Simplicial Complexes of Compact Homogeneous Spaces

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Abstract

Let $H < G$ be compact Lie groups with Lie algebra $\mathfrak{h}$ and $\mathfrak{g}$, respectively. In this thesis a simplicial complex $\Delta^T_{G/H}$ will be considered whose simplices are chains of appropiate intermediate subalgebras $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$. It is known that the compact homogeneous space G/H carries a G-invariant Einstein metric if $\Delta^T_{G/H}$ is not contractible. Therefore, it will be shown that under certain circumstances the existence of a non-trivial homology class of $\Delta^T_{G/H}$ can be deduced from non-trivial homology classes of $\Delta^T_{G/K}$ for appropiate intermediate subgroups $H < K < G$. Thus, in these cases the non-contractibility of $\Delta^T_{G/H}$ can be deduced from the non-contractibility of $\Delta^T_{G/K}$. This method will be used to determine the (non-)contractibility for all $\Delta^T_{G/H}$ with H, G connected and $\operatorname{rank} H = \operatorname{rank} G$.

Contents

Introduction

In [Böh04], Böhm introduced for every compact homogeneous space G/H a simplicial complex to obtain G-invariant Einstein metrics on G/H. More precisely, let $H < G$ be compact Lie groups and denote the Lie algebras of H and G with $\mathfrak{h}$ and $\mathfrak{g}$, respectively. An intermediate subalgebra $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$ is called an H-subalgebra, if it is $\mathrm{Ad}(H)$-invariant. For connected H, this condition is satisfied by all intermediate subalgebras. Moreover, let Q be a fixed $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$. Such an inner product exists since G is compact. Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to Q and let

$$\mathfrak{n}(\mathfrak{h}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{h}] \subseteq \mathfrak{h}\}$$

be the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. If $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$, then by [Böh04, Lemma 6.2] there exist only finitely many *minimal* H-subalgebras. By definition, the vertices of the simplicial complex $\Delta_{G/H}^{\min}$ are all subalgebras which are generated by minimal H-subalgebras. For $n \geq 1$, the n-simplices of $\Delta_{G/H}^{\min}$ are given by all chains $(\mathfrak{k}_0 < \ldots < \mathfrak{k}_n)$ with $\mathfrak{k}_i$ a vertex of $\Delta_{G/H}^{\min}$ for $0 \leq i \leq n$.

In general, there may exist infinitely many H-subalgebras. However, if only finitely many H-subalgebras exist, one can define the *extended* simplicial complex $\hat{\Delta}_{G/H}$ analogously to $\Delta_{G/H}^{\min}$ by taking *all* H-subalgebras as vertices. By [Böh04, Corollary 6.12], $\hat{\Delta}_{G/H}$ is homotopy equivalent to $\Delta_{G/H}^{\min}$.

If $\mathfrak{n}(\mathfrak{h}) \neq \mathfrak{h}$, let $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \perp \mathfrak{m}_0$. By [Böh04, Lemma 4.27], $\mathfrak{m}_0$ is the Lie algebra of a compact Lie group. So, fix a maximal torus T of a compact Lie group with Lie algebra $\mathfrak{m}_0$. By [Böh04, Corollary 7.2], there exist only finitely many H-subalgebras $\mathfrak{k}$ which are minimal among all H-subalgebras with the following properties:

1. $\mathfrak{k}$ is *non-toral*, i.e. for $\mathfrak{m}_{\mathfrak{k}} := \mathfrak{m} \cap \mathfrak{k}$, it holds $[\mathfrak{m}_{\mathfrak{k}}, \mathfrak{m}_{\mathfrak{k}}] \neq 0$.

2. $\mathfrak{k}$ is *T-adapted*, i.e. $\mathfrak{k}$ is $\mathrm{Ad}(T)$-invariant.

The simplcial complex $\Delta_{G/H}^{T}$ is defined analogously to $\Delta_{G/H}^{\min}$ by taking all subalgebras as vertices which are generated by minimal non-toral T-adapted H-subalgebras. This is a generalization of the first definition. In fact, for $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ it is $\mathfrak{m}_0 = \{0\}$, $T = \{e\}$ and $\Delta_{G/H}^{\{e\}} = \Delta_{G/H}^{\min}$. Now, Böhm proved the following theorem:

Theorem A. *Let G/H be a compact homogeneous space. If $\Delta_{G/H}^{T}$ is not contractible, then G/H admits a G-invariant Einstein metric.*

See [Böh04, Theorem 8.1]. Hence, the question arises how the (non-)contractibility of $\Delta^T_{G/H}$ can be determined for given compact Lie groups $H < G$. Suppose that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and that the extended complex $\hat{\Delta}_{G/H}$ is well-defined. For every compact connected Lie group $H < K < G$ with Lie algebra $\mathfrak{k}$ it then follows $\mathfrak{n}(\mathfrak{k}) = \mathfrak{k}$. Moreover, if H is connected, then $\hat{\Delta}_{G/K} \subseteq \hat{\Delta}_{G/H}$ is just the subcomplex induced by all H-subalgebras greater than $\mathfrak{k}$. In this thesis, it will be proved that for *minimal* H-subalgebras $\mathfrak{k}$ under certain circumstances a non-trivial homology class $[\theta] \in \tilde{H}_k(\hat{\Delta}_{G/K}, \mathbb{Q}) \setminus \{0\}$ can be "lifted"to a non-trivial homology class $[\theta_{\text{new}}] \in \tilde{H}_{k+1}(\hat{\Delta}_{G/H}, \mathbb{Q}) \setminus \{0\}$. Hence, if $(H < K_N < \ldots < K_0 < G)$ is a maximal chain of compact connected Lie groups for some $N \in \mathbb{N}_0$, this method can be applied iteratively to determine non-trivial homology classes in $\tilde{H}_{i-1}(\hat{\Delta}_{G/K_i}, \mathbb{Q})$ for $0 \leq i \leq N$ and finally to determine a non-trivial homology class in $\tilde{H}_N(\hat{\Delta}_{G/H}, \mathbb{Q})$. In detail, this works as follows:

K_0 is maximal, so $\hat{\Delta}_{G/K_0} = \emptyset$ and $\tilde{H}_{-1}(\emptyset, \mathbb{Q}) \neq 0$. Moreover, $(\mathfrak{k}_0)$ is a maximal simplex of $\hat{\Delta}_{G/K_1}$, also called a *facet*. If there exists another K_1-subalgebra $\mathfrak{k}_0^1 \neq \mathfrak{k}_0$, then $\hat{\Delta}_{G/K_1}$ is disconnected, i.e. $\tilde{H}_0(\hat{\Delta}_{G/K_1}, \mathbb{Q}) \neq 0$. More precisely, $[\theta] := [\mathfrak{k}_0^1 - \mathfrak{k}_0]$ is a non-zero element of $\tilde{H}_0(\hat{\Delta}_{G/K_1}, \mathbb{Q})$.

Now, suppose that for some $i \geq 1$, $[\theta] \in \tilde{H}_{i-1}(\hat{\Delta}_{G/K_i}, \mathbb{Q}) \setminus \{0\}$ is given with a representative $\theta = \sum_{l=1}^r q_l \cdot s_l$, $q_l \in \mathbb{Q} \setminus \{0\}$, s_l an $(i-1)$-simplex, such that θ satisfies the following properties:

1. θ is supported by a facet, i.e. at least on of the s_l is a facet of $\hat{\Delta}_{G/K_i}$.

2. A lower bound set (l.b.s.) of the vertex support of θ is given by minimal K_i-subalgebras $\mathfrak{k}_{i-1} =: \mathfrak{k}_{i-1}^0, \mathfrak{k}_{i-1}^1, \ldots, \mathfrak{k}_{i-1}^i$, i.e. for each vertex $\mathfrak{l}$ of each s_l it holds $\mathfrak{l} \geq \mathfrak{k}_{i-1}^j$ for some $0 \leq j \leq i$.

Suppose, there exist K_{i+1}-subalgebras $\mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1}$ not equal to $\mathfrak{k}_i =: \mathfrak{k}_i^0$ and $\mathfrak{k}_{i-1}^{j-1} \geq \mathfrak{k}_i^j$ for $1 \leq j \leq i+1$. Let

$$C_j := \hat{\Delta}_{G/K_i^j} * \mathfrak{k}_i^j = \{\sigma \in \hat{\Delta}_{G/H} \mid \min \sigma \geq \mathfrak{k}_i^j\}, \quad 0 \leq j \leq i+1$$

be the cone over $\mathfrak{k}_i^j$. If $\langle \mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1} \rangle < \mathfrak{g}$, then $D := \cup_{j=1}^{i+1} C_j$ is a contractible subcomplex and the boundary operator ∂_* of the Mayer-Vietoris sequence for reduced homology with respect to the triple $(C_0 \cup D; C_0, D)$ is an isomorphism. Thus, there is some $[\theta_{\text{new}}] \in \tilde{H}_i(C_0 \cup D, \mathbb{Q}) \setminus \{0\}$ with $\partial_*([\theta_{\text{new}}]) = [\theta]$. From the construction of θ_{new} it follows that it is supported by a facet of $\hat{\Delta}_{G/K_{i+1}}$ and a l.b.s. is given by $\mathfrak{k}_i^0, \ldots, \mathfrak{k}_i^{i+1}$. Since cycles which are supported by a facet cannot be boundaries, it follows $[\theta_{\text{new}}] \in \tilde{H}_i(\hat{\Delta}_{G/K_{i+1}}, \mathbb{Q}) \setminus \{0\}$. If this method can be applied iteratively, it follows $\tilde{H}_N(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$. In summary, $\tilde{H}_N(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ if subalgebras $\mathfrak{k}_i^j$ can be found satisfying the following properties:

1. $\mathfrak{k}_i^j \neq \mathfrak{k}_i =: \mathfrak{k}_i^0$ for all $0 \leq i \leq N$, $1 \leq j \leq i+1$.

2. $\mathfrak{k}_{i-1}^{j-1} \geq \mathfrak{k}_i^j > \mathfrak{k}_{i+1}^0$ for all $0 \leq i \leq N$, $1 \leq j \leq i+1$, with $\mathfrak{k}_{-1}^0 := \mathfrak{g}$, $\mathfrak{k}_{N+1}^0 := \mathfrak{h}$.

3. $\langle \mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1} \rangle < \mathfrak{g}$ for all $0 \leq i \leq N$.

Now, let $\mathfrak{g}$ be semisimple and $\mathfrak{h}$ a subalgebra of maximal rank, i.e. $\mathfrak{t} \leq \mathfrak{h}$ for a maximal abelian subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Then $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\hat{\Delta}_{G/H}$ is well-defined. In this thesis the (non-)contractibility of $\hat{\Delta}_{G/H}$ for all these cases will be determined. If G is not simple, then $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ for simple ideals $\mathfrak{g}_i$, $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$ with $\operatorname{rank} \mathfrak{h}_i = \operatorname{rank} \mathfrak{g}_i$ and

$$\hat{\Delta}_{G/H} \simeq \hat{\Delta}_{G_1/H_2} * \ldots * \hat{\Delta}_{G_k/H_k} * S^{k-2}.$$

It will turn out that $\hat{\Delta}_{G/H}$ is non-contractible if and only if $\hat{\Delta}_{G_i/H_i}$ is non-contractible for all i. Hence, it is sufficient to consider the case when $\mathfrak{g}$ is simple.

Note, that this already classifies $\Delta_{G/H}^T$ when $\mathfrak{h}$ is *regular*, i.e. when there exists an abelian subalgebra $\mathfrak{t}' \subseteq \mathfrak{m}_0$ such that $\mathfrak{h} \oplus \mathfrak{t}'$ has maximal rank. In this case, $\mathfrak{t}'$ is maximal abelian in $\mathfrak{m}_0$ and by [Böh04, Proposition 7.3], it is $\Delta_{G/H}^{T'} = \Delta_{G/HT'}^{\min}$.

So, let $\mathfrak{g}$ be simple with rank n. Up to covering, G is one of the classical Lie gruops, i.e. $G = SU(n+1)$, $SO(2n)$, $SO(2n+1)$ or $Sp(n)$ or G is an exceptional Lie group, i.e. $G = G_2$, F_4, E_6, E_7 or E_8. Let $T < G$ be a fixed maximal torus. As a first step, all T-subalgebras have to be determined. If

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{m}_\alpha$$

is the root space decompostion of $\mathfrak{g}$ with respect to T, then all T-subalgebras $\mathfrak{t} < \mathfrak{h} < \mathfrak{g}$ are precisely given by $\mathfrak{t} \oplus \bigoplus_{\alpha \in I} \mathfrak{m}_\alpha$ with $I \,\dot{\cup}\, {-I} \subseteq R$ being a closed subroot system of R. The *maximal* closed subroot systems of irreducible root systems are given by the Borel - de Siebenthal theorem up to conjugacy, see [Wol77, Theorem 8.10.9]. Applying Borel - de Siebenthal iteratively yields *all* closed subroot systems and hence, all T-subalgebras.

In chapter 4 of this thesis, it will be determined if $\hat{\Delta}_{G/H}$ is contractible or not for all cases where G is classical and H has maximal rank. In chapter 5, the same will be done for exceptional G. The results of chapter 4 can be summarized as follows:

$G = SU(n)$,	$H \cong S(U(n_1) \times \ldots \times U(n_k))$.
$G = SO(2n)$,	$H \cong U(n_1) \times \ldots \times U(n_k)$ or
	$H \cong SO(2n_1) \times \ldots \times SO(2n_k)$.
$G = SO(2n+1)$,	$H \cong SO(2n_1) \times \ldots \times SO(2n_k)$ or
	$H \cong SO(2n_1) \times \ldots \times SO(2n_{k-1}) \times SO(2n_k+1)$.
$G = Sp(n)$,	$H \cong U(n_1) \times \ldots \times U(n_k)$ or
	$H \cong Sp(n_1) \times \ldots \times Sp(n_k)$.

On the other hand, if H consists of block matrices of "mixed type", e.g. $G = SO(8)$, $H = U(2) \times SO(4)$, then $\hat{\Delta}_{G/H}$ is contractible. Note, that $U(1) = SO(2)$. In particular, $\hat{\Delta}_{G/T}$ is always non-contractible for classical G. As an example, for $\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{h} = \mathfrak{t} = \mathfrak{s}(\oplus_{i=1}^{n} \mathfrak{u}(1))$, the subalgebras $\mathfrak{k}_i^j$ are given by

$$\mathfrak{k}_i^j := \mathfrak{s}\left(\mathfrak{u}(n-1-i)_{1,\ldots,n-2-i,n-1-i+j} \oplus \bigoplus_{\substack{l=1 \\ l \neq n-1-i+j}} \mathfrak{u}(1)_l\right)$$

for $i \in \{0, \ldots, n-3\}$, $j \in \{0, \ldots, i+1\}$. The indices at the bottom specify which submatrices of $\mathfrak{su}(n)$ are given by the $\mathfrak{u}(n-1-i)$-block and the $\mathfrak{u}(1)$-blocks. Now, property 1 and 2 follow directly from the definition and

$$\langle \mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1} \rangle = \mathfrak{s}(\mathfrak{u}(n-1)_{1,\ldots,\widehat{n-1-i},\ldots,n} \oplus \mathfrak{u}(1)_{n-1-i}) < \mathfrak{su}(n).$$

Hence, $\tilde{H}_{n-3}(\hat{\Delta}_{SU(n)/T}, \mathbb{Q}) \neq 0$. In the contractible case, well-known facts about poset-topology will be used to show contractibility. For example, let $G = SO(2n)$ and $H = U(n_1) \times SO(2n_2) \times \ldots \times SO(2n_r)$, $n_1, n_2 \geq 2$. The H-subalgebras form a partially ordered set (poset) $\hat{P}_{G/H}$ with respect to $\subseteq$. Let $\mathfrak{k} := \oplus_{i=1}^{r} \mathfrak{so}(2n_i) > \mathfrak{h}$. For every H-subalgebra $\mathfrak{l}$, it is $\langle \mathfrak{k}, \mathfrak{l} \rangle < \mathfrak{so}(2n)$, since $\mathfrak{l} \not\supseteq \mathfrak{u}(n)$. Hence, $\mathfrak{k}$ and $\mathfrak{l}$ have a least upper bound in $\hat{P}_{G/H}$ for all $\mathfrak{l}$. So, $\mathfrak{k}$ has no complements in the poset $\hat{P}_{G/H}$. By a standard argument of poset-topology, see [BW83, Theorem 3.2], the contractibility of $\hat{\Delta}_{G/H}$ follows.

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Chapter 1

Preliminaries

The first chapter of this thesis is a recap of the basic properties of geometric and abstract simplicial complexes and their homology groups. At the end of this chapter, the complexes $\Delta^T_{G/H}$, $\Delta^{\min}_{G/H}$ and $\hat{\Delta}_{G/H}$ will be introduced.

1.1 Geometric and Abstract Simplicial Complexes

In this section, finite simplicial complexes of geometric and abstract type will be introduced, see also section 2.3 of [Mau70] for details.

Definition 1.1. A finite set of points $v_0, \ldots, v_n \in \mathbb{R}^m$ is called *independent* if the following condition holds:

$$\forall \lambda_0, \ldots, \lambda_n \in \mathbb{R} : \quad \sum_{i=0}^{n} \lambda_i v_i = 0 \ \wedge \ \sum_{i=0}^{n} \lambda_i = 0 \quad \Longrightarrow \quad \lambda_0 = \ldots = \lambda_n = 0$$

If $\{v_0, \ldots, v_n\} \in \mathbb{R}^m$ is independent, the convex span

$$\sigma := \operatorname{Conv}(v_0, \ldots, v_n) := \left\{ \sum_{i=0}^{n} \lambda_i v_i \mid 0 \leq \lambda_i \leq 1, \ \sum_{i=0}^{n} \lambda_i = 1 \right\},$$

is the smallest convex subset of $\mathbb{R}^m$ which contains $v_0, \ldots, v_n$ and is called a *geometric n-simplex*. $v_0, \ldots, v_n$ are called the *vertices of σ*. $\dim \sigma := n$ is called the *dimension of σ*. This is well-defined since the vertices of a geometric simplex are unique, see [Mau70, Proposition 2.3.3]. Moreover, the independence of $v_0, \ldots, v_n$ ensures that for $p \in \sigma$ the representation $p = \sum_{i=0}^{n} \lambda_i(p) v_i$ with $0 \leq \lambda_i(p) \leq 1$, $\sum_{i=0}^{n} \lambda_i(p) = 1$, is unique.

The *interior of σ*, $\operatorname{Int} \sigma$, is the subset of all points $p = \sigma$ such that $\lambda_i(p) > 0$ for all $i \in \{0, \ldots, n\}$. Its complement $\operatorname{Bd} \sigma := \sigma \setminus \operatorname{Int} \sigma$ is called the *boundary of σ*.

If $\tau = \operatorname{Conv}(v_{i_0}, \ldots, v_{i_k})$ for some non-empty subset $\{i_0, \ldots, i_k\} \subseteq \{0, \ldots, n\}$, then τ is called a *face of σ*.

A *geometric simplicial complex* K *of* $\mathbb{R}^m$ is a finite set of geometric simplices of $\mathbb{R}^m$ such that the following two conditions hold:

1. If $\sigma \in K$ and if τ is a face of σ, then $\tau \in K$.

2. If $\sigma, \tau \in K$, then $\sigma \cap \tau$ is either empty or a common face of σ and τ.

The *dimension of* K is $\dim K := \sup\{\dim \sigma \mid \sigma \in K\}$ for $K \neq \emptyset$ and $\dim K := -1$ for $K = \emptyset$. A subset L of K which satisfies 1. and 2. is called a *subcomplex of* K. The *polyhedron of* K is the topological space

$$\|K\| := \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^m$$

endowed with the subspace topology of $\mathbb{R}^m$. If $L \subseteq K$ is a subcomplex, then $\|L\|$ is called a *subpolyhedron*.

Remark 1.2. Some elementary examples of geometric simplicial complexes are given by the following list:

1. If $L_1,\ L_2$ are subcomplexes of K, then so are $L_1 \cap L_2$ and $L_1 \cup L_2$.

2. For any subset $S \subseteq K$, there exists a smallest subcomplex $\langle S \rangle$ of K which contains S. $\langle S \rangle$ is called the *subcomplex of* K *generated by* S. For $\sigma \in K$, the subcomplex $\langle \sigma \rangle = \{\emptyset \subsetneq \tau \subseteq \sigma\}$ is also denoted by σ.

3. For $m \in \mathbb{N}_0$, the subcomplex $K^m := \{\sigma \in K \mid \dim \sigma \leq m\}$ is called the *m-skeleton of* K. K^0 is also called the *vertex set of* K. If $\sigma = \{v\} \in K^0$, then v is called a *vertex of* K and one writes $v \in K$ instead of $\{v\} \in K$.

4. Let K_1 be a geometric simplicial complex of $\mathbb{R}^{m_1}$ and K_2 a geometric simplicial complex of $\mathbb{R}^{m_2}$ for some $m_1, m_2 \geq 1$. For any $\sigma = \mathrm{Conv}(v_0, \ldots, v_{n_1}) \in K_1$ and $\tau = \mathrm{Conv}(w_0, \ldots, w_{n_2}) \in K_2$, the set

$$\{(v_0, 0, 0), \ldots, (v_{n_1}, 0, 0), (0, w_0, 1), \ldots, (0, w_{n_2}, 1)\}$$

is an independent subset of $\mathbb{R}^{m_1+m_2+1}$. So, let

$$\sigma * \tau := \mathrm{Conv}((v_0, 0, 0), \ldots, (v_{n_1}, 0, 0), (0, w_0, 1), \ldots, (0, w_{n_2}, 1)).$$

Furthermore, let

$$\begin{aligned} \sigma * \emptyset &:= \mathrm{Conv}((v_0, 0, 0), \ldots, (v_{n_1}, 0, 0)) \text{ and} \\ \emptyset * \tau_{n_2} &:= \mathrm{Conv}((0, w_0, 1), \ldots, (0, w_{n_2}, 1)). \end{aligned}$$

Then,

$$K_1 * K_2 := \{\sigma * \tau \mid \sigma \in K_1,\ \tau \in K_2\} \cup \{\sigma * \emptyset \mid \sigma \in K_1\} \cup \{\emptyset * \tau \mid \tau \in K_2\}$$

is a geometric simplicial complex of $\mathbb{R}^{m_1+m_2+1}$ of dimension $\dim K_1 + \dim K_2 + 1$, called the *join of* K_1 *and* K_2. Its polyhedron is

$$\|K_1 * K_2\| = \{(tp, (1-t)q, t) \mid p \in \|K_1\|,\ q \in \|K_2\|,\ t \in [0,1]\}.$$

Remark 1.3. The following list gives some elementary properties of the polyhedron $\|K\|$ of a geometric simplicial complex K, see also [Mau70, p. 33f.]:

1. $\|K\|$ is compact.
2. Every point $p \in \|K\|$ lies in the interior of exactly one geometric simplex of K.
3. $A \subseteq \|K\|$ is closed in $\|K\|$ if and only if $A \cap \|\sigma\|$ is closed in $\|\sigma\|$ for all $\sigma \in K$.
4. If X is any topological space, then a map $f : \|K\| \to X$ is continous if and only if $f_{|\,\|\sigma\|} : \|\sigma\| \to X$ is continous for every $\sigma \in K$.

Definition 1.4. Let K, L be geometric simplicial complexes and let $f^0 : K^0 \to L^0$ be a map such that if $v_0, \ldots, v_n$ are the vertices of a simplex of K, then the distinct elements of $f(v_0), \ldots, f(v_n)$ are the vertices of a simplex of L. The induced map

$$f : K \longrightarrow L;\ \operatorname{Conv}(v_0, \ldots, v_n) \mapsto \operatorname{Conv}(f^0(v_0), \ldots, f^0(v_n))$$

is called a *simplicial map*. If, in addition, $f^0 : K^0 \to L^0$ is a bijection whose inverse also induces a simplicial map, then $f : K \to L$ is called a *simplicial isomorphism* and the complexes K and L are called *isomorphic*, denoted by $K \cong L$. Furthermore, f induces a map $\|f\| : \|K\| \to \|L\|$ which is also called a *simplicial map* in the following way:

$$\|f\| : \|K\| \longrightarrow \|L\|;\ \sum_{i=0}^{n} \lambda_i v_i \mapsto \sum_{i=0}^{n} \lambda_i f(v_i) \tag{1.1}$$

$\|f\|$ is well-defined and continous by the properties 2 and 4 given above. Moreover, $\|f\|$ is a homeomorphism if and only if f is a simplicial isomorphism.

By the next definition, abstract simplicial complexes will be introduced, which allow any finite set being a vertex set.

Definition 1.5. An *abstract simplicial complex* or a *simplicial complex* Δ is a finite set whose elements are non-empty finite sets such that if $\sigma \in \Delta$, then $\tau \in \Delta$ for all non-empty subsets $\tau \subseteq \sigma$. A subset $\Gamma \subseteq \Delta$ is called a *subcomplex of* Δ if Γ itself is a simplicial complex.

If $\sigma \in \Delta$ and $n := \#\sigma - 1$, then σ is called an *n-simplex of* Δ and $\dim \sigma := n$ is the *dimension of* σ. The elements of σ are called the *vertices* of σ. As in the geometric case, the *dimension of* Δ is $\dim \Delta := \sup\{\dim \sigma \mid \sigma \in \Delta\}$ for $\Delta \neq \emptyset$ and $\dim \Delta := -1$ for $\Delta = \emptyset$.

A simplex $\tau \subseteq \sigma$ is called a *face* of σ. Moreover, if σ is not a face of any other simplex, then it is called a *maximal simplex* or a *facet*.

Remark 1.6. The following list gives some elementary examples of abstract simplicial complexes:

1. If $\Gamma_1,\ \Gamma_2$ are subcomplexes of Δ, then so are $\Gamma_1 \cap \Gamma_2$ and $\Gamma_1 \cup \Gamma_2$.

2. As in the geometric case, for any subset $S \subseteq \Delta$, the subcomplex of Δ which is generated by S is denoted by $\langle S \rangle$. For $\sigma \in \Delta$, the subcomplex $\langle \sigma \rangle = \{\emptyset \subsetneq \tau \subseteq \sigma\}$ is denoted by σ.

3. The subcomplex $\Delta^m := \{\sigma \in \Delta \mid \dim \sigma \leq m\},\ m \in \mathbb{N}_0$, is called the *m-skeleton of* Δ and Δ^0 is called the *vertex set of* Δ. If $\sigma = \{v\} \in \Delta^0$, then v is called a *vertex of* Δ and one writes $v \in \Delta$ instead of $\{v\} \in \Delta$.

4. Let $\Delta_1,\ \Delta_2$ be any simplicial complexes. Then
$$\Delta_1 * \Delta_2 := \{\sigma \sqcup \tau \mid \sigma \in \Delta_1 \cup \{\emptyset\},\ \tau \in \Delta_2 \cup \{\emptyset\},\ \sigma \sqcup \tau \neq \emptyset\}$$
is a simplicial complex of dimension $\dim \Delta_1 + \dim \Delta_2 + 1$, called the *join of* Δ_1 *and* Δ_2. In particular, $\Delta * \emptyset = \Delta = \emptyset * \Delta$. If $\{v\}$ is a singleton, $\Delta * \{v\}$ is called a *cone over* Δ. The join $\Delta * S^0$ is called the *suspension over* Δ. Moreover, $*$ is commutative and associative, i.e. $\Delta_1 * \Delta_2 = \Delta_2 * \Delta_1$ and $(\Delta_1 * \Delta_2) * \Delta_3 = \Delta_1 * (\Delta_2 * \Delta_3)$.

Definition 1.7. Let $\Delta,\ \Gamma$ be abstract simplicial complexes. Let $f^0 : \Delta^0 \to \Gamma^0$ be a map such that if $\{v_0, \ldots, v_n\} \in \Delta$, then $\{f(v_0), \ldots, f(v_n)\} \in \Gamma$. Again, the induced map
$$f : \Delta \longrightarrow \Gamma;\ \{v_0, \ldots, v_n\} \mapsto \{f(v_0), \ldots, f(v_n)\}$$
is called a *simplicial map*. If, in addition, $f^0 : \Delta^0 \to \Gamma^0$ is a bijection whose inverse induces a simplicial map, then $f : \Delta \to \Gamma$ is called a *simplicial isomorphism* and Δ and Γ are called *isomorphic*, denoted by $\Delta \cong \Gamma$.

There is the following correspondence between abstract and geometric simplicial complexes:

Let K be a geometric simplicial complex of $\mathbb{R}^m$ and $\sigma = \mathrm{Conv}(v_0, \ldots, v_n) \in K$ any simplex of K. Since all subsets $\mathrm{Conv}(v_{i_0}, \ldots, v_{i_k}),\ \emptyset \neq \{i_0, \ldots, i_k\} \subseteq \{0, \ldots, n\}$, are elements of K, the set
$$\mathcal{K} := \{\{v_0, \ldots, v_n\} \subseteq \mathbb{R}^m \text{ independent} \mid \mathrm{Conv}(v_0, \ldots, v_n) \in K\}$$
is an abstract simplicial complex, called the *abstraction of* K. On the other hand, let Δ be an abstract simplicial complex. A *realization* $K(\Delta)$ *of* Δ is a geometric simplicial complex whose abstraction is isomorphic to Δ.

Lemma 1.8 ([Mau70, p. 37-40]). *Let Δ be an abstract simplicial complex of dimension n. Then Δ possesses a realization $K(\Delta) \subseteq \mathbb{R}^{2n+1}$. Furthermore, if $K_1(\Delta) \subseteq \mathbb{R}^{m_1}$ and $K_2(\Delta) \subseteq \mathbb{R}^{m_2}$ are two realizations of Δ, then $K_1(\Delta)$ is isomorphic to $K_2(\Delta)$. In particular, their polyhedrons $\|K_1(\Delta)\|$ and $\|K_2(\Delta)\|$ are homeomorphic. Moreover, for abstract simplicial complexes Δ_1, Δ_2, $K(\Delta_1) * K(\Delta_2)$ is a realization of $\Delta_1 * \Delta_2$.*

If Δ is an abstract simplicial complex, Lemma 1.8 assigns a topological space $\|\Delta\| := \|K(\Delta)\|$ to Δ which is unique up to homeomorphism. Thus, one may assign topological properties and topological invariants of the space $\|\Delta\|$ to Δ. For instance, Δ is called *contractible*, if and only if $\|\Delta\|$ is contractible. Furthermore, if $v \in \Delta$ is a vertex and $k \in \mathbb{N}$, one writes $\pi_k(\Delta, v)$ for the k-th homotopy group $\pi_k(\|\Delta\|, \|v\|)$ of $\|\Delta\|$ with base point $\|v\|$. The group $\pi_k(\Delta, v)$ is unique up to isomorphism. As the next section will show, the singular homology groups of $\|\Delta\|$ can be computed directly from the simplicial homology groups of Δ without considering any realization of Δ.

1.2 Homology

This section introduces all required properties of the singular and the simplicial homology groups including the *reduced* homology groups and the homology groups *with coefficients*. At the end of this section, the Mayer-Vietoris sequence for simplicial homology will be introduced.

1.2.1 Singular Homology

Definition 1.9. For $n \in \mathbb{N}_0$ let $\Delta^n_{\text{std}} := \text{Conv}(e_0 := 0, e_1, \dots, e_n)$ be the *standard n-simplex* of $\mathbb{R}^n$ with $\mathbb{R}^0 := \Delta^0_{\text{std}} := \{0\}$.

Definition 1.10. Let X be a topological space. For $n \in \mathbb{N}_0$, a *singular n-simplex σ_n of X* is a continous map

$$\sigma_n : \Delta^n_{\text{std}} \longrightarrow X.$$

The free abelian group generated by all singular n-simplices is denoted by $S_n(X)$ and its elements are called *singular n-chains of X*. For $n \in \mathbb{Z}$, $n < 0$, one defines $S_n(X) := 0$ and

$$S_*(X) := \bigoplus_{n \in \mathbb{Z}} S_n(X).$$

For $n \in \mathbb{N}$, $i \in \{0, \dots, n\}$, the *face map* $\delta^n_i : \Delta^{n-1}_{\text{std}} \to \Delta^n_{\text{std}}$ is defined by

$$\delta^n_i(e_k) := \begin{cases} e_k, & 0 \le k \le i-1 \\ e_{k+1}, & i \le k \le n-1 \end{cases}$$

and continously extended as in (1.1). The *boundary operator* $\partial : S_n(X) \to S_{n-1}(X)$ is the group homomorphism defined by

$$\partial(\sigma_n) := \sum_{i=0}^{n} (-1)^i \sigma_n \circ \delta_i^n \in S_{n-1}(X) \quad \text{for all singular } n\text{-simplices } \sigma_n.$$

For $n \in \mathbb{Z},\ n \leq 0$, one defines $\partial : S_n(X) \to S_{n-1}(X)$ being the zero map. By [Mau70, Proposition 4.2.7],

$$\partial^2 = 0 : S_*(X) \longrightarrow S_*(X),$$

i.e., $(S_*(X), \partial)$ is a *chain complex*. Thus, for $n \in \mathbb{Z}$ one defines

$$\begin{aligned} Z_n(X) &:= \ker\{\partial : S_n(X) \to S_{n-1}(X)\}, && \text{the group of } \textit{singular } n\textit{-cycles of } X, \\ B_n(X) &:= \operatorname{Im}\{\partial : S_{n+1}(X) \to S_n(X)\}, && \text{the group of } \textit{singular } n\textit{-boundaries of } X, \\ H_n(X) &:= Z_n(X)/B_n(X), && \text{the } n\textit{-th singular homology group of } X. \end{aligned}$$

The *singular homology group of* X is

$$H_*(X) := \bigoplus_{n \in \mathbb{Z}} H_n(X).$$

Lemma 1.11 ([Mau70, Corollary 4.2.17, 4.2.23]). *Let X, Y be topological spaces and let $f : X \to Y$ be a continous map. The group homomorphism $f_\star : S_*(X) \to S_*(Y)$ defined by $f_\star(\sigma_n) := f \circ \sigma_n$ for all singular n-simplices σ_n of X satisfies $\partial \circ f_\star = f_\star \circ \partial$. Hence, it induces a homomorphism*

$$f_\bullet : H_n(X) \longrightarrow H_n(Y)$$

for each $n \in \mathbb{Z}$. The assignment $f \mapsto f_\bullet$ is functorial, i.e. $(g \circ f)_\bullet = g_\bullet \circ f_\bullet$ and $\mathrm{id}_\bullet = \mathrm{id}$. *Moreover, if $f, g : X \to Y$ are homotopic maps, then $f_\bullet = g_\bullet$. In particular, $f_\bullet$ is a group isomorphism if f is a homotopy equivalence.*

By Lemma 1.11, all contractible spaces have the singular homology type of the one-point space, i.e.

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases} \tag{1.2}$$

for all contractible spaces X, see [Mau70, Example 4.2.12].

1.2.2 Simplicial Homology

Definition 1.12. Let Δ be a simplicial complex. For $n \in \mathbb{N}_0$ let $C_n(\Delta)$ be the abelian group generated by all $(n+1)$-tuples

$$(v_0, \ldots, v_n), \quad \{v_0, \ldots, v_n\} \text{ is } n\text{-simplex of } \Delta,$$

subject to the relations

$$(v_0,\dots,v_n) - \operatorname{sgn}(\rho)\cdot(v_{\rho(0)},\dots,v_{\rho(n)}),\quad \begin{array}{l}\{v_0,\dots,v_n\} \text{ is } n\text{-simplex of } \Delta,\\ \rho \text{ is permutation on } \{0,\dots,n\}.\end{array}$$

Furthermore, for $n \in \mathbb{Z},\ n < 0$, let $C_n(\Delta) := 0$ and let

$$C_*(\Delta) := \bigoplus_{n\in\mathbb{Z}} C_n(\Delta).$$

The coset of $(v_0,\dots,v_n)$ in $C_n(\Delta)$ is denoted by $[v_0,\dots,v_n]$. Since $[v_0,\dots,v_n] = \operatorname{sgn}(\rho)\cdot[v_{\rho(0)},\dots,v_{\rho(n)}]$ for any permutation ρ, it follows that $C_n(\Delta)$ is isomorphic to the free abelian group generated by all n-simplices $\sigma_n \in \Delta$. In fact, if for each n-simplex σ_n there is a fixed ordering $v_0^{\sigma_n} < \dots < v_n^{\sigma_n}$ of its vertices, the corresponding cosets $[v_0^{\sigma_n},\dots,v_n^{\sigma_n}]$ form a $\mathbb{Z}$-basis of $C_n(\Delta)$. The boundary operator $\partial : C_n(\Delta) \to C_{n-1}(\Delta)$ for $n \in \mathbb{N}$ is the group homomorphism defined by

$$\partial([v_0,\dots,v_n]) := \sum_{i=0}^{n}(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_n] \in C_{n-1}(\Delta) \tag{1.3}$$

and the zero map for $n \in \mathbb{Z},\ n \le 0$. In fact, $\partial([v_{\rho(0)},\dots,v_{\rho(n)}]) = \operatorname{sgn}(\rho)\cdot\partial([v_0,\dots,v_n])$ for each permutation ρ, so (1.3) yields a well-defined group homomorphism, see [Mau70, Proposition 4.3.7]. Again, $\partial^2 = 0 : C_*(\Delta) \to C_*(\Delta)$ and for $n \in \mathbb{Z}$ let

$$\begin{array}{ll}
Z_n(\Delta) := \ker\{\partial : C_n(\Delta) \to C_{n-1}(\Delta)\}, & \text{the group of } n\textit{-cycles of } \Delta,\\
B_n(\Delta) := \operatorname{Im}\{\partial : C_{n+1}(\Delta) \to C_n(\Delta)\}, & \text{the group of } n\textit{-boundaries of } \Delta,\\
H_n(\Delta) := Z_n(\Delta)/B_n(\Delta), & \text{the } n\textit{-th homology group of } \Delta.
\end{array}$$

The *homology group of* Δ is

$$H_*(\Delta) := \bigoplus_{n\in\mathbb{Z}} H_n(\Delta).$$

The link between singular and simplicial homology groups is given by the following theorem.

Theorem 1.13 ([Mau70, Theorem 4.3.9])**.** *Let Δ be a simplicial complex and let $\|\Delta\|$ be the polyhedron of a realization of Δ. Then $H_n(\Delta)$ and $H_n(\|\Delta\|)$ are isomorphic groups for all $n \in \mathbb{Z}$. More precisely, let $v_1 < \dots < v_N$ be any ordering of all vertices of Δ and for any n-simplex σ_n let $v_0^{\sigma_n} < \dots < v_n^{\sigma_n}$ be the induced ordering of the vertices of σ_n. The homomorphism $\alpha : C_*(\Delta) \to S_*(\|\Delta\|)$ which maps the coset $[v_0^{\sigma_n},\dots,v_n^{\sigma_n}]$ to the corresponding simplicial map, i.e.*

$$\alpha([v_0^{\sigma_n},\dots,v_n^{\sigma_n}]) := \left\{\Delta^n_{\text{std}} \longrightarrow \|\Delta\|;\ \sum_{i=0}^{n}\lambda_i\cdot e_i \mapsto \sum_{i=0}^{n}\lambda_i\cdot\|v_i^{\sigma_n}\|\right\}, \tag{1.4}$$

is a chain homotopy equivalence. In other words, α induces a group isomorphism

$$\alpha_\Delta : H_n(\Delta) \longrightarrow H_n(\|\Delta\|)$$

for each $n \in \mathbb{Z}$.

Corollary 1.14 ([Mau70, p. 117])**.** *Let Δ, Γ be simplicial complexes and let $f : \|\Delta\| \to \|\Gamma\|$ be continous. Then f induces a group homomorphism*

$$f_* : H_n(\Delta) \longrightarrow H_n(\Gamma)$$

for each $n \in \mathbb{Z}$. Moreover, $f \mapsto f_*$ is functorial, $f_* = g_*$ for homotopic maps $f, g : \|\Delta\| \to \|\Gamma\|$ and f_* is an isomorphism if f is a homotopy equivalence.

Proof. Let

$$f_* := (\alpha_\Gamma)^{-1} \circ f_\bullet \circ \alpha_\Delta : H_n(\Delta) \longrightarrow H_n(\Gamma),$$

where α_Δ, α_Γ are defined as in Theorem 1.13 and $f_\bullet : H_n(\|\Delta\|) \to H_n(\|\Gamma\|)$ is defined as in Lemma 1.11. Now, the claim follows directly from Lemma 1.11. □

The next subsection will introduce the *reduced* homology groups. Using reduced homology appears to be more natural than using non-reduced homology, since the reduced homology groups vanish for contractible spaces.

1.2.3 Reduced Homology Groups

The definitions and lemmas of this subsection are stated in [Mun84, p. 71ff.].

Definition 1.15. Let $\mathcal{C} = \{C_*, \partial_*\}_{n\in\mathbb{Z}}$ be a nonnegative chain complex, i.e. $C_n = 0$ for $n < 0$. The *augmented chain complex* $\tilde{\mathcal{C}} = \{\tilde{C}_*, \tilde{\partial}_*\}_{n\in\mathbb{Z}}$ is the chain complex defined by

$$\tilde{C}_n := \begin{cases} \mathbb{Z}, & n = -1 \\ C_n, & n \neq -1 \end{cases}$$

and $\tilde{\partial}_n := \partial_n$ for $n \notin \{-1, 0\}$, $\tilde{\partial}_0 : C_0 \to \mathbb{Z}$ being an epimorphism which satisfies $\tilde{\partial}_0 \circ \tilde{\partial}_1 = 0$ and $\tilde{\partial}_{-1} : \mathbb{Z} \to 0$ being the zero map. The homology groups of $\tilde{\mathcal{C}}$ are called *reduced homology groups* and are denoted by $\tilde{H}_n(\mathcal{C})$, $n \in \mathbb{Z}$. Furthermore, one defines

$$\tilde{H}_*(\mathcal{C}) := \bigoplus_{n\in\mathbb{Z}} \tilde{H}_n(\mathcal{C}).$$

Lemma 1.16. *For augmented chain complexes the following properties hold:*

1. $H_n(\mathcal{C}) = \begin{cases} \tilde{H}_n(\mathcal{C}) \oplus \mathbb{Z}, & n = 0 \\ \tilde{H}_n(\mathcal{C}), & n \neq 0. \end{cases}$

2. *Let* $f : \mathcal{C} = \{C_n, \partial_n^C\}_{n\in\mathbb{Z}} \to \mathcal{D} = \{D_n, \partial_n^D\}_{n\in\mathbb{Z}}$ *be a chain map between nonnegative chain complexes with augmentations* $\tilde{\mathcal{C}} = \{\tilde{C}_n, \tilde{\partial}_n^C\}_{n\in\mathbb{Z}}$ *and* $\tilde{\mathcal{D}} = \{\tilde{D}_n, \tilde{\partial}_n^D\}_{n\in\mathbb{Z}}$ *such that* f *preserves augmentation, i.e.* $\tilde{\partial}_0^C = \tilde{\partial}_0^D \circ f_{|C_0}$. *Then* f *induces a chain map* $\tilde{f} : \tilde{\mathcal{C}} \to \tilde{\mathcal{D}}$ *and therefore, it induces a homomorphism*
$$\tilde{f}_* : \tilde{H}_n(\mathcal{C}) \longrightarrow \tilde{H}_n(\mathcal{D}) \tag{1.5}$$
for $n \in \mathbb{Z}$.

3. *Under the conditions of 2, if* f *is in addition a chain homotopy equivalence with chain homotopy inverse* $g : \mathcal{D} \to \mathcal{C}$, *then* g *also preserves augmentation, i.e.* $\tilde{\partial}_0^D = \tilde{\partial}_0^C \circ g_{|D_0}$, *and* $\tilde{f}$ *is a chain homotopy equivalence with chain homotopy inverse* $\tilde{g} : \tilde{\mathcal{D}} \to \tilde{\mathcal{C}}$, *i.e.* $\tilde{f}_*$ *in* (1.5) *is an isomorphism.*

Now, let Δ be a simplicial complex. Since $C_0(\Delta)$ is freely generated by all vertices of Δ, $C_*(\Delta)$ can be augmented by setting $\tilde{\partial}_0(v) := 1$ for any vertex $v \in \Delta$. Similarly, $S_*(\|\Delta\|)$ can be augmented by setting $\tilde{\partial}_0(\sigma_0) := 1$ for any singular 0-simplex σ_0. The chain homotopy equivalence α defined in (1.4) is then augmentation preserving, i.e. $\widetilde{\alpha_\Delta} : \tilde{H}_n(\Delta) \to \tilde{H}_n(\|\Delta\|)$ is an isomorphism by Lemma 1.16, 3. Moreover, let $f : \|\Delta\| \to \|\Gamma\|$ be continous. Then the induced chain map $f_\star : S_*(\|\Delta\|) \to S_*(\|\Gamma\|)$, see Lemma 1.11, is augmentation preserving. As in Corollary 1.14, it follows that f induces a group homomorphism
$$\tilde{f}_* : \tilde{H}_n(\Delta) \longrightarrow \tilde{H}_n(\Gamma)$$
for each $n \in \mathbb{Z}$ such that $f \mapsto \tilde{f}_*$ is functorial, $\tilde{f}_* = \tilde{g}_*$ if $f, g : \|\Delta\| \to \|\Gamma\|$ are homotopic maps and $\tilde{f}_*$ is an isomorphism if f is a homotopy equivalence. Note, that $\tilde{H}_0(\Delta) = 0$ if and only if Δ is connected and
$$\tilde{H}_n(\emptyset) := \begin{cases} \mathbb{Z}, & n = -1 \\ 0 & n \neq -1. \end{cases}$$
So, $\tilde{H}_{-1}(\Delta) = 0$ if and only if $\Delta \neq \emptyset$. Therefore, non-empty simplicial complexes are also called (-1)-connected.

1.2.4 Homoloy with Coefficients

In this subsection, the homology with coefficients in any abelian group will be introduced, see [Mun84, p. 308ff.].

Lemma 1.17. *Let* $\mathcal{C} := \{C_n, \partial_n\}_{n\in\mathbb{Z}}$ *be a chain complex and* G *an abelian group. Then*
$$\mathcal{C} \otimes G := \{C_n \otimes G, \partial_n \otimes \mathrm{id}_G\}_{n\in\mathbb{Z}}$$
is also a chain complex. Moreover, if $f : \mathcal{C} \to \mathcal{D}$ *is a chain map (chain homotopy equivalence), then so is* $f \otimes \mathrm{id}_G : \mathcal{C} \otimes G \to \mathcal{D} \otimes G$.

Let Δ be a simplicial complex and let G be any abelian group. One defines

$$H_n(\Delta, G) := H_n(C_*(\Delta) \otimes G), \qquad \tilde{H}_n(\Delta, G) := H_n(\tilde{C}_*(\Delta) \otimes G)$$

for $n \in \mathbb{Z}$ and

$$H_*(\Delta, G) := \bigoplus_{n\in\mathbb{Z}} H_n(\Delta, G), \qquad \tilde{H}_*(\Delta, G) := \bigoplus_{n\in\mathbb{Z}} \tilde{H}_n(\Delta, G).$$

Similarly, for topological spaces X the groups $H_*(X, G)$ and $\tilde{H}_*(X, G)$ are defined as above by replacing $C_*(\Delta)$ and $\tilde{C}_*(\Delta)$ with $S_*(X)$ and $\tilde{S}_*(X)$, respectively. Note, that $G = \mathbb{Z}$ yields the ordinary homology groups. Again, $H_n(\|\Delta\|, G) \cong H_n(\Delta, G)$, $\tilde{H}_n(\|\Delta\|, G) \cong \tilde{H}_n(\Delta, G)$ for $n \in \mathbb{Z}$ and if $f : \|\Delta\| \to \|\Gamma\|$ is continous, it induces for each $n \in \mathbb{Z}$ a group homomorphism

$$f_* : H_n(\Delta, G) \longrightarrow H_n(\Gamma, G) \quad (\tilde{f}_* : \tilde{H}_n(\Delta, G) \longrightarrow \tilde{H}_n(\Gamma, G))$$

such that $f \mapsto f_*$ is functorial, $f_* = g_*$ for homotopic maps $f, g : \|\Delta\| \to \|\Gamma\|$ and f_* is an isomorphism if f is a homotopy equivalence. The same holds true for $f \mapsto \tilde{f}_*$.

All homology groups with coefficients in G are already determined by the ordinary homology groups in the following way:

$$H_n(\Delta, G) \cong (H_n(\Delta) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(\Delta), G) \tag{1.6}$$

and similarly for the reduced homology groups. Here, Tor denotes the Tor-functor, see [Mau70, p. 156] or [Mun84, Thm. 55.1, p. 332].

If $G = \mathbb{F}$ is a field, then $H_n(\Delta, \mathbb{F})$ and $\tilde{H}_n(\Delta, \mathbb{F})$ are not just groups but also $\mathbb{F}$-vector spaces and f_*, $\tilde{f}_*$ are $\mathbb{F}$-linear. From (1.6), it follows $\tilde{H}_n(\Delta, \mathbb{F}) \neq 0$ for any field $\mathbb{F}$, if the group $\tilde{H}_n(\Delta)$ has positive rank. Moreover, $\tilde{H}_n(\Delta, \mathbb{Q}) \neq 0$ if and only if $\tilde{H}_n(\Delta)$ has positive rank. Therefore, non-vanishing homology with rational coefficients ensures non-vanishing homology with coefficients in any field. For this reason, in chapters 4 and 5 homology with rational coefficients will be considered. In fact, it will be shown that $\tilde{H}_*(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, whenever $\hat{\Delta}_{G/H}$ is non-contractible. So, $\tilde{H}_*(\hat{\Delta}_{G/H}, \mathbb{F}) \neq 0$ for any field $\mathbb{F}$ in this case.

1.2.5 Mayer-Vietoris Sequence

Now, the Mayer-Vietoris sequence for the reduced simplicial homology groups will be introduced. This tool will be needed to prove the main theorem of this thesis.

Theorem 1.18. *Let Δ be a simplicial complex and let Δ_1, Δ_2 be subcomplexes such that $\Delta_1 \cup \Delta_2 = \Delta$ and $\Delta_1 \cap \Delta_2 \neq \emptyset$. Let*

$$i_1 : \Delta_1 \cap \Delta_2 \to \Delta_1, \quad i_2 : \Delta_1 \cap \Delta_2 \to \Delta_2, \quad i_3 : \Delta_1 \to \Delta, \quad i_4 : \Delta_2 \to \Delta$$

denote the corresponding inclusions. Furthermore, let R *be a commutative unitary ring,* $n \in \mathbb{Z}$ *and*

$$\begin{aligned} i_* &:= \tilde{i}_{1*} - \tilde{i}_{2*} : \tilde{H}_n(\Delta_1 \cap \Delta_2, R) \longrightarrow \tilde{H}_n(\Delta_1, R) \oplus \tilde{H}_n(\Delta_2, R), \\ j_* &:= \tilde{i}_{3*} + \tilde{i}_{4*} : \tilde{H}_n(\Delta_1, R) \oplus \tilde{H}_n(\Delta_2, R) \longrightarrow \tilde{H}_n(\Delta, R). \end{aligned}$$

Then there exists an R*-linear homomorphism* $\partial_* : \tilde{H}_n(\Delta, R) \to \tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, R)$ *such that*

$$\begin{aligned} \ldots \to \tilde{H}_n(\Delta_1 \cap \Delta_2, R) &\xrightarrow{i_*} \tilde{H}_n(\Delta_1, R) \oplus \tilde{H}_n(\Delta_2, R) \\ &\xrightarrow{j_*} \tilde{H}_n(\Delta, R) \xrightarrow{\partial_*} \tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, R) \to \ldots \end{aligned}$$

is a long exact sequence, called the Mayer-Vietoris sequence for reduced homology of the triple $(\Delta; \Delta_1, \Delta_2)$.

Proof. See [Mun84, Thm. 25.1, p. 142]. □

In particular, if Δ_1 and Δ_2 are contractible, the homology of $\Delta_1 \cup \Delta_2$ can be obtained from the homology of $\Delta_1 \cap \Delta_2$. This will be used to prove the main theorem of this thesis in section 2.3.

1.3 The Simplicial Complexes $\Delta^{\min}_{G/H}$, $\hat{\Delta}_{G/H}$ and $\Delta^{T}_{G/H}$

In this section, the simplicial complexes $\Delta^{\min}_{G/H}$, $\hat{\Delta}_{G/H}$ and $\Delta^{T}_{G/H}$ will be introduced. Let $H < G$ be compact Lie groups such that G/H is connected with finite fundamental group and G acts almost effectively on G/H. Moreover, let Q be a fixed $\mathrm{Ad}(G)$-invariant inner product on $\mathfrak{g}$ which exists by [War71, p. 152]. The orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ will be denoted by $\mathfrak{m}$. Furthermore, let

$$\mathfrak{m}_0 := \{X \in \mathfrak{m} \mid [X, \mathfrak{h}] = 0\}.$$

The Lie algebra of $N_{G_0}(H_0)$ is given by

$$\mathfrak{n}(\mathfrak{h}) = \mathfrak{h} \perp \mathfrak{m}_0,$$

see [Böh04, Lemma 4.26].

Definition 1.19. With the notation as above, a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is called an H-subalgebra, if the following two conditions hold:

1. $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$.
2. $\mathfrak{k}$ is $\mathrm{Ad}(H)$-invariant.

Furthermore, let $\mathfrak{m}_\mathfrak{k} := \mathfrak{m} \cap \mathfrak{k}$, i.e. $\mathfrak{k} = \mathfrak{h} \perp \mathfrak{m}_\mathfrak{k}$. $\mathfrak{k}$ is called *toral*, if $[\mathfrak{m}_\mathfrak{k}, \mathfrak{m}_\mathfrak{k}] = 0$, otherwise $\mathfrak{k}$ is called *non-toral*.

Note that for connected H every intermediate Lie subalgebra $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$ is $\mathrm{Ad}(H)$-invariant, thus an H-subalgebra. To define the simplicial complexes $\Delta^{\min}_{G/H}$, $\hat{\Delta}_{G/H}$ and $\Delta^T_{G/H}$ one has to distinguish the cases $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{n}(\mathfrak{h}) \neq \mathfrak{h}$.

1.3.1 $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$

Let $H < G$ be compact Lie groups as above such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$. In particular, $\mathfrak{m}_0 = \{0\}$ and every H-subalgebra is non-toral, since $[\mathfrak{m}_\mathfrak{k}, \mathfrak{m}_\mathfrak{k}] = 0$ implies $[\mathfrak{h}, \mathfrak{m}_\mathfrak{k}] = 0$.

Let $\hat{P}_{G/H}$ be the set of all H-subalgebras. $\hat{P}_{G/H}$ might be infinite. However, by [Böh04, Lemma 6.2], there exist at most finitely many *minimal* H-subalgebras. Hence, let $\{\mathfrak{k}^*_1, \ldots, \mathfrak{k}^*_N\}$ be the set of all minimal H-subalgebras and let $P^{\min}_{G/H}$ be the set of all H-subalgebras which are generated by minimal ones, i.e.

$$P^{\min}_{G/H} = \{\langle \mathfrak{k}^*_{i_1}, \ldots, \mathfrak{k}^*_{i_l} \rangle \mid 1 \leq l \leq N,\ 1 \leq i_1 < \ldots < i_l \leq N\}.$$

$\hat{P}_{G/H}$ and $P^{\min}_{G/H}$ are partially ordered by "$\subseteq$". Furthermore, the chains, i.e. the totally ordered subsets of $P^{\min}_{G/H}$, form an abstract simplicial complex, since $P^{\min}_{G/H}$ is finite and if $\sigma \subseteq P^{\min}_{G/H}$ is totally ordered, then so is τ for every $\tau \subseteq \sigma$. The same holds true for the chains of $\hat{P}_{G/H}$, if $\hat{P}_{G/H}$ is finite. This yields the following definition.

Definition 1.20. With the notation as above, the simplicial complex of the chains of $P^{\min}_{G/H}$ is called the *simplicial complex of* G/H and is denoted by $\Delta^{\min}_{G/H}$. If, in addition, $\hat{P}_{G/H}$ is finite, the simplicial complex of the chains of $\hat{P}_{G/H}$ is called the *extended* simplicial complex of G/H and is denoted by $\hat{\Delta}_{G/H}$.

Remark 1.21. The following properties of $\Delta^{\min}_{G/H}$ and $\hat{\Delta}_{G/H}$ are of further interest.

1. If H is connected, then $\Delta^{\min}_{G/H}$ and $\hat{\Delta}_{G/H}$ only depend on the Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ instead of the Lie groups H and G. In particular, if $\pi : \tilde{G} \to G$ is a covering map and $\tilde{H} := \pi^{-1}(H)_0$, it follows $\Delta^{\min}_{G/H} = \Delta^{\min}_{\tilde{G}/\tilde{H}}$ ($\hat{\Delta}_{G/H} = \hat{\Delta}_{\tilde{G}/\tilde{H}}$).

2. $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{h} < \mathfrak{k}$ imply $\mathfrak{n}(\mathfrak{k}) = \mathfrak{k}$. So, $\Delta^{\min}_{G/K}$ is well-defined for every compact Lie subgroup $H_0 < K < G$.

   If, in addition, H is connected and $\hat{P}_{G/H}$ is finite, i.e. if there exist only finitely many intermediate subalgebras $\mathfrak{h} < \mathfrak{k} < \mathfrak{g}$, then $\hat{P}_{G/K}$ is finite for every compact Lie subgroup $H < K < G$ and $\hat{\Delta}_{G/K}$ is well-defined. Moreover, every K-subalgebra is also an H-subalgebra. Hence, $\hat{P}_{G/K}$ can be identified as a subposet of $\hat{P}_{G/H}$ and $\hat{\Delta}_{G/K}$ can be identified as a subcomplex of $\hat{\Delta}_{G/H}$.

1.3.2 $\mathfrak{n}(\mathfrak{h}) \neq \mathfrak{h}$

Let $H < G$ be compact Lie groups as above such that $\mathfrak{n}(h) \neq \mathfrak{h}$, i.e. $\mathfrak{m}_0 \neq 0$. By [Böh04, Lemma 4.26], $\mathfrak{m}_0 \cong \mathfrak{n}(\mathfrak{h})/\mathfrak{h}$ is a compact Lie algebra. So, fix a maximal torus T of the compact connected Lie group $N_{G_0}(H_0)_0/H_0$ with Lie algebra $\mathfrak{m}_0$ and let $\mathfrak{t}$ the Lie algebra of T. This yields the following definition, see [Böh04, p. 153].

Definition 1.22. With the notation as above, an H-subalgebra $\mathfrak{k}$ is called *T-adapted*, if it is invariant under the adjoint action of T, i.e. if $[\mathfrak{t}, \mathfrak{k}] \subseteq \mathfrak{k}$. A non-toral T-adapted H-subalgebra is called *T-minimal non-toral*, if it is minimal among all non-toral T-adapted H-subalgebras.

By [Böh04, Corollary 7.2], there exist at most finitely many T-minimal non-toral H-subalgebras. As above, let $P^T_{G/H}$ be the set of all non-toral T-adapted H-subalgebras which are generated by minimal ones. Again, $P^T_{G/H}$ is a finite, partially ordered set.

Definition 1.23. The simplicial complex of the chains of $P^T_{G/H}$ is called the *simplicial complex of* G/H and is denoted by $\Delta^T_{G/H}$.

Remark 1.24. The following properties of $\Delta^T_{G/H}$ are of interest, see [Böh04, p. 154].

1. $\Delta^T_{G/H}$ is a generalization of $\Delta^{\min}_{G/H}$, since both definitions coincide for $\mathfrak{m}_0 = \{0\}$ and $T = \{e\}$.

2. If H is connected, then $\Delta^T_{G/H}$ only depends on $\mathfrak{g}$ and $\mathfrak{h}$ and not on the choice of T up to isomorphism. In particular, $\Delta^T_{G/H} = \Delta^T_{\tilde{G}/\tilde{H}}$ for covering maps $\pi : \tilde{G} \to G$ and $\tilde{H} := \pi^{-1}(H)_0$.

3. If both G and H are connected and $|\pi_1(G/H)| < \infty$, then there is a one-to-one correspondence between the minimal non-toral TH-subalgebras and the T-minimal non-toral H-subalgebras. Furthermore, $\mathfrak{n}(\mathfrak{t} \oplus \mathfrak{h}) = \mathfrak{t} \oplus \mathfrak{h}$. By [Böh04, Proposition 7.3], it follows $\Delta^T_{G/H} \cong \Delta^{\min}_{G/TH}$.

Note, that one always may assume the following conditions:

1. G/H is connected (but G and H might be disconnected.)

2. The group action of G on G/H is almost effective, i.e. any normal subgroup of G which is contained in H is discrete.

3. $\pi_1(G/H, eH)$ is finite.

In fact, if $\hat{G}$ is the union of all connected components of G which intersect H, then $\hat{G}$ is a compact subgroup of G such that $\hat{G}/H$ is the connected component of G/H which contains eH. Moreover, $\Delta^T_{\hat{G}/H} = \Delta^T_{G/H}$, since the complex only depends on $\mathfrak{h}$, $\mathfrak{g}$, H and T.

Furthermore, if $N \trianglelefteq G$, $N \leq H$, one may consider the homogeneous space $M := (G/N)/(H/N) \cong G/H$. Then $\Delta^T_{G/H} \cong \Delta^T_M$, since $\mathfrak{k}$ is $\mathrm{Ad}(H)$-invariant if and only if $\mathfrak{k}/\mathfrak{n}$ is $\mathrm{Ad}(H/N)$-invariant.

The third assumption is no restriction since $|\pi(G/H, eH)| = \infty$ implies that $\Delta^T_{G/H}$ is contractible or $\Delta^T_{G/H} = \emptyset$ and G/H is a torus, see [Böh04, Proposition 7.5].

In this thesis, all homogeneous spaces which are considered will satisfy the conditions of Remark 1.24 3, so one may always assume assume $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ by considering G/TH instead of G/H. For product spaces $G_1 \times G_2/H_1 \times H_2$, $\Delta^{T_1\times T_2}_{G_1\times G_2/H_1\times H_2}$ arises from $\Delta^{T_1}_{G_1/H_1}$ and $\Delta^{T_2}_{G_2/H_2}$ in the following manner.

Lemma 1.25. *For $i \in \{1, 2\}$ let $H_i < G_i$ be compact Lie groups as above and let T_i be a maximal torus as above. Then:*

$$\Delta^{T_1\times T_2}_{G_1\times G_2/H_1\times H_2} \cong \Delta^{T_1}_{G_1/H_1} * \Delta^{T_2}_{G_2/H_2} * S^0 \tag{1.7}$$

A proof is given by [Böh04, Theorem 3.2]. Moreover, Corollary 2.18 yields a second proof for the homotopy equivalence in (1.7). Since $\underbrace{S^0 * \ldots * S^0}_{n \text{ times}} = S^{n-1}$, it follows

$$\Delta^{\prod_{i=1}^n T_i}_{\prod_{i=1}^n G_i/\prod_{i=1}^n H_i} \cong \Delta^{T_1}_{G_1/H_1} * \ldots * \Delta^{T_n}_{G_n/H_n} * S^{n-2} \tag{1.8}$$

by induction for compact Lie groups $H_i < G_i$ and maximal tori T_i as above.

Chapter 2

The Homotopy Type of $\hat{\Delta}_{G/H}$

In this chapter several tools will be introduced which are useful to determine wether the complex $\hat{\Delta}_{G/H}$ is contractible or not. In the first section, the theorem of Whitehead will be used to obtain a useful corollary that yields some conditions under which the union of contractible complexes is still contractible. In the second section, contractible carriers, which are a useful tool to obtain informations about the topology of order complexes, will be introduced. In the last section, the main theorem of this thesis will be proved. It describes a way to find non-trivial homology classes of $\hat{\Delta}_{G/H}$.

2.1 The Theorem of Whitehead

By [Mau70, Prop. 7.3.2, p. 274], the polyhedron of a simplicial complex is also a CW-complex. Thus, one can apply the following theorem of Whitehead to simplicial complexes. Note that if X is any connected CW-complex, then it is path-connected and the k-homotopy group $\pi_k(X, x_0)$ does not depend on the basepoint x_0 up to isomorphism. Hence, one can write $\pi_k(X)$ for the k-th homotopy group. In particular, $\pi_k(\Delta)$ is unique up to isomorphism for any connected simplicial complex. For $k \in \mathbb{N}_0$, Δ is called *k-connected*, if $\Delta \neq \emptyset$ and $\pi_i(\Delta) = 0$ for all $0 \leq i \leq k$.

Theorem 2.1 (Theorem of Whitehead, [Hat02, Theorem 4.5])**.** *Let X, Y be non-empty connected CW-complexes and let $f : X \to Y$ be a weak homotopy equivalence, i.e. f is continous and $f_* : \pi_n(X) \to \pi_n(Y)$ is an isomorphism for all $n \in \mathbb{N}$. Then f is a homotopy equivalence. If, in addition, X is a subcomplex of Y and $f : X \hookrightarrow Y$ is the inclusion map, then X is a strong deformation retract of Y.*

Corollary 2.2. *Let X be a CW-complex. If X is simply connected and $\tilde{H}_*(X, \mathbb{Z}) = 0$, then X is contractible.*

Proof. X is a non-empty connected CW-complex by assumption. By Theorem 2.1, it remains to prove that for any vertex x_0 of X the inclusion $\iota : \{x_0\} \to X$ is a weak homotopy equivalence. So, it remains to prove that $\pi_n(X) = 0$ for all $n \geq 1$.

For $n = 1$, this is true by assumption. Now, suppose $n \geq 2$ and $\pi_k(X) = 0$ for all $1 \leq k \leq n-1$. By a theorem of Hurewicz, see [Hat02, Theorem 4.32], it follows $\pi_n(X) \cong \tilde{H}_n(X, \mathbb{Z}) = 0$ and the claim follows by induction. $\square$

Now, the theorem of Whitehead can be used to prove that the union of contractible complexes is contractible, if the complexes are glued together in an appropiate way. This will be needed in section 2.3 to prove the main theorem.

Lemma 2.3. *Let $\Delta_1,\ \Delta_2$ be contractible subcomplexes of a simplicial complex Δ such that $\Delta_1 \cap \Delta_2$ is contractible. Then $\Delta_1 \cup \Delta_2$ is also contractible.*

Proof. $\Delta_1 \cap \Delta_2$ is non-empty by assumption. So, let $v \in \Delta_1 \cap \Delta_2$ be any vertex. Since $\Delta_1,\ \Delta_2$ and $\Delta_1 \cap \Delta_2$ are path-connected and since $\pi_1(\Delta_1, v) = \pi_1(\Delta_2, v) = 0$, it follows $\pi_1(\Delta_1 \cup \Delta_2, v) = 0$ by van Kampen's theorem, see [Hat02, Theorem 1.20]. By Corollary 2.2, it remains to prove that $\tilde{H}_*(\Delta_1 \cup \Delta_2, \mathbb{Z}) = 0$. This follows from the Mayer-Vietoris sequence for reduced homology, see Theorem 1.18. More precisely, for all $n \in \mathbb{Z}$ there is an exact sequence:

$$\tilde{H}_n(\Delta_1, \mathbb{Z}) \oplus \tilde{H}_n(\Delta_2, \mathbb{Z}) \xrightarrow{j_*} \tilde{H}_n(\Delta_1 \cup \Delta_2, \mathbb{Z}) \xrightarrow{\partial_*} \tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, \mathbb{Z})$$

But $\tilde{H}_n(\Delta_1, \mathbb{Z}) \oplus \tilde{H}_n(\Delta_2, \mathbb{Z}) = 0$ and $\tilde{H}_{n-1}(\Delta_1 \cap \Delta_2, \mathbb{Z}) = 0$. It follows $\tilde{H}_n(\Delta_1 \cup \Delta_2, \mathbb{Z}) = 0$ for all $n \in \mathbb{Z}$ and $\Delta_1 \cup \Delta_2$ is contractible. $\square$

Corollary 2.4. *Let $n \in \mathbb{N},\ n \geq 2$ and let Δ be a simplicial complex with subcomplexes $\Delta_1, \ldots, \Delta_n$ such that*

$$\Delta = \bigcup_{i=1}^{n} \Delta_i.$$

If $\Delta_{i_1} \cap \ldots \cap \Delta_{i_l}$ is contractible for all non-empty subsets $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n\}$, then Δ is contractible.

Proof. If $n = 2$, then $\Delta_1,\ \Delta_2$ and $\Delta_1 \cap \Delta_2$ are contractible by assumption. Hence, $\Delta_1 \cup \Delta_2$ is contractible by Lemma 2.3. Now, let $n \geq 3$ and let the claim be true for all $n' \in \{2, \ldots, n-1\}$. In particular, $\bigcup_{i=1}^{n-1} \Delta_i$ is contractible. By Lemma 2.3, it remains to prove that

$$\Gamma := \left(\bigcup_{i=1}^{n-1} \Delta_i \right) \cap \Delta_n = \bigcup_{i=1}^{n-1} \Delta_i \cap \Delta_n$$

is contractible. For $i \in \{1, \ldots, n-1\}$ let $\Gamma_i := \Delta_i \cap \Delta_n$. Then $\Gamma = \bigcup_{i=1}^{n-1} \Gamma_i$ and by assumption $\Gamma_{i_1} \cap \ldots \cap \Gamma_{i_l} = \Delta_{i_1} \cap \ldots \cap \Delta_{i_l} \cap \Delta_n$ is contractible for all non-empty subsets $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, n-1\}$. Hence, by the induction hypothesis, Γ is contractible. This proves the claim. $\square$

2.2 Order Complexes and Contractible Carriers

This section introduces the basic properties of order complexes and the contractible carriers. These carriers will be needed to prove Theorem 2.17 and Corollary 2.19, which will be the most important tools to prove contractibility in chapters 4 and 5. At the end of this section, the nerve of a homogeneous space will be defined. The nerve gives a lower bound for $n \in \mathbb{N}_0$ such that $\pi_n(\hat{\Delta}_{G/H}) \neq 0$ or $\tilde{H}_n(\hat{\Delta}_{G/H}, \mathbb{Z}) \neq 0$.

Definition 2.5 ([Bjö95, 9.2, p. 1843f.], [BW96, p. 1312])**.** Let $P = (P, \leq)$ be a *poset*, i.e. a partially ordered set. A finite totally ordered subset $C := \{x_0 < \ldots < x_k\}$ is called a *chain* of P. $l(C) := k$ is called the *length of* C. A *maximal chain* is a chain $C = \{x_0 < \ldots < x_k\}$ such that there exists no $\tilde{x} \in P$ with $\tilde{x} < x_0$, $x_{i-1} < \tilde{x} < x_i$ for some $1 \leq i \leq k$ or $x_k < \tilde{x}$.

For a given poset P let $\hat{0}$ and $\hat{1}$ be two distinct elements not contained in P. Then $\hat{P} := P \,\dot{\cup}\, \{\hat{0}, \hat{1}\}$ becomes a poset by $\hat{0} < x < \hat{1}$ for all $x \in P$. $\hat{0}$ and $\hat{1}$ are called the *bottom element* and the *top element* of $\hat{P}$, respectively. P is called the *proper part* of $\hat{P}$. Moreover, $\hat{P}$ is called a *lattice*, if for all $x, y \in P$ there exists a least upper bound (join) in $\hat{P}$, denoted by $x \vee y$, and a greatest lower bound (meet) in $\hat{P}$, denoted by $x \wedge y$. Furthermore, for all $x \in P$ let

$$P_{\leq x} := \{z \in P \mid z \leq x\}$$

and similarly $P_{<x}$, $P_{\geq x}$ and $P_{>x}$.

Definition 2.6 ([Bjö95, p. 1844])**.** Let $(P, \leq)$ be a finite poset. The *order complex* $\Delta(P, \leq) = \Delta(P)$ of P is the simplicial complex whose k-simplices are the chains of length k of P for $k \geq 0$. A polyhedron of $\Delta(P)$ will be denoted by $\|P\|$ instead of $\|\Delta(P)\|$.

If Δ is any abstract simplicial complex, then $P(\Delta) := (\Delta, \subseteq)$ is called the *face poset* of Δ.

Remark 2.7. Some important properties of order complexes are given by the following list:

1. If $(P, \leq^{\text{op}})$ is the dual poset of P, i.e.

$$\forall x, y \in P: \ x \leq^{\text{op}} y \Leftrightarrow y \leq x,$$

   then $\Delta(P, \leq) = \Delta(P, \leq^{\text{op}})$.

2. Let P, Q be finite posets and let $f : P \to Q$ be a monotonic map, i.e. f is either *order-preserving* or *order-reversing*. Then for any chain C of P, $f(C)$ is a chain of Q. Thus, it induces a simplicial map from $\Delta(P)$ to $\Delta(Q)$ which is again denoted by f. Two monotonic maps $f, g : P \to Q$ are called *homotopic*, denoted by $f \sim g$, if the induced maps $\|f\|, \|g\| : \|P\| \to \|Q\|$ are homotopic.

3. For any simplicial complex Δ, the order complex $\Delta(P(\Delta))$ of the face poset $P(\Delta)$ is isomorphic to the barycentric subdivision of Δ, see [Bjö95, p. 1844]. This implies that $\|P(\Delta)\|$ is homeomorphic to $\|\Delta\|$.

The most important application is the order complex of compact Lie groups $H < G$.

Lemma 2.8. *Let G/H be a compact homogeneous space with $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ such that the poset $P = \hat{P}_{G/H}$ as in section 1.3 is finite. Then $\hat{P} := P \,\dot{\cup}\, \{\hat{0}, \hat{1}\}$ is a lattice.*

Proof. The bottom element $\hat{0}$ may be identified with $\mathfrak{h}$ while the top element $\hat{1}$ may be identified with $\mathfrak{g}$. Let $\mathfrak{k}_1, \mathfrak{k}_2 \in P$. Then the subalgebra $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is either an H-subalgebra or equal to $\mathfrak{h}$. Hence, $\mathfrak{k}_1 \wedge \mathfrak{k}_2 = \mathfrak{k}_1 \cap \mathfrak{k}_2 \in \hat{P}$. Moreover, the subalgebra $\langle \mathfrak{k}_1, \mathfrak{k}_2 \rangle$ is either an H-subalgebra or equal to $\mathfrak{g}$. Thus, $\mathfrak{k}_1 \vee \mathfrak{k}_2 = \langle \mathfrak{k}_1, \mathfrak{k}_2 \rangle \in \hat{P}$. □

Now, the contractible carriers can be introduced, see also [Wal81, p. 374].

Definition 2.9. Let Δ be a simplicial complex, X a topological space and $\mathcal{P}(X)$ its power set. A map

$$\mathcal{C} : \Delta \longrightarrow \mathcal{P}(X)$$

is called a *contractible carrier from Δ to X*, if the following two conditions hold:

1. $\mathcal{C}(\sigma)$ is contractible for all $\sigma \in \Delta$.
2. If $\tau, \sigma \in \Delta$ with $\tau \subseteq \sigma$, then $\mathcal{C}(\tau) \subseteq \mathcal{C}(\sigma)$.

Furthermore, a continous map

$$g : \|\Delta\| \longrightarrow X$$

is *carried by $\mathcal{C}$*, if $g(\|\sigma\|) \subseteq \mathcal{C}(\sigma)$ for all $\sigma \in \Delta$.

Lemma 2.10 ([Wal81, 2.1]). *Let $\mathcal{C}$ be a contractible carrier from Δ to X. Then:*

1. *There exists a continous map $g : \|\Delta\| \to X$ which is carried by $\mathcal{C}$.*
2. *If $g_1, g_2 : \|\Delta\| \to X$ are both carried by $\mathcal{C}$, then g_1 is homotopic to g_2.*

Proof. 1. The map g will be constructed iteratively on the n-skeleton of Δ for $n \geq 0$. First, let σ be a vertex of Δ. Since $\mathcal{C}(\sigma) \neq \emptyset$ by assumption, one can define $g(\|\sigma\|) := p$ for any point $p \in \mathcal{C}(\sigma)$. Now, suppose $g : \|\Delta^n\| \to X$ is carried by $\mathcal{C}_{|\Delta^n}$ for some $0 \leq n < \dim \Delta$ and let σ be an $(n+1)$-simplex of Δ. Since $g(\|\tau\|) \subseteq \mathcal{C}(\tau) \subseteq \mathcal{C}(\sigma)$ for every proper face τ of σ, it follows

$$g_{|\,\mathrm{Bd}\,\|\sigma\|} : \mathrm{Bd}\,\|\sigma\| \longrightarrow \mathcal{C}(\sigma).$$

But $(\|\sigma\|, \mathrm{Bd}\,\|\sigma\|) \cong (D^{n+1}, S^n)$ and by assumption, $\pi_n(\mathcal{C}(\sigma)) = 0$. Hence, by [Hat02, p. 346], there exists a continous extension of $g_{|\,\mathrm{Bd}\,\|\sigma\|}$ to a map $\|\sigma\| \to \mathcal{C}(\sigma) \subseteq X$. By [Mun84, Lemma 2.3, p. 9], these maps can be glued together to a continous map

$g : \|\Delta^{n+1}\| \to X$. Hence, g is carried by $\mathcal{C}_{|\Delta^{n+1}}$ and for $n+1 = \dim \Delta$, the claim follows.

2. Similarly to 1, the homotopy between g_1 and g_2 will be constructed iteratively on the n-skeleton of Δ for $n \geq 0$. First, let

$$H : \|\Delta\| \times \{0,1\} \longrightarrow X$$

such that $H(-,0) = g_1$ and $H(-,1) = g_2$. If $\sigma \in \Delta^0$ is a vertex, then by assumption $g_1(\|\sigma\|),\ g_2(\|\sigma\|) \in \mathcal{C}(\sigma)$. Since $\mathcal{C}(\sigma)$ is path-connected, it contains a continous path $c_\sigma : [0,1] \to \mathcal{C}(\sigma)$ from $g_1(\|\sigma\|)$ to $g_2(\|\sigma\|)$. By [Mun84, Lemma 2.3, p. 9], H can be extended continously to

$$H : (\|\Delta\| \times \{0,1\}) \cup (\|\Delta^0\| \times [0,1]) \longrightarrow X$$

by the rule $H(\|\sigma\|, t) := c_\sigma(t),\ 0 \leq t \leq 1$. In particular, $H(\|\sigma\| \times [0,1]) \subseteq \mathcal{C}(\sigma)$ for every vertex σ. Now, let σ be an n-simplex of Δ for some $n \geq 1$ and suppose

$$H : (\|\Delta\| \times \{0,1\}) \cup (\|\Delta^{n-1}\| \times [0,1]) \longrightarrow X$$

is already defined and $H(\|\tau\| \times [0,1]) \subseteq \mathcal{C}(\tau)$ for every $(n-1)$-simplex τ of Δ. Since $g_1(\|\sigma\|),\ g_2(\|\sigma\|) \subseteq \mathcal{C}(\sigma)$, it follows $H(S_\sigma) \subseteq \mathcal{C}(\sigma)$ for

$$S_\sigma := (\|\sigma\| \times \{0,1\}) \cup (\operatorname{Bd} \|\sigma\| \times [0,1]).$$

But $S_\sigma \cong S^n$ and $\pi_n(\mathcal{C}(\sigma)) = 0$, so $H_{|S_\sigma}$ can be extended continously to a map $\|\sigma\| \times [0,1] \to \mathcal{C}(\sigma)$. Again, these maps can be glued together to a continous map

$$H : (\|\Delta\| \times \{0,1\}) \cup (\|\Delta^n\| \times [0,1]) \to X$$

and for $n = \dim \Delta$, the claim follows. □

Corollary 2.11 ([Wal81, 2.2]). *Let P, Q be finite posets and let $f : P \to Q$ be a monotonic map. Suppose that either $\Delta(f^{-1}(Q_{\leq q}))$ is a contractible subcomplex of $\Delta(P)$ for all $q \in Q$ or $\Delta(f^{-1}(Q_{\geq q}))$ is a contractible subcomplex for all $q \in Q$. Then*

$$\|f\| : \|P\| \longrightarrow \|Q\|$$

is a homotopy equivalence.

Proof. Without loss of generality, one may assume that $f^{-1}(Q_{\leq q})$ is contractible for all $q \in Q$. Otherwise, one may replace Q by its dual poset Q^{op}. So, consider the contractible carriers

$$\mathcal{C}_1 : \Delta(Q) \longrightarrow \mathcal{P}(\|P\|);\ \sigma \mapsto \|f\|^{-1}(\|Q_{\leq \max \sigma}\|)$$

and

$$\mathcal{C}_2 : \Delta(Q) \longrightarrow \mathcal{P}(\|Q\|);\ \sigma \mapsto \|Q_{\leq \max \sigma}\|.$$

In fact, if $\tau, \sigma \in \Delta(Q)$, then $\tau \subseteq \sigma$ implies $\max \tau \leq \max \sigma$. Hence, $\mathcal{C}_i(\tau) \subseteq \mathcal{C}_i(\sigma)$ for $i \in \{1,2\}$. Furthermore, $\mathcal{C}_1(\sigma)$ is contractible by assumption and $\mathcal{C}_2(\sigma)$ is contractible since it is a cone.

By Lemma 2.10, there exists a continous map $g : \|Q\| \to \|P\|$ such that $g(\|\sigma\|) \subseteq \mathcal{C}_1(\sigma) = \|f\|^{-1}(\|Q_{\leq \max \sigma}\|)$ for all $\sigma \in \Delta(Q)$. Hence, $(\|f\| \circ g)(\|\sigma\|) \subseteq \|Q_{\max \leq \sigma}\| = \mathcal{C}_2(\sigma)$. But also $\|\sigma\| \subseteq \mathcal{C}_2(\sigma)$ for $\sigma \in \Delta(Q)$. It follows that $\|f\| \circ g$ and $\mathrm{id}_{\|Q\|}$ are both carried by $\mathcal{C}_2$. Thus, $\|f\| \circ g \sim \mathrm{id}_{\|Q\|}$ by Lemma 2.10. Now, consider the contractible carrier

$$\mathcal{D} : \Delta(P) \longrightarrow \mathcal{P}(\|P\|);\ \mu \mapsto \mathcal{C}_1(f(\mu)).$$

In fact, $f(\mu) \in \Delta(Q)$, since f is monotonic and for $\nu, \mu \in \Delta(P)$, $\nu \subseteq \mu$ implies $f(\nu) \subseteq f(\mu)$. So, $\mathcal{D}(\nu) \subseteq \mathcal{D}(\mu)$. Hence, $\mathcal{D}$ is a contractible carrier. For $\mu \in \Delta(P)$, it follows $(g \circ \|f\|)(\|\mu\|) = g(\|f(\mu)\|) \subseteq \mathcal{D}(\mu)$. On the other hand, $\|\mu\| \subseteq \|f\|^{-1}(\|Q_{\leq \max f(\mu)}\|) = \mathcal{D}(\mu)$. Hence, $g \circ \|f\|$ and $\mathrm{id}_{\|P\|}$ are both carried by $\mathcal{D}$ and $g \circ \|f\| \sim \mathrm{id}_{\|P\|}$ by Lemma 2.10. □

As a first application of contractible carriers, the next theorem shows that for compact Lie groups $H < G$ $\hat{\Delta}_{G/H}$ is a strong deformation retract of $\Delta^{\min}_{G/H}$ if both complexes are defined.

Theorem 2.12 ([Böh04, Cor. 6.12], [Sha01, Cor. 2.5]). *Let G/H be a compact homogeneous space such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\hat{P}_{G/H}$ is finite. Then $\Delta^{\min} := \Delta^{\min}_{G/H}$ is a strong deformation retract of $\hat{\Delta} := \hat{\Delta}_{G/H}$.*

Proof. Let $P^{\min} := P^{\min}_{G/H}$ and $\hat{P} := \hat{P}_{G/H}$. Consider the inclusion map

$$\iota : P^{\min} \hookrightarrow \hat{P}.$$

ι is monotonic and for each $\mathfrak{k} \in \hat{P}$, the set $\iota^{-1}(\hat{P}_{\leq \mathfrak{k}})$ is contractible. In fact, if $\mathfrak{l} \in P^{\min}$, then $\mathfrak{l} \leq \mathfrak{k}$ if and only if $\mathfrak{l}$ is generated by minimal H-subalgebras which are all contained in $\mathfrak{k}$. For any $\mathfrak{k} \in \hat{P}$, define

$$\mathfrak{min}(\mathfrak{k}) := \langle \bar{\mathfrak{l}} \mid \bar{\mathfrak{l}} \text{ minimal } H\text{-subalgebra}, \bar{\mathfrak{l}} \leq \mathfrak{k} \rangle.$$

It follows

$$\iota^{-1}(\hat{P}_{\leq \mathfrak{k}}) = P^{\min}_{\leq \mathfrak{min}(\mathfrak{k})}.$$

Hence, $\Delta(\iota^{-1}(\hat{P}_{\leq \mathfrak{k}})) = \Delta(P^{\min}_{\leq \mathfrak{min}(\mathfrak{k})})$ is a cone over $\mathfrak{min}(\mathfrak{k})$. Thus, by Corollary 2.11, $\|\iota\| : \|\Delta^{\min}\| \to \|\hat{\Delta}\|$ is a homotopy equivalence. From the theorem of Whitehead, it follows that $\Delta^{\min}$ is a strong deformation retract of $\hat{\Delta}$. □

The following lemmas will be needed to prove Theorem 2.17, Corollary 2.18 and Corollary 2.19, which show that under certain circumstances the complex $\hat{\Delta}_{G/H}$ may be constructed from the complexes of the form $\hat{\Delta}_{G/K}$ and $\hat{\Delta}_{K/H}$ for some $H < K < G$, see p. 14 ff. of [BW83]. In chapters 4 and 5, this will be used to prove contractibility.

Lemma 2.13. *Let P, Q be finite posets and let $f, g : P \to Q$ be order preserving maps. Suppose that for all $p \in P$, $f(p)$ and $g(p)$ are comparable, i.e.*

$$\forall p \in P : \quad f(p) \geq g(p) \ \vee \ f(p) \leq g(p)$$

Then f and g are homotopic maps.

Proof. Let $\sigma \in \Delta(P)$ be a chain of P. Then $q := \min\{f(\min \sigma), g(\min \sigma)\}$ is well-defined and the minimal element of the subposet $f(\sigma) \cup g(\sigma)$. Let

$$\mathcal{C} : \Delta(P) \longrightarrow \|Q\|; \ \sigma \mapsto \|f(\sigma) \cup g(\sigma)\|.$$

Since $\mathcal{C}(\sigma)$ is a cone over q, it is contractible. Moreover, $\mathcal{C}(\tau) \subseteq \mathcal{C}(\sigma)$ for $\tau \subseteq \sigma$, so $\mathcal{C}$ is a contractible carrier from $\Delta(P)$ to $\|Q\|$. Furthermore, $\|f(\sigma)\|$, $\|g(\sigma)\| \subseteq \mathcal{C}(\sigma)$ for $\sigma \in \Delta(P)$. Thus, f and g are homotopic by Lemma 2.10. □

Definition 2.14. Let P be a finite poset and $p \in P$. P is called *join-contractible (with respect to p)*, if for all $x \in P$ the join $x \vee p$ exists in P. Similarly, P is called *meet-contractible (with respect to p)*, if for all $x \in P$ the meet $x \wedge p$ exists in P.

Lemma 2.15. *Let P be a join-contractible (meet-contractible) poset with respect to some $p \in P$. Then $\Delta(P)$ is contractible.*

Proof. The map

$$f : P \longrightarrow P; \ x \mapsto x \vee p \quad (f : P \longrightarrow P; \ x \mapsto x \wedge p)$$

is a well-defined order preserving map and comparable to id_P. Furthermore, f is comparable to the constant map $c_p(x) := p$, $x \in P$. By Lemma 2.13, it follows $\mathrm{id}_P \sim f \sim c_p$. Hence, $\Delta(P)$ is contractible. □

Definition 2.16. Let P be a finite poset such that $\hat{P} := P \,\dot{\cup}\, \{\hat{0}, \hat{1}\}$ is a lattice. For $p \in P$, the *complement of p* is

$$c(p) := \{x \in P \mid x \wedge p = \hat{0} \text{ and } x \vee p = \hat{1}\}.$$

Now, the following theorem shows that the order complex of $P \backslash c(p)$ is contractible.

Theorem 2.17. *Let P be a finite poset such that $\hat{P} = P \,\dot{\cup}\, \{\hat{0}, \hat{1}\}$ is a lattice. For a fixed $p \in P$ let $Q := P \setminus c(p)$. Then $\Delta(Q)$ is contractible.*

Proof. As a first step, it has to be proved that for all $x, y \in Q,\ x < y$, the following conditions hold:

1. $x \wedge p$ exists in Q or $x \vee p$ exists in Q (or both).

2. If $y \wedge p$ exists in Q, then $x \vee (y \wedge p)$ exists in Q.

To prove 1, note that for each $x \in Q$ at least one of the elements $x \wedge p,\ x \vee p$ lies in P. Assume $x \wedge p \in P$. Since $(x \wedge p) \wedge p = x \wedge p$, $x \wedge p$ is not complementary to p, so $x \wedge p \in Q$. Similarly, $x \vee p \in P$ implies $x \vee p \in Q$.

Now, let $x < y$ such that $y \wedge p$ exists in Q. Since y is an upper bound for x and for $y \wedge p$, it follows

$$\hat{1} > y \geq x \vee (y \wedge p) \geq x > \hat{0}.$$

So, $x \vee (y \wedge p) \in P$. Furthermore, $y \wedge p$ is a lower bound for $x \vee (y \wedge p)$ and for p. It follows

$$(x \vee (y \wedge p)) \wedge p \geq y \wedge p > \hat{0}.$$

This implies $x \vee (y \wedge p) \in Q$. Now, let

$$M := \{x \in Q \mid x \wedge p \text{ exists in } Q\}.$$

For any chain $\sigma \in \Delta(Q)$ let

$$\begin{aligned} \mathcal{C}(\sigma) := \sigma &\cup \{p\} \cup \{x \wedge p \mid x \in \sigma \cap M\} \cup \{x \vee p \mid x \in \sigma \cap M^c\} \\ &\cup \{x \vee (y \wedge p) \mid x < y,\ x \in \sigma,\ y \in \sigma \cap M\}. \end{aligned}$$

By 1 and 2, $\mathcal{C}(\sigma)$ is a well-defined subposet of Q . Furthermore, $\Delta(\mathcal{C}(\sigma))$ is contractible. In fact, let $z := \min \sigma$. If $z \in M$, then $z \wedge p$ is a lower bound for each element of $\mathcal{C}(\sigma)$. Hence, $(z \wedge p) \wedge c = z \wedge p \in \mathcal{C}(\sigma)$ for $c \in \mathcal{C}(\sigma)$ and $\mathcal{C}(\sigma)$ is meet-contractible with respect to $z \wedge p$.

If $z \notin M$, then $\mathcal{C}(\sigma)$ is join-contractible with respect to z. In fact, for $x \in \sigma$, $z \vee x = x \in \mathcal{C}(\sigma)$ and $z \vee p \in \mathcal{C}(\sigma)$ by 1. Moreover, if $x \in \sigma \cap M$, then $z < x$ and $z \vee (x \wedge p) \in \mathcal{C}(\sigma)$ by definition. For all other elements $c \in \mathcal{C}(\sigma)$, z is a lower bound. So, $z \vee c = c \in \mathcal{C}(\sigma)$.

By Lemma 2.15, $\Delta(\mathcal{C}(\sigma))$ is contractible. Furthermore, the definition of $\mathcal{C}(\sigma)$ implies $\mathcal{C}(\tau) \subseteq \mathcal{C}(\sigma)$ for all $\tau \subseteq \sigma$. It follows that $\sigma \mapsto \|\mathcal{C}(\sigma)\|$ is a contractible carrier from $\Delta(Q)$ to $\|Q\|$. Since $\sigma \subseteq \mathcal{C}(\sigma)$ and $p \in \mathcal{C}(\sigma)$, $\mathrm{id}_{\|Q\|}$ and the constant map $c_p \equiv \|p\|$ are both carried by this carrier. Hence, $\mathrm{id}_{\|Q\|} \sim c_p$ by Lemma 2.10 and $\Delta(Q)$ is contractible. □

For the case that $c(p)$ is an antichain, this theorem also yields informations about the topology of $\Delta(P)$, as the following corollary will show.

Corollary 2.18 ([BW83, Thm. 4.2])**.** *Let P be a finite poset such that $\hat{P} = P\dot{\cup}\{\hat{0}, \hat{1}\}$ is a lattice. Furthermore, let $p \in P$ such that $c(p)$ is an antichain, i.e. for all $x, y \in c(p)$, neither $x < y$ nor $y < x$ holds. Then*

$$\|P\| \simeq \bigvee_{x\in c(p)} \|P_{<x}\| * \|P_{>x}\| * S^0. \tag{2.1}$$

Proof. Since $c(p)$ is an antichain, $P_x := \|P_{<x} \cup P_{>x}\| = \|P_{<x}\| * \|P_{>x}\|$ is a subcomplex of $\|P \setminus c(p)\|$ for $x \in c(p)$. In particular, $\|P\|$ can be obtained from $\|P \setminus c(p)\|$ by attaching the cone $P_x * x$ for all $x \in c(P)$.

Now, let $Q := \|P\|/\|P \setminus c(p)\|$. The quotient map $\pi : \|P\| \to Q$ deforms the cones $P_x * x$, to the suspensions $P_x * S^0$ for $x \in c(P)$, which are glued together at the base point $[\|P \setminus c(p)\|]$. Hence, Q is homeomorphic to the wedge sum in (2.1). Moreover, since $\|P \setminus c(P)\|$ is contractible, the quotient map $\pi : \|P\| \to Q$ is a homotopy equivalence, see [Hat02, p. 11]. This proves the claim. □

To apply these results to complexes of type $\hat{\Delta}_{G/H}$, the following corollary is needed.

Corollary 2.19. *Let $H < G$ be compact Lie groups such that H is connected, $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\hat{P}_{G/H}$ is finite. If $\mathfrak{k}_0$ is an H-subalgebra such that $c(\mathfrak{k}_0)$ contains only minimal H-subalgebras $\mathfrak{k}_1, \dots, \mathfrak{k}_l$, then*

$$\hat{\Delta}_{G/H} \simeq \bigvee_{i=1}^{l} \hat{\Delta}_{G/K_i} * S^0.$$

In particular, $\hat{\Delta}_{G/H}$ is contractible if all of the complexes $\hat{\Delta}_{G/K_i}$, $1 \le i \le l$, are contractible.

Proof. Since $c(\mathfrak{k}_0)$ consists of minimal H-subalgebras, it is an antichain. Furthermore, for $i \in \{1, \dots, l\}$, it is $(\hat{P}_{G/H})_{<\mathfrak{k}_i} = \emptyset$ and $(\hat{P}_{G/H})_{>\mathfrak{k}_i} = \hat{P}_{G/K_i}$, where $\hat{P}_{G/K_i}$ is considered as a subposet of $\hat{P}_{G/H}$, see Remark 1.21. Hence, $\|(\hat{P}_{G/H})_{<\mathfrak{k}_i}\| * \|(\hat{P}_{G/H})_{>\mathfrak{k}_i}\| * S^0 = \|\hat{\Delta}_{G/K_i}\| * S^0$. The claim follows from Lemma 2.8 and Corollary 2.18. □

As mentioned above, Corollary 2.18 yields an alternative proof that the spaces $\hat{\Delta}_{G_1\times G_2/H_1\times H_2}$ and $\hat{\Delta}_{G_1/H_1} * \hat{\Delta}_{G_2/H_2} * S^0$ as in (1.7) are at least homotopy equivalent.

Corollary 2.20 ([BW83, Prop. 4.3], [Böh04, Thm.3.2, Cor.3.3])**.** *Let G_1, G_2 be compact Lie groups, $H_i < G_i$ compact connected subgroups such that $\mathfrak{n}(\mathfrak{h}_i) = \mathfrak{h}_i$ and $\hat{P}_{G/H_i}$ is finite for $i \in \{1, 2\}$. Then*

$$\hat{\Delta}_{G_1\times G_2/H_1\times H_2} \simeq \hat{\Delta}_{G_1/H_1} * \hat{\Delta}_{G_2/H_2} * S^0.$$

Proof. Consider the $H_1 \times H_2$-subalgebra $\mathfrak{k}_0 := \mathfrak{g}_1 \oplus \mathfrak{h}_2$ and let $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ be any $H_1 \times H_2$-subalgebra. Then $\mathfrak{k}_0 \cap \mathfrak{k} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ if and only if $\mathfrak{k}_1 = \mathfrak{h}_1$ and $\langle \mathfrak{k}_0, \mathfrak{k}\rangle = \mathfrak{g}_1 \oplus \mathfrak{g}_2$

if and only if $\mathfrak{k}_2 = \mathfrak{g}_2$. It follows $c(\mathfrak{k}_0) = \{\mathfrak{h}_1 \oplus \mathfrak{g}_1\}$. Since there are canonical poset isomorphisms

$$(\hat{P}_{G_1 \times G_2/H_1 \times H_2})_{>\mathfrak{h}_1 \oplus \mathfrak{g}_2} \cong \hat{\Delta}_{G_1/H_1} \text{ and}$$
$$(\hat{P}_{G_1 \times G_2/H_1 \times H_2})_{<\mathfrak{h}_1 \oplus \mathfrak{g}_2} \cong \hat{\Delta}_{G_2/H_2},$$

the claim follows from Lemma 2.8 and Corollary 2.18. □

As another application for contractible carriers, the nerve of a homogeneous space will be introduced.

Definition 2.21. [Bjö95, p. 1849] Let $\mathcal{A} = (A_i)_{i \in I}$ be a finite family of non-empty sets. The *nerve* $\mathcal{N}(\mathcal{A})$ of $\mathcal{A}$ is the simplicial complex with vertex set I such that a subset $\emptyset \subsetneq \sigma \subseteq I$ satisfies $\sigma \in \mathcal{N}(\mathcal{A})$ if and only if $\cap_{i \in \sigma} A_i \neq \emptyset$.

Let G/H be a compact homogeneous space and let $\Delta = \Delta^{\min}_{G/H}$ or $\Delta = \Delta^{T}_{G/H}$. Let $\mathfrak{k}_1, \dots, \mathfrak{k}_N$ denote all $(T-)$minimal non-toral H-subalgebras, so

$$\Delta = \bigcup_{i=1}^{N} C_i,$$

where $C_i := \{\sigma \in \Delta \mid \min \sigma \geq \mathfrak{k}_i\}$ is the cone over $\mathfrak{k}_i$. The nerve $\mathcal{N}((C_i)_{i \in \{1,\dots,N\}})$ will be denoted by $\mathcal{N}(\Delta)$.

Theorem 2.22 (Nerve Theorem)**.** *[Bjö95, p. 1850] Let Δ be a simplicial complex and let $(\Delta_i)_{i \in I}$ be a finite family of contractible subcomplexes such that $\Delta = \cup_{i \in I} \Delta_i$. If $\Delta_{i_1} \cap \ldots \cap \Delta_{i_l}$ is either empty or contractible for every subset $\{i_1, \dots, i_l\} \subseteq I$, then Δ and $\mathcal{N}((\Delta_i)_{i \in I})$ are homotopy equivalent.*

Proof. Consider the face posets $Q := P(\Delta)$ and $R := P(\mathcal{N}((\Delta_i)_{i \in I}))$ of Δ and $\mathcal{N}((\Delta_i)_{i \in I})$, respectively, and let

$$f : Q \longrightarrow R;\ \sigma \mapsto \{i \in I \mid \sigma \subseteq \Delta_i\}.$$

In fact, $f(\sigma) \neq \emptyset$, since $\Delta = \cup_{i \in I} \Delta_i$ and $f(\sigma)$ is a simplex of $\mathcal{N}((\Delta_i)_{i \in I})$, since $\emptyset \neq \sigma \subseteq \cap_{i \in f(\sigma)} \Delta_i$. Hence, f is well-defined. Moreover, if τ is a face of σ, then $f(\sigma) \subseteq f(\tau)$. Thus, f is monotonic. Finally, for each simplex $\mu \in \mathcal{N}((\Delta_i)_{i \in I})$ it follows

$$\begin{aligned} f^{-1}(R_{\geq \mu}) &= \{\sigma \in \Delta \mid \mu \subseteq f(\sigma)\} \\ &= \{\sigma \in \Delta \mid \sigma \subseteq \Delta_i \text{ for all } i \in \mu\} \\ &= \bigcap_{i \in \mu} \Delta_i. \end{aligned}$$

This set is non-empty, since $\mu \in \mathcal{N}((\Delta_i)_{i \in I})$. Thus, it is contractible by assumption and $\Delta(Q) \simeq \Delta(R)$ by Corollary 2.11. But $\Delta(Q)$ and $\Delta(R)$ are the barycentric subdivisions of Δ and $\mathcal{N}((\Delta_i)_{i \in I})$, respectively. Hence, $\Delta \simeq \mathcal{N}((\Delta_i)_{i \in I})$. □

Corollary 2.23. *Let G/H be a compact homogeneous space and let Δ, $\mathcal{N}(\Delta)$ as in Definition 2.21. Then, $\Delta \simeq \mathcal{N}(\Delta)$. Moreover, for $\Delta \neq \emptyset$ let*

$$r := \inf\{s \in \mathbb{N} \mid \exists\ (T-)\textit{minimal non-toral } H\textit{-subalgebras } \mathfrak{k}_1, \ldots, \mathfrak{k}_s : \langle \mathfrak{k}_1, \ldots, \mathfrak{k}_s \rangle = \mathfrak{g}\}$$

with $\inf \emptyset := \infty$. *Then Δ is $(r-3)$-connected.*

Proof. Let $\mathfrak{k}_1, \ldots, \mathfrak{k}_N$ be all $(T-)$minimal non-toral H-subalgebras and let C_i be as in Definition 2.21. For any $\emptyset \neq I \subseteq \{1, \ldots, N\}$ let $\mathfrak{l}_I := \langle \mathfrak{k}_i \mid i \in I \rangle$. Then

$$\bigcap_{i \in I} C_i = \{\sigma \in \Delta \mid \min \sigma \geq \mathfrak{k}_i \text{ for all } i \in I\} = \{\sigma \in \Delta \mid \min \sigma \geq \mathfrak{l}_I\}.$$

Hence, if $\mathfrak{l}_I < \mathfrak{g}$, then $\cap_{i \in I} C_i$ is a cone over $\mathfrak{l}_I$. Otherwise, $\cap_{i \in I} C_i = \emptyset$. It follows that I is a simplex of $\mathcal{N}(\Delta)$ if and only if $\mathfrak{l}_I < \mathfrak{g}$ and $\Delta \simeq \mathcal{N}(\Delta)$ by Theorem 2.22.

Now, let $\mathfrak{l} := \langle \mathfrak{k}_i \mid 1 \leq i \leq N \rangle$ be the subalgebra generated by all $(T-)$minimal non-toral H-subalgebras. So, $r = \infty$ is equivalent to $\mathfrak{l} < \mathfrak{g}$. But in this case, Δ is a cone over $\mathfrak{l}$ and therefore, it is contractible. Hence, one may assume $r < \infty$.

Now, $r \geq 2$, since $\mathfrak{k} < \mathfrak{g}$ for all H-subalgebras. By assumption, $\Delta \neq \emptyset$, i.e. Δ is (-1)-connected. So, assume $r \geq 3$. Every subset $\{i_1 < \ldots < i_s\} \subseteq \{1, \ldots, N\}$ with $s \leq r-1$ is an $(s-1)$-simplex of $\mathcal{N}(\Delta)$, since $\cap_{j=1}^{s} C_{i_j}$ is a cone over $\langle \mathfrak{k}_1, \ldots, \mathfrak{k}_s \rangle < \mathfrak{g}$. It follows that the $(r-2)$-skeleton of $\mathcal{N}(\Delta)$ coincides with the $(r-2)$-skeleton of the standard $(N-1)$-simplex. But for $k \leq r-3$, the k-th homotopy group is determined by the $(r-2)$-skeleton. Thus, $\pi_k(\Delta) \cong \pi_k(\mathcal{N}(\Delta)) = 0$. □

2.3 Main Theorem

In this section, the main theorem of this thesis will be proved. The theorem describes how a non-zero homology class $[\theta_{\text{new}}] \in \tilde{H}_m(\hat{\Delta}_{G/H}, R) \setminus \{0\}$ can be constructed under appropiate circumstances, if a non-zero homology class $[\theta] \in \tilde{H}_{m-1}(\hat{\Delta}_{G/K}, R) \setminus \{0\}$ is given, where $m \in \mathbb{N}$, $H < K < G$, H maximal in K and R is a commutative unitary ring. The beginning of such an iteration usually starts with a 0-cycle defined as in the following lemma.

Lemma 2.24. *Let $H < G$ be compact connected Lie groups such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\hat{P}_{G/H}$ is finite and let R be any commutative unitary ring. Furthermore, assume that there exists a maximal H-subalgebra $\mathfrak{k}_0$ of $\mathfrak{g}$ such that $\mathfrak{h}$ is maximal in $\mathfrak{k}_0$. Then $\tilde{H}_0(\hat{\Delta}_{G/H}, R) \neq 0$ if and only if there exists another H-subalgebra $\mathfrak{k}_1 \neq \mathfrak{k}_0$.*

Proof. The above conditions imply that $\mathfrak{k}_0$ is a facet of $\hat{\Delta}_{G/H}$. Hence, the existence of another vertex ensures that $\hat{\Delta}_{G/H}$ is disconnected. More precisely, $\theta_0 := \mathfrak{k}_0 - \mathfrak{k}_1$ is a 0-cycle with $[\theta_0] \in \tilde{H}_0(\hat{\Delta}_{G/H}, R) \setminus \{0\}$. □

To formulate the main theorem, the following definition is needed.

Definition 2.25. Let Δ be any simplicial complex and R a commutative unitary ring. Furthermore, let $\theta = \sum_{i=1}^{N} r_i \cdot [v_0^i, \ldots, v_n^i] \in C_n(\Delta) \otimes R$ be an n-chain with coefficients in R, $r_i \in R \setminus \{0\}$ for $1 \leq i \leq N$. The *support of* θ is then given by

$$\operatorname{supp}(\theta) := \{\{v_0^1, \ldots, v_n^1\}, \ldots, \{v_0^N, \ldots, v_n^N\}\},$$

i.e. the set of all n-simplices whose coset is a non-zero summand of θ. If $\operatorname{supp}(\theta)$ contains a facet of Δ, then θ is called *supported by a facet.* Moreover, the *vertex-support of* θ is given by

$$\operatorname{vsupp}(\theta) := \bigcup_{i=1}^{N} \{v_0^i, \ldots, v_n^i\},$$

i.e. the underlying vertex set of $\operatorname{supp}(\theta)$, see also [Kah09, p. 1665]. Now, let $\Delta = \Delta(P)$ be an order complex. A subset $L \subseteq P$ is called a *lower bound set (l.b.s.) of* θ, if for all vertices $v \in \operatorname{vsupp}(\theta)$ there exists an $l \in L$ with $l \leq v$.

Theorem 2.26 (Main Theorem, Part I)**.** *Let $H < G$ be compact connected Lie groups such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\hat{P}_{G/H}$ is finite and let R be any commutative unitary ring. Furthermore, let $m \in \mathbb{N}$, $\mathfrak{k}_0$ a minimal H-subalgebra and $[\theta] \in \tilde{H}_{m-1}(\hat{\Delta}_{G/K_0}, R) \setminus \{0\}$ a non-zero homology class with a representative θ such that the following holds:*

For a given l.b.s. $\{\mathfrak{l}_1, \ldots, \mathfrak{l}_N\}$ of θ in $\hat{P}_{G/K_0}$, there exist H-subalgebras $\mathfrak{k}_1, \ldots, \mathfrak{k}_N$, not necessarily distinct, with the following properties:

$$\forall i \in \{1, \ldots, N\}: \quad \mathfrak{k}_0 \neq \mathfrak{k}_i \tag{2.2}$$

$$\forall i \in \{1, \ldots, N\}: \quad \mathfrak{l}_i \geq \mathfrak{k}_i \tag{2.3}$$

$$\langle \mathfrak{k}_1, \ldots, \mathfrak{k}_N \rangle < \mathfrak{g} \tag{2.4}$$

Moreover, suppose that θ is supported by a facet of $\hat{\Delta}_{G/K_0}$. Then

$$\tilde{H}_m(\hat{\Delta}_{G/H}, R) \neq 0.$$

More precisely, there exists an m-cycle θ_{new} with $[\theta_{\text{new}}] \in \tilde{H}_m(\hat{\Delta}_{G/H}, R) \setminus \{0\}$, θ_{new} is supported by a facet of $\hat{\Delta}_{G/H}$ and a l.b.s. of θ_{new} is given by $\{\mathfrak{k}_0, \ldots, \mathfrak{k}_N\}$.

Proof. Let $\mathfrak{k}_1, \ldots, \mathfrak{k}_N$ be H-subalgebras satisfying (2.2), (2.3) and (2.4). For any simplex $\sigma = (\mathfrak{k}_0^\sigma < \ldots < \mathfrak{k}_{\dim \sigma}^\sigma) \in \hat{\Delta}_{G/H}$ let $\mathfrak{k}_{\sigma_{\min}} := \mathfrak{k}_0^\sigma$ be its minimal element. Moreover, for $l \in \{0, \ldots, N\}$, define

$$C_l := \{\sigma \in \hat{\Delta}_{G/H} \mid \mathfrak{k}_{\sigma_{\min}} \geq \mathfrak{k}_l\} \text{ and}$$

$$D := \bigcup_{l=1}^{N} C_l.$$

C_l is contractible for $0 \leq l \leq N$ as it is a cone over $\mathfrak{k}_l$. Since $\langle \mathfrak{k}_1, \ldots, \mathfrak{k}_N \rangle < \mathfrak{g}$ by (2.4), $\langle \mathfrak{k}_{i_1}, \ldots, \mathfrak{k}_{i_s} \rangle$ is an H-subalgebra for every non-empty subset $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, N\}$. Thus,

$$C_{i_1} \cap \ldots \cap C_{i_s} = \{\sigma \in \hat{\Delta}_{G/H} \mid \mathfrak{k}_{\sigma_{\min}} \geq \langle \mathfrak{k}_{i_1}, \ldots, \mathfrak{k}_{i_s} \rangle\}$$

is also a cone. By Corollary 2.4, D is contractible. Thus, $\tilde{H}_k(C_0, R) \oplus \tilde{H}_k(D, R) = 0$ for all $k \in \mathbb{Z}$. Moreover, $C_0 \cap D \neq \emptyset$, since $\mathfrak{l}_i \in C_0 \cap D$ for $1 \leq i \leq N$. Therefore, the Mayer-Vietoris sequence for reduced homology of the triple $(C_0 \cup D; C_0, D)$, see Theorem 1.18, yields the following exact sequence for $k \in \mathbb{Z}$:

$$0 \xrightarrow{j_*} \tilde{H}_k(C_0 \cup D, R) \xrightarrow{\partial_*} \tilde{H}_{k-1}(C_0 \cap D, R) \xrightarrow{i_*} 0$$

In particular, $\partial_* : \tilde{H}_m(C_0 \cup D, R) \to \tilde{H}_{m-1}(C_0 \cap D, R)$ is an isomorphism.

Now, let $\sigma \in \mathrm{supp}(\theta)$, so $\mathfrak{k}_{\sigma_{\min}} \in \mathrm{vsupp}(\theta)$. Since by (2.3) $\mathfrak{k}_{\sigma_{\min}} \geq \mathfrak{l}_i \geq \mathfrak{k}_i$ for some $i \in \{1, \ldots, N\}$, it follows $\sigma \in C_i \subseteq D$. Furthermore, $\sigma \in \hat{\Delta}_{G/K_0} \subseteq C_0$. Thus, $\mathrm{supp}(\theta) \subseteq C_0 \cap D$ and θ represents an element $[\theta] \in \tilde{H}_{m-1}(C_0 \cap D, R)$. Moreover, every vertex $\mathfrak{l} \in C_0 \cap D$ satisfies $\mathfrak{l} \geq \langle \mathfrak{k}_0, \mathfrak{k}_i \rangle$ for some $i \in \{1, \ldots, N\}$. By (2.2), $\mathfrak{l} > \mathfrak{k}_0$. Hence, $C_0 \cap D \subseteq \hat{\Delta}_{G/K_0}$. It follows $[\theta] \in \tilde{H}_{m-1}(C_0 \cap D, R) \setminus \{0\}$, since θ is not a boundary of $\hat{\Delta}_{G/K_0}$. Now, let

$$[\theta'] := \partial_*^{-1}([\theta]) \in \tilde{H}_m(C_0 \cup D, R) \setminus \{0\},$$

where θ' is a fixed representative. By the definition of ∂_*, see [Mun84, p. 137], θ' can be constructed the following way:

θ is a boundary of C_0 and D, since $\tilde{H}_{m-1}(C_0, R) \oplus \tilde{H}_{m-1}(D, R) = 0$. So, there are m-chains τ_1 of C_0 and τ_2 of D, such that $\partial(\tau_1) = \theta$, $\partial(\tau_2) = -\theta$. Then

$$\theta' := \tau_1 + \tau_2$$

is an m-cycle of $C_0 \cup D$ and $\partial_*([\theta']) = [\theta]$. Now, let $s \in \mathrm{supp}(\theta)$ be a facet of $\hat{\Delta}_{G/K_0}$, i.e. $s = \{\tilde{\mathfrak{l}}_0 < \ldots < \tilde{\mathfrak{l}}_{m-1}\}$ is a maximal chain of K_0-subalgebras. Since $\mathfrak{k}_0$ is minimal over $\mathfrak{h}$, $t := \{\mathfrak{k}_0 < \tilde{\mathfrak{l}}_0 < \ldots < \tilde{\mathfrak{l}}_{m-1}\}$ is a maximal chain of H-subalgebras, i.e. a facet of $\hat{\Delta}_{G/H}$, and t is the only simplex of C_0 such that s is a proper face. This implies $t \in \mathrm{supp}(\tau_1)$. On the other hand, (2.2) and the minimality of $\mathfrak{k}_0$ imply $\mathfrak{k}_0 \not\geq \mathfrak{k}_i$ for $i \in \{1, \ldots, N\}$. Hence, t is not a simplex of D. So, $t \notin \mathrm{supp}(\tau_2)$ and therefore, $t \in \mathrm{supp}(\theta')$. Now, the inclusion map $\iota : C_0 \cup D \hookrightarrow \hat{\Delta}_{G/H}$ induces the inclusion of chain complexes $\iota_\star : \tilde{C}_*(C_0 \cup D) \otimes R \hookrightarrow \tilde{C}_*(\hat{\Delta}_{G/H}) \otimes R$ with $\partial \circ \iota_\star = \iota_\star \circ \partial$. Hence,

$$\theta_{\mathrm{new}} := \iota_\star(\theta')$$

is an m-cycle of $\hat{\Delta}_{G/H}$ with $\mathrm{supp}(\theta_{\mathrm{new}}) = \mathrm{supp}(\theta')$, so $\mathrm{supp}(\theta_{\mathrm{new}})$ contains the facet t. This implies θ_{new} cannot be a boundary of $\hat{\Delta}_{G/H}$, so $[\theta_{\mathrm{new}}] \in \tilde{H}_m(\hat{\Delta}_{G/H}, R) \setminus \{0\}$. Moreover, for every $\mathfrak{k} \in \mathrm{vsupp}(\theta_{\mathrm{new}}) \subseteq C_0 \cup D$ there exists an $i \in \{0, \ldots, N\}$ with $\mathfrak{k} \geq \mathfrak{k}_i$. Thus, $\{\mathfrak{k}_0, \ldots, \mathfrak{k}_N\}$ is a l.b.s. of θ_{new}. □

Theorem 2.26 yields an algorithm to detect non-zero homology classes iteratively along maximal chains of H-subalgebras by applying purely algebraic methods. This will be shown in the second part of the main theorem.

Theorem 2.27 (Main Theorem, Part II)**.** *Let $H < G$ be compact connected Lie groups such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$, $\hat{P}_{G/H}$ is finite and let $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_N^0)$ be a maximal chain of H-subalgebras for some $N \in \mathbb{N}_0$. Furthermore, let $\mathfrak{k}_{N+1}^0 := \mathfrak{h}$ and assume that for each $m \in \{1, \ldots, N+1\}$ there exist K_m^0-subalgebras $\mathfrak{k}_{m-1}^1, \ldots, \mathfrak{k}_{m-1}^m$, not necessarily distinct, with the following properties:*

1. $\mathfrak{k}_0^1 \neq \mathfrak{k}_0^0$
2. *If $N \geq 1$, then for each $m \in \{1, \ldots, N\}$ the K_{m+1}^0-subalgebras $\mathfrak{k}_m^1, \ldots, \mathfrak{k}_m^{m+1}$ satisfy the following properties:*
   - *(a)* $\forall i \in \{1, \ldots, m+1\}: \quad \mathfrak{k}_m^0 \neq \mathfrak{k}_m^i$
   - *(b)* $\forall i \in \{1, \ldots, m+1\}: \quad \mathfrak{k}_{m-1}^{i-1} \geq \mathfrak{k}_m^i$
   - *(c)* $\langle \mathfrak{k}_m^1, \ldots, \mathfrak{k}_m^{m+1} \rangle < \mathfrak{g}$.

Then $\tilde{H}_N(\hat{\Delta}_{G/H}, R) \neq 0$ for any commutative unitary ring R.

Proof. Let $\theta_0 := \mathfrak{k}_0^0 - \mathfrak{k}_0^1 \neq 0$. Then $[\theta_0] \in \tilde{H}_0(\Delta_{G/K_1^0}, R) \setminus \{0\}$ as in the proof of Lemma 2.24. Furthermore, $\operatorname{supp}(\theta)$ contains the facet $\mathfrak{k}_0^0$ of $\hat{\Delta}_{G/K_1^0}$ and a l.b.s. of θ_0 is given by $\{\mathfrak{k}_0^0, \mathfrak{k}_0^1\}$.

Now, suppose that for $m \in \{1, \ldots, N\}$ there exists an $(m-1)$-cycle θ_{m-1} of $\hat{\Delta}_{G/K_m^0}$ with $[\theta_{m-1}] \in \tilde{H}_{m-1}(\hat{\Delta}_{G/K_m^0}, R) \setminus \{0\}$, θ_{m-1} is supported by a facet of $\hat{\Delta}_{G/K_m^0}$ and a l.b.s. of θ_{m-1} is given by $\{\mathfrak{k}_{m-1}^0, \ldots, \mathfrak{k}_{m-1}^m\}$. By assumption, the K_{m+1}^0-subalgebras $\mathfrak{k}_m^1, \ldots, \mathfrak{k}_m^{m+1}$ satisfy the properties (2.2), (2.3) and (2.4) of Theorem 2.26 with respect to θ_{m-1}. Hence, Theorem 2.26 yields an m-cycle θ_m of $\hat{\Delta}_{G/K_{m+1}^0}$ such that $[\theta_m] \in \tilde{H}_m(\hat{\Delta}_{G/K_{m+1}^0}, R) \setminus \{0\}$, θ_m is supported by a facet of $\hat{\Delta}_{G/K_{m+1}^0}$ and $\{\mathfrak{k}_m^0, \ldots, \mathfrak{k}_m^{m+1}\}$ is a l.b.s. of θ_m. By iteration, the claim follows from the case $m = N$. □

In chapters 4 and 5 Theorem 2.26 and Theorem 2.27 will be used to show that $\tilde{H}_*(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ whenever they are applicable for the cases with G simple and $\operatorname{rank} G = \operatorname{rank} H$. For this purpose, root systems have to be introduced first. This will be done in the next chapter.

Chapter 3

Root Systems and Subalgebras of Maximal Rank

As mentioned in the introduction, the main goal of this thesis is to determine, wether $\Delta^T_{G/H}$ is contractible or not for all compact semisimple Lie groups G and $H < G$ a compact connected Lie subgroup of maximal rank. By Remark 1.24 2, $\Delta^T_{G/H}$ only depends on the corresponding Lie algebras. So, as a first step, all possibilities of real compact semisimple Lie algebras $\mathfrak{g}$ have to be classified. For this purpose, root systems have to be introduced.

3.1 Abstract Root Systems

Definition 3.1. A *root system* R of a finite-dimensional Euclidean vector space $(V, \langle\cdot,\cdot\rangle)$ is a finite subset $R \subseteq V \setminus \{0\}$ such that the following conditions hold:

1. $V = \langle R\rangle_{\mathbb{R}}$.
2. $s_\alpha(\beta) = \beta - 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \cdot \alpha \in R$ for all $\alpha,\ \beta \in R$, where $s_\alpha : V \to V$ denotes the orthogonal reflection at α with respect to $\langle\cdot,\cdot\rangle$.
3. $n_{\alpha\beta} := 2\frac{\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbb{Z}$ for all $\alpha,\ \beta \in R$.
4. $R \cap \langle\alpha\rangle_{\mathbb{R}} = \{-\alpha, \alpha\}$ for all $\alpha \in R$.

The elements of R are called *roots* and $\dim_{\mathbb{R}} V$ is called the *rank of* R.

Let $W = W(R)$ denote the (finite) subgroup of $O(V, \langle\cdot,\cdot\rangle)$ generated by all reflections $s_\alpha,\ \alpha \in R$. W is called the *Weyl group of* R.

Two root systems $R \subseteq V$ and $R' \subseteq V'$ are called *isomorphic*, denoted by $R \cong R'$, if there exists a linear isomorphism $\phi : V \to V'$ such that $\phi(R) = R'$ and $n_{\phi(\alpha)\phi(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in R$.

Moreover, R is called *irreducible*, if there exists no decompostion of R into non-empty proper subsets $R_1,\ R_2$ such that $\langle\alpha,\beta\rangle = 0$ for all $\alpha \in R_1,\ \beta \in R_2$. Otherwise, R is called *reducible*. If R is reducible, there exists a unique decompostion $R = R_1 \dot{\cup} \ldots \dot{\cup} R_k$ of pairwise orthogonal subsets such that R_i is an irreducible root system of $V_i := \langle R_i\rangle_{\mathbb{R}}$ for $1 \le i \le k$. These subsets are called the *irreducible components of* R and R will be written as $R_1 + \ldots + R_k$. Moreover, if R is the sum of isomorphic root system one writes $2R_1,\ 3R_1$ and so on as abbreviation for $R_1 + R_1,\ R_1 + R_1 + R_1$ and so on.

Definition 3.2. Let $R \subseteq V$ be a root system. A subset $F = \{\alpha_1, \ldots, \alpha_n\} \subseteq R$ is called a set of *simple roots*, if the following two conditions hold:

1. F is an $\mathbb{R}$-basis of V.
2. Each $\alpha \in R$ can be (uniquely) written as

$$\alpha = \sum_{i=1}^{n} k_i \cdot \alpha_i \tag{3.1}$$

   such that the coefficients k_i are either all non-negative integrals or all non-positive integrals.

A root α is called a *positive root (negative root) (with respect to F)*, if the coefficients in (3.1) satisfy $k_i \ge 0$ ($k_i \le 0$).

Each root system contains a set of simple roots. Moreover all sets of simple roots are conjugate under W, i.e. if a set of simple roots $F = \{\alpha_1, \ldots, \alpha_n\}$ is given, all sets of simple roots are precisely given by $w.F = \{w(\alpha_1), \ldots, w(\alpha_n)\},\ w \in W$, and $w.F = F$ if and only if $w = \mathrm{id}_V$, see [Hum72, p. 48, 51]. Moreover, the simple roots determine R up to isomorphism as the following lemma shows.

Lemma 3.3 ([Hum72, p. 55])**.** *Let $R \subseteq V$ and $R' \subseteq V'$ be root systems with simple roots $F = \{\alpha_1, \ldots, \alpha_n\}$ and $F' = \{\alpha'_1, \ldots, \alpha'_n\}$. If $n_{\alpha_i\alpha_j} = n_{\alpha'_i\alpha'_j}$ for all $1 \le i, j \le n$, then $R \cong R'$.*

Lemma 3.4 ([Hum72, p. 53, Lemma C])**.** *Let R be an irreducible root system. Then there occur at most two different lengths among the roots. Moreover, roots of the same length are conjugate under W.*

If two different lengths occur, the roots will be called *short* and *long* in dependence of their length.

Lemma 3.5 ([Hum72, p. 52, Lemma A])**.** *Let R be an irreducible root system and $F = \{\alpha_1, \ldots, \alpha_n\}$ a set of simple roots. Then there exists a unique root $\alpha_0 \in R$ with $\alpha_0 = \sum_{i=1}^{n} n_i \cdot \alpha_i$ such that each root $\beta \in R,\ \beta = \sum_{i=1}^{n} k_i \cdot \alpha_i$ satisfies $k_i \le n_i$ for all $1 \le i \le n$. α_0 is called the* highest root *or the* maximal root (with respect to F).

In the next step, Dynkin diagrams will be introduced which classify all root systems. For this purpose, let R be a root system and F be any set of simple roots. Then $n_{\alpha_i\alpha_j} \cdot n_{\alpha_j\alpha_i} \in \{0, 1, 2, 3\}$ for all $1 \leq i < j \leq n$. The *Coxeter graph of* R is the graph with n vertices such that for $i \neq j$ the i-th and the j-th vertex are connected by $n_{\alpha_i\alpha_j} \cdot n_{\alpha_j\alpha_i}$ edges. If two vertices are connected by a multiple edge, they represent roots of different length. In this case, an arrow points towards the short root. The resulting figure is called the *Dynkin diagram of* R, denoted by $D(R)$.

Note, that $D(R)$ is independent of the choice of F. Moreover, two isomorphic root systems have the same Dynkin diagram. Hence, root systems are classified by their Dynkin diagrams. Since R is irreducible if and only if $D(R)$ is connected, it remains to classify all connected Dynkin diagrams. By [Hel78, p. 470, Theorem 3.21], these are given by four infinite series A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, the *classical* Dynkin diagrams, and five *exceptional* Dynkin diagrams E_6, E_7, E_8, F_4, G_2, see Figure 3.1.

3.2 Root Systems of Semisimple Lie Algebras

In this section, the root systems of semisimple Lie algebras will be introduced. First, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$ with Kiling form

$$\kappa(X, Y) := \operatorname{trace}(\operatorname{ad}_X \circ \operatorname{ad}_Y), \ X, Y \in \mathfrak{g}.$$

κ is a symmetric bilinear form on $\mathfrak{g}$, see [Hel78, p. 131]. $\mathfrak{g}$ is called *semisimple*, if its Killing form κ is non-degenerate. Moreover, $\mathfrak{g}$ is called *simple*, if $\mathfrak{g}$ is non-abelian and if it has no non-trivial ideals. Since $\ker \kappa$ is an ideal of $\mathfrak{g}$, simple Lie algebras are semisimple. A Lie group G is called *semisimple (simple)*, if its Lie algebra is semisimple (simple).

Now, let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{h}$ is a maximal abelian subalgebra of $\mathfrak{g}$ and $\operatorname{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is a semisimple endomorphism for all $X \in \mathfrak{h}$. Let $\alpha \in \mathfrak{h}^*$ be any linear form on $\mathfrak{h}$ and

$$L_\alpha := \{X \in \mathfrak{g} \mid [h, X] = \alpha(h) \cdot X \text{ for all } h \in \mathfrak{h}\}.$$

α is called a *root of* $\mathfrak{g}$ *with respect to* $\mathfrak{h}$, if $\alpha \neq 0$ and $L_\alpha \neq \{0\}$. The set of all roots with respect to $\mathfrak{h}$ will be denoted by $R(\mathfrak{g}, \mathfrak{h})$. Since $\mathfrak{h}$ is maximal abelian, $L_0 = \mathfrak{h}$. Furthermore, the endomorphisms ad_X, $X \in \mathfrak{h}$, are simultaneously diagonalizable, since $[\operatorname{ad}_X, \operatorname{ad}_Y] = \operatorname{ad}_{[X,Y]} = 0$ for $X, Y \in \mathfrak{h}$. It follows

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})} L_\alpha. \tag{3.2}$$

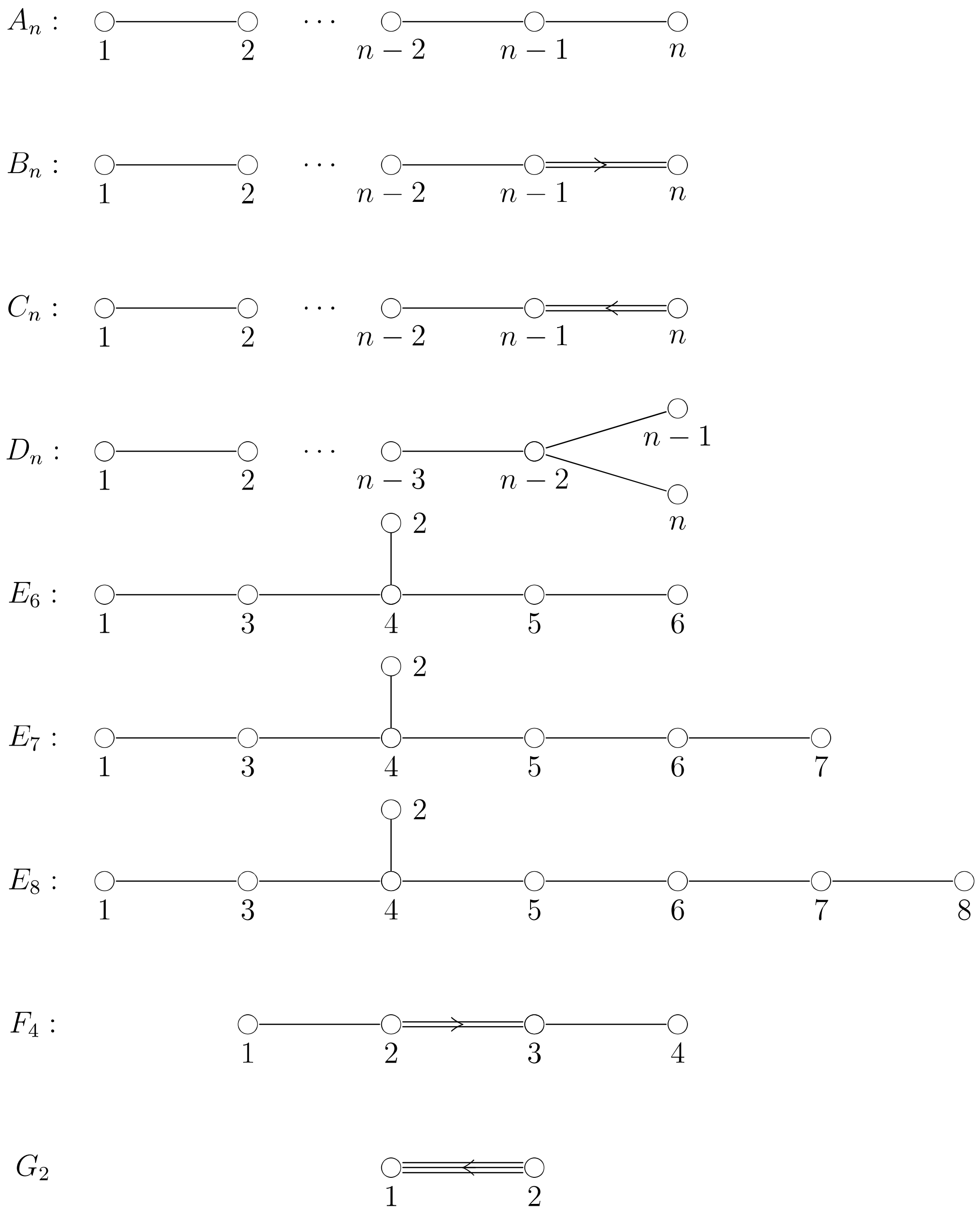

Figure 3.1: Dynkin diagrams

Lemma 3.6 ([Hel78, Thm. 4.2, p. 166, Thm. 44, p. 170])**.** *The restriction of the Killing form to* $\mathfrak{h}$*,* $\kappa_{|\mathfrak{h}\times\mathfrak{h}}$*, is non-degenerate. In particular, for each* $\alpha \in \mathfrak{h}^*$ *there exists a unique* $\mathfrak{h}_\alpha \in \mathfrak{h}$ *such that*

$$\kappa(\mathfrak{h}_\alpha, h) = \alpha(h) \text{ for all } h \in \mathfrak{h}.$$

This induces a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ *on* $\mathfrak{h}^*$ *by*

$$\langle\alpha,\beta\rangle := \kappa(\mathfrak{h}_\alpha, \mathfrak{h}_\beta) \text{ for all } \alpha,\beta \in \mathfrak{h}^*.$$

Moreover, κ *is real-valued and positive-definite on*

$$\mathfrak{h}_{\mathbb{R}} := \langle \mathfrak{h}_\alpha \mid \alpha \in R(\mathfrak{g},\mathfrak{h})\rangle_{\mathbb{R}},$$

so, $\langle\cdot,\cdot\rangle$ *is real-valued and positive-definite on*

$$\mathfrak{h}^*_{\mathbb{R}} = \langle \alpha \mid \alpha \in R(\mathfrak{g},\mathfrak{h})\rangle_{\mathbb{R}}.$$

Now, the following theorem can be formulated:

Theorem 3.7. *Let* $\langle\cdot,\cdot\rangle$ *be the inner product on* $\mathfrak{h}^*_{\mathbb{R}}$ *as in Lemma 3.6. Then,* $R := R(\mathfrak{g},\mathfrak{h})$ *is a root system of* $\mathfrak{h}^*_{\mathbb{R}}$ *as in Definition 3.1. If* $\mathfrak{g}$ *is simple, then* R *is irreducible. Otherwise, let* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ *be a decompostion of* $\mathfrak{g}$ *into simple ideals. Then* $\mathfrak{h} = \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$ *with* $\mathfrak{h}_i$ *a Cartan subalgebra of* $\mathfrak{g}_i$ *for* $1 \le i \le k$ *and* $R = R_1 + \ldots + R_k$ *with* $R_i := R(\mathfrak{g}_i, \mathfrak{h}_i)$ *is the decompostion of* R *into its irreducible components.*

Proof. By [Hum72, Thm. 5.2, p. 23], every semisimple Lie algebra has a unique decompostion into simple ideals. The claim follows from [Hum72, Thm. 14.1, Cor. 14.1, p. 73f.]. □

By [Hel78, Thm. 4.1, p. 165] each semisimple complex Lie algebra has a Cartan subalgebra and all Cartan subalgebras of $\mathfrak{g}$ are conjugate, see [Hum72, Thm. 16.2, p. 82]. So, let $\sigma \in \mathrm{Inn}(\mathfrak{g})$, $\tilde{\mathfrak{h}} := \sigma(\mathfrak{h})$. Then $\alpha \in R(\mathfrak{g},\tilde{\mathfrak{h}})$ if and only if $\alpha \circ \sigma \in R(\mathfrak{g},\mathfrak{h})$ and $\widetilde{L_\alpha} = \sigma(L_{\alpha\circ\sigma})$. Moreover, $\tilde{\mathfrak{h}}_\alpha = \sigma(\mathfrak{h}_{\alpha\circ\sigma})$. Hence, the κ-isometry $\sigma_{|\mathfrak{h}_{\mathbb{R}}} : \mathfrak{h}_{\mathbb{R}} \to \tilde{\mathfrak{h}}_{\mathbb{R}}$ induces an isomorphism between the root systems $R(\mathfrak{g},\mathfrak{h})$ and $R(\mathfrak{g},\tilde{\mathfrak{h}})$. Thus, the root system $R(\mathfrak{g}) := R(\mathfrak{g},\mathfrak{h})$ of a complex semisimple Lie algebra is unique up to isomorphism. Its Dynkin diagram will be denoted by $D(\mathfrak{g})$. By Theorem 3.7, $D(\mathfrak{g})$ is connected if and only if $\mathfrak{g}$ is simple.

On the other hand, the Dynkin diagram $D(\mathfrak{g})$ determines $\mathfrak{g}$ up to isomorphism. In fact, if $\tilde{\mathfrak{g}}$ is another complex semisimple Lie algebra with Cartan subalgebra $\tilde{\mathfrak{h}}$ and $D(\tilde{\mathfrak{g}}) = D(\mathfrak{g})$, then their root systems are isomorphic, i.e. there exists an $\mathbb{R}$-linear isomorphism $\phi : \mathfrak{h}^*_{\mathbb{R}} \to \tilde{\mathfrak{h}}^*_{\mathbb{R}}$ such that $\phi(R(\mathfrak{g},\mathfrak{h})) = R(\tilde{\mathfrak{g}},\tilde{\mathfrak{h}})$ and $\langle\alpha,\beta\rangle = \langle\phi(\alpha),\phi(\beta)\rangle$ for all $\alpha,\beta \in \mathfrak{h}^*_{\mathbb{R}}$. Thus, ϕ induces an isometry $\bar{\phi} : (\mathfrak{h}_{\mathbb{R}},\kappa) \to (\tilde{\mathfrak{h}}_{\mathbb{R}},\tilde{\kappa})$ and by [Hel78, Thm. 5.4, p. 173], $\bar{\phi}$ can be extended to a Lie algebra isomorphism $\bar{\phi} : \mathfrak{g} \to \tilde{\mathfrak{g}}$.

Moreover, by [Hel78, Thm. 4.15, p. 490], for any Dynkin diagram D, there exists a finite-dimensional complex semisimple Lie algebra $\mathfrak{g}$ such that $D(\mathfrak{g}) = D$. From this, it follows:

Theorem 3.8. *The assignment $\mathfrak{g} \mapsto D(\mathfrak{g})$ yields a one-to-one correspondence between the isomorphism classes of complex semisimple (simple) Lie algebras and (connected) Dynkin diagrams.*

3.3 Compact Real Forms

Let $\mathfrak{k}$ be a real compact semisimple Lie algebra, i.e. $\mathfrak{k} = T_eK$ for a compact semisimple Lie group K. The complexification of $\mathfrak{k}$ is $\mathfrak{g} := \mathfrak{k} \otimes \mathbb{C}$. $\mathfrak{g}$ becomes a complex semisimple Lie algebra by

$$[X_1 \otimes z_1, X_2 \otimes z_2] := [X_1, X_2] \otimes z_1 z_2 \quad \text{for all } X_1, X_2 \in \mathfrak{k},\ z_1, z_2 \in \mathbb{C}$$

and $\mathfrak{g}$ is simple if and only if $\mathfrak{k}$ is simple. $\mathfrak{k}$ is called a *compact real form of* $\mathfrak{g}$. Now, let $\mathfrak{t} \subset \mathfrak{k}$ be a maximal abelian subalgebra. Then $\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$ and the root system of $\mathfrak{k}$ with respect to $\mathfrak{t}$ is

$$R(\mathfrak{k}) := R(\mathfrak{k}, \mathfrak{t}) := R(\mathfrak{g}, \mathfrak{h}).$$

Since the root system $R(\mathfrak{g})$ does not depend on the choice of the Cartan subalgebra up to isomorphism, the root system $R(\mathfrak{k})$ does not depend on the choice of $\mathfrak{t}$ up to isomorphism. Moreover, $R(\mathfrak{k})$ is irreducible if and only if $\mathfrak{k}$ is simple and

$$\operatorname{rank} \mathfrak{k} := \dim_{\mathbb{R}} \mathfrak{t} = \dim_{\mathbb{R}} \mathfrak{h}^*_{\mathbb{R}} = \operatorname{rank} R(\mathfrak{k}).$$

The Dynkin diagram of $R(\mathfrak{k})$ is $D(\mathfrak{k}) := D(\mathfrak{g})$. On the other hand, $D(\mathfrak{k})$ determines $\mathfrak{k}$ up to isomorphism. In fact, $D(\mathfrak{k} \otimes \mathbb{C}) = D(\tilde{\mathfrak{k}} \otimes \mathbb{C})$ implies $\mathfrak{k} \otimes \mathbb{C} \cong \tilde{\mathfrak{k}} \otimes \mathbb{C}$ by Theorem 3.8. But the compact real form of a complex semisimple Lie algebra is unique up to isomorphism, see [Hel78, Cor. 7.3, p. 184]. Thus, $\mathfrak{k} \cong \tilde{\mathfrak{k}}$. Moreover, by [Hel78, Cor. 6.3, p. 181], every complex semisimple Lie algebra has a compact real form. It follows from Theorem 3.8:

Theorem 3.9. *The assignment $\mathfrak{k} \mapsto D(\mathfrak{k})$ yields a one-to-one correspondence between the isomorphism classes of real compact semisimple (simple) Lie algebras and (connected) Dynkin diagrams.*

A compact simple Lie algebras is called *classical*, if its Dynkin diagram is of classical type and *exceptional*, if its Dynkin diagram is of exceptional type. The classical Lie algebras of rank n are $\mathfrak{su}(n+1)$ $(n \geq 1)$, $\mathfrak{so}(2n+1)$ $(n \geq 2)$, $\mathfrak{sp}(n)$ $(n \geq 3)$ and $\mathfrak{so}(2n)$ $(n \geq 4)$ with Dynkin diagrams A_n, B_n, C_n and D_n, respectively. The exceptional Lie algebras will be denoted by $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$, respectively, see [Hel78, p. 516, Table IV].

A compact simple Lie group is called *classical* or *exceptional*, if its Lie algebra is classical or exceptional, respectively. The *simply-connected* classical Lie groups of rank

n are $SU(n+1)$, $Spin(2n+1)$, $Sp(n)$ and $Spin(2n)$ with Dynkin diagram A_n, B_n, C_n and D_n, respectively. Since by Remark 1.24 2, for connected H, $\hat{\Delta}^T_{G/H}$ only depends on $\mathfrak{g}$ instead of G, in chapter 4 the Lie group $SO(N)$ may be considered instead of $Spin(N)$.

The simply-connected exceptional Lie groups will be denoted by their Dynkin diagrams E_6, E_7, E_8, F_4 and G_2, respectively. A construction of these groups can be found in [Yok09].

3.4 Subalgebras of Maximal Rank

Now, let G be any semisimple compact Lie group with real compact semisimple Lie algebra $\mathfrak{g}$ and let $T < G$ be a maximal torus with Lie algebra $\mathfrak{t} \leq \mathfrak{g}$. To determine the simplicial complex of G/H for all connected compact Lie subgroups $T \leq H < G$, one has to determine all possibilities of H in the first place. By [Djo81, Lemma 3], every subgroup $T \leq H < G$ is closed in G and hence a compact Lie subgroup. Thus, there is a one-to-one correspondence between all connected compact Lie subgroups $T \leq H < G$ and all Lie subalgebras $\mathfrak{t} \leq \mathfrak{h} < \mathfrak{g}$. These Lie subalgebras are given by the following lemma.

Lemma 3.10. *Let $\mathfrak{g}$ and $\mathfrak{t}$ be as above. Furthermore, let a basis of $R(\mathfrak{g})$ and hence, a set of positive roots $R(\mathfrak{g})^+$ be given. Then*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R(\mathfrak{g})^+} \mathfrak{m}_\alpha$$

with $\mathfrak{m}_\alpha := \mathfrak{g} \cap L_\alpha \oplus L_{-\alpha}$ and L_α as in (3.2). All subalgebras $\mathfrak{t} < \mathfrak{h} < \mathfrak{g}$ are given by

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in I} \mathfrak{m}_\alpha \tag{3.3}$$

where $I \subseteq R(\mathfrak{g})^+$ is any non-empty subset with the following property:

$$\alpha, \beta \in I,\ \alpha \pm \beta \in R(\mathfrak{g}) \quad \Longrightarrow \quad \alpha \pm \beta \in I \,\dot{\cup}\, -I \tag{3.4}$$

Proof. Let $\alpha, \beta \in R(\mathfrak{g})^+$. Using the notation $\mathfrak{m}_{-\alpha} := \mathfrak{m}_\alpha$, it follows

$$[\mathfrak{m}_\alpha, \mathfrak{m}_\beta] = \begin{cases} \mathfrak{m}_{\alpha+\beta} \oplus \mathfrak{m}_{\alpha-\beta}, & \alpha+\beta,\ \alpha-\beta \in R(\mathfrak{g}) \\ \mathfrak{m}_{\alpha+\beta}, & \alpha+\beta \in R(\mathfrak{g}),\ \alpha-\beta \notin R(\mathfrak{g}) \\ \mathfrak{m}_{\alpha-\beta}, & \alpha-\beta \in R(\mathfrak{g}),\ \alpha+\beta \notin R(\mathfrak{g}) \\ \{0\}, & \alpha+\beta,\ \alpha-\beta \notin R(\mathfrak{g}), \end{cases} \tag{3.5}$$

see [Böh04, p. 89]. Hence, the vector space $\mathfrak{h}$ as in (3.3) is closed under Lie brackets. On the other hand, any subalgebra $\mathfrak{t} < \mathfrak{h} < \mathfrak{g}$ is also an $\mathrm{Ad}(T)$-submodule, so $\mathfrak{h}$ is the sum of $\mathfrak{t}$ and root spaces $\mathfrak{m}_\alpha$, $\alpha \in I$, for some $I \subseteq R(\mathfrak{g})^+$, see [Böh04, p. 89]. Now, equation (3.5) implies that I has to satisfy property (3.4). This proves the claim. □

Note, that Lemma 3.10 implies that $\mathfrak{g}$ contains only finitely many T-subalgebras. Moreover, $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ for every $\mathfrak{t} \leq \mathfrak{h} < \mathfrak{g}$ by [Djo81, Lemma 1]. Hence, the extended simplicial complex $\hat{\Delta}_{G/H}$ is well-defined for all $T \leq H < G$.

In the following chapters, it will be proved that if G is simple and $\hat{\Delta}_{G/H}$ is non-contractible, then $\tilde{H}_*(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$. But this already determines, if $\hat{\Delta}_{G/H}$ is contractible or not for all *semi*simple compact Lie groups G. In fact, by Remark 1.21 1, one may assume that G is simply-connected, so $G = G_1 \times \ldots \times G_k$ with G_i simply-connected and simple for all $1 \leq i \leq k$. Let T_i be a maximal torus of G_i for $1 \leq i \leq k$. Then $T := T_1 \times \ldots \times T_k$ is a maximal torus of G and $T \leq H < G$ is a subgroup of maximal rank if and only if $H = H_i \times \ldots \times H_k$ with $T_i \leq H_i \leq G_i$. Furthermore, the assumption that G acts almost effectively on G/H implies $H_i \neq G_i$ for all $1 \leq i \leq k$. By (1.8), it follows

$$\hat{\Delta}_{G/H} \simeq \hat{\Delta}_{G_1/H_1} * \ldots * \hat{\Delta}_{G_k/H_k} * S^{k-2}.$$

Thus, $\hat{\Delta}_{G/H}$ is contractible, if $\hat{\Delta}_{G_i/H_i}$ is contractible for at least one $1 \leq i \leq k$. Moreover, by Milnor, see [Mil56, Lemma 2.1], $\tilde{H}_*(X * Y, \mathbb{F}) \neq 0$, if $\tilde{H}_*(X, \mathbb{F}) \neq 0$ and $\tilde{H}_*(Y, \mathbb{F}) \neq 0$ for any spaces X, Y and any field $\mathbb{F}$. In particular, $\tilde{H}_*(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, if $\tilde{H}_*(\hat{\Delta}_{G_i/H_i}, \mathbb{Q}) \neq 0$ for all $1 \leq i \leq k$. In other words, it follows:

$$\hat{\Delta}_{G/H} \text{ non-contractible } \Leftrightarrow \hat{\Delta}_{G_i/H_i} \text{ non-contractible for all } i \in \{1, \ldots, k\}$$

Hence, it remains to determine $\hat{\Delta}_{G/H}$ for G simple.

Chapter 4

$\hat{\Delta}_{G/H}$ for G of Classical Type

In this chapter the root systems of the classical simple Lie algebras will be studied to determine all subalgebras of maximal rank by using Lemma 3.10 and to determine for which cases the complex $\hat{\Delta}_{G/H}$ is non-contractible. Moreover, it will be shown that for a maximal torus T of a classical group G the complex $\hat{\Delta}_{G/T}$ is homotopy equivalent to a wedge sum of spheres of dimension $\mathrm{rank}(G)-2$. Note, that for a one-dimensional Lie subalgebra of $\mathfrak{g}$, the three notations $\mathbb{R}$, $\mathfrak{u}(1)$ and $\mathfrak{so}(2)$ will be used in this chapter.

4.1 $SU(n+1),\ n \geq 1$

By [Hel78, p. 186, 187], it is

$$\begin{aligned}\mathfrak{su}(n+1) &= \{A \in \mathbb{C}^{(n+1)\times(n+1)} \mid A = -\overline{A^T} \ \wedge \ \mathrm{trace}(A) = 0\},\\ \mathfrak{su}(n+1)_{\mathbb{C}} &= \{A \in \mathbb{C}^{(n+1)\times(n+1)} \mid \mathrm{trace}(A) = 0\}.\end{aligned}$$

A maximal abelian subalgebra is given by the diagonal matrices of $\mathfrak{su}(n+1)$, i.e.

$$\mathfrak{t} := \left\{ \mathrm{diag}(i\alpha_1, \ldots, i\alpha_{n+1}) \mid \alpha_k \in \mathbb{R},\ 1 \leq k \leq n+1,\ \sum_{k=1}^{n+1} \alpha_k = 0 \right\} \text{ and} \tag{4.1}$$

$$\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C} = \left\{ \mathrm{diag}(z_1, \ldots, z_{n+1}) \mid z_k \in \mathbb{C},\ 1 \leq k \leq n+1,\ \sum_{k=1}^{n+1} z_k = 0 \right\}$$

is the corresponding Cartan subalgebra of $\mathfrak{su}(n+1)_{\mathbb{C}}$. For $1 \leq k \leq n+1$ consider $\nu_k \in \mathfrak{h}^*$ defined by $\nu_k(\mathrm{diag}(z_1, \ldots, z_n)) := z_k$. Moreover, for $1 \leq k, l \leq n+1,\ k \neq l$, let $E_{k,l} := (\delta_{ik} \cdot \delta_{jl})_{1 \leq i,j \leq n} \in \mathfrak{su}(n+1)_{\mathbb{C}}$. It follows

$$[\mathrm{diag}(z_1, \ldots, z_n), E_{kl}] = (z_k - z_l) \cdot E_{kl}.$$

Thus, $\nu_k-\nu_l$ is a root with root space $\langle E_{kl}\rangle_{\mathbb{C}}$ and $\mathfrak{su}(n+1)_{\mathbb{C}}=\mathfrak{h}\oplus\bigoplus_{k\neq l}\langle E_{kl}\rangle_{\mathbb{C}}$. Hence, all roots are given this way. Furthermore, for indices $i,j,k,l\in\{1,\dots,n+1\}$ such that $i\neq j,\ k\neq l,\ \{i,j\}\neq\{k,l\}$, the following equivalence holds:

$$\alpha:=(\nu_i-\nu_j)+(\nu_k-\nu_l)\text{ is a root }\ \Leftrightarrow i=l\text{ or }j=k \tag{4.2}$$

More precisely, if α is a root, then $\alpha\in\{\pm(\nu_p-\nu_q)\}$, where $p<q,\ \{p,q\}=\{i,j\}\triangle\{k,l\}$ and $\triangle$ denotes the symmetric difference. From (4.2), it follows that a set of simple roots is given by

$$\{\nu_i-\nu_{i+1}\mid 1\leq i\leq n\}$$

with Dynkin diagram A_n and a set of positive roots is given by

$$\{\nu_k-\nu_l\mid 1\leq k<l\leq n+1\},$$

see [Hel78, p. 462]. So, for $k<l$, let

$$\mathfrak{m}_{kl}:=\mathfrak{m}_{lk}:=\mathfrak{m}_{\{k,l\}}:=\mathfrak{m}_{\nu_k-\nu_l}=\langle E_{kl},E_{lk}\rangle_{\mathbb{C}}\cap\mathfrak{su}(n+1).$$

This subspace consists of all matrices of $\mathfrak{su}(n+1)$ whose entries are all zero except the $(kl)-$th and the $(lk)-$th one, i.e.

$$\mathfrak{m}_{kl}\cong\left\{\begin{pmatrix}0 & z\\ -\overline{z} & 0\end{pmatrix}\mid z\in\mathbb{C}\right\}$$

and $\mathfrak{su}(n+1)=\mathfrak{t}\oplus\bigoplus_{k<l}\mathfrak{m}_{kl}$. It follows:

Proposition 4.1. *Let* $\mathfrak{t}$ *be as in* (4.1). *Moreover, let* $r\in\{2,\dots,n\}$ *and* $I_1\dot\cup\dots\dot\cup I_r=\{1,\dots,n+1\}$ *be a partition of the index set* $\{1,\dots,n+1\}$. *Let* $n_i:=|I_i|,\ 1\leq i\leq r$. *Then*

$$\mathfrak{s}(\oplus_{i=1}^r\mathfrak{u}(n_i)_{I_i}):=\mathfrak{t}\oplus\bigoplus_{\substack{i,j\in I_1,\\ i<j}}\mathfrak{m}_{ij}\oplus\dots\oplus\bigoplus_{\substack{i,j\in I_r,\\ i<j}}\mathfrak{m}_{ij} \tag{4.3}$$

is a T*-subalgebra of* $\mathfrak{su}(n+1)$. *Moreover, every* T*-subalgebra is of this type.*

Proof. For indices $i<j,\ k<l,\ (i,j)\neq(k,l)$ it follows from (4.2) and (3.5) that

$$[\mathfrak{m}_{ij},\mathfrak{m}_{kl}]=\begin{cases}\mathfrak{m}_{\{i,j\}\triangle\{k,l\}}, & \{i,j\}\cap\{k,l\}\neq\emptyset\\ \{0\} & \{i,j\}\cap\{k,l\}=\emptyset.\end{cases} \tag{4.4}$$

So, if $\mathfrak{m}_{I_s}$ denotes $\bigoplus_{i,j\in I_s,\ i<j}\mathfrak{m}_{ij}$ for $1\leq s\leq r$, then $[\mathfrak{m}_{I_s},\mathfrak{m}_{I_s}]\subseteq\mathfrak{t}\oplus\mathfrak{m}_{I_s}$ and $[\mathfrak{m}_{I_s},\mathfrak{m}_{I_{s'}}]=0$ for $s\neq s'$. It follows that $\mathfrak{s}(\oplus_{i=1}^r\mathfrak{u}(n_i)_{I_i})$ is closed under Lie brackets.

Now, let $\mathfrak{k}$ be any T-subalgebra. Consider $\{1,\ldots,n+1\}$ as the vertex set of a graph Γ where i and j are connected by an edge if and only if $\mathfrak{m}_{ij} \subseteq \mathfrak{k}$. By (4.4), if i and j are connected and if j and k are connected, then i and k are also connected. It follows that $\mathfrak{k}$ is of type (4.3) where $I_1,\ldots,I_r$ denote the connected components of Γ. Moreover, the number r of connected components satisfies $r \geq 2$ since $\mathfrak{k} \neq \mathfrak{su}(n+1)$ and $r \leq n$ since $\mathfrak{k} \neq \mathfrak{t}$. □

Note, that after conjugation the subalgebra $\mathfrak{k} = \mathfrak{s}(\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i})$ just consists of all block matrices

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}, \quad A_i \in \mathfrak{u}(n_i),\ \sum_{i=1}^r \operatorname{trace}(A_i) = 0. \tag{4.5}$$

In fact, if $n_0 := 0$ and $\sigma \in S_n$ with $I_i = \sigma(\{\sum_{j=0}^{i-1} n_j + 1, \ldots, \sum_{j=0}^{i} n_j\})$ for $1 \leq i \leq r$, then $\mathrm{Ad}_P(\mathfrak{k})$ is of type (4.5), where $P = \begin{pmatrix} \operatorname{sgn}(\sigma) & 0 \\ 0 & I_n \end{pmatrix} P_\sigma \in SU(n+1)$ and P_σ is the permutation matrix.

Using the notation $\mathfrak{t} := \mathfrak{s}(\oplus_{i=1}^{n+1} \mathfrak{u}(1)_{\{i\}})$, the non-contractibility of $\hat{\Delta}_{G/H}$ for $G = SU(n+1)$ and $H < G$ connected and of maximal rank is given by the following theorem.

Theorem 4.2. *Let $\mathfrak{h} = \mathfrak{s}(\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i})$ as in Proposition 4.1. Then*

$$\tilde{H}_{r-3}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0.$$

Proof. If $r = 2$, then $\mathfrak{h}$ is maximal by Proposition 4.1. Hence, $\hat{\Delta}_{G/H} = \emptyset$ which implies $\tilde{H}_{-1}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$. Thus, one may assume $r \geq 3$.

To simplify notation, for $s \in \{1,\ldots,r\}$ and $1 \leq i_1 < \ldots < i_s \leq r$ let $I_{i_1,\ldots,i_s} := \cup_{j=1}^s I_{i_j}$ and $n_{i_1,\ldots,i_s} := \sum_{j=1}^s n_{i_j}$. Now, for $p \in \{0,\ldots,r-3\}$ and $q \in \{0,\ldots,p+1\}$ let

$$\mathfrak{k}_p^q := \mathfrak{s}\left(\mathfrak{u}(n_{1,\ldots,r-2-p,r-1-p+q})_{I_{1,\ldots,r-2-p,r-1-p+q}} \oplus \bigoplus_{\substack{l=r-1-p \\ l \neq r-1-p+q}}^{r} \mathfrak{u}(n_l)_{I_l}\right).$$

It follows that $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{r-3}^0)$ is a maximal chain of H-subalgebras. Furthermore, for $q \neq 0$ it holds

$\mathfrak{k}_p^q \neq \mathfrak{k}_p^0$ and

$\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0$ with $\mathfrak{k}_{-1}^0 := \mathfrak{g},\ \mathfrak{k}_{r-2}^0 := \mathfrak{h}$.

Moreover, for all $p \geq 1$ it holds:

$$\langle \mathfrak{k}_p^1, \ldots, \mathfrak{k}_p^{p+1} \rangle = \mathfrak{s}(\mathfrak{u}(n_{1,\ldots,r-2-p,r-p,\ldots,r})_{I_{1,\ldots,r-2-p,r-p,\ldots,r}} \oplus \mathfrak{u}(n_{r-1-p})_{I_{r-1-p}}) < \mathfrak{g}.$$

Hence, $\tilde{H}_{r-3}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.27. □

4.2 $SO(N),\ N \geq 3$

By [Hel78, p. 186], it is

$$\mathfrak{so}(N) = \{A \in \mathbb{R}^{N\times N} \mid A = -A^T\},$$
$$\mathfrak{so}(N)_{\mathbb{C}} = \{A \in \mathbb{C}^{N\times N} \mid A = -A^T\}.$$

The cases N even and N odd have to be considered seperatedly. Moreover, for N even one may assume $N \geq 8$, since $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ is not simple and $\mathfrak{so}(6) \cong \mathfrak{su}(4)$ is covered by the the upper case.

4.2.1 N even, $N \geq 8$

For any $z \in \mathbb{C}$ let $I(z) := \left(\begin{smallmatrix} 0 & z \\ -z & 0 \end{smallmatrix}\right) \in Mat(2,\mathbb{C})$. A maximal abelian subalgebra of $\mathfrak{so}(2n)$ and a Cartan subalgebra of $\mathfrak{so}(2n)_{\mathbb{C}}$ for $n \geq 4$ is then given by

$$\mathfrak{t} := \left\{ \begin{pmatrix} I(\alpha_1) & & \\ & \ddots & \\ & & I(\alpha_n) \end{pmatrix} = \mathrm{diag}(I(\alpha_1),\dots,I(\alpha_n)) \mid \alpha_k \in \mathbb{R},\ 1 \leq k \leq n \right\} \text{ and} \tag{4.6}$$

$$\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C} = \{\mathrm{diag}(I(z_1),\dots,I(z_n)) \mid z_k \in \mathbb{C},\ 1 \leq k \leq n\}.$$

By [Hel78, p. 186ff.], the root system is given as follows: For $1 \leq k \leq n$ let $\nu_k \in \mathfrak{h}^*$ defined by $\nu_k(\mathrm{diag}(I(z_1),\dots,I(z_n))) := i \cdot z_k$. Now, consider the following matrices:

$$M_{++} := \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix},\ M_{--} := \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix},\ M_{+-} := \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \text{ and } M_{-+} := \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$$

For $1 \leq k < l \leq n$ let $E_{kl}^{++} \in \mathfrak{so}(2n)_{\mathbb{C}}$ be defined as follows: All entries of E_{kl}^{++} are zero but its submatrix induced by the indices $2k-1,\ 2k,\ 2l-1$ and $2l$ is of type

$$\begin{pmatrix} 0 & M_{++} \\ -M_{++}^T & 0 \end{pmatrix}.$$

Let $E_{kl}^{--},\ E_{kl}^{+-},\ E_{kl}^{-+} \in \mathfrak{so}(2n)_{\mathbb{C}}$ be defined similarly. For $z_1 \dots, z_n \in \mathbb{C}$ and $H := \mathrm{diag}(I(z_1),\dots,I(z_n))$ it then follows:

$$\begin{aligned} [H, E_{kl}^{++}] &= \ \ i(z_k + z_l) \cdot E_{kl}^{++} \\ [H, E_{kl}^{--}] &= -i(z_k + z_l) \cdot E_{kl}^{--} \\ [H, E_{kl}^{+-}] &= \ \ i(z_k - z_l) \cdot E_{kl}^{+-} \\ [H, E_{kl}^{-+}] &= -i(z_k - z_l) \cdot E_{kl}^{-+} \end{aligned}$$

Hence, $\pm\nu_k \pm \nu_l,\ k < l$, are roots and since $\mathfrak{so}(2n)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{k<l} \langle E_{kl}^{++}, E_{kl}^{--}, E_{kl}^{+-}, E_{kl}^{-+} \rangle_{\mathbb{C}}$, all roots are given this way. Again, for indices $i, j, k, l \in \{1, \ldots, n\}$ with $i \neq j,\ k \neq l$, $\{i, j\} \neq \{k, l\}$, the following equivalences hold:

$$\begin{aligned} \alpha &:= (\nu_i - \nu_j) + (\nu_k - \nu_l) \text{ is a root } &&\Leftrightarrow i = l \text{ or } j = k \\ \beta &:= (\nu_i + \nu_j) - (\nu_k + \nu_l) \text{ is a root } &&\Leftrightarrow \{i, j\} \cap \{k, l\} \neq \emptyset \\ \gamma &:= (\nu_i - \nu_j) + (\nu_k + \nu_l) \text{ is a root } &&\Leftrightarrow j \in \{k, l\} \\ \delta &:= (\nu_i - \nu_j) - (\nu_k + \nu_l) \text{ is a root } &&\Leftrightarrow i \in \{k, l\} \end{aligned} \tag{4.7}$$

More precisely, if the left-hand side is a root, then $\alpha, \beta \in \{\pm(\nu_p - \nu_q)\}$ and $\gamma, \delta \in \{\pm(\nu_p + \nu_q)\}$, where $p < q,\ \{p, q\} = \{i, j\} \triangle \{k, l\}$. Note, that neither $(\nu_i + \nu_j) + (\nu_k + \nu_l)$ nor $(\nu_i + \nu_j) \pm (\nu_i - \nu_j)$ is a root. It follows from [Hel78, p. 464] that a set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_{n-1} + \nu_n\}$$

with Dynkin diagram D_n and a set of positive roots is given by

$$\{\nu_k \pm \nu_l \mid 1 \leq k < l \leq n\}.$$

Furthermore, for $k < l$, let

$$\mathfrak{m}_{kl}^{+} := \mathfrak{m}_{lk}^{+} := \mathfrak{m}_{\{k,l\}}^{+} = \mathfrak{m}_{\nu_k + \nu_l} = \langle E_{kl}^{++}, E_{kl}^{--} \rangle_{\mathbb{C}} \cap \mathfrak{so}(2n)$$

and

$$\mathfrak{m}_{kl}^{-} := \mathfrak{m}_{lk}^{-} := \mathfrak{m}_{\{k,l\}}^{-} = \mathfrak{m}_{\nu_k - \nu_l} = \langle E_{kl}^{+-}, E_{kl}^{-+} \rangle_{\mathbb{C}} \cap \mathfrak{so}(2n).$$

In other words,

$$\mathfrak{m}_{kl}^{+} \cong \left\{ \begin{pmatrix} 0 & \begin{smallmatrix} \alpha & \beta \\ \beta & -\alpha \end{smallmatrix} \\ \begin{smallmatrix} -\alpha & -\beta \\ -\beta & \alpha \end{smallmatrix} & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\} \quad \text{and} \quad \mathfrak{m}_{kl}^{-} \cong \left\{ \begin{pmatrix} 0 & \begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix} \\ \begin{smallmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{smallmatrix} & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

To determine the T-subalgebras of $\mathfrak{so}(2n) = \mathfrak{t} \oplus \bigoplus_{k<l} (\mathfrak{m}_{kl}^{+} \oplus \mathfrak{m}_{kl}^{-})$, the following lemmas are needed.

Lemma 4.3. *Let $i, j, k, l \in \{1, \ldots, n\},\ i < j,\ k < l,\ (i, j) \neq (k, l)$. Then for any signs $\epsilon_{ij},\ \epsilon_{kl} \in \{-, +\}$ it follows*

$$[\mathfrak{m}_{ij}^{\epsilon_{ij}}, \mathfrak{m}_{kl}^{\epsilon_{kl}}] = \begin{cases} \mathfrak{m}_{pq}^{-}, & \epsilon_{ij} = \epsilon_{kl},\ \{i, j\} \triangle \{k, l\} = \{p, q\} \\ \mathfrak{m}_{pq}^{+}, & \epsilon_{ij} \neq \epsilon_{kl},\ \{i, j\} \triangle \{k, l\} = \{p, q\} \\ \{0\}, & \{i, j\} \cap \{k, l\} = \emptyset. \end{cases} \tag{4.8}$$

If a multiplication on $\{-, +\}$ is given by $- \circ - := + \circ + := -$ and $- \circ + := + \circ - := +$, i.e. $(\{-, +\}, \circ) \cong \mathbb{Z}_2$, then (4.8) can be simplified to

$$[\mathfrak{m}_{ij}^{\epsilon_{ij}}, \mathfrak{m}_{kl}^{\epsilon_{kl}}] = \begin{cases} \mathfrak{m}_{pq}^{\epsilon_{ij} \circ \epsilon_{kl}}, & \{i, j\} \triangle \{k, l\} = \{p, q\} \\ \{0\}, & \{i, j\} \cap \{k, l\} = \emptyset. \end{cases} \tag{4.9}$$

Proof. (4.8) follows directly from (4.7) and (3.5). □

Lemma 4.4. *Let $r \in \{2, \ldots, n\}$ and $I = \{i_1 < \ldots < i_r\} \subseteq \{1, \ldots, n\}$. For any $(r-1)$-tuple $(\epsilon_1, \ldots, \epsilon_{r-1}) \in \{-, +\}^{r-1}$ of signs, there exist unique signs $\epsilon_{pq} \in \{-, +\}$, $1 \leq p < q \leq r$, satisfying:*

1. *$\epsilon_{p,p+1} = \epsilon_p$ for all $1 \leq p \leq r-1$.*
2. *The subspace*
$$\mathfrak{u}(r)_I^{\epsilon_1,\ldots,\epsilon_{r-1}} := \mathfrak{t} \oplus \bigoplus_{1\leq p<q\leq r} \mathfrak{m}_{i_p i_q}^{\epsilon_{pq}} \tag{4.10}$$
is a T-subalgebra.

Proof. If $r = 2$, then $\epsilon_{1,2} := \epsilon_1$ and $\mathfrak{t} \oplus \mathfrak{m}_{i_1 i_2}^{\epsilon_{1,2}} \cong \mathfrak{u}(2) \oplus \mathfrak{so}(2)^{n-1}$ is a subalgebra. Now, let $r \geq 3$ and let $1 \leq k < l \leq r$ such that $|k - l| \geq 2$. If signs ϵ_{pq} for $1 \leq p < q \leq n$ are given such that $\mathfrak{u}(r)_I^{\epsilon_1,\ldots,\epsilon_{r-1}}$ as in (4.10) is a subalgebra, then (4.9) yields

$$[\mathfrak{m}_{i_k,i_{k+1}}^{\epsilon_{k,k+1}}, \mathfrak{m}_{i_{k+1},i_l}^{\epsilon_{k+1,l}}] = \mathfrak{m}_{i_k,i_l}^{\epsilon_{k,k+1}\circ\epsilon_{k+1,l}} = \mathfrak{m}_{i_k,i_l}^{\epsilon_{kl}}$$

which implies

$$\epsilon_{kl} = \epsilon_{k,k+1} \circ \epsilon_{k+1,l} = \epsilon_k \circ \epsilon_{k+1,l}.$$

By iteration, it follows

$$\epsilon_{kl} = \epsilon_k \circ \ldots \circ \epsilon_{l-1}. \tag{4.11}$$

Hence, the sign ϵ_{kl} is unique. It remains to prove that $\mathfrak{u}(r)_I^{\epsilon_1,\ldots,\epsilon_{r-1}}$ is closed under Lie brackets with ϵ_{kl} defined as in (4.11). For this purpose, let $i, j, k, l \in I$, $i < j$, $k < l$, $(i, j) \neq (k, l)$. By (4.9), it remains to prove $\epsilon_{ij} \circ \epsilon_{kl} = \epsilon_{pq}$, if $\{i, j\} \triangle \{k, l\} = \{p < q\}$. Since $\epsilon_{ij} \circ \epsilon_{kl} = \epsilon_{kl} \circ \epsilon_{ij}$, one may assume $i < k$ or $i = k$ and $j < l$. Thus, the following three cases remain:

$$\begin{aligned}
j = k: \quad p = i,\ q = l \ &\Rightarrow \quad \epsilon_{ij} \circ \epsilon_{kl} = (\epsilon_p \circ \ldots \circ \epsilon_{j-1}) \circ (\epsilon_j \circ \ldots \circ \epsilon_{q-1}) = \epsilon_{pq}. \\
j = l: \quad p = i,\ q = k \ &\Rightarrow \quad \epsilon_{ij} \circ \epsilon_{kl} = (\epsilon_p \circ \ldots \circ \epsilon_{q-1} \circ \epsilon_q \circ \ldots \circ \epsilon_{j-1}) \circ (\epsilon_q \circ \ldots \circ \epsilon_{j-1}) \\
&\qquad = \epsilon_{pq},
\end{aligned}$$

since $\epsilon_s^{-1} = \epsilon_s$ for $1 \leq s \leq n$, so the factors $\epsilon_q, \ldots, \epsilon_{j-1}$ cancel out. Similarly,

$$i = k: \quad p = j,\ q = l \ \Rightarrow \quad \epsilon_{ij} \circ \epsilon_{kl} = \epsilon_i \circ \ldots \circ \epsilon_{p-1} \circ \epsilon_i \circ \ldots \circ \epsilon_{q-1} = \epsilon_{pq},$$

since the factors $\epsilon_i, \ldots, \epsilon_{p-1}$ cancel out. Hence, $\mathfrak{u}(r)_I^{\epsilon_1,\ldots,\epsilon_{r-1}}$ is a subalgebra. □

The T-subalgebras of $\mathfrak{so}(2n)$ are now given by the following proposition.

Proposition 4.5. *Let $\mathfrak{t}$ be as in (4.6). Moreover, let $r\in\{1,\dots,n-1\}$, $I_1\dot\cup\dots\dot\cup I_r=\{1,\dots,n\}$ be a partition of the index set $\{1,\dots,n\}$ and $n_i:=|I_i|,\ 1\leq i\leq r$. Then the T-subalgebras of $\mathfrak{so}(2n)$ are precisely given by*

$$\begin{aligned}&\mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1\dots\epsilon_{n_1-1}^1}\oplus\dots\oplus\mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l\dots\epsilon_{n_l-1}^l}\oplus\mathfrak{so}(2n_{l+1})_{I_{l+1}}\oplus\dots\oplus\mathfrak{so}(2n_r)_{I_r}\\ :=&\mathfrak{t}\oplus\bigoplus_{\substack{i,j\in I_1\\ i<j}}\mathfrak{m}_{ij}^{\epsilon_{ij}^1}\oplus\dots\oplus\bigoplus_{\substack{i,j\in I_l\\ i<j}}\mathfrak{m}_{ij}^{\epsilon_{ij}^l}\oplus\bigoplus_{\substack{i,j\in I_{l+1}\\ i<j}}(\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-)\oplus\dots\oplus\bigoplus_{\substack{i,j\in I_r\\ i<j}}(\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-)\end{aligned}\tag{4.12}$$

for some given $l\in\{0,\dots,r\}$ and $\epsilon_1^k,\dots,\epsilon_{n_k-1}^k\in\{-,+\}$ for $1\leq k\leq l$. The signs $\epsilon_{ij}^k\in\{-,+\}$, $i<j$, $i,j\in I_k$ are given as in (4.11). For $n_s=1$, this notation means that $\mathfrak{u}(1)_{I_s}=\mathfrak{so}(2)_{I_s}=\mathbb{R}$ is contained in $\mathfrak{t}$. Moreover, $n_s\geq 2$ for at least one $1\leq s\leq r$ and if $l=0$, then $r\geq 2$.

Proof. For $s\in\{1,\dots,r\}$, let

$$\mathfrak{m}_{I_s}:=\bigoplus_{i,j\in I_s,\ i<j}\mathfrak{m}_{ij}^{\epsilon_{ij}^s}\text{ for }s\leq l\text{ and }\mathfrak{m}_{I_s}:=\bigoplus_{i,j\in I_s,\ i<j}(\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-)\text{ for }s>l.$$

By Lemma 4.4, $[\mathfrak{m}_{I_s},\mathfrak{m}_{I_s}]\subseteq\mathfrak{t}\oplus\mathfrak{m}_{I_s}$ for $s\leq l$. Furthermore, from (4.8), it follows $[\mathfrak{m}_{I_s},\mathfrak{m}_{I_s}]\subseteq\mathfrak{t}\oplus\mathfrak{m}_{I_s}$ for $s>l$ and $[\mathfrak{m}_{I_s},\mathfrak{m}_{I_{s'}}]=0$ for $s\neq s'$. Hence, (4.12) defines a T-subalgebra.

Now, let $\mathfrak{k}$ be any T-subalgebra of $\mathfrak{so}(2n)$. Similarly to the proof of Proposition 4.1, let $\{1,\dots,n\}$ be the vertex set of a graph G where i and j are connected if and only $\mathfrak{m}_{ij}^+\subseteq\mathfrak{k}$ or $\mathfrak{m}_{ij}^-\subseteq\mathfrak{k}$. By (4.8), if i and j are connected and if j and k are connected, then so are i and k. Thus, if $I_1,\dots,I_r$ denote the connected components of G, then

$$\mathfrak{k}=\mathfrak{t}\oplus\bigoplus_{\substack{s=1\\ n_s\geq 2}}^{r}\left(\bigoplus_{\substack{i,j\in I_s\\ i<j}}\mathfrak{k}_{ij}\right),$$

where $\mathfrak{k}_{ij}\in\{\mathfrak{m}_{ij}^+,\mathfrak{m}_{ij}^-,\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-\}$. Now, fix some $s\in\{1,\dots,r\}$ such that $n_s\geq 2$. If $\mathfrak{k}_{ij}\in\{\mathfrak{m}_{ij}^+,\mathfrak{m}_{ij}^-\}$ for all $i<j,\ i,j\in I_s$, then by the uniqueness statement of Lemma 4.4, $\mathfrak{t}\oplus\bigoplus_{i,j\in I_s,\ i<j}\mathfrak{k}_{ij}$ must be of type $\mathfrak{u}(n_s)_{I_s}^{\epsilon_1^1,\dots,\epsilon_{n_s-1}^s}$. If $\mathfrak{k}_{ij}=\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-$ for any $i<j$, $i,j\in I_s$, then $\mathfrak{k}_{pq}=\mathfrak{m}_{pq}^+\oplus\mathfrak{m}_{pq}^-$ for all $p<q,\ p,q\in I_s$. In fact, if $i\neq p$, then by (4.8),

$$\mathfrak{m}_{ip}^+\oplus\mathfrak{m}_{ip}^-=[\mathfrak{m}_{ij}^+\oplus\mathfrak{m}_{ij}^-,\mathfrak{k}_{jp}]\subseteq\mathfrak{k}$$

and

$$\mathfrak{m}_{pq}^+\oplus\mathfrak{m}_{pq}^-=[\mathfrak{m}_{ip}^+\oplus\mathfrak{m}_{ip}^-,\mathfrak{k}_{iq}]\subseteq\mathfrak{k}$$

and similarly $\mathfrak{m}_{pq}^+\oplus\mathfrak{m}_{pq}^-\subseteq\mathfrak{k}$ for the case $j\neq p$. This proves that $\mathfrak{k}$ is of type (4.12). Moreover, $n_s\geq 2$ for at least one s since $\mathfrak{k}\neq\mathfrak{t}$ and if $l=0$, then $r\geq 2$ since $\mathfrak{k}\neq\mathfrak{so}(2n)$. ∎

Using the embedding

$$Mat(m,\mathbb{C}) \hookrightarrow Mat(2m,\mathbb{R});$$

$$\begin{pmatrix} x_{1,1}+i\cdot y_{1,1} & \cdots & x_{1,m}+i\cdot y_{1,m} \\ \vdots & & \vdots \\ x_{m,1}+i\cdot y_{m,1} & \cdots & x_{m,m}+i\cdot y_{m,m} \end{pmatrix} \mapsto \begin{pmatrix} \begin{smallmatrix} x_{1,1} & -y_{1,1} \\ y_{1,1} & x_{1,1} \end{smallmatrix} & \cdots & \begin{smallmatrix} x_{1,m} & -y_{1,m} \\ y_{1,m} & x_{1,m} \end{smallmatrix} \\ \vdots & & \vdots \\ \begin{smallmatrix} x_{m,1} & -y_{m,1} \\ y_{m,1} & x_{m,1} \end{smallmatrix} & \cdots & \begin{smallmatrix} x_{m,m} & -y_{m,m} \\ y_{m,m} & x_{m,m} \end{smallmatrix} \end{pmatrix}$$

for $m \in \mathbb{N}$ with $x_{k,l}, y_{k,l} \in \mathbb{R},\ 1 \le k,l \le m$, $\mathfrak{u}(m)$ can be considered as a subalgebra of $\mathfrak{so}(2m)$. Moreover, $\mathfrak{k} = \mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1\dots\epsilon_{n_1-1}^1} \oplus\ldots\oplus \mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l\dots\epsilon_{n_l-1}^l} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus\ldots\oplus \mathfrak{so}(2n_r)_{I_r}$ consists of all block matrices of type

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix}, \quad A_i \in \mathfrak{u}(n_i),\ i \le l,\ A_i \in \mathfrak{so}(2n_i),\ i > l, \tag{4.13}$$

up to automorphism. More precisely, for $\sigma \in S_n$ let $P_\sigma \in SO(2n)$ be the permutation matrix acting on the 2×2-blocks in a canonical way. As above, after conjugation with some P_σ, one may assume $I_i = \{\sum_{j=0}^{i-1} n_j + 1, \ldots, \sum_{j=0}^{i} n_j\}$ for $n_0 := 0$, $1 \le i \le r$. Moreover, if $A := \operatorname{diag}\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), I_2, \ldots, I_2\right) \in O(2n)$, then the (outer) automorphism Ad_A maps $\mathfrak{m}_{1,2}^{\pm}$ to $\mathfrak{m}_{1,2}^{\mp}$ and leaves $\mathfrak{m}_{p,p+1}^{\pm}$ invariant for all $p \ge 2$. Combining A with appropiate cyclic permutations, it follows that $\mathfrak{u}(n_i)^{\epsilon_1^i\dots\epsilon_{n_i-1}^i}$ is $\mathrm{Aut}(SO(2n_i))$-conjugate to $\mathfrak{u}(n_i)^{-\dots-}$. Hence, $\mathfrak{k}$ is of type (4.13) up to automorphism.

With $\mathfrak{t} := \oplus_{i=1}^n \mathfrak{u}(1)_{\{i\}}$, the (non-)contractibility of $\hat{\Delta}_{G/H}$ for $G = SO(2n)$ and $H < G$ connected and of maximal rank follows from the following theorem.

Theorem 4.6. *Let* $\mathfrak{h} = \mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1\dots\epsilon_{n_1-1}^1} \oplus\ldots\oplus \mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l\dots\epsilon_{n_l-1}^l} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus\ldots\oplus \mathfrak{so}(2n_r)_{I_r}$ *as in Proposition 4.5. If* $n_s = 1$ *for some* $s \in \{1,\ldots,r\}$*, the corresponding summand will be written as* $\mathfrak{so}(2)_{I_s}$*, whenever there exists a summand of type* $\mathfrak{so}(2n_{s'})_{I_{s'}}$ *for some* $n_{s'} \ge 2$*. Otherwise, it will be written as* $\mathfrak{u}(1)_{I_s}$*. Using this notation, the following statements hold:*

1. *If* $l = 0$*, i.e.* $\mathfrak{h} \cong \oplus_{i=1}^r \mathfrak{so}(2n_i)$*, then* $\tilde{H}_{r-3}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.
2. *If* $l = r$*, i.e.* $\mathfrak{h} \cong \oplus_{i=1}^r \mathfrak{u}(n_i)$*, then* $\tilde{H}_{r-2}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.
3. *If* $l \notin \{0, r\}$*, then* $\hat{\Delta}_{G/H}$ *is contractible.*

Proof. Let $l = 0$. If $r = 2$, then $\mathfrak{h}$ is maximal. Hence, $\tilde{H}_{-1}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$. So, let $r \ge 3$. With the same notation as in the proof of Theorem 4.2, let

$$\mathfrak{k}_p^q := \mathfrak{so}(2n_{1,\ldots,r-2-p,r-1-p+q})_{I_{1,\ldots,r-2-p,r-1-p+q}} \oplus \bigoplus_{\substack{l=r-1-p \\ l\neq r-1-p+q}}^{r} \mathfrak{so}(2n_l)_{I_l}$$

for $p \in \{0,\dots,r-3\}$, $q \in \{0,\dots,p+1\}$. Without loss of generality, let $n_1 \geq 2$. Under this assumption, $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{r-3})$ is again a maximal chain of H-subalgebras and for $q \neq 0$, it holds

$$\begin{aligned}&\mathfrak{k}_p^q \neq \mathfrak{k}_p^0,\\ &\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0 \text{ with } \mathfrak{k}_{-1}^0 := \mathfrak{g},\ \mathfrak{k}_{r-2}^0 := \mathfrak{h} \text{ and}\\ &\langle \mathfrak{k}_p^1,\dots,\mathfrak{k}_p^{p+1}\rangle = \mathfrak{so}(2n_{1,\dots,r-2-p,r-p,\dots,r})_{I_{1,\dots,r-2-p,r-p,\dots,r}} \oplus \mathfrak{so}(2n_{r-1-p})_{I_{r-1-p}} < \mathfrak{g}.\end{aligned}$$

Hence, $\tilde{H}_{r-3}(\hat{\Delta}_{G/H},\mathbb{Q}) \neq 0$ by Theorem 2.27.

Now, let $l = r$, i.e. $\mathfrak{h} = \mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1\dots\epsilon_{n_1-1}^1} \oplus \ldots \oplus \mathfrak{u}(n_r)_{I_r}^{\epsilon_1^r\dots\epsilon_{n_r-1}^r}$. As mentioned above, one may assume $I_i = \{\sum_{l=1}^{i-1} n_l + 1,\dots,\sum_{l=1}^{i} n_l\}$ with $n_0 := 0$ and $\epsilon_j^i = -$ for all $i \in \{1,\dots,r\}$, $j \in \{1,\dots,n_i-1\}$. If $r = 1$, then $\mathfrak{h} = \mathfrak{u}(n)^{-\dots-}$ is maximal and $\tilde{H}_{-1}(\hat{\Delta}_{G/H},\mathbb{Q}) \neq 0$. So, let $r \geq 2$. For $p \in \{0,\dots,r-2\}$ let

$$\mathfrak{k}_p^q := \mathfrak{u}(n_{1,\dots,r-p-1,r-p+q})_{I_{1,\dots,r-p-1,r-p+q}}^{-\dots-} \oplus \bigoplus_{\substack{l=r-p\\ l\neq r-p+q}}^{r} \mathfrak{u}(n_l)_{I_l}^{-\dots-} \text{ for } q \in \{0,\dots,p\} \text{ and}$$

$$\mathfrak{k}_p^{p+1} := \mathfrak{u}(n_{1,\dots,r-p-1,r})_{I_{1,\dots,r-p-1,r}}^{-\dots-+} \oplus \bigoplus_{l=r-p}^{r-1} \mathfrak{u}(n_l)_{I_l}^{-\dots-},$$

i.e. $\mathfrak{k}_p^{p+1}$ arises from $\mathfrak{u}(n_{1,\dots,r-p-1})_{I_{1,\dots,r-p-1}}^{-\dots-}$ and $\mathfrak{u}(n_r)_{I_r}^{-\dots-}$, extended by the root space $\mathfrak{m}_{1,n}^+$. Moreover, $\mathfrak{k}_p^0 = \mathfrak{u}(n_{1,\dots,r-p})_{I_{1,\dots,r-p}}^{-\dots-}$, so $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{r-2}^0)$ is a maximal chain of H-subalgebras and with $\mathfrak{k}_{-1}^0 := \mathfrak{g}$ and $\mathfrak{k}_{r-1}^0 := \mathfrak{h}$ it follows for $q \neq 0$:

$$\begin{aligned}&\mathfrak{k}_p^q \neq \mathfrak{k}_p^0,\\ &\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0.\end{aligned}$$

Furthermore, for $p \geq 1$ it follows

$$\begin{aligned}\langle \mathfrak{k}_p^1,\dots,\mathfrak{k}_p^{p+1}\rangle &= \langle \mathfrak{u}(n_{1,\dots,\widehat{r-p},\dots,r})_{I_{1,\dots,\widehat{r-p},\dots,r}}^{-\dots-} \oplus \mathfrak{u}(n_{r-p})_{I_{r-p}}^{-\dots-}, \mathfrak{k}_p^{p+1}\rangle\\ &= \mathfrak{so}(2n_{1,\dots,\widehat{r-p},\dots,r})_{I_{1,\dots,\widehat{r-p},\dots,r}} \oplus \mathfrak{u}(n_{r-p})_{I_{r-p}}^{-\dots-} < \mathfrak{g}.\end{aligned}$$

Thus, $\tilde{H}_{r-2}(\hat{\Delta}_{G/H},\mathbb{Q}) \neq 0$ by Theorem 2.27.

Now, let $l \notin \{0,r\}$ and assume $\mathfrak{h} = \mathfrak{u}(n_1)_{I_1}^{-\dots-} \oplus \ldots \oplus \mathfrak{u}(n_l)_{I_l}^{-\dots-} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus \ldots \oplus \mathfrak{so}(2n_r)_{I_r}$. Consider the H-subalgebra

$$\mathfrak{k} := \mathfrak{so}(2n_1)_{I_1} \oplus \ldots \oplus \mathfrak{so}(2n_l)_{I_l} \oplus \mathfrak{so}(2n_{l+1})_{I_{l+1}} \oplus \ldots \oplus \mathfrak{so}(2n_r)_{I_r}.$$

In fact, $\mathfrak{k} \neq \mathfrak{h}$, since $n_i \geq 2$ for at least one $i \leq l$. To show that $\hat{\Delta}_{G/H}$ is contractible, it suffices to show that $\mathfrak{k}$ has no complements in the poset $\hat{P}_{G/H}$, i.e. that there exists no H-subalgebra $\mathfrak{l}$ with $\mathfrak{l} \cap \mathfrak{k} = \mathfrak{h}$ and $\langle \mathfrak{l}, \mathfrak{k} \rangle = \mathfrak{g}$. So, let $\mathfrak{l}$ be any H-subalgebra. Since $n_i \geq 2$ for at least one $i > l$, $\mathfrak{l} \not\cong \mathfrak{u}(n)$. Hence,

$$\mathfrak{l} = \mathfrak{u}(m_1)_{J_1}^{\epsilon_1^1 \dots \epsilon_{m_1-1}^1} \oplus \dots \oplus \mathfrak{u}(m_{l'})_{J_{l'}}^{\epsilon_1^{l} \dots \epsilon_{m_{l'}-1}^{l'}} \oplus \mathfrak{so}(2m_{l'+1})_{J_{l'+1}} \oplus \dots \oplus \mathfrak{so}(2m_{r'})_{J_{r'}}$$

for some $r' \geq 2$. Moreover, for all $i \in \{1, \dots, r\}$ there exists some $j \in \{1, \dots, r'\}$ with $I_i \subseteq J_j$. In particular,

$$\langle \mathfrak{k},\ \mathfrak{l} \rangle \leq \mathfrak{so}(2m_1)_{J_1} \oplus \dots \oplus \mathfrak{so}(2m_{r'})_{J_{r'}} < \mathfrak{g}.$$

Hence, $\mathfrak{k}$ has no complements and $\hat{\Delta}_{G/H}$ is contractible by Theorem 2.17. □

4.2.2 N odd, $N \geq 3$

First, consider $\mathfrak{so}(3)$. A maximal abelian subalgebra is given by

$$\mathfrak{t} = \left\{ \left(\begin{smallmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right) \mid \alpha \in \mathbb{R} \right\}.$$

Hence, $\mathfrak{so}(3)$ has rank 1. It follows from Lemma 3.10, that there exist no T-subalgebras of $\mathfrak{so}(3)$, i.e. $\hat{\Delta}_{SO(3)/T} = \emptyset$.

So, assume $N \geq 5$, i.e. $SO(N) = SO(2n+1)$ for some $n \geq 2$. Subalgebras of $\mathfrak{so}(2n)$, $\mathfrak{so}(2n)_{\mathbb{C}}$ can be considered as subalgebras of $\mathfrak{so}(2n+1)$, $\mathfrak{so}(2n+1)_{\mathbb{C}}$ in a canonical way. Let $\mathfrak{t}$ be the maximal abelian subalgebra of $\mathfrak{so}(2n)$ as in (4.6). By [Hel78, p. 187ff.], $\mathfrak{t}$ is also a maximal abelian subalgebra of $\mathfrak{so}(2n+1)$ and $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{so}(2n+1)_{\mathbb{C}}$. For $1 \leq k < l \leq n$ let $\pm\nu_k \pm \nu_l$ be as above. These are roots with root space $\langle E_{kl}^{++} \rangle_{\mathbb{C}}$, $\langle E_{kl}^{--} \rangle_{\mathbb{C}}$, $\langle E_{kl}^{+-} \rangle_{\mathbb{C}}$ and $\langle E_{kl}^{-+} \rangle_{\mathbb{C}} \subseteq \mathfrak{so}(2n)_{\mathbb{C}} \subseteq \mathfrak{so}(2n+1)_{\mathbb{C}}$. Furthermore, for $1 \leq k \leq n$, let $E_k^+ \in \mathfrak{so}(2n+1)_{\mathbb{C}}$ be the matrix whose entries are all zero but its submatrix induced by the indices $2k-1,\ 2k$ and $2n+1$ is of type

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ -1 & -i & 0 \end{pmatrix}.$$

Analogously, let $E_k^- \in \mathfrak{so}(2n+1)_{\mathbb{C}}$ be the matrix whose entries are all zero but its submatrix induced by the indices $2k-1,\ 2k$ and $2n+1$ is of type

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ -1 & i & 0 \end{pmatrix}.$$

It follows for $z_1, \dots, z_n \in \mathbb{C},\ H = \operatorname{diag}(I(z_1), \dots, I(z_n), 0) \in \mathfrak{h}$:

$$[H, E_k^+] = \ \ iz_k \cdot E_k^+$$
$$[H, E_k^-] = -iz_k \cdot E_k^-$$

Hence, ν_k and $-\nu_k$ are roots with root spaces $\langle E_k^+ \rangle_{\mathbb{C}}, \langle\ E_k^- \rangle_{\mathbb{C}}$, respectively. Since $\mathfrak{so}(2n+1)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{1 \leq k < l \leq n} \langle E_{kl}^{++}, E_{kl}^{--}, E_{kl}^{+-}, E_{kl}^{-+} \rangle_{\mathbb{C}} \oplus \bigoplus_{1 \leq k \leq n} \langle E_k^+, E_k^- \rangle_{\mathbb{C}}$, all roots are given by $\pm\nu_k$ and $\pm\nu_k \pm \nu_l,\ 1 \leq k < l \leq n$. In addition to (4.7), for indices $i, k, l \in \{1, \dots, n\},\ k \neq l$, the following equivalences hold:

$$\begin{aligned} \epsilon &:= \nu_i + (\nu_k - \nu_l) \text{ is a root } \Leftrightarrow i = l \\ \zeta &:= -\nu_i + (\nu_k - \nu_l) \text{ is a root } \Leftrightarrow i = k \\ \eta &:= \pm\nu_i \mp (\nu_k + \nu_l) \text{ is a root } \Leftrightarrow i \in \{k, l\} \end{aligned} \tag{4.14}$$

In particular, a set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{\nu_n\}$$

with Dynkin diagram B_n and a set of positive roots is given by

$$\{\nu_k \pm \nu_l \mid 1 \leq k < l \leq n\} \cup \{\nu_k \mid 1 \leq k \leq n\},$$

see [Hel78, 462f.]. For $1 \leq k < l \leq n$ let $\mathfrak{m}_{kl}^+ = \mathfrak{m}_{\nu_k + \nu_l},\ \mathfrak{m}_{kl}^- = \mathfrak{m}_{\nu_k - \nu_l}$ as above and

$$\mathfrak{m}_k := \mathfrak{m}_{\nu_k} = \langle E_k^+, E_k^- \rangle_{\mathbb{C}} \cap \mathfrak{so}(2n+1) \cong \left\{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & \beta \\ -\alpha & -\beta & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}.$$

Thus, the T-subalgebras of $\mathfrak{so}(2n+1)$ are given by the following proposition.

Proposition 4.7. *With the notation as above, all T-subalgebras $\mathfrak{k}$ of $\mathfrak{so}(2n+1)$ are precisely given by the following three cases:*

1. $\mathfrak{k} = \mathfrak{so}(2n)$.
2. $\mathfrak{k} < \mathfrak{so}(2n)$ *and $\mathfrak{k}$ given as in* (4.12).
3. $\mathfrak{k} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \dots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r-1} \mathfrak{so}(2n_j)_{I_j} \oplus \mathfrak{so}(2n_r+1)_{I_r} := \mathfrak{k}' \oplus \bigoplus_{k \in I_r} \mathfrak{m}_k$, *where* $\mathfrak{k}' = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \dots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r} \mathfrak{so}(2n_j)_{I_j}$ *is equal to $\mathfrak{t}$ or a T-subalgebra of $\mathfrak{so}(2n)$ as in* (4.12) *with* $0 \leq l < r \leq n$.

Proof. Every subalgebra of $\mathfrak{so}(2n)$ is also a subalgebra of $\mathfrak{so}(2n+1)$. Hence, 1 and 2 yield T-subalgebras. Now, let $\mathfrak{k} = \mathfrak{k}' \oplus \bigoplus_{k\in I_r} \mathfrak{m}_k$ as in 3. For $i, j, k \in \{1, \dots, n\},\ i < j$, (3.5) and (4.14) imply

$$[\mathfrak{m}_k, \mathfrak{m}_{ij}^{\pm}] = \begin{cases} \mathfrak{m}_i, & k = j \\ \mathfrak{m}_j, & k = i \\ \{0\}, & k \notin \{i, j\} \end{cases} \tag{4.15}$$

and furthermore,

$$[\mathfrak{m}_i, \mathfrak{m}_j] = \mathfrak{m}_{ij}^{+} \oplus \mathfrak{m}_{ij}^{-}. \tag{4.16}$$

Let $\mathfrak{m}_{I_s}$ be defined as in the proof of Proposition 4.5 for $1 \le s \le r$. For all $k, l \in I_r$, $k < l$, it follows $[\mathfrak{m}_k, \mathfrak{m}_l] \subseteq \mathfrak{m}_{I_r},\ [\mathfrak{m}_k, \mathfrak{m}_k] \subseteq \mathfrak{t},\ [\mathfrak{m}_k, \mathfrak{m}_{I_s}] = 0$ for $s < r$ and $[\mathfrak{m}_k, \mathfrak{m}_{I_r}] \subseteq \bigoplus_{i\in I_r} \mathfrak{m}_i$. Thus, 3 defines a subalgebra of $\mathfrak{so}(2n+1)$.

Now, let $\mathfrak{k}$ be any T-subalgebra of $\mathfrak{so}(2n+1)$. One may assume that $\mathfrak{m}_k \subseteq \mathfrak{k}$ for at least one k, since otherwise $\mathfrak{k} = \mathfrak{so}(2n)$ or $\mathfrak{k}$ would a T-subalgebra of $\mathfrak{so}(2n)$ and thus, it would be of type (4.12). By (4.15) and (4.16), $[\mathfrak{m}_k, \mathfrak{so}(2n)] = \mathfrak{so}(2n+1)$ and $[\mathfrak{m}_k, \mathfrak{u}(n)^{\epsilon_1,\dots,\epsilon_{n-1}}] = \mathfrak{so}(2n+1)$ for any $1 \le k \le n$ and $\epsilon_1, \dots, \epsilon_{n-1} \in \{-, +\}$. Thus, for $\mathfrak{k}' := \mathfrak{k} \cap \mathfrak{so}(2n)$, it follows $\mathfrak{k}' = \mathfrak{t}$ or $\mathfrak{k}' = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \cdots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r} \mathfrak{so}(2n_j)_{I_j}$ as in (4.12) for some $0 \le l \le r,\ r \ge 2$. Now, let

$$I := \{k \in \{1, \dots, n\} \mid \mathfrak{m}_k \subseteq \mathfrak{k}\}.$$

By (4.16), $\mathfrak{m}_{ij}^{+} \oplus \mathfrak{m}_{ij}^{-} \subseteq \mathfrak{k}'$ for all $i, j \in I,\ i < j$. Hence, if $\mathfrak{k}' = \mathfrak{t}$, then $I = \{k\}$ for some $k \in \{1, \dots, n\}$ and $\mathfrak{k} = \mathfrak{t} \oplus \mathfrak{m}_k$ is of type 3. Now, assume $\mathfrak{k}' \neq \mathfrak{t}$. By (4.16), there is an $s > l$ such that $I \subseteq I_s$, so assume $I \subseteq I_r$. But for any $i \in I,\ j \in I_r,\ i \neq j$, it follows from (4.15) that $\mathfrak{k} \supseteq [\mathfrak{m}_i, \mathfrak{m}_{ij}^{\pm}] = \mathfrak{m}_j$. Hence, $I = I_r$ and $\mathfrak{k}$ is of type 3. □

Let $\mathfrak{k} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)^{\epsilon_1^i \cdots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r-1} \oplus \mathfrak{so}(2n_j)_{I_j} (\oplus \mathfrak{so}(2n_r)_{I_r}) < \mathfrak{so}(2n+1)$ be any T-subalgebra of $\mathfrak{so}(2n+1)$. Similarly to the case $\mathfrak{g} = \mathfrak{so}(2n)$, after conjugation with appropiate matrices of type $P_\sigma \subseteq SO(2n) \subseteq SO(2n+1),\ \sigma \in S_n$, and $\operatorname{diag}\left(\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), I_2, \dots, I_2, -1\right) \in SO(2n+1)$, one may assume that $\mathfrak{k}$ consists of all block matrices of type

$$\begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_{r-1} & \\ 0 & & & 0 \end{pmatrix} \text{ or } \begin{pmatrix} A_1 & & & 0 \\ & \ddots & & \\ & & A_r & \\ 0 & & & B \end{pmatrix}$$

with $A_i \in \mathfrak{u}(n_i),\ i \le l,\ A_i \in \mathfrak{so}(2n_i),\ i > l$ and $B \in \mathfrak{so}(2n_r+1)$. The (non)-contractibility of $\hat{\Delta}_{G/H}$ for $G = SO(2n+1)$ and $H < G$ a connected subgroup of maximal rank is now given by the following theorem.

Theorem 4.8. *With the notation as above, let* $\mathfrak{h} = \mathfrak{so}(2n)$ *or* $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i \ldots \epsilon_{n_i-1}^i} \oplus \bigoplus_{j=l+1}^{r-1} \mathfrak{so}(2n_j)_{I_j} \oplus \mathfrak{so}(2n_r+1)_{I_r}$ *or* $\mathfrak{h} = \mathfrak{t}$. *Here,* $n_r = 0,\ I_r = \emptyset$ *means that* $\mathfrak{h} \leq \mathfrak{so}(2n)$.

As above, if $n_s = 1$ *for some* $s \in \{1, \ldots, r\}$*, the corresponding summand will be written as* $\mathfrak{so}(2)_{I_s}$*, whenever there exists a summand of type* $\mathfrak{so}(2n_{s'})_{I_{s'}}$ *for some* $n_{s'} \geq 2$ *or a summand of type* $\mathfrak{so}(2n_r+1)_{I_r}$ *with* $n_r \geq 1$*. Otherwise, it will be written as* $\mathfrak{u}(1)_{I_s}$*. Using this notation, the following statements hold:*

1. *If* $\mathfrak{h} = \mathfrak{so}(2n)$*, then* $\hat{\Delta}_{G/H} = \emptyset$.

2. *If* $l = 0$*, then* $\tilde{H}_{r-3}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

3. *If* $\mathfrak{h} = \mathfrak{t}$*, then* $\tilde{H}_{n-2}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

4. *If* $l \neq 0$ *and* $\mathfrak{h} \neq \mathfrak{t}$*, then* $\hat{\Delta}_{G/H}$ *is contractible.*

Proof. $\mathfrak{h} = \mathfrak{so}(2n)$ is maximal in $\mathfrak{so}(2n+1)$, hence $\hat{\Delta}_{SO(2n+1)/SO(2n)} = \emptyset$.

Now, assume $l = 0$. Under this assumption, $r = 2$ implies that $\mathfrak{h}$ is of type $\mathfrak{so}(2n_1)_{I_1} \oplus \mathfrak{so}(2n_2+1)_{I_2}$. Hence, $\mathfrak{h}$ is maximal, i.e. $\hat{\Delta}_{G/H} = \emptyset$ and $\tilde{H}_{-1}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$. So, assume $r \geq 3$. With the notation as in Theorem 4.6 let $p \in \{0, \ldots, r-3\}$, $q \in \{0, \ldots, p+1\}$ and

$$\mathfrak{k}_p^q := \mathfrak{so}(2n_{1,\ldots,r-3-p,r-2-p+q,r} + 1)_{I_{1,\ldots,r-3-p,r-2-p+q,r}} \oplus \bigoplus_{\substack{l=r-2-p \\ l \neq r-2-p+q}}^{r-1} \mathfrak{so}(2n_l)_{I_l}. \tag{4.17}$$

This yields a maximal chain of H-subalgebras $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{r-3}^0)$. Furthermore, for $q \neq 0$, $\mathfrak{k}_{-1}^0 := \mathfrak{g}$ and $\mathfrak{k}_{r-2}^0 := \mathfrak{h}$ it follows

$\mathfrak{k}_p^q \neq \mathfrak{k}_p^0$,
$\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0$

and for $p \geq 1$:

$$\langle \mathfrak{k}_p^1, \ldots, \mathfrak{k}_p^{p+1} \rangle = \mathfrak{so}(2n_{1,\ldots,r-3-p,r-1-p,\ldots,r} + 1)_{I_{1,\ldots,r-3-p,r-1-p,\ldots,r}} \oplus \mathfrak{so}(2n_{r-2-p})_{I_{r-2-p}} < \mathfrak{g}.$$

So, $\tilde{H}_{r-3}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.27.

Let $\mathfrak{h} = \mathfrak{t}$. Then $\mathfrak{h} = \bigoplus_{j=1}^{r-1} \mathfrak{so}(2n_j)_{I_j} \oplus \mathfrak{so}(2n_r+1)_{I_r}$ with $r = n+1$, $I_i = \{i\}$, $n_i = 1$ for $1 \leq i \leq n$ and $I_r = \emptyset$, $n_r = 0$. As above, (4.17) yields H-subalgebras $\mathfrak{k}_p^q \cong \mathfrak{so}(2(n-1-p)+1) \oplus \mathfrak{so}(2)^{p+1}$ and $\tilde{H}_{n-2}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.27.

Now, let $l \neq 0$. If $\mathfrak{h} \cong \mathfrak{u}(n)$, then $\hat{\Delta}_{G/H} = \{\mathfrak{so}(2n)\}$ is a singleton. So, let $\mathfrak{h} \ncong \mathfrak{u}(n)$. After conjugation one may assume $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)^{-\cdots-} \oplus \bigoplus_{j=1}^{r-1} \mathfrak{so}(2n_j)_{I_j} \oplus \mathfrak{so}(2n_r+1)_{I_r}$, where $n_r = 0,\ I_r = \emptyset$ is possible. Since $\mathfrak{h} \neq \mathfrak{t}$, there exists at least one $i \leq l$ with $n_i \geq 2$. So,

$$\mathfrak{k} := \bigoplus_{i=1}^{r-1} \mathfrak{so}(2n_i)_{I_i} \oplus \mathfrak{so}(2n_r + 1)_{I_r}$$

is an H-subalgebra. As in the proof of Theorem 4.6, any H-subalgebra is of type

$$\mathfrak{l} = \bigoplus_{i=1}^{l'} \mathfrak{u}(m_i)_{J_i}^{\epsilon_1^i \ldots \epsilon_{m_i-1}^i} \oplus \bigoplus_{i=l'+1}^{r'-1} \mathfrak{so}(2m_i)_{J_i} \oplus \mathfrak{so}(2m_{r'})_{J_{r'}}$$

for some $r' \geq 2$ and for all $i \in \{1, \ldots, r\}$ there is a $j \in \{1, \ldots, r'\}$ with $I_i \subseteq J_j$. Hence,

$$\langle \mathfrak{k}, \mathfrak{l} \rangle \leq \bigoplus_{i=1}^{r'-1} \mathfrak{so}(2m_i)_{J_i} \oplus \mathfrak{so}(2m_{r'} + 1)_{J_{r'}} < \mathfrak{g}.$$

Thus, $\mathfrak{k}$ has no complements in the poset $\hat{P}_{G/H}$ and $\hat{\Delta}_{G/H}$ is contractible. □

4.3 $Sp(n),\ n \geq 1$

By [Hel78, p. 186, 189f.], it is

$$\mathfrak{sp}(n) = \left\{ \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathbb{C}^{2n \times 2n} \mid A, B \in \mathbb{C}^{n \times n},\ A = -\overline{A^T}\ \wedge\ B = B^T \right\} \text{ and}$$

$$\mathfrak{sp}(n)_{\mathbb{C}} = \left\{ \begin{pmatrix} U & W \\ V & -U^T \end{pmatrix} \in \mathbb{C}^{2n \times 2n} \mid U, V, W \in \mathbb{C}^{n \times n},\ V = V^T\ \wedge\ W = W^T \right\}.$$

A maximal abelian subalgebra of $\mathfrak{sp}(n)$ and a Cartan subalgebra of $\mathfrak{sp}(n)_{\mathbb{C}}$ are given by

$$\mathfrak{t} := \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \mid A = \operatorname{diag}(i\alpha_1, \ldots, i\alpha_n),\ \alpha_i \in \mathbb{R},\ 1 \leq i \leq n \right\} \text{ and} \tag{4.18}$$

$$\mathfrak{h} := \mathfrak{t} \otimes \mathbb{C} = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \mid A = \operatorname{diag}(z_1, \ldots, z_n),\ z_i \in \mathbb{C},\ 1 \leq i \leq n \right\}.$$

For $k \in \{1, \ldots, n\}$ and $H := \operatorname{diag}(z_1, \ldots, z_n, -z_1, \ldots, -z_n)) \in \mathfrak{h}$ let $\nu_k \in \mathfrak{h}^*$ be defined by $\nu_k(H) := z_k$. Moreover, for $1 \leq k, l \leq n$, let $E_{kl} := (\delta_{ik} \cdot \delta_{jl})_{1 \leq i,j \leq n}$. It follows:

$$\begin{aligned} [H, E_{k,n+l} + E_{l,n+k}] &= \ (z_k + z_l) \cdot E_{k,n+l} + E_{l,n+k}, \quad k \leq l \\ [H, E_{n+k,l} + E_{n+l,k}] &= -(z_k + z_l) \cdot E_{n+k,l} + E_{n+l,k}, \quad k \leq l \\ [H, E_{kl} - E_{n+l,n+k}] &= \ (z_k - z_l) \cdot E_{kl} - E_{n+l,n+k}, \quad k \neq l. \end{aligned}$$

Since

$$\mathfrak{sp}(n)_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{k \leq l} \langle E_{k,n+l} + E_{l,n+k} \rangle_{\mathbb{C}} \oplus \bigoplus_{k \leq l} \langle E_{n+k,l} + E_{n+l,k} \rangle_{\mathbb{C}} \oplus \bigoplus_{k \neq l} \langle E_{kl} - E_{n+l,n+k} \rangle_{\mathbb{C}},$$

the roots are given by $\pm\nu_k \pm \nu_l,\ 1 \leq k < l \leq n$ and $\pm 2\nu_k,\ 1 \leq k \leq n$. For indices $i, j, k, l \in \{1, \dots, n\},\ i \neq j,\ k \neq l,\ \{i, j\} \neq \{k, l\}$, the equivalences in (4.7) hold and, in addition, the following equivalences hold:

$$\begin{aligned} \epsilon &:= \pm((\nu_i + \nu_j) - 2\nu_k) \text{ is a root } &&\Leftrightarrow k \in \{i, j\} \\ \zeta &:= \quad (\nu_i - \nu_j) - 2\nu_k \quad \text{is a root } &&\Leftrightarrow k = i \\ \eta &:= \quad (\nu_i - \nu_j) + 2\nu_k \quad \text{is a root } &&\Leftrightarrow k = j \end{aligned} \tag{4.19}$$

Moreover, $(\nu_i - \nu_j) \pm (\nu_i + \nu_j)$ is always a root. It follows that a set of simple roots is given by

$$\{\nu_i - \nu_{i+1} \mid 1 \leq i \leq n-1\} \cup \{2\nu_n\}$$

with Dynkin diagram C_n and a set of positive roots is given by

$$\{\nu_k \pm \nu_l \mid 1 \leq k < l \leq n\} \cup \{2\nu_k \mid 1 \leq k \leq n\},$$

see [Hel78, p. 463]. Furthermore, for $k < l$ let

$$\begin{aligned} \mathfrak{m}^+_{kl} := \mathfrak{m}^+_{lk} := \mathfrak{m}^+_{\{k,l\}} &:= \mathfrak{m}_{\nu_k + \nu_l} = \langle E_{k,n+l} + E_{l,n+k}, E_{n+k,l} + E_{n+l,k} \rangle_{\mathbb{C}} \cap \mathfrak{sp}(n), \\ \mathfrak{m}^-_{kl} := \mathfrak{m}^-_{lk} := \mathfrak{m}^-_{\{k,l\}} &:= \mathfrak{m}_{\nu_k - \nu_l} = \langle E_{kl} - E_{n+l,n+k}, E_{lk} - E_{n+k,n+l} \rangle_{\mathbb{C}} \cap \mathfrak{sp}(n) \end{aligned}$$

and

$$\mathfrak{m}_k := \mathfrak{m}_{2\nu_k} = \langle E_{k,n+k}, E_{n+k,k} \rangle_{\mathbb{C}} \cap \mathfrak{sp}(n).$$

In other words,

$$\mathfrak{m}^+_{kl} \cong \left\{ \begin{pmatrix} 0 & \begin{smallmatrix} & -\overline{z} \\ -\overline{z} & \end{smallmatrix} \\ \begin{smallmatrix} & z \\ z & \end{smallmatrix} & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\},\ \mathfrak{m}^-_{kl} \cong \left\{ \begin{pmatrix} \begin{smallmatrix} & -\overline{z} \\ z & \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} & -z \\ \overline{z} & \end{smallmatrix} \end{pmatrix} \mid z \in \mathbb{C} \right\} \text{ and}$$

$$\mathfrak{m}_k \cong \left\{ \begin{pmatrix} 0 & -\overline{z} \\ z & 0 \end{pmatrix} \mid z \in \mathbb{C} \right\}.$$

Since (4.7) holds for roots of $\mathfrak{sp}(n)$, Lemma 4.3 and Lemma 4.4 also hold for subalgebras of $\mathfrak{sp}(n)$, i.e. $\mathfrak{u}(r)^{\epsilon_1,\dots,\epsilon_{r-1}}_I$ defined as in (4.10) is a T-subalgebra of $\mathfrak{sp}(n)$. It then follows:

Proposition 4.9. *Let $\mathfrak{t}$ be as in (4.18). Furthermore, let $r \in \{1, \ldots, n\}$, $I_1 \dot{\cup} \ldots \dot{\cup} I_r = \{1, \ldots, n\}$ be a partition of the index set $\{1, \ldots, n\}$ and $n_i := |I_i|$, $1 \le i \le r$. Then the T-subalgebras of $\mathfrak{sp}(n)$ are precisely given by*

$$\begin{aligned}
&\mathfrak{u}(n_1)_{I_1}^{\epsilon_1^1 \ldots \epsilon_{n_1-1}^1} \oplus \ldots \oplus \mathfrak{u}(n_l)_{I_l}^{\epsilon_1^l \ldots \epsilon_{n_l-1}^l} \oplus \mathfrak{sp}(n_{l+1})_{I_{l+1}} \oplus \ldots \oplus \mathfrak{sp}(n_r)_{I_r} \\
:=&\mathfrak{t} \oplus \bigoplus_{\substack{i,j \in I_1 \\ i<j}} \mathfrak{m}_{ij}^{\epsilon_{ij}^1} \oplus \ldots \oplus \bigoplus_{\substack{i,j \in I_l \\ i<j}} \mathfrak{m}_{ij}^{\epsilon_{ij}^l} \\
&\oplus \left(\bigoplus_{\substack{i,j \in I_{l+1} \\ i<j}} (\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-) \oplus \bigoplus_{i \in I_{l+1}} \mathfrak{m}_i \right) \oplus \ldots \oplus \left(\bigoplus_{\substack{i,j \in I_r \\ i<j}} (\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-) \oplus \bigoplus_{i \in I_r} \mathfrak{m}_i \right) \qquad (4.20)
\end{aligned}$$

for some given $l \in \{0, \ldots, r\}$ and $\epsilon_1^k, \ldots, \epsilon_{n_k-1}^k \in \{-, +\}$ for $1 \le k \le l$. The signs $\epsilon_{ij}^k \in \{-, +\}$, $i < j$, $i, j \in I_k$ are given as in (4.10). For $s \le l$ and $n_s = 1$, this notation means that $\mathfrak{u}(1)_{I_s} = \mathbb{R}$ is contained in $\mathfrak{t}$. Moreover, if $l = 0$, then $r \ge 2$ and if $l = r$, then $n_s \ge 2$ for at least one $s \in \{1, \ldots, r\}$.

Proof. Fix some $s \in \{1, \ldots, r\}$. If $s \le l$, then let $\mathfrak{m}_{I_s} := \bigoplus_{i,j \in I_s,\ i<j} \mathfrak{m}_{ij}^{\epsilon_{ij}^s}$, otherwise let $\mathfrak{m}_{I_s} := \bigoplus_{i,j \in I_s,\ i<j} (\mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-) \oplus \bigoplus_{i \in I_s} \mathfrak{m}_i$. In particular, $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ for $s \le l$ by Lemma 4.4. Moreover, (4.19) and (3.5) imply

$$[\mathfrak{m}_{ij}^{\pm}, \mathfrak{m}_k] = \begin{cases} \mathfrak{m}_{ij}^{\mp}, & k \in \{i, j\} \\ \{0\}, & k \notin \{i, j\} \end{cases} \qquad (4.21)$$

and

$$[\mathfrak{m}_{ij}^-, \mathfrak{m}_{ij}^+] = \mathfrak{m}_i \oplus \mathfrak{m}_j. \qquad (4.22)$$

Now, (4.8), (4.21), (4.22) and $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$ for $i \ne j$ yield $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_s}] \subseteq \mathfrak{t} \oplus \mathfrak{m}_{I_s}$ for $s > l$ and $[\mathfrak{m}_{I_s}, \mathfrak{m}_{I_{s'}}] = 0$ for $s \ne s'$. Thus, (4.20) is a T-subalgebra.

On the other hand, let $\mathfrak{k}$ be any T-subalgebra. As above, let $\{1, \ldots, n\}$ be the vertex set of a graph Γ where i and j are connected by an edge if and only if $\mathfrak{m}_{ij}^+ \subseteq \mathfrak{k}$ or $\mathfrak{m}_{ij}^- \subseteq \mathfrak{k}$. Let $I_1, \ldots, I_r$ be the connected components of G and let $s \in \{1, \ldots, r\}$. If $n_s = 1$, $I_s = \{i\}$, then $\mathfrak{k}$ contains either $\mathfrak{sp}(1)_{I_s}$ or $\mathfrak{u}(1)_{I_s}$ depending on wether $\mathfrak{k}$ contains $\mathfrak{m}_i$ or not. So, assume $n_s \ge 2$. By (4.8), if i and j are connected and if j and k are connected then so are i and k. Thus, for all $i, j \in I_s$, $i < j$, it is $\mathfrak{k}_{ij} \subseteq \mathfrak{k}$ for some $\mathfrak{k}_{ij} \in \{\mathfrak{m}_{ij}^-, \mathfrak{m}_{ij}^+, \mathfrak{m}_{ij}^- \oplus \mathfrak{m}_{ij}^+\}$. If for all $i, j \in I_s$, $i < j$, there exists a unique sign $\epsilon_{ij} \in \{-, +\}$ such that $\mathfrak{k}_{ij} = \mathfrak{m}_{ij}^{\epsilon_{ij}}$, then $\mathfrak{m}_i \not\subseteq \mathfrak{k}$ for $i \in I_s$ by (4.21) and the uniqueness statement of Lemma 4.4 implies that $\mathfrak{t} \oplus \mathfrak{m}_{I_s}$ is of type $\mathfrak{u}(n_s)_{I_s}^{\epsilon_1^s, \ldots, \epsilon_{n_s-1}^s}$. If $\mathfrak{k}_{ij} = \mathfrak{m}_{ij}^+ \oplus \mathfrak{m}_{ij}^-$ for some $i, j \in I_s$, $i < j$, then it follows as in the proof of Proposition 4.5 that $\mathfrak{k}_{pq} = \mathfrak{m}_{pq}^+ \oplus \mathfrak{m}_{pq}^-$ for all $p, q \in I_s$, $p < q$. Moreover, by (4.22), $\mathfrak{m}_i \subseteq \mathfrak{k}$ for all $i \in I_s$.

Thus, $\mathfrak{t}\oplus\mathfrak{m}_{I_s}$ is of type $\mathfrak{sp}(n_s)_{I_s}$. It follows that $\mathfrak{k}$ is of type (4.20). Furthermore, if $l=0$, then $r\geq 2$ since $\mathfrak{k}\neq\mathfrak{sp}(n)$ and if $l=r$, then $n_s\geq 2$ for at least one s, since $\mathfrak{k}\neq\mathfrak{t}$. □

Again, for $\mathfrak{k}=\bigoplus_{i=1}^{l}\mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i,\dots,\epsilon_{n_i-1}^i}\oplus\bigoplus_{i=l+1}^{r}\mathfrak{sp}(n_i)_{I_i}$ one may assume that $I_i=\{\sum_{j=0}^{i-1}n_j+1,\dots,\sum_{j=0}^{i}n_j\}$ for $1\leq i\leq r$, $n_0:=0$, after conjugation with an appropiate element of type $\left(\begin{smallmatrix}P_\sigma & 0\\ 0 & P_\sigma\end{smallmatrix}\right)\in Sp(n)$ for some $\sigma\in S_n$. Moreover, for $1\leq l\leq n$ let

$$P_l:=E_{n+l,l}-E_{l,n+l}+\sum_{\substack{k=1\\k\neq l}}^{n}E_{kk}+E_{n+k,n+k}\in Sp(n).$$

Then $\mathrm{Ad}_{P_l}(\mathfrak{m}_{ij}^{\pm})=\mathfrak{m}_{ij}^{\mp}$ for $l\in\{i,j\}$ and $\mathrm{Ad}_{P_l}(\mathfrak{m}_{ij}^{\pm})=\mathfrak{m}_{ij}^{\pm}$ for $l\notin\{i,j\}$. Hence, after conjugation with appropiate elements of type P_l one may assume $\epsilon_j^i=-$ for all $1\leq i\leq l,\ 1\leq j\leq n_i$.

Using the notation $\mathfrak{t}=\oplus_{i=1}^{n}\mathfrak{u}(1)_{\{i\}}$, the (non-)contractibility of $\hat{\Delta}_{G/H}$ for $G=Sp(n)$ and $H<G$ a connected subgroup of maximal rank is now given by the following theorem.

Theorem 4.10. *Let* $\mathfrak{h}=\bigoplus_{i=1}^{l}\mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i,\dots,\epsilon_{n_i-1}^i}\oplus\bigoplus_{i=l+1}^{r}\mathfrak{sp}(n_i)_{I_i}$ *be as in Proposition 4.9. Then, the following statement holds:*

1. *If* $l=0$*, then* $\tilde{H}_{r-3}(\hat{\Delta}_{G/H},\mathbb{Q})\neq 0$.
2. *If* $l=r$*, then* $\tilde{H}_{r-2}(\hat{\Delta}_{G/H},\mathbb{Q})\neq 0$.
3. *If* $l\notin\{0,r\}$*, then* $\hat{\Delta}_{G/H}$ *is contractible.*

Proof. First, let $l=0$, i.e. $2\leq r\leq n$ and $\mathfrak{h}=\bigoplus_{i=1}^{r}\mathfrak{sp}(n_i)_{I_i}$. Then the claim follows as in Theorem 4.2. In fact, every H-subalgebra is of type $\mathfrak{l}=\oplus_{j=1}^{r'}\mathfrak{sp}(m_j)_{I_j}$, hence $\mathfrak{l}$ is already determined by the partition $J_1\dot{\cup}\dots\dot{\cup}J_{r'}=\{1,\dots,n\}$. Thus, if $\mathfrak{h}':=\mathfrak{s}(\oplus_{i=1}^{r}\mathfrak{u}(n_i)_{I_i})$, then

$$\hat{P}_{Sp(n)/H}\longrightarrow\hat{P}_{SU(n)/H'};\ \oplus_{j=1}^{r'}\mathfrak{sp}(m_j)_{I_j}\mapsto\mathfrak{s}(\oplus_{j=1}^{r'}\mathfrak{u}(m_j)_{I_j})$$

is an isomorphism of posets. Therefore, $\hat{\Delta}_{Sp(n)/H}\cong\hat{\Delta}_{SU(n)/H'}$. By Theorem 4.2, it follows $\tilde{H}_{r-3}(\hat{\Delta}_{G/H},\mathbb{Q})\cong\tilde{H}_{r-3}(\hat{\Delta}_{SU(n)/H'},\mathbb{Q})\neq 0$.

Now, let $l=r$. If $r=1$, i.e. $\mathfrak{h}\cong\mathfrak{u}(n)$, then $\hat{\Delta}_{G/H}=\emptyset$ and $\tilde{H}_{-1}(\hat{\Delta}_{G/H},\mathbb{Q})\neq 0$. So, let $r\geq 2$ and assume $\mathfrak{h}=\oplus_{i=1}^{r}\mathfrak{u}(n_i)_{I_i}^{-\dots-}$. For $p\in\{0,\dots,r-2\}$ let

$$\mathfrak{k}_p^q:=\mathfrak{u}(n_{1,\dots,r-p-1,r-p+q})_{I_{1,\dots,r-p-1,r-p+q}}^{-\dots-}\oplus\bigoplus_{\substack{l=r-p\\l\neq r-p+q}}^{r}\mathfrak{u}(n_l)_{I_l}^{-\dots-},\ 0\leq q\leq p,\ \text{and}$$

$$\mathfrak{k}_p^{p+1}:=\mathfrak{u}(n_{1,\dots,r-p-1,r})_{I_{1,\dots,r-p-1,r}}^{-\dots-+}\oplus\bigoplus_{l=r-p}^{r-1}\mathfrak{u}(n_l)_{I_l}^{-\dots-}$$

as in the proof of Theorem 4.6, so Again, $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{r-2}^0)$ is a maximal chain of H-subalgebras and with $\mathfrak{k}_{-1}^0 := \mathfrak{g}$ and $\mathfrak{k}_{r-1}^0 := \mathfrak{h}$ it follows

$$\begin{aligned}&\mathfrak{k}_p^q \neq \mathfrak{k}_p^0,\\ &\mathfrak{k}_{p-1}^{q-1} > \mathfrak{k}_p^q > \mathfrak{k}_{p+1}^0\end{aligned}$$

for $q \neq 0$ and

$$\langle \mathfrak{k}_p^1, \ldots, \mathfrak{k}_p^{p+1} \rangle = \mathfrak{sp}(2n_{1,\ldots,\widehat{r-p},\ldots,r})_{I_{1,\ldots,\widehat{r-p},\ldots,r}} \oplus \mathfrak{u}(n_{r-p})_{I_{r-p}}^{-\cdots-} < \mathfrak{g}$$

for $p \geq 1$. So, $\tilde{H}_{r-2}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.27.

Now, let $l \notin \{0, r\}$ and assume $\mathfrak{h} = \bigoplus_{i=1}^{l} \mathfrak{u}(n_i)_{I_i}^{-\cdots-} \oplus \bigoplus_{i=l+1}^{r} \mathfrak{sp}(n_i)_{I_i}$. The contractibility of $\hat{\Delta}_{G/H}$ follows as in the proof of Theorem 4.6. More precisely,

$$\mathfrak{k} := \bigoplus_{i=1}^{r} \mathfrak{sp}(n_i)_{I_i}$$

is an H-subalgebra. Note, that, in contrast to the case $\mathfrak{g} = \mathfrak{so}(2n)$, no conditions for the indices n_i are needed, since $\mathfrak{u}(1)$ is a proper subalgebra of $\mathfrak{sp}(1)$. Now, if $\mathfrak{l} = \bigoplus_{j=1}^{l'} \mathfrak{u}(m_j)_{J_j}^{\epsilon_1^j \ldots \epsilon_{m_j-1}^j} \oplus \bigoplus_{j=l'+1}^{r'} \mathfrak{sp}(m_j)_{J_j}$, is any H-subalgebra, then $r' \geq 2$ and

$$\langle \mathfrak{k}, \mathfrak{l} \rangle = \bigoplus_{j=1}^{r'} \mathfrak{sp}(m_j)_{I_j} < \mathfrak{g}.$$

Hence, $\mathfrak{k}$ has no complements in the poset $\hat{P}_{G/H}$ and $\hat{\Delta}_{G/H}$ is contractible. □

4.4 The Topology of $\hat{\Delta}_{G/T}$ for G of Classical Type

In this section, the homotopy type of $\hat{\Delta}_{G/T}$ will be determined. For this purpose, the *reduced* Euler characteristic has to be defined.

Definition 4.11. Let Δ be any finite simplicial complex. The *reduced Euler characteristic of* Δ is given by

$$\tilde{\chi}(\Delta) := \sum_{\sigma \in \Delta \cup \{\emptyset\}} (-1)^{\dim \sigma}.$$

Here, $\dim \emptyset := -1$. For any field $\mathbb{F}$, $\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim \Delta} (-1)^i \dim_{\mathbb{F}} \tilde{H}_i(\Delta, \mathbb{F})$. In particular, $\tilde{\chi}(\Delta)$ is a homotopy invariant of Δ. Now, the homotopy type of $\hat{\Delta}_{G/T}$ for classical G is given by the following theorem.

Theorem 4.12. *Let G be classical Lie group and T a maximal torus. Then*

$$\hat{\Delta}_{G/T} \simeq \bigvee_{i=1}^{|\tilde{\chi}(\hat{\Delta}_{G/T})|} S_i^{\mathrm{rank}(G)-2},$$

where $\bigvee_{i=1}^{1} S_i^{-1} := \emptyset$. More precisely, if $R(\mathfrak{g})$ denotes the root system which corresponds to G, then $\tilde{\chi}(\hat{\Delta}_{G/T})$ is given by the following table:

$R(\mathfrak{g})$	$\tilde{\chi}(\hat{\Delta}_{G/T})$
A_n	$a_n := (-1)^n \cdot n!$
B_n	$b_n := (-1)^n \cdot n!$
C_n	$c_n := (-1)^n \cdot 2^{n-1} \cdot (n-1)!$
$D_n,\ n \geq 2$	$d_n := (-1)^n \cdot (2^{n-1}-1) \cdot (n-1)!$

The values a_n, d_n will also be needed in chapter 5 to prove that $\hat{\Delta}_{E_8/T^8}$ is non-contractible. To prove Theorem 4.12, the following two lemmas will be needed to compute the reduced Euler characteristic.

Lemma 4.13. *For $N \in \mathbb{N}$, $i \in \{1, \dots, N\}$, let $H_i < G_i$ be compact connected Lie groups such that $\mathfrak{n}(\mathfrak{h}_i) = \mathfrak{h}_i$ and $\hat{P}_{G_i/H_i}$ is finite. Then*

$$\tilde{\chi}(\hat{\Delta}_{G_1\times\ldots\times G_N/H_1\times\ldots\times H_N}) = \prod_{i=1}^{N} \tilde{\chi}(\hat{\Delta}_{G_i/H_i}).$$

Proof. For any two simplicial complexes Δ and Γ, the simplices of $\Delta * \Gamma$ are given by $\sigma \sqcup \tau$, $\sigma \in \Delta \cup \{\emptyset\}$, $\tau \in \Gamma \cup \{\emptyset\}$ and $\dim \sigma \sqcup \tau = \dim \sigma + \dim \tau + 1$, see Remark 1.6. This still holds true for the case $\sigma = \tau = \sigma \sqcup \tau = \emptyset$. It follows:

$$\begin{aligned} \tilde{\chi}(\Delta * \Gamma) &= \sum_{\sigma\in\Delta\cup\{\emptyset\}} \sum_{\tau\in\Gamma\cup\{\emptyset\}} (-1)^{\dim\sigma+\dim\tau+1} \\ &= \tilde{\chi}(\Delta) \sum_{\tau\in\Gamma\cup\{\emptyset\}} (-1)^{\dim\tau+1} \\ &= -\tilde{\chi}(\Delta)\cdot\tilde{\chi}(\Gamma) \end{aligned}$$

Since $\hat{\Delta}_{G_1\times\ldots\times G_N/H_1\times\ldots\times H_N} \simeq \hat{\Delta}_{G_1/H_1} * \ldots * \hat{\Delta}_{G_N/H_N} * S^{N-2}$ by Corollary 2.20, this implies

$$\begin{aligned} \tilde{\chi}(\hat{\Delta}_{G_1\times\ldots\times G_N/H_1\times\ldots\times H_N}) &= (-1)^N \cdot \prod_{i=1}^{N} \tilde{\chi}(\hat{\Delta}_{G_i/H_i}) \cdot \tilde{\chi}(S^{N-2}) \\ &= \prod_{i=1}^{N} \tilde{\chi}(\hat{\Delta}_{G_i/H_i}), \end{aligned}$$

since $\tilde{\chi}(S^m) = (-1)^m$ for all $m \in \mathbb{N}_0$. □

Lemma 4.14. *If $H < G$ are compact connected Lie groups such that $\mathfrak{n}(\mathfrak{h}) = \mathfrak{h}$ and $\hat{P}_{G/H}$ is finite, then*

$$\tilde{\chi}(\hat{\Delta}_{G/H}) = -\left(1 + \sum_{\mathfrak{h}<\mathfrak{k}<\mathfrak{g}} \tilde{\chi}(\hat{\Delta}_{K/H})\right),$$

where K denotes the connected Lie subgroup of G with $T_eK = \mathfrak{k}$.

Proof. Since the non-empty simplices $\sigma \in \hat{\Delta}_{G/H}$ are chains of H-subalgebras, they can be ordered by their maximums, i.e.

$$\tilde{\chi}(\hat{\Delta}_{G/H}) = -1 + \sum_{\sigma\in\hat{\Delta}_{G/H}} (-1)^{\dim\sigma} = -1 + \sum_{\mathfrak{h}<\mathfrak{k}<\mathfrak{g}} \sum_{\substack{\sigma\in\hat{\Delta}_{G/H}: \\ \mathfrak{k} \text{ maximum of } \sigma}} (-1)^{\dim\sigma}.$$

For $\sigma \in \hat{\Delta}_{G/H}$, the maximum of σ is given by $\mathfrak{k}$ if and only if $\sigma = \sigma' * \mathfrak{k}$ for some $\sigma' \in \hat{\Delta}_{K/H} \cup \{\emptyset\}$. Hence,

$$\sum_{\substack{\sigma\in\hat{\Delta}_{G/H}: \\ \mathfrak{k} \text{ maximum of } \sigma}} (-1)^{\dim\sigma} = \sum_{\sigma'\in\hat{\Delta}_{K/H}\cup\{\emptyset\}} (-1)^{\dim\sigma' * \mathfrak{k}} = -\tilde{\chi}(\hat{\Delta}_{K/H}).$$

This proves the claim. □

Furthermore, the following lemma will help to determine the homotopy type of $\hat{\Delta}_{G/T}$.

Lemma 4.15. *Let Δ be a finite simplicial complex of dimension n. If Δ is $(n-1)$-connected, then $\Delta \simeq \bigvee_{i=1}^m S_i^n$ for some $m \in \mathbb{N}_0$, where $\bigvee_{i=1}^0 S_i^n := \{\cdot\}$.*

Proof. If $n = 0$, then $\Delta \neq \emptyset$ by assumption and if m is the number of vertices, Δ is the wedge sum of $m-1$ S^0's.

If $n = 1$, then Δ is a connected graph. By [Hat02, Proposition 1 A.2], $\pi_1(\Delta)$ is a finitely generated free group. The higher homotopy groups of a connected graph are trivial, since the universal cover is contractible, see [Hat02, Proposition 1 A.2, Lemma 1 A.3, Proposition 4.1]. So, the generators $a_1, \ldots, a_m$ of $\pi_1(\Delta)$ induce a weak homotopy equivalence from $\bigvee_{i=1}^m S_i^1$ to Δ. Hence, $\bigvee_{i=1}^m S_i^1 \simeq \Delta$ by Theorem 2.1.

Now, let $n \geq 2$. Then $\pi_k(\Delta) = \tilde{H}_k(\Delta, \mathbb{Z}) = 0$ for $0 \leq k \leq n-1$ and $\pi_n(\Delta) \cong \tilde{H}_n(\Delta, \mathbb{Z})$ by Hurewicz, see [Hat02, Theorem 4.32]. Since $\dim \Delta = n$, $\tilde{H}_n(\Delta, \mathbb{Z})$ is a finitely generated free abelian group and $\tilde{H}_k(\Delta, \mathbb{Z}) = 0$ for $k > n$. So, the generators $a_1, \ldots, a_m$ of $\tilde{H}_n(\Delta, \mathbb{Z})$ induce a map $f : \bigvee_{i=1}^m S_i^n \to \Delta$ such that the map $f_* : \tilde{H}_k(\bigvee_{i=1}^m S_i^n, \mathbb{Z}) \to \tilde{H}_k(\Delta, \mathbb{Z})$ is an isomorphism for all k. Since Δ is simply-connected, this condition already implies that f is a homotopy equivalence, see [Hat02, Corollary 4.33]. □

Proof of Theorem 4.12. As a first step, the reduced Euler characteristic of $\hat{\Delta}_{G/T}$ will be computed using Lemma 4.13 and Lemma 4.14.

If $\operatorname{rank}(G) = 1$, then $\hat{\Delta}_{G/T} = \emptyset$ and $\tilde{\chi}(\emptyset) = -1 = a_1 = b_1 = c_1$. Furthermore, $\hat{\Delta}_{SO(4)/T} = \{\mathfrak{u}(2)^-, \mathfrak{u}(2)^+\}$ consists of two vertices. Thus, $\tilde{\chi}(\hat{\Delta}_{SO(4)/T}) = 1 = d_2$. Moreover, $\hat{\Delta}_{SO(5)/T} \cong \hat{\Delta}_{Sp(2)/T}$ consists of five vertices and two edges. So, its reduced Euler characteristic is $2 = b_2 = c_2$.

Now, let $n \geq 3$ and $\mathfrak{g} = \mathfrak{su}(n)$, $\mathfrak{so}(2n+1)$, $\mathfrak{sp}(n)$ or $\mathfrak{so}(2n)$. Fix any T-subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. Then there exists a partition $I_1 \dot{\cup} \ldots \dot{\cup} I_{l_\mathfrak{k}} = \{1, \ldots, n\}$ such that

$$\mathfrak{k} = \bigoplus_{i=1}^{l} \mathfrak{k}_i \text{ for some } \mathfrak{k}_i = \mathfrak{su}(n_i)_{I_i},\ \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i, \ldots, \epsilon_{n_i-1}^i},\ \mathfrak{so}(2n_i)_{I_i},\ \mathfrak{so}(2n_i+1)_{I_i} \text{ or } \mathfrak{sp}(n_i)_{I_i}$$

with $n_i = |I_i|$. Without loss of generality, assume $1 \in I_1$. Let $n_\mathfrak{k} := n - |I_1|$. Then

$$-\tilde{\chi}(\Delta_{G/T}) = 1 + \underbrace{\sum_{\mathfrak{k}:\ n_\mathfrak{k}=0} \tilde{\chi}(\hat{\Delta}_{K/T})}_{=:A} + \underbrace{\sum_{\mathfrak{k}:\ n_\mathfrak{k}=1} \tilde{\chi}(\hat{\Delta}_{K/T})}_{=:B} + \underbrace{\sum_{\mathfrak{k}:\ n_\mathfrak{k}>1} \tilde{\chi}(\hat{\Delta}_{K/T})}_{=:C}.$$

First,

$$A = \begin{cases} 0, & \mathfrak{g} = \mathfrak{su}(n) \\ 2^{n-1} \cdot a_{n-1} + d_n, & \mathfrak{g} = \mathfrak{so}(2n+1) \\ 2^{n-1} \cdot a_{n-1}, & \mathfrak{g} = \mathfrak{sp}(n) \\ 2^{n-1} \cdot a_{n-1}, & \mathfrak{g} = \mathfrak{so}(2n). \end{cases} \tag{4.23}$$

In fact, if $\mathfrak{g} = \mathfrak{su}(n)$ there exists no T-subalgebra $\mathfrak{k}$ with $n_\mathfrak{k} = 0$. If $\mathfrak{g} \neq \mathfrak{su}(n)$, then $n_\mathfrak{k} = 0$ if and only if $\mathfrak{k} = \mathfrak{u}(n)^{\epsilon_1, \ldots, \epsilon_{n-1}}$ for some signs $\epsilon_i^j \in \{-, +\}$ or $\mathfrak{k} = \mathfrak{so}(2n)$ in the case $\mathfrak{g} = \mathfrak{so}(2n+1)$. By assumption, $\tilde{\chi}(\hat{\Delta}_{K/T}) = a_{n-1}$ in the first case and $\tilde{\chi}(\hat{\Delta}_{K/T}) = d_n$ in the second case. Since, there are 2^{n-1} possible choices of signs, (4.23) follows. Moreover,

$$B = \begin{cases} (n-1) \cdot a_{n-2}, & \mathfrak{g} = \mathfrak{su}(n) \\ (n-1) \cdot b_{n-1}, & \mathfrak{g} = \mathfrak{so}(2n+1) \\ 0, & \mathfrak{g} = \mathfrak{sp}(n) \\ (n-1) \cdot (2^{n-2} \cdot a_{n-2} + d_{n-1}), & \mathfrak{g} = \mathfrak{so}(2n). \end{cases} \tag{4.24}$$

In fact, $n_\mathfrak{k} = 1$ implies $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ with $\mathfrak{k}_2 = \mathfrak{u}(1)_{\{i\}}$, $\mathfrak{so}(2)_{\{i\}}$, $\mathfrak{so}(3)_{\{i\}}$ or $\mathfrak{sp}(1)_{\{i\}}$ for some $i \in \{2, \ldots, n\}$. Nox fix i.

If $\mathfrak{g} = \mathfrak{su}(n)$, then $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(n-1)_{I_1} \oplus \mathfrak{u}(1)_{\{i\}})$ and $\tilde{\chi}(\hat{\Delta}_{K/T}) = a_{n-2}$. Taking the sum over all i yields B.

If $\mathfrak{g} = \mathfrak{so}(2n+1)$, then $\mathfrak{k}_1 = \mathfrak{u}(n-1)_{I_1}^{\epsilon_1, \ldots, \epsilon_{n-2}}$, $\mathfrak{so}(2(n-1))_{I_1}$ or $\mathfrak{so}(2(n-1)+1)_{I_1}$. If $\mathfrak{k}_1 \in \{\mathfrak{u}(n-1)_{I_1}^{\epsilon_1, \ldots, \epsilon_{n-2}}, \mathfrak{so}(2(n-1))_{I_1}\}$, then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{so}(2)_{\{i\}}$ or $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{so}(3)_{\{i\}}$.

But $\tilde{\chi}(\hat{\Delta}_{SO(3)/T}) = -1$, so $\tilde{\chi}(\hat{\Delta}_{K_1\times SO(3)/T}) = -\tilde{\chi}(\hat{\Delta}_{K_1/T})$ by Lemma 4.13 and these summands cancel each other out. If $\mathfrak{k}_1 = \mathfrak{so}(2(n-1)+1)_{I_1}$, then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{so}(2)_{\{i\}}$. Hence, taking the sum over alll i yields B.

If $\mathfrak{g} = \mathfrak{sp}(n)$, then for any choice of $\mathfrak{k}_1$ there occur two different T-subalgebras, namely $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{u}(1)_{\{i\}}$ and $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{sp}(1)_{\{i\}}$. Since $\tilde{\chi}(\hat{\Delta}_{Sp(1)/T}) = -1$, it follows $\tilde{\chi}(\hat{\Delta}_{K_1\times Sp(1)/T}) = -\tilde{\chi}(\hat{\Delta}_{K_1/T})$ by Lemma 4.13. Hence, taking the sum over all i and $\mathfrak{k}_1$ yields $B = 0$.

If $\mathfrak{g} = \mathfrak{so}(2n)$, then $\mathfrak{k} = \mathfrak{so}(2(n-1))_{I_1} \oplus \mathfrak{so}(2)_{\{i\}}$ or $\mathfrak{k} = \mathfrak{u}(n-1)_{I_1}^{\epsilon_1,\ldots,\epsilon_{n-2}} \oplus \mathfrak{so}(2)_{\{i\}}$. So, $\tilde{\chi}(\hat{\Delta}_{K/T}) = d_{n-1}$ in the first case and $\tilde{\chi}(\hat{\Delta}_{K/T}) = a_{n-2}$ in the second case. Taking the sum over all i and all possible signs yields B. This proves (4.24).

Furthermore, $C = -1$ in all cases. In fact, fix any subset $I_1 \subseteq \{1,\ldots,n\}$ with $1 \in I_1$, $|I_1| \le n-2$ and fix any choice of $\mathfrak{k}_1$.

If $I_1 = \{1\}$ and $\mathfrak{k}_1 = \mathfrak{u}(1)_{\{1\}}$, then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{l}$ for some $\mathfrak{t} < \mathfrak{l} \le \mathfrak{g}(n-1)$ with $\mathfrak{g}(n-1) = \mathfrak{su}(n-1)$, $\mathfrak{so}(2(n-1)+1)$, $\mathfrak{sp}(n-1)$ or $\mathfrak{so}(2(n-1))$), respectively. Taking the sum over all $\mathfrak{l}$ yields

$$\sum_{\mathfrak{t}<\mathfrak{l}\le\mathfrak{g}(n-1)} \tilde{\chi}(\hat{\Delta}_{L/T}) = \tilde{\chi}(\hat{\Delta}_{G(n-1)/T}) + \sum_{\mathfrak{t}<\mathfrak{l}<\mathfrak{g}(n-1)} \tilde{\chi}(\hat{\Delta}_{L/T}) = -1$$

by Lemma 4.14. If $\mathfrak{k}_1 \ne \mathfrak{u}(1)_{\{1\}}$, then $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{l}$ is a T-subalgebra for all $\mathfrak{t} \le \mathfrak{l} \le \mathfrak{g}(n-|I_1|)$, where $\mathfrak{g}(n-|I_1|)$ is defined similarly to $g(n-1)$. Now, taking the sum over all $\mathfrak{l}$ yields

$$\tilde{\chi}(\hat{\Delta}_{K_1/T}) + \sum_{\mathfrak{t}<\mathfrak{l}\le\mathfrak{g}(n-|I_1|)} \tilde{\chi}(\hat{\Delta}_{K_1\times L/T}) = \tilde{\chi}(\hat{\Delta}_{K_1/T}) \cdot \left(1 + \sum_{\mathfrak{t}<\mathfrak{l}\le\mathfrak{g}(n-|I_1|)} \tilde{\chi}(\hat{\Delta}_{L/T})\right) = 0$$

by Lemma 4.13 and Lemma 4.14. In summary, $\tilde{\chi}(\hat{\Delta}_{G/T}) = -(1+A+B+C) = -A-B$. Since $a_1 = b_1 = c_1 = -1$, $d_2 = 1$, it follows by induction:

$$\begin{aligned}
a_n = \tilde{\chi}(\hat{\Delta}_{SU(n+1)/T}) \quad &= -0 - n \cdot a_{n-1} \\
&= (-1)^n \cdot n \cdot (n-1)! = (-1)^n \cdot n!,\ n \ge 2. \\
d_n = \tilde{\chi}(\hat{\Delta}_{SO(2n)/T}) \quad &= -2^{n-1} \cdot a_{n-1} - (n-1) \cdot (2^{n-2} \cdot a_{n-2} + d_{n-1}) \\
&= (-1)^n \cdot 2^{n-1} \cdot (n-1)! - (n-1) \cdot (-1)^{n-2} \cdot 2^{n-2} \cdot (n-2)! \\
&\quad - (n-1) \cdot (-1)^{n-1} \cdot (2^{n-2} - 1) \cdot (n-2)! \\
&= (-1)^n \cdot 2^{n-1} \cdot (n-1)! - (n-1) \cdot (-1)^n \cdot (n-2)! \\
&= (-1)^n \cdot (2^{n-1} - 1) \cdot (n-1)!,\ n \ge 3. \\
b_n = \tilde{\chi}(\hat{\Delta}_{SO(2n+1)/T}) \quad &= -2^{n-1} \cdot a_{n-1} - d_n - (n-1) \cdot b_{n-1} \\
&= (-1)^n \cdot 2^{n-1} \cdot (n-1)! - (-1)^n \cdot (2^{n-1} - 1) \cdot (n-1)! \\
&\quad - (n-1) \cdot (-1)^{n-1} \cdot (n-1)! \\
&= (-1)^n \cdot (n-1)! \cdot (1 + n - 1) = (-1)^n \cdot n!,\ n \ge 2.
\end{aligned}$$

$$\begin{aligned} c_n = \tilde{\chi}(\hat{\Delta}_{Sp(n)/T}) \quad &= -2^{n-1} \cdot a_{n-1} - 0 \\ &= (-1)^n \cdot 2^{n-1} \cdot (n-1)!, \ n \geq 2. \end{aligned}$$

It remains to prove that $\hat{\Delta}_{G/T}$ is a wedge sum of spheres up to homotopy. Let $\mathrm{rank}(G) \geq 2$ and $m := \mathrm{rank}(G) - 2$. Furthermore, let $\mathfrak{k}_1, \ldots, \mathfrak{k}_N$ be the minimal T-subalgebras of $\mathfrak{g}$. For any subalgebra $\mathfrak{l} = \langle \mathfrak{k}_{i_1}, \ldots, \mathfrak{k}_{i_r} \rangle$, the root system $R(\mathfrak{l})$ is contained in the $\mathbb{R}$-linear span of the root systems $R(\mathfrak{k}_{i_j})$. But $\mathrm{rank}\, R(\mathfrak{k}_{i_j}) = 1$ for all $1 \leq j \leq r$, so $\mathrm{rank}\, R(\mathfrak{l}) \leq r$. In particular, $\mathfrak{l} = \mathfrak{g}$ implies $r \geq \mathrm{rank}(G)$. By Corollary 2.23, $\hat{\Delta}_{G/T}$ is $(\mathrm{rank}(G) - 3)$-connected, i.e. $(m-1)$-connected. By Lemma 4.15, it remains to prove that $\hat{\Delta}_{G/T}$ is homotopy equivalent to a simplicial complex of dimension m. For this purpose, consider the subposet

$$P^{\max}_{G/T} := \{\mathfrak{k} \mid \exists \text{ maximal } T\text{-subalgebras } \mathfrak{l}_1, \ldots, \mathfrak{l}_s : \ \mathfrak{l}_1 \cap \ldots \cap \mathfrak{l}_s = \mathfrak{k}\} \subseteq \hat{P}_{G/T}$$

and let $\Delta^{\max}_{G/T}$ be its order complex. Similarly to the proof of Theorem 2.12, it follows that for the inclusion map $\iota : P^{\max} \hookrightarrow \hat{P}_{G/T}$ the fibres $\iota^{-1}((\hat{P}_{G/T})_{\geq \mathfrak{k}})$ are contractible, so $\Delta^{\max}_{G/T}$ is a strong deformation retract of $\hat{\Delta}_{G/T}$. It remains to prove $\dim \Delta^{\max}_{G/H} = m$, i.e. every maximal chain in $P^{\max}_{G/T}$ has length $\leq m$.

First, let $\mathfrak{g} = \mathfrak{su}(n)$. The maximal T-subalgebras are of type $\mathfrak{s}(\mathfrak{u}(n_1)_{I_1} \oplus \mathfrak{u}(n_2)_{I_2})$. So, any T-subalgebra $\mathfrak{k} = \mathfrak{s}(\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i})$ can be written as the intersection of maximal ones, namely $\mathfrak{k} = \cap_{i=1}^r \mathfrak{s}(\mathfrak{u}(n_i)_{I_i} \oplus \mathfrak{u}(n-n_i)_{\{1,\ldots,n\} \setminus I_i})$. Hence, $\Delta^{\max}_{G/T} = \hat{\Delta}_{G/T}$. Let $(\mathfrak{k}_0 > \ldots > \mathfrak{k}_l)$ be any maximal chain of T-subalgebras. In each step, two blocks get merged together to one block, i.e. $\mathfrak{k}_i = \mathfrak{s}(\oplus_{j=1}^{r_i} \mathfrak{u}(n^i_j)_{I^i_j})$ with $r_i = i+2$ for all $0 \leq i \leq l$. Since $\mathfrak{k}_l \cong \mathfrak{s}(\mathfrak{u}(2) \oplus \mathfrak{u}(1)^{n-2})$, it follows $r_l = n-1$, so $l = n-3 = m$.

Now, let $\mathfrak{g} = \mathfrak{so}(2n)$. The maximal T-subalgebras are of type $\mathfrak{so}(2n_1)_{I_1} \oplus \mathfrak{so}(2n_2)_{I_2}$ and $\mathfrak{u}(n)^{\epsilon_1,\ldots,\epsilon_{n-1}}$. As above, intersections of maximal T-subalgebras of first type are precisely given by all subalgebras of type $\oplus_{i=1}^r \mathfrak{so}(2n_i)_{I_i}$. If $\mathfrak{k} = \cap_{i=1}^p \mathfrak{l}_i$ is the intersection of some maximal T-subalgebras with $\mathfrak{l}_1 = \mathfrak{u}(n)^{\epsilon_1,\ldots,\epsilon_{n-1}}$, then $\mathfrak{k}$ must be a subalgebra of $\mathfrak{l}_1$. On the other hand, every T-subalgebra of $\mathfrak{l}_1$ can be obtained as an intersection of maximal ones, since $\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i}^{\epsilon^i_1,\ldots,\epsilon^i_{n_i-1}} = \oplus_{i=1}^r \mathfrak{so}(2n_i)_{I_i} \cap \mathfrak{u}(n)^{\delta_1,\ldots,\delta_{n-1}}$ for an appropiate choice of signs δ_i. In summary, the vertices of $\Delta^{\max}_{G/T}$ are precisely given by

$$\oplus_{i=1}^r \mathfrak{so}(2n_i)_{I_i} \text{ and } \oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i}^{\epsilon^i_1,\ldots,\epsilon^i_{n_i-1}}.$$

In other words, T-subalgebras of "mixed"type are not vertices of $\Delta^{\max}_{G/T}$. Now, let $(\mathfrak{k}_0 > \ldots > \mathfrak{k}_l)$ be a maximal chain of T-subalgebras in $P^{\max}_{G/T}$. Then $\mathfrak{k}_l \cong \mathfrak{u}(2) \oplus \mathfrak{u}(1)^{n-2}$. If $\mathfrak{k}_0 = \mathfrak{u}(n)^{\epsilon_1,\ldots,\epsilon_{n-1}}$, then $n-2$ steps are needed to merge all blocks to a single block, i.e. $l = n-2$. If $\mathfrak{k}_0 = \mathfrak{so}(2n_1)_{I_1} \oplus \mathfrak{so}(2n_2)_{I_2}$, then $n-3$ step are needed to merge the blocks. But a further step is needed to switch from $\oplus_{i=1}^r \mathfrak{u}(n_i)_{I_i}^{\epsilon^i_1,\ldots,\epsilon^i_{n_i-1}}$ to $\oplus_{i=1}^r \mathfrak{so}(2n_i)_{I_i}$. So, $l = n-2 = m$ in each case.

If $\mathfrak{g} = \mathfrak{sp}(n)$, the same argument as above yields that the vertices of $\dim \Delta^{\max}_{G/T}$ are given by

$$\oplus_{i=1}^{r} \mathfrak{sp}(n_i)_{I_i} \text{ and } \oplus_{i=1}^{r} \mathfrak{u}(n_i)_{I_i}^{\epsilon_1^i,\ldots,\epsilon_{n_i-1}^i}.$$

Let $(\mathfrak{k}_0 > \ldots > \mathfrak{k}_l)$ be any maximal chain of T-subalgebras in $P^{\max}_{G/T}$. If $\mathfrak{k}_l \cong \mathfrak{u}(2) \oplus \mathfrak{u}(1)^{n-2}$, then $l = n-2 = m$ as above. If $\mathfrak{k}_l \cong \mathfrak{sp}(1)^n$, then $\mathfrak{k}_0 = \mathfrak{sp}(n_1)_{I_1} \oplus \mathfrak{sp}(n_2)_{I_2}$ and $n-2$ steps are needed to merge the blocks. Hence, $l = n-2 = m$ in each case.

If $\mathfrak{g} = \mathfrak{so}(2n+1)$, the maximal T-subalgebras are given by $\mathfrak{so}(2n_1+1)_{I_1} \oplus \mathfrak{so}(2n_2)_{I_2}$ and $\mathfrak{so}(2n)$. The vertices of $\Delta^{\max}_{G/T}$ are now given by

$$\oplus_{i=1}^{r} \mathfrak{so}(2n_i)_{I_i} \text{ and } \mathfrak{so}(2n_1+1)_{I_1} \oplus \oplus_{i=2}^{r} \mathfrak{so}(2n_i)_{I_i}.$$

Let $(\mathfrak{k}_0 > \ldots > \mathfrak{k}_l)$ be a maximal chain of T-subalgebras in $P^{\max}_{G/T}$. If $\mathfrak{k}_l \cong \mathfrak{so}(3) \oplus \mathfrak{so}(2)^{n-1}$, then $\mathfrak{k}_0 = \mathfrak{so}(2n_1+1)_{I_1} \oplus \mathfrak{so}(2n_2)_{I_2}$ and $n-2$ steps are needed to merge the blocks. So, let $\mathfrak{k}_l \cong \mathfrak{so}(4) \oplus \mathfrak{so}(2)^{n-1}$. If $\mathfrak{k}_0 = \mathfrak{so}(2n)$, then $n-2$ steps are needed to merge all blocks to a single block. If $\mathfrak{k}_0 = \mathfrak{so}(2n_1+1)_{I_1} \oplus \mathfrak{so}(2n_2)_{I_2}$, then $n-3$ steps are needed to merge the blocks and a further step is needed to switch from $\mathfrak{so}(2n_1)_{I_1} \oplus \oplus_{i=2}^{r} \mathfrak{so}(2n_i)_{I_i}$ to $\mathfrak{so}(2n_1+1)_{I_1} \oplus \oplus_{i=2}^{r} \mathfrak{so}(2n_i)_{I_i}$. It follows $l = n-2 = m$ in each case. Hence, $\dim \Delta^{\max}_{G/H} = m$ in all cases. This proves the claim. □

Chapter 5

$\hat{\Delta}_{G/H}$ for G of Exceptional Type

5.1 General Properties

Let G be a compact simple Lie group of exceptional type, i.e. $\mathfrak{g} = \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8,\ \mathfrak{f}_4$ or $\mathfrak{g}_2$ and $G = E_6,\ E_7,\ E_8,\ F_4$ or G_2, respectively. Let $T < G$ be a fixed maximal torus of G with Lie algebra $\mathfrak{t}$. The T-subalgebras of G can be determined by applying the Borel - de Siebenthal theorem. For this purpose, closed subroot systems have to be introduced.

Definition 5.1. Let R be a root system. A subset $S \subseteq R$ is called a *closed subroot system*, denoted by $S < R$, if S itself is a root system of $\langle S \rangle_{\mathbb{R}}$ and if the following condition holds:

$$\forall \alpha, \beta \in S: \quad \alpha + \beta \in R \quad \Longrightarrow \quad \alpha + \beta \in S$$

Equivalently, a subset $S \subseteq R$ is a closed subroot system, if the following two conditions hold:

$$\begin{aligned} \forall \alpha \in S: &\quad -\alpha \in S \\ \forall \alpha, \beta \in S: &\quad \alpha + \beta \in R \quad \Longrightarrow \quad \alpha + \beta \in S \end{aligned}$$

For any subset $S \subseteq R$, $\langle S \rangle$ denotes the closed subroot system of R which is generated by S.

From Lemma 3.10, it directly follows:

Corollary 5.2. *Let $\mathfrak{t}$ and $\mathfrak{g}$ as above and let a set $R(\mathfrak{g})^+$ of positive roots and a subset $I \subseteq R(\mathfrak{g})^+$ be given. Then*

$$\mathfrak{h} = \mathfrak{t} \oplus \bigoplus_{\alpha \in I} \mathfrak{m}_\alpha$$

is a Lie subalgebra if and only if $I \,\dot{\cup}\, -I$ is a closed subroot system.

In particular, there exists a one-to-one correspondence between all T-subalgebras of $\mathfrak{g}$ and all closed subroot systems of $R(\mathfrak{g})$. Since $\mathfrak{g}$ is simple, it remains to classify all closed subroot systems of R for any irreducible root system R. The *maximal* closed subroot systems are classified by the Borel - de Siebenthal theorem.

Theorem 5.3 (Borel - de Siebenthal, [Wol77, Theorem 8.10.9]). *Let R be an irreducible root system with simple roots $F = \{\alpha_1, \dots, \alpha_n\}$. Let $\alpha_0 := \sum_{i=1}^n n_i\alpha_i$ be the maximal root as in Lemma 3.5. Then, up to $W(R)$-conjugacy, all maximal closed subroot systems $S < R$ are given by:*

1. *$S = \langle \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n, -\alpha_0\rangle$ for some $i \in \{1, \dots, n\}$ such that n_i is prime.*
2. *$S = \langle \alpha_1, \dots, \widehat{\alpha}_i, \dots, \alpha_n\rangle$ for some $i \in \{1, \dots, n\}$ such that $n_i = 1$.*

Now, let $\widetilde{D(R)}$ be the *extended Dynkin diagram of* R which is defined as $D(R)$ but with an additional vertex representing $-\alpha_0$. This vertex will be denoted by $\odot$. Furthermore, for $i \in \{1, \dots, n\}$, the vertex representing α_i will be labeled with n_i, see Figure 5.1. Theorem 5.3 states that the Dynkin diagram $D(S)$ of any maximal closed subroot system $S < R$ can be obtained by one of the following two ways:

1. Delete a vertex v of $\widetilde{D(R)}$ which is labeled by a prime number.
2. Delete a vertex v of $\widetilde{D(R)}$ which is labeled by 1 and delete the vertex $\odot$.

Hence, the maximal closed subroot systems $S < R$ are given by Table 5.1.

Table 5.1: Maximal closed subroot systems

R	S maximal, rank(S)=rank(R)	S maximal, rank(S)=rank(R)-1
A_n		$A_k + A_{n-k},\ 1 \le k \le \lfloor \frac{n}{2} \rfloor$
B_n	$B_k + D_{n-k},\ 1 \le k \le n-2$ D_n	B_{n-1}
C_n	$C_k + C_{n-k},\ 1 \le k \le \lfloor \frac{n}{2} \rfloor$	A^S_{n-1}
D_n	$D_k + D_{n-k},\ 1 \le k \le \lfloor \frac{n}{2} \rfloor$	A_{n-1} D_{n-1}
E_6	$A_1 + A_5,\ 3A_2$	D_5
E_7	$A_7,\ A_2 + A_5$ $A_1 + D_6$	E_6

Table 5.1: Maximal closed subroot systems

R	S maximal, rank(S)=rank(R)	S maximal, rank(S)=rank(R)-1
E_8	$A_8,\ 2A_4$ $A_2+E_6,\ A_1+E_7$ D_8	
F_4	$A_2^L+A_2^S$ $A_1^L+C_3$ B_4	
G_2	$A_2^L,\ A_1^L+A_1^S$	

In root systems with roots of different length, a subroot system of type A_k may either consist of long roots or consist of short roots which will be denoted by A_k^L or A_k^S, respectively. Note, that the root system of rank 1, A_1, contains no proper closed subroot system except $\emptyset$.

Now, if $R = R_1 + \ldots + R_l$ for irreducible $R_i,\ 1 \leq i \leq l$, then each maximal closed subroot system $S < R$ is of type $S = S_1 + \ldots + S_l$ with $S_i < R_i$ maximal for a unique $i \in \{1, \ldots, l\}$ and $S_j = R_j$ for all $j \neq i$. It follows that *all* closed subroot systems of $R(\mathfrak{g})$ can be determined up to $W(\mathfrak{g})$-conjugacy by applying the Borel - de Siebenthal theorem iteratively.

Note, that a given root system S may appear for several times in the above described algorithm up to isomorphism. It is useful to know whether closed subroot systems of $R(\mathfrak{g})$, which are isomorphic as root systems, are also conjugate under $W(\mathfrak{g})$ or not. This question is answered by the following lemma, see [Dyn57, p. 147f.].

Lemma 5.4. *Let R be an irreducible root system of exceptional type and $S < R$ a closed subroot system. Then all closed subroot systems $\tilde{S} < R$ with $\tilde{S}$ isomorphic to S as root system are conjugate to S under $W(R)$, except for the following 11 cases:*

1. *$R = E_7$ and $S = A_1 + A_5,\ A_5,\ 2A_1 + A_3,\ A_1 + A_3,\ 3A_1$ or $4A_1$,*
2. *$R = E_8$ and $S = A_7,\ A_1 + A_5,\ 2A_3,\ 2A_1 + A_3$ or $4A_1$.*

In all these exceptional cases, the isomorphism class of S splits into two Weyl orbits. More precisely, if $G = E_n,\ n = 7, 8$, one Weyl orbit contains all $\tilde{S} \cong S$ which satisfy $\tilde{S} \subseteq A_n \subseteq E_n$. The other Weyl orbit contains all $\tilde{S} \cong S$ which are not conjugate to a subroot system of $A_n \subseteq E_n$.

To distinguish the two Weyl orbits in these exceptional cases, the subroot system S will be denoted by S', if $S \subseteq A_7 \subseteq E_7$ ($S \subseteq A_8 \subseteq E_8$) and by S'' otherwise.

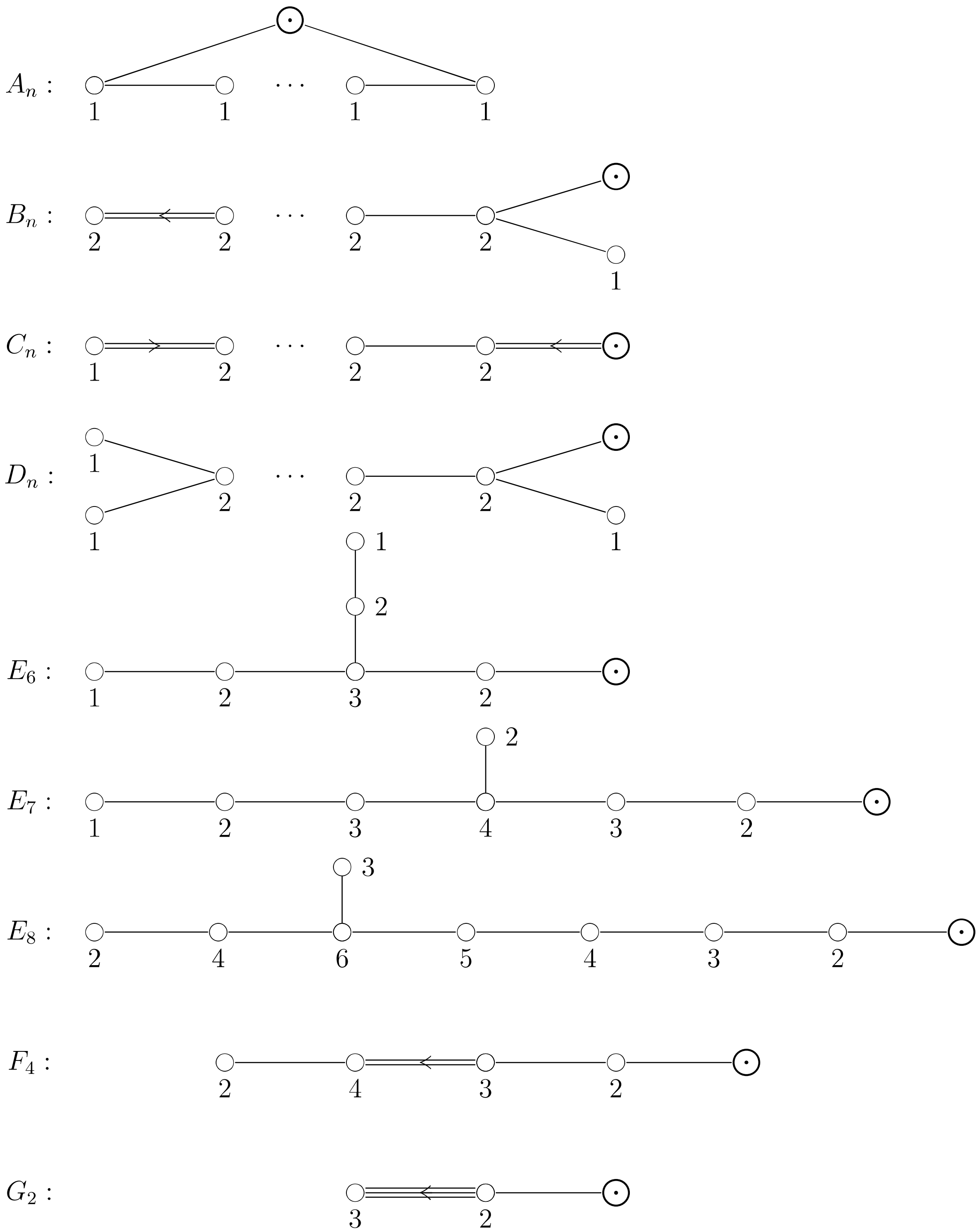

Figure 5.1: Extended Dynkin Diagrams

Now, the determination of all closed subroot systems up to $W(\mathfrak{g})$-conjugacy is sufficient to determine all Lie subalgebras $\mathfrak{t} \leq \mathfrak{h} < \mathfrak{g}$. In fact, by [BD85, Thm. 3.12, p. 200], there is a canonical isomorphism from $W(\mathfrak{g})$ to $N(T)/T$. Thus, closed subroot systems $S,\ S' < R$ are conjugate by some $w \in W(\mathfrak{g})$ if and only if the corresponding subalgebras $\mathfrak{h},\ \mathfrak{h}'$ are conjugate by some $n \in N(T)$ and Ad_n induces an isomorphism $\hat{\Delta}_{G/H} \cong \hat{\Delta}_{G/H'}$.

In the next sections, the (non)-contractibility of $\hat{\Delta}_{G/H}$ will be determined, where G runs over all exceptional Lie groups and $R(\mathfrak{h})$, the root system which corresponds to H, runs over all $W(\mathfrak{g})$-conjugacy classes of closed subroot systems of $R(\mathfrak{g})$. To apply Theorem 2.27 to $\hat{\Delta}_{G/H}$, the following theorem will be useful in many cases.

Theorem 5.5. *Let G be semisimple with maximal torus T, $l \in \mathbb{N}$ and $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_l^0)$ a non-extendable chain of T-subalgebras with root systems $R(\mathfrak{k}_i^0)$, $1 \leq i \leq l$. If*

$$\operatorname{rank} R(\mathfrak{g}) > \operatorname{rank} R(\mathfrak{k}_0^0) > \ldots > \operatorname{rank} R(\mathfrak{k}_l^0)$$

and if $\mathfrak{k}_0^0$ is maximal in $\mathfrak{g}$, then $\tilde{H}_{l-1}(\hat{\Delta}_{G/K_l^0}, \mathbb{Q}) \neq 0$.

Proof. Note, that the assumptions imply that $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{l-1}^0)$ is a maximal chain of K_l^0-subalgebras. Thus, it is sufficient to show that this chain satisfies the conditions of Theorem 2.27. For this purpose, choose any root $r_1 \in R(\mathfrak{g}) \backslash R(\mathfrak{k}_0^0)$ and let $\mathfrak{k}_0^1 := \langle \mathfrak{k}_1^0, r_1 \rangle$, i.e. $\mathfrak{k}_0^1$ is the subalgebra with root system $\langle R(\mathfrak{k}_1^0), r_1 \rangle$. This implies

$$\operatorname{rank} R(\mathfrak{k}_0^1) \leq \operatorname{rank} R(\mathfrak{k}_1^0) + 1 < \operatorname{rank} R(\mathfrak{k}_0^0) + 1 \leq \operatorname{rank} R(\mathfrak{g}),$$

i.e. $\mathfrak{k}_0^1 < \mathfrak{g}$ is a K_1-subalgebra. Moreover, $\mathfrak{k}_0^1 \neq \mathfrak{k}_0^0$, since $r_1 \in R(\mathfrak{k}_0^1) \setminus R(\mathfrak{k}_0^0)$. By Lemma 2.24, it follows $\tilde{H}_0(\hat{\Delta}_{G/K_1^0}, \mathbb{Q}) \neq 0$.

Now, let $l \geq 2$ and $r_1,\ \mathfrak{k}_0^1$ as above. Furthermore, choose roots $r_s \in R(\mathfrak{k}_{s-2}^0) \backslash R(\mathfrak{k}_{s-1}^0)$ for $s \in \{2, \ldots, l\}$. For every $i \in \{1, \ldots, l-1\}$ and $j \in \{1, \ldots, i+1\}$ let

$$\mathfrak{k}_i^j := \langle \mathfrak{k}_{i+1}^0, r_{i+2-j} \rangle. \tag{5.1}$$

Then for all $i \in \{1, \ldots, l-1\}$ the following conditions hold:

1. $\mathfrak{k}_i^0 \neq \mathfrak{k}_i^j$ for $j \in \{1, \ldots, i+1\}$, since $r_{i+2-j} \notin R(\mathfrak{k}_i^0)$.
2. $\mathfrak{k}_{i-1}^{j-1} \geq \mathfrak{k}_i^j$ for $j \in \{1, \ldots, i+1\}$.
   In fact, for $j \geq 2$ this follows directly from (5.1), since $\mathfrak{k}_{i+1}^0 < \mathfrak{k}_i^0$. For $j = 1$, the inequality holds since $r_{i+1} \in R(\mathfrak{k}_{i-1}^0)$.
3. $\langle \mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1} \rangle < \mathfrak{g}$ for $i \in \{1, \ldots, l-1\}$.
   In fact, $\langle \mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1} \rangle = \langle \mathfrak{k}_{i+1}^0, r_{i+1}, \ldots, r_1 \rangle$, hence:

$$\operatorname{rank} R(\langle \mathfrak{k}_i^1, \ldots, \mathfrak{k}_i^{i+1} \rangle) \leq \operatorname{rank} R(\mathfrak{k}_{i+1}^0) + i + 1 < \operatorname{rank} R(\mathfrak{k}_{i+1}^0) + i + 2 \leq \operatorname{rank}(\mathfrak{g})$$

It follows that the subalgebras $\mathfrak{k}_i^j$ satisfy the conditions of Theorem 2.27 with respect to the chain $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_{l-1}^0)$ of K_l^0-subalgebras. Thus, $\tilde{H}_{l-1}(\hat{\Delta}_{G/K_l^0}, \mathbb{Q}) \neq 0$. □

Moreover, to find the appropiate subalgebras for applying Theorem 2.27 or Corollary 2.19, the following lemma and corollary are needed.

Lemma 5.6. *Let R be an irreducible root system and S be any closed subroot system of R. Denote*

$$S^\perp := \{\alpha \in R \mid \alpha \perp S\}.$$

Then $S^\perp$ is a closed subroot system of S. If, in addition, all roots of R have the same length or if $R = G_2$, then $S + S^\perp$ is also a closed subroot system.

Proof. If $\alpha \in S^\perp$, then $-\alpha \in S^\perp$. Moreover, let $\alpha, \beta \in S^\perp$ such that $\alpha + \beta \in R$. Then $\alpha + \beta \in S^\perp$, i.e. $S^\perp$ is closed.

Now, suppose that all roots of R have the same length or $R = G_2$. If $\alpha \in S + S^\perp$, then $-\alpha \in S + S^\perp$. So, let $\alpha, \beta \in S + S^\perp$ such that $\alpha + \beta \in R$. If $\alpha, \beta \in S$, then $\alpha + \beta \in S \subset S + S^\perp$, since S is closed. Similarly for $\alpha, \beta \in S^\perp$. If $\alpha \in S,\ \beta \in S^\perp$, it follows $\|\alpha + \beta\|^2 = \|\alpha\|^2 + \|\beta\|^2$, since $\alpha \perp \beta$. Hence, $\alpha + \beta$ must be a long root and α, β must be short roots. This is a contradiction, if all roots have the same length. If $R = G_2$, then $\frac{\|\alpha+\beta\|^2}{\|\alpha\|^2} = 3$ which is again a contradiction. Hence, $\alpha + \beta \notin R$ in this case and $S + S^\perp$ is closed. □

Note, that $S^\perp = \emptyset$ is possible even if $\operatorname{rank} S < \operatorname{rank} R$, for instance $A_{n-1}^\perp = \emptyset$ for $A_{n-1} < A_n$.

Corollary 5.7. *With the same notaion as in Lemma 5.6, assume that all closed subroot systems of R which are isomorphic to S are also $W(R)$-conjugate to S. Then $S^\perp$ is the largest root system of all root systems R' with the property that R contains a closed subroot system isomorphic to $R' + S$.*

Proof. Let R'' be any root system such that R contains a closed subroot system isomorphic to $R'' + S$. Hence, $R'' + \tilde{S} < R$ for some $\tilde{S}$ isomorphic to S. By assumption, there exists an $w \in W(R)$ with $w(\tilde{S}) = S$, thus $w(R'') \subseteq S^\perp$. □

At last, the following lemma will be needed to determine $\dim \hat{\Delta}_{G/H}$ and $m_{G/H}$, the minimal dimension of all facets of $\hat{\Delta}_{G/H}$. It will be shown that if $\hat{\Delta}_{G/H}$ is non-contractible, then in most cases $\tilde{H}_{m_{G/H}}(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$. An example for a complex $\hat{\Delta}_{G/H}$ with $\tilde{H}_k(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ for *more than one* k is given by $G = E_7$, $R(\mathfrak{h}) = A_1 + A_2$, see section 5.5.

Lemma 5.8. *Let $H < G$ be compact connected Lie groups with* $\operatorname{rank} H = \operatorname{rank} G$. *Furthermore, let the set of all minimal H-subalgebras be given by $\{\mathfrak{k}_1, \ldots, \mathfrak{k}_N\}$. For*

$$m_{G/H} := \min\{\dim s \mid s \textit{ is a facet of } \hat{\Delta}_{G/H}\},$$

it follows

$$m_{G/H} = \min\{m_{G/K_1}, \dots, m_{G/K_N}\} + 1$$

with $m_{G/K_i} := -1$*, if* K_i *is maximal in* G*. Moreover,*

$$\dim \hat{\Delta}_{G/H} = \max\{\dim \hat{\Delta}_{G/K_1}, \dots, \dim \hat{\Delta}_{G/K_1}\} + 1.$$

Proof. If some K_i is maximal, then $\mathfrak{k}_i$ is an isolated vertex of $\hat{\Delta}_{G/H}$ and $m_{G/H} = 0$. So, assume that no K_i is maximal. All maximal chains of H-subalgebras are of type $(\mathfrak{k}_i < \mathfrak{l}_0, < \dots < \mathfrak{l}_m)$ for some $1 \leq i \leq N$ and $m_{G/K_i} \leq m \leq \dim \hat{\Delta}_{G/K_i}$. Hence, choosing i such that m_{G/K_i} is minimal yields $m_{G/H} = m_{G/K_i} + 1$.

Similarly, if all K_i are maximal, then $\dim \hat{\Delta}_{G/H} = 0$. Otherwise, choosing i such that $\dim \hat{\Delta}_{G/K_i}$ is maximal yields $\dim \hat{\Delta}_{G/H} = \dim \hat{\Delta}_{G/K_i} + 1$. □

5.2 G_2

As described above, applying the Borel - de Siebenthal theorem iteratively yields all closed subroot systems of G_2 up to W-conjugacy, see also [Dyn57, p. 149]. The closed subroot systems of G_2 are given by Table 5.2. To simplify notation, in the following a connected subgroup $H < G$ of maximal rank or its Lie algebra $\mathfrak{h}$ may be written as the root system $R(\mathfrak{h})$, e.g. A_1 may denote the root system A_1, the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{t}^{\mathrm{rank}(G)-1}$ or the Lie group $SU(2) \times T^{\mathrm{rank}(G)-1}$ (or $SO(3) \times T^{\mathrm{rank}(G)-1}$).

Table 5.2: $G = G_2$

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
1	A_2^L	–	-1	-1	yes
2	$A_1^L + A_1^S$	–	-1	-1	yes
3	A_1^L	$1, 2$	0	0	yes
4	A_1^S	2	0	0	no
5	$\emptyset$	$3, 4$	1	1	no

Note, that Table 5.2 also yields the following informations about a given subroot system H:

1. All closed subroot systems S, in which H is maximally contained in. This will also be denoted by "$H \to S$".

2. The values $\dim \hat{\Delta}_{G/H}$ and $m_{G/H}$.

3. The answer to the question, if $\hat{\Delta}_{G/H}$ is non-contractible.

Since $\hat{\Delta}_{G/H} = \emptyset$ for maximal H, it remains to check whether $\hat{\Delta}_{G/H}$ is contractible or not for all non-maximal H.

$H = A_1^L$: Since $A_1^L \to A_1^L,\ A_1^S + A_1^S$, $\hat{\Delta}_{G/H}$ is disconnected, By Lemma 2.24, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1^S$: $H \to A_1^L + A_1^S$. Since G_2 has rank 2, there can only exist one positive root perpendicular to A_1^S. Hence, $\hat{\Delta}_{G/H} = \{A_1^L + A_1^S\}$ is a singleton.

$H = \emptyset$: $H \to A_1^L,\ A_1^S$. Note, that for an irreducible root system with roots of different length, the long roots form a proper closed subroot system. For G_2, this is $K = A_2^L$. Now, let $c(K)$ denote the complement of K in the poset $\hat{P}_{G/H}$. Since K contains all long roots, so $A_1^L \notin c(K)$ and $A_1^L + A_1^S \notin c(K)$. On the other hand, for any A_1^S it is $A_1^S \cap K = \emptyset$ and $\langle A_1^S, K\rangle = G_2$, since K is maximal in G_2. Hence, $c(K)$ consists of all A_1^S. But $\hat{\Delta}_{G_2/A_1^S}$ is contractible. Thus, $\hat{\Delta}_{G/H}$ is contractible by Corollary 2.19.

5.3 F_4

Borel - de Siebenthal's theorem yields the following closed subroot systems of F_4 up to W-conjugacy, see [Dyn57, p. 149].

Table 5.3: $G = F_4$

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
1	$A_1^L + C_3$	–	-1	-1	yes
2	C_3	1	0	0	no
3	B_4	–	-1	-1	yes
4	B_3	3	0	0	no
5	$2A_1^L + B_2$	1, 3	0	0	yes
6	$A_1^L + B_2$	2, 5	1	1	no
7	B_2	4, 6	2	1	no

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
8	D_4	3	0	0	yes
9	$A_3^L + A_1^S$	3	0	0	no
10	A_3^L	$4, 8, 9$	1	1	no
11	$A_2^L + A_2^S$	$-$	-1	-1	yes
12	$A_2^L + A_1^S$	$9, 11$	1	0	yes
13	A_2^L	$10, 12$	2	1	yes
14	$A_1^L + A_2^S$	$1, 11$	0	0	yes
15	A_2^S	$2, 14$	1	1	no
16	$2A_1^L + A_1^S$	$4, 5, 9$	1	1	no
17	$A_1^L + A_1^S$	$6, 12, 14, 16$	2	1	yes
18	$4A_1^L$	$5, 8$	1	1	yes
19	$3A_1^L$	$6, 18$	2	2	no
20	$2A_1^L$	$7, 10, 16, 19$	3	2	no
21	A_1^L	$13, 17, 20$	4	2	yes
22	A_1^S	$7, 15, 17$	3	2	no
23	$\emptyset$	$21, 22$	5	3	no

Again, consider all non-maximal T-subalgebras. If different root systems of the same type have to be considered, they will be numbered consecutively, e.g. $A_1(1)$, $A_1(2)$, $A_1(3)$ and so on.

$H = C_3$: $H \to A_1^L + C_3$. Since $\operatorname{rank} F_4 - \operatorname{rank} C_3 = 1$, there is at most one positive root A_1^L perpendicular to C_3. Hence, $\hat{\Delta}_{G/H} = \{A_1^L + C_3\}$ is a singleton.

$H = B_3$: $H \to B_4$. To prove that $\hat{\Delta}_{G/H}$ is contractible, it is sufficient to prove that there is a unique B_4 containing H, so that $\hat{\Delta}_{G/H}$ is a singleton. So, let $H < B_4$. Now, the long roots of F_4 form a proper closed subroot system which is of type D_4. In particular, there is a unique subroot system of type D_4. Moreover, $D_4 < B_4$, so $B_4 = \langle D_4, H \rangle$ is unique.

$H = 2A_1^L + B_2$: $H \to A_1^L + C_3,\ B_4$, so $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1^L + B_2$: $H \to C_3,\ 2A_1^L + B_2$. Since $\operatorname{rank} H^\perp = 1$, there is a unique $K := 2A_1^L + B_2 > H$. Let L be any H-subalgebra. Then $L \leq B_4$ or $A_1^L + C_3$. But B_4 and $A_1^L + C_3$ contain the positive root which is perpendicular to H. Thus, $\langle K, L\rangle \leq B_4$ or $A_1^L + C_3$, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = B_2$: $H \to A_1^L + B_2,\ B_3$. $K := 2A_1^L + B_2 > H$ is unique, since $\operatorname{rank} H^\perp = 2$. Any H-subalgebra L satisfies $L \leq B_4$ or $A_1^L + C_3$. But B_4 and $A_1^L + C_3$ contain $2A_1^L \perp H$, so $\langle K, L\rangle \leq B_4$ or $A_1^L + C_3$ and $c(K) = \emptyset$. Hence, $\hat{\Delta}_{G/H}$ is contractible.

$H = D_4$: $H \to B_4$. Since $H = D_4$ is unique, every subroot system of type B_4 must contain H. So, $\tilde{H}_0(\hat{\Delta}_{G/H}) \neq 0$, if there exist more than one subroot systems of type B_4. F_4 contains 24 short roots but a root system of type B_4 contains only 8 short roots. So, fix subroot systems $B_4(1)$, $A_1^S(1) < B_4(1)$, $A_1^S(2) \not< B_4(1)$ and some $w \in W(F_4)$, $w(A_1^S(1)) = A_1^S(2)$. Then $B_4(2) := w(B_4(1))$ contains $A_1^S(2)$, i.e. $B_4(2) \neq B_4(1)$.

$H = A_1^S + A_3^L$: $H \to B_4$. If $H < B_4$, it follows as above that $B_4 = \langle D_4, H\rangle$ is unique, hence $\hat{\Delta}_{G/H}$ is contractible.

$H = A_L^3$: $H \to B_3,\ D_4,\ A_1^S + A_3^L$. Consider $K = D_4$ and let L be any H-subalgebra. If $L \in c(K)$, then $L \geq A_1^S + A_3^L$ or $L \geq B_3$. But this implies that L is contained in some B_4. Now, all B_4 contain the unique $K = D_4$, so $\langle K, L\rangle \leq B_4$. Hence, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1^S + A_2^L$: $H \to A_1^S + A_3^L,\ A_2^L + A_2^S$, so $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_2^L$: $H \to A_3^L,\ A_1^S + A_2^L$. Choose an H-subalgebra $K_0 := A_1^S(1) + A_2^L$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2^L + A_2^S,\ L_2 = A_1^S(1) + A_3^L\}.$$

Let

$$\begin{aligned} K_1 &:= A_2^L + A_1^S(2) < L_1 \text{ and} \\ K_2 &:= A_3^L < L_2. \end{aligned}$$

Since D_4 contains all long roots, it is $K_1,\ K_2 \leq \langle D_4, A_1^S(2)\rangle = B_4$, so $\langle K_1, K_2\rangle < F_4$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.26.

$H = A_1^L + A_2^S$: $H \to A_1^L + C_3,\ A_2^L + A_2^S$, so $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_2^S$: $H \to C_3,\ A_1^L + A_2^S$. Since $\operatorname{rank} H^\perp = 2$, $K := A_2^L + A_2^S$ is unique. Now, let L be any H-subalgebra. If $L \geq A_1^L + A_2^S$, then $K \cap L \geq A_1^L + A_2^S$, so $L \notin c(K)$.

If $L \geq C_3$ then $L = C_3$ or $A_1^L + C_3$. The second case implies $K \cap L = A_1^L + A_2^S > H$. Hence, $c(K)$ consists of $C_3's$. But $\hat{\Delta}_{F_4/C_3}$ is contractible, so is $\hat{\Delta}_{G/H}$ by Corollary 2.19.

$H = A_1^S + 2A_1^L$: $H \to B_3,\ 2A_1^L + B_2,\ A_1^S + A_3^L$. Since $\operatorname{rank}(2A_1^L)^\perp = 2$, there exists a unique $K := 2A_1^L + B_2 > H$. Let L be any H-subalgebra. Then $L \leq B_4$ or $L \leq A_1^L + C_3$ and these subalgebras must contain K. Thus, $\langle K, L\rangle \leq B_4$ or $A_1^L + C_3$, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1^L + A_1^S$: $H \to A_1^L + B_2,\ A_1^S + 2A_1^L,\ A_1^L + A_2^S,\ A_2^L + A_1^S$. Let $K_0 := A_1^L(1) + A_2^S$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1^L(1) + C_3,\ L_2 = A_2^L + A_2^S\}.$$

Now choose H-subalgebras $K_1,\ K_2$ in the following way:

$$K_1 := A_1^L(1) + (A_1^S + A_1^L(2)) < L_1 \text{ and}$$
$$K_2 := A_2^L + A_1^S < L_2.$$

Since K_1 and K_2 consist of long roots added to the short root A_1^S of H, both are contained in $B_4 = \langle A_1^S, D_4\rangle$, i.e. $\langle K_1, K_2\rangle < F_4$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 4A_1^L$: $H \to 2A_1^L + B_2,\ D_4$. Write H as $A_1^L(1) + A_1^L(2) + A_1^L(3) + A_1^L(4)$ and let $K_0 = A_1^L(1) + A_1^L(2) + B_2(1)$. By Lemma 2.24, $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1^L(1) + C_3,\ L_2 = A_1^L(2) + C_3\}.$$

Since all T-subalgebras of type $2A_1^L$ are conjugate, there are H-subalgebras of type $2A_1^L + B_2$ given by:

$$K_1 = A_1^L(1) + A_1^L(3) + B_2(2) < L_1 \text{ and}$$
$$K_2 = A_1^L(2) + A_1^L(3) + B_2(3) < L_2.$$

Altough Lemma 5.6 is not applicable in this case, $A_1^L(3) + (A_1^L(3))^\perp$ is a T-subalgebra and $K_1, K_2 \leq A_1^L(3) + (A_1^L(3))^\perp$. In fact, there exists a maximal T-subalgebra of type $A_1^L(3) + C_3$. But C_3 is a maximal root system of rank 3, hence $(A_1^L(3))^\perp = C_3$ and $A_1^L(3) + (A_1^L(3))^\perp = A_1^L(3) + C_3$. It follows $\langle K_1, K_2\rangle \leq A_1^L(3) + C_3$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 2.26.

$H = 3A_1^L$: $H \to A_1^L + B_2,\ 4A_1^L$. Let $K = 4A_1^L$. Since $\operatorname{rank} H^\perp = 1$, K is unique. Denote the positive root which is perpendicular to H by $A_1^L(1)$. Let L be any H-subalgebra. If $L \leq B_4$ for some H-subalgebra of type B_4, then $K < D_4 < B_4$, so $\langle K, L\rangle \leq B_4$ and $L \notin c(K)$. Hence, $L \in c(K)$ implies $L = C_3$ or $A_1^L(2) + C_3$. If $L = C_3$, then $L < A_1^L(2) + C_3$ with $A_1^L(2) \perp H$. Thus, $A_1^L(2) = A_1^L(1)$ and $\langle K, L\rangle = A_1^L(1) + C_3$.

If $L = A_1^L(2) + C_3$, one may assume $A_1^L(2) \not\leq H$, so $H \not< C_3$. Hence, $H \cap C_3 = 2A_1^L$ and the root $A_1^L(2)$ is a summand of H. But then, C_3 contains a root $A_1^L(3) \perp H \cap C_3$, hence $A_1^L(3) \perp H$. It follows $A_1^L(3) = A_1^L(1)$ and $K < L$. In summary, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = 2A_1^L$: $H \to B_2,\ A_3^L,\ 2A_1^L + A_1^S,\ 3A_1^L$. Let $K = D_4$, which is unique. If L is any H-subalgebra and $L \leq B_4$ for some H-subalgebra of type B_4, then $\langle K, L\rangle \leq B_4$ and $L \notin c(K)$. Again, $L \in c(K)$ implies $L = C_3$ or $A_1^L + C_3$. In both cases, L contains at least 3 positive long roots which are perpendicular to each other, i.e. $K \cap L \geq 3A_1^L > H$. Hence, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1^L$: $H \to A_2^L,\ A_1^L + A_1^S,\ 2A_1^L$. Let $K_0 := A_2^L$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2^L + A_1^S(1),\ L_2 = A_2^L + A_1^S(2),\ L_3 = A_3^L\}.$$

Since $2A_1^L < A_3^L$, there is a long root $A_1^L(2) < A_3^L$ perpendicular to H. So, let

$$\begin{aligned} K_1 &:= A_1^L + A_1^S(1) < L_1, \\ K_2 &:= A_1^L + A_1^S(2) < L_2 \text{ and} \\ K_3 &:= A_1^L + A_1^L(2) < L_3. \end{aligned}$$

As above, $A_1^L + (A_1^L)^\perp = A_1^L + C_3$ and $K_1, K_2, K_3 \leq A_1^L + (A_1^L)^\perp = C_3$. Hence, $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1^S$: $H \to B_2,\ A_2^S,\ A_1^L + A_1^S$. Let $K = D_4$. Then $L \in c(K)$ implies that L contains no long roots. Hence, $L = A_2^S$. But $\hat{\Delta}_{F_4/A_2^S}$ is contractible, so is $\hat{\Delta}_{G/H}$ by Corollary 2.19.

$H = \emptyset$: $H \to A_1^L,\ A_1^S$. Let $K = D_4$. As above, if L is a T-subalgebra containing any long root, then $L \notin c(K)$. If $L = A_1^S$, then $\langle K, L\rangle = B_4$. Hence, $c(K)$ consists of T-subalgebras of type A_2^S and since $\hat{\Delta}_{F_4/A_2^S}$ is contractible, so is $\hat{\Delta}_{G/H}$ by Corollary 2.19.

5.4 E_6

By Borel - de Siebenthal, the closed subroot systems of E_6 are given up to W-conjugacy by the following table, see [Dyn57, p. 149]. Note, that if R is a root system with roots of equal length, then all subroot systems are closed, i.e. if $S < R$ is a root system of $\langle S\rangle_{\mathbb{R}}$, then $\alpha,\ \beta \in S$, $\alpha+\beta \in R$ always implies $\alpha+\beta \in S$. Hence, the term "closed" will be omitted from now on.

Table 5.4: $G = E_6$

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
1	D_5	–	-1	-1	yes
2	D_4	1	0	0	yes
3	$A_1 + A_5$	–	-1	-1	yes
4	A_5	3	0	0	no
5	$A_1 + A_4$	3	0	0	no
6	A_4	$1, 4, 5$	1	0	yes
7	$2A_1 + A_3$	$1, 3$	0	0	yes
8	$A_1 + A_3$	$4, 5, 7$	1	1	no
9	A_3	$2, 6, 8$	2	1	yes
10	$3A_2$	–	-1	-1	yes
11	$A_1 + 2A_2$	$3, 10$	0	0	yes
12	$2A_2$	$4, 11$	1	1	no
13	$2A_1 + A_2$	$5, 7, 11$	1	1	yes
14	$A_1 + A_2$	$6, 8, 12, 13$	2	1	yes
15	A_2	$9, 14$	3	2	yes
16	$4A_1$	$2, 7$	1	1	yes
17	$3A_1$	$8, 13, 16$	2	2	yes
18	$2A_1$	$9, 14, 17$	3	2	yes
19	A_1	$15, 18$	4	3	yes
20	$\emptyset$	19	5	4	yes

$H = D_4$: $H \to D_5$. Since $\operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} D_4$, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 5.5.

$H = A_5$: $H \to A_1 + A_5$. Since $A_5^{\perp} = A_1$ by Corollary 5.7, there exists a unique $A_1 + A_5 > A_5$ and $\hat{\Delta}_{G/H}$ is a singleton.

$H = A_1 + A_4$: $H \to A_1 + A_5$. By Corollary 5.7, $A_1^\perp = A_5$. Hence, there is a unique $A_1 + A_5 > H$ and $\hat{\Delta}_{G/H} = \{A_1 + A_5\}$ is a singleton.

$H = A_4$: $H \to D_5,\ A_5,\ A_1 + A_4$. Again, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\hat{\Delta}_{G/H}$ is disconnected.

$H = 2A_1 + A_3$: $H \to D_5,\ A_1 + A_5$. Since there exist at least two different maximal H-subalgebras, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_3$: $H \to A_5,\ A_1 + A_4,\ 2A_1 + A_3$. By Corollary 5.7, $H^\perp = A_1$. Hence, the H-subalgebra $K = 2A_1 + A_3$ is unique. So, $L \in c(K)$ implies $L \geq A_1 + A_4$ or A_5, and hence, $L \leq A_1 + A_5$. Suppose $L = A_1 + A_5$. If $H < A_5 < L$, then the A_1-summand of L is perpendicular to H, i.e. $K < L$. If $H < A_1 + A_4 < L$, then the A_5-summand of L consists a root $A_1(1)$ perpendicular to $A_3 = H \cap A_4$. Moreover, the A_1-summand of L is the A_1-summand of H. Thus, $A_1(1) \perp H$ and $K < L$. It follows that all H-subalgebras of type $A_1 + A_5$ contain K. Therefore, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_3$: $H \to D_4,\ A_5,\ A_1 + A_3$. By Theorem 5.5, $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_3$.

$H = A_1 + 2A_2$: $H \to A_1 + A_5,\ 3A_2$. Thus, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ since $\hat{\Delta}_{G/H}$ is disconnected.

$H = 2A_2$: $H \to A_5,\ A_1 + 2A_2$. By Corollary 5.7, $H^\perp = A_2$, so $K = 3A_2$ is a unique H-subalgebra. If $L \geq A_1 + 2A_2$, then $L \leq K$. So, let $L \geq A_5$. If $L = A_1 + A_5$, H must be contained in the A_5-summand of L and the A_1-summand of L is perpendicular to H. Hence, $L \cap K = A_1 + 2A_2 > H$. It follows that $c(K)$ consists of H-subalgebras of type A_5. Hence, by Corollary 2.19, $\hat{\Delta}_{G/H}$ is contractible, since $\hat{\Delta}_{E_6/A_5}$ is contractible.

$H = 2A_1 + A_2$: $H \to A_1 + A_4,\ 2A_1 + A_3,\ A_1 + 2A_2$. Write $H = A_1(1) + A_1(2) + A_2(3)$. Let $K_0 := A_1(1) + A_1(2) + A_3$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_5(1),\ L_2 = A_1(2) + A_5(2)\}.$$

Let

$$K_1 := A_1(1) + A_2(2) + A_2(3) < L_1 \text{ and}$$
$$K_2 := A_2(1) + A_1(2) + A_2(3) < L_2,$$

where $A_1(1) < A_2(1)$ and $A_1(2) < A_2(2)$. It follows $K_1, K_2 \leq A_2(3) + A_2(3)^\perp < E_6$. Thus, $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_2$: $H \to A_4,\ A_1 + A_3,\ 2A_2,\ 2A_1 + A_2$. Again, $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_1 + A_2$.

$H = A_2$: $H \to A_3,\ A_1 + A_2$. $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_3 > \operatorname{rank} A_2$.

$H = 4A_1$: $H \to D_4,\ 2A_1 + A_3$. Write H as $A_1(1) + A_1(2) + A_1(3) + A_1(4)$. Let $K_0 := A_1(1) + A_1(2) + A_3(1) > H$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_5(1),\ L_2 = A_1(2) + A_5(2)\}.$$

Let

$$K_1 := A_1(1) + A_1(3) + A_3(2) < L_1 \text{ and}$$
$$K_2 := A_1(2) + A_1(3) + A_3(3) < L_2.$$

Thus, $\langle K_1, K_2 \rangle \leq A_1(3) + A_1(3)^\perp < E_6$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1$: $H \to A_1 + A_3,\ 2A_1 + A_2,\ 4A_1$. Write H as $A_1(1) + A_1(2) + A_1(3)$ and let $K_0 := A_1(1) + A_1(2) + A_2(3)$, where $A_1(3) < A_2(3)$. Then $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_1(2) + A_3,\ L_2 = A_1(1) + A_2(2) + A_2(3),\ L_3 = A_2(1) + A_1(2) + A_2(3)\},$$

where $A_1(1) < A_2(1)$, $A_1(2) < A_2(2)$. Now, let $A_3 \ominus A_1(3)$ denote the orthogonal complement of $A_1(3)$ in A_3. By Corollary 5.7, this is a subroot system of type A_1, i.e. there exists $A_1(4) < A_3$ with $A_1(4) \perp A_1(3)$. So, let

$$K_1 := A_1(1) + A_1(2) + A_1(3) + A_1(4) < L_1,$$
$$K_2 := A_1(1) + A_2(2) + A_1(3) < L_2 \text{ and}$$
$$K_3 := A_2(1) + A_1(2) + A_1(3) < L_3.$$

It follows $K_1, K_2, K_3 \leq A_1(3) + A_1(3)^\perp < E_6$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1$: $H \to A_3,\ A_1 + A_2,\ 3A_1$. Again, $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_3 > \operatorname{rank} 2A_1$.

$H = A_1$: $H \to A_2,\ 2A_1$. It follows $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by considering the chain $\operatorname{rank} E_6 > \ldots > \operatorname{rank} 2A_1 > \operatorname{rank} A_1$.

$H = \emptyset$: $H \to A_1$. Since $\operatorname{rank} E_6 > \ldots > \operatorname{rank} A_1 > \operatorname{rank} \emptyset$, it is $\tilde{H}_4(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

5.5 E_7

Borel - de Siebenthal yields the following subroot systems of E_7 up to W-conjugacy by the following table, see [Dyn57, p. 149]. Note, that for $G = E_7$ and E_8 there are subroot systems S which are isomorphic but not W-conjugate. As described in Lemma 5.4, the Weyl obrits are denoted by S' and S''.

Table 5.5: $G = E_7$

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
1	E_6	–	-1	-1	yes
2	$A_1 + D_6$	–	-1	-1	yes
3	D_6	2	0	0	no
4	$A_1 + D_5$	2	0	0	no
5	D_5	$1, 3, 4$	1	0	yes
6	$3A_1 + D_4$	2	0	0	yes
7	$2A_1 + D_4$	$3, 6$	1	1	no
8	$A_1 + D_4$	$4, 7$	2	1	yes
9	D_4	$5, 8$	3	1	yes
10	A_7	–	-1	-1	yes
11	A_6	10	0	0	no
12	$A_2 + A_5$	–	-1	-1	yes
13	$(A_1 + A_5)'$	$1, 2, 10$	0	0	yes
14	$(A_1 + A_5)''$	$2, 12$	0	0	yes
15	$(A_5)'$	$3, 13$	1	1	no
16	$(A_5)''$	$3, 14$	1	1	no
17	$A_2 + A_4$	$10, 12$	0	0	yes
18	$A_1 + A_4$	$4, 11, 13, 14, 17$	1	1	yes
19	A_4	$5, 18, 15, 16$	2	1	yes
20	$A_1 + 2A_3$	2	0	0	no
21	$2A_3$	$3, 10, 20$	1	0	yes
22	$A_1 + A_2 + A_3$	$12, 20$	1	0	yes
23	$A_2 + A_3$	$11, 17, 21, 22$	2	1	yes
24	$3A_1 + A_3$	$4, 6, 20$	1	1	no

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
25	$(2A_1 + A_3)'$	$5, 7, 13, 21, 24$	2	1	yes
26	$(2A_1 + A_3)''$	$7, 14, 22, 24$	2	1	yes
27	$(A_1 + A_3)'$	$8, 15, 18, 23, 25, 26$	3	2	yes
28	$(A_1 + A_3)''$	$8, 16, 26$	3	2	no
29	A_3	$9, 19, 27, 28$	4	2	yes
30	$3A_2$	$1, 12$	0	0	yes
31	$A_1 + 2A_2$	$13, 14, 17, 22, 30$	2	1	yes
32	$2A_2$	$15, 16, 23, 31$	3	2	yes
33	$3A_1 + A_2$	$22, 24$	2	1	yes
34	$2A_1 + A_2$	$18, 23, 25, 26, 31, 33$	3	2	yes
35	$A_1 + A_2$	$19, 27, 28, 32, 34$	4	2	yes
36	A_2	$29, 35$	5	3	yes
37	$7A_1$	6	1	1	yes
38	$6A_1$	$7, 37$	2	2	no
39	$5A_1$	$8, 24, 38$	3	2	no
40	$(4A_1)'$	$9, 25, 39$	4	2	yes
41	$(4A_1)''$	$26, 33, 39$	4	2	yes
42	$(3A_1)'$	$27, 34, 40, 41$	5	3	yes
43	$(3A_1)''$	$28, 41$	5	3	no
44	$2A_1$	$29, 35, 42, 43$	6	3	yes
45	A_1	$36, 44$	7	4	yes
46	$\emptyset$	45	8	5	yes

$H = D_6$: $H \to A_1 + D_6$. Since $D_6^{\perp} = A_1$, $\hat{\Delta}_{G/H} = \{A_1 + D_6\}$ is a singleton.

$H = A_1 + D_5$: $H \to A_1 + D_6$. Since $A_1^{\perp} = D_6$, there is a unique $A_1 + D_6 > H$, so $\hat{\Delta}_{G/H}$ is a singleton.

$H = D_5$: Since $H \to E_6,\ D_6,\ A_1 + D_5$, $\hat{\Delta}_{G/H}$ is disconnected and by Lemma 2.24. Thus, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1 + D_4$: $H \to A_1 + D_6$. Write H as $A_1(1) + A_1(2) + A_1(3) + D_4$. Then $H < A_1(1) + D_6,\ A_1(2) + D_6,\ A_1(3) + D_6$. It follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1 + D_4$: $H \to D_6,\ 3A_1 + D_4$. By Corollary 5.7, $H^\perp = A_1$, so there exists a unique $K = 3A_1 + D_4 > H$. Hence, $L \in c(K)$ implies $L \geq D_6$, so $L = A_1 + D_6$ or D_6. If $L = A_1 + D_6$, then L contains the positive root $A_1 \perp H$, so $K \leq L$. If $L = D_6$, then $L < A_1 + D_6$ and again, this $A_1 + D_6$ must contain K. Hence, $\langle K, L \rangle = A_1(1) + D_6$, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1 + D_4$: $H \to A_1 + D_5,\ 2A_1 + D_4$. By Corollary 5.7, $H^\perp = 2A_1$. So, $K = 3A_1 + D_4 > H$ is unique. Now, if $L \geq A_1 + D_5$, it follows $L \leq A_1 + D_6$. But the complement of H in $A_1 + D_6$ is $2A_1$, so $K \leq A_1 + D_6$ and $\langle K, L \rangle < E_7$. If $L \geq 2A_1 + D_4$, then L contains a root perpendicular to H. So, $K \cap L > H$. It follows $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = D_4$: $H \to D_5,\ A_1 + D_4$. By Theorem 5.5, $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_7 > \operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} D_4$.

$H = A_6$: $H \to A_7$. First, let $A_4 < H$ be fixed. If $H < A_7$, then there is a unique A_2 such that $A_4 < A_2 + A_4 < A_7$. It follows $A_7 = \langle A_2 + A_4, H \rangle$. Hence, the H-subalgebra A_7 is unique and $\hat{\Delta}_{G/H} = \{A_7\}$ is a singleton.

$H = (A_1 + A_5)'$: Since $H \to E_6,\ A_1 + D_6$, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = (A_1 + A_5)''$: $H \to A_1 + D_6,\ A_2 + A_5$. Again, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_5'$: $H \to D_6,\ (A_1 + A_5)'$. By Corollary 5.7, $H^\perp = A_1$, so the H-subalgebra $K = (A_1 + A_5)'$ is unique. Hence, $L \in c(K)$ implies $L \geq D_6 > H$, so $L = A_1 + D_6$ or D_6. As in the case $H = 2A_1 + D_4$, it follows $K \leq L$ for $L = A_1 + D_6$ and $\langle K, L \rangle = A_1 + D_6$ for $L = D_6$. Hence, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_5''$: $H \to D_6,\ (A_1 + A_5)''$. Since $A_5'' < (A_1 + A_5)'' < A_2 + A_5$, it follows $H^\perp = A_2$ by Corollary 5.7 so $K = A_2 + A_5 > H$ is unique. Furthermore, all H-subalgebras of type $(A_1 + A_5)''$ are contained in $H + H^\perp = K$. So, $L \in c(K)$ implies $L \geq D_6 > H$, i.e. $L = A_1 + D6$ or D_6. In the first case, L contains a positive root $A_1 \perp H$, so $K \cap L \geq A_1 + H > H$. It follows that $c(K)$ consists of H-subalgebras of type D_6. Thus, $\hat{\Delta}_{G/H}$ is contractible by Corollary 5.7, since $\hat{\Delta}_{E_7/D_6}$ is contractible.

$H = A_2 + A_4$: $H \to A_7,\ A_2 + A_5$. It follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1 + A_4$: $H \to A_1 + D_5,\ A_6,\ (A_1 + A_5)',\ (A_1 + A_5)'',\ A_2 + A_4$. Write $H = A_1(1) + A_4$. Let $K_0 := A_2 + A_4$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by facet with l.b.s.

$$\{L_1 = A_7,\ L_2 = A_2 + A_5\}.$$

Now, let $A_5(1) = A_7 \ominus A_1(1) \cong A_5'$ and $A_5(2) = (A_2 + A_5) \ominus A_1(1) \cong A_5''$. Let

$$K_1 := A_1(1) + A_5(1) \cong (A_1 + A_5)' < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_5(2) \cong (A_1 + A_5)'' < L_2.$$

It follows, $K_1, K_2 \leq A_1(1) + A_1(1)^\perp < E_7$, so $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_4$: $H \to D_5,\ A_5',\ A_5'',\ A_1 + A_4$. Since $\operatorname{rank} E_7 > \operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4$, it is $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + 2A_3$: $H \to A_1 + D_6$. Since $A_1^\perp = D_6$, there is a unique $A_1 + D_6 > H$ and $\hat{\Delta}_{G/H}$ is a singleton.

$H = 2A_3$: $H \to D_6,\ A_7,\ A_1 + 2A_3$. Hence, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1 + A_2 + A_3$: $H \to A_2 + A_5,\ A_1 + 3A_3$. By Lemma 2.24, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_2 + A_3$: $H \to A_6,\ A_2 + A_4,\ 2A_3,\ A_1 + A_2 + A_3$. For $K_0 := A_2 + A_4$, $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by facet with l.b.s.

$$\{L_1 = A_7,\ L_2 = A_2 + A_5\}.$$

Let

$$K_1 := A_3 + A_3 < L_1 \text{ and}$$
$$K_2 := A_2 + A_3 + A_1 < L_2.$$

By considering the A_3-summand of H, it follows $K_1, K_2 \leq A_3 + A_3^\perp < E_7$, so $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1 + A_3$: $H \to A_1 + D_5,\ 3A_1 + D_4,\ A_1 + 2A_3$. By Corollary 5.7, $3A_1^\perp = D_4$, so there is a unique $K = 3A_1 + D_4 > H$. Now, write $H = A_1(1) + A_1(2) + A_1(3) + A_3$. All minimal H-subalgebras but K are of type $A_1(k) + D_5$ or $A_1(k) + 2A_3$ for some $1 \leq k \leq 3$ and thus, all maximal H-subalgebras are of type $A_1(k) + D_6$. It follows that all maximal H-subalgebras contain K and therefore, $\langle K, L\rangle < E_7$ for all H-subalgebras L. This shows $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = (2A_1 + A_3)'$: $H \to D_5,\ 2A_1 + D_4,\ (A_1 + A_5)',\ 2A_3,\ 3A_1 + A_3$. Let $H = A_1(1) + A_1(2) + A_3$ and $K_0 := A_1(1) + A_5(1) \cong (A_1 + A_5)'$. Then $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + D_6,\ L_2 = E_6\}.$$

Let $A_5(2) := E_6 \ominus A_1(2)$, so $A_5(2) \neq E_6 \ominus A_1(1) = A_5(1)$. It follows

$$K_1 := A_1(1) + A_1(2) + D_4 < L_1 \text{ and}$$
$$K_2 := A_1(2) + A_5(2) \cong (A_1 + A_5)' < L_2.$$

Thus, $\langle K_1, K_2 \rangle \leq A_1(2) + A_1(2)^\perp < E_7$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (2A_1 + A_3)''$: $H \to 2A_1 + D_4,\ 3A_1 + A_3,\ A_1 + A_2 + A_3$. Let $H = A_1(1) + A_1(2) + A_3$ and $K_0 := A_1(1) + A_5 \cong (A_1 + A_5)''$. $\hat{\Delta}_{G/K_0}$ contains a non-trivial 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + D_6,\ L_2 = A_2 + A_5\}.$$

Let

$$K_1 := A_1(1) + A_1(2) + D_4 < L_1 \text{ and}$$
$$K_2 := A_2 + A_1(2) + A_3 < L_2.$$

Since $K_1, K_2 < A_1(2) + A_1(2)^\perp < E_7$, it follows $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (A_1 + A_3)'$: $H \to A_1 + D_4,\ A_5',\ A_1 + A_4,\ A_2 + A_3,\ (2A_1 + A_3)',\ (2A_1 + A_3)''$,. Let $H = A_1(1) + A_3$ and $K_0 := A_1(1) + A_4$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2 + A_4,\ L_2 = A_1(1) + A_5(1) \cong (A_1 + A_5)',\ L_3 = A_1(1) + A_5(2) \cong (A_1 + A_5)''\}.$$

Furthermore, let $A_1(2) := A_5(1) \ominus A_3$ and $A_1(3) := A_5(2) \ominus A_3$. Then $A_1(2), A_1(3) \neq A_1(1)$ and

$$K_1 := A_2 + A_3 < L_1,$$
$$K_2 := A_1(1) + A_1(2) + A_3 \cong (2A_1 + A_3)' < L_2 \text{ and}$$
$$K_3 := A_1(1) + A_1(3) + A_3 \cong (2A_1 + A_3)'' < L_3.$$

It follows $\langle K_1, K_2, K_3 \rangle \leq A_3 + A_3^\perp < E_7$. Hence, $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (A_1 + A_3)''$: $H \to A_1 + D_4,\ A_5'',\ (2A_1 + A_3)''$. Let $H = A_1(1) + A_3$ and $K := A_1(1) + A_1(1)^\perp = A_1 + D_6$. Let L be any H-subalgebra. $L \notin c(K)$. If $L \geq A_1 + D_4$ or $L \geq (2A_1 + A_3)''$, then L contains a root perpendicular to $A_1(1)$, i.e.

$K \cap L > H$ and $L \notin c(K)$. So, assume $L \geq A_5''$. If $L > A_5''$, then $L \geq (A_1 + A_5)''$ or $L \geq D_6$. In both cases, L contains a root perpendicular to H, so $K \cap L > H$. It follows that $c(K)$ consists of subroot systems of type A_5''. But $\hat{\Delta}_{E_7/A_5''}$ is contractible, so is $\hat{\Delta}_{G/H}$.

$H = A_3$: $H \to D_4,\ A_4,\ (A_1 + A_3)',\ (A_1 + A_3)''$. Since $\operatorname{rank} E_7 > \operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_3$, it follows $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Theorem 5.5.

$H = 3A_2$: $H \to E_6,\ A_2 + A_5$. Thus, by Lemma 2.24, it is $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + 2A_2$: $H \to (A_1 + A_5)',\ (A_1 + A_5)'',\ A_2 + A_4,\ A_1 + A_2 + A_3,\ 3A_2$. Write $H = A_1(1) + A_2(2) + A_2(3)$ and let $K_0 := 3A_2 = A_2(1) + A_2(2) + A_2(3)$, i.e. $A_1(1) < A_2(1)$. $\hat{\Delta}_{G/H}$ contains a 0-cycle supported by facet with l.b.s.

$$\{L_1 = A_2(1) + A_5(1),\ L_2 = E_6\}.$$

Now, let $A_5(2) := E_6 \ominus A_1(1)$ and let

$$\begin{aligned} K_1 &:= A_1(1) + A_5(1) = (A_1 + A_5)'' < L_1, \\ K_2 &:= A_1(1) + A_5(2) = (A_1 + A_5)' < L_2. \end{aligned}$$

It follows $K_1, K_2 < A_1(1) + A_1(1)^{\perp} < E_7$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_2$: $H \to A_5',\ A_5'',\ A_2 + A_3,\ A_1 + 2A_2$. Let $H = A_2(1) + A_2(2)$. If $K_0 := A_3(1) + A_2(2)$, i.e. $A_2(1) < A_3(1)$, then $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2(2) + A_4,\ L_2 = A_3(1) + A_3(2),\ L_3 = A_3(1) + A_2(2) + A_1(1)\}$$

with $A_2(2) < A_3(2)$. Now, let $A_1(2) := A_4 \ominus A_2(1)$ and

$$\begin{aligned} K_1 &:= A_2(1) + A_2(2) + A_1(2) < L_1, \\ K_2 &:= A_2(1) + A_3(2) < L_2 \text{ and} \\ K_3 &:= A_2(1) + A_2(2) + A_1(1) < L_3. \end{aligned}$$

Thus, $\langle K_1, K_2, K_3 \rangle \leq A_2(1) + A_2(1)^{\perp} < E_7$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1 + A_2$: $H \to A_1 + A_2 + A_3,\ 3A_1 + A_3$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_2(1)$ and $K_0 = A_1(1) + A_3(1) + A_2(1)$, i.e. $A_1(2),\ A_1(3) < A_3(1)$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_5 + A_2(1),\ L_2 = A_1(1) + A_3(1) + A_3(2)\},$$

i.e. $A_2(1) < A_3(2)$. Now, let $A_5 \ominus A_1(2) := A_3(3)$ and $L_2 \ominus A_1(1) + A_1(3) + A_3$, so let

$$K_1 := A_1(2) + A_3(3) + A_2(1) < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(2) + A_1(3) + A_3(2) < L_2.$$

It follows $\langle K_1, K_2 \rangle < A_1(2) + A_1(2)^\perp < E_6$, so $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1+A_2$: $H \to A_1+A_4,\ A_2+A_3,\ (2A_1+A_3)',\ (2A_1+A_3)'',\ A_1+2A_2,\ 3A_1+A_2$. Let $H = A_1(1) + A_1(2) + A_2$ and $K_0 := A_1(1) + A_2 + A_2$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = 3A_2,\ L_2 = A_1(1) + A_5(1) \cong (A_1 + A_5)',\ L_3 = A_1(1) + A_5(2) \cong (A_1 + A_5)''\}.$$

Note, that for $i \in \{1, 2\}$, $A_5(i) \ominus A_1(2) \cong A_3$. So, denote this orthogonal complement by $A_3(i)$. Let

$$K_1 := A_1(2) + A_2 + A_2 < L_1,$$
$$K_2 := A_1(1) + A_1(2) + A_3(1) \cong (2A_1 + A_3)' < L_2 \text{ and}$$
$$K_3 := A_1(1) + A_1(2) + A_3(2) \cong (2A_1 + A_3)'' < L_3.$$

So, $\langle K_1, K_2, K_3 \rangle A_1(2) + A_1(2)^\perp < E_7$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_2$: $H \to A_4,\ (A_1 + A_3)',\ (A_1 + A_3)'',\ 2A_1 + A_2,\ 2A_2$. As mentioned above, this case in an example for $\tilde{H}_k(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ for more than one k.

On one hand, $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_7 > \operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_1 + A_2$. On the other hand, let $H = A_1(1) + A_2(2)$ and consider $K_0 := 2A_2 = A_2(1) + A_2(2)$, so $A_1(1) < A_2(1)$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_3(1) + A_2(2),\ L_2 = A_2(1) + A_3(2),$$
$$L_3 = A_2(1) + A_2(2) + A_1(2),\ L_4 = A_2(1) + A_2(2) + A_1(3)\}$$

with $A_1(2),\ A_1(3) \neq A_1(1)$. If $A_1(4) := A_3(1) \ominus A_1(1)$, then

$$K_1 := A_1(1) + A_1(4) + A_2(2) < L_1,$$
$$K_2 := A_1(1) + A_3(2) < L_2,$$
$$K_3 := A_1(1) + A_2(2) + A_1(2) < L_3 \text{ and}$$
$$K_4 := A_1(1) + A_2(2) + A_1(3) < L_4.$$

Since $\langle K_1, K_2, K_3, K_4 \rangle \leq A_1(1) + A_1(1)^\perp < E_8$, it follows $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_2$: $H \to A_3,\ A_1 + A_2$. It is $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\operatorname{rank} E_7 > \operatorname{rank} E_6 > \operatorname{rank} D_5 > \operatorname{rank} A_4 > \operatorname{rank} A_3 > \operatorname{rank} A_2$.

$H = 7A_1$: $H \to 3A_1 + D_4$. Write $H = A_1(1) + \ldots + A_1(7)$ such that $K_0 := A_1(1)+A_1(2)+A_1(3)+D_4(1)$ is an H-subalgebra. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + D_6(1),\ L_2 = A_1(2) + D_6(1)\}.$$

Now, there exists a $p \in \{5, 6, 7\}$ and a $D_4(2)$ such that $A_1(4)+A_1(p)+D_4(2) < D_6(1)$. To see this, first note, that the T-subalgebras of type $2A_1$ of D_6 are not $W(D_6)$-conjugate, see [Dyn57, p. 146]. In fact, there are two orbits represented by $\mathfrak{so}(4) = \mathfrak{t} \oplus (\mathfrak{m}_{12}^+ \oplus \mathfrak{m}_{12}^-)$ and $\mathfrak{u}(2)^2 = \mathfrak{t} \oplus (\mathfrak{m}_{12}^- \oplus \mathfrak{m}_{34}^-)$. In the first case, $D_6 \ominus 2A_1 = D_4$ and in the second case, $D_6 \ominus 2A_1 = 4A_1$.

Now, $A_1(2) + A_1(3) < D_6(1)$ represents a subalgebra of type $\mathfrak{so}(4)$. Furthermore, $A_1(2)+\ldots+A_1(7) = 6A_1 < D_6(1)$ represents a subalgebra of type $\mathfrak{so}(4)^3$. Hence, there is a (unique) $p \in \{5, 6, 7\}$ such that $A_1(4) + A_1(p) < D_6(1)$ represents a subalgebra of type $\mathfrak{so}(4)$.

Similarly, there exists a $q \in \{5, 6, 7\}$ and a $D_4(3)$ such that $A_1(4)+A_1(q)+D_4(3) < D_6(2)$. So, let

$$K_1 := A_1(1) + A_1(4) + A_1(p) + D_4(2) < L_1, \text{ and}$$
$$K_2 := A_1(2) + A_1(4) + A_1(q) + D_4(3) < L_2.$$

Thus, $\langle K_1, K_2\rangle \leq A_1(4) + A_1(4)^\perp < E_7$, so $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 6A_1$: $H \to 2A_1 + D_4,\ 7A_1$. It is $H^\perp = A_1$ by Corollary 5.7, so there exists a unique $K = 7A_1 > H$. Now, let L be any H-subalgebra. If $L \in c(K)$, then $L \geq 2A_1 + D_4$. Thus, L must be contained in a maximal subroot system of type A_1+D_6. But $A_1+D_6\ominus 6A_1 = A_1$, i.e. A_1+D_6 contains the positive root perpendicular to H and $K < A_1 + D_6$. Hence, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = 5A_1$: $H \to A_1 + D_4,\ 3A_1 + A_3,\ 6A_1$. Since $H^\perp = 2A_1$ by Corollary 5.7, $K = 7A_1 > H$ is unique. Let L be any H-subalgebra. If $L \geq A_1 + D_4$ or $L \geq 3A_1 + A_3$, then L must be contained in a maximal subroot system of type $A_1 + D_6$. But $A_1 + D_6 \ominus 5A_1 = 2A_1$, so $K < A_1 + D_6$ and hence, $L \notin c(K)$. Otherwise, if $L \geq 6A_1$, then $K \cap L \geq 6A_1 > H$. It follows $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = (4A_1)'$: $H \to D_4,\ 5A_1,\ (2A_1 + A_3)'$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_1(4)$ and $K_0 := A_1(1) + A_1(2) + A_3(1) \cong (2A_1 + A_3)'$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_5(1) \cong (A_1 + A_5)',\ L_2 = A_1(2) + A_5(2) \cong (A_1 + A_5)',$$
$$L_3 = A_1(1) + A_1(2) + D_4\}.$$

Now, let $A_3(2) := A_5(1) \ominus A_1(3)$. Then $A_1(2) < A_3(2)$. In particular, $A_3(2) \neq A_3(1)$. Similarly, $A_3(3) := A_5(2) \ominus A_1(3)$ satisfies $A_1(1) < A_3(3) \neq A_3(1)$. Furthermore, let $A_1(5)$ be any root contained in D_4 and perpendicular to $A_1(3) + A_1(4)$. Then

$$\begin{aligned} K_1 &:= A_1(1) + A_1(3) + A_3(2) \cong (2A_1 + A_3)' < L_1, \\ K_2 &:= A_1(2) + A_1(3) + A_3(3) \cong (2A_1 + A_3)' < L_2 \text{ and} \\ K_3 &:= H + A_1(5) \cong 5A_1 < L_3 \end{aligned}$$

satisfy $\langle K_1, K_2, K_3\rangle A_1(3) + A_1(3)^\perp < E_7$. Thus, $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (4A_1)''$: $H \to (2A_1 + A_3)''$, $3A_1 + A_2$, $5A_1$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_1(4)$, $K_0 := A_1(1) + A_1(2) + A_1(3) + A_2$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1)+A_3(1)+A_2,\ L_2 = A_1(2)+A_3(2)+A_2,\ L_3 = A_1(1)+A_1(2)+A_1(3)+A_3(3)\}.$$

Let $A_1(5)$ be any root contained in $A_3(3)$ and perpendicular to $A_1(4)$, i.e. $A_1(5) \perp H$. Let

$$\begin{aligned} K_1 &:= A_1(1) + A_3(1) + A_1(4) \cong (2A_1 + A_3)'' < L_1, \\ K_2 &:= A_1(2) + A_3(2) + A_1(4) \cong (2A_1 + A_3)'' < L_2 \text{ and} \\ K_3 &:= H + A_1(5) \cong 5A_1 < L_3. \end{aligned}$$

It follows $\langle K_1, K_2, K_3\rangle A_1(4) + A_1(4)^\perp < E_7 \cong A_1 + D_6$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (3A_1)'$: $H \to (A_1 + A_3)'$, $2A_1 + A_2$, , $(4A_1)'$, $(4A_1)''$. Let $H = A_1(1) + A_1(2) + A_1(3)$ and $K_0 := A_1(1) + A_1(2) + A_2(3)$ with $A_1(3) < A_2(3)$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\begin{aligned} \{L_1 &= A_1(1) + A_2(2) + A_2(3),\ L_2 = A_1(2) + A_2(1) + A_2(3), \\ L_3 &= A_1(1) + A_1(2) + A_3(1) \cong (2A_1 + A_3)', \\ L_4 &= A_1(1) + A_1(2) + A_3(2) \cong (2A_1 + A_3)''\}, \end{aligned}$$

where $A_1(1) < A_2(1)$ and $A_1(2) < A_2(2)$. Furthermore, let $A_1(4) := A_3(1) \ominus A_1(3)$ and $A_1(5) := A_3(2) \ominus A_1(3)$. It follows

$$\begin{aligned} K_1 &:= A_1(1) + A_2(2) + A_1(3) < L_1, \\ K_2 &:= A_2(1) + A_1(2) + A_1(3) < L_2, \\ K_3 &:= H + A_1(4) \cong (4A_1)' < L_3 \text{ and} \\ K_4 &:= H + A_1(5) \cong (4A_1)'' < L_4. \end{aligned}$$

Thus, $\langle K_1, K_2, K_3, K_4\rangle \leq A_1(3) + A_1(3)^\perp < E_7$ and $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (3A_1)''$: $H \to (A_1 + A_3)''$, $(4A_1)''$. Let $H = A_1(1) + A_1(2) + A_1(3)$ and $K := A_1(1) + A_1(1)^\perp = A_1 + D_6$ and let L be any H-subalgebra. If $L \geq (4A_1)''$, it contains a root perpendicular to H. In particular, $K \cap L > H$. So, let $L \geq (A_1 + A_3)''$. If $L > (A_1 + A_3)''$, then $L \geq A_1 + D_4$, $(A_5)''$ or $(2A_1 + A_3)''$. Hence, $\mathrm{rank}(L \ominus A_1(1)) \geq 3$. In particular, L contains a root $A_1(4) \neq A_1(2)$, $A_1(3)$, $A_1(4) \perp A_1(1)$. Thus, $K \cap L > H$. It follows that $c(K)$ consists of subroot systems of type $(A_1 + A_3)''$ and $\hat{\Delta}_{G/H}$ is contractible, since $\hat{\Delta}_{E_7/(A_1+A_3)''}$ is contractible.

$H = 2A_1$: $H \to A_3$, $A_1 + A_2$, $(3A_1)'$, $(3A_1)''$. Since $\mathrm{rank}\, E_7 > \mathrm{rank}\, E_6 > \mathrm{rank}\, D_5 > \mathrm{rank}\, A_4 > \mathrm{rank}\, A_3 > \mathrm{rank}\, 2A_1$, it is $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1$: $H \to A_2$, $2A_1$. It follows $\tilde{H}_4(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\mathrm{rank}\, E_7 > \ldots > \mathrm{rank}\, 2A_1 > \mathrm{rank}\, A_1$.

$H = \emptyset$: $H \to A_1$. As above, $\tilde{H}_5(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since $\mathrm{rank}\, E_7 > \ldots > \mathrm{rank}\, A_1 > \mathrm{rank}\, \emptyset$.

5.6 E_8

By Borel - de Siebenthal, the subroot systems of E_8 are given up to W-conjugacy by the following table, see [Dyn57, p. 149]. Note, that Theorem 5.5 is not applicable for T-subalgebras of E_8, since every *maximal* subroot systems $S < E_8$ has rank 8.

Moreover, the case $G = E_8$, $H = T^8$ is the only case where the non-contractibility of $\hat{\Delta}_{G/H}$ cannot be shown by Theorem 2.27. In fact, since every maximal subroot system has rank 8, every maximal chain $(\mathfrak{k}_0^0 > \ldots > \mathfrak{k}_l^0)$ of T-subalgebras has length $l \geq 7$. Suppose, Theorem 2.27 would be applicable to such a chain. Let $\mathfrak{k}_l^1, \ldots, \mathfrak{k}_l^{l+1}$ be minimal T-subalgebras as in Theorem 2.27. For $i \in \{0, \ldots, l+1\}$ it is $R(\mathfrak{k}_l^i) = \{r_i, -r_i\}$ for a root r_i, since the $\mathfrak{k}_l^i$ are minimal. Now, let $\mathfrak{k}_j^i$, $0 \leq j \leq l-1$, $1 \leq i \leq j+1$, be as in Theorem 2.27. The conditions of Theorem 2.27 imply $\mathfrak{k}_j^0 \geq \langle \mathfrak{k}_l^0, \ldots, \mathfrak{k}_l^{l-j} \rangle$ and $\mathfrak{k}_j^i \geq \langle \mathfrak{k}_{j+1}^0, \mathfrak{k}_j^{l-j+i} \rangle$ for $i \geq 1$. Since $\langle \mathfrak{k}_j^1, \ldots, \mathfrak{k}_j^{j+1} \rangle < \mathfrak{e}_8$ for all $0 \leq j \leq l-1$, it follows

$$E_8 \neq R(\langle \mathfrak{k}_j^1, \ldots, \mathfrak{k}_j^{j+1} \rangle) \geq \langle R(\mathfrak{k}_{j+1}^0), r_{l-j+1}, \ldots, r_{l+1} \rangle \geq \langle r_0, \ldots, \widehat{r_{l-j}}, \ldots, r_{l+1} \rangle.$$

Moreover, $\langle \mathfrak{k}_l^l, \ldots, \mathfrak{k}_l^{l+1} \rangle < \mathfrak{e}_8$ implies $\langle r_1, \ldots, r_{l+1} \rangle \neq E_8$ and from $\mathfrak{k}_0^0 \neq \mathfrak{e}_8$ it follows $\langle r_0, \ldots, r_l \rangle \neq E_8$. So, $\langle r_0, \ldots, \widehat{r_k}, \ldots, r_{l+1} \rangle \neq E_8$ for all $k \in \{0, \ldots, l+1\}$. On the other hand, $\langle r_0, \ldots, r_{l+1} \rangle = E_8$, since otherwise, $\langle \mathfrak{k}_l^0, \ldots, \mathfrak{k}_l^{l+1} \rangle < \mathfrak{e}_8$. But then, the subcomplex $\cap_{i=0}^{l+1} \{\sigma \in \hat{\Delta}_{G/H} \mid \min \sigma \geq \mathfrak{k}_l^i\}$ would be contractible by Corollary 2.4 and the l-cycle θ constructed in Theorem 2.26 would be a cycle of this subcomplex, hence, a boundary. By [Kan01, p. 138], it follows

$$\langle r_0, \ldots, r_{l+1} \rangle_{\mathbb{Z}} = \mathbb{Z}E_8 \text{ and}$$
$$\langle r_0, \ldots, \widehat{r_k}, \ldots, r_{l+1} \rangle_{\mathbb{Z}} \neq \mathbb{Z}E_8 \text{ for all } k.$$

But Theorem 1.1 of [Goo07] states, that each set of roots which generates the root lattice contains a $\mathbb{Z}$-basis of the root lattice. Since $l+1 \geq 8$, there must exist at least one k with $\langle r_0, \ldots, \widehat{r_k}, \ldots, r_{l+1}\rangle_{\mathbb{Z}} = \mathbb{Z}E_8$. This is a contradiction, so Theorem 2.27 is not applicable. However, at the end of this section, the reduced Euler characteristic of $\hat{\Delta}_{E_8/T^8}$ will computed to prove the non-contractibility.

Table 5.6: $G = E_8$

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
1	$A_1 + E_7$	–	-1	-1	yes
2	E_7	1	0	0	no
3	$A_2 + E_6$	–	-1	-1	yes
4	$A_1 + E_6$	$1, 3$	0	0	yes
5	E_6	$2, 4$	1	1	no
6	D_8	–	-1	-1	yes
7	D_7	6	0	0	no
8	$2A_1 + D_6$	$1, 6$	0	0	yes
9	$A_1 + D_6$	$2, 8$	1	1	no
10	D_6	$7, 9$	2	1	no
11	$A_3 + D_5$	6	0	0	no
12	$A_2 + D_5$	$3, 11$	1	0	yes
13	$2A_1 + D_5$	$7, 8, 11$	1	1	no
14	$A_1 + D_5$	$4, 9, 12, 13$	2	1	yes
15	D_5	$5, 10, 14$	3	2	no
16	$2D_4$	6	0	0	yes
17	$A_3 + D_4$	$7, 11, 16$	1	1	no
18	$A_2 + D_4$	$12, 17$	2	1	yes
19	$4A_1 + D_4$	$8, 16$	1	1	yes
20	$3A_1 + D_4$	$9, 19$	2	2	no

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
21	$2A_1 + D_4$	$10, 13, 17, 20$	3	2	no
22	$A_1 + D_4$	$14, 18, 21$	4	2	yes
23	D_4	$15, 22$	5	3	no
24	A_8	–	-1	-1	yes
25	$A_1 + A_7$	1	0	0	no
26	A_7'	$6, 24$	0	0	yes
27	A_7''	$2, 6, 25$	1	0	yes
28	$A_1 + A_6$	$24, 25$	1	0	yes
29	A_6	$7, 26, 27, 28$	2	1	yes
30	$A_1 + A_2 + A_5$	$1, 3$	0	0	yes
31	$A_2 + A_5$	$2, 24, 30$	1	0	yes
32	$2A_1 + A_5$	$4, 8, 25, 30$	1	1	yes
33	$(A_1 + A_5)'$	$9, 26, 28, 31, 32$	2	1	yes
34	$(A_1 + A_5)''$	$5, 9, 27, 32$	2	1	yes
35	A_5	$10, 29, 33, 34$	3	2	yes
36	$2A_4$	–	-1	-1	yes
37	$A_3 + A_4$	$11, 24, 36$	1	0	yes
38	$A_1 + A_2 + A_4$	$25, 30, 36$	1	0	yes
39	$A_2 + A_4$	$12, 26, 27, 31, 37, 38$	2	1	yes
40	$2A_1 + A_4$	$13, 28, 32, 37, 38$	2	1	yes
41	$A_1 + A_4$	$14, 29, 33, 34, 39, 40$	3	2	yes
42	A_4	$15, 35, 41$	4	3	yes
43	$2A_1 + 2A_3$	$8, 11$	1	1	no
44	$A_1 + 2A_3$	$9, 25, 43$	2	1	no
45	$(2A_3)'$	$17, 26, 37$	2	1	yes
46	$(2A_3)''$	$10, 17, 27, 44$	3	1	yes

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
47	$2A_1 + A_2 + A_3$	$12, 30, 43$	2	1	yes
48	$A_1 + A_2 + A_3$	$28, 31, 37, 38, 44, 47$	3	1	yes
49	$A_2 + A_3$	$18, 29, 39, 45, 46, 48$	4	2	yes
50	$4A_1 + A_3$	$13, 17, 19, 43$	2	2	no
51	$3A_1 + A_3$	$14, 20, 32, 44, 47, 50$	3	2	yes
52	$(2A_1 + A_3)'$	$21, 33, 40, 45, 48, 51$	4	2	yes
53	$(2A_1 + A_3)''$	$15, 21, 34, 46, 51$	4	2	yes
54	$A_1 + A_3$	$22, 35, 41, 49, 52, 53$	5	3	yes
55	A_3	$23, 42, 54$	6	4	yes
56	$4A_2$	3	0	0	yes
57	$A_1 + 3A_2$	$4, 30, 56$	1	1	yes
58	$3A_2$	$5, 31, 57$	2	1	yes
59	$2A_1 + 2A_2$	$32, 38, 47, 57$	3	1	yes
60	$A_1 + 2A_2$	$33, 34, 39, 48, 58, 59$	4	2	yes
61	$2A_2$	$35, 49, 60$	5	3	yes
62	$4A_1 + A_2$	$18, 47, 50$	3	2	yes
63	$3A_1 + A_2$	$40, 48, 51, 59, 62$	4	2	yes
64	$2A_1 + A_2$	$41, 49, 52, 53, 60, 63$	5	3	yes
65	$A_1 + A_2$	$42, 54, 61, 64$	6	4	yes
66	A_2	$55, 65$	7	5	yes
67	$8A_1$	19	2	2	yes
68	$7A_1$	$20, 67$	3	3	no
69	$6A_1$	$50, 21, 68$	4	3	no
70	$5A_1$	$22, 51, 62, 69$	5	3	yes
71	$(4A_1)'$	$52, 63, 70$	6	3	yes
72	$(4A_1)''$	$23, 53, 70$	6	3	yes

No.	H	maximally contained in	$\dim \hat{\Delta}_{G/H}$	$m_{G/H}$	$\hat{\Delta}_{G/H}$ is non-contractible
73	$3A_1$	$54, 64, 71, 72$	7	4	yes
74	$2A_1$	$55, 65, 73$	8	5	yes
75	A_1	$66, 74$	9	6	yes
76	$\emptyset$	75	10	7	yes

$H = E_7$: $H \to A_1 + E_7$. Since $E_7^\perp = A_1$, there exists a unique $A_1 \perp H$. Hence, $\hat{\Delta}_{G/H} = \{A_1 + E_7\}$ is a singleton.

$H = A_1 + E_6$: Since $H \to A_1 + E_7,\ A_2 + E_6$, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, Q) \neq 0$ by Lemma 2.24.

$H = E_6$: $H \to E_7,\ A_1 + E_6$. Since $H^\perp = A_2$, $K := A_2 + E_6 > H$ is unique. Let L be any H-subalgebra. If $L \geq A_1 + E_6$, it contains a root perpendicular to H and $K \cap L > H$. So, let $L \geq E_7$, i.e. $L = E_7$ or $A_1 + E_7$. In the second case, it follows $K \cap L > H$ as above. Hence, $c(K)$ consists of $E_7's$ and by Corollary 2.19, $\hat{\Delta}_{G/H}$ is contractible since $\hat{\Delta}_{E_8/E_7}$ is contractible.

$H = D_7$: $H \to D_8$. Fix a $D_4 < H$ and let $K := D_4 + D_4^\perp = 2D_4$. If $D_4 < H < D_8$, then $K < D_8$, since $D_8 \ominus D_4 = D_4$. Therefore, $\langle H, K \rangle = D_8$. In particular, the subroot system $D_8 > H$ is unique and $\hat{\Delta}_{G/H}$ is a singleton.

$H = 2A_1 + D_6$: $H \to A_1 + E_7,\ D_8$. Thus, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1 + D_6$: $H \to E_7,\ 2A_1 + D_6$. Since $H^\perp = A_1$, there exists a unique $K := 2A_1 + D_6 > H$. $L \in c(K)$ implies $L \geq E_7$, so $L = E_7$ or $L = A_1 + E_7$. If $L = A_1 + E_7$, it contains $H^\perp$, so $K < L$. If $L = E_7$, it follows $\langle K, L \rangle = \langle H^\perp, L \rangle = A_1 + E_7$. Thus, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = D_6$: $H \to D_7,\ A_1 + D_6$. Since $H^\perp = 2A_1$, there exists a unique $K := 2A_1 + D_6 > H$. Hence, if L is an H-subalgebra with $L \geq A_1 + D_6$, then $K \cap L \geq A_1 + D_6$ and $L \notin c(K)$. If $L \geq D_7$, then L is contained in a subroot system of type D_8. But $D_8 \ominus D_6 = 2A_1$, so $K < D_8$. It follows $\langle K, L \rangle \leq D_8$, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_3 + D_5$: $H \to D_8$. Fix a $D_4 < H$ and let $K := D_4 + D_4^\perp = 2D_4$. As in the case $H = D_7$, $D_4 < H < D_8$ implies $K < D_8$ and $\langle H, K \rangle = D_8$ is unique. Thus, $\hat{\Delta}_{G/H}$ is a singleton.

$H = A_2 + D_5$: $H \to A_2 + E_6,\ A_3 + D_5$: Since $A_2 + E_6$ is maximal, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = 2A_1 + D_5$: $H \to D_7,\ 2A_1 + D_6,\ A_3 + D_5$. Since $2A_1^\perp = D_6$, there exists a unique $K := 2A_1 + D_6 > H$. Now, $L \in c(K)$ implies $L \geq A_3 + D_5$ or $L \geq D_7$, so $L = A_3 + D_5,\ D_7$ or D_8. If $L = D_8$, then L must contain K, so $L \notin c(K)$. If $L = A_3 + D_5$ or D_7 it must be contained in a subroot system of type D_8. Therefore, $\langle K, L\rangle = D_8$. Hence, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1 + D_5$: $H \to A_1 + E_6,\ A_1 + D_6,\ A_2 + D_5,\ 2A_1 + D_5$. Write $H = A_1(1) + D_5$ and let $K_0 := A_1(1) + E_6$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2 + E_6,\ L_2 = A_1(1) + E_7\}.$$

Let $A_1(2) := E_7 \ominus D_5$. Then

$$\begin{aligned} K_1 &:= A_2 + D_5 < L_1 \text{ and} \\ K_2 &:= A_1(1) + A_1(2) + D_5 < L_2. \end{aligned}$$

Since $\langle K_1, K_2\rangle \leq D_5 + D_5^\perp < E_8$, it follows $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = D_5$: $H \to E_6,\ D_6,\ A_1 + D_5$. It is $H^\perp = A_3$, so there exists a unique $K := A_3 + D_5 > H$. Let L be any H-subalgebra. This implies $L \leq D_8$ except for the cases $L = E_6,\ A_1 + E_6,\ A_2 + E_6,\ E_7$ and $A_1 + E_7$.

First, let $L \leq D_8$. By Corollary 5.7, it is $D_8 \ominus D_5 = A_3$. Hence, $K < D_8$, so $\langle K, L\rangle \leq D_8$. If $L = A_1 + E_6,\ A_2 + E_6$ or $A_1 + E_7$, then $H < E_6$ ($H < E_7$) and L contains roots perpendicular to H. Thus, $K \cap L > H$. If $L = E_7$, then $L \ominus H = A_1$ and again, $K \cap L > H$. It follows that $c(K)$ consists of subroot systems of type E_6. But $\hat{\Delta}_{E_8/E_6}$ is contractible. Thus, $\hat{\Delta}_{G/H}$ is contractible.

$H = 2D_4$: $H \to D_8$. Write H as $D_4(1) + D_4(2)$. By Lemma 2.24, it remains to show that there exist at least two H-subalgebras of type D_8 containing H. For this purpose, fix a $H < D_8(1)$ and fix a $2A_1(1) < D_4(1) < H < D_8(1)$.

The T-subalgebras of type $2A_1$ of $D_8(1)$ split into two $W(D_8)$-orbits represented by $\mathfrak{so}(4) = \mathfrak{t} \oplus (\mathfrak{m}_{12}^+ \oplus \mathfrak{m}_{12}^-)$ and $\mathfrak{u}(2)^2 = \mathfrak{t} \oplus (\mathfrak{m}_{12}^- \oplus \mathfrak{m}_{34}^-)$. In the first case, $D_8(1) \ominus 2A_1 \cong D_6$ and in the second case, $D_8(1) \ominus 2A_1 \cong 2A_1 + D_4$.

Moreover, $D_4(1) < D_8(1)$ contains both, subroot systems $2A_1$ of type $\mathfrak{so}(4)$ and subroot systems $2A_1$ of type $\mathfrak{u}(2)^2$. So, without loss of generality, assume that $D_8(1) \ominus 2A_1(1) \cong D_6$. Moreover, let $2A_1(2) < D_4(1) < D_8(1)$ be another subroot system satisfying $D_8(1) \ominus 2A_1(2) \cong 2A_1 + D_4$. Now, there exists a $w \in W(E_8)$ such that

$$w(H) = H, \tag{5.2}$$
$$w(2A_1(1)) = 2A_1(2). \tag{5.3}$$

More precisely, the subroot systems $2A_1$ of E_8 are all $W(E_8)$-conjugate, so there exists a $\tau \in W(E_8)$ with $\tau(2A_1(1)) = 2A_1(2)$. Thus, $D_4(2)$ and $\tau(D_4(2))$ are both perpendicular to $2A_1(2)$, i.e. $D_4(2),\ \tau(D_4(2)) < E_8 \ominus 2A_1(2) \cong D_6$. Since all subroot systems $D_4 < D_6$ are conjugate under $W(D_6)$, there exists $\mu \in W(D_6) \subseteq W(E_8)$ satisfying $\mu \circ \tau(D_4(2)) = D_4(2)$. This implies $\mu \circ \tau(D_4(1)) = D_4(1)$, so $\mu \circ \tau(H) = H$. Furthermore, $\mu \in W(D_6) = W(2A_1(2)^\perp)$ leaves $2A_1(2)$ invariant. Hence, $w := \mu \circ \tau$ satisfies (5.2) and (5.3).

Now, $H < D_8(1)$ and $H < w(D_8(1)) =: D_8(2)$ by (5.2). Furthermore, $D_8(1) \ominus 2A_1(2) = 2A_1 + D_4$ and $D_8(2) \ominus 2A_1(2) = w(D_8(1) \ominus 2A_1(1)) = D_6$ by (5.3). This implies $D_8(1) \neq D_8(2)$ and therefore $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_3{+}D_4$: $H \to D_7,\ A_3{+}D_5,\ 2D_4$: Since $D_4^\perp$, there is a unique $K := 2D_4 > H$. Each H-subalgebra L must be contained in a subroot system of type D_8. But this implies $K < L$ since $D_8 \ominus D_4 = D_4$. Hence, $\langle K, L\rangle \leq D_8$, so $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_2 + D_4$: $H \to A_2 + D_5,\ A_3 + D_4$. Let $K_0 := A_2 + D_5(1)$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2 + E_6,\ L_2 = A_3 + D_5(1)\}.$$

As shown above, the complex $\hat{\Delta}_{E_6/D_4}$ is disconnected, i.e. there exist at least two subroot systems of type D_5 satisfying $D_4 < D_5 < E_6$. So, let

$$K_1 := A_2 + D_5(2) < L_1 \text{ and}$$
$$K_2 := A_3 + D_4 < L_2,$$

with $D_5(1) \neq D_5(2)$. Now, let $L := D_8$ be a subroot system of type D_8 satisfying $K_1 < L$. In fact, L is unique, since by Corollary 5.7, there exists a unique $2D_4 > H$ and L must contain this $2D_4$. Hence, $L = \langle 2D_4, K_1\rangle$. Moreover, $K_2 < 2D_4$ by Corollary 5.7. It follows $\langle K_1, K_2\rangle \leq L < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 4A_1{+}D_4$: $H \to 2A_1{+}D_6,\ 2D_4$. Let $H = A_1(1){+}A_1(2){+}A_1(3){+}A_1(4){+}D_4(1)$ and let $K_0 := 2D_4 = D_4(1) + D_4(2)$.

As described in the case "$H = 2D_4$", the subroot systems $2A_1$ of D_8 are either of type $\mathfrak{so}(4)$ or type $\mathfrak{u}(2)^2$. Since there is a unique root $A_1 \perp D_4$ such that $A_1(1) + A_1$ is of type $\mathfrak{so}(4)$, one may assume that $A_1(1) + A_1(2)$ is of type $\mathfrak{so}(4)$ and $A_1(1) + A_1(3)$ is of type $\mathfrak{u}(2)^2$. Hence, if a subroot system $D_8 > H$ is given, one may assume $D_8 \ominus (A_1(1) + A_1(2)) \cong D_6$ and $D_8 \ominus (A_1(1) + A_1(3)) \cong 2A_1 + D_4$. For the case "$H = 2D_4$", it was shown that there exists $w \in W(E_8)$ such that $w(D_4(i)) = D_4(i),\ i = 1, 2$ and $w(A_1(1){+}A_1(2)) = A_1(1){+}A_1(3)$. Furthermore, it was shown that $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = w(D_8)\}.$$

It follows $w(D_8) \ominus (A_1(1) + A_1(3)) = w(D_8 \ominus (A_1(1) + A_1(2))) \cong D_6$. Hence,

$$K_1 := A_1(1) + A_1(2) + D_6(1) < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(3) + D_6(2) < L_2.$$

Since $\langle K_1, K_2 \rangle \leq A_1(1) + A_1(1)^\perp < E_8$, it follows $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1 + D_4$: $H \to A_1 + D_6,\ 4A_1 + D_4$. Since $H^\perp = A_1$, there is a unique $K := 4A_1 + D_4 > H$. So, let L be any H-subalgebra with $L \geq A_1 + D_6$. This implies $L \leq D_8$ or $L \leq A_1 + E_7$. First, let $L \leq D_8$. By Corollary 5.7, it is $D_8 \ominus H = A_1$. So, $K < D_8$ and $\langle K, L \rangle \leq D_8$. Now, let $L \leq A_1 + E_7$. If $H \leq E_7$, the A_1-summand of $A_1 + E_7$ is perpendicular to H and $\langle K, L \rangle \leq A_1 + E_7$. If $H \not\leq E_7$, then $H \cap E_7 = 2A_1 + D_4$ and by Corollary 5.7, $E_7 \ominus (2A_1 + D_4) = A_1$. Again, this A_1-summand is perpendicular to H and $\langle K, L \rangle \leq A_1 + E_7$. It follows $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = 2A_1 + D_4$: $H \to D_6,\ 2A_1 + D_5,\ A_3 + D_4,\ 3A_1 + D_4$. Since $H^\perp = 2A_1$, there exists a unique $K := 4A_1 + D_4 > H$. Let L be any H-subalgebra. Then $L \leq D_8$ or $A_1 + E_7$ since these are the only maximal subroot systems of E_8 containing H.

First, let $L \leq D_8$. Again, the $2A_1$-summand is either of type $\mathfrak{so}(4)$ or type $\mathfrak{u}(2)^2$. Hence, a subgroup of $SO(16)$ with corresponding root system $\tilde{H} \cong H$ is either of type $SO(8)SO(4)SO(2)^2$ or type $SO(8)U(2)^2$. But in both cases, $D_8 \ominus \tilde{H} \cong 2A_1$. So, if $L \leq D_8$, then $K \leq D_8$ and $\langle K, L \rangle \leq D_8$.

Now, let $L \leq A_1 + E_7$. If $H < E_7$, then $E_7 \ominus H = A_1$ by Corollary 5.7, so $(A_1 + E_7) \ominus H = 2A_1$. If $H \not\leq E_7$, then $H \cap E_7 = A_1 + D_4$, $E_7 \ominus (A_1 + D_4) = 2A_1$ and this $2A_1$ is perpendicular to H. In both cases, $K \leq A_1 + E_7$, so $\langle K, L \rangle \leq A_1 + E_7$. This shows $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1 + D_4$: $H \to A_1 + D_5,\ A_2 + D_4,\ 2A_1 + D_4$. Let $H = A_1(1) + D_4$ and $K_0 := A_1(1) + D_5(1)$. Then $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + E_6,\ L_2 = A_2 + D_5(1),\ L_3 = A_1(1) + A_1(2) + D_5(1)\}.$$

Since $\hat{\Delta}_{E_6/D_4}$ is disconnected, there exists a subroot system $D_4 < D_5(2) < E_6$ with $D_5(1) \neq D_5(2)$. So, let

$$K_1 := A_1(1) + D_5(2) < L_1,$$
$$K_2 := A_2 + D_4 < L_2 \text{ and}$$
$$K_3 := A_1(1) + A_1(2) + D_4 < L_3.$$

Now, $K_2, K_3 \leq D_4 + D_4^\perp = 2D_4$. Furthermore, let L be a K_1-subalgebra of type D_8. Since $D_8 \ominus D_4 = D_4$ and $H < L$, it follows $D_4 + D_4^\perp < L$. In summary, $\langle K_1, K_2, K_3 \rangle \leq L < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = D_4$: $H \to D_5,\ A_1 + D_4$. Since $H^\perp = D_4$, there is a unique $K := 2D_4 > H$. Now, let L be any H-subalgebra. Then $L \le D_8$ except for the cases $L = E_6,\ A_1 + E_6,\ A_2 + E_6,\ E_7$ or $A_1 + E_7$. If $L \le D_8$, then this D_8 also contains K and $\langle K, L\rangle \le D_8$. If $L = A_1 + E_6,\ A_2 + E_6$ or $A_1 + E_7$, then L contains a root perpendicular to H and therefore, $K \cap L > H$. The same holds true for the case $L = E_7$, since $E_7 \ominus D_4 = 3A_1$ by Corollary 5.7. It follows that $c(K)$ consists of L-subalgebras of type E_6.

Note, that Corollary 2.19 is not applicable here, since E_6 is not a minimal H-subalgebra. However, by Corollary 2.18, it follows

$$\hat{\Delta}_{G/H} \simeq \wedge_{E_6 > H} \hat{\Delta}_{E_8/E_6} * \hat{\Delta}_{E_6/D_4} * S^0.$$

But $\hat{\Delta}_{E_8/E_6}$ is contractible and so is $\hat{\Delta}_{G/H}$.

$H = A_1 + A_7$: $H \to A_1 + E_7$. Since $A_1^\perp = E_7$, there exists a unique $A_1 + E_7 > H$ and $\hat{\Delta}_{G/H}$ is a singleton.

$H = A_7'$: $H \to D_8,\ A_8$. By Lemma 2.24, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_7''$: $H \to E_7,\ D_8,\ A_1 + A_7$. Since D_8 is maximal, it follows $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1 + A_6$: $H \to A_8,\ A_1 + A_7$. It is $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$, since A_8 is maximal.

$H = A_6$: $H \to D_7,\ A_7',\ A_7'',\ A_1 + A_6$. Let $K_0 := A_7'$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = A_8\}.$$

Now, let

$$\begin{aligned} K_1 &:= A_7'' < L_1 \text{ and} \\ K_2 &:= A_1 + A_6 < L_2. \end{aligned}$$

By Corollary 5.7, $A_6^\perp = A_1$, so there is a unique root $A_1(1)$ perpendicular to H, so $K_2 = A_1(1) + A_6$. Moreover, $K_1 = A_7'' < A_1 + A_7$. But this A_1-summand is also perpendicular to H, hence $K_1 < A_1(1) + A_7$. It follows $\langle K_1, K_2\rangle \le A_1(1) + A_1(1)^\perp < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_2 + A_5$: $H \to A_1 + E_7,\ A_2 + E_6$, so $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_2 + A_5$: $H \to A_8,\ A_7,\ A_1 + A_2 + A_5$: As above, $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ since A_8 is maximal.

$H = 2A_1 + A_5$: $H \to A_1 + E_6,\ 2A_1 + D_6,\ A_1 + A_7,\ A_1 + A_2 + A_5$: Let $H = A_1(1) + A_1(2) + A_5$ and $K_0 := A_1(1) + E_6$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2 + E_6,\ L_2 = A_1(1) + E_7\}.$$

Let $D_6 := E_7 \ominus A_1(2)$. Then

$$K_1 := A_2 + A_1(2) + A_5 < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(2) + D_6 < L_2.$$

Since $\langle K_1, K_2\rangle \leq A_1(2) + A_1(2)^{\perp} < E_8$, it follows $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (A_1+A_5)'$: $H \to A_1+D_6,\ A_7',\ A_1+A_6,\ A_2+A_5,\ 2A_1+A_5$. Let $H = A_1(1)+A_5$ and $K_0 := A_7'$. Then $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = A_8\}.$$

Let

$$K_1 := A_1(1) + D_6 < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_6 < L_2.$$

It follows $\langle K_1, K_2\rangle \leq A_1(1) + A_1(1)^{\perp} < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (A_1 + A_5)''$: $H \to E_6,\ A_1 + D_6,\ A_7'',\ 2A_1 + A_5$. Let $H = A_1(1) + A_5$ and $K_0 := A_7''$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = A_1(2) + A_7\}.$$

So, let

$$K_1 := A_1(1) + D_6 < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(2) + A_5 < L_2.$$

It follows $\langle K_1, K_2\rangle \leq A_(1) + A_1(1)^{\perp} < E_8$ and therefore, $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_5$: $H \to D_6,\ A_6,\ (A_1 + A_5)',\ (A_1 + A_5)''$. Let $K_0 := A_6$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_7',\ L_2 = A_7'',\ L_3 = A_1(1) + A_6\}.$$

Since $A_7 \ominus A_5 = A_1$, there exist roots $A_1(2)$ and $A_1(3)$ such that

$$K_1 := A_1(2) + A_5 \cong (A_1 + A_5)' < L_1,$$
$$K_2 := A_1(3) + A_5 \cong (A_1 + A_5)'' < L_2 \text{ and}$$
$$K_3 := A_1(1) + A_5 \cong (A_1 + A_5)' < L_3.$$

Hence, $\langle K_1, K_2, K_3\rangle \leq A_5 + A_5^{\perp} < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_3 + A_4$: $H \to A_8,\ 2A_4,\ A_3 + D_5$, so $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1 + A_2 + A_4$: $H \to A_1 + A_7,\ A_1 + A_2 + A_5,\ 2A_4$: Since $2A_4$ is maximal, it is $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_2 + A_4$: $H \to A_2 + D_5,\ A_7',\ A_7'',\ A_2 + A_5,\ A_3 + A_4,\ A_1 + A_2 + A_4$. Let $H = A_2(1) + A_4,\ K_0 := A_7'$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = A_8\}.$$

Since $D_8 \ominus A_2 = D_5$ and $A_8 \ominus A_2 = A_5$, let

$$K_1 := A_2(1) + D_5 < L_1 \text{ and}$$
$$K_2 := A_2(1) + A_5 < L_2.$$

Then $\langle K_1, K_2\rangle \leq A_2(1) + A_2(1)^{\perp} < E_8$. It follows $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1 + A_4$: $H \to 2A_1 + D_5,\ A_1 + A_6,\ A_2 + A_5,\ 2A_1 + A_5,\ A_3 + A_4$. Let $H = A_1(1) + A_1(2) + A_4$ and $K_0 := A_1(1) + A_6(1)$. Then $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_8,\ L_2 = A_1(1) + A_7\}.$$

So, let

$$K_1 := A_1(2) + A_6(2) < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(2) + A_5 < L_2.$$

It follows $\langle K_1, K_2\rangle \leq A_1(2) + A_1(2)^{\perp} < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_4$: $H \to A_1 + D_5,\ A_6,\ (A_1 + A_5)',\ (A_1 + A_5)'',\ A_2 + A_4,\ 2A_1 + A_4$. Let $H = A_1(1) + A_4$ and $K_0 := A_6$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_7',\ L_2 = A_7'',\ L_3 = A_1(2) + A_6\}.$$

Let $A_2(1) := A_7' \ominus A_4$ and $A_2(2) := A_7'' \ominus A_4$. Then

$$K_1 := A_2(1) + A_4 < L_1,$$
$$K_2 := A_2(2) + A_4 < L_2 \text{ and}$$
$$K_3 := A_1(1) + A_1(2) + A_4 < L_3.$$

So, $\langle K_1, K_2, K_3\rangle \leq A_4 + A_4^{\perp} < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_4$: $H \to D_5,\ A_5,\ A_1 + A_4$. Let $K_0 := A_5$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_6,\ L_2 = A_1(1) + A_5 \cong (A_1 + A_5)',\\ L_3 = A_1(2) + A_5 \cong (A_1 + A_5)',\ L_4 = A_1(3) + A_5 \cong (A_1 + A_5)''\}.$$

Let $A_1(4) := A_6 \ominus H$ and

$$K_1 := A_1(4) + A_4 < L_1,\\ K_2 := A_1(1) + A_4 < L_2,\\ K_3 := A_1(2) + A_4 < L_3 \text{ and}\\ K_4 := A_1(3) + A_4 < L_4.$$

Then $\langle K_1, K_2, K_3, K_4\rangle \leq A_4 + A_4^{\perp}$ and $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1 + 2A_3$: $H \to 2A_1 + D_6,\ A_3 + D_5$. Since $2A_1^{\perp} = D_6$ by Corollary 5.7, $K := 2A_1 + D_6 > H$ is unique. Hence, $L \in c(K)$ implies $L \geq A_3 + D_5$, so $L = A_3 + D_5$ or D_8. If $L = D_8$, then $K < L$ and $L \notin c(K)$. If $L = A_3 + D_5$, then L is contained in some D_8 and $\langle K, L\rangle = D_8$. Thus, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = A_1 + 2A_3$: $H \to A_1 + D_6,\ A_1 + A_7,\ 2A_1 + 2A_3$. Since $H^{\perp} = A_1$, $K := 2A_1 + 2A_3 > H$ is unqiue. Let L be any H-subalgebra. Then $L \leq A_1 + E_7$ or $L \leq D_8$. First, suppose $L \leq A_1 + E_7$. Then $A_1 + E_7$ contains a root perpendicular to H and $\langle K, L\rangle \leq A_1 + E_7$. In fact, if $H < E_7$, then the A_1-summand of $A_1 + E_7$ satisfies $A_1 \perp H$. If $H \not< E_7$, then $2A_3 < E_7$, $E_7 \ominus 2A_3 = A_1$ and this root is perpendicular to H. Now, let $L \leq D_8$. By Corollary 5.7, it is $D_8 \ominus H = A_1$. Hence, D_8 also contains a root perpendicular to H. It follows $\langle K, L\rangle \leq D_8$, so $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = (2A_3)'$: $H \to A_3 + D_4,\ A_7',\ A_3 + A_4$. Let $H = A_3(1) + A_3(2)$ and $K_0 := A_7'$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = A_8\}.$$

Now, let

$$K_1 := A_3(1) + D_4 < L_1 \text{ and}\\ K_2 := A_3(1) + A_4 < L_2.$$

Since $\langle K_1, K_2\rangle \leq A_3(1) + A_3(1)^{\perp} < E_8$, it is $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (2A_3)''$: $H \to D_6,\ A_3 + D_4,\ A_7'',\ A_1 + 2A_3$. Let $H = A_3(1) + A_3(2)$ and $K_0 := A_7''$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = D_8,\ L_2 = A_1 + A_7\}.$$

So, let

$$K_1 := A_3(1) + D_4 < L_1 \text{ and}$$
$$K_2 := A_1 + A_3(1) + A_3(2) < L_2.$$

It follows $\langle K_1, K_2\rangle \le A_3(1) + A_3(1)^\perp < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1 + A_2 + A_3$: $H \to A_2 + D_5,\ A_1 + A_2 + A_5,\ 2A_1 + 2A_3$. Let $H = A_1(1) + A_1(2) + A_2 + A_3(1)$ and $K_0 := A_2 + D_5$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2 + E_6,\ L_2 = A_3(2) + D_5\}.$$

Let $A_5 := E_6 \ominus A_1(1)$. Then

$$K_1 := A_2 + A_1(1) + A_5 < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(2) + A_3(1) + A_3(2) < L_2.$$

Thus, $\langle K_1, K_2\rangle \le A_1(1) + A_1(1)^\perp < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_2 + A_3$: $H \to A_1 + A_6,\ A_2 + A_5,\ A_3 + A_4,\ A_1 + A_2 + A_4,\ 2A_1 + A_2 + A_3,\ A_1 + 2A_3$. Let $K_0 := A_1 + A_6$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_8,\ L_2 = A_1 + A_7\}.$$

Let

$$K_1 := A_2 + A_5 < L_1 \text{ and}$$
$$K_2 := A_1 + A_2 + A_4 < L_2.$$

Hence, $\langle K_1, K_2\rangle \le A_2 + A_2^\perp < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_2 + A_3$: $H \to A_6,\ A_2 + D_4,\ A_2 + A_4,\ (2A_3)',\ (2A_3)'',\ A_1 + A_2 + A_3$. Let $K_0 := A_6$. Then $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_7',\ L_2 = A_7'',\ L_3 = A_1 + A_6\}.$$

Let $A_4(1) := A_7' \ominus A_2$ and $A_4(2) := A_7'' \ominus A_2$. Then

$$K_1 := A_2 + A_4(1) < L_1,$$
$$K_2 := A_2 + A_4(2) < L_2 \text{ and}$$
$$K_3 := A_1 + A_2 + A_3 < L_3.$$

Thus, $\langle K_1, K_2, K_3\rangle \le A_2 + A_2^\perp < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 4A_1 + A_3$: $H \to 2A_1 + D_5,\ A_3 + D_4,\ 4A_1 + D_4,\ 2A_1 + 2A_3$. Since $4A_1^\perp = D_4$, there exists a unique $K := 4A_1 + D_4 > H$. Let L be any H-subalgebra with $L \geq 2A_1 + D_5,\ A_3 + D_4$ or $2A_1 + 2A_3$. Then $L \leq D_8$ or $L \leq A_1 + E_7$.

First, suppose $L \leq D_8$. A subgroup of $SO(16)$ with corresponding root system $\tilde{H} \cong H$, is either of type $SO(6)SO(4)^2SO(2)$ or $U(4)SO(4)^2$. In both cases, it is contained in $SO(8)SO(4)^2$, i.e. D_8 contains a subroot system of type D_4 perpendicular to the $4A_1$-summand of H. This shows $\langle K, L\rangle \leq D_8$.

Now, let $L \leq A_1 + E_7$. Since $H \not\leq E_7$, it follows $H \cap E_7 = 3A_1 + A_3$. By Corollary 5.7, $E_7 \ominus 3A_1 = D_4$ and this D_4 is perpendicular to the $4A_1$-summand of H, so $K < A_1 + E_7$. It follows $\langle K, L\rangle \leq A_1 + E_7$, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = 3A_1 + A_3$: $H \to A_1 + D_5,\ 3A_1 + D_4,\ 2A_1 + A_5,\ A_1 + 2A_3,\ 2A_1 + A_2 + A_3,\ 4A_1 + A_3$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_3$ and $K_0 := A_1(1) + A_1(2) + A_5(1)$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + E_6,\ L_2 = A_1(2) + A_2 + A_5(1),\ L_3 = A_1(1) + A_1(2) + D_6\}.$$

Let $A_5(2) := E_6 \ominus A_1(3)$ and $D_4 := D_6 \ominus A_1(3)$. So,

$$\begin{aligned} K_1 &:= A_1(1) + A_1(3) + A_5(2) < L_1, \\ K_2 &:= A_1(2) + A_2 + A_1(3) + A_3 < L_2 \text{ and} \\ K_3 &:= A_1(1) + A_1(2) + A_1(3) + D_4 < L_3. \end{aligned}$$

Hence, $\langle K_1, K_2, K_3\rangle \leq A_1(3) + A_1(3)^\perp < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (2A_1 + A_3)'$: $H \to 2A_1 + D_4,\ (A_1 + A_5)',\ 2A_1 + A_4,\ (2A_3)',\ A_1 + A_2 + A_3,\ 3A_1 + A_3$. Let $H = A_1(1) + A_1(2) + A_3$ and $K_0 := A_1(1) + A_5 \cong (A_1 + A_5)'$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_7',\ L_2 = A_1(1) + D_6,\ L_3 = A_1(1) + A_6\}.$$

Now, let

$$\begin{aligned} K_1 &:= A_1(2) + A_5 \cong (A_1 + A_5)' < L_1, \\ K_2 &:= A_1(1) + A_1(2) + D_4 < L_2 \text{ and} \\ K_3 &:= A_1(1) + A_1(2) + A_4 < L_3. \end{aligned}$$

Hence, $\langle K_1, K_2, K_3\rangle \leq A_1(2) + A_1(2)^\perp < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (2A_1 + A_3)''$: $H \to D_5,\ 2A_1 + D_4,\ (A_1 + A_5)'',\ (2A_3)'',\ 3A_1 + A_3$. Let $H = A_1(1) + A_1(2) + A_3(1)$ and $K_0 := A_1(1) + A_5 \cong (A_1 + A_5)''$. Then $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_7'',\ L_2 = A_1(1) + D_6,\ L_3 = A_1(1) + A_1(3) + A_5\}$$

for some $A_1(3) \neq A_1(1),\ A_1(2)$. If $\mathfrak{l} < \mathfrak{so}(12)$ is a T-subalgebra with root system $A_1 + A_3$, then the corresponding subgroup is $SO(6)SO(2)U(2)$ or $U(4)U(2)$. In both cases, $D_6 \ominus R(\mathfrak{l}) = A_1$. So, let $A_1(4) := D_6 \ominus (A_1(2) + A_3(1))$. Moreover, let $A_3(2) := A_7'' \ominus A_3(1)$. Then

$$\begin{aligned} K_1 &:= A_3(1) + A_3(2) \cong (2A_3)'' < L_1, \\ K_2 &:= A_1(1) + A_1(2) + A_1(4) + A_3(1) < L_2 \text{ and} \\ K_3 &:= A_1(1) + A_1(2) + A_1(3) + A_3(1) < L_3. \end{aligned}$$

Thus, $\langle K_1, K_2, K_3 \rangle \leq A_3(1) + A_3(1)^{\perp} < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_3$: $H \to A_1 + D_4,\ A_5,\ A_1 + A_4,\ A_2 + A_3,\ (2A_1 + A_3)',\ (2A_1 + A_3)''$. Let $H = A_1(1) + A_3$ and $K_0 := A_5$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\begin{aligned} \{L_1 = A_6,\ L_2 = A_1(2) + A_5 \cong (A_1 + A_5)', \\ L_3 = A_1(3) + A_5 \cong (A_1 + A_5)',\ L_4 = A_1(4) + A_5 \cong (A_1 + A_5)''\}. \end{aligned}$$

For $A_4 := A_6 \ominus A_1(1)$, let

$$\begin{aligned} K_1 &:= A_1(1) + A_4 < L_1, \\ K_2 &:= A_1(1) + A_1(2) + A_3 \cong (2A_1 + A_3)' < L_2, \\ K_3 &:= A_1(1) + A_1(3) + A_3 \cong (2A_1 + A_3)' < L_3 \text{ and} \\ K_4 &:= A_1(1) + A_1(4) + A_3 \cong (2A_1 + A_3)'' < L_4. \end{aligned}$$

Since, $\langle K_1, K_2, K_3, K_4 \rangle \leq A_1(1) + A_1(1)^{\perp} < E_8$, it follows $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_3$: $H \to D_4,\ A_4,\ A_1 + A_3$. Let $K_0 := A_4$. Then $\hat{\Delta}_{G/K_0}$ contains a 3-cycle supported by a facet with l.b.s.

$$\{L_1 = A_5\} \cup \{L_i = A_1(i-1) + A_4 \mid 2 \leq i \leq 5\}.$$

Let $A_1(5) := A_5 \ominus A_3$. So,

$$\begin{aligned} K_1 &:= A_1(5) + A_3 < L_1, \text{ and} \\ K_i &:= A_1(i-1) + A_3 < L_i \text{ for } 2 \leq i \leq 5. \end{aligned}$$

Hence, $\langle K_1, \ldots, K_5 \rangle \leq A_3 + A_3^{\perp} < E_8$, so $\tilde{H}_4(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 4A_2$: $H \to A_2 + E_6$. Let $H = A_2(1) + A_2(2) + A_2(3) + A_2(4)$. For $i \in \{1, 2, 3, 4\}$ let $E_6(i) := A_2(i)^{\perp}$. So, $H < A_2(i) + E_6(i)$ for each i and $\tilde{H}_0(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$ by Lemma 2.24.

$H = A_1+3A_2$: $H \to A_1+E_6,\ A_1+A_2+A_5,\ 4A_2$. Let $H = A_1+A_2(1)+A_2(2)+A_2(3)$ and $K_0 := A_1 + E_6$. $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2(4) + E_6,\ L_2 = A_1 + E_7\}.$$

with $A_2(4) \neq A_2(1),\ A_2(2),\ A_2(3)$. Let $A_5 := E_7 \ominus A_2(1)$. Then

$$K_1 := 4A_2 < L_1 \text{ and}$$
$$K_2 := A_1 + A_2(1) + A_5 < L_2.$$

Thus, $\langle K_1, K_2\rangle \leq A_2(1) + A_2(1)^{\perp} < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_2$: $H \to E_6,\ A_2 + A_5,\ A_1 + 3A_2$. Let $H = A_2(1) + A_2(2) + A_2(3)$. If $K_0 := A_2(1) + A_5(1)$, then $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = A_8,\ L_2 = A_1 + A_2(1) + A_5\}.$$

Let $A_5(2) := A_8 \ominus A_2(2)$. Then

$$K_1 := A_2(2) + A_5(2) < L_1 \text{ and}$$
$$K_2 := A_1 + 3A_2 < L_2.$$

Thus, $\langle K_1, K_2\rangle \leq A_2(2) + A_2(2)^{\perp} < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1 + 2A_2$: $H \to 2A_1 + A_5,\ A_1 + A_2 + A_4,\ A_1 + 3A_2,\ 2A_1 + A_2 + A_3$. Let $H = A_1(1) + A_1(2) + A_2(1) + A_2(2)$ and $K_0 := A_1(1) + A_2(1) + A_4(2)$ with $A_1(2) + A_2(2) < A_4(2)$. Then $\hat{\Delta}_{G/K_0}$ contains a 0-cycle supported by a facet with l.b.s.

$$\{L_1 = 2A_4 = A_4(1) + A_4(2),\ L_2 = A_1(1) + A_7\}.$$

with $A_1(1) + A_2(1) < A_4(1)$. Let $A_5 := A_7 \ominus A_1(2)$. Then

$$K_1 := A_1(2) + A_2(2) + A_4(1) < L_1 \text{ and}$$
$$K_2 := A_1(1) + A_1(2) + A_5 < L_2.$$

Thus, $\langle K_1, K_2\rangle \leq A_1(2) + A_1(2)^{\perp} < E_8$ and $\tilde{H}_1(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1+2A_2$: $H \to (A_1+A_5)',\ ,\ (A_1+A_5)'',\ A_2+A_4,\ A_1+A_2+A_3,\ 3A_2,\ 2A_1+2A_2$. Let $H = A_1+A_2(1)+A_2(2)$ and $K_0 := (A_1+A_5)'$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_7',\ L_2 = A_1 + D_6,\ L_3 = A_1 + A_6\}.$$

Let $A_4 := A_7' \ominus A_2(1)$, $A_3(1) := D_6 \ominus A_2(1)$ and $A_3(2) := A_6 \ominus A_2(1)$. So,

$$\begin{aligned} K_1 &:= A_2(1) + A_4 < L_1, \\ K_2 &:= A_1 + A_2(1) + A_3(1) < L_2 \text{ and} \\ K_3 &:= A_1 + A_2(1) + A_3(2) < L_3. \end{aligned}$$

Hence, $\langle K_1, K_2, K_3 \rangle \leq A_2(1) + A_2(1)^{\perp} < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_2$: $H \to A_5,\ A_2 + A_3,\ A_1 + 2A_2$. Let $H = A_2(1) + A_2(2)$ and $K_0 := A_5$, so $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_6,\ L_2 = A_1(1) + A_5 \cong (A_1 + A_5)',\\ L_3 = A_1(2) + A_5 \cong (A_1 + A_5)',\ L_4 = A_1(3) + A_5 \cong (A_1 + A_5)''\}.$$

Let

$$\begin{aligned} K_1 &:= A_2(1) + A_3 < L_1, \\ K_2 &:= A_1(1) + A_2(1) + A_2(2) < L_2, \\ K_3 &:= A_1(2) + A_2(1) + A_2(2) < L_3 \text{ and} \\ K_4 &:= A_1(3) + A_2(1) + A_2(2) < L_4. \end{aligned}$$

It follows $\langle K_1, K_2, K_3, K_4 \rangle \leq A_2(1) + A_2(1)^{\perp} < E_8$. Hence, $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 4A_1 + A_2$: $H \to A_2 + D_4,\ 2A_1 + A_2 + A_3,\ 4A_1 + A_3$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_1(4) + A_2$ and $K_0 := A_2 + D_4$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_2 + D_5(1),\ L_2 = A_2 + D_5(2),\ L_3 = A_3(1) + D_4\}.$$

If $\mathfrak{l} < \mathfrak{so}(10)$ is a T-subalgebra with root system $4A_1$, the corresponding subgroup of $SO(10)$ is of type $SO(4)^2 SO(2)$. Hence, for a given $A_1 < 4A_1$, there exists a unique $\widetilde{A_1} < 4A_1$ such that the subgroup which corresponds to $A_1 + \widetilde{A_1}$ is of type $SO(4)SO(2)^3$. Since $SO(4)SO(3)^2 < SO(4)SO(6) < SO(10)$, it is $D_5 \ominus (A_1 + \widetilde{A_1}) = A_3$.

This implies that there exist $p, q \in \{2, 3, 4\}$ and subroot systems $A_3(2),\ A_3(3)$ with $A_1(1) + A_1(p) + A_3(2) < D_5(1)$ and $A_1(1) + A_1(q) + A_3(3) < D_5(2)$. So, let

$$\begin{aligned} K_1 &:= A_1(1) + A_1(p) + A_3(2) + A_2 < L_1, \\ K_2 &:= A_1(1) + A_1(q) + A_3(3) + A_2 < L_2 \text{ and} \\ K_3 &:= A_1(1) + A_1(2) + A_1(3) + A_1(4) + A_3(1) < L_2. \end{aligned}$$

It follows $\langle K_1, K_2, K_3 \rangle \leq A_1(1) + A_1(1)^{\perp} < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1 + A_2$: $H \to 2A_1 + A_4,\ A_1 + A_2 + A_3,\ 3A_1 + A_3,\ 2A_1 + 2A_2,\ 4A_1 + A_2$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_2$ and $K_0 := A_1(1) + A_1(2) + A_4(1)$. Then $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_6(1),\ L_2 = A_1(2) + A_6(2),\ L_3 = A_1(1) + A_1(2) + A_5\}.$$

Let $A_4(2) := A_6(1) \ominus A_1(3)$, $A_4(3) := A_6(2) \ominus A_1(3)$ and $A_3 := A_5 \ominus A_1(3)$. Then

$$\begin{aligned} K_1 &:= A_1(1) + A_1(3) + A_4(2) < L_1,\\ K_2 &:= A_1(2) + A_1(3) + A_4(3) < L_2 \text{ and}\\ K_3 &:= A_1(1) + A_1(2) + A_1(3) + A_3 < L_3. \end{aligned}$$

It follows $\langle K_1, K_2, K_3\rangle \leq A_1(3) + A_1(3)^\perp < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1{+}A_2$: $H \to A_1{+}A_4,\ A_2{+}A_3,\ (2A_1{+}A_3)',\ (2A_1{+}A_3)'',\ A_1{+}2A_2,\ 3A_1{+}A_2$. Let $H = A_1(1){+}A_1(2){+}A_2(1)$ and $K_0 := A_2(1){+}A_3$. Then, $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_6,\ L_2 = A_2(1) + A_4(1),\ L_3 = A_2(1) + A_4(2),\ L_4 = A_2(1) + A_3 + A_1(3)\}.$$

for some $A_1(3) \neq A_1(1),\ A_1(2)$. With $A_4(3) := A_6 \ominus A_1(1)$, $A_2(2) := A_4(1) \ominus A_1(1)$ and $A_2(3) := A_4(2) \ominus A_1(1)$ let

$$\begin{aligned} K_1 &:= A_1(1) + A_4(3) < L_1,\\ K_2 &:= A_2(1) + A_1(1) + A_2(2) < L_2,\\ K_3 &:= A_2(1) + A_1(1) + A_2(3) < L_3, \text{ and}\\ K_4 &:= A_2(1) + A_1(1) + A_1(2) + A_1(3) < L_4. \end{aligned}$$

Hence, $\langle K_1, K_2, K_3, K_4\rangle \leq A_1(1) + A_1(1)^\perp < E_8$ and $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1 + A_2$: $H \to A_4,\ A_1 + A_3,\ 2A_2,\ 2A_1 + A_2$. Let $H = A_1(1) + A_2$ and $K_0 := A_4$. $\hat{\Delta}_{G/K_0}$ contains a 3-cycle supported by a facet with l.b.s.

$$\{L_1 = A_5\} \cup \{L_i = A_1(i) + A_4 \mid 2 \leq i \leq 5\}.$$

Furthermore, let $A_3 = A_5 \ominus A_1(1)$. So,

$$\begin{aligned} K_1 &:= A_1(1) + A_3 < L_1 \text{ and}\\ K_i &:= A_1(1) + A_1(i) + A_2 < L_i \text{ for } 2 \leq i \leq 5. \end{aligned}$$

It follows $\langle K_1, \ldots, K_5\rangle \leq A_1(1) + A_1(1)^\perp < E_8$ and $\tilde{H}_4(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_2$: $H \to A_3,\ A_1 + A_2$. Let $K_0 := A_3$. Then $\hat{\Delta}_{G/K_0}$ contains a 4-cycle supported by a facet with l.b.s.

$$\{L_1 = A_4\} \cup \{L_i = A_1(i-1) + A_3 \mid 2 \leq i \leq 6\}.$$

Now, let $A_1(6) := A_4 \ominus A_2$. It follows

$$K_1 := A_1(6) + A_2 < L_1 \text{ and}$$
$$K_i := A_1(i-1) + A_2 < L_i \text{ for } 2 \leq i \leq 6.$$

It follows $\langle K_1, \ldots, K_6 \rangle \leq A_2 + A_2^\perp < E_8$ and $\tilde{H}_5(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 8A_1$: $H \to 4A_1 + D_4$. Let $H = A_1(1) + \ldots + A_1(8)$ and $K_0 := A_1(1) + A_1(2) + A_1(3) + A_1(4) + D_4(1)$. $\hat{\Delta}_{G/K_0}$ contains a 1-cycle supported by a facet with l.b.s.

$$\{L_1 = D_4(1) + D_4(2),\ L_2 = A_1(1) + A_1(2) + D_6(1),\ L_3 = A_1(1) + A_1(3) + D_6(2)\}.$$

Now, if $\mathfrak{l} < \mathfrak{so}(12)$ is a T-subalgebra with root system $6A_1$, the corresponding subgroup of $SO(12)$ is of type $SO(4)^3$. In particular, for a given $A_1 < 6A_1$ there exists a unique $\widetilde{A_1} < 6A_1$ such that the subgroup which corresponds to $A_1 + \widetilde{A_1}$ is of type $SO(4)SO(2)^4$. Since $SO(4)SO(2)^4 < SO(4)SO(8) < SO(12)$, it follows $D_6 \ominus (A_1 + \widetilde{A_1}) = D_4$. In particular, there exists a unique $p \in \{3,4,5,6,7\}$ with $A_1(p) + A_1(8) + D_4(3) < D_6(1)$ and a unique $q \in \{2,4,5,6,7\}$ with $A_1(q) + A_1(8) + D_4(4) < D_6(2)$. So, let

$$K_2 := A_1(5) + A_1(6) + A_1(7) + A_1(8) + D_4(2) < L_1,$$
$$K_1 := A_1(1) + A_1(2) + A_1(p) + A_1(8) + D_4(3) < L_2 \text{ and}$$
$$K_3 := A_1(1) + A_1(3) + A_1(q) + A_1(8) + D_4(4) < L_3.$$

It follows $\langle K_1, K_2, K_3 \rangle \leq A_1(8) + A_1(8)^\perp < E_8$ and $\tilde{H}_2(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 7A_1$: $H \to 3A_1 + D_4,\ 8A_1$. Since $H^\perp = A_1$, there is a unique $8A_1 > H$. Let L be any H-subalgebra. This implies $L \leq D_8$ or $A_1 + E_7$. All subroot systems of type $7A_1$ of D_8 are $W(D_8)$-conjugate, so $D_8 \ominus H = A_1$ by Corollary 5.7. Hence, $L \leq D_8$ implies $K \leq D_8$, so $\langle K, L \rangle \leq D_8$. Now, let $L \leq A_1 + E_7$. If $L < E_7$, the A_1-summand of $A_1 + E_7$ is perpendicular to H, so $K \leq A_1 + E_7$. If $L \not< E_7$, then $H \cap E_7 = 6A_1$. But $E_7 \ominus 6A_1 = A_1$ and this A_1-summand is perpendicular to H. Again, it follows $K \leq A_1 + E_7$, i.e. $\langle K, L \rangle \leq A_1 + E_7$. Hence, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = 6A_1$: $H \to 2A_1 + D_4,\ 4A_1 + A_3,\ 7A_1$. It is $H^\perp = 2A_1$, so there exists a unique $K := 8A_1 > H$. Let L be any H-subalgebra. Then $L \leq D_8$ except for the cases $L = E_7$ and $L = A_1 + E_7$.

First, suppose $L \leq D_8$. A subgroup of $SO(16)$ with corresponding root system $\tilde{H} \cong H$ is of type $SO(4)^3 SO(2)^2$ or $SO(4)^2 U(2)^2$. But in both cases, it follows $D_8 \ominus \tilde{H} = 2A_1$. In particular, $L \leq D_8$ implies $K < D_8$, so $\langle K, L \rangle \leq D_8$.

If $L = E_7$ or $L = A_1 + E_7$, then L contains at least one root perpendicular to H, since $E_7 \ominus 6A_1 = A_1$. It follows $K \cap L > H$. Thus, $c(K) = \emptyset$ and $\hat{\Delta}_{G/H}$ is contractible.

$H = 5A_1$: $H \to A_1 + D_4,\ 3A_1 + A_3,\ 4A_1 + A_2,\ 6A_1$. Let $H = A_1(1) + \ldots + A_1(5)$ and $K_0 := A_1(1) + A_1(2) + A_1(3) + A_3(1)$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_1(2) + A_5(1),\ L_2 = A_1(1) + A_1(3) + A_5(2),\\ L_3 = A_1(2) + A_1(3) + A_2 + A_3(1),\ L_4 = A_1(1) + A_1(2) + A_1(3) + D_4\}.$$

Note, that if $\mathfrak{l} < \mathfrak{so}(8)$ is a T-subalgebra with root system $2A_1$, then the corresponding subgroup of $SO(8)$ is of type $SO(4)SO(2)^2$ or $U(2)^2$. In both cases, it follows $D_4 \ominus 2A_1 = 2A_1$. So, let $A_1(6)$ a root of D_4 perpendicular to $A_1(4) + A_1(5)$. Furthermore, let $A_3(2) := A_5(1) \ominus A_1(5)$ and $A_3(3) := A_5(2) \ominus A_1(5)$. Then

$$\begin{aligned} K_1 &:= A_1(1) + A_1(2) + A_1(5) + A_3(2) < L_1, \\ K_1 &:= A_1(1) + A_1(3) + A_1(5) + A_3(3) < L_2, \\ K_3 &:= A_1(2) + A_1(3) + A_2 + A_1(4) + A_1(5) < L_3 \text{ and} \\ K_4 &:= A_1(1) + \ldots + A_1(6) < L_4. \end{aligned}$$

Thus, $\langle K_1, K_2, K_3, K_4 \rangle \leq A_1(5) + A_1(5)^\perp < E_8$ and $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (4A_1)'$: $H \to (2A_1 + A_3)',\ 3A_1 + A_2,\ 5A_1$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_1(4)$ and $K_0 := A_1(1) + A_1(2) + A_1(3) + A_2(1)$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_1(2) + A_4(1),\ L_2 = A_1(1) + A_1(3) + A_4(2),\\ L_3 = A_1(2) + A_1(3) + A_4(3),\ L_4 = A_1(1) + A_1(2) + A_1(3) + A_3\}.$$

Let $A_1(5) := A_3 \ominus A_1(4)$ and $A_2(i) := A_4(i-1) \ominus A_1(4)$ for $2 \leq i \leq 4$. Then

$$\begin{aligned} K_1 &:= A_1(1) + A_1(2) + A_1(4) + A_2(2) < L_1, \\ K_2 &:= A_1(1) + A_1(3) + A_1(4) + A_2(3) < L_2, \\ K_3 &:= A_1(2) + A_1(3) + A_1(4) + A_2(4) < L_3 \text{ and} \\ K_4 &:= A_1(1) + \ldots + A_1(5) < L_4. \end{aligned}$$

It follows $\langle K_1, K_2, K_3, K_4 \rangle \leq A_1(4) + A_1(4)^\perp < E_8$ and $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = (4A_1)''$: $H \to D_4,\ (2A_1 + A_3)'',\ 5A_1$. Let $H = A_1(1) + A_1(2) + A_1(3) + A_1(4)$ and $K_0 := A_1(1) + A_1(2) + A_3(1) \cong (2A_1 + A_3)''$. $\hat{\Delta}_{G/K_0}$ contains a 2-cycle supported by a facet with l.b.s.

$$\{L_1 = A_1(1) + A_5 \cong (A_1 + A_5)'',\ L_2 = A_3(1) + A_3(2) \cong (2A_3)'',\\ L_3 = A_1(1) + A_1(2) + A_1(5) + A_3(1),\ L_4 = A_1(1) + A_1(2) + A_1(6) + A_3(1)\}$$

with $A_1(5),\ A_1(6) \perp H$. With $A_3(3) := A_5 \ominus A_1(4)$, let

$$\begin{aligned} K_1 &:= A_1(1) + A_1(4) + A_3(3) \cong (2A_1 + A_3)'' < L_1, \\ K_2 &:= A_1(3) + A_1(4) + A_3(2) \cong (2A_1 + A_3)'' < L_2, \\ K_3 &:= A_1(1) + A_1(2) + A_1(3) + A_1(4) + A_1(5) < L_3 \text{ and} \\ K_4 &:= A_1(1) + A_1(2) + A_1(3) + A_1(4) + A_1(6) < L_4. \end{aligned}$$

Hence, $\langle K_1, K_2, K_3, K_4 \rangle \leq A_1(4) + A_1(4)^{\perp} < E_8$ and $\tilde{H}_3(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 3A_1$: $H \to A_1 + A_3,\ 2A_1 + A_2,\ (4A_1)',\ (4A_1)''$. Let $H = A_1(1) + A_1(2) + A_1(3)$ and $K_0 := A_1(1) + A_3(1)$. $\hat{\Delta}_{G/K_0}$ contains a 3-cycle supported by a facet with l.b.s.

$$\begin{aligned} \{L_1 = A_5,\ L_2 = A_1(1) + A_4,\ L_3 = A_1(1) + A_1(4) + A_3(1), \\ L_4 = A_1(1) + A_1(5) + A_3(1),\ L_5 = A_1(1) + A_1(6) + A_3(1)\}, \end{aligned}$$

where $L_3,\ L_4 \cong (2A_1 + A_3)'$, $L_5 \cong (2A_1 + A_3)''$ and $A_1(4),\ A_1(5),\ A_1(6) \perp H$. Let $A_2 := A_4 \ominus A_1(2)$ and $A_3(2) := A_5 \ominus A_1(2)$. Then

$$\begin{aligned} K_1 &:= A_1(2) + A_3(2) < L_1, \\ K_2 &:= A_1(1) + A_1(2) + A_2 < L_2, \\ K_3 &:= A_1(1) + A_1(2) + A_1(3) + A_1(4) \cong (4A_1)' < L_3, \\ K_4 &:= A_1(1) + A_1(2) + A_1(3) + A_1(5) \cong (4A_1)' < L_4 \text{ and} \\ K_5 &:= A_1(1) + A_1(2) + A_1(3) + A_1(6) \cong (4A_1)'' < L_5. \end{aligned}$$

Thus, $\langle K_1, \ldots, K_5 \rangle \leq A_1(2) + A_1(2)^{\perp} < E_8$ and $\tilde{H}_4(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = 2A_1$: $H \to A_3,\ A_1 + A_2,\ 3A_1$. Let $H = A_1(1) + A_1(2)$ and $K_0 := A_3$. $\hat{\Delta}_{G/K_0}$ contains a 4-cycle supported by a facet with l.b.s.

$$\{L_1 = A_4\} \cup \{L_i = A_1(i+1) + A_3 \mid 2 \leq i \leq 6\}.$$

Let $A_2 := A_4 \ominus A_1(1)$. Then

$$\begin{aligned} K_1 &:= A_1(1) + A_2 < L_1 \text{ and} \\ K_i &:= A_1(1) + A_1(2) + A_1(i+1) < L_i \text{ for } 2 \leq i \leq 6. \end{aligned}$$

So, $\langle K_1, \ldots, K_6 \rangle \leq A_1(1) + A_1(1)^{\perp} < E_8$ and $\tilde{H}_5(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = A_1$: $H \to A_2,\ 2A_1$. Let $H = A_1(1)$ and $K_0 := A_2$. $\hat{\Delta}_{G/K_0}$ contains a 5-cycle supported by a facet with l.b.s.

$$\{L_1 = A_3\} \cup \{L_i = A_1(i) + A_2 \mid 2 \leq i \leq 7\}.$$

If $A_1(8) := A_3 \ominus A_1(1)$, then

$$K_1 := A_1(1) + A_1(8) < L_1 \text{ and}$$
$$K_i := A_1(1) + A_1(i) \text{ for } 2 \leq i \leq 7.$$

Hence, $\langle K_1, \ldots, K_7 \rangle \leq A_1(1) + A_1(1)^\perp < E_8$ and $\tilde{H}_6(\hat{\Delta}_{G/H}, \mathbb{Q}) \neq 0$.

$H = \emptyset$ $H \to A_1$. As mentioned above, Theorem 2.27 is not applicable but $\tilde{\chi}(\hat{\Delta}_{E_8/T})$ can be computed. By Lemma 4.14, it is

$$\tilde{\chi}(\hat{\Delta}_{E_8/T}) = -\left(1 + \sum_{\mathfrak{t}<\mathfrak{k}<\mathfrak{g}} \tilde{\chi}(\hat{\Delta}_{K/T})\right), \tag{5.4}$$

where K is the Lie subgroup with $T_e K = \mathfrak{k}$. Let $\tilde{\chi}(R(\mathfrak{k})) := \tilde{\chi}(\hat{\Delta}_{K/T})$. To calculate the sum on the right-hand side, one needs $\tilde{\chi}(R(\mathfrak{k}))$ for every $R(\mathfrak{k}) \neq \emptyset$ given by the table above and the order of its Weyl orbit.

First, if $R(\mathfrak{k})$ is reducible, i.e. if the semisimple part $\mathfrak{k}_s := \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_r$ is not simple, then by Lemma 4.13, $\tilde{\chi}(R(\mathfrak{k})) = \prod_{i=1}^r \tilde{\chi}(R(\mathfrak{k}_i))$. Hence, to calculate each $\tilde{\chi}(R(\mathfrak{k}))$, one needs the values a_n, $1 \leq n \leq 8$, d_n, $4 \leq n \leq 8$, $x := \tilde{\chi}(\hat{\Delta}_{E_6/T})$ and $y := \tilde{\chi}(\hat{\Delta}_{E_7/T})$. It will turn out, that x and y do not have to be calculated explicitly, altough they could be calculated in a similar way.

It remains to calculate the order of the Weyl orbits $W.S$ for each subroot system $S < E_8$. Since $|W| = |W(E_8)| = 696,729,600$ is known, see [Hum72, p. 66], this is equivalent to calculating the order of the isotropy group

$$W_S := \{w \in W(E_8) \mid w(S) = S\},$$

since $|W.S| = |W|/|W_S|$. So, consider

$$\mathrm{Aut}(S, E_8) := \{\sigma \in \mathrm{Aut}(S) \mid \exists \mu \in W(E_8),\ \mu_{|S} = \sigma\},$$

the group of automorphisms of S which arise from some Weyl group element of E_8. There is a surjective group homomorphism

$$\Phi_S : W_S \longrightarrow \mathrm{Aut}(S, E_8);\ w \mapsto w_{|S}.$$

So,

$$|W_S| = |\ker \Phi_S| \cdot |\mathrm{Aut}(S, E_8)| = \frac{|\ker \Phi_S| \cdot |Aut(S)|}{\#(S)},$$

where $\#(S)$ denotes the order of the quotient group $|\mathrm{Aut}(S)/\mathrm{Aut}(S, E_8)|$. Now, by [Hum90, p. 22], $\ker \Phi_S = \{w \in W(E_8) \mid w_{|S} = \mathrm{id}_S\}$ is given by the subgroup

$$\langle s_\alpha \mid \alpha \perp S \rangle = \langle s_\alpha \mid \alpha \in S^\perp \rangle \cong W(S^\perp),$$

where $W(\emptyset) := \{1\}$. Moreover, $|\operatorname{Aut}(S)| = |\operatorname{Out}(S)| \cdot |W(S)|$ is given by $|W(S)|$ times the order of the symmetry group of the Dynkin diagram of S. For irreducible S, $|W(S)|$ is given in [Hum72, p. 66]. For $S = S_1 + \ldots + S_k$ it is $|W(S)| = \prod_{i=1}^{k} |W(S_k)|$. Furthermore, $\#(S)$ was calculated by Oshima and is given by the table in [Osh06, p. 45f.]. In summary,

$$|W.S| = \frac{|W(E_8)| \cdot \#(S)}{|\operatorname{Aut}(S)| \cdot |W(S^{\perp})|}.$$

Now, by (5.4), $-(1 + \tilde{\chi}(\hat{\Delta}_{E_8/T}))$ is given by the sum over the right column of the following table. Note, that $S^{\perp}$ is given by Corollary 5.7 and also by the table in [Osh06, p. 45f.].

Table 5.7: $\tilde{\chi}(\hat{\Delta}_{E_8/T})$

$S = R(\mathfrak{k})$	$S^{\perp}$	$\tilde{\chi} := \tilde{\chi}(S)$	$\lvert\operatorname{Aut}(S)\rvert$	$\lvert W(S^{\perp})\rvert$	$\#(S)$	$\lvert W(S)\rvert \cdot \tilde{\chi}$
$A_1 + E_7$	$\emptyset$	$-y$	$5,806,080$	1	1	$-120y$
E_7	A_1	y	$2,903,040$	2	1	$120y$
$A_2 + E_6$	$\emptyset$	$2x$	$1,244,160$	1	2	$2,240x$
$A_1 + E_6$	$\emptyset$	$-x$	$207,360$	1	1	$-3,360x$
E_6	A_2	x	$103,680$	6	1	$1,120x$
D_8	$\emptyset$	$640,080$	$10,321,920$	1	2	$86,410,800$
D_7	$\emptyset$	$-45,360$	$645,120$	1	1	$-48,988,800$
$2A_1 + D_6$	$\emptyset$	$3,720$	$368,640$	1	2	$14,061,600$
$A_1 + D_6$	A_1	$-3,720$	$92,160$	2	2	$-28,123,200$
D_6	$2A_1$	$3,720$	$46,080$	4	1	$14,061,600$
$A_3 + D_5$	$\emptyset$	$2,160$	$184,320$	1	2	$16,329,600$
$A_2 + D_5$	$\emptyset$	-720	$46,080$	1	2	$-21,772,800$
$2A_1 + D_5$	$\emptyset$	-360	$30,720$	1	1	$-8,164,800$
$A_1 + D_5$	A_1	360	$7,680$	2	1	$16,329,600$
D_5	A_3	-360	$3,840$	24	1	$-2,721,600$
$2D_4$	$\emptyset$	$1,764$	$2,654,208$	1	6	$2,778,300$
$A_3 + D_4$	$\emptyset$	-252	$55,296$	1	3	$-9,525,600$
$A_2 + D_4$	$\emptyset$	84	$13,824$	1	1	$4,233,600$

$S = R(\mathfrak{k})$	$S^\perp$	$\tilde{\chi} := \tilde{\chi}(S)$	$\lvert \mathrm{Aut}(S)\rvert$	$\lvert W(S^\perp)\rvert$	$\#(S)$	$\lvert W(S)\rvert \cdot \tilde{\chi}$
$4A_1 + D_4$	$\emptyset$	42	442,368	1	6	396,900
$3A_1 + D_4$	A_1	−42	55,296	2	6	−1,587,600
$2A_1 + D_4$	$2A_1$	42	9,216	4	3	2,381,400
$A_1 + D_4$	$3A_1$	−42	2,304	8	1	−1,587,600
D_4	D_4	42	1,152	192	1	132,300
A_8	$\emptyset$	40,320	725,760	1	1	38,707,200
$A_1 + A_7$	$\emptyset$	5,040	161,280	1	1	21,772,800
A_7'	$\emptyset$	−5,040	80,640	1	1	−43,545,600
A_7''	A_1	−5,040	80,640	2	1	−21,772,800
$A_1 + A_6$	$\emptyset$	−720	20,160	1	1	−24,883,200
A_6	A_1	720	10,080	2	1	24,883,200
$A_1 + A_2 + A_5$	$\emptyset$	240	34,560	1	2	9,676,800
$A_2 + A_5$	A_1	−240	17,280	2	2	−9,676,800
$2A_1 + A_5$	$\emptyset$	−120	11,520	1	2	−14,515,200
$(A_1 + A_5)'$	A_1	120	2,880	2	1	14,515,200
$(A_1 + A_5)''$	A_2	120	2,880	6	1	4,838,400
A_5	$A_1 + A_2$	−120	1,440	12	1	−4,838,400
$2A_4$	$\emptyset$	576	115,200	1	2	6,967,296
$A_3 + A_4$	$\emptyset$	−144	11,520	1	2	−17,418,240
$A_1 + A_2 + A_4$	$\emptyset$	−48	5,760	1	2	−11,612,160
$A_2 + A_4$	A_1	48	2,880	2	2	11,612,160
$2A_1 + A_4$	$\emptyset$	24	1,920	1	1	8,709,120
$A_1 + A_4$	A_2	−24	480	6	1	−5,806,080
A_4	A_4	24	240	120	1	580,608
$2A_1 + 2A_3$	$\emptyset$	36	36,864	1	2	1,360,800
$A_1 + 2A_3$	A_1	−36	9,216	2	2	−2,721,600
$(2A_3)'$	$\emptyset$	36	4,608	1	1	5,443,200

$S = R(\mathfrak{k})$	$S^\perp$	$\tilde{\chi} := \tilde{\chi}(S)$	$\lvert \mathrm{Aut}(S)\rvert$	$\lvert W(S^\perp)\rvert$	$\#(S)$	$\lvert W(S)\rvert \cdot \tilde{\chi}$
$(2A_3)''$	$2A_1$	36	4,608	4	1	1,360,800
$2A_1 + A_2 + A_3$	$\emptyset$	−12	4,608	1	2	−3,628,800
$A_1 + A_2 + A_3$	A_1	12	1,152	2	2	7,257,600
$A_2 + A_3$	$2A_1$	−12	576	4	1	−3,628,800
$4A_1 + A_3$	$\emptyset$	−6	18,432	1	3	−680,400
$3A_1 + A_3$	A_1	6	2,304	2	3	2,721,600
$(2A_1 + A_3)'$	$2A_1$	−6	384	4	1	−2,721,600
$(2A_1 + A_3)''$	A_3	−6	384	24	1	−453,600
$A_1 + A_3$	$A_1 + A_3$	6	96	48	1	907,200
A_3	D_5	−6	48	1,920	1	−45,360
$4A_2$	$\emptyset$	16	497,664	1	8	179,200
$A_1 + 3A_2$	$\emptyset$	−8	20,736	1	4	−1,075,200
$3A_2$	A_2	8	10,368	6	4	358,400
$2A_1 + 2A_2$	$\emptyset$	4	2,304	1	2	2,419,200
$A_1 + 2A_2$	A_2	−4	576	6	2	−1,612,800
$2A_2$	$2A_2$	4	288	36	1	268,800
$4A_1 + A_2$	$\emptyset$	2	4,608	1	1	302,400
$3A_1 + A_2$	A_1	−2	576	2	1	−1,209,600
$2A_1 + A_2$	A_3	2	96	24	1	604,800
$A_1 + A_2$	A_5	−2	24	720	1	−80,640
A_2	E_6	2	12	51,840	1	2,240
$8A_1$	$\emptyset$	1	10,321,920	1	30	2,025
$7A_1$	A_1	−1	645,120	2	30	−16,200
$6A_1$	$2A_1$	1	46,080	4	15	56,700
$5A_1$	$3A_1$	−1	3,840	8	5	−113,400
$(4A_1)'$	$4A_1$	1	384	16	1	113,400
$(4A_1)''$	D_4	1	384	192	1	9,450

$S = R(\mathfrak{k})$	$S^{\perp}$	$\tilde{\chi} := \tilde{\chi}(S)$	$\lvert \mathrm{Aut}(S) \rvert$	$\lvert W(S^{\perp}) \rvert$	$\#(S)$	$\lvert W(S) \rvert \cdot \tilde{\chi}$
$3A_1$	$A_1 + D_4$	-1	48	384	1	$-37,800$
$2A_1$	D_6	1	8	$23,040$	1	$3,780$
A_1	E_7	-1	2	$2,903,040$	1	-120

Taking the sum over all numbers of the right column yields:

$$-(1 + \tilde{\chi}(\hat{\Delta}_{E_8/T})) = y(120 - 120) + x(2,240 + 1,120 - 3,360) + 28,183,679.$$

Hence,

$$\tilde{\chi}(\hat{\Delta}_{E_8/T}) = -28,183,680 \neq 0.$$

It follows that $\hat{\Delta}_{E_8/T}$ is non-contractible.

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