

On an index theorem of Chang, Weinberger and Yu

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Abstract. In this paper we prove a strengthening of a theorem of Chang, Weinberger and Yu on obstructions to the existence of positive scalar curvature metrics on compact manifolds with boundary. They construct a relative index for the Dirac operator, which lives in a relative K -theory group, measuring the difference between the fundamental group of the boundary and of the full manifold.

Whenever the Riemannian metric has product structure and positive scalar curvature near the boundary, one can define an absolute index of the Dirac operator taking value in the K -theory of the C^* -algebra of fundamental group of the full manifold. This index depends on the metric near the boundary. We prove that (a slight variation of) the relative index of Chang, Weinberger and Yu is the image of this absolute index under the canonical map of K -theory groups.

This has the immediate corollary that positive scalar curvature on the whole manifold implies vanishing of the relative index, giving a conceptual and direct proof of the vanishing theorem of Chang, Weinberger and Yu (rather: a slight variation). To take the fundamental groups of the manifold and its boundary into account requires working with maximal C^* -completions of the involved $*$ -algebras. A significant part of this paper is devoted to foundational results regarding these completions. On the other hand, we introduce and propose a more conceptual and more geometric completion, which still has all the required functoriality.

1. INTRODUCTION

In [2] Chang, Weinberger and Yu define a relative index of the Dirac operator on a compact spin manifold M with boundary N as an element of $K_*(C^*(\pi_1(M), \pi_1(N)))$, where this relative K -theory group measures the difference between the two fundamental groups. The main geometric theorem of [2] then says that the existence of a positive scalar curvature metric on M which is collared at the boundary implies the vanishing of this index. The argument for this vanishing theorem is rather complicated and indeed contains a gap. We address this gap in this paper. After the first version of the present article was made public, [8] was posted, which also attempts to fix this gap.

More explicitly, the K -theory groups of the absolute and relative group C^* -algebras of the manifold and its boundary fit in a long exact sequence

$$(1) \quad \rightarrow K_*(C^*(\pi_1(N))) \rightarrow K_*(C^*(\pi_1(M))) \xrightarrow{j} K_*(C^*(\pi_1(M), \pi_1(N))) \rightarrow .$$

The relative index $\mu([M, N])$ is defined as the image of a relative fundamental class $[M, N] \in K_{\dim M}(M, N)$ under a relative index map $\mu: K_*(M, N) \rightarrow K_*(C^*(\pi_1(M), \pi_1(N)))$. Here, $K_*(M, N)$ is the relative K -homology and $[M, N]$ is constructed with the help of the Dirac operator on M . Indeed, in this paper we mainly deal with a small variant of the construction of [2] by choosing a slightly different C^* -completion. We discuss this in more detail below and, throughout the introduction, we work with this modification.

Our main goal is to better understand the vanishing theorem of Chang, Weinberger and Yu, and to prove a strengthening of it, at the same time giving a new and more conceptual proof.

For our approach, recall that one has a perfectly well-defined K -theoretic index of the Dirac operator on a Riemannian manifold with boundary provided the boundary operator is invertible, for example, if the metric is collared and of positive scalar curvature near the boundary (see, e.g., [17]). This index takes values in $K_*(C^*(\pi_1(M)))$ and explicitly depends on the boundary operator (i.e., on the positive scalar curvature metric g of the boundary). In the latter case we denote it by $\text{Ind}^{\pi_1(M)}(g) \in K_*(C^*(\pi_1(M)))$. Our main result states that a slight variant of the relative index of Chang–Weinberger–Yu is the image of the absolute index defined with invertible boundary operator under the natural homomorphism j of (1) (whenever this absolute index is defined):

Theorem 1.1. *We have*

$$j(\text{Ind}^{\pi_1(M)}(g)) = \mu([M, N]).$$

The absolute index $\text{Ind}^{\pi_1(M)}(g)$ vanishes whenever we have positive scalar curvature on all of M , implying immediately the corresponding vanishing result for the relative index of Chang, Weinberger and Yu.

Relative index theory has recently been the subject of considerable activity. In [3], Deeley and Goffeng define a relative index map using geometric K -homology instead of coarse geometry and prove index and vanishing results similar to the main result of our paper. However, this relies and uses the full package of higher Atiyah–Patodi–Singer index theory (like [14]), which we consider technically very demanding and somewhat alien to the spirit of large scale index theory. Indeed, in [3] it is not even proved in general that the constructions coincide with the ones of [2]. Yet another approach to relative index theory and the results of [2] is given by Kubota in [13]. There, the new concepts of relative Mishchenko bundles and Mishchenko–Fomenko index theory are introduced, and heavy use is made of the machinery of KK -theory. In [13], a careful identification of the different approaches is carried out.

The main point of our paper is its very direct and rather easy approach to the index theorems as described above. We work entirely in the realm of

large scale index theory, and just rely on the basic properties of the Dirac operator (locality, finite propagation of the wave operator, ellipticity). We avoid APS boundary conditions and we avoid deep KK -techniques. Such a direct approach is relevant also because it is more likely to allow for the construction of secondary invariants, to be used for classification rather than obstruction purposes.

In [2], fundamental use is made of the *maximal* Roe and localization algebras to obtain the required functoriality needed, e.g., in the sequence (1). The identification of its K -theory with K -homology of the space is needed for the *maximal* localization algebra and reference is given to [19] for the proof. However, that reference only deals with the *reduced* setting. Working out the details to extend the known results to the maximal setting turned out to be rather nontrivial. The first part of the present paper is devoted to the careful development of foundational issues of maximal Roe and localization algebras. For us, this complete and careful discussion of the properties of maximal completions in the context of coarse index theory is the second main contribution of this paper. Our results on this are used, e.g., in [3].

The maximal Roe algebra is defined in a rather ad hoc and ungeometric way: one comes up with the (somewhat arbitrary) algebraic Roe algebra, a $*$ -subalgebra of bounded operators on a Hilbert space which is *not* closed, and then passes to the maximal C^* -closure. This is hard to control and to compute (there are very few cases of actual computation), and geometric arguments are very delicate. It required the whole additional unpublished preprint [8], which appeared after the first version of this paper was posted, to prove the claim of [2] that the Schrödinger–Lichnerowicz vanishing theorem applies also to in the maximal Roe algebra. This claim was unjustified in [2], as the authors of [8] also observe.

Our approach is going in a different direction. We propose to use instead of the ad hoc maximal completion a much more geometric completion C_q^* , which we introduce in Section 3. Problems with the standard (reduced) Roe algebra arise in the equivariant setting of the group Γ acting on the space X due to lack of functoriality. Our completion takes all normal quotients Γ/N acting on X/N into account. This restores full functoriality, but is completely geometric. The Schrödinger–Lichnerowicz formula and other geometric arguments apply effortlessly.

The precise formulation of Theorem 1.1 and of (1) requires to specify which completion is used. In our approach, this becomes $C_q^*(\pi_1(M), \pi_1(N))$, involving the completions of the group algebras in the direct sum of the regular representation of all its quotients. Formally, the relative index in this K -theory group is weaker than the relative index obtained by using the maximal completion. However, not a single case is known where extra information on obstructions and classification has been obtained from the difference of the K -theory of the maximal and the reduced group C^* -algebras, and the Novikov conjecture suggests that this should not be possible. In any event, it seems extremely hard to exploit such a difference for geometric means. So we believe

that our approach and our completion is a very good choice: full functoriality, no extra effort for geometric arguments, in practice no loss of information.

Remark 1.2. Our approach works for arbitrary, also non-cocompact situations. In the cocompact case, there is another way for geometric constructions: one works with the compact space, and with the infinite-dimensional Mishchenko bundle. Here, one has the choice to use arbitrary group algebra completions, including the maximal one, which is used in [3] and [13].

Remark 1.3. We present details of the construction and manipulation of the relative index and the vanishing theorem only in the case that the dimension of the manifold is even. We chose to do this because this is the most classical set-up, and the constructions are particularly explicit and direct. This also means that we remain close to the original treatment of [2].

We discuss in Remark 5.3 how one can reduce the general case to the even-dimensional situation. We also discuss there how one could use the techniques of Zeidler [27] combined with our set-up to uniformly treat all dimensions and even the case of real C^* -algebras.

In parts of the present paper we give missing arguments for some of the results of the master thesis of Seyedhosseini [23].

1.4. Structure of the paper. In Section 2 we present our foundational results on *maximal* Roe algebras. In Section 3 we introduce our geometric functorial completed Roe algebra and establish its main properties. Section 4 recalls the construction of the relative index, following [2]. We try to motivate the construction, give additional details and fix small glitches in [2]. Section 5 gives the proof of Theorem 1.1.

2. THE MAXIMAL ROE ALGEBRA

In the following, we will only consider separable and proper metric spaces with bounded geometry. We recall that a locally compact metric X space has bounded geometry if one can find a discrete subset Y of X such that:

- There exists $c > 0$ such that every $x \in X$ has distance less than c to some $y \in Y$.
- For all $r > 0$, there is N_r such that $|Y \cap B_r(x)| \leq N_r$ for all $x \in X$.

A covering of a compact Riemannian manifold with the lifted metric obviously has bounded geometry.

2.1. Roe algebras. Let X be a separable and proper metric space endowed with a free and proper action of a discrete group Γ by isometries. In this section, we will recall the definition of the Roe algebra associated to X . Let $\rho: C_0(X) \rightarrow L(H)$ be an ample, nondegenerate representation of $C_0(X)$ on some separable Hilbert space H . A representation of $C_0(X)$ is called ample if no nonzero element of $C_0(X)$ acts as a compact operator on H . The representation ρ is called covariant for a unitary representation $\pi: \Gamma \rightarrow U(H)$ of Γ if $\rho(f_\gamma) = \text{Ad}_{\pi(\gamma)} \rho(f)$ for all $\gamma \in \Gamma$. Here f_γ denotes the function $x \mapsto f(\gamma^{-1}x)$.

From now on we will assume that ρ is an ample and covariant representation of $C_0(X)$ as above. By an abuse of notation we will denote $\rho(f)$ simply by f . We will later use representations of $C_0(X)$ which are an infinite direct sum of copies of an ample representation. Such representations are called very ample.

Definition 2.2. An operator $T \in L(H)$ is called a finite propagation operator if there exists an $r > 0$ such that $fTg = 0$ for all those $f, g \in C_0(X)$ with the property $d(\text{supp}(f), \text{supp}(g)) \geq r$. The smallest such r is called the propagation of T and is denoted by $\text{prop} T$. An operator $T \in L(H)$ is called locally compact if Tf and fT are compact for all $f \in C_0(X)$.

Definition 2.3. Denote by $\mathbb{R}_\rho(X)^\Gamma$ the $*$ -algebra of finite propagation, locally compact operators in $L(H)$ which are furthermore invariant under the action of the group Γ . We will call $\mathbb{R}_\rho(X)^\Gamma$ the algebraic Roe algebra of X . The maximal Roe algebra associated to the space X is the maximal C^* -completion of $\mathbb{R}_\rho(X)^\Gamma$, i.e., the completion of $\mathbb{R}_\rho(X)^\Gamma$ with respect to the supremum of all C^* -norms. This supremum is finite for spaces of bounded geometry by Proposition 2.4. It will be denoted by $C_{\rho, \max}^*(X)^\Gamma$. The reduced Roe algebra is the completion of the latter $*$ -algebra using the norm in $L(H)$. We denote this algebra by $C_{\rho, \text{red}}^*(X)^\Gamma$.

Proposition 2.4. *Suppose X has bounded geometry. For every $R > 0$, there is a constant C_R such that for every $T \in \mathbb{R}_\rho(X)^\Gamma$ with propagation less than R and every $*$ -representation $\pi: \mathbb{R}(X)^\Gamma \rightarrow L(H')$, we have*

$$\|\pi(T)\|_{L(H')} \leq C_R \|T\|_{C_{\rho, \text{red}}^*(X)^\Gamma}.$$

In particular, $\|T\|_{C_{\rho, \max}^(X)^\Gamma} \leq C_R \|T\|_{C_{\rho, \text{red}}^*(X)^\Gamma}$ and the bounded geometry assumption on X implies that the maximal Roe algebra is well defined.*

Proof. This follows from [7, Lem. 3.4] and [5, Thm. 2.7]. \square

Note that Proposition 2.4 implies that restricted to the subset of operators of propagation bounded by R , the reduced and the maximal norms are equivalent.

Proposition 2.5. *The K -theory groups of the reduced and maximal Roe algebra are independent of the chosen ample and covariant representation up to a canonical isomorphism.*

Proof. In the reduced case, this is the content of [11, Cor. 6.3.13]. For the maximal case we just note that conjugation by the isometries of the kind handled in [11, Section 6.3] gives rise to $*$ -homomorphisms of the algebraic Roe algebra and thus extend to morphisms of the maximal Roe algebras. Up to stabilization, any two such morphisms can be obtained from each other by conjugation by a unitary making the induced map in K -theory canonical. \square

Remark 2.6. As a consequence of Proposition 2.5 we will drop ρ in our notation for the Roe algebras. Later we will introduce a new completion of $\mathbb{R}(X)^\Gamma$, which sits between the reduced and maximal completions and denote it by $C_q^*(X)^\Gamma$. Moreover, if Γ is the trivial group, we will denote the Roe algebra by $C_d^*(X)$, where d stands for the chosen completion.

Proposition 2.7. *The K -theory of the maximal Roe algebra is functorial for coarse maps between locally compact metric spaces.*

Proof. The proof is similar to that of Proposition 2.5 and makes use of it. In the reduced case, this is proved by constructing an appropriate isometry between the representation spaces. Conjugation with the latter isometry gives rise to a $*$ -homomorphism of the algebraic Roe algebra and thus extends to a morphism of the reduced and maximal Roe algebra. The latter then gives rise to homomorphisms of the K -theory groups of the Roe algebra. As in the proof of Proposition 2.5, the induced map in K -theory is canonical which also implies functoriality. See [11, Section 6.3] for a more detailed discussion. \square

In the case where Γ acts cocompactly on X , we have the following theorem.

Theorem 2.8. *Suppose that Γ acts cocompactly on X . Then $K_*(C_{\max}^*(X)^\Gamma) \cong K_*(C_{\max}^*(\Gamma))$.*

Proof. See [7, Sections 3.12, 3.14] for the isomorphism $C_{\max}^*(|\Gamma|)^\Gamma \cong C_{\max}^*(\Gamma) \otimes K(H)$, where $C_{\max}^*(|\Gamma|)^\Gamma$ is the equivariant Roe algebra of Γ seen as a metric space using some word metric. The action of Γ on itself is given by left multiplication. Since the action of Γ on X is cocompact, the Γ -space X is coarsely equivalent to Γ . This implies that $K_*(C_{\max}^*(X)^\Gamma) \cong K_*(C_{\max}^*(|\Gamma|)^\Gamma)$. The claim then follows from the stability of K -theory. \square

For a Γ -invariant closed subset Y of X , we would like to define its Roe algebra relative to X as a closure of a space of operators in $C_{\max}^*(X)^\Gamma$, which are suitably supported near Y . The next two definitions make this precise.

Definition 2.9. For an operator $T \in L(H)$ we define the support $\text{supp } T$ of T as the complement of the union of all open sets $U_1 \times U_2 \subset X \times X$ with the property that $fTg = 0$ for all f and g with $\text{supp } f \subset U_1$ and $\text{supp } g \subset U_2$. T is said to be *supported near* $Y \subset X$ if there exists $r > 0$ such that $\text{supp } T \subset B_r(Y) \times B_r(Y)$. Here and afterwards $B_r(Y)$ denotes the open r -neighborhood of Y .

Definition 2.10. For a Γ -invariant closed subset Y of X as above, denote by $\mathbb{R}(Y \subset X)^\Gamma$ the $*$ -algebra of operators in $\mathbb{R}(X)^\Gamma$ which are supported near Y . The relative Roe algebra of Y in X is defined as the closure of $\mathbb{R}(Y \subset X)^\Gamma$ in $C_{\max}^*(X)^\Gamma$ and is an ideal inside the latter C^* -algebra. It is denoted by $C_{\max}^*(Y \subset X)^\Gamma$.

Since Y is a locally compact metric space with an action of Γ , it has its own (absolute) equivariant Roe algebra $C_{\max}^*(Y)^\Gamma$. Theorem 2.12 identifies the K -theory of the relative and absolute equivariant Roe algebras in the case, where the action of Γ on the subset is cocompact. However, for its proof we need further conditions on the group action.

Definition 2.11. Let Γ act freely and properly by isometries on X . Γ is said to *act conveniently* if there exists a fundamental domain F for the action of Γ satisfying:

- For each $R > 0$, there exist $\gamma_1, \dots, \gamma_{N_R} \in \Gamma$ such that $B_R(F) \subset \bigcup_{i=1}^{N_R} \gamma_i \cdot F$
- For each $\gamma \in \Gamma$ and $R > 0$, there exists $S(R, \gamma) > 0$ such that $\gamma^{-1}B_R(x) \cap F \subset B_{S(R, \gamma)}(x)$ for all $x \in F$.

Theorem 2.12. *Let Y and X be as above and suppose that Γ acts conveniently on X and cocompactly on Y . The inclusion $Y \rightarrow X$ induces an isomorphism $K_*(C_{\max}^*(Y)^\Gamma) \cong K_*(C_{\max}^*(Y \subset X)^\Gamma)$.*

Remark 2.13. A representation $\rho: C_0(X) \rightarrow L(H_X)$ gives rise to a spectral measure which can be used to extend ρ to the C^* -algebra $B_\infty(X)$ of bounded Borel functions on X (see [15, Thm. 2.5.5]). Given $Y \subset X$, we get a representation $C_0(Y) \rightarrow L(\chi_Y H_X)$. This is what is meant in the following Lemma 2.14 by “compressing the representation space of $C_0(X)$ in order to obtain a representation of $C_0(Y)$ ”. Given Y as above, we can choose ρ such that it and its compression to Y are both ample; for example, by choosing the ample representation of X to be given by multiplication of functions with square summable sequences on some countable dense subset of X whose intersection with Y is a dense subset of Y . We will need Lemma 2.14 for the proof of Theorem 2.12. Indeed, the novel difficulty in Theorem 2.12 is to relate the $*$ -representations used in the definition of $C_{\max}^*(Y)^\Gamma$ with the $*$ -representations used to define $C_{\max}^*(X)^\Gamma$ —of which $C_{\max}^*(Y \subset X)^\Gamma$ by definition is an ideal. Note that, at the moment, we only manage to do this if Y is cocompact and the Γ -action is convenient. It is an interesting challenge to generalize Theorem 2.12 to arbitrary pairs (X, Y) and arbitrary free and proper actions.

In the following lemma we choose an ample representation of X which can be compressed to an ample representation of Z . As pointed out in the previous remark, this can always be done.

Lemma 2.14. *Let Γ act conveniently on X , let $Z \subset X$ be Γ -invariant and suppose that the action of Γ on Z is cocompact. Construct $\mathbb{R}(Z)^\Gamma$ by compressing the representation space of $C_0(X)$, so that $\mathbb{R}(Z)^\Gamma$ is naturally a $*$ -subalgebra of $\mathbb{R}(X)^\Gamma$. Then an arbitrary non-degenerate $*$ -representation of $\mathbb{R}(Z)^\Gamma$ on a Hilbert space can be extended to a non-degenerate $*$ -representation of $\mathbb{R}(X)^\Gamma$. In particular, the inclusion $\mathbb{R}(Z)^\Gamma \rightarrow \mathbb{R}(X)^\Gamma$ extends to an injection $C_{\max}^*(Z)^\Gamma \rightarrow C_{\max}^*(X)^\Gamma$.*

Proof. Choose an ample representation $\rho: C_0(X) \rightarrow L(H_X)$ as above. By compressing the Hilbert space H_X and restricting the representation, we obtain an ample representation of $C_0(Z)$, i.e., $\rho|_{C_0(Z)}: C_0(Z) \rightarrow L(H_Z)$, where H_Z denotes the space $\chi_Z H_X$. Choose $D_Z \subset D_X$ as fundamental domains of Z and X for the action of Γ . Similar to the proof of [11, Lem. 12.5.3], one has $\mathbb{R}(Z)^\Gamma \cong \mathbb{C}[\Gamma] \odot K(\tilde{H}_Z)$, where $\tilde{H}_Z = \chi_{D_Z} H_Z$. The latter isomorphism is obtained using the isomorphisms $H_Z \cong \bigoplus_{\gamma \in \Gamma} \tilde{H}_Z \cong l^2(\Gamma) \otimes \tilde{H}_Z$. Denote by \tilde{H}_X the Hilbert

space $\chi_{D_X} H_X$. The isomorphism constructed in the proof can be extended to an injective map $\mathbb{C}[\Gamma] \odot L(\tilde{H}_X) \rightarrow L(H_X)$. The convenience of the action implies that its image contains the algebra $\mathbb{F}(X)^\Gamma$ of finite propagation Γ -invariant operators on X . This injection makes the diagram

$$\begin{CD} \mathbb{C}[\Gamma] \odot K(\tilde{H}_Z) @>\cong>> \mathbb{R}(Z)^\Gamma \\ @VVV @VVV \\ \mathbb{C}[\Gamma] \odot L(\tilde{H}_X) @>>> L(H_X) \end{CD}$$

commutative. We show that an arbitrary non-degenerate $*$ -representation of $\mathbb{C}[\Gamma] \odot K(\tilde{H}_Z)$ on a Hilbert space H_0 can be extended to a non-degenerate $*$ -representation of $\mathbb{C}[\Gamma] \odot L(\tilde{H}_X)$. This implies the lemma since $\mathbb{R}(X)^\Gamma \subset \mathbb{F}(X)^\Gamma$. Suppose that $\pi: \mathbb{C}[\Gamma] \odot K(\tilde{H}_Z) \rightarrow L(H_0)$ is a non-degenerate $*$ -representation of $\mathbb{C}[\Gamma] \odot K(\tilde{H}_Z)$ on a Hilbert space H_0 . The representation π extends to a representation of $C_{\max}^*(\Gamma) \otimes K(\tilde{H}_Z)$, which we denote by π . Note that since the C^* -algebra of compact operators is nuclear, the C^* -algebra tensor product above is unique. $C_{\max}^*(\Gamma) \otimes K(\tilde{H}_Z)$ is a C^* -subalgebra of $C_{\max}^*(\Gamma) \otimes K(\tilde{H}_X)$ and π can thus be extended to a non-degenerate representation of $C_{\max}^*(\Gamma) \otimes K(\tilde{H}_X)$ on a possibly bigger Hilbert space H , which we denote by $\tilde{\pi}$. From [15, Thm. 6.3.5], it follows that there exist unique non-degenerate representations $\tilde{\pi}_1$ and $\tilde{\pi}_2$ of $C_{\max}^*(\Gamma)$ and $K(\tilde{H}_X)$ on H , respectively, such that $\tilde{\pi}(a \otimes b) = \tilde{\pi}_1(a)\tilde{\pi}_2(b) = \tilde{\pi}_2(b)\tilde{\pi}_1(a)$ for all $(a, b) \in C_{\max}^*(\Gamma) \times K(\tilde{H}_X)$. The representation $\tilde{\pi}_2$ can be extended to a representation $\hat{\pi}_2$ of $L(\tilde{H}_X)$ on H by [4, Lem. 2.10.3], and from the same lemma, it follows that $\tilde{\pi}_2(K(\tilde{H}_X))$ is strongly dense in $\hat{\pi}_2(L(\tilde{H}_X))$. From the double commutant theorem, it follows that the commutant of a C^* -subalgebra of $L(H)$ is strongly closed. This in turn implies that $\tilde{\pi}_1(a)\hat{\pi}_2(b) = \hat{\pi}_2(b)\tilde{\pi}_1(a)$ for $(a, b) \in C_{\max}^*(\Gamma) \times L(\tilde{H}_X)$. Now restrict $\tilde{\pi}_1$ to $\mathbb{C}[\Gamma]$. From [15, Rem. 6.3.2], it follows that there is a unique $*$ -representation $\hat{\pi}: \mathbb{C}[\Gamma] \odot L(\tilde{H}_X) \rightarrow L(H)$ with the property $\hat{\pi}(a \otimes b) = \tilde{\pi}_1(a)\hat{\pi}(b)$. It is clear that $\hat{\pi}$ is an extension of π . □

Proof of Theorem 2.12. The proof is analogous to that of [12, Lem. 5.1]. As in Lemma 2.14, construct the algebras $C^*(\overline{B_r(Y)})^\Gamma$ by compressing the representation space of $C_0(X)$. The inclusions $\mathbb{R}(\overline{B_r(Y)})^\Gamma \rightarrow \mathbb{R}(\overline{B_R(Y)})^\Gamma$ for $r \leq R$ induce maps $C_{\max}^*(\overline{B_r(Y)})^\Gamma \rightarrow C_{\max}^*(\overline{B_R(Y)})^\Gamma$. We will show that $\varinjlim C_{\max}^*(\overline{B_r(Y)})^\Gamma = C_{\max}^*(Y \subset X)^\Gamma$. Let A be a C^* -algebra and let $\phi_r: C_{\max}^*(\overline{B_r(Y)})^\Gamma \rightarrow A$ be C^* -algebra morphisms such that all the diagrams of the form

$$\begin{CD} C_{\max}^*(\overline{B_r(Y)})^\Gamma @>>> C_{\max}^*(\overline{B_R(Y)})^\Gamma \\ @VVV @. \\ A @<<< @. \end{CD}$$

with $r < R$ commute. The above compatibility condition implies the existence of a unique morphism of $*$ -algebras $\phi: \mathbb{R}(Y \subset X)^\Gamma \rightarrow A$, such that all the diagrams

$$\begin{array}{ccc} \mathbb{R}(\overline{B_r(Y)})^\Gamma & \longrightarrow & \mathbb{R}(Y \subset X)^\Gamma \\ \downarrow & \swarrow & \\ A & & \end{array}$$

are commutative. Lemma 2.14 then implies that the map ϕ is continuous if $\mathbb{R}(Y \subset X)^\Gamma$ is endowed with the norm of $C^*(X)^\Gamma$. To see this, note that Lemma 2.14 implies that for $a \in \mathbb{R}(\overline{B_r(Y)})$, $\|a\|_{C^*_{\max}(\overline{B_r(Y)})^\Gamma} = \|a\|_{C^*_{\max}(X)^\Gamma}$. Hence, $\|\phi(a)\| = \|\phi_r(a)\| \leq \|a\|_{C^*_{\max}(\overline{B_r(Y)})^\Gamma} = \|a\|_{C^*_{\max}(X)^\Gamma}$. Thus, ϕ can be extended uniquely to a morphism $C^*(Y \subset X)^\Gamma \rightarrow A$ of C^* -algebras. The universal property of the direct limit of C^* -algebras implies $\varinjlim C^*_{\max}(\overline{B_r(Y)})^\Gamma = C^*_{\max}(Y \subset X)^\Gamma$. The claim of the theorem then follows from the continuity of K -theory and the coarse equivalence of $\overline{B_r(Y)}$ and $\overline{B_R(Y)}$ for arbitrary $r, R \in \mathbb{N}$ (recall that the K -theory groups of the Roe algebras of coarsely equivalent spaces are isomorphic). \square

2.15. The structure algebra and Paschke duality. Let X be as in the previous section. A representation $\rho: C_0(X) \rightarrow L(H)$ of $C_0(X)$ is called very ample if it is an infinite sum of copies of an ample representation. Construct $\mathbb{R}(X)^\Gamma$ and $C^*(X)^\Gamma$ using some very ample representation. In this section we will define a C^* -algebra associated to X which contains $C^*_{\max}(X)^\Gamma$ as an ideal and such that the K -theory of the quotient provides a model for K -homology of X .

Definition 2.16. We recall that an operator $T \in L(H)$ is called pseudolocal if it commutes with the image of ρ up to compact operators; i.e., $[f, T] \in K(H)$ for all $f \in C_0(X)$.

Definition 2.17. Denote by $\mathbb{S}_\rho(X)^\Gamma$ the $*$ -algebra of finite propagation, pseudolocal operators in $L(H)$ which are furthermore invariant under the action of the group Γ . The maximal structure algebra associated to the space X is the maximal C^* -completion of $\mathbb{S}_\rho(X)^\Gamma$. It will be denoted by $D^*_{\rho, \max}(X)^\Gamma$. The reduced structure algebra is the completion of the latter $*$ -algebra using the norm in $L(H)$. We denote this algebra by $D^*_{\rho, \text{red}}(X)^\Gamma$.

Remark 2.18. From now on, we will drop ρ from our notation. Later we will introduce a new completion of $\mathbb{S}(X)^\Gamma$, which sits between the reduced and maximal completions and denote it by $D^*_q(X)^\Gamma$. If the action of Γ is trivial, we denote the structure algebra by $D^*_d(X)$, where d stands for the chosen completion.

In comparison to the well-known $D^*_{\text{red}}(X)^\Gamma$, the definition and properties of the maximal structure algebra $D^*_{\max}(X)^\Gamma$ are trickier than one might think in the first place. First of all, one has to establish its existence; i.e., an upper bound on the C^* -norms. Secondly, we want $C^*_{\max}(X)^\Gamma$ to be an ideal in

$D_{\max}^*(X)^\Gamma$, and for this, one has to control the a priori different C^* -representations which are used in the definitions. Only then does it make sense to form $D_{\max}^*(X)/C_{\max}^*(X)$. Paschke duality states that its K -theory is canonically isomorphic to the locally finite K -homology of X . All of this will be done in the remainder of this section. We now introduce the so-called dual algebras, which are larger counterparts of the Roe and structure algebra.

Definition 2.19. Denote by $\mathfrak{C}^*(X)^\Gamma$ the C^* -algebra of Γ -invariant locally compact operators in $L(H)$. Denote by $\mathfrak{D}^*(X)^\Gamma$ the C^* -algebra of Γ -invariant pseudolocal operators in $L(H)$.

It is clear that $\mathfrak{C}^*(X)^\Gamma$ is an ideal of $\mathfrak{D}^*(X)^\Gamma$. We have the following.

Theorem 2.20. *There is an isomorphism $K_{*+1}(\frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}) \cong K_*^{lf}(X)$, where the right-hand side is the locally finite K -homology of X , given as the Kasparov group $KK_*(C_0(X), \mathbb{C})$.*

Proof. This is proven in [24, Prop. 3.4.11]. □

Lemma 2.21. *The map $\frac{\mathbb{S}(X)}{\mathbb{R}(X)} \rightarrow \frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}$ induced by the inclusion $\mathbb{S}(X) \rightarrow \mathfrak{D}^*(X)$ is an isomorphism. In particular, $\frac{\mathbb{S}(X)}{\mathbb{R}(X)}$ is a C^* -algebra. The corresponding statement holds for the Γ -equivariant versions.*

Proof. In [11, Lem. 12.3.2], the isomorphism $\frac{D_{\text{red}}^*(X)}{C_{\text{red}}^*(X)} \cong \frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}$ is proven. The truncation argument used in the proof shows that $\mathfrak{D}^*(X) = \mathbb{S}(X) + \mathfrak{C}^*(X)$, which implies the surjectivity of the map $\frac{\mathbb{S}(X)}{\mathbb{R}(X)} \rightarrow \frac{\mathfrak{D}^*(X)}{\mathfrak{C}^*(X)}$. Injectivity is clear. An analogous argument using a suitable invariant open covering and partition of unity gives the isomorphism $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \cong \frac{\mathfrak{D}^*(X)^\Gamma}{\mathfrak{C}^*(X)^\Gamma}$. □

Proposition 2.22. *For $a \in \mathbb{S}(X)^\Gamma$, there exists $C_a > 0$ such that, for an arbitrary non-degenerate representation π of $\mathbb{S}(X)^\Gamma$, we have $\|\pi(a)\| \leq C_a$.*

We need a few lemmas before proving Proposition 2.22. This proposition shows that the maximal structure algebra is well defined. Since the structure algebra depends on both the coarse and topological structure of the space, the coarse geometric property of having bounded geometry alone does not guarantee the existence of the maximal structure algebra. This is where the properness of the metric is needed. More precisely, this is used in Lemma 2.21, which in turn is used in the proof of Proposition 2.22.

Lemma 2.23. *There exists a C^* -algebra $A \subset \mathbb{R}(X)^\Gamma$ which contains an approximate identity for $C_{\max}^*(X)^\Gamma$.*

Proof. Let D be a fundamental domain for the action of Γ on X . Choose a discrete subset Y_D of D as provided by the bounded geometry condition. Denote the set obtained by transporting Y_D by the action of Γ by Y . Y is then clearly Γ -invariant. By [5, Prop. 2.7], extended straightforwardly to the equivariant case, it suffices to show that there exists a C^* -algebra $B \subset \mathbb{R}(Y)^\Gamma$ which contains an approximate identity for $C_{\max}^*(Y)^\Gamma$. Here, as the

representation space, we choose $l^2(Y) \otimes l^2(\mathbb{N})$, where the action of $C_0(Y)$ is given by multiplication. By [5, Prop. 2.19], $l^\infty(Y; C_0(\mathbb{N}))^\Gamma \subset \mathbb{R}(Y)^\Gamma$ is a C^* -algebra which contains an approximate unit of $\mathbb{R}(Y)$ endowed with the reduced norm and, by Proposition 2.4, of $\mathbb{R}(Y)$ endowed with the maximal norm. The claim then follows from density of $\mathbb{R}(Y)^\Gamma$ in $C^*(Y)^\Gamma$. \square

Lemma 2.24. *Let ρ be an arbitrary non-degenerate $*$ -representation of $\mathbb{R}(X)^\Gamma$ on some Hilbert space H . It extends in a unique way to a $*$ -representation of $\mathbb{S}(X)^\Gamma$ on H .*

More generally, let $\mathbb{M}(X)^\Gamma$ be the algebra of bounded multipliers of $\mathbb{R}(X)^\Gamma$, i.e., all bounded operators on the defining Hilbert space which preserve $\mathbb{R}(X)^\Gamma$ by left and right multiplication. Note that $\mathbb{M}(X)^\Gamma$ contains $\mathbb{S}(X)^\Gamma$. The representation ρ extends in a unique way to a $$ -representation of $\mathbb{M}(X)^\Gamma$.*

Proof. Let $\pi: \mathbb{R}(X)^\Gamma \rightarrow L(H)$ be a non-degenerate $*$ -representation of $\mathbb{R}(X)^\Gamma$. It extends to a non-degenerate representation of $C_{\max}^*(X)^\Gamma$. Pick a C^* -subalgebra A of $C_{\max}^*(X)^\Gamma$ which contains an approximate identity for $C_{\max}^*(X)^\Gamma$ and sits inside $\mathbb{R}(X)^\Gamma$. The restriction of π to A is thus also non-degenerate. It follows from the Cohen–Hewitt factorization theorem [10, Thm. 2.5] that, for all $w \in H$, there exist $T \in A$ and $v \in H$ with $\pi(T)v = w$. Furthermore, $\pi(S)v = 0$ for all $S \in \mathbb{R}(X)^\Gamma$ implies that v is in the orthogonal complement of $\pi(\mathbb{R}(X))H$; hence, $v = 0$ by the nondegeneracy of π . It follows from [6, Prop. IV.3.18] that $\hat{\pi}(T)(\pi(S)v) := \pi(TS)v$ for $T \in \mathbb{S}(X)^\Gamma$ gives a well-defined algebraic representation $\hat{\pi}: \mathbb{M}(X)^\Gamma \rightarrow \mathbb{L}(H)$. Here $\mathbb{L}(H)$ denotes the vector space of linear maps on H . It is clear that $\hat{\pi}$ is an extension of π . We show that $\hat{\pi}$ is actually a $*$ -representation of $\mathbb{M}(X)^\Gamma$. The equalities

$$\begin{aligned} \langle \hat{\pi}(T)(\pi(S)v), \pi(S')v' \rangle &= \langle \pi(TS)v, \pi(S')v' \rangle = \langle \pi((S^*T^*)^*)v, \pi(S')v' \rangle \\ &= \langle v, \pi(S^*T^*S')v' \rangle = \langle \pi(S)v, \pi(T^*S')v' \rangle \\ &= \langle \pi(S)v, \hat{\pi}(T^*)(\pi(S')v') \rangle \end{aligned}$$

imply that the operator $\hat{\pi}(T)$ is formally selfadjoint if T is selfadjoint. Furthermore, since $\hat{\pi}(T)$ is defined everywhere on H , it follows from the Hellinger–Toeplitz theorem that it is bounded. Since every element of a $*$ -algebra is a linear combination of selfadjoint elements, this implies that the image of $\hat{\pi}$ is actually contained in $L(H)$. The previous computation then shows that $\hat{\pi}$ respects the involution; thus, it is a $*$ -representation. Uniqueness of the extension follows from the fact that every extension $\hat{\pi}$ of π has to satisfy $\hat{\pi}(T)(\pi(S)v) = \pi(TS)v$ for $T \in \mathbb{M}(X)^\Gamma$ and $S \in \mathbb{R}(X)^\Gamma$, but this determines $\hat{\pi}$ since all elements of H are of the form $\pi(S)v$ for some $S \in \mathbb{R}(X)^\Gamma$ and $v \in H$. \square

Lemma 2.25. *An arbitrary non-degenerate $*$ -representation π of $\mathbb{S}(X)^\Gamma$ can be decomposed as $\pi = \pi_1 \oplus \pi_2$, where both π_1 and its restriction to $\mathbb{R}(X)^\Gamma$ are non-degenerate representations on some Hilbert space H_1 and π_2 is a non-degenerate representation of $\mathbb{S}(X)^\Gamma$ vanishing on $\mathbb{R}(X)^\Gamma$.*

Proof. The proof follows from Lemma 2.24 and the discussion prior to [1, Thm. 1.3.4]. □

Proof of Proposition 2.22. We denote by S the set of cyclic representations of $\mathbb{S}(X)^\Gamma$ on some Hilbert space with the property that their restriction to $\mathbb{R}(X)^\Gamma$ is a non-degenerate representation of $\mathbb{R}(X)^\Gamma$ on the same space. For $\pi \in S$, denote by $\pi_{\mathbb{R}}$ its restriction to $\mathbb{R}(X)^\Gamma$. The bounded geometry condition on X (see Proposition 2.4) implies that $\bigoplus_{\pi \in S} \pi_{\mathbb{R}}$ is a well-defined non-degenerate representation of $\mathbb{R}(X)^\Gamma$. Lemma 2.24 implies that $\Pi = \bigoplus_{\pi \in S} \pi$ is a well-defined Hilbert space representation of $\mathbb{S}(X)^\Gamma$. For $a \in \mathbb{S}(X)^\Gamma$, set $C_1^a = \|\Pi(a)\|$. It is shown in Lemma 2.21 that $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$ is a C^* -algebra. Set $C_2^a = \|[a]\|_{\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}}$ and $C_a = \max\{C_1^a, C_2^a\}$. Now let π be an arbitrary non-degenerate representation of $\mathbb{S}(X)^\Gamma$ with a decomposition $\pi_1 \oplus \pi_2$, as provided by Lemma 2.25. Obviously, $\|\pi(a)\| \leq \max\{\|\pi_1(a)\|, \|\pi_2(a)\|\}$. The claim now follows from the facts that π_1 is a subrepresentation of Π and π_2 factors through $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$. □

Proposition 2.26. *As with the Roe algebra, the K -theory groups of the structure algebra are independent of the choice of the very ample representation. Furthermore, the assignment $X \mapsto K_*(D_{\max}^*(X)^\Gamma)$ is functorial for uniform (i.e., coarse and continuous) maps.*

Proof. See the discussion in [11, Ch. 12.4] □

Lemma 2.24 immediately implies the following.

Proposition 2.27. $C_{\max}^*(X)^\Gamma$ is an ideal of $D_{\max}^*(X)^\Gamma$.

Proposition 2.28. *The inclusion $\mathbb{S}(X)^\Gamma \rightarrow D_{\max}^*(X)^\Gamma$ gives rise to an isomorphism $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \cong \frac{D_{\max}^*(X)^\Gamma}{C_{\max}^*(X)^\Gamma}$.*

Proof. Since $D_{\max}^*(X)^\Gamma$ is the maximal C^* -completion of $\mathbb{S}(X)^\Gamma$, the projection $\mathbb{S}(X)^\Gamma \rightarrow \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$ gives rise to a morphism of C^* -algebras $D_{\max}^*(X)^\Gamma \rightarrow \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$. Continuity of this map and the fact that its kernel contains $\mathbb{R}(X)^\Gamma$ imply that it induces a morphism $\frac{D_{\max}^*(X)^\Gamma}{C_{\max}^*(X)^\Gamma} \rightarrow \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$. The composition $\frac{D_{\max}^*(X)^\Gamma}{C_{\max}^*(X)^\Gamma} \rightarrow \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \rightarrow \frac{D^*(X)^\Gamma}{C^*(X)^\Gamma}$ is the identity on the set of classes of $\frac{D_{\max}^*(X)^\Gamma}{C_{\max}^*(X)^\Gamma}$ which have a representative from $\mathbb{S}(X)^\Gamma$. Since the latter set is dense, it follows that the composition is injective. On the other hand, by construction, the composition $\frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma} \rightarrow \frac{D_{\max}^*(X)^\Gamma}{C_{\max}^*(X)^\Gamma} \rightarrow \frac{\mathbb{S}(X)^\Gamma}{\mathbb{R}(X)^\Gamma}$ is the identity and the claim follows. □

Corollary 2.29. *There is an isomorphism $K_{*+1}(\frac{D_{\max}^*(X)}{C_{\max}^*(X)}) \cong K_*^{lf}(X)$.*

2.30. Yu’s localization algebras and K -homology.

Definition 2.31 ([19, Section 2]). Let A be a normed $*$ -algebra. By \mathfrak{A} denote the normed $*$ -algebra of functions $f: [1, \infty) \rightarrow A$ which are bounded and uniformly continuous.

Clearly, if A is a C^* -algebra, so is $\mathfrak{A}A$. Important examples for us will be the algebras $\mathfrak{A}D_{\max}^*(X)$ and $\mathfrak{A}C_{\max}^*(X)$, defined using some very ample representation of $C_0(X)$. Now we are in the position to define the localization algebra associated to a locally compact metric space X .

Definition 2.32 ([19, Section 2]). The C^* -algebra generated by functions $f \in \mathfrak{A}C_{\max}^*(X)^\Gamma$, with the properties

- $\text{prop } f(t) < \infty$ for all $t \in [1, \infty)$,
- $\text{prop } f(t) \rightarrow 0$ as $t \rightarrow \infty$,

is called the localization algebra of X and is denoted by $C_{L,\max}^*(X)^\Gamma$.

Remark 2.33. In analogy to the fact that $C_{\max}^*(X)^\Gamma$ is contained as an ideal in the C^* -algebra $D_{\max}^*(X)^\Gamma$, one can define a C^* -algebra denoted by $D_{L,\max}^*(X)^\Gamma$, which contains $C_{L,\max}^*(X)^\Gamma$ as an ideal. This is the C^* -algebra generated by the elements in $\mathfrak{A}D_{\max}^*(X)^\Gamma$ with the two properties of Definition 2.32.

Yu’s theorem states that the K -theory groups of the localization algebra are isomorphic to the locally finite K -homology groups.

Theorem 2.34 ([19, Thm. 3.4]). *Let X be a locally compact metric space and suppose $C_{L,\max}^*(X)$ is defined using a very ample representation. Then the local index map $\text{ind}_L: K_*^{lf}(X) \rightarrow K_*(C_{L,\max}^*(X))$ of [19, Def. 2.4] is an isomorphism. Furthermore, the diagram*

$$\begin{array}{ccc}
 K_*^{lf}(X) & \xrightarrow{\text{ind}_L} & K_*(C_{L,\max}^*(X)) \\
 & \searrow \mu & \downarrow (\text{ev}_1)_* \\
 & & K_*(C_{\max}^*(X))
 \end{array}$$

is commutative. Here μ denotes the index map $K_*^{lf}(X) \cong K_{*+1}(\frac{D_{\max}^*(X)}{C_{\max}^*(X)}) \rightarrow K_*(C_{\max}^*(X))$.

Proof. First note that the local index map as defined in [19, Thm. 3.4] can be defined analogously in the maximal case. In [19] the theorem is proven for the reduced localization algebra and uses the isomorphism $K_{*+1}(\frac{D_{\text{red}}^*(X)}{C_{\text{red}}^*(X)}) \cong K_*^{lf}(X)$. However, Corollary 2.29 states that the isomorphism still holds if we replace the reduced Roe and structure algebra with the maximal ones. Thus, the argument of [19] can be used literally. \square

Having the above theorem in mind, we will, from now on, use the notation $K_*^L(X)$ for the group $K_*(C_{L,\max}^*(X))$. Given a closed subset Y of X , we are now going to define the relative K -homology groups using localization algebras and discuss the existence of a long exact sequence for pairs. Chang, Weinberger and Yu define the relative groups by using a concrete very ample representation, which we will now describe.

Let $Y \subset X$ be as above. Choose a countable dense set Γ_X of X such that $\Gamma_Y := \Gamma_X \cap Y$ is dense in Y . Define $C_{L,\max}^*(X)$ and $C_{L,\max}^*(Y)$ using the

very ample representations $H_X = l^2(\Gamma_X) \otimes l^2(\mathbb{N})$ and $H_Y = l^2(\Gamma_Y) \otimes l^2(\mathbb{N})$, respectively. The constant family of isometries $V_t := \iota$, where $\iota: H_Y \rightarrow H_X$ is the inclusion covers the inclusion $Y \rightarrow X$ in the sense of [19, Def. 3.1]. Hence, applying $\text{Ad}(V_t)$ pointwise, we obtain a C^* -algebra morphism $C_{L,\max}^*(Y) \rightarrow C_{L,\max}^*(X)$, which we will denote by $\iota(X, Y)$. Note that on elements with finite propagation, this map, for each t , is just the extension by zero of an operator on H_Y to an operator on H_X . We get a map $\iota(X, Y)_*: K_*^L(Y) \rightarrow K_*^L(X)$.

Now denote by $K_*^L(X, Y)$ the group $K_{*-1}(C_{\iota(X,Y)})$, where S denotes the suspension and $C_{\iota(X,Y)}$ the mapping cone of $\iota(X, Y)$. The short exact sequence

$$0 \rightarrow SC_{L,\max}^*(X) \rightarrow C_{\iota(X,Y)} \rightarrow C_{L,\max}^*(Y) \rightarrow 0$$

gives rise to a long exact sequence

$$\cdots \rightarrow K_*(C_{L,\max}^*(Y)) \rightarrow K_{*-1}(SC_{L,\max}^*(X)) \rightarrow K_{*-1}(C_{\iota(X,Y)}) \rightarrow \cdots$$

of K -theory groups. Using the canonical isomorphism $K_{*-1}(S(\cdot)) = K_*(\cdot)$, this sequence becomes the desired long exact sequence of a pair

$$\cdots \rightarrow K_*^L(Y) \rightarrow K_*^L(X) \rightarrow K_*^L(X, Y) \rightarrow \cdots,$$

constructed solely using localization algebras.

2.34.1. Relative localization algebra. Let X and Y be as above. We would like to extend $C_{L,\max}^*(Y)^\Gamma \subset C_{L,\max}^*(X)^\Gamma$ to an ideal with the same K -theory.

Definition 2.35. Denote by $C_{L,\max}^*(Y \subset X)^\Gamma$ the ideal in $C_{L,\max}^*(X)^\Gamma$ generated by functions $f \in \mathfrak{T}C^*(X)^\Gamma$ such that for all $t \in [1, \infty)$, $f(t)$ is supported in an $S(t)$ -neighborhood of Y , where $S: [1, \infty) \rightarrow \mathbb{R}$ is some function with $S(t) \rightarrow 0$ as $t \rightarrow \infty$.

Lemma 2.36 ([26, Lem. 1.4.18]). *Let Y and X be as above. The inclusion $Y \rightarrow X$ induces isomorphisms $K_*(C_{L,\max}^*(Y)^\Gamma) \cong K_*(C_{L,\max}^*(Y \subset X)^\Gamma)$.*

Proof. In [26, Lem. 1.4.18], this is proven in the reduced case. However in light of the discussion in Section 2.1, the modification of the arguments for use in the maximal setting is straight-forward. \square

2.37. Relative group C^* -algebra. Let X be a proper path-connected metric space and Y a path-connected subset of X . The inclusion $Y \rightarrow X$ induces a map $\pi_1(Y) \rightarrow \pi_1(X)$, where we choose a point $y_0 \in Y$ to construct the fundamental groups and the latter map. This map in turn induces a morphism $\varphi: C_{\max}^*(\pi_1(Y)) \rightarrow C_{\max}^*(\pi_1(X))$. The *relative group C^* -algebra* is defined as

$$C_{\max}^*(\pi_1(X), \pi_1(Y)) := SC_\varphi.$$

The short exact sequence

$$0 \rightarrow SC_{\max}^*(\pi_1(X)) \rightarrow C_\varphi \rightarrow C_{\max}^*(\pi_1(Y)) \rightarrow 0$$

and the Bott periodicity isomorphism gives a long exact sequence

$$\rightarrow K_*(C_{\max}^*(\pi_1(Y))) \rightarrow K_*(C_{\max}^*(\pi_1(X))) \rightarrow K_*(C_{\max}^*(\pi_1(X), \pi_1(Y))) \rightarrow .$$

Remark 2.38. Note that the above C^* -algebras are independent of the chosen point y_0 up to an isomorphism which is well defined up to conjugation by a unitary, and therefore is canonical on K -theory.

Remark 2.39. Recall that unless $\varphi: \pi_1(Y) \rightarrow \pi_1(X)$ is injective, it does not necessarily induce a map of the reduced group C^* -algebras. Thus, the relative group C^* -algebra does not always have a reduced counterpart.

2.40. The relative index map. The index of Chang, Weinberger and Yu is the image of a fundamental class in $K_*^L(X, Y)$ under a mapping $\mu: K_*^L(X, Y) \rightarrow K_*(C_{\max}^*(\pi_1(X), \pi_1(Y)))$, which they call the relative Baum–Connes map. In this subsection we present the definition of this map along the lines of [2, Section 2]. There the authors relate the K -theory groups of the localization algebras and their equivariant counterparts and exploit Theorem 2.8 to relate the latter K -theory groups with those of the group C^* -algebras of the fundamental groups.

Let X be a locally compact, path-connected, separable metric space and Y be a closed path-connected subset of X . We suppose that the universal coverings $p: \tilde{X} \rightarrow X$ and $p': \tilde{Y} \rightarrow Y$ of these spaces exist (e.g., suppose X and Y are CW -complexes) and are endowed with an invariant metric and that the metrics on X and Y are the pushdowns of these metrics, i.e., the projections are local isometries. In the case of smooth manifolds we can start with Riemannian metrics on X and Y and take their pullbacks to be the invariant Riemannian metrics on \tilde{X} and \tilde{Y} . Pick countable dense subsets Γ_X and Γ_Y of X and Y such that $\Gamma_Y \subset \Gamma_X$ as before. Denote by $\Gamma_{\tilde{X}}$ and $\Gamma_{\tilde{Y}}$ the preimages of Γ_X and Γ_Y , respectively. Construct the (equivariant) Roe algebras and the (equivariant) localization algebras using the representations $l^2(\Gamma) \otimes l^2(\mathbb{N})$. We recall that the equivariant algebras are constructed using the action of fundamental groups by deck transformations.

Proposition 2.41 ([2, Prop. 2.8]). *Let X and \tilde{X} be as above. Suppose furthermore that X is compact. Then there exists an $\epsilon > 0$ depending on X such that for finite propagation locally compact operators T with $\text{prop}(T) < \epsilon$, the kernel \tilde{k} defined in the following defines an element of $C_{\max}^*(\tilde{X})^{\pi_1(X)}$, which we will denote by $L(T)$.*

Observe for the definition of $L(T)$ that a finite propagation locally compact operator T on $l^2(\Gamma_X) \otimes H$ with $\text{prop}(T) = r$ is given by a matrix $\Gamma_X \times \Gamma_X \xrightarrow{k} K(H)$ such that $k(x, x')$ is 0 for all $(x, x') \in \Gamma_X \times \Gamma_X$ with $d_X(x, x') \geq r$. Define the lifted operator on $l^2(\Gamma_{\tilde{X}}) \otimes H$ using the matrix $(\tilde{x}, \tilde{x}') \xrightarrow{\tilde{k}} k(p(\tilde{x}), p(\tilde{x}'))$ if $d_{\tilde{X}}(\tilde{x}, \tilde{x}') < r$ and 0 otherwise.

Vice versa, every equivariant kernel $\tilde{T} \in C_{\max}^*(\tilde{X})^{\pi_1(X)}$ of propagation $< \epsilon$ is such a lift, and this in a unique way, defining the push-down $\pi(\tilde{T}) \in C_{\max}^*(X)$ as the inverse of the lift.

For the appropriate choice of ϵ , the covering $\tilde{X} \rightarrow X$ should be trivial when restricted to balls say of radius 2ϵ .

Remark 2.42. Later we will need a slight generalization of Proposition 2.41 for manifolds obtained by attaching an infinite cylinder to a compact manifold with boundary. It is evident that the ϵ obtained for the manifold with boundary also works for the manifold with the infinite cylinder attached, and then the construction indeed goes through without any modification.

Definition 2.43. Let $T: s \mapsto T_s$ be an element of $\mathbb{R}_L(X)$, i.e., T_s is locally compact and has finite propagation which tends to 0 as $s \rightarrow \infty$. Therefore, $\text{prop}(T_s) < \epsilon$ for all $s \geq s_T$ with some $s_T \in [1, \infty)$. Define the lift

$$L(T): s \mapsto \begin{cases} L(T_{s_T}), & s \leq s_T, \\ L(T_s), & s \geq s_T, \end{cases}$$

to obtain an element in $C_{L,\max}^*(\tilde{X})^{\pi_1(X)}$.

Similarly, for $\tilde{T}: s \mapsto \tilde{T}_s$ an element of $\mathbb{R}_L(\tilde{X})^{\pi_1(X)}$ such that \tilde{T}_s is locally compact, equivariant and has finite propagation which tends to 0 as $t \rightarrow \infty$ (in particular $\text{prop}(\tilde{T}_s) < \epsilon$ for all $s \geq s_{\tilde{T}}$ for some $s_{\tilde{T}} \in [1, \infty)$), define its *push-down*

$$\pi(\tilde{T}): s \mapsto \begin{cases} \pi(\tilde{T}_{s_{\tilde{T}}}), & s \leq s_{\tilde{T}}, \\ \pi(\tilde{T}_s), & s \geq s_{\tilde{T}}. \end{cases}$$

Proposition 2.44. Set $C_0^*(\tilde{X})^{\pi_1(X)} := C_0([1, \infty), C_{\max}^*(\tilde{X})^{\pi_1(X)})$, the ideal of $C_{L,\max}^*(\tilde{X})^\Gamma$ consisting of functions whose norm tends to 0 as $s \rightarrow \infty$. The assignments of Definition 2.43 give rise to continuous $*$ -homomorphisms

$$\begin{aligned} L: \mathbb{R}_L(X) &\rightarrow C_{L,\max}^*(\tilde{X})^{\pi_1(X)} / C_0^*(\tilde{X})^{\pi_1(X)}, \\ \pi: \mathbb{R}_L(\tilde{X})^{\pi_1(X)} &\rightarrow C_{L,\max}^*(X) / C_0^*(X), \end{aligned}$$

where we use that the algebra of functions vanishing at ∞ is an ideal of the localization algebra. Being continuous, they extend to the C^* -completions, and they evidently map the ideal $C_0([1, \infty), C_{\max}^*(X))$ or $C_0([1, \infty), C_{\max}^*(\tilde{X})^{\pi_1(X)})$ to 0, so that we get C^* -algebra homomorphisms

$$\begin{aligned} L: C_{L,\max}^*(X) / C_0^*(X) &\rightarrow C_{L,\max}^*(\tilde{X})^{\pi_1(X)} / C_0^*(\tilde{X})^{\pi_1(X)}, \\ \pi: C_{L,\max}^*(\tilde{X})^{\pi_1(X)} / C_0^*(\tilde{X})^{\pi_1(X)} &\rightarrow C_{L,\max}^*(X) / C_0^*(X). \end{aligned}$$

By construction, these two homomorphisms are inverse to each other.

Being cones, $C_0([1, \infty), C_{\max}^*(\tilde{X})^{\pi_1(X)})$ and $C_0(1, \infty), C_{\max}^*(X)$ have vanishing K -theory and by the 6-term exact sequence, the projections induce isomorphisms in K -theory

$$\begin{aligned} K_*(C_{L,\max}^*(\tilde{X})^{\pi_1(X)}) &\rightarrow K_*(C_{L,\max}^*(\tilde{X})^{\pi_1(X)} / C_0^*(\tilde{X})^{\pi_1(X)}), \\ K_*(C_{L,\max}^*(X)) &\rightarrow K_*(C_{L,\max}^*(X) / C_0^*(X)). \end{aligned}$$

We therefore get a well-defined induced isomorphism in K -theory

$$L_*: K_*^L(X) = K_*(C_{L,\max}^*(X)) \rightarrow K_*(C_{L,\max}^*(\tilde{X})^{\pi_1(X)})$$

with inverse π_* .

The proof of Proposition 2.44 is not trivial, as we have to come to grips with the potentially different representations which enter the definition of the maximal C^* -norms for $C_{\max}^*(X)$ and $C_{\max}^*(\tilde{X})^{\pi_1(X)}$. To do this, we use the following lemma.

Lemma 2.45. *Let ϵ be as in Proposition 2.41. There exists $K \in \mathbb{N}$ such that for all $T \in \mathbb{R}(X)$ and $\tilde{T} \in \mathbb{R}(\tilde{X})^{\pi_1(X)}$ with propagation less than ϵ , we have $\|L(T)\|_{C_{\max}^*(\tilde{X})^{\pi_1(X)}} \leq K\|T\|_{C_{\max}^*(X)}$ and $\|\pi(\tilde{T})\|_{C_{\max}^*(X)} \leq K\|\tilde{T}\|_{C_{\max}^*(\tilde{X})^{\pi_1(X)}}$.*

Proof. By assumption, X has bounded geometry. Consequently, we can and do choose, for some fixed $c > 0$, a c -dense uniformly discrete subset D of Γ_X and denote by $C_{\max}^*(D)$ and $C_{\max}^*(\tilde{D})^{\pi_1(X)}$ the Roe algebras of D constructed using $l^2(D) \otimes H$ and $l^2(\tilde{D}) \otimes H$ as before. The proof of [7, Lem. 3.4] guarantees the existence of a $K \in \mathbb{N}$ such that for all $T \in C_{\max}^*(D)$ with $\text{prop}(T) < \epsilon$, there exist operators $T_{i \in \{1, \dots, K\}} \in C_{\max}^*(D)$ such that $\|T_i\| \leq \|T\|$, $T_i^* T_i \in l^\infty(D; K(H))$, i.e., $T_i^* T_i$ are operators of propagation 0, and such that $\sum T_i = T$. Moreover, the lift \tilde{T}_i satisfies that

$$\tilde{T}_i^* \tilde{T}_i = \widetilde{T_i^* T_i} \in l^\infty(\tilde{D}; K(H))^{\pi_1(X)} \stackrel{L}{\cong} l^\infty(D; K(H)).$$

Hence the norm of $\tilde{T}_i^* \tilde{T}_i$ is exactly $\|T_i\|^2$.

We thus have $\|L(T)\| \leq K\|T\|$. With a completely analogous argument, we get $\|\pi(\tilde{T})\| \leq K\|\tilde{T}\|$.

Note that there are isomorphisms

$$C_{\max}^*(X) \rightarrow C_{\max}^*(D), \quad C_{\max}^*(\tilde{X})^{\pi_1(X)} \rightarrow C_{\max}^*(\tilde{D})^{\pi_1(X)},$$

which can be constructed explicitly (compare [7, Section 4.4]). These isomorphisms can be chosen so as to make the diagrams

$$\begin{array}{ccc} \mathbb{R}(\tilde{X})_\epsilon^{\pi_1(X)} & \longrightarrow & \mathbb{R}(\tilde{D})_\epsilon^{\pi_1(X)} & & \mathbb{R}(\tilde{X})_\epsilon^{\pi_1(X)} & \longrightarrow & \mathbb{R}(\tilde{D})_\epsilon^{\pi_1(X)} \\ & & \uparrow L & & \downarrow \pi & & \downarrow \pi \\ \mathbb{R}(X)_\epsilon & \longrightarrow & \mathbb{R}(D)_\epsilon & & \mathbb{R}(X)_\epsilon & \longrightarrow & \mathbb{R}(D)_\epsilon \end{array}$$

commute. Here the subscript ϵ means that we are only considering operators with propagation less than ϵ .

The latter commutative diagrams complete the proof. □

Proof of Proposition 2.44. Recall that for $(\tilde{T}: s \rightarrow \tilde{T}_s) \in C_{L, \max}^*(\tilde{X})^{\pi_1(X)}$, we use the supremum norm: $\|\tilde{T}\| = \sup_{s \in [1, \infty)} \|\tilde{T}_s\|$. It follows that the norm of the image of \tilde{T} in $C_{L, \max}^*(\tilde{X})^{\pi_1(X)} / C_0([1, \infty); C^*(\tilde{X})^{\pi_1(X)})$ under the projection map is $\|\tilde{[\tilde{T}]}\| = \limsup_{s \in [1, \infty)} \|\tilde{T}_s\|$ (specifically, multiplication of \tilde{T} with a cutoff function $\rho: [1, \infty) \rightarrow [0, 1]$ which vanishes on $[1, R]$ and is identically 1 on $[R + 1, \infty)$ produces representative of $[\tilde{T}]$ whose norm in $C_L^*(\tilde{X})^{\pi_1(X)}$ approaches $\limsup_{s \in [1, \infty)} \|\tilde{T}_s\|$ as $R \rightarrow \infty$).

The assertion then follows immediately from Lemma 2.45. □

Until the end of Section 2.5 we are going to suppose that X is compact and that Y is a closed subset of X . Recall that φ denotes the map $\pi_1(Y) \rightarrow \pi_1(X)$ induced by the inclusion. Following the notation introduced in [2, Section 2], we denote by Y' the set $p^{-1}(Y)$ and by $p'' : Y'' \rightarrow Y$ the covering of Y associated to the subgroup $\ker \varphi$; hence, $Y' = \pi_1(X) \times_{\pi_1(Y)/\ker \varphi} Y''$. Now construct the equivariant Roe and localization algebras for Y' and Y'' using the sets $p^{-1}(\Gamma_Y)$ and $(p'')^{-1}(\Gamma_Y)$ similarly as before.

Theorem 2.46 ([2, Lem. 2.12]). *There is a map*

$$\psi'' : C_{\max}^*(\tilde{Y})^{\pi_1(Y)} \rightarrow C_{\max}^*(Y'')^{\pi_1(Y)/\ker \varphi}$$

with the property that there exists $\epsilon > 0$ such that, given an operator $T \in C_{\max}^*(\tilde{Y})^{\pi_1(Y)}$ with $\text{prop}(T) < \epsilon$ and kernel k on $(p')^{-1}(\Gamma_Y)$, the pushdown of k gives a unique well-defined kernel k_Y on Γ_Y and $\psi''(T)$ is given by the kernel $(x, y) \mapsto k_Y(p''(x), p''(y))$ for $x, y \in Y''$ with $d_{Y''}(x, y) < \epsilon$.

Remark 2.47. It can be observed from the proof of Theorem 2.46 that the result can be generalized to obtain a map $C_{\max}^*(Z)^\Gamma \rightarrow C_{\max}^*(Z/N)^\Gamma/N$, where Z is a bounded geometry space satisfying the properties mentioned in the beginning of the paper, Γ is a discrete group acting freely and properly on Z via isometries, $N \subset \Gamma$ is a normal subgroup and there exists an ϵ such that the coverings $Z \rightarrow Z/N'$ are trivial when restricted to ϵ -balls for any normal subgroup $N' \subset \Gamma$.

Remark 2.48. For the proof of Theorem 2.46, Chang, Weinberger and Yu use that the push-down of operators with small propagation as defined in Definition 2.43 can be extended to an honest $*$ -homomorphism. Doing it partially gives a morphism of $*$ -algebras $\psi'' : \mathbb{R}(\tilde{Y})^{\pi_1(Y)} \rightarrow \mathbb{R}(Y'')^{\pi_1(Y)/\ker \varphi}$, and then maximality of the norms provides the extension to the desired C^* -homomorphism $C_{\max}^*(\tilde{Y})^{\pi_1(Y)} \rightarrow C_{\max}^*(Y'')^{\pi_1(Y)/\ker \varphi}$. Note that, in general, this is not possible if we use the reduced equivariant Roe algebras.

Using $Y' = Y'' \times_{\pi_1(Y)/\ker \varphi} \pi_1(X)$, we get a C^* -algebra morphism

$$\psi' : C_{\max}^*(Y'')^{\pi_1(Y)/\ker \varphi} \rightarrow C_{\max}^*(Y')^{\pi_1(X)} \subset C_{\max}^*(\tilde{X})^{\pi_1(X)},$$

where the first map repeats the operators on the different copies of Y'' inside Y' . Composing ψ' and ψ'' , we obtain the map $\psi : C_{\max}^*(\tilde{Y})^{\pi_1(Y)} \rightarrow C_{\max}^*(\tilde{X})^{\pi_1(X)}$. Application of the maps pointwise defines the corresponding maps for localization algebras, which we denote with the same symbols with subscript L .

Theorem 2.49. *The constructions just described fit into the following commutative diagram of C^* -algebras, where the composition in the third row is the map ψ_L , in the fourth row is ψ , and in the last row is φ . The projection maps in the second row of vertical maps are K -theory isomorphism. The last vertical maps induce the canonical isomorphism in K -theory of Theorem 2.8. The Roe and localization algebras are constructed using the maximal completion (we*

write m instead of \max for typesetting reasons).

$$\begin{array}{ccccccc}
 C_{L,m}^*(Y) & \xlongequal{\quad} & C_{L,m}^*(Y) & \xlongequal{\quad} & C_{L,m}^*(Y) & \xrightarrow{\iota} & C_{L,m}^*(X) \\
 \downarrow L & & \downarrow L & & \downarrow L & & \downarrow L \\
 \frac{C_{L,m}^*(\tilde{Y})\pi_1(Y)}{C_0^*(\tilde{Y})\pi_1(Y)} & \xrightarrow{\psi''_L} & \frac{C_{L,m}^*(Y'')\pi_1(Y)/\ker\varphi}{C_0^*(Y'')\pi_1(Y)/\ker\varphi} & \xrightarrow{\psi'_L} & \frac{C_{L,m}^*(Y')\pi_1(X)}{C_0^*(Y')\pi_1(X)} & \xrightarrow{\subset} & \frac{C_{L,m}^*(\tilde{X})\pi_1(X)}{C_0^*(\tilde{X})\pi_1(X)} \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 C_{L,m}^*(\tilde{Y})\pi_1(Y) & \xrightarrow{\psi''_L} & C_{L,m}^*(Y'')\pi_1(Y)/\ker\varphi & \xrightarrow{\psi'_L} & C_{L,m}^*(Y')\pi_1(X) & \xrightarrow{\subset} & C_{L,m}^*(\tilde{X})\pi_1(X) \\
 \downarrow \text{ev}_1 & & \downarrow \text{ev}_1 & & \downarrow \text{ev}_1 & & \downarrow \text{ev}_1 \\
 C_m^*(\tilde{Y})\pi_1(Y) & \xrightarrow{\psi''} & C_m^*(Y'')\pi_1(Y)/\ker\varphi & \xrightarrow{\psi'} & C_m^*(Y')\pi_1(X) & \xrightarrow{\subset} & C_m^*(\tilde{X})\pi_1(X) \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 C_m^*(\pi_1(Y)) & \xrightarrow{pr^*} & C_m^*(\pi_1(Y)/\ker\varphi) & \longrightarrow & C_m^*(\pi_1(X)) & \xlongequal{\quad} & C_m^*(\pi_1(X)).
 \end{array}$$

Proof. If in the first row $C_L^*(Y)$ is replaced by $\mathbb{R}_L(Y)$, then the definition of L , the behavior of the push-down map ψ'' and the (trivial) lifting map ψ' on operators of small propagation and the definition of ι and \subset imply the commutativity of the first two rows of the diagram. The continuity of the involved maps then implies the commutativity of the first two rows. In order to show the commutativity of the last two rows we recall the isomorphisms $K_*(C_{\max}^*(\pi_1(\cdot))) \rightarrow K_*(C_{\max}^*(\cdot)^{\pi_1(\cdot)})$. For this we need the isomorphisms $C_{\max}^*(\pi_1(\cdot)) \otimes K(H) \cong C_{\max}^*(\cdot)^{\pi_1(\cdot)}$. Here we modify the proof of [11, Lem. 12.5.3] slightly to suit our choice of the representation space. Choose a countable dense subset D of the fundamental domain of \tilde{Y} such that D and gD are disjoint for $g \neq e$ in $\pi_1(Y)$. With $\Gamma_{\tilde{Y}} = \bigsqcup_{g \in \pi_1(Y)} gD$, we get an isomorphism $l^2(\Gamma_{\tilde{Y}}) \otimes l^2(\mathbb{N}) \cong l^2(\pi_1(Y)) \otimes (\bigoplus_{n \in \mathbb{N}} l^2(D))$. Using this isomorphism we then obtain a $*$ -isomorphism between $\mathbb{C}[\pi_1(Y)] \otimes K(\bigoplus_{n \in \mathbb{N}} l^2(D))$ and the algebra of invariant, finite propagation and locally compact operators. This induces the desired isomorphism $C_{\max}^*(\pi_1(Y)) \otimes K(\bigoplus_{n \in \mathbb{N}} l^2(D)) \cong C_{\max}^*(\tilde{Y})\pi_1(Y)$. Furthermore, we note (see [22, Prop. 6.4.1 and Proposition 8.2.8]) that the standard isomorphisms $K_p(A) \rightarrow K_p(A \otimes K(H))$ for a C^* -algebra A and a separable infinite-dimensional Hilbert space H is induced by the morphism $a \mapsto a \otimes p$, with p a rank one projection. Now consider the rank one projection $p_{x_0} \otimes p_1$ on $\bigoplus_{n \in \mathbb{N}} l^2(D) \cong l^2(D) \otimes l^2(\mathbb{N})$ for some $x_0 \in D$ and p_1 the operator on $l^2(\mathbb{N})$ projecting to the first component. The composition gives the desired map $C_{\max}^*(\pi_1(Y)) \rightarrow C_{\max}^*(\tilde{Y})\pi_1(Y)$ which induces the K -theory isomorphism of Theorem 2.8. We can perform the same procedure for $Y'' = \tilde{Y}/(\ker\varphi)$. Considering the above D (or rather its image under $\tilde{Y} \rightarrow Y''$) as a subset of Y'' and using $\Gamma_{Y''} = \bigsqcup_{g \in \frac{\pi_1(Y)}{\ker\varphi}} gD$, we get the corresponding isomorphism $l^2(\Gamma_{Y''}) \otimes l^2(\mathbb{N}) \cong l^2(\frac{\pi_1(Y)}{\ker\varphi}) \otimes (\bigoplus_{n \in \mathbb{N}} l^2(D))$. Choosing the same p as above our procedure defines the desired $C_{\max}^*(\frac{\pi_1(Y)}{\ker\varphi}) \rightarrow C_{\max}^*(Y'')\frac{\pi_1(Y)}{\ker\varphi}$, which is a K -theory isomorphism and which makes the lower left corner of the diagram

of Theorem 2.49 commutative. Similarly we construct the corresponding map for Y' , which is the associated bundle to Y'' with fibre $\pi_1(X)$ (we can consider the above D as a subset of Y'). The construction gives rise to the morphism $C_{\max}^*(\pi_1(X)) \rightarrow C_{\max}^*(Y')^{\pi_1(X)}$, which is a K -theory isomorphism and which makes the lower middle square of the diagram of Theorem 2.49 commutative. Finally, considering D as a subset of Y' and extending it to a dense subset of a fundamental domain of \tilde{X} , we obtain, similarly as above, a corresponding map for \tilde{X} , the morphism $C_{\max}^*(\pi_1(X)) \rightarrow C_{\max}^*(\tilde{X})^{\pi_1(X)}$, which is a K -theory isomorphism such that also the lower right corner of the diagram of Theorem 2.49 commutes. This finishes the proof of the theorem. \square

Definition 2.50. The commutative diagram of Theorem 2.49 defines a zig-zag of maps between the mapping cones of the compositions of the maps from left to right. Using in addition that the two wrong way vertical maps induce isomorphisms in K -theory, we obtain the map

$$\mu: K_*(SC_{\iota(X,Y)}) \rightarrow K_*(SC_{\varphi}) \stackrel{\text{Def}}{=} K_*(C_{\max}^*(\pi_1(X), \pi_1(Y))),$$

which we call the *relative index map*. In [2] it is called the *maximal relative Baum-Connes map*.

3. A GEOMETRIC AND FUNCTORIAL COMPLETION OF THE EQUIVARIANT ROE ALGEBRA

3.1. Maximal Roe algebra and functions of the Dirac operator. Before describing our geometric completion of the algebraic Roe algebra, we discuss issues arising in coarse index theory when one uses maximal completions of the relevant C^* -algebras, which lead to gaps in [2]. A crucial role in coarse index theory is played by functions of the Dirac operator (via functional calculus). If we work with the usual (reduced) Roe algebras, the latter are defined as algebras of bounded operators on L^2 -spinors, and the Dirac operator is an unbounded operator on the same Hilbert space. Ellipticity and finite propagation of the wave operator then are used to show that certain functions of the Dirac operator satisfy the defining conditions for the reduced Roe algebra and of the reduced structure algebra.

However, if one uses the maximal versions this is highly nontrivial:

- (i) The functions $f(D)$ which do have finite propagation are by the very definition elements of the algebraic Roe algebra (if f vanishes at infinity) or of the algebraic structure algebra (if f is a normalizing function). The wave operators e^{itD} are bounded multipliers of the maximal Roe algebra and by Lemma 2.24 act as bounded operators on the defining representation of the maximal Roe algebra.
- (ii) However, it is not obvious at all that the one parameter group $t \mapsto e^{itD}$ is strongly continuous on any Hilbert space on which the maximal Roe algebra is represented faithfully, i.e., is obtained from an (unbounded) selfadjoint operator D on such a Hilbert space. Thus one needs to have a

reasonable definition of $f(D)$ in the maximal Roe and structure algebra for f without a compactly supported Fourier transform.

- (iii) Even if one manages to construct the selfadjoint unbounded operator D on the maximal representation, it remains to show that this maximal Dirac operator is invertible if the underlying manifold has uniformly positive scalar curvature: one has to make sense also of a (geometric) Schrödinger–Lichnerowicz formula for this non-geometric representation?

Chang, Weinberger and Yu’s article [2] takes all these necessary constructions and properties for granted, without any justification. We propose a way around by passing to a slightly different and much more convenient completion. Later, Guo, Xie and Yu posted the preprint [8] where they also identify these gaps in [2] and propose positive answers to the above questions.

3.2. The quotient completion. Our suggestion to overcome the problems addressed in Section 3.1 is to work with another functorial completion of the equivariant Roe algebra which is more geometric. We are studying the case that a group Γ acts freely and properly discontinuously by isometries on a proper metric space X .

For every normal subgroup $N \subset \Gamma$, we then can form the metric space X/N on which the quotient group $Q := \Gamma/N$ acts as before. Indeed, typically we obtain X as a Γ -covering of a space X/Γ and the X/N are then other normal coverings of X/Γ .

In the usual way, the purely algebraically defined algebras $\mathbb{R}(X)^\Gamma$ and $\mathbb{S}(X)^\Gamma$ act via their images in $\mathbb{R}(X/N)^{\Gamma/N}$ and $\mathbb{S}(X/N)^{\Gamma/N}$ on all these quotients (see Theorem 2.46), and we complete with respect to all these norms at once. Denote the corresponding completions by $C_q^*(X)^\Gamma$ and $D_q^*(X)^\Gamma$. It is clear that the former is an ideal in the latter. It is also clear that this has the usual functoriality properties for Γ -equivariant maps for fixed Γ , but now in addition is functorial (this is built in) for the quotient maps $X \rightarrow X/N$, giving $C_q^*(X)^\Gamma \rightarrow C_q^*(X/N)^{\Gamma/N}$ and $D_q^*(X)^\Gamma \rightarrow D_q^*(X/N)^{\Gamma/N}$.

Finally, for inclusion of groups $\iota: \Gamma \rightarrow G$ induces an *induction map* $C_q^*(X)^\Gamma \rightarrow C_q^*(X \times_\Gamma G)^G$, because for every quotient G/N , we get the associated induction $\mathbb{R}(X/\Gamma \cap N)^{\Gamma/(\Gamma \cap N)} \rightarrow \mathbb{R}(X/(\Gamma \cap N) \times_{\Gamma/(\Gamma \cap N)} G/N)^{G/N} = \mathbb{R}(X \times_\Gamma G/N)^{G/N}$.

The corresponding construction works for D_q^* and for the localization algebras.

Putting this together, we get the expected functoriality of C_q^* and D_q^* and the localization algebras for maps equivariant for any homomorphism $\alpha: \Gamma_1 \rightarrow \Gamma_2$.

Lemma 3.3. *Suppose Γ acts cocompactly on X . Then $C_q^*(X)^\Gamma$ is isomorphic to $C_q^*(\Gamma) \otimes K(H)$. Here, $C_q^*(\Gamma)$ is the C^* -completion of $\mathbb{C}[\Gamma]$ in the representation $\bigoplus_{N \triangleleft \Gamma} l^2(\Gamma/N)$, where the sum is over all normal subgroups N of Γ .*

Proof. The proof is precisely along the lines of the one of Theorem 2.8. \square

Proposition 3.4. *Let X/Γ be a complete Riemannian spin manifold with Γ -covering X . The Dirac operator on the different normal coverings X/N for*

the normal subgroups N of Γ gives rise to a selfadjoint unbounded operator in the defining representation of $C_q^*(X)^\Gamma$. If $f \in C_0(\mathbb{R})$, we get $f(D) \in C_q^*(X)^\Gamma$, if $\Psi: \mathbb{R} \rightarrow [-1, 1]$ is a normalizing function, we get $\Psi(D) \in D_q^*(X)^\Gamma$.

This construction is functorial for the quotient maps $X \rightarrow X/N$ for normal subgroups $N \triangleleft \Gamma$.

The Schrödinger–Lichnerowicz argument applies: if X/Γ has uniformly positive scalar curvature, then the spectrum of the operator D in the defining representation of $C_q^*(X)^\Gamma$ does not contain 0.

Let $A \subset X$ be a Γ -invariant measurable subset. Then χ_A , the operator of multiplication with the characteristic function of A is an element of $D_q^*(X)^\Gamma$, in particular a multiplier of $C_q^*(X)^\Gamma$. Under the quotient map $X \rightarrow X/N$, for a normal subgroup $N \triangleleft \Gamma$, it is mapped to $\chi_{A/N}$. Similarly, a function of the Dirac operator on X is mapped to the same function of the Dirac operator on X/N .

Proof. The statements about the Dirac operator are just an application of the usual arguments to all the quotients X/N simultaneously, using Lemma 4.9.

The statement about χ_A is a direct consequence of the definitions. \square

Remark 3.5. We note that all the statements in Section 2 have a counterpart when we use the quotient completion instead of the maximal completion of the equivariant algebras and their proofs are completely analogous to (and often easier than) the proofs for the maximal completions. In particular, we have a relative index map in this case. Furthermore, we would like to emphasize that Theorem 2.12 holds for the quotient completion. Given a map $\phi: \Gamma \rightarrow \pi$, we get by functoriality a morphism $\phi: C_q^*(\Gamma) \rightarrow C_q^*(\pi)$, and $C_q^*(\pi, \Gamma)$ will denote SC_ϕ as before.

4. HIGHER INDICES OF DIRAC OPERATORS ON MANIFOLDS WITH BOUNDARY

4.1. Construction of the relative index. Throughout this section, we consider only even-dimensional spin manifolds. We define the relative index of the Dirac operator of a manifold M with boundary N in the following groups:

- in $C_{\max}^*(\pi_1(M), \pi_1(N))$,
- in $C_q^*(\pi_1(M), \pi_1(N))$ and
- in $C_{\text{red}}^*(\pi_1(M), \pi_1(N))$ if $\pi_1(N) \rightarrow \pi_1(M)$ is injective.

In what follows the subscript d stands for one of the mentioned completions. Before defining the relative index of the Dirac operator on a manifold with boundary, we recall the explicit image of the fundamental class under the local index map. Given a complete Riemannian spin manifold X with a free and proper action of Γ by isometries, denote by D_X the Dirac operator on X . Let Ψ_t be a sup-norm continuous family of normalizing functions, i.e., each Ψ_t is an odd, smooth function $\Psi_t: \mathbb{R} \rightarrow [-1, 1]$ such that $\Psi_t(s) \xrightarrow{s \rightarrow \infty} 1$. Suppose furthermore that for $t \geq 1$, the distributional Fourier transform of Ψ_t is supported in a $\frac{1}{t}$ -neighborhood of 0. Choose an isometry α between $L^2(\mathcal{S}^+)$ and $L^2(\mathcal{S}^-)$ induced from a measurable bundle isometry, set $\Psi_t(D_X)^+ :=$

$\Psi_t(D_X)|_{L^2(\mathcal{S}^+)}$ and $F_X(t) := \alpha^* \circ \Psi_t(D_X)^+$. Set $e_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Note that the presence of α^* implies that $F_X(t)$ is an operator on $L^2(\mathcal{S}^+)$.

Definition 4.2. In the above situation, the (locally finite) fundamental class $[D_X] \in K_0(C_{L,d}^*(X)) = K_0^L(X)$ is given explicitly by $[P_X] - [e_{11}]$, with

$$P_X := \begin{pmatrix} FF^* + (1 - FF^*)FF^* & F(1 - F^*F) + (1 - FF^*)F(1 - F^*F) \\ (1 - F^*F)F^* & (1 - F^*F)^2 \end{pmatrix}.$$

In this formula F denotes $F_X(\cdot)$ and P_X is an idempotent in $M_2(C_{L,d}^*(X)^+)$. Here, A^+ denotes the unitalization of A .

Remark 4.3. Note that since Ψ_t is assumed to have compactly supported Fourier transform, $\Psi_t(D_X)$ has finite propagation, which means that P_X is a matrix over the unitalization of $\mathbb{R}_L^*(X) \subset C_{L,\max}^*(X)$.

Now let M be a compact spin manifold with boundary N . Denote by N_∞ the cylinder $N \times [0, \infty)$ and by M_∞ the manifold $M \cup_N N_\infty$. Given a Riemannian metric on M which is collared at the boundary, we will equip N_∞ with the product metric. Taking the image of $[D_{M_\infty}]$ in $K_*^L(M_\infty, N_\infty)$ and then under the excision isomorphism defines the relative fundamental class $[M, N] \in K_*^L(M, N)$. For the index calculations, which we have to carry out, we need an explicit representative of this class, and this in the model of relative K -homology as the K -theory of the mapping cone algebra $C_\iota(M, N)$. Therefore, we recall the construction of [2], referring for further details to [2]—see also [11, Prop. 4.8.2 and 4.8.3].

As the relative K -homology groups are constructed as mapping cones which come with a built-in shift of degree, we have to use Bott periodicity to shift the fundamental class to the suspension algebra (with degree shift). To implement this, denote by v the Bott generator of $K_1(C_0(\mathbb{R}))$. Following [2], define the invertible element

$$\tau_D := v \otimes P_{M_\infty} + I \otimes (I - P_{M_\infty})$$

in a matrix algebra over $C(S^1) \otimes C_{L,d}^*(M_\infty)^+$ with inverse given by $\tau_D^{-1} = v^{-1} \otimes P_{M_\infty} + I \otimes (I - P_{M_\infty})$ (see [11, Prop. 4.8.3] for more details). Next, we map to the relative K -homology of the pair (M, N) , which requires applying the inverse of the excision isomorphism $K_*(M, N) \rightarrow K_*(M_\infty, N_\infty)$. This is implemented for our K -theory cycles by multiplication with a cut-off. For technical reasons, we observe that instead of $N \subset M$, we can use the homeomorphic $N_R := N \times \{R\} \subset M_R := M \cup N \times [0, R]$ for each $R \geq 0$. We use localization algebras, and then we can use the K -theory isomorphism $C_{L,d}^*(M_R) \rightarrow C_{L,d}^*(M \subset M_\infty)$ and work with $C_{L,d}^*(M \subset M_\infty)$, which is independent of R . Similarly, we use the K -theory isomorphism $C_{L,d}^*(N_R) \rightarrow C_{L,d}^*(N \subset N_\infty)$ and replace $C_{L,d}^*(N_R)$ by the R -independent $C_{L,d}^*(N \subset N_\infty)$. This causes slight differences to the construction of [2].

For the cut-off, set $\chi_R := \chi_{M_R}$, the characteristic function of M_R . Consider

$$\begin{aligned} \tau_{D,R} &:= v \otimes (\chi_R P_{M_\infty} \chi_R + (1 - \chi_R) e_{11} (1 - \chi_R)) \\ &\quad + I \otimes (I - (\chi_R P_{M_\infty} \chi_R + (1 - \chi_R) e_{11} (1 - \chi_R))) \end{aligned}$$

and define $\tau_{D,R}^{-1}$ in the same way with v replaced by v^{-1} . Note that these two operators are in general *not* inverse to each other. Define, for $s \in [0, 1]$,

$$w_{D,R}(s) := \begin{pmatrix} I & (1-s)\tau_{D,R} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ -(1-s)(\tau_{D,R})^{-1} & I \end{pmatrix} \begin{pmatrix} I & (1-s)\tau_{D,R} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Finally, set

$$(2) \quad q_{D,R}(s) := w_{D,R}(s) e_{11} w_{D,R}(s)^{-1}.$$

Applying the same procedure not to τ_D but to $v \otimes e_{11} + I \otimes e_{22}$, we obtain a curve $q_p(s)$. Note that by construction of $\tau_{D,R}$, all operators, in particular $q_{D,R}(s)$, are diagonal for the decomposition $L^2(M_\infty) = L^2(M_R) \oplus L^2(N \times [R, \infty))$ and are of standard form on $L^2(N \times [R, \infty))$. This summand does not appear in [2] but has to be there to construct the appropriate operators in $C_{L,d}^*(M \subset M_\infty)$.

Lemma 4.4. *Assume that the operator $F_{M_\infty}(t)$ has propagation $\leq L$ for some $L \in [0, \infty)$. Then $q_{D,R}(s)(t)$ (recall that we always have an additional $t \in [1, \infty)$ -dependency) has propagation $\leq 30L$. It is diagonal with respect to the decomposition $L^2(M_\infty) = L^2(M_R) \oplus L^2(N \times [R, \infty))$ and coincides with $q_p(s)$ on $L^2(N \times [R, \infty))$. It is obtained via finitely many algebraic operations (addition, composition) from $\Psi_t(D_{M_\infty})$, the measurable bundle isometry α , the Bott element v and χ_R .*

If $R > 30L$, then $q_{M,R}(0)(t)$ differs from $q_p(0)(t)$ by an operator Q supported on $N \times [0, R]$. More precisely, for suitable operators A, B ,

$$Q = \chi_R A \circ I \otimes [\chi_R, P_\infty] \circ B \chi_R,$$

where the commutator $[\chi_R, P_\infty]$ is supported on $N \times [R - 5L, R + 5L]$ and Q has propagation $\leq 30L$.

Like $q_{D,R}(s)(t)$, the operator $Q(t)$ is obtained via finitely many algebraic operations from $\Psi_t(D_{M_\infty})$, α , v , v^{-1} , and χ_R .

Due to the local nature of all constructions and because of the support property of the commutator $[\Psi_R, P_\infty]$ (using Lemma 4.9 for $\Psi_t(D)$), the operator Q on $L^2(N \times [0, R])$ is equal to the operator constructed correspondingly, where D_{M_∞} is replaced by $D_{N \times \mathbb{R}}$ and χ_R by $\chi_{N \times (-\infty, R]}$.

Proof. The explicit formulas show that $q_{D,M}(s)(t)$ is an algebraic combination of $\Psi_t(D_{M_\infty})$, α , etc. as claimed, where all building blocks either have propagation 0 or are $\Psi_t(D_{M_\infty})$, and we compose at most 30 of the latter. The claim about the propagation follows.

As it can be seen from the formula in the proof of [2, Claim 2.19], $q_{D,R}(0)$ would be equal to $q_p(0)$ if $\tau_{D,R}$ was invertible with inverse $\tau_{D,R}^{-1}$, which would happen if $\chi_R P_{M_\infty} \chi_R$ was an idempotent. To compare with this situation one

has to commute P_{M_∞} and χ_R , which produces the shape of Q as claimed. The rest then follows as for $q_{D,R}(s)$. \square

Denote by ι'_R the inclusion of $C_{L,d}^*(N \subset N_\infty)$ in $C_{L,d}^*(M \subset M_\infty)$, the image consisting of those operators which act only on $L^2(N_\infty)$.

The relative fundamental class $[M, N] \in K_0(C_{S\iota'_R}) \cong K_0(SC_{\iota'_R}) \cong K_0^L(M, N)$ is defined as

$$(3) \quad [M, N] := [(q_{D,R}(0), q_{D,R}(\cdot))] - [(q_p(0), q_p(\cdot))].$$

It is implicit in [2] that the K -theory class is independent of R and the family of normalizing functions Ψ_t .

Definition 4.5 (The relative index). The relative index of the Dirac operator is defined as

$$\mu([M, N]) \in K_0(C_d^*(\pi_1(M), \pi_1(N))).$$

The explicit K -theory cycle defining $[M, N]$ and the description of the map μ of Definition 2.50 gives us an explicit cycle for the relative index:

We have to lift the operators $q_{D,M}(s)$ involved in the construction of $[M, N]$ to equivariant operators on the $\pi_1(M)$ -cover \widetilde{M}_∞ and those involved in $q_{D,M}(0)$ to equivariant operators on the $\pi_1(N)$ -cover \widetilde{N}_∞ . This is possible here and the operators are given as the corresponding functions of the Dirac operator on the coverings. For this, we use that, by Lemma 4.4, the operator $q_{D,M}(t)$ is obtained as an expression in functions of the Dirac operator which lift to the corresponding functions of the Dirac operator by Lemma 4.9.

Similarly, by Lemma 4.4 and if $R > 30L$, where the propagation of $\Psi_t(D)$ is bounded by L for all $t \in [1, \infty)$, the operator $q_{D,R}(0)$ is obtained as an algebraic combination of functions of $D_{N \times \mathbb{R}}$ and the cut-off function $\chi_{N \times (-\infty, R]}$ which lift by Lemma 4.9 to $\pi_1(N)$ -equivariant operators on $\widetilde{N} \times [0, \infty)$ defined by the same expressions. Thus if we denote by $\tilde{q}_{D,R}$ the element constructed as above using the Dirac operator of \widetilde{M}_∞ and $\chi_{\widetilde{M}_R}$ and by $\tilde{q}_{D,R}^N$ the element constructed using the Dirac operator on $\widetilde{N} \times \mathbb{R}$ and $\chi_{N \times (-\infty, R]}$, then we have the following.

Lemma 4.6. *The expression $[(\tilde{q}_{D,R}^N(0), \tilde{q}_{D,R}(\cdot))] - [(q_p(0), q_p(\cdot))]$ defines an element of $K_0(SC_{C_{L,d}^*(\widetilde{N} \subset \widetilde{N}_\infty)^{\pi_1(N)} \rightarrow C_{L,d}^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}}$ which identifies under the canonical isomorphism of the latter group with $K_0(M, N)$ with $[M, N]$.*

Hence, under these conditions on R and the propagation of $\Psi_t(D)$, the relative index is the obtained by evaluation at $t = 1$, or by homotopy invariance at any $t \geq 1$:

$$(4) \quad \begin{aligned} \mu([M, N]) &= [(\tilde{q}_{D,R}^N(0)(t), \tilde{q}_{D,R}(\cdot)(t))] - [(q_p(0), q_p(\cdot))] \\ &\in K_0(SC_{C^*(\widetilde{N} \subset \widetilde{N}_\infty)^{\pi_1(N)} \rightarrow C_d^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}}) \\ &\cong K_0(C_d^*(\pi_1(M), \pi_1(N))). \end{aligned}$$

As $q_p(\cdot)$ is independent of t , we omit specifying the evaluation at t here.

4.7. The localized fundamental class and coarse index. Suppose X is a smooth even-dimensional spin manifold with free and proper action by Γ . Let Z be a closed Γ -invariant subset of X . Suppose that there exists a complete Γ -invariant Riemannian metric on X which has uniformly positive scalar curvature outside Z . In [20] and in more detail in [21], Roe defines a localized coarse index of the Dirac operator in $K_*(C_{\text{red}}^*(Z \subset X)^\Gamma)$. In the course of the proof of [9, Thm. 3.11], the construction of the latter localized index is generalized to the case of a Dirac operator twisted with a Hilbert C^* -module bundle. In [26, Ch. 2], Zeidler defines this index using localization algebras. There, he also shows that under certain assumptions on a manifold X with boundary Y , the localized coarse index can be used to define an obstruction to the extension of a uniformly positive scalar curvature metric on the boundary to a uniformly positive scalar curvature metric on the whole manifold. In this section we follow the approach in [26] to define the localized fundamental class and coarse index.

Definition 4.8. Denote by $C_{L,0,d}^*(X)^\Gamma$ the kernel of the evaluation homomorphism $\text{ev}_1: C_{L,d}^*(X)^\Gamma \rightarrow C_d^*(X)^\Gamma$. Denote by $C_{L,Z,d}^*(X)^\Gamma$ the preimage of $C_d^*(Z \subset X)^\Gamma$ under ev_1 . The symbol d here stands for the chosen completion (red, max, or q).

Suppose that g is a Γ -invariant metric on X with uniformly positive scalar curvature outside of a Γ -invariant set Z . In [26, Def. 2.2.6], in this situation, the so-called partial ρ -invariant $\rho_{Z,\text{red}}^\Gamma(g) \in C_{L,Z,\text{red}}^*(X)^\Gamma$ is constructed, which is a lift of $[D_X]$ under the morphism $K_*(C_{L,Z,\text{red}}^*(X)^\Gamma) \rightarrow K_*(C_{L,\text{red}}^*(X)^\Gamma)$ induced by the inclusion.

Recall the explicit representative for $[D_X] \in K_0(C_{L,d}^*(X)^\Gamma)$ of Section 4. We next recall the construction of [26, Def. 2.2.6] and show that it also works for C_q^* .

Lemma 4.9. *If $f_2 \in C_b(\mathbb{R})$ has Fourier transform with support in $[-r, r]$, then $f_2(D)$ is r -local and depends only on the r -local geometry in the following sense: if $A \subset X$ is a Γ -invariant measurable subset, then $\chi_A f_2(D)(1 - \chi_{B_r(A)}) = 0$ and $\chi_A f_2(D)$ depends only on the Riemannian metric on $\overline{B_r(A)}$.*

Proof. This is the usual unit propagation statement in the form that $f_2(D)$ is the integral of $\hat{f}_2(t)e^{itD}$, where e^{itD} not only has propagation $|t|$ but also is well known to depend only on the r -local geometry. The latter fact is a consequence of [11, Cor. 10.3.4]. \square

Lemma 4.10 ([21, Lem. 2.3], [9, Prop. 3.15]). *Suppose as above that the scalar curvature of g outside Z is bounded from below by $4\epsilon^2$. If $f \in C_0(\mathbb{R})$ has support in $(-\epsilon, \epsilon)$, then $f(D)$ lies in $C_d^*(Z \subset X)^\Gamma$.*

Proof. By [9, Prop. 3.15] the statement holds for all quotients X/N and their reduced Roe algebra, which implies by definition of the quotient completion that it holds for $C_q^*(X)^\Gamma$. \square

Because of the geometric nature of the completion of the Roe algebra we use, Lemmas 4.9 and 4.10 allow to define the localized coarse index using the completion C_q^* as follows.

Definition 4.11. Choose a sup-norm continuous family of normalizing functions Ψ_t for $t \geq 1$ such that $\Psi_1^2 - 1$ has support in $(-\epsilon, \epsilon)$, the Fourier transform of Ψ_t has compact support for each $t > 1$ and the Fourier transform of Ψ_t has support in $[-\frac{1}{t}, \frac{1}{t}]$ for $t \geq 2$. Note that the support condition on Ψ_1 implies that its Fourier transform is *not* compactly supported. For the existence, note that we have to approximate the Fourier transform of Ψ_1 by compactly supported functions (with a singularity at 0) such that the error is small in L^1 -norm. This is possible, as can be seen from the discussion in the proof of [9, Lem. 3.6].

Define $F_X(t)$ and P_X as in Section 4. Observe, however, that by Lemma 4.9, $F_X(1)F_X(1)^* - 1 \in C_q^*(Z \subset X)^\Gamma$. It follows that now the cycle $[P_X] - [e_{11}]$ defines a class

$$\rho_Z^\Gamma(g) \in K_0(C_{L,Z,d}^*(X)^\Gamma),$$

which is of course a lift of $[D_X]$.

Corollary 4.12. *The construction shows that if we have uniform positive scalar curvature not only on $X \setminus Z$ but on all of X , there is a further lift of $\rho_Z^\Gamma(g)$ to $\rho^\Gamma(g) \in K_0(C_{L,0,d}^*(X)^\Gamma)$, the usual rho-invariant.*

Definition 4.13. Let $Z \subset X$ and g be as above. Suppose furthermore that the action of Γ on Z is cocompact so that Lemma 2.12 holds for Z . The equivariant localized coarse index $\text{Ind}_Z^\Gamma(g)$ of g with respect to Z is defined as the image of $\rho_Z^\Gamma(g)$ under the composition

$$K_0(C_{L,Z,d}^*(X)^\Gamma) \rightarrow K_0(C_d^*(Z \subset X)^\Gamma) \cong K_0(C_d^*(Z)^\Gamma),$$

where the first map is induced by evaluation at 1.

The long exact sequence in K -theory associated to the short exact sequence

$$0 \rightarrow C_{L,0}^*(X)^\Gamma \rightarrow C_{L,Z}^*(X)^\Gamma \rightarrow C^*(Z \subset X)^\Gamma \rightarrow 0,$$

along with Corollary 4.12 imply that if g has uniformly positive scalar curvature on all of X , then $\text{Ind}_Z^\Gamma(g)$ vanishes.

4.14. Application to the case of a compact manifold with boundary. Suppose M is a compact even-dimensional spin manifold with boundary N . In this case we cannot directly define an index for the Dirac operator on M with value in $K_*(C_q^*(\pi_1(M)))$. However, given a metric g with positive scalar curvature and product structure near the boundary, we can use the above localized coarse index to define an index in $K_0(C_q^*(\widetilde{M})^{\pi_1(M)}) \cong K_0(C_q^*(\pi_1(M)))$. Note that this index *does* in general depend on the chosen metric of positive scalar curvature near the boundary. Let us review the construction of the latter index.

As in Section 4, denote by N_∞ the cylinder $N \times [0, \infty)$ and by M_∞ the manifold $M \cup_N N_\infty$. Denote by $[D_{M_\infty}]$ the fundamental class of the Dirac

operator in $K_*(C_{L,q}^*(M))$ associated to some metric g on M_∞ (not necessarily collared on the cylindrical end) and by $[\tilde{D}_{M_\infty}]$ the fundamental class of the Dirac operator in $K_*(C_{L,q}^*(\tilde{M})^{\pi_1(M)})$ on \tilde{M}_∞ associated to the pullback of g , which we denote by \tilde{g} . As observed in Remark 2.42, Proposition 2.41 extends to M_∞ and the pointwise lifting procedure of operators with small propagation gives rise to an isomorphism $K_*^L(M_\infty) \cong K_*(C_L^*(\tilde{M}_\infty)^{\pi_1(M)})$ under which $[D_{M_\infty}]$ is mapped to $[\tilde{D}_{M_\infty}]$. If g has positive scalar curvature on N , then its pullback has uniformly positive scalar curvature on $N'_\infty \subset \tilde{M}_\infty$, i.e., outside the cocompact subset \tilde{M} of \tilde{M}_∞ . This allows us to define the localized coarse index $\text{Ind}^{\pi_1(M)}(g) := \text{Ind}_{\tilde{M}}^{\pi_1(M)}(\tilde{g}) \in K_0(C^*(\tilde{M})^{\pi_1(M)}) \cong K_0(C^*(\pi_1(M)))$. The latter index is an obstruction to \tilde{g} , and thus g , having positive scalar curvature.

5. STATEMENT AND PROOF OF THE MAIN THEOREM

Finally, we are in the position to state the main theorem of this paper. Throughout this section we will assume all the manifolds and their boundary to be path-connected.

Theorem 5.1. *Let M be a compact spin manifold with boundary N . We have the commutative diagram*

$$\begin{CD} \rightarrow K_*^L(N) @>>> K_*^L(M) @>>> K_*^L(M, N) @>>> \\ @VV\mu_N V @VV\mu_M V @VV\mu V \\ \rightarrow K_*(C_q^*(\pi_1(N))) @>>> K_*(C_q^*(\pi_1(M))) @>j>> K_*(C_q^*(\pi_1(M), \pi_1(N))) @>>> \end{CD}$$

where the vertical maps are the index maps and relative index maps.

Assume that M has a metric g which is collared at the boundary and has positive scalar curvature there. Then

$$j(\text{Ind}^{\pi_1(M)}(g)) = \mu([M, N])$$

under the canonical map $j: K_*(C_q^*(\pi_1(M))) \rightarrow K_*(C_q^*(\pi_1(M), \pi_1(N)))$.

The above theorem has as a corollary the following vanishing theorem of Chang, Weinberger and Yu for the relative index constructed in the mapping cone of the quotient completion of the group ring:

Theorem 5.2. *Let M be a compact spin manifold with boundary N . Suppose that M admits a metric of uniformly positive scalar curvature which is collared at the boundary. Then $\mu([M, N]) = 0$.*

Proof of the Theorem 5.1. Proposition 2.49 implies the commutativity of the diagram. To see this, note that the discussion there relies only on the functoriality properties of the maximal completions which are also satisfied by the quotient completions. It remains to show that given a metric with positive scalar curvature at the boundary, $\text{Ind}^{\pi_1(M)}(g)$ is mapped to $\mu([M, N])$ under the canonical map. Let us analyze the situation with the strategy of proof and

the difficulties involved. For the notation used, we refer to Sections 4 and 4.7 on the relative index and the localized coarse index.

Both index classes are defined using explicit expressions involving functions of the Dirac operator. For $\text{Ind}^{\pi_1(M)}(g)$, we only use the manifold \widetilde{M} and $\pi_1(M)$ -equivariant constructions, which, however, are necessarily non-local to make use of the invertibility of the Dirac operator on the boundary. For $\mu([\widetilde{M}, N])$, on the other hand, one has to use a $\pi_1(M)$ -equivariant operator on \widetilde{M} and a further lift to a $\pi_1(N)$ -equivariant operator on \widetilde{N} , which is only possible if all the functions of the Dirac operator involved are sufficiently local. To show that the two classes are mapped to each other, we need to reconcile these two points.

First, observe that in the construction of the relative fundamental class and relative index, we use the explicit implementation of the Bott periodicity map. We apply this now to our representative of the local index: with our choice of Ψ_1 , $P_{\widetilde{M}_\infty}(1)$ is an idempotent in $C_q^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$ representing $\text{Ind}^{\pi_1(M)}(g) \in K_0(C_q^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}) \cong K_0(C^*(\pi_1(M)))$. Next,

$$\tau := v \otimes P_{\widetilde{M}_\infty}(1) + I \otimes (I - P_{\widetilde{M}_\infty}(1))$$

is an invertible element in $C_0(\mathbb{R}) \otimes C^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$ representing the K_1 -class corresponding to the localized index under the suspension isomorphism. Finally, if we define $q(s)$ as in equation (2), with $\tau_{D,R}$ replaced by τ , then

$$a := [q(0)(1), q(\cdot)(1)] - [q_p(0), q_p(\cdot)] \in K_0(SC_{\{0\} \rightarrow C^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}}$$

defines the class corresponding to $\text{Ind}^{\pi_1(M)}(g)$ under the Bott periodicity isomorphism, where we use that the cone of the inclusion of $\{0\}$ into A is the suspension of A . Of course, here $q(0)(1) = q_p(0)$.

We now have to show that, under the canonical map to the suspension of the cone of $C^*(\widetilde{N} \subset \widetilde{N}_\infty)^{\pi_1(N)} \rightarrow C^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$ induced by the inclusion $\{0\} \rightarrow C^*(\widetilde{N}, \widetilde{N}_\infty)^{\pi_1(N)}$, the class a is mapped to the relative index $\mu[M, N]$. Recall from (4) that the latter is represented by any cycle of the form

$$[\tilde{q}_{D,R_t}^N(0)(t), \tilde{q}_{D,R_t}(\cdot)(t)] - [(q_p(0), q_p(\cdot))]$$

for $t > 1$, such that the support of $\widehat{\Psi}_t$ is contained in $[-L_t, L_t]$ for $L_t \in \mathbb{R}$ and therefore $\Psi_t(D)$ has propagation $\leq L_t$, where we must choose $R_t > 30L_t$. The construction of $\tilde{q}_{D,R_t}(\cdot)(t)$ involves the same steps as the one of $q(\cdot)$, but we use $\Psi_t(D)$ instead of $\Psi_1(D)$ and moreover apply cut-off with χ_{R_t} . Note that now $\tilde{q}_{D,R_t}^N(0)(t) - q_p(0) \neq 0$, but rather $\tilde{q}_{D,R_t}^N(0)(t) - q_p(0) \in C^*(\widetilde{N} \subset \widetilde{N}_\infty)^{\pi_1(N)}$, so that this is not a class in the suspension of $SC^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$ but in the mapping cone.

We claim now that for each $\epsilon > 0$, there is (t_ϵ, R_ϵ) such that

$$(5) \quad \|\tilde{q}_{D,R_\epsilon}^N(0)(t_\epsilon) - q_p(0)\| + \|\tilde{q}_{D,R_\epsilon}(\cdot)(t_\epsilon) - q(\cdot)(1)\| \leq \epsilon.$$

This implies by standard properties of the K -theory of Banach algebras the desired result (as $q(0)(1) = q_p(0)$),

$$\mu([M, N]) = c(\text{Ind}^{\pi_1(M)}(g)).$$

To prove (5) we make use of Lemma 4.4 which explicitly describes the operators involved. This implies

$$(6) \quad \|\tilde{q}_{D,R}(\cdot)(t) - \tilde{q}_{D,R}(\cdot)(1)\| \xrightarrow{t \rightarrow 1} 0$$

uniformly in R , as the two expressions are obtained via algebraic operations involving $\Psi_t(D)$, and by the sup-norm continuity of Ψ_t , $\Psi_t(D)$ converges to $\Psi_1(D)$ in norm (and this again uniformly, independent of the complete Riemannian manifold for which D is considered).

Next by the uniformly positive scalar curvature on N_∞ , we have $P_{\widetilde{M}_\infty}(1) - e_{11} \in C^*(\widetilde{M} \subset \widetilde{M}_\infty)^{\pi_1(M)}$. This implies (convergence in norm)

$$\chi_R(P_{\widetilde{M}_\infty}(1) - e_{11})\chi_R \xrightarrow{R \rightarrow \infty} P_{\widetilde{M}_\infty}(1) - e_{11}$$

or, equivalently,

$$(7) \quad \chi_R P_{\widetilde{M}_\infty}(1)\chi_R + (1 - \chi_R)e_{11}(1 - \chi_R) \xrightarrow{R \rightarrow \infty} P_{\widetilde{M}_\infty}(1).$$

Because of Lemma 4.4, (7) implies that

$$(8) \quad \|\tilde{q}_{D,R}(\cdot)(1) - q(\cdot)(1)\| \xrightarrow{R \rightarrow \infty} 0$$

as these operators are obtained as a fixed algebraic expression of either

$$\chi_R P_{\widetilde{M}_\infty}(1)\chi_R + (1 - \chi_R)e_{11}(1 - \chi_R) \quad \text{or} \quad P_{\widetilde{M}_\infty}(1).$$

Next, (6) together with (8) imply the assertion of (5) for the second summand. Here, we can and have to choose R_ϵ depending on t_ϵ such that $R_\epsilon > R_{t_\epsilon}$ (depending on the propagation of $\Psi_{t_\epsilon}(D)$).

Then, the lift $\tilde{q}_{D,R_\epsilon}^N(0)(t_\epsilon)$ to $C^*(\tilde{N} \subset \tilde{N}_\infty)^{\pi_1(N)}$ actually exists and is defined in terms of the Dirac operator on $\tilde{N} \times \mathbb{R}$, and we have to show that by choosing t_ϵ sufficiently close to 1 it is close to $q_p(0)$.

This, as we have already shown, it is a special case of (6) and (8), now applied to the Dirac operator on $\tilde{N} \times \mathbb{R}$. Note that because of the invertibility of the Dirac operator on $N \times \mathbb{R}$ and our appropriate choice of the normalizing function Ψ_1 , we have on the nose

$$\tilde{q}^N(0)(1) = q_p(0),$$

where q^N is defined like q but using the Dirac operator on $\tilde{N} \times \mathbb{R}$. This finishes the proof of (5) and therefore of our main Theorem 5.1. \square

Remark 5.3. We decided to present the details of the index constructions and proofs only for even-dimensional manifolds.

The case of odd-dimensional manifolds can easily be reduced to this case via a ‘‘suspension construction’’, as also done in [2]. More precisely, if we have an odd-dimensional compact manifold M , we pass to the even dimensional

manifold $M \times S^1$. Correspondingly, the covering space \tilde{M} with action by $\pi_1(M)$ is replaced by $\tilde{M} \times \mathbb{R}$ with action of $\pi_1(M) \times \mathbb{Z}$.

It is now a standard result that we have Künneth isomorphisms for the K -theory groups relevant to us, in particular, for a group homomorphism $\Lambda \rightarrow \Gamma$

$$(9) \quad K_0(C_d^*(\Gamma \times \mathbb{Z}, \Lambda \times \mathbb{Z})) \xrightarrow{\cong} K_0(C_d^*(\Gamma, \Lambda)) \oplus K_1(C_d^*(\Gamma, \Lambda)).$$

The ad hoc definition of the relative index $\mu(M, N) \in K_1(C_d^*(\pi_1(M), \pi_1(N)))$, generalizing Definition 4.5 to odd dimensional M , is now just the image of $\mu([M \times S^1, N \times S^1])$ under the Künneth map (9) (and indeed, the K_0 -component is zero).

Because positive scalar curvature of M implies positive scalar curvature of $M \times S^1$, Theorem 5.2 for odd-dimensional M follows from its version for the even-dimensional $M \times S^1$.

In the same way, using Künneth and suspension isomorphisms for the whole diagram of Theorem 5.1 (using along the way, e.g., [27, Section 5]), the statement and proof of Theorem 5.1 for odd-dimensional M follows from the corresponding one for the even-dimensional $M \times S^1$.

More systematically, Zeidler [27] develops a set-up of Cl_n -linear Roe algebras and localization algebras and Cl_n -equivariant Dirac operators on n -dimensional spin manifolds. Our constructions and arguments should carry through in this set-up, given a uniform treatment for all dimensions, and working with real group C^* -algebras. As this requires a bit more notation and additional concepts, and as we were striving for a down to earth exposition, we decided to stick to the classical set-up and leave it to the interested reader to work out the details of such an approach.

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