show *iii)* in the case that $n = 2$. One computes that

$$\begin{aligned} \|[u, v_1 v_2]\| &\leq \|u v_1 v_2 - v_1 u v_2\| + \|v_1 u v_2 - v_1 v_2 u\| \\ &= \|u v_1 - v_1 u\| \quad\quad + \|u v_2 - v_2 u\| \\ &< 2. \end{aligned}$$

Hence, the Bott element $\kappa(u, v_1 v_2) \in K_0(B)$ is well-defined. Also observe that $e(u, 1) = \mathrm{diag}(1, 0)$, which leads to $\kappa(u, 1) = 0 \in K_0(B)$. Using *ii)*, this implies

$$\kappa(u, v_1 v_2) = \kappa(\mathrm{diag}(u, u), \mathrm{diag}(v_1 v_2, 1)).$$

Let $w_t \in \mathcal{U}_2(\mathbb{C})$ be a homotopy with

$$w_0 = 1_2 \qquad \text{and} \qquad w_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $\mathrm{diag}(v_1, 1) w_t \,\mathrm{diag}(v_2, 1) w_t^*$ defines a homotopy between $\mathrm{diag}(v_1 v_2, 1)$ and $\mathrm{diag}(v_1, v_2)$. Since w_t and $\mathrm{diag}(u, u)$ commute, a similar computation as above shows that

$$\begin{aligned} &\|[\mathrm{diag}(v_1, 1) w_t \,\mathrm{diag}(v_2, 1) w_t^*, \mathrm{diag}(u, u)]\| \\ &\qquad\qquad \leq \|[\mathrm{diag}(v_1, 1) w_t, \mathrm{diag}(u, u)]\| + \|[\mathrm{diag}(v_2, 1) w_t^*, \mathrm{diag}(u, u)]\| \\ &\qquad\qquad < 2. \end{aligned}$$

Now *iii)* follows from *i)*. Since $\kappa(1, u) = 0$, one can use basically the same arguments to show the analogous statement of *iii)* with the rôles of the two coordinates reversed.

Applying *iii)* to the coordinate functions $z_1, z_2 \in C(\mathbb{T}^2)$, one gets that

$$\kappa(z_1, z_2) + \kappa(z_1, z_2^*) = \kappa(z_1, z_2 z_2^*) = \kappa(z_1, 1) = 0.$$

This shows that $\kappa(u_\varepsilon, v_\varepsilon) = -\kappa(u_\varepsilon, v_\varepsilon^*) \in K_0(A_\varepsilon)$ for all $\varepsilon < 2$, and hence that $\kappa(u, v) = -\kappa(u, v^*)$ for all unitaries u, v with $\|[u, v]\| < 2$. Similarly, we see that $\kappa(u, u) = 0$ for every unitary u. An application of *ii)* therefore yields

$$\begin{aligned} \kappa(u, v) + \kappa(v, u) &= \kappa(u, v) + \kappa(v, v) + \kappa(v, u) + \kappa(u, u) \\ &= \kappa(uv, v) + \kappa(uv, u) \\ &= \kappa(uv, uv) \\ &= 0. \end{aligned}$$

□

As a consequence of Proposition 2.1, we get that $\kappa(u, v) = 0$ whenever u and v commute and v does not have full spectrum. In fact, there is a homotopy between v and 1 consisting of unitaries commuting with u. Moreover, we recover the following well-known fact.

Corollary 2.2. *Let $n \geq 2$ and $\sigma \in \Sigma_n$ be a permutation. Consider the automorphism $\varphi_\sigma \in C_0(\mathbb{R}^n)$ given by*

$$\varphi_\sigma(f)(t_1, \ldots, t_n) = f(t_{\sigma(1)}, \ldots, t_{\sigma(n)}).$$

Then $K_*(\varphi_\sigma) = \mathrm{sgn}(\sigma) \cdot \mathrm{id}$.

Proof. For $n = 2$, the claim follows from part *iv)* of Proposition 2.1. Given an arbitrary permutation $\sigma \in \Sigma_n$, we write it as a product of elementary transpositions $\sigma = \tau_1 \circ \ldots \circ \tau_k$. Naturality of the Künneth formula yields $K_*(\varphi_{\tau_j}) = \mathrm{sgn}(\tau_j) \cdot \mathrm{id}$ for $j = 1, \ldots, k$. The proof is complete since

$$K_*(\varphi_\sigma) = K_*(\varphi_{\tau_1}) \circ \ldots \circ K_*(\varphi_{\tau_k}) = \mathrm{sgn}(\tau_1) \cdot \ldots \cdot \mathrm{sgn}(\tau_k) \cdot \mathrm{id} = \mathrm{sgn}(\sigma) \cdot \mathrm{id}\,.$$

□

In the case that A is a unital, purely infinite and simple C^*-algebra, Elliott and Rørdam showed in [13, 2.2.1] that every element $x \in K_0(A)$ is a Bott element $x = \kappa(u, v)$ for some pair of commuting unitaries $u, v \in A$ (with full spectrum).

On the other hand, if the unital C^*-algebra A has a tracial state τ, then there are certain restrictions to the existence of Bott elements in $K_0(A)$. Indeed, if τ_n denotes the induced (unnormalized) trace on $M_n(A)$, then all unitaries $u, v \in A$ satisfy

$$\tau_2(e(u, v)) = \tau(f(u)) + 1 - \tau(f(u)) = 1,$$

Hence, by Loring's definition of the Bott elements, we see that $\kappa(u, v)$ vanishes under the induced state $K_0(\tau) : K_0(A) \to \mathbb{R}$ whenever u and v commute. In particular, in the presence of a trace, $[1] \in K_0(A)$ is never a Bott element associated with two exactly commuting unitaries.

For non-commuting unitaries, the situation is different, as the following example illustrates. For $n \in \mathbb{N}$, let $u_n, v_n \in M_n(\mathbb{C})$ denote the *Voiculescu matrices* [48] which are given by $u_n(e_k) = e_{k+1}$ (we count modulu n) and $v_n = \operatorname{diag}(\zeta, \zeta^2, \ldots, \zeta^n)$ with $\zeta = \exp(\frac{2\pi i}{n}) \in \mathbb{C}$. It is clear that $\|[u_n, v_n]\|$ converges to 0. On the other hand, it was shown in [26] that for large n,

$$\tau_{2n}(\chi_{[1/2,\infty)}(e(u_n, v_n))) = n - 1.$$

In other words, if n is big enough, then $\chi_{[1/2,\infty)}(e(u_n, v_n)) \in M_{2n}(\mathbb{C})$ is a projection of dimension $n - 1$. In particular, for such a number $n \in \mathbb{N}$, the associated Bott element $\kappa(u_n, v_n) \in K_0(\mathbb{C}) \cong \mathbb{Z}$ is a generator. Loring used this to recover Voiculescu's result [48] that the commuting unitaries

$$(u_n),\ (v_n) \in \frac{\prod M_n(\mathbb{C})}{\sum M_n(\mathbb{C})}$$

do not lift to commuting unitaries in $\prod M_n(\mathbb{C})$, which shows that $C(\mathbb{T}^2)$ is not weakly semiprojective.

Chapter 3

The Pimsner-Voiculescu sequence

This chapter is devoted to the Pimsner-Voiculescu exact sequence. In the first section we recall the main ideas of the original proof of Pimsner and Voiculescu in [36], where the Pimsner-Voiculescu sequence is derived from the six-term sequence of an extension of the crossed product $A \rtimes_\alpha \mathbb{Z}$ by $A \otimes \mathcal{K}$, the so-called *Toeplitz extension.*

In the second section we give a concrete description for preimages under the boundary map $\rho_* : K_*(A \rtimes_\alpha \mathbb{Z}) \to K_{*+1}(A)$ of the Pimsner-Voiculescu sequence. The lifts for ρ_1 are well-known, and we obtain them by using the partial isometry picture of the index map. However, more work is required in order to get lifts for ρ_0. The standard picture of the K_1-group in terms of stable homotopy classes of unitaries seems to be inappropriate for this purpose. Instead, we use a deep result by Dadarlat [12], which implies that two unitaries describe the same K_1-class if and only if they are stably approximately unitarily equivalent. This enables us to find lifts for ρ_0 in terms of Bott elements associated with almost commuting unitaries.

3.1 Recalling the Pimsner-Voiculescu sequence

Let A be a unital C^*-algebra and $\alpha \in \mathrm{Aut}(A)$ a $*$-automorphism on A. Consider its *crossed Toeplitz-algebra* [10]

$$\mathcal{T}(A, \alpha) := C^*(1 \otimes a,\ v \otimes u\ :\ a \in A) \subseteq \mathcal{T} \otimes (A \rtimes_\alpha \mathbb{Z}),$$

where $u \in A \rtimes_\alpha \mathbb{Z}$ is the canonical unitary implementing α. The natural surjection $\mathcal{T} \to \mathbb{C}$ gives rise to the *Toeplitz extension*

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \otimes A \arrow[r] & \mathcal{T}(A, \alpha) \arrow[r] & A \rtimes_\alpha \mathbb{Z} \arrow[r] & 0. \end{tikzcd} \tag{3.1}$$

Here, $\mathcal{K} \otimes A$ is identified with the ideal in $\mathcal{T}(A, \alpha)$ generated by all elements of the form $(1 - vv^*) \otimes a$ with $a \in A$. Let $e \in \mathcal{K}$ denote the minimal projection which has the property that $e \otimes 1$ corresponds to $(1 - vv^*) \otimes 1$ under this identification.

One crucial step in the proof is to recognize that the canonical embedding

$$\iota_\alpha : A \longrightarrow \mathcal{T}(A, \alpha), \quad \iota_\alpha(a) = 1 \otimes a,$$

induces an isomorphism in K-theory. In fact, for separable C^*-algebras, this map is even a KK-equivalence [10]. Using this isomorphism, we can replace $K_*(\mathcal{T}(A,\alpha))$ by $K_*(A)$ in the six-term exact sequence associated with (3.1). This yields the *Pimsner-Voiculescu exact sequence*

$$\begin{tikzcd}
K_0(A) \arrow[r, "K_0(\alpha)-\mathrm{id}"] & K_0(A) \arrow[r, "K_0(j)"] & K_0(A \rtimes_\alpha \mathbb{Z}) \arrow[d, "\rho_0"] \\
K_1(A \rtimes_\alpha \mathbb{Z}) \arrow[u, "\rho_1"] & K_1(A) \arrow[l, "K_1(j)"] & K_1(A) \arrow[l, "K_1(\alpha)-\mathrm{id}"]
\end{tikzcd}$$

in the unital case. Here, $j : A \to A \rtimes_\alpha \mathbb{Z}$ denotes the canonical embedding.

The Pimsner-Voiculescu sequence has the following naturality property.

Proposition 3.1.1. *Let $\varphi : (A, \alpha, \mathbb{Z}) \to (B, \beta, \mathbb{Z})$ be a unital equivariant $*$-homomorphism between unital C^*-algebras. Then the following diagram commutes:*

$$\begin{tikzcd}
K_*(A) \arrow[r, "K_*(\alpha)-\mathrm{id}"] \arrow[d, "K_*(\varphi)"] & K_*(A) \arrow[r] \arrow[d, "K_*(\varphi)"] & K_*(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\rho_*"] \arrow[d, "K_*(\check{\varphi})"] & K_{*+1}(A) \arrow[d, "K_{*+1}(\varphi)"] \\
K_*(B) \arrow[r, "K_*(\beta)-\mathrm{id}"] & K_*(B) \arrow[r] & K_*(B \rtimes_\beta \mathbb{Z}) \arrow[r, "\rho_*"] & K_{*+1}(B)
\end{tikzcd}$$

Proof. The $*$-homomorphism $\psi : \mathcal{T}(A, \alpha) \to \mathcal{T}(B, \beta)$ given by

$$\psi(1 \otimes a) = 1 \otimes \varphi(a) \quad \text{and} \quad \psi(v \otimes u) = v \otimes u$$

induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes A \arrow[r] \arrow[d, "\psi'"] & \mathcal{T}(A, \alpha) \arrow[r] \arrow[d, "\psi"] & A \rtimes_\alpha \mathbb{Z} \arrow[r] \arrow[d, "\check{\varphi}"] & 0 \\
0 \arrow[r] & \mathcal{K} \otimes B \arrow[r] & \mathcal{T}(B, \beta) \arrow[r] & B \rtimes_\beta \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

For all $a \in A$, we have that $\psi'(e \otimes a) = e \otimes \varphi(a)$, and hence the stabilization isomorphism intertwines $K_*(\psi')$ and $K_*(\varphi)$. Furthermore, the diagram

$$\begin{tikzcd}
A \arrow[r, "\iota_\alpha"] \arrow[d, "\varphi"] & \mathcal{T}(A, \alpha) \arrow[d, "\psi"] \\
B \arrow[r, "\iota_\beta"] & \mathcal{T}(B, \beta)
\end{tikzcd}$$

commutes, so that the claim now follows by the naturality of K-theory and the fact that $K_*(\iota_\alpha)$ and $K_*(\iota_\beta)$ are isomorphisms. $\square$

Using the naturality property, one can now deduce the Pimsner-Voiculescu sequence for arbitrary C^*-dynamical systems $(A, \alpha, \mathbb{Z})$. By passing to the unitization $(A^{\sim}, \alpha^{\sim}, \mathbb{Z})$, we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & A^{\sim} \arrow[r] \arrow[d, "\alpha^{\sim}"] & \mathbb{C} \arrow[r] \arrow[l, bend right] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r] & A^{\sim} \arrow[r] & \mathbb{C} \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

This leads to a split-exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A \rtimes_{\alpha} \mathbb{Z} \arrow[r] & A^{\sim} \rtimes_{\alpha^{\sim}} \mathbb{Z} \arrow[r] & \mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z} \arrow[r] \arrow[l, bend right] & 0.
\end{tikzcd}$$

Since K-theory is split-exact, we end up with the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K_*(A) \arrow[r] \arrow[d, "K_*(\alpha)-\mathrm{id}"] & K_*(A^{\sim}) \arrow[r] \arrow[d, "K_*(\alpha^{\sim})-\mathrm{id}"] & K_*(\mathbb{C}) \arrow[r] \arrow[l, bend right] \arrow[d, "0"] & 0 \\
0 \arrow[r] & K_*(A) \arrow[r] \arrow[d] & K_*(A^{\sim}) \arrow[r] \arrow[d] & K_*(\mathbb{C}) \arrow[r] \arrow[l, bend right] \arrow[d] & 0 \\
0 \arrow[r] & K_*(A \rtimes_{\alpha} \mathbb{Z}) \arrow[r] \arrow[d, "\rho_*"] & K_*(A^{\sim} \rtimes_{\alpha^{\sim}} \mathbb{Z}) \arrow[r] \arrow[d, "\rho_*"] & K_*(\mathbb{C} \rtimes_{\mathrm{id}} \mathbb{Z}) \arrow[r] \arrow[l, bend right] \arrow[d, "\rho_*"] & 0 \\
0 \arrow[r] & K_{*+1}(A) \arrow[r] & K_{*+1}(A^{\sim}) \arrow[r] & K_{*+1}(\mathbb{C}) \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

Hence, the Pimsner-Voiculescu sequence exists for every C^*-dynamical system $(A, \alpha, \mathbb{Z})$. Proposition 3.1.1 also implies that the Pimsner-Voiculescu sequence is natural with respect to arbitrary equivariant $*$-homomorphisms.

In general, the K-theory of a crossed product $A \rtimes_{\alpha} \mathbb{Z}$ does not only depend on $K_*(\alpha)$, see [40, 10.6]. However, it is true that $K_*(A \rtimes_{\alpha} \mathbb{Z})$ is determined by the homotopy class of α. Paschke [32] showed this by using that homotopic automorphisms give rise to isomorphic mapping tori. The following is a well-known generalization of this result to $\mathbb{Z}^n$-actions.

Proposition 3.1.2. *Let α and β be homotopic $\mathbb{Z}^n$-actions on a C^*-algebra A. Then $K_*(A \rtimes_{\alpha} \mathbb{Z}^n) \cong K_*(A \rtimes_{\beta} \mathbb{Z}^n)$.*

Proof. Let γ be a homotopy between α and β, and consider the induced extension of C^*-algebras

$$\begin{tikzcd}
0 \arrow[r] & C_0((0,1], A) \rtimes_{\gamma} \mathbb{Z}^n \arrow[r] & C([0,1], A) \rtimes_{\gamma} \mathbb{Z}^n \arrow[r, "\mathrm{ev}_0 \rtimes \mathbb{Z}^n"] & A \rtimes_{\alpha} \mathbb{Z}^n \arrow[r] & 0.
\end{tikzcd}$$

As $C_0((0,1], A)$ is contractive, and hence has trivial K-theory, an iterative use of the Pimsner-Voiculescu sequence shows that

$$K_*(C_0((0,1], A) \rtimes_{\gamma} \mathbb{Z}^n) = 0.$$

The six-term exact sequence associated with the above extension therefore reveals that $K_*(\mathrm{ev}_0 \rtimes \mathbb{Z}^n)$ is an isomorphism. The analogous statement also holds for $K_*(\mathrm{ev}_1 \rtimes \mathbb{Z}^n)$, so that the composition

$$K_*(\mathrm{ev}_1 \rtimes \mathbb{Z}^n) \circ K_*(\mathrm{ev}_0 \rtimes \mathbb{Z}^n)^{-1} : K_*(A \rtimes_\alpha \mathbb{Z}^n) \xrightarrow{\cong} K_*(A \rtimes_\beta \mathbb{Z}^n)$$

yields the desired isomorphism. □

It is worth noticing that the above proof works in more generality. In fact, Proposition 3.1.2 is still true when we substitute $\mathbb{Z}^n$ by a locally compact group G with the property that for any C^*-dynamical system (C, γ, G) with a contractive C^*-algebra C we have that $K_*(C \rtimes_\gamma G) = 0$.

3.2 Lifts for the boundary maps of the Pimsner-Voiculescu sequence

Let us fix some notation, first. For an automorphism α on a C^*-algebra A, we write $\alpha^{(n)} := \alpha \otimes \mathrm{id} \in \mathrm{Aut}(M_n(A))$. In a similar fashion, we define $a^{(n)} := a \otimes 1 \in M_n(A)$ for a given element $a \in A$.

For any nuclear C^*-algebra B, consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes (A \otimes B) \arrow[r] \arrow[d, "\psi'"] & \mathcal{T}(A \otimes B, \alpha \otimes \mathrm{id}) \arrow[r] \arrow[d, "\psi"] & (A \otimes B) \rtimes_{\alpha \otimes \mathrm{id}} \mathbb{Z} \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & (\mathcal{K} \otimes A) \otimes B \arrow[r] & \mathcal{T}(A, \alpha) \otimes B \arrow[r] & A \rtimes_\alpha \mathbb{Z} \otimes B \arrow[r] & 0
\end{tikzcd} \tag{3.2}$$

where $\psi : \mathcal{T}(A \otimes B, \alpha \otimes \mathrm{id}) \to \mathcal{T}(A, \alpha) \otimes B$ is given by

$$\psi(1 \otimes (a \otimes b)) = (1 \otimes a) \otimes b \quad \text{and} \quad \psi(v \otimes u) = (v \otimes u) \otimes 1.$$

It can be shown that η is always an isomorphism, see [50, 2.75]. For $B = M_k(\mathbb{C})$, this induces a commutative diagram

$$\begin{tikzcd}
K_*(M_k(A) \rtimes_{\alpha^{(k)}} \mathbb{Z}) \arrow[r, "\rho_*^{(k)}"] \arrow[d, "K_*(\eta)"'] & K_{*+1}(A) \arrow[d, "\mathrm{id}"] \\
K_*(A \rtimes_\alpha \mathbb{Z}) \arrow[r, "\rho_*"'] & K_{*+1}(A)
\end{tikzcd} \tag{3.3}$$

relating the respective boundary maps. This allows one to transfer results holding for $(A, \alpha, \mathbb{Z})$ to $(M_n(A), \alpha^{(n)}, \mathbb{Z})$, and vice versa.

Let $p \in A$ be a projection and let $u \in A$ be a unitary commuting with p. Then $pup + 1 - p \in A$ is a unitary, and we define the *Bott element* associated with p and u as

$$\kappa(p, u) := [pup + 1 - p] \in K_1(A).$$

This notation is justified since the Bott isomorphism $K_0(A) \xrightarrow{\cong} K_1(SA)$ sends $[p]$ to $\kappa(p, z)$.

Given a unital C^*-algebra A, every element $g \in K_0(A)$ can be expressed as $g = [p] - [1_n]$ for some projection $p \in M_m(A)$ and $n \geq 0$. It is obvious that $g \in \ker(K_0(\alpha) - \mathrm{id})$ if and only if $[p] \in \ker(K_0(\alpha) - \mathrm{id})$. Hence, it suffices to describe lifts for elements of the form $[p] \in \mathrm{im}(\rho_1) = \ker(K_0(\alpha) - \mathrm{id})$. The following result is well-known.

Proposition 3.2.1. *Let A be a unital C^*-algebra, $\alpha \in \mathrm{Aut}(A)$, and let $p \in \mathcal{P}_k(A)$ be a projection satisfying $[p] \in \ker(K_0(\alpha) - \mathrm{id})$. By the standard picture of $K_0(A)$, we find $l, m \geq 0$ and a unitary $w \in \mathcal{U}_n(A)$ such that*

$$\alpha^{(n)}(p \oplus 1_l \oplus 0_m) = w(p \oplus 1_l \oplus 0_m)w^*,$$

where $n := k + l + m$. If $q := p \oplus 1_l \oplus 0_m \in \mathcal{P}_n(A)$, then

$$\rho_1(\kappa(q, w^* u^{(n)}) - [u^{(l)}]) = [p].$$

Proof. Assume first that $k = 1$ and $l = m = 0$. It is easy to verify that

$$y := v \otimes (pw^*up) + 1 \otimes (1 - p) \in \mathcal{T}(A, \alpha)$$

is an isometry and a lift for $pwu^*p + 1 - p \in A \rtimes_\alpha \mathbb{Z}$. Using the partial isometry picture of the index map, one computes

$$\begin{aligned} \rho_1(\kappa(p, w^*u)) &= [1 - yy^*] - [1 - y^*y] = [1 - yy^*] \\ &= [(1 - vv^*) \otimes p] \in K_0(\mathcal{K} \otimes A). \end{aligned}$$

By the stabilization isomorphism $K_0(A) \cong K_0(\mathcal{K} \otimes A)$, we deduce that

$$\rho_1(\kappa(p, w^*u)) = [p] \in K_0(A).$$

Now, let $q \in \mathcal{P}_n(A)$ be as in the statement, and recall the homomorphisms

$$\eta : M_n(A) \rtimes_{\alpha^{(k)}} \mathbb{Z} \to M_n(A \rtimes_\alpha \mathbb{Z}) \quad \text{and} \quad \rho_1^{(n)} : K_1(M_n(A) \rtimes_{\alpha^{(n)}} \mathbb{Z}) \to K_0(A)$$

from (3.2) and (3.3), respectively. Then

$$\rho_1(\kappa(q, w^*u^{(k)})) = (\rho_1 \circ K_1(\eta))(\kappa(q, w^*u)) = \rho_1^{(k)}(\kappa(q, w^*u)) = [q].$$

It follows that

$$\rho_1(\kappa(q, w^*u^{(k)}) - [u^l]) = [q] - [1_l] = [p \oplus 1_l] - [1_l] = [p],$$

and the proof is complete. □

The lifts for the boundary map $\rho_0 : K_0(A \rtimes_\alpha \mathbb{Z}) \to K_1(A)$ require an alternative picture for the K_1-group of a unital C^*-algebra. For this, we use the natural identification $K_*(A) \cong KK(C(\mathbb{T}), A)$ and Dadarlat's result [12, Theorem A] in the special case that $X = \mathbb{T}$. We then obtain the following characterization.

Theorem 3.2.2 ([12])**.** *Two unitaries $u, v \in A$ represent the same K_1-class if and only if they are* stably approximately unitarily equivalent*, i.e. for any $\varepsilon > 0$, there exist $k \geq 1$, $\lambda_1, \dots, \lambda_k \in \mathbb{T}$, and a unitary $w \in M_{k+1}(A)$ such that*

$$\|w(\operatorname{diag}(u, \lambda_1, \dots, \lambda_k))w^* - \operatorname{diag}(v, \lambda_1, \dots, \lambda_k)\| \leq \varepsilon.$$

For $\varepsilon > 0$ define the universal C^*-algebra

$$T_\varepsilon := C^*(s \text{ isometry},\ u \text{ unitary } : \ \|[s, u]\| \leq \varepsilon \text{ and } u(1 - ss^*) = (1 - ss^*)u),$$

and recall the definition of the soft torus A_ε from Chapter 2. Consider the canonical surjection $\pi_\varepsilon : T_\varepsilon \to A_\varepsilon$ given by $\pi_\varepsilon(s) = u_\varepsilon$ and $\pi_\varepsilon(u) = v_\varepsilon$, and observe that the surjective $*$-homomorphism $\psi_\varepsilon : T_\varepsilon \to \mathcal{T} \otimes C(\mathbb{T})$ given by $\psi_\varepsilon(s) = v$ and $\psi_\varepsilon(u) = z$ fits into the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes C(\mathbb{T}) \arrow[r] \arrow[d, "\mathrm{id}"] & T_\varepsilon \arrow[r, "\pi_\varepsilon"] \arrow[d, "\psi_\varepsilon"] & A_\varepsilon \arrow[r] \arrow[d, "\varphi_\varepsilon"] & 0 \\
0 \arrow[r] & \mathcal{K} \otimes C(\mathbb{T}) \arrow[r] & \mathcal{T} \otimes C(\mathbb{T}) \arrow[r] & C(\mathbb{T}) \otimes C(\mathbb{T}) \arrow[r] & 0
\end{tikzcd}$$

Naturality of K-theory allows us to compare the occurring boundary maps

$$\begin{tikzcd}
K_0(A_\varepsilon) \arrow[rd, "\rho_\varepsilon"] \arrow[dd, "K_0(\varphi_\varepsilon)"'] & \\
& K_1(C(\mathbb{T})) \\
K_0(C(\mathbb{T}) \otimes C(\mathbb{T})) \arrow[ru, "\rho"] &
\end{tikzcd}$$

For $\varepsilon < 2$, we therefore get that $\rho_\varepsilon(\mathfrak{b}_\varepsilon) = \rho(\mathfrak{b}) = [z] \in K_1(C(\mathbb{T}))$.

Proposition 3.2.3. *Let A be a unital C^*-algebra, $\alpha \in \operatorname{Aut}(A)$, and let $x \in \mathcal{U}_k(A)$ be a unitary satisfying $[x] \in \ker(K_1(\alpha) - \mathrm{id})$. An application of Theorem 3.2.2 yields $l \geq 0$, $\lambda_1, \dots, \lambda_l \in \mathbb{T}$, and $w \in \mathcal{U}_m(A)$ satisfying*

$$\left\|\alpha^{(m)}(\operatorname{diag}(x, \lambda_1, \dots, \lambda_l)) - w(\operatorname{diag}(x, \lambda_1, \dots, \lambda_l))w^*\right\| < 2,$$

where $m := k + l$. If $y := x \oplus \operatorname{diag}(\lambda_1, \cdots, \lambda_l) \in \mathcal{U}_m(A)$, then

$$\rho_0(\kappa(w^* u^{(k)}, y)) = [x].$$

Proof. First assume that $k = 1$ and that $l = 0$. For suitably chosen $\varepsilon < 2$, there is a $*$-homomorphism

$$\psi : T_\varepsilon \longrightarrow \mathcal{T}(A, \alpha) \quad \text{with} \quad \psi(s) = v \otimes w^* u \;\text{ and }\; \psi(u) = 1 \otimes x.$$

This homomorphism fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \otimes C(\mathbb{T}) \arrow[r] \arrow[d, "\mathrm{id}_{\mathcal{K}} \otimes \nu"] & T_\varepsilon \arrow[r] \arrow[d, "\psi"] & A_\varepsilon \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & \mathcal{K} \otimes A \arrow[r] & \mathcal{T}(A, \alpha) \arrow[r] & A \rtimes_\alpha \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

with φ and ν given by $\varphi(u_\varepsilon) = w^* u$, $\varphi(v_\varepsilon) = x$, and $\nu(z) = x$, respectively. By stability of K-theory, one gets that

$$\begin{aligned}
\rho_0(\kappa(w^* u, x)) &= (\rho_0 \circ K_0(\varphi))(\mathfrak{b}_\varepsilon) &&= (K_1(\mathrm{id}_{\mathcal{K}} \otimes \nu) \circ \rho_\varepsilon)(\mathfrak{b}_\varepsilon) \\
&= K_1(\nu)([z]) &&= [x] \;\in\; K_1(A).
\end{aligned}$$

If $y \in \mathcal{P}_m(A)$ is as in the statement, then we use (3.2) and (3.3) to conclude that

$$\rho_0(\kappa(w^* u^{(k)}, y)) = (\rho_0 \circ K_0(\eta))(\kappa(w^* u, y)) = \rho_0^{(k)}(\kappa(w^* u, y)) = [y] = [x].$$

□

Chapter 4

The K-theory of crossed products by $\mathbb{Z}^2$-actions

In this chapter, we define the obstruction homomorphism $d_*(\alpha)$ associated with a C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, which is essential for the study of the K-theory for the corresponding crossed product. Due to the naturality of the Pimsner-Voiculescu sequence, $K_*(\check{\alpha}_2) - \mathrm{id} : K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \to K_*(A \rtimes_{\alpha_1} \mathbb{Z})$ induces an endomorphism of the Pimsner-Voiculescu for $(A, \alpha_1, \mathbb{Z})$, which we consider as an extension of $\ker(K_{*+1}(\alpha_1) - \mathrm{id})$ by $\mathrm{coker}(K_*(\alpha_1) - \mathrm{id})$. We then obtain $d_*(\alpha)$ as the Snake Lemma homomorphism of the resulting diagram. Using the Snake Lemma exact sequence, we conclude that $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ is determined by $(K_*(\alpha), K_*(\alpha), d_*(\alpha))$ up to group extension problems. If A is unital, then we also give a concrete description for $d_*(\alpha)$ in terms of Bott elements.

The second section is concerned with the special case of a pointwise inner $\mathbb{Z}^2$-action α on a unital C^*-algebra A. The very nature of inner automorphisms allows us to simplify the concrete description of $d_*(\alpha)$ given in the first section. It turns out that $d_*(\alpha)$ highly depends on the central unitary $u(\alpha) := v^* w^* v w$, where v and w are unitaries implementing α_1 and α_2, respectively. In particular, $d_*(\alpha) = 0$ whenever $u(\alpha)$ does not have full spectrum. As another special property of pointwise inner $\mathbb{Z}^2$-actions, we show that $d_{*+1}(\alpha) \circ d_*(\alpha) = 0$.

In the third section, we use the K-theoretical isomorphism between the mapping torus $\mathcal{M}_\alpha(A)$ and the crossed product $A \rtimes_\alpha \mathbb{Z}^2$ to derive an alternative characterization of the obstruction homomorphism $d_*(\alpha)$. For the special case that α_1 is homotopic to the identity on A, we present a third description for $d_*(\alpha)$ which is based on the fact that in this case the C^*-algebras $\mathcal{M}_{\alpha_1}(A)$ and $A \otimes C(\mathbb{T})$ are isomorphic.

4.1 The obstruction homomorphism associated with a $\mathbb{Z}^2$-action

For the sake of better readability, we write $d_*(\beta) := K_*(\beta) - \mathrm{id}$ for an automorphism $\beta \in \mathrm{Aut}(B)$. Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, we denote by

$$\mathrm{k}(d_*(\alpha_2)) \in \mathrm{End}(\ker(d_*(\alpha_1))) \quad \text{and} \quad \mathrm{co}(d_*(\alpha_2)) \in \mathrm{End}(\mathrm{coker}(d_*(\alpha_1)))$$

the respective natural homomorphisms induced by $d_*(\alpha_2)$

By the naturality of the Pimsner-Voiculescu sequence, the equivariant automorphism $\alpha_2 : (A, \alpha_1, \mathbb{Z}) \to (A, \alpha_1, \mathbb{Z})$ gives rise to the following commuting diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker}(d_*(\alpha_1)) \arrow[r] \arrow[d, "\operatorname{co}(d_*(\alpha_2))"] & K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \arrow[r, "\rho_*"] \arrow[d, "d_*(\breve{\alpha}_2)"] & \ker(d_{*+1}(\alpha_1)) \arrow[r] \arrow[d, "\operatorname{k}(d_{*+1}(\alpha_2))"] & 0 \\
0 \arrow[r] & \operatorname{coker}(d_*(\alpha_1)) \arrow[r] & K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \arrow[r, "\rho_*"] & \ker(d_{*+1}(\alpha_1)) \arrow[r] & 0
\end{tikzcd} \tag{4.1}$$

Recall that we would like to compute $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ via the Pimsner-Voiculescu sequence for $(A \rtimes_{\alpha_1} \mathbb{Z}, \breve{\alpha}_2, \mathbb{Z})$. We are therefore particularly interested in the kernel and the cokernel of $K_*(\breve{\alpha}_2)$. The Snake Lemma provides the right tool for this purpose, see also [49, 1.3.2].

Lemma 4.1.1 (Snake Lemma). *Consider the following commutative diagram of abelian groups with exact rows*

$$\begin{tikzcd}
 & H \arrow[r, "i"] \arrow[d, "h"] & G \arrow[r, "p"] \arrow[d, "g"] & Q \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & H' \arrow[r, "i'"] & G' \arrow[r, "p'"] & Q' &
\end{tikzcd}$$

There is a homomorphism $d : \operatorname{coker}(q) \to \ker(h)$ *such that the sequence*

$$\begin{tikzcd}
\ker(h) \arrow[r] & \ker(g) \arrow[r] & \ker(q) \arrow[r, "d"] & \operatorname{coker}(h) \arrow[r] & \operatorname{coker}(g) \arrow[r] & \operatorname{coker}(q)
\end{tikzcd}$$

is exact. Moreover, if i is injective, then $\ker(h) \to \ker(g)$ *is injective, and if p is surjective, then* $\operatorname{coker}(g) \to \operatorname{coker}(q)$ *is surjective. The Snake Lemma homomorphism is natural in the following sense. If*

$$\begin{tikzcd}
 & & H_2 \arrow[rr] \arrow[dd, "h_2"] & & G_2 \arrow[rr] \arrow[dd, "g_2"] & & Q_2 \arrow[r] \arrow[dd, "q_2"] & 0 \\
 & H_1 \arrow[ur] \arrow[rr] \arrow[dd, "h_1"] & & G_1 \arrow[ur] \arrow[rr] \arrow[dd, "g_1"] & & Q_1 \arrow[ur] \arrow[rr] \arrow[dd, "q_1"] & & 0 \\
0 \arrow[rr] & & H_2' \arrow[rr] & & G_2' \arrow[rr] & & Q_2' & \\
0 \arrow[r] & H_1' \arrow[ur] \arrow[rr] & & G_1' \arrow[ur] \arrow[rr] & & Q_1' \arrow[ur] & &
\end{tikzcd}$$

is a commutative diagram with exact rows, then the induced diagram

$$\begin{tikzcd}
\ker(q_1) \arrow[r, "d_1"] \arrow[d] & \operatorname{coker}(h_1) \arrow[d] \\
\ker(q_2) \arrow[r, "d_2"] & \operatorname{coker}(h_2)
\end{tikzcd}$$

commutes as well.

The Snake Lemma homomorphism $d : \ker(q) \to \operatorname{coker}(h)$ admits the following explicit description. Given $x \in \ker(q)$, we can find $y \in G$ such that $p(y) = x$. Moreover, $p'(g(y)) = 0$, and so there is a unique $z \in H'$ satisfying $i'(z) = g(y)$. Then, $d(x) = [z] \in \operatorname{coker}(h)$.

By applying the Snake Lemma to (4.1), we get a group homomorphism

$$d_{*+1}(\alpha) : S_{*+1}(\alpha) \longrightarrow T_*(\alpha),$$

where $S_*(\alpha)$ and $T_*(\alpha)$ are defined as

$$\begin{aligned} S_*(\alpha) &:= \ker(d_*(\alpha_1)) \cap \ker(d_*(\alpha_2)), \\ T_*(\alpha) &:= K_*(A) \Big/ \langle \operatorname{im}(d_*(\alpha_1)), \operatorname{im}(d_*(\alpha_2)) \rangle . \end{aligned}$$

We call $d_*(\alpha)$ the *obstruction homomorphism* associated with $(A, \alpha, \mathbb{Z}^2)$. The Snake Lemma exact sequence associated with (4.1) splits into two extensions, namely

$$\begin{tikzcd} 0 \arrow[r] & \ker(\operatorname{co}(d_*(\alpha_2))) \arrow[r] & \ker(d_*(\check{\alpha}_2)) \arrow[r] & \ker(d_{*+1}(\alpha)) \arrow[r] & 0, \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(d_{*+1}(\alpha)) \arrow[r] & \operatorname{coker}(d_*(\check{\alpha}_2)) \arrow[r] & \operatorname{coker}(\mathrm{k}(d_{*+1}(\alpha_2))) \arrow[r] & 0. \end{tikzcd}$$

Moreover, the Pimsner-Voiculecsu sequence for $(A \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$ gives rise to a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(d_*(\check{\alpha}_2)) \arrow[r] & K_*(A \rtimes_\alpha \mathbb{Z}^2) \arrow[r, "\rho_*"] & \ker(d_{*+1}(\check{\alpha}_2)) \arrow[r] & 0. \end{tikzcd}$$

This shows that $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ is determined by $(K_*(A), K_*(\alpha), d_*(\alpha))$ up to group extension problems. So, $d_*(\alpha)$ really is an obstruction for that $K_*(A \rtimes_\alpha \mathbb{Z})$ only depends on $K_*(A)$ and $K_*(\alpha)$ up to group extension problems.

The obstruction homomorphism has the following naturality property, which easily follows from the naturality of the Pimsner-Voiculescu sequence and of the Snake Lemma.

Proposition 4.1.2. *If $\varphi : (A, \alpha, \mathbb{Z}^2) \to (B, \beta, \mathbb{Z}^2)$ denotes an equivariant $*$-homomorphism, then the following diagram commutes*

$$\begin{tikzcd} S_*(\alpha) \arrow[r, "K_*(\varphi)"] \arrow[d, "d_*(\alpha)"'] & S_*(\beta) \arrow[d, "d_*(\beta)"] \\ T_{*+1}(\beta) \arrow[r, "K_{*+1}(\varphi)"] & T_{*+1}(\beta) \end{tikzcd}$$

We proceed with a concrete description of $d_*(\alpha)$, given that the underlying C^*-algebra A is unital. Here, we make use of the lifts for the boundary map $\rho_* : K_*(A \rtimes_{\alpha_1} \mathbb{Z}) \to K_{*+1}(A)$ from Section 3. Unitality is not really a restriction since Proposition 4.1.2 allows one to derive a concrete description of $d_*(\alpha)$ in the non-unital case, accordingly.

Proposition 4.1.3. *Let $(A, \alpha, \mathbb{Z}^2)$ be a C^*-dynamical system on a unital C^*-algebra A, and let $p \in \mathcal{P}_k(A)$ be projection satisfying $[p] \in S_0(\alpha)$. Find $l, m \geq 0$ and $v, w \in \mathcal{U}_n(A)$ with*

$$\begin{aligned} \alpha_1^{(n)}(\mathrm{diag}(p, 1_l, 0_m)) &= v(\mathrm{diag}(p, 1_l, 0_m))v^*, \\ \alpha_2^{(n)}(\mathrm{diag}(p, 1_l, 0_m)) &= w(\mathrm{diag}(p, 1_l, 0_m))w^*, \end{aligned}$$

where $n := k + l + m$, and set $q := \mathrm{diag}(p, 1_l, 0_m)$. Then

$$d_0(\alpha)([p]) = \left[\kappa(q, w^*\alpha_2^{(n)}(v)^*\alpha_1^{(n)}(w)v)\right] \in T_1(\alpha).$$

Proof. If $\rho_1 : K_1(A \rtimes_{\alpha_1} \mathbb{Z}) \to K_0(A)$ denotes the index map of the corresponding Pimsner-Voiculescu sequence, then Proposition 3.2.1 yields

$$\rho_0(\kappa(q, v^*u^{(n)}) - [u^{(m)}]) = [p].$$

One computes that

$$\begin{aligned} (K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(q, v^*u^{(n)}) - [u^{(m)}]) &= (K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(q, v^*u^{(n)})) \\ &= \kappa(\alpha_2^{(n)}(q), \alpha_2^{(n)}(v)^*u^{(n)}) - \kappa(q, v^*u^{(n)}) \\ &= \kappa(wqw^*, \alpha_2^{(n)}(v)^*u^{(n)}) + \kappa(q, {u^{(n)}}^*v) \\ &= \kappa(q, w^*\alpha_2^{(n)}(v)^*u^{(n)}w) + \kappa(q, {u^{(n)}}^*v) \\ &= \kappa(q, w^*\alpha_2^{(n)}(v)^*u^{(n)}w{u^{(n)}}^*v) \\ &= \kappa(q, w^*\alpha_2^{(n)}(v)^*\alpha_1^{(n)}(w)v) \in K_1(A). \end{aligned}$$

We therefore get that $d_0(\alpha)([p]) = \left[\kappa(q, w^*\alpha_2^{(n)}(v)^*\alpha_1^{(n)}(w)v)\right] \in T_1(\alpha)$. □

Observe that $d_0(\alpha)$ is completely determined by Proposition 4.1.3. In fact, given $g \in S_0(\alpha)$, there is a projection $p \in M_m(A)$ and some $n \geq 0$ such that $g = [p] - [1_n] \in K_0(A)$. In this situation, $[p] \in S_0(\alpha)$ as well, and it holds that $d_0(\alpha)(g) = d_0(\alpha)([p]) \in T_1(\alpha)$.

For the description of $d_1 : S_1(\alpha) \to T_0(\alpha)$, we need the following perturbation result.

Lemma 4.1.4. *Let $0 < \varepsilon < \frac{2}{3}$ and let $u, \bar{u}, v \in A$ be unitaries satisfying*

$$\|u - \bar{u}\|, \ \|[u, v]\| \leq \varepsilon.$$

Then there is a homotopy $u_t \in \mathcal{U}(A)$ between u and $\bar{u}$ which satisfies $\|[u_t, v]\| \leq 3\varepsilon$ for all $t \in [0, 1]$.

Proof. Since $\|u-\bar{u}\| < \frac{2}{3}$, the spectrum of $u^*\bar{u}$ does not contain -1. Therefore, we can define $h := -i\log(u^*\bar{u}) \in A$, where log denotes the principal branch of the logarithm. This yields a continuous path of unitaries $u_t := u\exp(ith) \in A$, $t \in [0,1]$, with $u_0 = u$ and $u_1 = u\exp(\log(u^*\bar{u})) = \bar{u}$. Given $s, t \in [0,1]$, it holds that

$$\|u_s - u_t\| = \|1 - \exp(i(s-t)h)\| \leq \|1-\exp(ih)\| = \|u-\bar{u}\| \leq \varepsilon.$$

One now computes

$$\|[u_t, v]\| \leq \|u_t v - uv\| + \|vu_t - vu\| + \|[u,v]\| \leq 3\varepsilon.$$

□

Proposition 4.1.5. *Let $(A, \alpha, \mathbb{Z}^2)$ be a C^*-dynamical system with a unital C^*-algebra A. Let $v \in \mathcal{U}_k(A)$ be a unitary satisfying $[v] \in S_1(\alpha)$. By Theorem 3.2.2, there are $l \geq 0$, $\lambda_1, \ldots, \lambda_l \in \mathbb{T}$, and unitaries $x, y \in \mathcal{U}_m(A)$ such that*

$$\|\alpha_1^{(m)}(w) - xwx^*\|,\ \|\alpha_2^{(m)}(w) - ywy^*\| < \frac{1}{2},$$

where $m := k + l$ and $w := \operatorname{diag}(v, \lambda_1, \ldots, \lambda_l) \in \mathcal{U}_m(A)$. Then

$$d_1(\alpha)([v]) = \left[\kappa(y^*\alpha_2^{(m)}(x)^*\alpha_1^{(m)}(y)x, w)\right] \in T_0(\alpha).$$

Proof. Using the isomorphism $M_m(A \rtimes_{\alpha_1} \mathbb{Z}) \cong M_m(A) \rtimes_{\alpha_1^{(m)}} \mathbb{Z}$, we compute that

$$\|[x^*u^{(m)}, w]\| = \|u^{(m)}wu^{(m)^*} - xwx^*\| = \|\alpha_1^{(m)}(w) - xwx^*\| < \frac{1}{2}.$$

If $\rho_1 : K_1(A \rtimes_{\alpha_1} \mathbb{Z}) \to K_0(A)$ denotes the boundary map of the respective Pimsner-Voiculescu sequence, Proposition 3.2.3 implies that

$$\rho_1(\kappa(x^*u^{(m)}, w)) = [v].$$

Naturality of the Bott elements and part *iv)* of Proposition 2.1 yield

$$\begin{aligned}(K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(x^*u^{(m)}, w)) &= \kappa(\alpha_2^{(m)}(x)^*u^{(m)}, \alpha_2^{(m)}(w)) - \kappa(x^*u^{(m)}, w) \\ &= \kappa(\alpha_2^{(m)}(x)^*u^{(m)}, \alpha_2^{(m)}(w)) + \kappa(u^{(m)^*}x, w).\end{aligned}$$

Since

$$\|\alpha_2^{(m)}(w) - ywy^*\|,\ \|[\alpha_2^{(m)}(x)^*u^{(m)}, \alpha_2^{(m)}(w)]\| < \frac{1}{2},$$

we can apply Lemma 4.1.4 and find a homotopy $w_t \in \mathcal{U}_m(A)$ between $\alpha_2^{(m)}(w)$ and ywy^* such that

$$\|[w_t, \alpha_2^{(m)}(x)^*u^{(m)}]\| < \frac{3}{2} \quad \text{for all } t \in [0,1].$$

By part *i)* of Proposition 2.1 and the naturality of the Bott elements, we obtain that

$$\kappa(\alpha_2^{(m)}(x)^*u^{(m)}, \alpha_2^{(m)}(w)) = \kappa(\alpha_2^{(m)}(x)^*u^{(m)}, ywy^*) = \kappa(y^*\alpha_2^{(m)}(x)^*u^{(m)}y, w).$$

Moreover,

$$\|[u^{(m)^*}x, w]\| + \|[y^*\alpha_2^{(m)}(x)^*u^{(m)}y, w]\| < \frac{1}{2} + \frac{3}{2} = 2,$$

and therefore part *iii)* of Proposition 2.1 yields

$$\begin{aligned}(K_0(\check{\alpha}_2) - \mathrm{id})(\kappa(x^*u^{(m)}, w)) &= \kappa(y^*\alpha_2^{(m)}(x)^*u^{(m)}y, w) + \kappa(u^{(m)^*}x, w)\\ &= \kappa(y^*\alpha_2^{(m)}(x)^*u^{(m)}yu^{(m)^*}x, w)\\ &= \kappa(y^*\alpha_2^{(m)}(x)^*\alpha_1^{(m)}(y)x, w) \in K_0(A).\end{aligned}$$

By the definition of the obstruction homomorphism, it follows that

$$d_1(\alpha)([v]) = \left[\kappa(y^*\alpha_2^{(m)}(x)^*\alpha_1^{(m)}(y)x, w)\right] \in T_0(\alpha).$$

□

For a given $\mathbb{Z}^2$-action α, we denote by $\overline{\alpha}$ the action satisfying $\overline{\alpha}_{(1,0)} = \alpha_2$ and $\overline{\alpha}_{(0,1)} = \alpha_1$. As one may expect, $d_*(\alpha)$ and $d_*(\overline{\alpha})$ coincide up to a minus sign.

Proposition 4.1.6. *For any C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, we have that $d_*(\overline{\alpha}) = -d_*(\alpha)$.*

Proof. Naturality of the obstruction homomorphism allows us to reduce to the unital case. Moreover, we only show that $d_0(\overline{\alpha}) = -d_0(\alpha)$ since the other case is similar.

Let $p \in A$ be a projection satisfying $\alpha_1(p) = vpv^*$ and $\alpha_2(p) = wpw^*$ for some unitaries $v, w \in A$. Then

$$\begin{aligned}d_0(\overline{\alpha})([p]) &= \kappa(p, v^*\alpha_1(w)^*\alpha_2(v)w) &&= \kappa(p, (w^*\alpha_2(v)^*\alpha_1(w)v)^*)\\ &= -\kappa(p, w^*\alpha_2(v)^*\alpha_1(w)v) &&= -d_0(\alpha)([p]).\end{aligned}$$

Standard arguments show that this already yields $d_0(\overline{\alpha}) = -d_0(\alpha)$. □

4.2 The special case of pointwise inner $\mathbb{Z}^2$-actions

We call a $\mathbb{Z}^2$-action α on a unital C^*-algebra A *pointwise inner* if the two generators α_1 and α_2 are both inner automorphisms. In this case, there are unitaries $v, w \in A$ satisfying $\alpha_1 = \mathrm{Ad}(v)$, $\alpha_2 = \mathrm{Ad}(w)$, and

$$vwaw^*v^* = wvav^*w^* \quad \text{for all} \quad a \in A.$$

Hence, two inner automorphisms $\alpha_1 = \mathrm{Ad}(v)$ and $\alpha_2 = \mathrm{Ad}(w)$ define a pointwise inner $\mathbb{Z}^2$-action α if and only if $v^*w^*vw \in A$ is a central unitary. Observe that the commutator v^*w^*vw does not depend on the specific choice of the implementing unitaries v and w. In fact, if we have another presentation $\alpha_1 = \mathrm{Ad}(\widetilde{v})$ and $\alpha_2 = \mathrm{Ad}(\widetilde{w})$, then the unitaries $\widetilde{v}v^*$ and $\widetilde{w}w^*$ are central, and

$$(\widetilde{v}^*\widetilde{w}^*\widetilde{v}\widetilde{w})^*v^*w^*vw = \widetilde{w}^*\widetilde{v}^*\widetilde{w}w^*\widetilde{v}w = \widetilde{w}^*\widetilde{v}^*\widetilde{v}\widetilde{w} = 1.$$

We therefore call $u(\alpha) := v^*w^*vw$ the *commutator* associated with α.

Proposition 4.2.1. *Let* α *denote a pointwise inner* $\mathbb{Z}^2$*-action on a unital* C^**-algebra* A*. Let* $n \geq 1$*,* $x \in \mathcal{U}_n(A)$*, and* $p \in \mathcal{P}_n(A)$*. Then the associated obstruction homomorphism* $d_*(\alpha) : K_*(A) \to K_{*+1}(A)$ *is given by*

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)^{(n)}) \quad \textit{and} \quad d_1(\alpha)([x]) = \kappa(u(\alpha)^{(n)}, x).$$

Proof. The generating automorphisms $\alpha_1 = \mathrm{Ad}(v)$ and $\alpha_2 = \mathrm{Ad}(w)$ certainly satisfy

$$\alpha_1^{(n)}(p) = v^{(n)} p v^{(n)*} \quad \text{and} \quad \alpha_2^{(n)}(p) = w^{(n)} p w^{(n)*}.$$

One computes that

$$w^* \alpha_2(v)^* \alpha_1(w) v = w^* w v^* w^* v w v^* v = v^* w^* v w = u(\alpha),$$

and thus by Proposition 4.1.3, the obstruction homomorphism $d_0(\alpha)$ satisfies

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)^{(n)}).$$

The proof for $d_1(\alpha)([x])$ is similar. □

This description also reveals that $d_*(\alpha)$ can only be non-trivial if the unitary $u(\alpha)$ has full spectrum. Otherwise, $u(\alpha)$ is connected to 1 by unitaries in $C^*(u(\alpha)) \subseteq \mathcal{Z}(A)$. Thus, if $C(\mathbb{T})$ does not embed into the center of A, then $d_*(\alpha) = 0$. In fact, we have the following result.

Proposition 4.2.2. *Let* α *be a* $\mathbb{Z}^n$*-action by inner automorphism on a unital* C^**-algebra* A*. For* $i, j = 1, \ldots, n$*, let* $v(i, j)$ *denote the commutator associated with the* $\mathbb{Z}^2$*-action generated by* α_i *and* α_j*. Assume that all* $v(i, j)$ *are homotopic to* 1 *in* $\mathcal{U}(\mathcal{Z}(A))$*. Then* $K_*(A \rtimes_\alpha \mathbb{Z}^n) \cong K_*(A \otimes C(\mathbb{T}^n))$*.*

Proof. The proof goes by induction over $n \in \mathbb{N}$. For a single automorphism, this is trivial since $A \rtimes_{\mathrm{Ad}(v)} \mathbb{Z} \cong A \otimes C(\mathbb{T})$. Assume now that the statement is true for $n - 1$. Find unitaries $v_1, \ldots, v_n \in A$ implementing the inner automorphisms $\alpha_1, \ldots, \alpha_n$, and denote by $\breve{\alpha}$ the $\mathbb{Z}^{n-1}$-action on $A \rtimes_{\alpha_1} \mathbb{Z}$ induced by $\breve{\alpha}_2, \ldots, \breve{\alpha}_n$. As inner automorphisms fix the center pointwise, we get that

$$v(i, j) \in \mathcal{Z}(A) \subseteq \mathcal{Z}(A \rtimes_{\alpha_1} \mathbb{Z}) \quad \text{for all} \quad i, j = 1, \ldots, n.$$

Equivalently, the inner automorphisms $\mathrm{Ad}(v_2), \ldots, \mathrm{Ad}(v_n) \in \mathrm{Aut}(A \rtimes_{\alpha_1} \mathbb{Z})$ define a $\mathbb{Z}^{n-1}$-action α' on $A \rtimes_{\alpha_1} \mathbb{Z}$. Observe that the action α' is as in the statement, so that we can apply the induction hypothesis to it. By assumption, there are homotopies $w_{t,i} \in \mathcal{U}(\mathcal{Z}(A))$, $i = 2, \ldots, n$, connecting $w_{0,i} = v(i, 1)$ and $w_{1,i} = 1$. Since these homotopies lie in the center of $A \rtimes_{\alpha_1} \mathbb{Z}$, we can define automorphisms

$$\phi_{t,i} : A \rtimes_{\alpha_1} \mathbb{Z} \longrightarrow A \rtimes_{\alpha_1} \mathbb{Z}, \quad \phi_{t,i}(a) = v_i a v_i^*, \ \phi_{t,i}(u) = w_{t,i} u.$$

For fixed i, the family $\{\phi_{t,i}\}_{t \in [0,1]}$ defines a homotopy between $\phi_{0,i} = \mathrm{Ad}(v_i)$ and $\phi_{1,i} = \breve{\alpha}_i$. Moreover,

$$\phi_{s,i} \circ \phi_{t,j}(u) = w_{t,j} w_{s,i} u = w_{s,i} w_{t,j} u = \phi_{s,i} \circ \phi_{t,j}(u),$$

showing that $\phi_{s,i} \circ \phi_{t,j} = \phi_{s,i} \circ \phi_{t,j}$ for all $s, t \in [0,1]$ and $i, j = 2, \dots, n$. In particular, the actions $\breve{\alpha}$ and α' are homotopic, and Proposition 3.1.2 together with the induction hypothesis yields

$$\begin{array}{rclcl} K_*(A \rtimes_\alpha \mathbb{Z}^n) & \cong & K_*((A \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\breve{\alpha}} \mathbb{Z}^{n-1}) & \cong & K_*((A \rtimes_{\alpha_1} \mathbb{Z}) \rtimes_{\alpha'} \mathbb{Z}^{n-1}) \\ & \cong & K_*((A \rtimes_{\alpha_1} \mathbb{Z}) \otimes C(\mathbb{T}^{n-1})) & \cong & K_*(A \otimes C(\mathbb{T}^n)). \end{array}$$

□

As announced before, we will see in Chapter 5 that there are pointwise inner $\mathbb{Z}^2$-actions whose corresponding obstruction homomorphisms are non-trivial. Many of the crossed products arising from such C^*-dynamical systems have K-groups different from the ones of the respective crossed products by the trivial $\mathbb{Z}^2$-action. However, there are certain restrictions on the obstruction homomorphisms associated with pointwise inner $\mathbb{Z}^2$-actions.

Proposition 4.2.3. *Let α denote a pointwise inner $\mathbb{Z}^2$-action on a unital C^*-algebra A. Then the associated obstruction homomorphism $d_*(\alpha)$ always satisfies $d_{*+1}(\alpha) \circ d_*(\alpha) = 0$.*

Proof. Using the canonical isomorphism

$$K_*(C(\mathbb{T}) \oplus C(\mathbb{T})) \cong K_*(C(\mathbb{T})) \oplus K_*(C(\mathbb{T})),$$

we compute

$$\kappa(z \oplus z, z \oplus 1) = \kappa(z,z) \oplus \kappa(z,1) = 0 \in K_0(C(\mathbb{T})) \oplus K_0(C(\mathbb{T})).$$

Given a projection $p \in \mathcal{P}_n(A)$, the central unitary $u(\alpha)^{(n)} \in M_n(A)$ induces a $*$-homomorphism $\varphi : C(\mathbb{T}) \oplus C(\mathbb{T}) \to M_n(A)$ satisfying $\varphi(z \oplus z) = u(\alpha)^{(n)}$ and $\varphi(1 \oplus 0) = p$. By Proposition 4.2.1, we get that

$$\begin{array}{rcl} d_1(\alpha)(\kappa(p, u(\alpha)^{(n)})) & = & \kappa(u(\alpha)^{(n)}, pu(\alpha)^{(n)}p + 1_n - p) \\ & = & K_0(\varphi)(\kappa(z \oplus z, z \oplus 1)) \\ & = & 0, \end{array}$$

showing that $d_1(\alpha) \circ d_0(\alpha) = 0$. If we consider the suspended system $(SA, S\alpha, \mathbb{Z}^2)$, then we also get that $d_0(S\alpha) \circ d_1(S\alpha) = 0$. Hence, $d_0(\alpha) \circ d_1(\alpha) = 0$ by Bott periodicity. □

Given a $\mathbb{Z}^2$-action α on a C^*-algebra A and numbers $m, n \in \mathbb{N}$, we denote by $\alpha(m,n)$ the $\mathbb{Z}^2$-action induced by the commuting automorphisms α_1^m and α_2^n. In the case that α acts by pointwise inner actions, the following relationship between the associated obstruction homomorphisms holds.

Proposition 4.2.4. *Let α denote a pointwise inner $\mathbb{Z}^2$-action on a unital C^*-algebra A. For $m, n \in \mathbb{N}$, we then have that*

$$d_*(\alpha(m,n)) = mn \cdot d_*(\alpha).$$

Proof. If $\alpha_1 = \mathrm{Ad}(v)$ and $\alpha_2 = \mathrm{Ad}(w)$, then $\alpha_1^m = \mathrm{Ad}(v^m)$ and $\alpha_2^n = \mathrm{Ad}(w^n)$. Since wvw^*v^* is a central unitary, we have that

$$\begin{aligned} u(\alpha(1,n)) &= v^* w^{*n-1} v w^* (w v^* w^* v) w^n = (v^* w^{*n-1} v w^{n-1})(w v^* w^* v) \\ &= (w v^* w^* v)^n = \mathrm{Ad}(w)(u(\alpha))^n \\ &= u(\alpha)^n. \end{aligned}$$

By symmetry, we obtain that $u(\alpha(m,n)) = u(\alpha)^{mn}$. For $y \in \mathcal{U}_k(A)$,

$$d_1(\alpha(m,n))([y]) = \kappa(u(\alpha(m,n))^{(k)}, y) = mn \cdot d_1(\alpha)([y]).$$

Analogous considerations show that $d_0(\alpha(m,n)) = mn \cdot d_0(\alpha)$. □

4.3 Alternative descriptions of the obstruction homomorphism

Given a C^*-dynamical system $(A, \alpha, \mathbb{Z})$, we consider the associated mapping torus extension

$$\begin{tikzcd} 0 \arrow[r] & SA \arrow[r] & \mathcal{M}_\alpha(A) \arrow[r, "\mathrm{ev}_0"] & A \arrow[r] & 0. \end{tikzcd}$$

Using Bott periodicity, it can be verified that the corresponding six-term exact sequence is of the form

$$\begin{tikzcd} K_1(A) \arrow[r] & K_0(\mathcal{M}_\alpha(A)) \arrow[r, "K_0(\mathrm{ev}_0)"] & K_0(A) \arrow[d, "K_0(\alpha)-\mathrm{id}"] \\ K_1(A) \arrow[u, "K_1(\alpha)-\mathrm{id}"] & K_1(\mathcal{M}_\alpha(A)) \arrow[l, "K_1(\mathrm{ev}_0)"] & K_0(A) \arrow[l] \end{tikzcd}$$

By an application of the isomorphism $K_{*+1}(\mathcal{M}_\alpha(A)) \cong K_*(A \rtimes_\alpha \mathbb{Z})$, this sequence gets transformed into the Pimsner-Voiculescu sequence associated with $(A, \alpha, \mathbb{Z})$, see for example [32] and [4].

The obstruction homomorphism $d_*(\alpha) : S_*(\alpha) \to T_{*+1}(\alpha)$ is now given as the Snake Lemma homomorphism of the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{coker}(d_{*+1}(\alpha_1)) \arrow[r] \arrow[d, "\mathrm{co}(d_{*+1}(\alpha_2))"] & K_*(\mathcal{M}_{\alpha_1}(A)) \arrow[r, "K_*(\mathrm{ev}_0)"] \arrow[d, "d_*(\widetilde{\alpha}_2)"] & \ker(d_*(\alpha_1)) \arrow[r] \arrow[d, "\mathrm{k}(d_*(\alpha_2))"] & 0 \\ 0 \arrow[r] & \mathrm{coker}(d_{*+1}(\alpha_1)) \arrow[r] & K_*(\mathcal{M}_{\alpha_1}(A)) \arrow[r, "K_*(\mathrm{ev}_0)"] & \ker(d_*(\alpha_1)) \arrow[r] & 0 \end{tikzcd}$$

Let us give an explicit description of $d_1(\alpha)$ in the case that A is unital. The description of $d_0(\alpha)$ is similar, and for actions on non-unital C^*-algebras, one can pass to the unitized system.

Let $x \in S_1(\alpha)$ be represented by a unitary $v_0 \in \mathcal{U}_k(A)$. Find $l \geq 0$ such that $v_0 \oplus 1_l$ is homotopic to $\alpha_1^{(k+l)}(v_0 \oplus 1_l)$ in $\mathcal{U}_{k+l}(A)$. Fix such a homotopy and note that it defines a unitary lift

$$v \in M_{k+l}(\mathcal{M}_{\alpha_1}(A)) \cong \mathcal{M}_{\alpha_1^{(k+l)}}(M_{k+l}(A))$$

for $v_0 \oplus 1_l$. By exactness of the lower row extension,

$$K_0(\widetilde{\alpha}_2)([v]) - [v] = [\widetilde{\alpha}_2^{(k+l)}(v)v^*] \in K_1(\mathcal{M}_{\alpha_1}(A))$$

can be regarded as an element in $K_1(SA) \cong K_0(A)$. To do this more concretely, find some $m \geq 0$ and a homotopy $u_t \in \mathcal{U}(M_n(A))$ satisfying $u_0 = 1_n$ and $u_1 = \alpha_2^{(n)}(v_0 \oplus 1_m)(v_0^* \oplus 1_m)$, where $n := k+l+m$. Define a unitary $w \in M_n((SA)^\sim)$ via

$$w(t) := \begin{cases} u(3t) & , \quad \text{if } t \in [0, \frac{1}{3}], \\ \alpha_2^{(n)}(v(3t-1) \oplus 1_m)(v(3t-1)^* \oplus 1_m) & , \quad \text{if } t \in [\frac{1}{3}, \frac{2}{3}], \\ \alpha_1^{(n)}(u(3-3t)) & , \quad \text{if } t \in [\frac{2}{3}, 1]. \end{cases}$$

The natural inclusion $M_n((SA)^\sim) \to M_n(\mathcal{M}_{\alpha_1}(A))$ now yields that

$$K_0(\widetilde{\alpha}_2)([v]) - [v] = [w] \in K_1(\mathcal{M}_{\alpha_1}(A)).$$

Finally, by the definition of the Snake Lemma homomorphism, $d_1(\alpha)$ satisfies

$$d_1(\alpha)(x) = [g] \in T_0(\alpha),$$

where $g \in K_0(A)$ is the image of $[w] \in K_0(SA)$ under the Bott isomorphism.

Assume now that $(A, \alpha, \mathbb{Z}^2)$ is a C^*-dynamical system with the property that α_1 is homotopic to id_A in $\mathrm{Aut}(A)$. Fix a homotopy $\beta_t \in \mathrm{Aut}(A)$ between $\beta_0 = \alpha_1$ and $\beta_1 = \mathrm{id}_A$, and consider the induced $*$-automorphism $\phi \in \mathrm{Aut}(A \otimes C(\mathbb{T}))$ given by

$$\phi(f)(\exp(2\pi it)) = \left(\beta_t \circ \alpha_2 \circ \beta_t^{-1}\right)(f(\exp(2\pi it))), \; f \in A \otimes C(\mathbb{T}), \; t \in [0,1].$$

Note that ϕ is well-defined since α_1 and α_2 commute. Obviously, ϕ fits into the following commutative diagram with split-exact rows

$$\begin{tikzcd} 0 \arrow[r] & SA \arrow[r] \arrow[d, "\phi'"] & A \otimes C(\mathbb{T}) \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\phi"] & A \arrow[r] \arrow[d, "\alpha_2"] \arrow[l, bend right, "j"'] & 0 \\ 0 \arrow[r] & SA \arrow[r] & A \otimes C(\mathbb{T}) \arrow[r, "\mathrm{ev}_1"] & A \arrow[r] \arrow[l, bend left, "j"] & 0 \end{tikzcd} \tag{4.2}$$

where $j : A \to A \otimes C(\mathbb{T})$ denotes the canonical embedding. There is a $*$-isomorphism $\psi : \mathcal{M}_{\alpha_1}(A) \xrightarrow{\cong} A \otimes C(\mathbb{T})$ satisfying

$$\psi(f)(\exp(2\pi it)) = (\beta_t \circ \alpha_1^{-1})(f(t)), \quad f \in \mathcal{M}_{\alpha_1}(a), \; t \in [0,1].$$

Its restriction $\psi' \in \mathrm{Aut}(SA)$ is homotopic to id_{SA} via $\psi'_s \in \mathrm{Aut}(SA)$ given by

$$\psi'_s(f)(t) = (\beta_{st} \circ \alpha_1^{-1})(f(t)), \quad f \in SA, \; s,t \in [0,1].$$

Proposition 4.3.1. *Let $(A, \alpha, \mathbb{Z}^2)$ be a C^*-dynamical system such that α_1 is homotopic to id_A in $\mathrm{Aut}(A)$. Consider the commutative diagram with split-exact rows*

$$\begin{tikzcd}
0 \arrow[r] & K_{*+1}(A) \arrow[r] \arrow[d] & K_*(A \otimes C(\mathbb{T})) \arrow[r] \arrow[d, "K_*(\phi)-\mathrm{id}"] & K_*(A) \arrow[r] \arrow[d, "K_*(\alpha_2)-\mathrm{id}"] \arrow[l, bend right] & 0 \\
0 \arrow[r] & K_{*+1}(A) \arrow[r] & K_*(A \otimes C(\mathbb{T})) \arrow[r] & K_*(A) \arrow[r] \arrow[l, bend left] & 0
\end{tikzcd} \tag{4.3}$$

obtained from (4.2). *If $h_* : S_*(\alpha) \to T_{*+1}(\alpha)$ denotes its associated Snake Lemma homomorphism, then $h_* = d_*(\alpha)$.*

Proof. The natural isomorphism $\psi : \mathcal{M}_{\alpha_1}(A) \xrightarrow{\cong} A \otimes C(\mathbb{T})$ and (4.2) induce the following commutative diagram

$$\begin{tikzcd}
0 \arrow[rr] & & SA \arrow[rr] \arrow[dd, "\phi'"] & & A \otimes C(\mathbb{T}) \arrow[rr] \arrow[dd, "\phi"] & & A \arrow[r] \arrow[dd, "\alpha_2"] & 0 \\
0 \arrow[r] & SA \arrow[rr, crossing over] \arrow[ur, "\psi'"] \arrow[dd, "S\alpha_2"] & & \mathcal{M}_{\alpha_1}(A) \arrow[rr, crossing over] \arrow[ur, "\psi"] \arrow[dd, "\widetilde{\alpha}_2"] & & A \arrow[rr, crossing over] \arrow[ur, "\mathrm{id}_A"] \arrow[dd, "\alpha_2"] & & 0 \\
0 \arrow[rr] & & SA \arrow[rr] & & A \otimes C(\mathbb{T}) \arrow[rr] & & A \arrow[r] & 0 \\
0 \arrow[r] & SA \arrow[rr] \arrow[ur, "\psi'"] & & \mathcal{M}_{\alpha_1}(A) \arrow[rr] \arrow[ur, "\psi"] & & A \arrow[rr] \arrow[ur, "\mathrm{id}_A"] & & 0
\end{tikzcd}$$

Observe that all occurring rows are split-exact. Apply K-theory to the whole diagram and use Bott periodicity for the left hand square involving the suspensions of A. The Snake Lemma homomorphism of the resulting front diagram is the obstruction homomorphism $d_*(\alpha) : S_*(\alpha) \to T_{*+1}(\alpha)$, and the one of the back side diagram is $h_* : S_*(\alpha) \to T_{*+1}(\alpha)$. The naturality of the Snake Lemma and the fact that ψ' acts trivially on K-theory yield that the two Snake Lemma homomorphisms h_* and $d_*(\alpha)$ coincide. ☐

Chapter 5

Examples of non-trivial obstruction homomorphisms

In this chapter, we give examples of C^*-dynamical systems $(A, \alpha, \mathbb{Z}^2)$ whose associated obstruction homomorphisms $d_*(\alpha)$ are non-trivial. Furthermore, all actions we investigate act trivially on K-theory. First, we consider a certain class of $\mathbb{Z}^2$-actions on UCT Kirchberg algebras. In order to show that their associated obstruction homomorphisms do not vanish, we make use of Izumi and Matui's classification of locally KK-trivial outer $\mathbb{Z}^2$-actions on Kirchberg algebras [17]. Given a Kirchberg algebra A, Izumi and Matui show the existence of a bijection between $KK(A, SA)$ (or rather a certain large subgroup in the unital case) and equivalence classes of locally KK-trivial outer $\mathbb{Z}^2$-actions under a certain strong conjugacy relation. Our key observation then is that the descend homomorphism

$$\gamma_* : KK(A, SA) \longrightarrow \mathrm{Hom}(K_*(A), K_{*+1}(A))$$

maps the invariant corresponding to the class of α to the obstruction homomorphism $d_*(\alpha)$. As a consequence, we get that in the presence of the UCT every graded homomorphism $\eta_* : K_*(A) \to K_{*+1}(A)$ (with $\eta_0([1]) = 0$) is the obstruction homomorphism of some locally KK-trivial $\mathbb{Z}^2$-action on A.

Other examples of actions with non-trivial obstruction homomorphisms are found within the class of pointwise inner $\mathbb{Z}^2$-actions. In the second section, we investigate a concrete action α on the group C^*-algebra of the discrete Heisenberg group $C^*(H_3)$. To be more precise, α is the natural pointwise inner action induced by the generating unitaries of $C^*(H_3)$. The corresponding C^*-dynamical system $(C^*(H_3), \alpha, \mathbb{Z}^2)$ is interesting for its universality, which expresses in the fact that it admits an equivariant $*$-homomorphism into any C^*-dynamical system $(B, \beta, \mathbb{Z}^2)$ with pointwise inner action β. We investigate the associated obstruction homomorphism, and show that it is non-trivial. Then we compute the K-groups of the corresponding crossed product $C^*(H_3) \rtimes_\alpha \mathbb{Z}^2$, which turn out to be non-isomorphic to the ones of $C^*(H_3) \otimes C(\mathbb{T}^2)$.

In the last section, we present a general method of constructing pointwise inner actions which induce non-trivial obstruction homomorphisms. The C^*-algebras are all given as amalgamated free products of the form $A *_{C(\mathbb{T})} B$. The first C^*-algebra A is equipped with a pointwise inner action α whose associated commutator $u(\alpha)$

has full spectrum. The second C^*-algebra B also contains a central unitary with full spectrum, which gets identified with $u(\alpha)$ under the amalgamation process. The amalgamated free product henceforth admits a pointwise inner action extending α. Moreover, B is supposed to satisfy a certain K-theoretical condition which ensures the existence of an obstruction, i.e. a non-trivial element in the image of the obstruction homomorphism. To deal with the K-theory of amalgamated free products, we use Thomsen's six-term exact sequence [47] relating the K-theory of the amalgamated free product with the ones of A, B, and $C(\mathbb{T})$. Among the constructed examples, we find C^*-dynamical systems which are universal for obstructions coming from pointwise inner actions. We compute the K-theory of the crossed products associated with these universal systems.

5.1 Locally KK-trivial $\mathbb{Z}^2$-actions on Kirchberg algebras

We begin with the following definitions used in [17].

Definition 5.1.1. A $\mathbb{Z}^2$-action α on a C^*-algebra A is called *locally KK-trivial* if $KK(\alpha_1) = KK(\alpha_2) = 1_A$. Moreover, we call two $\mathbb{Z}^2$-actions α and β on A *KK-trivially cocycle conjugate* if there exists an α-cocyle $u : \mathbb{Z}^2 \to \mathcal{U}(\mathcal{M}(A))$ and an automorphism $\mu \in \mathrm{Aut}(A)$ with $KK(\mu) = 1_A$ such that $\alpha_i^u = \mu \circ \beta_i \circ \mu^{-1}$ for $i = 1, 2$.

It is obvious from the definition that KK-trivially cocycle conjugate actions give rise to isomorphic crossed products. Also recall that if A is a Kirchberg algebra and α is an outer $\mathbb{Z}^2$-action on A, i.e. $\alpha_1^m \circ \alpha_2^n$ is not inner for all $(m, n) \neq (0, 0)$, then the crossed product $A \rtimes_\alpha \mathbb{Z}^2$ is again a Kirchberg algebra.

For the convenience of the reader, we recall Izumi and Matui's result. Given a locally KK-trivial action α on a Kirchberg algebra A, we associate an element $\phi(\alpha) \in KK(A, SA)$ as follows. Since α_1 represents the element $1_A \in KK(A, A)$, [35, 4.1.1] yields a homotopy $\beta_t \in \mathrm{Aut}(A \otimes \mathcal{K})$ between $\beta_0 = \alpha_1 \otimes \mathrm{id}_{\mathcal{K}}$ and $\beta_1 = \mathrm{id}_{A\otimes\mathcal{K}}$. As in Section 4.3, we define the automorphism $\phi \in \mathrm{Aut}(A \otimes \mathcal{K} \otimes C(\mathbb{T}))$ by

$$\phi(f)(\exp(2\pi it)) = \left(\beta_t \circ (\alpha_2 \otimes \mathrm{id}_{\mathcal{K}}) \circ \beta_t^{-1}\right)(f(\exp(2\pi it)))$$

for $f \in A \otimes \mathcal{K} \otimes C(\mathbb{T})$ and $t \in [0, 1]$. Denote by $j : A \otimes \mathcal{K} \to A \otimes \mathcal{K} \otimes C(\mathbb{T})$ the canonical embedding. Using stability of the KK-bifunctor and the fact that $KK(\alpha_2) = 1_A$, we obtain that

$$\Phi(\alpha) := KK(\phi \circ j) - KK(j) \in KK(A, SA) \subseteq KK(A, A \otimes C(\mathbb{T})).$$

Recall the definition of the descend homomorphism

$$\gamma_* : KK(A, SA) \longrightarrow \mathrm{Hom}(K_*(A), K_{*+1}(A)),$$

where $\gamma_*(x) \in \mathrm{Hom}(K_*(A), K_{*+1}(A))$ is given by taking the Kasparov products with x, that is $\gamma_0(x)(z_0) = z_0 \otimes x$ and $\gamma_1(x)(z_1) = z_1 \otimes x$ for every $z_0 \in K_0(A) \cong KK(\mathbb{C}, A)$ and $z_1 \in K_1(A) \cong KK(S, A)$.

Theorem 5.1.2 ([17]). *Let A be a unital Kirchberg algebra. The aforementioned assignment $\alpha \mapsto \Phi(\alpha)$ induces a well-defined bijection between the following two sets:*

i) KK-trivially cocycle conjugacy classes of locally KK-trivial outer $\mathbb{Z}^2$-actions on A.

ii) $\{x \in KK(A, SA) \ : \ \gamma_0(x)([1]) = 0 \in K_1(A)\}$.

If A is a stable Kirchberg algebra, then the statement remains true when we take $KK(A, SA)$ as a classifying invariant.

By the definition of γ_*,

$$\gamma_*(\Phi(\alpha)) = K_*(\phi \circ j) - K_*(j) = (K_*(\phi) - \mathrm{id}) \circ K_*(j) \in \mathrm{Hom}(K_*(A), K_{*+1}(A))$$

for any locally KK-trivial $\mathbb{Z}^2$-action α. Hence, $\gamma_*(\Phi(\alpha))$ is the Snake Lemma homomorphism of diagram (4.3) applied to $A \otimes \mathcal{K}$ and $\alpha \otimes \mathrm{id}_{\mathcal{K}}$. Proposition 4.3.1 therefore yields that $\gamma_*(\Phi(\alpha)) = d_*(\alpha)$. Combining this observation with Theorem 5.1.2, we draw the following consequence.

Corollary 5.1.3. *Let A be a unital Kirchberg algebra satisfying the UCT. Let $\eta_* : K_*(A) \to K_{*+1}(A)$ be a homomorphisms with $\eta_0([1]) = 0$. Then there is a locally KK-trivial $\mathbb{Z}^2$-action α on A such that $K_*(A \rtimes_\alpha \mathbb{Z}^2)$ fits into a six-term exact sequence*

$$\begin{tikzcd}
K_1(A) \oplus K_0(A) \arrow[r, "\eta_1 \oplus 0"] & K_0(A) \oplus K_1(A) \arrow[r] & K_0(A \rtimes_\alpha \mathbb{Z}^2) \arrow[d] \\
K_1(A \rtimes_\alpha \mathbb{Z}^2) \arrow[u] & K_1(A) \oplus K_0(A) \arrow[l] & K_0(A) \oplus K_1(A) \arrow[l, "\eta_0 \oplus 0"']
\end{tikzcd}$$

If A is a stable Kirchberg algebra in the UCT-class, then the statement remains true if the condition on the class of the unit is removed.

Proof. Since A satisfies the UCT, we find some element $x \in KK(A, SA)$ satisfying $\gamma_*(x) = \eta_*$. Observe that if A is unital, then the condition $\gamma_*(x)([1]) = 0$ is satisfied by assumption. Theorem 5.1.2 yields a locally KK-trivial action α with $\Phi(\alpha) = x$, and hence

$$\eta_* = \gamma_*(x) = \gamma_*(\Phi(\alpha)) = d_*(\alpha).$$

We have already seen that $\alpha_1 \otimes \mathrm{id}_{\mathcal{K}}$ is homotopic to $\mathrm{id}_{A \otimes \mathrm{id}}$, and consequently $K_i(A \rtimes_{\alpha_1} \mathbb{Z}) \cong K_0(A) \oplus K_1(A)$ for $i = 0, 1$. The claim now follows from the Pimsner-Voiculescu sequence for $(A \rtimes_{\alpha_1} \mathbb{Z}, \breve{\alpha}_2, \mathbb{Z})$. $\square$

5.2 A natural action on the group C^*-algebra of the discrete Heisenberg group

Recall the discrete Heisenberg group $H_3 := \langle r, s \ : \ rsr^{-1}s^{-1} \text{ is central}\rangle$ and its associated group C^*-algebra

$$C^*(H_3) := C^*(u, v \text{ unitaries } : \ uvu^*v^* \text{ is central}).$$

Consider the pointwise inner $\mathbb{Z}^2$-action α on $C^*(H_3)$ given by $\alpha_1 = \mathrm{Ad}(u)$ and $\alpha_2 = \mathrm{Ad}(v)$. The associated C^*-dynamical system $(C^*(H_3), \alpha, \mathbb{Z}^2)$ is universal in the following sense. Let $(B, \beta, \mathbb{Z}^2)$ be some C^*-dynamical system with β acting by inner automorphisms on the unital C^*-algebra B. If $\beta_1 = \mathrm{Ad}(x)$ and $\beta_2 = \mathrm{Ad}(y)$, then there is an equivariant unital $*$-homomorphism $\varphi : C^*(H_3) \to B$ satisfying $\varphi(u) = x$ and $\varphi(v) = y$.

The Heisenberg group also admits the following description as a semidirect product

$$H_3 = \mathbb{Z}^2 \rtimes_{\widetilde{\sigma}} \mathbb{Z} \quad \text{with} \quad \widetilde{\sigma}(e_1) = e_1, \ \widetilde{\sigma}(e_2) = e_1 + e_2.$$

Hence, the $*$-automorphism $\sigma \in \mathrm{Aut}(C(\mathbb{T}^2))$ satisfying $\sigma(z_1) = z_1$ and $\sigma(z_2) = z_1z_2$ gives rise to an isomorphism

$$C^*(H_3) \xrightarrow{\cong} C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}, \quad u \mapsto u, \ v \mapsto z_2,$$

where $u \in C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}$ denotes the canonical unitary implementing β. Using this identification, the action α is given by $\alpha_1 = \mathrm{Ad}(u)$ and $\alpha_2 = \mathrm{Ad}(z_2)$, and the commutator associated with α satisfies

$$u(\alpha) = u^*z_2^*uz_2 = z_1z_2^*z_2 = z_1 \in C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}.$$

The Bott projection $e := e(z_1, z_2) \in M_2(C(\mathbb{T}^2))$ is unitarily equivalent to $\sigma^{(2)}(e)$, see [1]. So, we find a unitary $x \in \mathcal{U}_2(C(\mathbb{T}^2))$ with $\sigma^{(2)}(e) = xex^*$, and define the Bott element $\kappa(e, x^*u^{(2)}) \in K_1(C^*(H_3))$.

Let us recall the K-theory of $C^*(H_3)$, which was determined in [1, 1.4(a)].

Proposition 5.2.1. *The K-theory of the group C^*-algebra of the discrete Heisenberg group $C^*(H_3)$ is given by*

$$\begin{aligned} K_0(C^*(H_3)) &= \mathbb{Z}^3\left[\kappa(z_1, z_2), \kappa(u, z_1), [1]\right], \\ K_1(C^*(H_3)) &= \mathbb{Z}^3\left[[z_2], [u], \kappa(e, x^*u^{(2)})\right]. \end{aligned}$$

Proof. We have that $K_0(\sigma) = \mathrm{id}$, and thus the Pimsner-Voiculescu sequence associated with $(C(\mathbb{T}^2), \sigma, \mathbb{Z})$ is of the form

$$\begin{tikzcd} K_0(C(\mathbb{T}^2)) \arrow[r, "0"] & K_0(C(\mathbb{T}^2)) \arrow[r] & K_0(C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}) \arrow[d, "\rho_0"] \\ K_1(C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z}) \arrow[u, "\rho_1"] & K_1(C(\mathbb{T}^2)) \arrow[l] & K_1(C(\mathbb{T}^2)) \arrow[l, "K_1(\sigma)-\mathrm{id}"'] \end{tikzcd}$$

The kernel of $K_1(\sigma)$ is given by $\ker(K_1(\sigma) - \mathrm{id}) = \mathbb{Z}[z_1]$, showing that the cyclic subgroup $[z_2] \in K_1(C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z})$ is a non-trivial direct summand. Proposition 3.2.1 implies that $\rho_1([u]) = [1]$ and $\rho_1(\kappa(e, x^* u^{(2)})) = [e]$. This shows the assertion for $K_1(C^*(H_3))$.

Recall that $K_0(C(\mathbb{T})) \cong \mathbb{Z}^2$ is generated by $[1]$ and $\kappa(z_1, z_2)$. Moreover, the Pimsner-Voiculescu sequence shows that $K_0(C(\mathbb{T}^2)) \to K_0(C(\mathbb{T}^2) \rtimes_\sigma \mathbb{Z})$ is split-injective. By Proposition 3.2.3, we have that $\rho_0(\kappa(u, z_1)) = [z_1]$, and hence $\kappa(u, z_1)$ may be taken as a third generator. ☐

Theorem 5.2.2. *The K-theory of the crossed product $C^*(H_3) \rtimes_\alpha \mathbb{Z}^2$ is given as*

$$K_0(C^*(H_3)) \rtimes_\alpha \mathbb{Z}^2) \cong K_1(C^*(H_3)) \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^{10}.$$

In particular, $C^*(H_3) \rtimes_\alpha \mathbb{Z}^2$ and $C^*(H_3) \otimes C(\mathbb{T}^2)$ are not isomorphic in K-theory.

Proof. By applying Proposition 4.2.1 to $d_*(\alpha) : K_*(C^*(H_3) \to K_{*+1}(C^*(H_3)$, we obtain

$$d_1([z_2]) = \kappa(u(\alpha), z_2) = \kappa(z_1, z_2) \quad \text{and} \quad d_1([u])) = \kappa(z_1, u).$$

The discussion in Chapter 2 shows that the existence of a trace on $C^*(H_3)$ prevents $[1] \in K_0(C^*(H_3))$ from being a Bott element associated with two exactly commuting unitaries. Since α is pointwise inner, every element in the image of $d_1(\alpha))$ is representable as such a Bott element, and hence

$$\mathrm{im}(d_1(\alpha)) = \mathbb{Z}^2 \left[\kappa(z_1, z_2),\ \kappa(z_1, u)\right] \subseteq K_0(C^*(H_3)),$$

which sits inside $K_0(C^*(H_3))$ as a direct summand.

It holds that $d_0([1]) = 0$, and moreover, by Proposition 4.2.3, we have that $d_0(\alpha) \circ d_1(\alpha) = 0$. Since $K_0(C^*(H_3))$ is generated by $[1]$, $\kappa(z_1, z_2)$, and $\kappa(z_1, u)$, we conclude that $d_0(\alpha)$ vanishes.

If $G_0 := K_0(C^*(H_3))$ and $G_1 := K_1(C^*(H_3))$, then the Pimsner-Voiculescu sequence for $(C^*(H_3) \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$ is of the form

$$\begin{tikzcd}
G_1 \oplus G_0 \arrow[r, "d_1(\alpha)\oplus 0"] & G_0 \oplus G_1 \arrow[r] & K_0(C^*(H_3) \rtimes_\alpha \mathbb{Z}^2) \arrow[d] \\
K_1(C^*(H_3) \rtimes_\alpha \mathbb{Z}^2) \arrow[u] & G_1 \oplus G_0 \arrow[l] & G_0 \oplus G_0 \arrow[l, "0"']
\end{tikzcd} \tag{5.1}$$

The result now follows by splitting up this six-term exact sequence into two extension, and then comparing the ranks of the occurring abelian groups. ☐

We also find pointwise inner $\mathbb{Z}^2$-actions on $C^*(H_3)$ whose associated crossed products have torsion in K-theory.

Corollary 5.2.3. *For $m, n \in \mathbb{N}$, let $\alpha(m, n)$ denote the pointwise inner $\mathbb{Z}^2$-action on $C^*(H_3)$ generated by α_1^m and α_2^n. Then*

$$K_0(C^*(H_3) \rtimes_{\alpha(m,n)} \mathbb{Z}^2) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/mn\mathbb{Z} \oplus \mathbb{Z}/mn\mathbb{Z},$$
$$K_1(C^*(H_3) \rtimes_{\alpha(m,n)} \mathbb{Z}^2) \cong \mathbb{Z}^{10}.$$

Proof. Proposition 4.2.4 yields that $d_*(\alpha(m,n)) = mn \cdot d_*(\alpha)$. The proof now follows from the exact sequence (5.1) with $d_1(\alpha)$ replaced by $d_1(\alpha(m,n))$. $\square$

As another consequence of Theorem 5.2.2, we get the following sufficient criterion for pointwise inner $\mathbb{Z}^2$-actions to have non-trivial obstruction homomorphisms.

Corollary 5.2.4. *Let* $\varphi : (C^*(H_3), \alpha, \mathbb{Z}^2) \to (B, \beta, \mathbb{Z}^2)$ *be a unital equivariant* $*$*-homomorphism. If* $K_0(\varphi) : K_0(C^*(H_3)) \to K_0(B)$ *is injective, then* $d_1(\beta) \neq 0$.

Proof. By the proof of Theorem 5.2.2 and the naturality of the obstruction homomorphism, it follows that $d_1(\beta)([\varphi(z_2)]))$ and $d_1(\alpha')([\varphi(u)]))$ are non-trivial. $\square$

At this point we remark that in the situation of Corollary 5.2.4, we also get injectivity of $K_1(\varphi)$. To see this, it suffices to show that $K_1(\varphi)(\kappa(e, x^*u^{(2)})) \neq 0$. In fact, one can use the mapping torus picture of the obstruction homomorphism to show that $d_1(\alpha)(\kappa(e, x^*u^{(2)})) \in K_1(B)$ generates a direct summand isomorphic to $\mathbb{Z}$. The claim then follows by injectivity of $K_0(\varphi)$.

5.3 Certain pointwise inner actions on amalgamated free product C^*-algebras

Throughout this section, A denotes a unital separable C^*-algebra, and α a pointwise inner $\mathbb{Z}^2$-action on A whose associated commutator $u(\alpha)$ has full spectrum. Let $u, v \in A$ be unitaries satisfying $\alpha_1 = \mathrm{Ad}(u)$ and $\alpha_2 = \mathrm{Ad}(v)$.

We first give the construction of C^*-dynamical systems $(C, \alpha, \mathbb{Z}^2)$ with non-trivial obstruction homomorphisms $d_0(\alpha)$. Let B be a unital separable C^*-algebra whose K-groups both do not vanish. Also assume that B contains a central unitary w with full spectrum and a projection $p \in M_n(B)$ such that

$$\kappa(p, w^{(n)}) \neq k[w] \in K_1(B) \quad \text{for all} \quad k \in \mathbb{Z}. \tag{5.2}$$

Consider the two injective $*$-homomorphisms

$$i_1 : C(\mathbb{T}) \longrightarrow A, \quad i_1(z) := u(\alpha), \qquad \text{and} \qquad i_2 : C(\mathbb{T}) \longrightarrow B, \quad i_2(z) := w,$$

and form the amalgamated free product $C := A *_{C(\mathbb{T})} B$ (see [4, 10.11.11] for a definition). There are natural unital $*$-homomorphisms $j_1 : A \to C$ and $j_2 : B \to C$, which are also injective by [3, 3.1]. Since $u(\alpha) = w$ is central in C, the action on A extends to a pointwise inner $\mathbb{Z}^2$-action on C, which we also denote by α. The obstruction homomorphism $d_0(\alpha) : K_0(C) \to K_1(C)$ satisfies

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)^{(n)}) = \kappa(p, w^{(n)}).$$

Next, we prove that this element does not vanish.

Lemma 5.3.1. *We have that* $d_0(\alpha)([p]) = \kappa(p, w^{(n)}) \neq 0 \in K_1(C)$.

Proof. By [47, 6.1], there exists a six-term exact sequence

$$\begin{tikzcd}
K_0(C(\mathbb{T})) \arrow[r, "K_0(i_1)\oplus K_0(i_2)"] & K_0(A)\oplus K_0(B) \arrow[r, "K_0(j_1)-K_0(j_2)"] & K_0(C) \arrow[d] \\
K_1(C) \arrow[u] & K_1(A)\oplus K_1(B) \arrow[l, "K_1(j_1)-K_1(j_2)"'] & K_1(C(\mathbb{T})) \arrow[l, "K_1(i_1)\oplus K_1(i_2)"']
\end{tikzcd} \tag{5.3}$$

Since

$$(K_1(j_1) - K_1(j_2))(0 \oplus -\kappa(p, w^{(n)})) = \kappa(p, w^{(n)}),$$

it suffices to check that $0 \oplus -\kappa(p, w^{(n)})$ is not contained in the image of the map $K_1(i_1) \oplus K_1(i_2)$. We have that $[u(\alpha)] = 0 \in K_1(A)$, and therefore

$$\operatorname{im}(K_1(i_1) \oplus K_1(i_2)) = 0 \oplus \mathbb{Z}[w].$$

By assumption, it holds that $\kappa(p, w^{(n)}) \neq k[w]$ for all $k \in \mathbb{Z}$, and the proof is complete. □

As the conditions on A and B are very mild, Lemma 5.3.1 can be applied in many situation. We would like to discuss one example which is of particular interest. Take $A := C^*(H_3)$ and equip it with the action α from Section 5.2. Let $B := C(\mathbb{T}) \oplus C(\mathbb{T})$ and set $w := z \oplus z$ and $p := 1 \oplus 0$. Observe that these elements satisfy condition (5.2). Hence, by Lemma 5.3.1 the amalgamated free product $C_1 := A *_{C(\mathbb{T})} B$ satisfies $\kappa(p, w) \neq 0 \in K_1(C_1)$. The pointwise inner action α on C_1 is inherited by the natural action on A.

The C^*-dynamical system $(C_1, \alpha, \mathbb{Z}^2)$ is universal for K_1-obstructions coming from pointwise inner $\mathbb{Z}^2$-actions in the following sense. For every pointwise inner $\mathbb{Z}^2$-action γ on a unital C^*-algebra D with $\gamma_1 = \operatorname{Ad}(\bar{u})$ and $\gamma_2 = \operatorname{Ad}(\bar{v})$, and every projection $\bar{p} \in D$, there is a unital equivariant $*$-homomorphism

$$\varphi : C_1 \longrightarrow D, \quad u \mapsto \bar{u},\ v \mapsto \bar{v},\ p \mapsto \bar{p}.$$

By the naturality of the obstruction homomorphism,

$$K_1(\varphi)(d_0(\alpha)([p])) = d_0(\gamma)([\bar{p}]).$$

Therefore, we can think of $d_0(\alpha)([p]) = \kappa(p, u(\alpha)) \in K_1(C_1)$ as the universal K_1-obstruction for pointwise inner $\mathbb{Z}^2$-actions.

The universal property of $(C_1, \alpha, \mathbb{Z}^2)$ also yields that $\kappa(p, u(\alpha)) \in K_1(C_1)$ has infinite order and induces a split-injection $\mathbb{Z}\,[\kappa(p, u(\alpha))] \to K_1(C_1)$. To see this, consider the C^*-dynamical system $(C^*(H_3)\otimes\mathcal{O}^\infty, \alpha\otimes\mathrm{tr}, \mathbb{Z}^2)$, where $\mathcal{O}^\infty$ is the UCT Kirchberg algebra with $K_0(\mathcal{O}^\infty) = 0$ and $K_1(\mathcal{O}^\infty) \cong \mathbb{Z}$. The proof of Theorem 5.2.2 shows that there is a projection $q \in C^*(H_3) \otimes \mathcal{O}^\infty$ such that the cyclic subgroup induced by

$$d_0(\alpha \otimes \mathrm{tr})([q]) \neq 0 \in K_1(C^*(H_3) \otimes \mathcal{O}^\infty)$$

sits inside $K_1(C^*(H_3)\otimes\mathcal{O}^\infty) \cong \mathbb{Z}^3$ as a non-trivial direct summand. The claim now follows by considering the equivariant $*$-homomorphism $\varphi : C_1 \to C^*(H_3) \otimes \mathcal{O}^\infty$ satisfying $\varphi(u) = u \otimes 1$, $\varphi(v) = v \otimes 1$, and $\varphi(p) = q$.

Proposition 5.3.2. *The K-theory of C_1 satisfies $K_0(C_1) \cong K_1(C_1) \cong \mathbb{Z}^4$. More precisely, $K_*(j_1) : K_*(A) \to K_*(C_1)$ is split-injective and induces the following decompositions*

$$K_0(C_1) \cong K_0(A) \oplus \mathbb{Z}[p] \quad \textit{and} \quad K_1(C_1) \cong K_1(A) \oplus \mathbb{Z}[\kappa(p, u(\alpha))].$$

Proof. A short computation shows that

$$K_*(i_1) \oplus K_*(i_2) : K_*(C(\mathbb{T})) \longrightarrow K_*(A) \oplus K_*(B)$$

is split-injective, and hence the six-term exact sequence (5.3) associated with the amalgamated free product $C_1 = A *_{C(\mathbb{T})} B$ reduces to a split-extension

$$0 \longrightarrow K_*(C(\mathbb{T})) \xrightarrow{K_*(i_1) \oplus K_*(i_2)} K_*(A) \oplus K_*(B) \xrightarrow{K_*(j_1) - K_*(j_2)} K_*(C_1) \longrightarrow 0. \quad (5.4)$$

Consequently, $K_*(C_1)$ is torsion-free, and by recalling that the K-theory of the Heisenberg group C^*-algebra A satisfies $K_0(A) \cong K_1(A) \cong \mathbb{Z}^3$, we conclude that $K_0(C_1) \cong K_1(C_1) \cong \mathbb{Z}^4$.

The universal property of the amalgamated free product yields a homomorphism $\varphi : C_1 \to A$ satisfying $\varphi \circ j_1 = \mathrm{id}_A$, $(\varphi \circ j_2)(p) = 1$, and $(\varphi \circ j_2)(w) = u(\alpha)$. Obviously, φ is surjective with splitting $j_1 : A \to C_1$. This shows that $K_*(j_1)$ is split-injective, and thus $K_*(A)$ sits in $K_*(C_1)$ as a direct summand. Moreover, we get that $[p] \in K_0(C_1)$ has infinite order and induces a split-injection $\mathbb{Z}[p] \to K_0(C_1)$. Since we already know that the analogous statement for $\kappa(p, u(\alpha)) \in K_1(C_1)$ is true as well, it remains to show that $[p]$ and $\kappa(p, u(\alpha))$ both do not lie in $K_*(j_1)(K_*(A)) \subseteq K_*(C_1)$.

Suppose that there is some $g \in K_0(A)$ with $K_0(j_1)(g) = [p]$. Another lift for $[p] \in K_0(C_1)$ is given by

$$(K_0(j_1) - K_0(j_2))(0 \oplus -[1 \oplus 0]) = [p].$$

Hence, exactness of (5.4) yields the existence of some $k \in \mathbb{Z}$ satisfying

$$k([1] \oplus [1 \oplus 1]) + g \oplus 0 = 0 \oplus -[1 \oplus 0].$$

This is a contradiction, and thus $[p] \notin K_0(j_1)(K_0(A))$. Basically the same proof works for $\kappa(p, u(\alpha)) \notin K_1(j_1)(K_1(A))$ if one uses that

$$(K_1(j_1) - K_1(j_2))(0 \oplus -[z \oplus 0]) = \kappa(p, u(\alpha)) \in K_1(C_1).$$

□

Theorem 5.3.3. *The K-theory of the crossed product $C_1 \rtimes_\alpha \mathbb{Z}^2$ satisfies*

$$K_0(C_1 \rtimes_\alpha \mathbb{Z}^2) \cong K_0(C_1 \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^{13}.$$

In particular, $K_*(C_1 \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C_1 \otimes C(\mathbb{T}^2))$.

Proof. By the naturality of the obstruction homomorphisms, there is a commutative diagram

$$\begin{tikzcd}
K_*(A) \arrow[r, "K_*(j_1)"] \arrow[d, "d_*(\alpha)"'] & K_*(C_1) \arrow[d, "d_*(\alpha)"] \\
K_{*+1}(A) \arrow[r, "K_{*+1}(j_1)"] & K_{*+1}(C_1)
\end{tikzcd}$$

Since $K_*(j_1)$ is split-exact, $d_*(\alpha) : K_*(C_1) \to K_{*+1}(C_1)$ is completely determined by the obstruction homomorphism associated with $(A, \alpha, \mathbb{Z}^2)$ and its values on $[p] \in K_0(C_1)$ and $\kappa(p, u(\alpha)) \in K_1(C_1)$. We have that

$$d_0(\alpha)([p]) = \kappa(p, u(\alpha)),$$

and moreover, by Proposition 4.2.3,

$$d_1(\alpha)(\kappa(p, u(\alpha))) = 0.$$

This together with the proof of Theorem 5.2.2 yields that the obstruction homomorphism $d_*(\alpha) : K_*(C_1) \to K_{*+1}(C_1)$ satisfies

$$\operatorname{coker}(d_0(\alpha)) \cong \ker(d_0(\alpha)) \cong \mathbb{Z}^3 \quad \text{and} \quad \operatorname{coker}(d_1(\alpha)) \cong \ker(d_1(\alpha)) \cong \mathbb{Z}^2.$$

As in the proof of Theorem 5.2.2, the statement now follows from the Pimsner-Voiculescu sequence associated with $(C_1 \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. ☐

Let us present another instance of a C^*-dynamical with non-trivial obstruction homomorphism arising from the above construction. Whereas the C^*-dynamical system $(C_1, \alpha, \mathbb{Z}^2)$ is interesting for its universal property, the next one is minimal concerning the K-groups of the underlying C^*-algebra.

Proposition 5.3.4. *There exists a unital separable C^*-algebra C with $K_0(C) \cong K_1(C) \cong \mathbb{Z}$ admitting a pointwise inner $\mathbb{Z}^2$-action α which is pointwise homotopic to the trivial action inside* $\operatorname{Inn}(A)$ *and satisfies*

$$K_0(C \rtimes_\alpha \mathbb{Z}^2) \cong K_1(C \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^3.$$

In particular, $K_*(C \rtimes_\alpha \mathbb{Z}^2) \ncong K_*(C \otimes C(\mathbb{T}^2))$

Proof. Define $A := C^*(H_3) \otimes \mathcal{O}_2$ and note that this C^*-algebra has trivial K-theory, see [9, 2.3]. By Kirchberg's absorbtion theorem [22], $A \cong A \otimes \mathcal{O}_\infty$, and hence A is K_1-injective, see [19, 2.10]. The unitaries $u \otimes 1$ and $v \otimes 1 \in A$ are therefore homotopic to $1 \in \mathcal{U}(A)$. By identifying $u(\alpha) \otimes 1 \in A$ with $z \oplus z \in C(\mathbb{T}) \oplus C(\mathbb{T})$, we can form the amalgamated free product

$$C := A *_{C(\mathbb{T})} (C(\mathbb{T}) \oplus C(\mathbb{T})).$$

Consider the pointwise inner $\mathbb{Z}^2$-action α on C induced by $\mathrm{Ad}(u\otimes 1)$ and $\mathrm{Ad}(v\otimes 1)$, which is obviously pointwise homotopic to the trivial action inside $\mathrm{Inn}(A)$. A similar calculation as in the proof of Proposition 5.3.2 shows that $K_0(C) = \mathbb{Z}[p]$ and $K_1(C) = \mathbb{Z}\,[\kappa(p, u(\alpha))]$. Moreover, the associated obstruction homomorphism $d_*(\alpha) : K_*(C) \to K_{*+1}(C)$ satisfies $d_0(\alpha)([p]) = \kappa(p, u(\alpha))$ and $d_1(\alpha) = 0$. The result now follows from the Pimsner-Voiculescu sequence for $(C \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. □

Next, we present the analogous construction of C^*-dynamical systems $(C, \alpha, \mathbb{Z}^2)$ with non-trivial obstruction homomorphisms $d_1(\alpha)$. Let B be a unital separable C^*-algebra whose K-groups both do not vanish. Assume further that there is a central unitary $w \in B$ with full spectrum and a unitary $x \in \mathcal{U}_n(B)$ such that

$$\kappa(w^{(n)}, x) \neq k[1] \in K_0(B) \quad \text{for all } k \in \mathbb{Z}. \tag{5.5}$$

The two injective $*$-homomorphisms

$$i_1 : C(\mathbb{T}) \longrightarrow A, \quad i_1(z) := u(\alpha), \qquad \text{and} \qquad i_2 : C(\mathbb{T}) \longrightarrow B, \quad i_2(z) := w,$$

give rise to an amalgamated free product $C := A *_{C(\mathbb{T})} B$. Here, $u(\alpha) = w$ is a central unitary in C, and hence α extends to a pointwise inner action on C, which we also denote by α. Note that the associated obstruction homomorphism $d_1(\alpha) : K_1(C) \to K_0(C)$ satisfies

$$d_1(\alpha)([x]) = \kappa(u(\alpha)^{(n)}, x) = \kappa(w^{(n)}, x).$$

The main observation is that this element does not vanish.

Lemma 5.3.5. *It holds that $\kappa(w^{(n)}, x) \neq 0 \in K_0(C)$.*

Proof. The proof is very similar to the one of Lemma 5.3.1. □

It is worth mentioning that if $[1] \in K_0(A)$ has infinite order, then Lemma 5.3.5 remains true if we replace (5.5) by the condition that $\kappa(w^{(n)}, x) \neq 0 \in K_1(B)$.

There is a C^*-dynamical system $(C_0, \alpha, \mathbb{Z}^2)$ which is universal for K_0-obstructions coming from pointwise inner $\mathbb{Z}^2$-actions. To define it, let again $A := C^*(H_3)$ and equip it with the $\mathbb{Z}^2$-action α from Section 5.2. Moreover, let $B := C(\mathbb{T}^2)$ and set $w := z_1$ and $x := z_2$. Observe that $\kappa(w, x) = \mathfrak{b} \in K_0(B)$ is the (classical) Bott element, and hence (5.2) is satisfied. Form the amalgamated free product $C_0 := A *_{C(\mathbb{T})} C(\mathbb{T}^2)$, which carries the induced pointwise inner $\mathbb{Z}^2$-action α.

Universality of this system expresses in the following property. For every pointwise inner $\mathbb{Z}^2$-action γ on a unital C^*-algebra D with $\gamma_1 = \mathrm{Ad}(\bar{u})$ and $\gamma_2 = \mathrm{Ad}(\bar{v})$, and every unitary $\bar{x} \in D$, there is a unital equivariant $*$-homomorphism

$$\varphi : C_0 \longrightarrow D, \quad u \mapsto \bar{u}, \; v \mapsto \bar{v}, \; x \mapsto \bar{x}.$$

By the naturality of the obstruction homomorphism,

$$K_0(\varphi)(d_1(\alpha)([x])) = d_1(\gamma)([\bar{x}]).$$

In this way, $d_1(\alpha)([x]) = \kappa(u(\alpha), x)$ can be considered as the universal K_1-obstruction for pointwise inner $\mathbb{Z}^2$-actions. Furthermore, the proof of Theorem 5.2.2 shows that $\kappa(u(\alpha), x) \in K_0(C_0)$ has infinite order and that $\mathbb{Z}[\kappa(u(\alpha), x)] \to K_0(C_0)$ is split-injective.

Proposition 5.3.6. *The K-theory of C_0 satisfies $K_0(C_0) \cong K_1(C_0) \cong \mathbb{Z}^4$. More precisely, $K_*(j_1) : K_*(A) \to K_*(C_0)$ is split-injective and induces the following decompositions*

$$K_0(C_0) \cong K_0(A) \oplus \mathbb{Z}[\kappa(u(\alpha), x)] \quad \textit{and} \quad K_1(C_0) \cong K_1(A) \oplus \mathbb{Z}[x].$$

Proof. The proof is similar to the one of Proposition 5.3.2. □

Theorem 5.3.7. *The K-theory of the crossed product $C_0 \rtimes_\alpha \mathbb{Z}^2$ satisfies*

$$K_0(C_0 \rtimes_\alpha \mathbb{Z}^2) \cong K_1(C_0 \rtimes_\alpha \mathbb{Z}^2) \cong \mathbb{Z}^{13}.$$

In particular, $K_*(C_0 \rtimes_\alpha \mathbb{Z}^2) \not\cong K_*(C_0 \otimes C(\mathbb{T}^2))$.

Proof. Considering Proposition 5.3.6 and the commutative diagram

$$\begin{tikzcd}
K_*(A) \arrow[r, "K_*(j_1)"] \arrow[d, "d_*(\alpha)"'] & K_*(C_0) \arrow[d, "d_*(\alpha)"] \\
K_{*+1}(A) \arrow[r, "K_{*+1}(j_1)"] & K_{*+1}(C_0)
\end{tikzcd}$$

we see that the restriction of $d_*(\alpha) : K_*(C_0) \to K_{*+1}(C_0)$ to the direct summand $K_*(A)$ coincides with the obstruction homomorphism associated with $(A, \alpha, \mathbb{Z}^2)$. Furthermore,

$$d_1(\alpha)([x]) = \kappa(u(\alpha), x) \qquad \text{and} \qquad d_0(\alpha)(\kappa(u(\alpha), x)) = 0,$$

where the second equality follows from Proposition 4.2.3. Altogether,

$$d_0(\alpha) = 0 \quad \text{and} \quad \ker(d_1(\alpha)) \cong \operatorname{coker}(d_1(\alpha)) \cong \mathbb{Z}.$$

Finally, we proceed as in the proof of Theorem 5.2.2, and consider the Pimsner-Voiculescu sequence for $(C_0 \rtimes_{\alpha_1} \mathbb{Z}, \check{\alpha}_2, \mathbb{Z})$. □

Chapter 6

Endomorphic homotopies between $\mathbb{Z}^n$-actions

After an extensive study of the K-theory for crossed products by $\mathbb{Z}^2$-actions, we now turn to C^*-dynamical systems $(A, \alpha, \mathbb{Z}^n)$ for arbitrary $n \in \mathbb{N}$. In the course of this chapter, we conclude that two $\mathbb{Z}^n$-actions on a C^*-algebra A give rise to crossed products with isomorphic K-theory if the actions are homotopic as $\mathbb{N}^n$-actions, i.e. if there is an $\mathbb{N}^n$-action on $C([0, 1], A)$ such that the actions on A are obtained by evaluation at the endpoints. This generalizes the well-known result that a crossed product by $\mathbb{Z}^n$ only depends on the automorphic homotopy class of the action, see Corollary 3.1.2. For the proof, we use dilation results by Laca [23] and by Pask, Raeburn, and Yeend [33].

All our semigroups are supposed to be discrete and have a unit. Moreover, we want semigroup homomorphisms to respect units.

A *semigroup dynamical system* (A, α, S) consists of a C^*-algebra A, a semigroup S, and a semigroup homomorphism $\alpha : S \to \mathrm{End}(A)$. A *covariant homomorphism* into the multiplier algebra $\mathcal{M}(D)$ of some C^*-algebra D is a pair (ϕ, V) consisting of a non-degenerate $*$-homomorphism $\phi : A \to \mathcal{M}(D)$ and a semigroup homomorphism $V : S \to \mathrm{Isom}(\mathcal{M}(D))$ into the isometries of $\mathcal{M}(D)$ satisfying the covariance condition

$$\phi(\alpha_s(a)) = V_s \phi(a) V_s^* \quad \text{for all } a \in A,\ s \in S.$$

Given a $\mathbb{Z}^n$-action α on a C^*-algebra A, we can consider its restriction to an $\mathbb{N}^n$-semigroup action, which we also denote by α. In this way, a C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$ gives rise to a semigroup dynamical system $(A, \alpha, \mathbb{N}^n)$.

There are several notions of semigroup crossed product C^*-algebras. The non-invertibility of endomorphisms and the corresponding isometries provides a source of freedom which does not exist in the case of group crossed products. We follow [23] and [25], and define semigroup crossed products as C^*-algebras which are universal with respect to covariant homomorphisms.

Definition 6.1. A *crossed product* for a semigroup dynamical system (A, α, S) is a triple (B, ι_A, ι_S) consisting of a C^*-algebra B, a non-degenerate $*$-homomorphism $\iota_A : A \to B$, and a semigroup homomorphism $\iota_S : S \to \mathrm{Isom}(\mathcal{M}(B))$ into the isometries of $\mathcal{M}(B)$ such that the following conditions are satisfied:

i) We have that $\iota_A(\alpha_s(a)) = \iota_S(s)\iota_A(a)\iota_S(s)^*$ for all $a \in A$ and $s \in S$.

ii) For any covariant homomorphism (ϕ, V) of (A, α, S) into $\mathcal{M}(D)$, there is a non-degenerate $*$-homomorphism $\phi \times V : B \to \mathcal{M}(D)$ with $\phi \times V \circ \iota_A = \phi$ and $\underline{\phi \times V} \circ \iota_S = V$.

iii) B is generated by $\{\iota_A(a)\iota_S(s) \ : a \in A, \ s \in S\}$ as a C^*-algebra.

As in the group case, the properties $i) - iii)$ uniquely determine the semigroup crossed product up to canonical isomorphism. We usually write $A \rtimes_\alpha S$ for the semigroup crossed product associated with (A, α, S). It may happen that a semigroup dynamical system (A, α, S) does not admit a non-trivial covariant homomorphism [45, 2.1], so that $A \rtimes_\alpha S = 0$ is possible even if $A \neq 0$. Let $\varphi : (A, \alpha, S) \to (B, \beta, S)$ denote a non-degenerate equivariant $*$-homomorphism and assume that the respective crossed products $(A, \rtimes_\alpha S, \iota_A, \iota_S)$ and $(B \rtimes_\beta S, j_B, j_S)$ both exist. Then $(j_B \circ \varphi, j_S)$ defines a covariant homomorphism for (A, α, S), and we obtain a $*$-homomorphism $\varphi \rtimes S : A \rtimes_\alpha S \to B \rtimes_\beta S$ satisfying the conditions in *ii)*.

Let S be a semigroup which admits an embedding into a group, and let α be an action of S by automorphisms on a C^*-algebra A. If G is an enveloping group for S, then the semigroup action of S extends naturally to a group action of G on A, which we may also call α. Since automorphisms are non-degenerate, every covariant homomorphism (ϕ, V) for (A, α, S) has the property that V_s is a unitary for every $s \in S$. Since V_s^* implements the automorphism $\alpha_{s^{-1}}$, (ϕ, V) gives rise to a non-degenerate covariant homomorphism for (A, α, G). On the other hand, every non-degenerate covariant homomorphism (ψ, U) for (A, α, G) induces a covariant homomorphism for (A, α, S) by restricting U to S. The universal properties of $A \rtimes_\alpha S$ and $A \rtimes_\alpha G$ now yield that these two crossed product C^*-algebras are canonically isomorphic.

A semigroup S is called *(left) Ore* if it admits an embedding into a group G such that $G = S^{-1}S$. Obviously, $\mathbb{N}^n \subseteq \mathbb{Z}^n$ is left Ore. Semigroup dynamical systems with actions of Ore semigroups admit dilations to C^*-dynamical systems coming from actions of their enveloping groups. This result is due to Laca, see [23, 2.1.1 and 2.2.2].

Theorem 6.2 (Minimal automorphic dilation)**.** *Let S be an Ore semigroup with enveloping group $G = S^{-1}S$. If (A, α, S) is a semigroup dynamical system, then there is a C^*-dynamical system (B, β, G) and a $*$-homomorphism $\iota : A \to B$ satisfying the following conditions:*

i) The action β dilates α, that is, $\beta_s \circ \iota = \iota \circ \alpha_s$ for all $s \in S$.

ii) The C^*-dynamical system (B, β, G) is minimal in the sense that $\bigcup_{s\in S} \beta_s^{-1}(\iota(A))$ is dense in B.

These conditions determine (B, β, G) up to canonical isomorphism, and we call $((B, \beta, G), \iota)$ the minimal automorphic dilation *for the semigroup dynamical system (A, α, S). If α acts by injective endomorphisms, then $\iota : A \to B$ is injective.*

Note that in Laca's result [23], the C^*-algebra A is supposed to have a unit. However, the proof reveals that this assumption is not needed. He also gives an explicit model $((A_\infty, \alpha_\infty, G), \iota_\infty)$ of the minimal automorphic dilation for (A, α, S). For $S = \mathbb{N}^n$, which is the case we are interested in, the C^*-algebra A_∞ is given as the inductive limit

$$\begin{tikzcd} A \arrow[r, "\psi"] & A \arrow[r, "\psi"] & A \arrow[r, "\psi"] & \ldots \arrow[r] & A_\infty \end{tikzcd} \qquad (6.1)$$

induced by the $*$-endomorphism $\psi := \alpha_{(1,\ldots,1)} = \alpha_1 \circ \ldots \circ \alpha_n$, and $\iota_\infty : A \to A_\infty$ is the canonical map corresponding to the first copy of A. Given an equivariant $*$-homomorphism $\varphi : (A, \alpha, \mathbb{N}^n) \to (B, \beta, \mathbb{N}^n)$, the naturality of the inductive limit gives rise to an equivariant $*$-homomorphism $\varphi_\infty : (A_\infty, \alpha_\infty, \mathbb{Z}^n) \to (B_\infty, \beta_\infty, \mathbb{Z}^n)$ which fits into the following commutative

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\iota_\infty"'] & B \arrow[d, "\iota_\infty"] \\ A_\infty \arrow[r, "\varphi_\infty"] & B_\infty \end{tikzcd}$$

Assume that (A, α, S) is a semigroup dynamical system with an Ore semigroup S acting by *extendible* endomorphisms, i.e. by endomorphisms $\alpha_s : A \to A$ admitting a strictly continuous extension $\underline{\alpha_s} : \mathcal{M}(A) \to \mathcal{M}(A)$. It was shown in [25, 1.3] that the corresponding crossed product $(A \rtimes_\alpha S, \iota_A, \iota_S)$ exists whenever there is a non-trivial covariant representation. Moreover, $A \rtimes_\alpha S$ is a full corner of the crossed product by the dilated group action. This was first shown in [23, 2.2.1 and 2.2.2] for the case that A is unital (where the acting endomorphisms are automatically extendible). After this, Pask, Raeburn, and Yeend pointed out that virtually the same proof as in *loc. cit.* works in the non-unital setting as well, provided the acting endomorphisms are extendible, see [33, 4.5].

Theorem 6.3. *Let S be an Ore semigroup with enveloping group $G = S^{-1}S$. Let (A, α, S) be a semigroup dynamical system with α acting by extendible endomorphisms. Denote the corresponding minimal automorphic dilation by $((B, \beta, G), \iota)$ and write $j_B : B \to B \rtimes_\beta G$ for the canonical embedding. Then $\iota : A \to B$ is an extendible $*$-homomorphism and induces a canonical isomorphism between $A \rtimes_\alpha S$ and $\underline{j_B \circ \iota}(1)(B \rtimes_\beta G)\underline{j_B \circ \iota}(1)$, which is a full corner in $B \rtimes_\beta G$.*

Brown's stabilization theorem [7] yields that the C^*-algebras $A \rtimes_\alpha S$ and $B \rtimes_\beta G$ are stably isomorphic and that the $*$-homomorphism

$$\kappa_\alpha : A \rtimes_\alpha S \xrightarrow{\cong} \underline{j_B \circ \iota}(1)(B \rtimes_\beta G)\underline{j_B \circ \iota}(1) \longrightarrow B \rtimes_\beta G$$

induces an isomorphism in K-theory. Moreover, the construction of the isomorphism $A \rtimes_\alpha S \cong \underline{j_B \circ \iota}(1)(B \rtimes_\beta G)\underline{j_B \circ \iota}(1)$ given in [23] reveals that κ_α is natural with respect to non-degenerate equivariant $*$-homomorphisms in the obvious sense.

Theorem 6.4. *Let α and β be two $\mathbb{N}^n$-actions by extendible endomorphisms on a C^*-algebra A, and assume that the corresponding crossed products both exist. If α and β are homotopic, then $K_*(A \rtimes_\alpha \mathbb{N}^n) \cong K_*(A \rtimes_\beta \mathbb{N}^n)$.*

Proof. Let γ be an $\mathbb{N}^n$-action on $C([0,1],A)$ with $\mathrm{ev}_0 \circ \gamma_i = \alpha_i$ and $\mathrm{ev}_1 \circ \gamma_i = \beta_i$ for $i = 1, \ldots, n$. The equivariant surjective $*$-homomorphism

$$\mathrm{ev}_0 : (C([0,1],A), \gamma, \mathbb{N}^n) \longrightarrow (A, \alpha, \mathbb{N}^n)$$

gives rise to a commutative diagram

$$\begin{tikzcd}
C([0,1],A) \rtimes_\gamma \mathbb{N}^n \arrow[r, "\mathrm{ev}_0 \rtimes \mathbb{N}^n"] \arrow[d, "\kappa_\gamma"'] & A \rtimes_\alpha \mathbb{N}^n \arrow[d, "\kappa_\alpha"] \\
C([0,1],A)_\infty \rtimes_{\gamma_\infty} \mathbb{Z}^n \arrow[r, "(\mathrm{ev}_0)_\infty \rtimes \mathbb{Z}^n"'] & A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}^n
\end{tikzcd}$$

An application of K-theory yields the following commutative diagram

$$\begin{tikzcd}
K_*(C([0,1],A) \rtimes_\gamma \mathbb{N}^n) \arrow[r, "K_*(\mathrm{ev}_0 \rtimes \mathbb{N}^n)"] \arrow[d, "K_*(\kappa_\gamma)"', "\cong"] & K_*(A \rtimes_\alpha \mathbb{N}^n) \arrow[d, "\cong"', "K_*(\kappa_\alpha)"] \\
K_*(C([0,1],A)_\infty \rtimes_{\gamma_\infty} \mathbb{Z}^n) \arrow[r, "\cong", "K_*((\mathrm{ev}_0)_\infty \rtimes \mathbb{Z}^n)"'] & K_*(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}^n)
\end{tikzcd}$$

Observe that the homotopy invariance of K-theory and the concrete description of $C([0,1],A)_\infty$ as an inductive limit (6.1) show that $K_*((\mathrm{ev}_0)_\infty)$ is an isomorphism. Hence, the naturality property of the Pimsner-Voiculescu sequence and the Five Lemma yield that $K_*((\mathrm{ev}_0)_\infty \rtimes \mathbb{Z}^n) : K_*(C([0,1],A)_\infty \rtimes_{\gamma_\infty} \mathbb{Z}^n) \xrightarrow{\cong} K_*(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}^n)$ is in fact an isomorphism. Analogous considerations for β show that

$$K_*(\mathrm{ev}_1 \rtimes \mathbb{N}^n) \circ K_*(\mathrm{ev}_0 \rtimes \mathbb{N}^n)^{-1} : K_*(A \rtimes_\alpha \mathbb{N}^n) \xrightarrow{\cong} K_*(A \rtimes_\beta \mathbb{N}^n)$$

yields the desired isomorphism. ☐

As a consequence, we obtain that the K-theory of a crossed product by $\mathbb{Z}^n$ is determined by the action's $\mathbb{N}^n$-homotopy class.

Corollary 6.5. *Let α and β be $\mathbb{Z}^n$-actions on a C^*-algebra A. If α and β are homotopic as $\mathbb{N}^n$-actions, then $K_*(A \rtimes_\alpha \mathbb{Z}^n) \cong K_*(A \rtimes_\beta \mathbb{Z}^n)$.*

Proof. Since $A \rtimes_\alpha \mathbb{Z}^n \cong A \rtimes_\alpha \mathbb{N}^n$ and $A \rtimes_\beta \mathbb{Z}^n \cong A \rtimes_\beta \mathbb{N}^n$, the claim follows by applying Theorem 6.4 to the semigroup dynamical systems $(A, \alpha, \mathbb{N}^n)$ and $(A, \beta, \mathbb{N}^n)$. ☐

Chapter 7

Spectral sequences associated with $\mathbb{Z}^n$-actions

In this chapter, we investigate a spectral sequence converging to $K_*(A \rtimes_\alpha \mathbb{Z}^n)$, which is a special case of Kasparov's construction [20, 6.10] and which already appeared in [2] and [43]. It also recovers the Pimsner-Voiculescu exact sequence.

In the first section, we recall the general procedure of constructing a spectral sequence associated with a given finite cofiltration of C^*-algebras. We also comment on its convergence and naturality property. However, it is not intended to formally define spectral sequences and we omit proofs. Further information on spectral sequences can be found in [49].

The second section is concerned with a spectral sequence $(E_k, d_k)_{k \geq 1}$ associated with a C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$. Using that the corresponding crossed product and mapping torus have isomorphic K-theory, we obtain this spectral sequence by a natural finite cofiltration of $\mathcal{M}_\alpha(A)$. In [43], Savignen and Bellissard describe (E_1, d_1) in terms of what they call the *Pimsner-Voiculescu complex* (C_{PV}, d_{PV}), which we identify as a certain Koszul complex over the integral group ring of $\mathbb{Z}^n$ with coefficients in $K_*(A)$. This is used to prove that the E_2-term coincides with the group cohomology of $\mathbb{Z}^n$ with coefficients in $K_*(A)$ (recovering Kasparov's result [20, 6.10] in the special case that the acting group is $\mathbb{Z}^n$). We show that the isomorphism between (E_1, d_1) and (C_{PV}, d_{PV}) is also obtained as a consequence of a result yielding a partial description of the differentials d_k. For $n = 2$, the second level differential d_2 is shown to coincide with the associated obstruction homomorphism $d_*(\alpha)$ defined in Chapter 4. If α is a $\mathbb{Z}^n$-action whose induced action on K-theory is trivial, then a combination of these results yields a complete description of the second level boundary map in terms of the obstruction homomorphisms of the natural $\mathbb{Z}^2$-subactions.

7.1 Spectral sequences arising from finite cofiltrations of C^*-algebras

Consider the following finite cofiltration of C^*-algebras

$$\begin{tikzcd} A = F_n \arrow[r, two heads, "\pi_n"] & F_{n-1} \arrow[r, two heads] & \cdots \arrow[r, two heads] & F_1 \arrow[r, two heads, "\pi_1"] & F_0 \arrow[r, two heads, "\pi_0"] & F_{-1} = 0. \end{tikzcd} \tag{7.1}$$

For convenience, we artificially extend it by defining $F_k := F_n$ for $k > n$, $F_k := 0$ for $k < -1$, and $\pi_k := \mathrm{id}_{F_k}$ in either case.

The ideal $I_k := \ker(\pi_k)$ induces a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & I_k \arrow[r, "\iota_k"] & F_k \arrow[r, "\pi_k"] & F_{k-1} \arrow[r] & 0, \end{tikzcd} \tag{7.2}$$

whose associated boundary map is denoted by $\rho_*^{(k)} : K_*(F_{k-1}) \to K_{*+1}(I_k)$.

The spectral sequence we construct allows, in principle, to determine the K-theory of A by means of $K_*(I_k)$ and $K_*(F_k)$. To this end, it combines all necessary information of the six-term exact sequences associated with (7.2) in an appropriate way. We use the following standard technique due to Massey [28], [29].

Definition + Proposition 7.1.1 (Exact couple)**.** An *exact couple* is a pair of abelian groups A, B together with group homomorphisms $f : A \to A$, $g : A \to B$, and $h : B \to A$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] & & A \arrow[dl, "g"] \\ & B \arrow[ul, "h"] & \end{tikzcd}$$

commutes and is exact in the sense that at each place the image of the in-going arrow coincides with the kernel of the out-going one. We shall denote such an exact couple by (A, B, f, g, h).

For a given exact couple (A, B, f, g, h), the *derived couple* is the induced exact couple $(\mathrm{im}(f), \ker(g \circ h)/\,\mathrm{im}(g \circ h), f, g', h')$, where

$$\begin{aligned} g'(a) &= [g(\bar{a})] \qquad \text{for } a = f(\bar{a}) \in \mathrm{im}(f), \\ h'([b]) &= h(b) \qquad \text{for } b \in \ker(g \circ h). \end{aligned}$$

A *morphism* of exact couples $(\varphi, \psi) : (A, B, f, g, h) \to (A', B', f', g', h')$ is a pair of group homomorphisms $\varphi : A \to A'$ and $\psi : B \to B'$ such that the following three commutation relations hold:

$$\varphi \circ f = f' \circ \varphi, \quad \varphi \circ h = h' \circ \psi, \quad \psi \circ g = g' \circ \varphi.$$

A morphism of exact couples naturally induces a morphism between the derived couples.

For $p, q \in \mathbb{Z}$, let

$$E_1^{p,q} := K_{p+q}(I_p) \quad \text{and} \quad D_1^{p,q} := K_{p+q}(F_p),$$

and set

$$E_1^{*,*} := \bigoplus_{p,q\in\mathbb{Z}} E_1^{p,q} \quad \text{and} \quad D_1^{*,*} := \bigoplus_{p,q\in\mathbb{Z}} D_1^{p,q}.$$

The long exact sequences associated with (7.2) admit the exact couple

$$\begin{tikzcd} D_1^{*,*} \arrow[rr, "K_*(\pi_*)", "(-1{,}1)"'] & & D_1^{*,*} \arrow[dl, "(1{,}0)"', "\rho_*^{(*)}"] \\ & E_1^{*,*} \arrow[ul, "(0{,}0)"', "K_*(\iota_*)"] & \end{tikzcd} \tag{7.3}$$

We have attached each arrow with a pair of numbers denoting the bidegree of the respective map. Often, the bigrading is suppressed and we just write E_1 and D_1.

To obtain the derived exact couple, define $d_1 : E_1 \to E_1$ by

$$d_1^{p,q} := \rho_{p+q}^{(p+1)} \circ K_{p+q}(\iota_p) : E_1^{p,q} \longrightarrow E_1^{p+1,q}.$$

Then, d_1 obviously has bidegree $(1, 0)$ and satisfies $d_1 \circ d_1 = 0$ by exactness of (7.3). Set

$$E_2^{p,q} := \ker(d_1^{p,q})/\operatorname{im}(d_1^{p-1,q}) \quad \text{and} \quad D_2^{p,q} := \operatorname{im}(K_{p+q}(\pi_{p+1})),$$

and define

$$\begin{aligned} \bar{\iota}_{p,q} &: E_2^{p,q} \longrightarrow D_2^{p,q}, & [y] &\mapsto K_{p+q}(\iota_p)(y), \\ \bar{\rho}_{p,q} &: D_2^{p,q} \longrightarrow E_2^{p+2,q-1}, & x &\mapsto [\rho_{p+q}^{(p+2)}(\bar{x})], \end{aligned}$$

where $K_{p+q}(\pi_{p+1})(\bar{x}) = x$. The derived couple of (7.3) is now given by

$$\begin{tikzcd} D_2^{*,*} \arrow[rr, "K_*(\pi_*)", "(-1{,}1)"'] & & D_2^{*,*} \arrow[dl, "(2{,}-1)"', "\bar{\rho}_{*,*}"] \\ & E_2^{*,*} \arrow[ul, "(0{,}0)"', "\bar{\iota}_{*,*}"] & \end{tikzcd}$$

We can use the endomorphism $d_2 : E_2 \to E_2$ given by $d_2^{p,q} = \bar{\rho}_{p,q} \circ \bar{\iota}_{p,q}$ to derive another exact couple. Observe that d_2 has bidegree $(2, -1)$.

By repeating this procedure, we obtain a family of pairs $(E_k, d_k)_{k\geq 1}$, the *spectral sequence* associated with the cofiltration (7.1). The abelian group E_k is called the *E_k-term*, and d_k is called the *k-th boundary map* or the *k-th level differential*, which has bidegree $(k, -k + 1)$.

Bott periodicity gives rise to isomorphisms $(E_k^{p,2q}, d_k^{p,2q}) \cong (E_k^{p,0}, d_k^{p,0})$, i.e. group isomorphisms respecting the respective differentials. However, we will not always use these identifications, since the bookkeeping of the occurring indices is easier using the original notation.

The inductive definition of (E_k, d_k) using exact couples admits the following description of the differential $d_k^{p,q} : E_k^{p,q} \to E_k^{p+k,q-k+1}$. Let $[x] \in E_k^{p,q}$ be represented by $x \in K_{p+q}(I_p)$, and consider its image $K_{p+q}(\iota_p)(x) \in K_{p+q}(F_p)$ under the map induced by the natural inclusion $\iota_p : I_p \to F_p$. Since we have started in E_k, there is a lift $y \in K_{p+q}(F_{p+k-1})$ for $K_{p+q}(\iota_p)(x)$ under $K_{p+q}(F_{p+k-1}) \to K_{p+q}(F_p)$. Then,

$$d_k^{p,q}([x]) = [\rho_{p+q}^{(p+k)}(y)] \in E_k^{p+k,q-k+1}.$$

For $m \geq n+1$ and for all $p, q \in \mathbb{Z}$, the differential

$$d_m^{p,q} : E_m^{p,q} \longrightarrow E_m^{p+m,q-m+1}$$

vanishes since either $E_m^{p,q} = 0$ or $E_m^{p+m,q-m+1} = 0$. Therefore, $E_m = E_{n+1}$, and we say that the spectral sequence *collapses*. We define the *E_∞-term* as $E_\infty^{p,q} := E_{n+1}^{p,q}$. It is connected to $K_*(A)$ in the following way. For $q = 0, 1$, consider the diagram

$$\begin{tikzcd} K_q(A) = K_q(F_n) \arrow[r] & K_q(F_{n-1}) \arrow[r] & \cdots \arrow[r] & K_q(F_0) \arrow[r] & K_q(F_{-1}) = 0. \end{tikzcd}$$

Define $\mathcal{F}_p K_q(A) := \ker(K_q(A) \to K_q(F_p))$ for $p = -1, \ldots, n$, and observe that this gives rise to a filtration of abelian groups

$$\begin{tikzcd} 0 = \mathcal{F}_n K_q(A) \arrow[r, hook] & \mathcal{F}_{n-1} K_q(A) \arrow[r, hook] & \cdots \arrow[r, hook] & \mathcal{F}_{-1} K_q(A) = K_q(A) \end{tikzcd}$$

One can now show the existence of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_p K_{p+q}(A) \arrow[r] & \mathcal{F}_{p-1} K_{p+q}(A) \arrow[r] & E_\infty^{p,q} \arrow[r] & 0, \end{tikzcd}$$

or in other words, there are isomorphisms

$$E_\infty^{p,q} \cong \mathcal{F}_{p-1} K_{p+q}(A) / \mathcal{F}_p K_{p+q}(A).$$

Hence, the E_∞-term determines the K-theory of A up to group extension problems. We say that the spectral sequence $(E_k, d_k)_{k\geq 1}$ *converges* to $K_*(A)$.

The spectral sequence associated with a finite cofiltration is natural in the following sense. Assume that we have a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, two heads] \arrow[d, "\varphi"] & F_{n-1} \arrow[r, two heads] \arrow[d, "\varphi_{n-1}"] & \cdots \arrow[r, two heads] & F_0 \arrow[r] \arrow[d, "\varphi_0"] & 0 \\ B \arrow[r, two heads] & G_{n-1} \arrow[r, two heads] & \cdots \arrow[r, two heads] & G_0 \arrow[r] & 0. \end{tikzcd} \tag{7.4}$$

We write $(E_{A,k}, d_{A,k})_{k\geq 1}$ and $(E_{B,k}, d_{B,k})_{k\geq 1}$ for the spectral sequence associated with the upper and lower cofiltration, respectively. Note that in the situation of (7.4), the φ_k are uniquely determined by φ, so that this diagram really only depends on φ. By the naturality of K-theory, φ induces a morphism between the exact couples (7.3) belonging to the upper and lower cofiltration, respectively. In this way, diagram (7.4) gives rise to a *morphism of spectral sequences*, i.e. a collection of homomorphisms with bidegree $(0, 0)$

$$E_k(\varphi) : E_{A,k} \longrightarrow E_{B,k}, \quad k \geq 1,$$

which are compatible with the differentials $d_{A,k}$ and $d_{B,k}$. This in turn yields a bigraded group homomorphism $E_\infty(\varphi) : E_{A,\infty} \to E_{B,\infty}$.

By the naturality of K-theory, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r, hook] & \mathcal{F}_{n-1}K_q(A) \arrow[r, hook] \arrow[d, "\mathcal{F}_{n-1}K_q(\varphi)"] & \cdots \arrow[r, hook] & K_q(A) \arrow[d, "K_q(\varphi)"] \\
0 \arrow[r, hook] & \mathcal{F}_{n-1}K_q(B) \arrow[r, hook] & \cdots \arrow[r, hook] & K_q(B)
\end{tikzcd}$$

One can show that $E_\infty(\varphi)$ is induced by $\mathcal{F}_*K_*(\varphi)$ in the sense that for every $p = -1, \ldots, n$ and $q = 0, 1$, there is a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}_pK_q(A) \arrow[r] \arrow[d, "\mathcal{F}_pK_q(\varphi)"] & \mathcal{F}_{p-1}K_q(A) \arrow[r] \arrow[d, "\mathcal{F}_{p-1}K_q(\varphi)"] & E_\infty^{p,q} \arrow[r] \arrow[d, "E_\infty^{p,q}(\varphi)"] & 0 \\
0 \arrow[r] & \mathcal{F}_pK_q(B) \arrow[r] & \mathcal{F}_{p-1}K_q(B) \arrow[r] & E_\infty^{p,q} \arrow[r] & 0
\end{tikzcd} \tag{7.5}$$

This yields a method to determining $K_*(\varphi)$ by means of the spectral sequences whenever $\varphi : A \to B$ is a cofiltration-respecting $*$-homomorphism.

A spectral sequence homomorphism $(E_k(\varphi))_{k\geq 1}$ is called an *isomorphism of spectral sequences* if there is some $k \geq 1$ such that $E_k(\varphi)$ is an isomorphism. Note that in this case, $E_l(\varphi)$ is an isomorphism for all $l \geq k$. In particular, the E_∞-terms are isomorphic via $E_\infty(\varphi)$.

Corollary 7.1.2. *Assume that we are in the situation of* (7.4)*, and assume further that* $\varphi : A \to B$ *induces an isomorphism between the associated spectral sequences. Then* $K_*(\varphi) : K_*(A) \to K_*(B)$ *is an isomorphism.*

Proof. For fixed $q = 0, 1$, one can show inductively that $\mathcal{F}_pK_q(\varphi)$ is an isomorphism. This is done by an iterative application of the Five Lemma to the respective diagram (7.5). □

7.2 A spectral sequence for the K-theory of crossed products by $\mathbb{Z}^n$

Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$, we construct a spectral sequence associated with a finite cofiltration converging to $K_*(A \rtimes_\alpha \mathbb{Z}^n)$. For this purpose, let us fix some notation first.

For a finite sequence of natural numbers $1 \leq \mu_1 < \ldots < \mu_k \leq n$, we write $\mu = (\mu_1, \ldots, \mu_k)$ for the induced k-tuple, and define $T(k,n)$ to be the set of all such ordered k-tuples. By convention, $T(0,n)$ is the set which contains only the empty tuple. We do not distinguish between a 1-tuple $\lambda \in T(1,n)$ and the corresponding number λ_1. If $k \leq l \leq n$ and if the underlying set of $\mu \in T(k,n)$ is contained in $\nu \in T(l,n)$, then we write $\mu \subseteq \nu$. In this situation, we define $\nu \setminus \mu \in T(l-k,n)$ as the unique element whose underlying set is equal to the set difference of the underlying sets of ν and μ. For $\mu \in T(k,n)$, let $\mu^\perp \in T(n-k,n)$ be the unique element which is disjoint to μ.

For $k = 1, \ldots, n$, let $\Lambda^k(\mathbb{Z}^n)$ be the k-th component of the exterior algebra over $\mathbb{Z}^n$. We use the convention that $\Lambda^0(\mathbb{Z}^n) = \mathbb{Z}$ and $\Lambda^k(\mathbb{Z}^n) = 0$ whenever $k < 0$ or $k > n$. As always, $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{Z}^n$, and we write $e_\mu := e_{\mu_1} \wedge \cdots \wedge e_{\mu_k} \in \Lambda^k(\mathbb{Z}^n)$ for $\mu \in T(k,n)$. If $\mu \in T(0,n)$ is the empty tuple, then we define $e_\mu := 1$. We also agree on the convention that $e_\mu \wedge 1 = 1 \wedge e_\mu = e_\mu$.

The set $\{e_\mu \;:\; \mu \in T(k,n)\}$ defines a $\mathbb{Z}$-basis for $\Lambda^k(\mathbb{Z}^n)$. Hence, if we equip $T(k,n)$ with the lexicographical ordering, then the natural order-preserving bijection $T(k,n) \cong \left\{1, \ldots, \binom{n}{k}\right\}$ yields a group isomorphism $\Lambda^k(\mathbb{Z}^n) \cong \mathbb{Z}^{\binom{n}{k}}$. Observe that this isomorphism exists for all $k \in \mathbb{Z}$ since $\binom{n}{k} = 0$ whenever $k < 0$ or $k > n$. We will use these identifications throughout this section.

Consider the following filtration of the n-cube

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = [0,1]^n,$$

where

$$X_k := \left\{ t \in [0,1]^n \;:\; t_{\mu_1} = \ldots = t_{\mu_{n-k}} = 0 \quad \text{for some } \mu \in T(n-k,n) \right\}.$$

This gives rise to a finite cofiltration of the mapping torus

$$\mathcal{M}_\alpha(A) = F_n \xrightarrow{\pi_n} F_{n-1} \xrightarrow{\pi_{n-1}} \cdots \longrightarrow F_0 = A \xrightarrow{\pi_0} F_{-1} = 0. \tag{7.6}$$

In the same way as in (and with the notation of) the last section, we extend this cofiltration trivially. We shall refer to F_p as the *p-skeleton* of $\mathcal{M}_\alpha(A)$. Moreover, we define $I_p := \ker(\pi_p)$ for $p \in \mathbb{Z}$. The machinery described in Section 7.1 gives rise to a spectral sequence $(E_k, d_k)_{k \geq 1}$ converging to $K_*(\mathcal{M}_\alpha(A))$. By Theorem 1.2.6, there is an isomorphism $K_*(\mathcal{M}_\alpha(A)) \cong K_{*+n}(A \rtimes_\alpha \mathbb{Z}^n)$, so that the spectral sequence $(E_k, d_k)_{k \geq 1}$ indeed converges to the K-theory of the crossed product $A \rtimes_\alpha \mathbb{Z}^n$.

The spectral sequence associated with the mapping torus cofiltration has the following naturality property. If $\varphi : (A, \alpha, \mathbb{Z}^n) \to (B, \beta, \mathbb{Z}^n)$ is an equivariant $*$-homomorphism, then the induced $*$-homomorphism

$$\widetilde{\varphi} : \mathcal{M}_\alpha(A) \longrightarrow \mathcal{M}_\beta(B), \quad \widetilde{\varphi}(f)(t) := \varphi(f(t)),$$

is cofiltration-preserving. By the discussion in the last section, we therefore obtain a morphism of spectral sequences $(E_k(\varphi))_{k\geq 1}$ between the spectral sequences associated with $(A, \alpha, \mathbb{Z}^n)$ and $(B, \beta, \mathbb{Z}^n)$, respectively. Since the isomorphism in Theorem 1.2.6 intertwines $K_*(\widetilde{\varphi})$ and $K_{*+n}(\varphi \rtimes \mathbb{Z}^n)$, we can, in principle, use $E_\infty(\varphi)$ to determine $K_*(\varphi \rtimes \mathbb{Z}^n)$.

Let us now turn to the E_1-term of the spectral sequence associated with (7.6). For $1 \leq p \leq n$ and $\mu \in T(p,n)$, let $\alpha(\mu)$ denote the $\mathbb{Z}^p$-action generated by $\alpha_{\mu_1}, \ldots, \alpha_{\mu_p}$. Consider the associated mapping torus $\mathcal{M}_{\alpha(\mu)}(A)$ and let $F(\mu)_k$ denote the k-skeleton of the respective cofiltration (7.6). Furthermore, we write $(E(\mu)_k, d(\mu)_k)_{k\geq 1}$ for the spectral sequence associated with $(A, \alpha(\mu), \mathbb{Z}^p)$.

We obtain $\mathcal{M}_{\alpha(\mu)}(A)$ as the quotient of F_p given by the restriction to the closed subset

$$X(\mu) := \Big\{ t \in [0,1]^n \quad : \quad t_{\mu_1^\perp} = \ldots = t_{\mu_{n-p}^\perp} = 0 \Big\} \subseteq X_p.$$

This induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & I_p \arrow[r] \arrow[d, "\chi(\mu)"] & F_p \arrow[r] \arrow[d] & F_{p-1} \arrow[r] \arrow[d, "\pi(\mu)"] & 0 \\
0 \arrow[r] & S^p A \arrow[r] & \mathcal{M}_{\alpha(\mu)}(A) \arrow[r] & F(\mu)_{p-1} \arrow[r] & 0
\end{tikzcd} \tag{7.7}$$

and we write $\rho(\mu)_* : K_*(F(\mu)_{p-1}) \to K_{*+1}(S^p A)$ for the boundary map associated with the lower row extension. Since

$$X_p = \bigcup_{\mu \in T(p,n)} X(\mu),$$

we can conclude from diagram (7.7) that F_p is an iterative pullback of the $\binom{n}{p}$ many mapping tori of the natural $\mathbb{Z}^p$-subactions glued together over the $\binom{n}{p-1}$ many mapping tori of the natural $\mathbb{Z}^{p-1}$-subactions. We also see that the $*$-homomorphism

$$\chi = (\chi(\mu))_{\mu \in T(p,n)} : I_p \xrightarrow{\cong} (S^p A)^{\binom{n}{p}} \tag{7.8}$$

is an isomorphism. With the convention that $S^0 A := A$, the E_1-term is given by

$$E_1^{p,q} := K_{p+q}(I_p) \cong \begin{cases} K_{p+q}(S^p A) \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^n) & , \text{ for } 0 \leq p \leq n, \\ 0 & , \text{ for } p < 0 \text{ and } p > n. \end{cases}$$

We proceed with a description of the differentials d_k of the associated spectral sequence.

Lemma 7.2.1. *Assume that $0 \leq p \leq n$ and $1 \leq k \leq n-p$. Let $g \in E_k^{p,q}$ be represented by $x \in K_{p+q}(S^p A) \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^n)$. Let $y \in K_{p+q}(F_{p+k-1})$ be a lift for $K_{p+q}(\iota_p)(x) \in K_{p+q}(F_p)$ under the map induced by the surjection $F_{p+k-1} \to F_p$. For $\mu \in T(p+k, n)$, set*

$$y_\mu := K_{p+q}(\pi(\mu))(y) \in K_{p+q}(F(\mu)_{p+k-1}).$$

Then the differential $d_k^{p,q}$ satisfies

$$d_k^{p,q}(g) = \left[\sum_{\mu \in T(p+k,n)} \rho(\mu)_{p+q}(y_\mu) \otimes e_\mu \right] \in E_k^{p+k,q-k+1}.$$

Proof. The definition of $(E_k, d_k)_{k \geq 1}$ in terms of exact couples yields that $d_k^{p,q}(g)$ is given by the class of $\rho_{p+q}(y)$ in $E_k^{p+k,q-k+1}$, where

$$\rho_{p+q} : K_{p+q}(F_{p+k-1}) \longrightarrow K_{p+q+1}(I_{p+k})$$

is the boundary map associated with the surjection $\pi_{p+k} : F_{p+k} \to F_{p+k-1}$. For every $\mu \in T(p+k,n)$, the respective diagram (7.7) together with the identification (7.8) yields a commutative diagram

$$\begin{tikzcd}
K_*(F_{p+k-1}) \arrow[r, "\rho_*"] \arrow[d, "K_*(\pi(\mu))"'] & K_{*+1}(S^{p+k}A) \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^n) \arrow[d, "\mathrm{pr}_\mu"] \\
K_*(F(\mu)_{p+k-1}) \arrow[r, "\rho(\mu)_*"] & K_{*+1}(S^{p+k}A)
\end{tikzcd}$$

where pr_μ is the canonical projection onto the coordinate labelled by μ. Thus,

$$\begin{aligned}
d_k^{p,q}(g) &= [\rho_{p+q}(y)] \\
&= \left[\sum_{\mu \in T(p+k,n)} (\rho(\mu)_{p+q} \circ K_{p+q}(\pi(\mu)))(y) \otimes e_\mu \right] \\
&= \left[\sum_{\mu \in T(p+k,n)} \rho(\mu)_{p+q}(y_\mu) \otimes e_\mu \right] \in E_k^{p+k,q-k+1}.
\end{aligned}$$

□

Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^n)$, Savignen and Bellissard [43] define the *Pimsner-Voiculescu complex* (C_{PV}, d_{PV}) as

$$\begin{aligned}
&C_{PV}^{p,q} := K_q(A) \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^n), \\
&d_{PV}^{p,q} : C_{PV}^{p,q} \longrightarrow C_{PV}^{p+1,q}, \qquad x \otimes e \mapsto \sum_{k=1}^{n} (K_q(\alpha_k) - \mathrm{id})(x) \otimes (e \wedge e_k)
\end{aligned}$$

for $p, q \in \mathbb{Z}$. Observe that Bott periodicity allows us to identify $E_1 \cong C_{PV}$, which we shall do for the remainder of this section. Savignen and Bellissard point out that this isomorphism actually intertwines the differentials d_1 and d_{PV}, so that the E_2-term is obtained as the cohomology of (C_{PV}, d_{PV}).

The identification of (E_1, d_1) with the Pimsner-Voiculescu complex also turns out to be a consequence of the next result, this section's main technical lemma. It yields, roughly speaking, a partial description of d_k in terms of the k-th level differentials of the spectral sequences associated with the natural $\mathbb{Z}^k$-subactions.

Lemma 7.2.2. *Let $0 \leq p \leq n$, $1 \leq k \leq n-p$, and $\mu \in T(p,n)$. Assume that $x \in K_q(A)$ represents an element $[x] \in E(\mu^\perp)_k^{0,q}$.*

Then $x \otimes e_\mu \in K_q(A) \otimes_{\mathbb{Z}} \Lambda^p(\mathbb{Z}^n)$ represents an element $[x \otimes e_\mu] \in E_k^{p,q}$ whose image under the differential $d_k^{p,q} : E_k^{p,q} \to E_k^{p+k,q-k+1}$ is given as follows. Let $y \in K_q(F(\mu^\perp)_{k-1})$ be a lift for $x \in K_q(A)$ under the map induced by the surjection $F(\mu^\perp)_{k-1} \to F(\mu^\perp)_0 = A$. For $\lambda \in T(k,n)$, we set $y_\lambda \in K_q(F(\lambda)_{k-1})$ to be the image of y under $K_q(F(\mu^\perp)_{k-1}) \to K_q(F(\lambda)_{k-1})$ if $\lambda \subseteq \mu^\perp$, and zero otherwise.

Then

$$d_k^{p,q}([x \otimes e_\mu]) = \left[\sum_{\lambda \in T(k,n)} \rho(\lambda)_q(y_\lambda) \otimes (e_\mu \wedge e_\lambda) \right] \in E_k^{p+k,q-k+1}.$$

Proof. Let $\sigma \in \Sigma_n$ be the permutation given by

$$\sigma^{-1}(l) := \begin{cases} \mu_l & , \text{ if } 1 \leq l \leq p, \\ \mu^\perp_{l-p} & , \text{ if } p+1 \leq l \leq n. \end{cases}$$

Consider the injective $*$-homomorphism

$$\begin{aligned} &\iota : C_0((0,1)^p \times [0,1]^{n-p}, A) \longrightarrow C([0,1]^n, A), \\ &\iota(f)(t_1, \dots, t_n) = f(t_{\sigma(1)}, \dots, t_{\sigma(n)}), \end{aligned}$$

which gives rise to the following commutative diagram of cofiltrations

$$\begin{tikzcd}
S^p\mathcal{M}_{\alpha(\mu^\perp)} \arrow[r, two heads] \arrow[d, hook, "\iota"] & S^pF(\mu^\perp)_{n-p-1} \arrow[r, two heads] \arrow[d, hook, "\iota_{n-p+1}"] & \cdots \arrow[r, two heads] & S^pA \arrow[r, two heads] \arrow[d, hook, "\iota_0"] & 0 \arrow[r] \arrow[d] & \cdots \arrow[r] & 0 \arrow[d] \\
\mathcal{M}_\alpha(A) \arrow[r, two heads] & F_{n-1} \arrow[r, two heads] & \cdots \arrow[r, two heads] & F_p \arrow[r, two heads] & F_{p-1} \arrow[r, two heads] & \cdots \arrow[r, two heads] & A
\end{tikzcd} \tag{7.9}$$

Write $(\widetilde{E}_k, \widetilde{d}_k)_{k\geq 1}$ for the spectral sequence associated with the upper row cofiltration after having applied Bott periodicity. Observe that ι gives rise to a morphism of spectral sequences $(E_k(\iota) : \widetilde{E_k} \to E_k)_{k\geq 1}$. By construction, it holds that $E_k(\iota)([x]) = [x \otimes e_\mu] \in E_k^{p,q}$.

Let J_k denote the kernel of the surjection $S^pF(\mu^\perp)_k \to S^pF(\mu^\perp)_{k-1}$. For every $\nu \in T(p+k,n)$ with $\mu \subseteq \nu$, (7.9) gives rise to a commutative diagram with exact rows

$$\begin{tikzcd}
J_k \arrow[rr, hook] \arrow[dr, two heads] \arrow[dd, hook, "\eta"'] & & S^pF(\mu^\perp)_k \arrow[rr, two heads] \arrow[dr, two heads] \arrow[dd, hook, "\iota_k"'] & & S^pF(\mu^\perp)_{k-1} \arrow[dr, two heads] \arrow[dd, hook, "\iota_{k-1}"'] & \\
& S^p(S^kA) \arrow[rr, hook] \arrow[dd, hook, "\eta(\nu)"] & & S^p\mathcal{M}_{\alpha(\nu\setminus\mu)}(A) \arrow[rr, two heads] \arrow[dd, hook, "\kappa"] & & S^pF(\nu \setminus \mu)_{k-1} \arrow[dd, hook] \\
I_{p+k} \arrow[rr, hook] \arrow[dr, two heads] & & F_{p+k} \arrow[rr, two heads] \arrow[dr, two heads] & & F_{p+k-1} \arrow[dr, two heads] & \\
& S^{p+k}A \arrow[rr, hook] & & \mathcal{M}_{\alpha(\nu)}(A) \arrow[rr, two heads] & & F(\nu)_{p+k-1}
\end{tikzcd} \tag{7.10}$$

As observed before, the maps

$$J_k \cong (S^p(S^kA))^{\binom{n-p}{k}} \longrightarrow S^p(S^kA) \text{ and } I_{p+k} \cong (S^{p+k}A)^{\binom{n}{p+k}} \longrightarrow S^{p+k}A$$

are the canonical surjections onto the coordinate labelled by $\nu \setminus \mu$ and ν, respectively. Let $\sigma(\nu) \in \Sigma_{p+k}$ be the permutation given by

$$\sigma(\nu)^{-1}(l) := \begin{cases} \mu_l & , \text{ if } 1 \leq l \leq p, \\ (\nu \setminus \mu)_{l-p} & , \text{ if } p+1 \leq l \leq p+k. \end{cases}$$

As above, $\kappa : S^p\mathcal{M}_{\alpha(\nu\setminus\mu)}(A) \to \mathcal{M}_{\alpha(\nu)}(A)$ is induced by the injective $*$-homomorphism

$$\kappa : C_0((0,1)^p \times [0,1]^{p+k}, A) \longrightarrow C([0,1]^n, A),$$
$$\kappa(f)(t_1, \ldots, t_n) = f(t_{\sigma(\nu)(1)}, \ldots, t_{\sigma(\nu)(n)}).$$

Hence, by using the canonical isomorphism $S^p(S^kA) \cong S^{p+k}A$, we see that the $*$-automorphism $\eta(\nu) : S^{p+k}A \to S^{p+k}A$ is induced by the homeomorphism of $\mathbb{R}^{p+k}$ which permutes the coordinates via $\sigma(\nu)$. Therefore,

$$K_*(\eta(\nu)) = \operatorname{sgn}(\sigma(\nu)) \cdot \operatorname{id}$$

by Corollary 2.2. Observe that this also shows that

$$K_*(\eta) : K_*(S^{p+k}A) \otimes_{\mathbb{Z}} \Lambda^k(\mathbb{Z}^{n-p}) \longrightarrow K_*(S^{p+k}A) \otimes_{\mathbb{Z}} \Lambda^{p+k}(\mathbb{Z}^n)$$

is injective. Using Lemma 7.2.1 and the fact that

$$e_\nu = \operatorname{sgn}(\sigma(\nu)) \cdot e_\mu \wedge e_{\nu\setminus\mu} \in \Lambda^{p+k}(\mathbb{Z}^n),$$

we conclude that

$$\begin{aligned}
d_k^{p,q}([x \otimes e_\mu]) &= d_k^{p,q}(E_k^{p,q}(\iota)([x])) \\
&= E_k^{p,q}(\iota)(\widetilde{d}_k^{p,q}([x])) \\
&= E_k^{p,q}(\iota)\left(\left[\sum_{\substack{\lambda \in T(k,n): \\ \lambda \subseteq \mu^\perp}} \rho(\lambda)_q(y_\lambda) \otimes e_\lambda\right]\right) \\
&= \left[K_{q+1}(\eta)\left(\sum_{\substack{\lambda \in T(k,n): \\ \lambda \subseteq \mu^\perp}} \rho(\lambda)_q(y_\lambda) \otimes e_\lambda\right)\right] \\
&= \left[\sum_{\substack{\nu \in T(p+k,n): \\ \mu \subseteq \nu}} K_{q+1}(\eta(\nu))(\rho(\nu \setminus \mu)_q(y_{\nu\setminus\mu})) \otimes e_\nu\right] \\
&= \left[\sum_{\lambda \in T(k,n)} \rho(\lambda)_q(y_\lambda) \otimes (e_\mu \wedge e_\lambda)\right] \in E_k^{p+k,q-k+1}.
\end{aligned}$$

□

Note that for $p = 0$, we just recover a special case of Lemma 7.2.1. Moreover, if $k = 1$, then Lemma 7.2.2 yields the desired identification $(C_{PV}, d_{PV}) \cong (E_1, d_1)$.

Corollary 7.2.3. *The isomorphism $E_1 \cong C_{PV}$ intertwines the differentials d_1 and d_{PV}.*

Proof. If $k = 1$, then the assumptions in Lemma 7.2.2 are satisfied for every $\mu \in T(1, n)$ and $x \in K_q(A)$. Moreover, for each $\lambda \in T(1, n)$, $\rho(\lambda)_*$ is the boundary map associated with the six-term exact sequence of the mapping torus extension

$$\begin{tikzcd} 0 \arrow[r] & SA \arrow[r] & \mathcal{M}_{\alpha_\lambda}(A) \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

Hence, by Bott periodicity, we obtain $\rho(\lambda)_* = K_*(\alpha_\lambda) - \mathrm{id}$, and Lemma 7.2.2 then yields

$$d_1^{p,q}(x \otimes e_\mu) = \sum_{\lambda=1}^{n} (K_q(\alpha_\lambda) - \mathrm{id})(x) \otimes (e_\mu \wedge e_\lambda).$$

The claim now follows by linearity of d_1. □

Consider the group ring $R := \mathbb{Z}[\mathbb{Z}^n]$, and let $t_i \in \mathbb{Z}^n \subseteq R$ denote the i-th canonical basis element for $i = 1, \ldots, n$. The *Koszul complex* (G_*, g_*) associated with the finite sequence $t_1 - 1, \ldots, t_n - 1 \in R$ is the $\mathbb{Z}$-graded R-complex given by

$$\begin{aligned} &G_p := \Lambda^p(R^n), \\ &g_p : G_p \longrightarrow G_{p-1}, \quad g_p(e_\mu) := \sum_{k=1}^{n} (-1)^k (t_k - 1) e_{\mu \setminus (\mu_k)}. \end{aligned}$$

For fixed $q \in \{0, 1\}$, α induces an R-module structure on $K_q(A)$. We pass to the corresponding cohomological Koszul complex (G^*, g^*) with coefficients in the R-module $K_q(A)$

$$\begin{aligned} &G^p := \mathrm{Hom}_R(G_p, K_q(A)), \\ &g^p : G^p \longrightarrow G^{p+1}, \quad g^p(f) := f \circ g_{p+1}. \end{aligned}$$

Using the R-module isomorphism

$$G^p \cong \mathrm{Hom}_R(G_p, R) \otimes_R K_q(A) \cong G_p \otimes_R K_q(A) \cong C_{PV}^{p,q},$$

one can check that the two complexes $(C_{PV}^{*,q}, d_{PV}^{*,q})$ and (G^*, g^*) are isomorphic, and hence give rise to the same cohomology groups. Using this, we recover Kasparov's result [20, 6.10] that the E_2-term is given as the group cohomology of $\mathbb{Z}^n$ with values in $K_*(A)$.

Corollary 7.2.4. *Let $(A, \alpha, \mathbb{Z}^n)$ be a C^*-dynamical system and let $(E_k, d_k)_{k \geq 1}$ denote its associated spectral sequence. Then the E_2-term satisfies*

$$E_2^{p,q} \cong H^p(\mathbb{Z}^n, K_q(A)), \quad p, q \in \mathbb{Z}.$$

Proof. By the definition of group cohomology, we only have to show that the Koszul complex (G_*, g_*) defines a projective R-resolution of $\mathbb{Z}$ (regarded as a trivial module over R). For this, it is sufficient to know that the finite sequence $t_1 - 1, \dots, t_n - 1$ is *regular*, that is, $(t_1 - 1, \dots, t_n - 1)R \neq R$, and for $i = 1, \dots, n$, the element $t_i - 1$ defines a non-zero-divisor in $R/(t_1 - 1, \dots, t_{i-1} - 1)R$, see [24, Chapter XXI, Theorem 4.6a)]. The first condition holds since $(t_1 - 1, \dots, t_n - 1)R$ is the augmentation ideal, which satisfies

$$R/(t_1 - 1, \dots, t_n - 1)R \cong \mathbb{Z}.$$

Concerning the second condition, note that for $i = 1, \dots, n$, there is an isomorphism

$$R/(t_1 - 1, \dots, t_{i-1} - 1)R \cong \mathbb{Z}[\mathbb{Z}^{n-i+1}].$$

However, for $k \in \mathbb{N}$, the group ring $\mathbb{Z}[\mathbb{Z}^k]$ is known to have no zero-divisors, see [27] and [30]. □

In the particular case of a $\mathbb{Z}$-action, the E_∞-term coincides with the E_2-term, and therefore reduces to

$$E_\infty^{0,q} = \ker(K_q(\alpha) - \mathrm{id}) \quad \text{and} \quad E_\infty^{1,q} = \operatorname{coker}(K_q(\alpha) - \mathrm{id}).$$

Convergence of the spectral sequence then gives rise to a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E_\infty^{1,q-1} \arrow[r] & K_q(\mathcal{M}_\alpha(A)) \arrow[r] & E_\infty^{0,q} \arrow[r] & 0. \end{tikzcd}$$

Using the isomorphism $K_*(\mathcal{M}_\alpha(A)) \cong K_{*-1}(A \rtimes_\alpha \mathbb{Z})$, we see that the spectral sequence yields an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker}(K_{*-1}(\alpha) - \mathrm{id}) \arrow[r] & K_{*-1}(A \rtimes_\alpha \mathbb{Z}) \arrow[r] & \ker(K_*(\alpha) - \mathrm{id}) \arrow[r] & 0. \end{tikzcd}$$

Hence, for $n = 1$ the spectral sequence and the Pimsner-Voiculescu sequence yield the same group extension problems for the K-theory of $A \rtimes_\alpha \mathbb{Z}$.

Although Lemma 7.2.2 is very useful, it does not provide a complete description of the differentials d_k. The first reason for this is that not every element $g \in E_k^{p,q}$ is decomposable in the sense that there are $x_\mu \in K_q(A)$, indexed by $\mu \in T(p, n)$, such that $[x \otimes e_\mu] \in E_k^{p,q}$ and

$$g = \sum_{\mu \in T(p,n)} [x_\mu \otimes e_\mu] \in E_k^{p,q}.$$

The second reason is that, even if $x \in K_q(A)$ satisfies $[x \otimes e_\mu] \in E_k^{p,q}$ for some $\mu \in T(p, n)$, it is not clear whether x defines an element $[x] \in E(\mu^\perp)_k^{0,q}$. In fact, we do not automatically obtain a lift for the corresponding element $y \in K_{p+q}(S^pA)$ to an element in $K_{p+q}(S^pF(\mu^\perp)_{k-1})$ if we know that $y \otimes e_\mu \in K_{p+q}(I_p)$ lifts to an element in $K_{p+q}(F_{p+k-1})$. However, this second problem does not occur for $k = 2$.

Corollary 7.2.5. *Let $0 \leq p \leq n-2$ and $\mu \in T(p,n)$. Assume that $x \in K_q(A)$ gives rise to an element $[x \otimes e_\mu] \in E_2^{p,q}$.*

Then $[x] \in E(\mu^\perp)_2^{0,q}$, and with the notation from Lemma 7.2.2, it follows that

$$d_2^{p,q}([x \otimes e_\mu]) = \left[\sum_{\lambda \in T(2,n)} \rho(\lambda)_q(y_\lambda) \otimes (e_\mu \wedge e_\lambda) \right] \in E_2^{p+2,q-1}.$$

Proof. Using the notation of the proof of Lemma 7.2.2, we consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S^p(SA)^{\binom{n-p}{1}} \arrow[r] \arrow[d, "\eta"] & S^pF(\mu^\perp)_1 \arrow[r] \arrow[d] & S^pA \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (S^{p+1}A)^{\binom{n}{p+1}} \arrow[r] & F_{p+1} \arrow[r] & F_p \arrow[r] & 0
\end{tikzcd}$$

Naturality of K-theory yields

$$\begin{tikzcd}
K_{p+q}(S^pA) \arrow[r, "\widetilde{\rho}_{p+q}"] \arrow[d] & K_{p+q+1}(S^{p+1}A) \otimes_{\mathbb{Z}} \Lambda^1(\mathbb{Z}^{n-p}) \arrow[d, "K_{p+q+1}(\eta)"] \\
K_{p+q}(F_p) \arrow[r, "\rho_{p+q}"] & K_{p+q+1}(S^{p+1}A) \otimes_{\mathbb{Z}} \Lambda^{p+1}(\mathbb{Z}^n)
\end{tikzcd}$$

Let $w \in K_{p+q}(S^pA)$ be the unique element corresponding to $x \in K_q(A)$ under the Bott isomorphism. By the definition of the E_2-term, we have that $\rho_{p+q}(w \otimes e_\mu) = 0$. The proof of Lemma 7.2.2 shows that $K_*(\eta)$ is injective, and hence $\widetilde{\rho}_{p+q}(w) = 0$ as well. Again by Bott periodicity, this gives rise to a lift $y \in K_q(F(\mu^\perp)_1)$ for $x \in K_q(A)$ under the map induced by the surjection $F(\mu^\perp)_1 \to F(\mu^\perp)_0 = A$. Hence, $[x] \in E(\mu^\perp)_2^{p,q}$, and the claim follows from Lemma 7.2.2. □

Given a C^*-dynamical system $(A, \alpha, \mathbb{Z}^2)$, the corresponding E_1-term is concentrated in $p = 0, 1, 2$. With the notation from Chapter 4, the Pimsner-Voiculescu complex yields

$$\begin{aligned}
E_2^{0,q} &= S_q(\alpha), \\
E_2^{1,q} &= \tfrac{\ker(-(K_q(\alpha_2)-\mathrm{id}) \oplus (K_q(\alpha_1)-\mathrm{id}))}{\mathrm{im}((K_q(\alpha_1)-\mathrm{id},\ K_q(\alpha_2)-\mathrm{id}))}, \\
E_2^{2,q} &= T_q(\alpha).
\end{aligned}$$

Since $d_2 : E_2 \to E_2$ has bidegree $(2, -1)$, it reduces to

$$d_2^{0,q} : S_q(\alpha) \longrightarrow T_{q-1}(\alpha), \quad q = 0, 1.$$

Next, we show that this differential actually coincides with the obstruction homomorphism $d_q(\alpha)$ that we investigated in the first part of this thesis.

Proposition 7.2.6. *Let* $(A, \alpha, \mathbb{Z}^2)$ *be a* C^**-dynamical system. Then the second level differential* d_2 *satisfies* $d_2^{0,q} = d_q(\alpha)$ *for* $q = 0, 1$*. Moreover, the* E_∞*-term is given by*

$$E_\infty^{0,q} = \ker(d_q(\alpha)), \quad E_\infty^{1,q} = E_2^{1,q}, \quad \textit{and} \quad E_\infty^{2,q} = \operatorname{coker}(d_{1-q}(\alpha)).$$

Proof. We only have to show the equality of the differentials. Recall the mapping torus cofiltration (7.6)

$$\begin{tikzcd} \mathcal{M}_\alpha(A) \arrow[r, two heads, "\pi_2"] & F_1 \arrow[r, two heads, "\pi_1"] & A \arrow[r] & 0. \end{tikzcd}$$

In this case, F_1 is given as the pullback of $\mathcal{M}_{\alpha_1}(A)$ and $\mathcal{M}_{\alpha_2}(A)$ along the respective evaluations at 0. Consider the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & S^2A \arrow[r] \arrow[d, hook] & \mathcal{M}_\alpha(A) \arrow[r, "\pi_2"] \arrow[d, "\mathrm{id}"] & F_1 \arrow[r] \arrow[d, "\pi"] & 0 \\ 0 \arrow[r] & S\mathcal{M}_{\alpha_1}(A) \arrow[r] & \mathcal{M}_\alpha(A) \arrow[r] & \mathcal{M}_{\alpha_1}(A) \arrow[r] & 0 \end{tikzcd}$$

Naturality of K-theory and Bott periodicity give rise to a commutative diagram

$$\begin{tikzcd} K_q(F_1) \arrow[r, "\rho_q"] \arrow[d, "K_q(\pi)"] & K_{q-1}(S^2A) \arrow[r, "\cong"] \arrow[d] & K_{q-1}(A) \arrow[r, two heads] \arrow[d] & T_{q-1}(\alpha) \\ K_q(\mathcal{M}_{\alpha_1}(A)) \arrow[r, "\rho_q"] \arrow[rr, bend right, "K_q(\widetilde{\alpha}_2)-\mathrm{id}"'] & K_{q-1}(S\mathcal{M}_{\alpha_1}(A)) \arrow[r, "\cong"] & K_q(\mathcal{M}_{\alpha_1}(A)) & \end{tikzcd}$$

Denote by $d : K_q(F_1) \to T_{q-1}(\alpha)$ the map that is obtained by following the upper row of the last diagram. Given $x \in S_q(\alpha) = \operatorname{im}(K_q(\pi_1)) \subseteq K_q(A)$, we choose $\bar{x} \in K_q(F_1)$ with $K_q(\pi_1)(\bar{x}) = x$. By the definition of the second level differential, $d_2^{0,q}(x) = d(\bar{x})$. On the other hand, it is clear that $K_q(\pi)(\bar{x})$ defines a lift for $x \in S_q(\alpha)$ under $K_q(\mathrm{ev}_0) : K_q(\mathcal{M}_{\alpha_1}(A)) \to K_q(A)$. Using the mapping torus picture of the obstruction homomorphism, we get that $d_q(\alpha)(x) = d(\bar{x})$. This concludes the proof. ☐

This result also shows that the examples discussed in Section 5 all give rise to spectral sequences with non-trivial second level differentials.

On the other hand, by combining Proposition 7.2.5 with Corollary 7.2.6, one can express the second level differential d_2 of a $\mathbb{Z}^n$-action in terms of Bott elements. Since the precise formulation for this is very lengthy, we only state this result for the important special case of K-theoretically trivial $\mathbb{Z}^n$-actions.

Corollary 7.2.7. *Let* $(A, \alpha, \mathbb{Z}^n)$ *be a* C^**-dynamical system with the property that* $K_*(\alpha_i) = \mathrm{id}$ *for* $i = 1, \ldots, n$*. Then*

$$d_2^{p,q}(x \otimes e) = \sum_{\mu \in T(2,n)} d_q(\alpha(\mu))(x) \otimes (e \wedge e_\mu)$$

for every $x \otimes e \in E_1^{p,q} = E_2^{p,q}$*.*

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