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Singularities and Quinn spectra

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Abstract. We introduce singularities to Quinn spectra. It enables us to talk about ads with prescribed singularities and to explicitly construct highly structured representatives for prominent spectra like Morava K-theories or for L-theory with singularities. We develop a spectral sequence for the computation of the associated bordism groups and investigate product structures in the presence of singularities.

1. Introduction

Manifolds with cone-like singularities were introduced by D. Sullivan in [10]. The concept was reformulated by Baas in [1] as manifolds with a higher order (multilevel) decomposition of its boundary. Based on this definition a theory of cobordisms with singularities was developed. Many interesting homology and cohomology theories were constructed based on this theory: for example the Morava K-theories, the Johnson–Wilson theories or versions of elliptic cohomology.

All these theories have played an important role in homotopy theory and algebraic topology during the last 30–40 years. However, it is surprising how many results could be obtained by just knowing their existence, not their construction. An explicit construction, however, can help in constructing important classes or in investigating the multiplicative structure of the representing spectra. Also, in order to obtain further results it seems to be important that the spectra are related to the original geometric category.

This is the goal of the present paper. The theory of 'ads' [8] is used to construct Quinn-spectra with singularities. They are symmetric spectra which come with the expected long exact sequences and a Bousfield–Kan spectral sequence for the computation of their coefficients. Moreover, it turns out that these spectra always give strict module spectra over the original Quinn spectra. In some cases they even have an explicit A_∞ structure or sometimes, as shown in [7], an E_∞-structure. If the Quinn spectrum is L-theory, the singularities spectrum seems to provide the natural surgery obstructions for manifolds with singularities.

This work is organized as follows: in Section 2 we recall from [7, 8] the main results on ad theories and Quinn spectra and give a few examples. In Section 3 we introduce the singularities in the context of ads and develop new ad theories this way. Then the exact sequence for the bordism groups are constructed. It relates the ad theories among each other in case of a sequence of singularities. Section 4 deals with the classical example of manifolds ads. An assembly map shows that the corresponding Quinn spectrum with singularities represents the homology of manifolds with singularities of [1]. Section 5 is devoted to a Bousfield–Kan type spectral sequence for ads with singularities. For complex bordism such a spectral sequence was developed by Morava in [9]. In Section 6 we discuss product structures. It is shown that the Quinn spectrum of an ad theory with singularities is a strict module spectrum over the original Quinn spectrum. Moreover, there always is an external product for ad theories with singularities which has all desired properties. Internal product structures are more difficult to obtain. We show that there is a way to produce an ad theory with an internal product which comes with a map from the original ad theory. In general, this map does not induce a homotopy equivalence on Quinn spectra in general but it does so under some conditions.

2. Ad theories and Quinn spectra

In this section we recall the basic notions of [8] which lead to spectra of Quinn type.

Recall from [8, Def. 3.3] that a $\mathbb{Z}$-graded category $\mathcal{A}$ is a category with an action of $\mathbb{Z}/2 = \{\pm 1\}$ and $\mathbb{Z}/2$-equivariant functors

$$d = \dim : \mathcal{A} \to \mathbb{Z}, \quad \varnothing : \mathbb{Z} \to \mathcal{A}$$

which satisfy $d\varnothing = \mathrm{id}$. Here, $\mathbb{Z}$ is regarded as a poset with the trivial action. The full subcategory of $\mathcal{A}$ of n-dimensional objects is denoted by $\mathcal{A}_n$. In abuse of notation, we will often write $\varnothing$ for the object $\varnothing_n$ of $\mathcal{A}_n$. A k-morphism between graded categories is a functor which decreases the dimension by k and strictly commutes with the involution -1 and $\varnothing$.

Let K be a ball complex in the sense of [4]. We write $\mathcal{C}\mathrm{ell}(K)$ for the category with the oriented cells (σ, o) of K and the empty cell $\varnothing_n$ as objects in dimension n. There are only identity morphisms in $\mathcal{C}\mathrm{ell}(K)_n$ and morphisms to higher dimensional cells are given by inclusions of cells with no requirements to the orientations. The category $\mathcal{C}\mathrm{ell}(K)$ is a graded category with the orientation reversing involution. Note that morphisms between ball complexes induce morphisms on the cellular categories. Moreover, if L is a subcomplex of K, we can form the quotient category $\mathcal{C}\mathrm{ell}(K, L)$ of $\mathcal{C}\mathrm{ell}(K)$ by identifying the cells of L with the empty cells.

Next we recall the definition of an ad theory from [8, Def. 3.8].

Definition 2.1. Let $\mathcal{A}$ be a category over $\mathbb{Z}$. A k-morphism from $\mathcal{C}\mathrm{ell}(K, L)$ to $\mathcal{A}$ is called a pre (K, L)-ad of degree k. We write $\mathrm{pre}^k(K, L)$ for the set of these pre ads. An *ad theory* is an i-invariant sub-functor ad^k of pre^k from ball complexes to sets for each k with the property $\mathrm{ad}^k(K,L) = \mathrm{pre}^k(K,L) \cap \mathrm{ad}^k(K)$ and which satisfies the following axioms:

- *(pointed)* The pre ad which takes every oriented cell to $\varnothing$ is an ad for every K.
- *(full)* Any pre K-ad which is isomorphic to a K-ad is a K-ad.
- *(local)* Every pre K-ad which restricts to a σ-ad for each cell σ of K is a K-ad.
- *(gluing)* For each subdivision K' of K and each K'-ad M there is a K-ad which agrees with M on each common subcomplex of K and K'.
- *(cylinder)* There is a natural transformation
$$J : \mathrm{ad}^n(K) \to \mathrm{ad}^n(K \times I)$$
with the property that for every K-ad M the restriction of $J(M)$ to $K \times 0$ and to $K \times 1$ coincides with M. It takes trivial ads to trivial ones.
- *(stable)* Let
$$\theta : \mathcal{C}\mathrm{ell}(K_0, L_0) \to \mathcal{C}\mathrm{ell}(K_1, L_1)$$
be a k-isomorphism with the property that it preserves all incidence numbers:
$$[o(\sigma), o(\sigma')] = [o(\theta\sigma), o(\theta\sigma')]$$
(see [12, p. 82]). Then the induced map of pre ads restricts to ads:
$$\theta^* : \mathrm{ad}^l(K_1, L_1) \to \mathrm{ad}^{k+l}(K_0, L_0).$$

A multiplicative ad theory in a graded symmetric monoidal category $\mathcal{A}$ is equipped with a natural transformation
$$\mathrm{ad}^p(K) \wedge \mathrm{ad}^q(L) \to \mathrm{ad}^{p+q}(K \times L)$$
and the object e in $\mathrm{ad}^0(*)$ which is associative and unital in the sense of [8, Def. 3.10 and 18.4]. A multiplicative ad theory is called commutative if the monoidal structure of $\mathcal{A}$ extends to a permutative structure (see [7, Def. 3.1 and 3.3]). In particular, there is a natural isomorphism
$$\gamma : x \otimes y \to (-1)^{d(x)d(y)} y \otimes x$$
for all $x, y \in \mathcal{A}$.

Example 2.2. Let R be a ring with unit. Consider R as the graded category for which the objects are the elements of R concentrated in dimension 0, there are only identity morphisms and the involution is the multiplication by -1. Then there is a multiplicative ad theory ad_R with K-ads M all pre K-ads with the property that, for all cells $\sigma \in K$ of dimension n,
$$\sum_{\substack{\dim(\sigma')=n-1 \\ \sigma' \subset \sigma}} [o(\sigma), o(\sigma')] M(\sigma', o(\sigma')) = 0,$$
where $[o(\sigma), o(\sigma')]$ is the incidence number. If R is commutative then so is ad_R.

Example 2.3. Let $\mathcal{STop}$ be the graded category of compact oriented topological manifolds. An ad theory over $\mathcal{STop}$ can be defined as follows: a pre K-ad M is an ad if for each $\sigma' \subset \sigma$ of one dimension lower, the map $M(\sigma', o') \to M(\sigma, o)$ factors through an orientation preserving map

$$M(\sigma', o') \to [o, o']\partial M(\sigma, o)$$

and $\partial M(\sigma, o)$ is the colimit of M restricted to $\partial\sigma$. See [8, §6] for details. Similarly, there is an ad theory over the graded category of compact unoriented topological manifolds $\mathcal{T}$op. For instance, a decomposed (oriented) manifold in the sense of [1] is a Δ^n-ad.

Example 2.4. Let W be the standard resolution of $\mathbb{Z}$ by $\mathbb{Z}[\mathbb{Z}/2]$ modules. Define the objects of $\mathcal{A}$ to be the quasi-symmetric complexes, that is, in dimension n we have pairs $(C; \varphi)$ where C is a quasi-finite complex of free abelian groups and

$$\varphi : W \to C \otimes C$$

is a $\mathbb{Z}/2$ equivariant map which raises the degree by n. The dimension increasing morphisms $f : (C; \varphi) \to (C'; \varphi')$ are the chain maps and for equal dimension of source and target one further assumes that

$$(f \otimes f)\varphi = \varphi'.$$

The involution changes the sign of φ. The K-ads of symmetric Poincaré complexes are those (balanced) functors which are

(i) closed, that is, for each cell σ of K the map from the cellular chain complex,

$$\mathrm{cl}(\sigma) \to \mathrm{Hom}(W, C \otimes C),$$

which takes (τ, o) to the composite,

$$W \xrightarrow{\varphi(\tau,o)} C_\tau \otimes C_\tau \to C_\sigma \otimes C_\sigma,$$

is a chain map;

(ii) well behaved, that is, each map $f_{\tau\subset\sigma}$ and

$$C_{\partial\sigma} = \operatorname*{colim}_{\tau \subsetneq \sigma} C_\tau \to C_\sigma$$

are cofibrations (split injective).

(iii) non-degenerate, that is, the induced map

$$H^*(\mathrm{Hom}(C, \mathbb{Z})) \to H_{\dim\sigma - \deg F - *}(C_\sigma / C_{\partial\sigma})$$

is an isomorphism.

For an ad theory the bordism groups Ω^n are obtained by identifying two $*$-ads of dimension n if there is an I-ad which restricts to the given ones on the ends.

Theorem 2.5 ([7, 8])**.** *The ads form the simplexes of the spaces in a positive Ω-spectrum $Q(\mathrm{ad})$ in a natural way. Its coefficients are given by the bordism groups. If the ad theory is multiplicative then the spectrum can be given the structure of a symmetric ring spectrum. If the ad theory is commutative then it is weakly equivalent to a commutative ring spectrum.*

In the example of a commutative ring one obtains the Eilenberg–MacLane spectrum with R-coefficients. In the example of oriented manifolds one obtains a spectrum which is homotopy equivalent to the Thom spectrum. In the example of symmetric Poincaré complexes the spectrum coincides with the symmetric L-theory spectrum. More examples can be found in [2].

3. Singularities

In this section we introduce the concept of singularities to ad theories. Bordism theories of manifolds with singularities have been studied by the first author in [1]. In ordinary bordism one works with closed manifolds and bordisms between them. In the case of singularities of type P_1 one considers manifolds M with a special boundary. (One may think that these objects are the results from removing the cones over P_1 from closed singular manifolds of type P_1.) More specifically, the 'closed' P_1-manifolds are those with boundary of the form $N \times P_1$ for some closed manifold N. A null bordism B of M comes with a decomposition into two boundary components which are glued along their common boundaries. One component is M and the other again comes with a homeomorphism to a product with one factor P_1. When studying more than one singularity at once, one is forced to look at further decompositions of manifolds. As we have seen before, such objects are provided by manifold Δ^n-ads. We will reconsider the bordism theory of manifolds with singularities in more detail in Section 4.

In order to generalize this concept to other ad theories, suppose we are given a commutative multiplicative ad theory ad over $\mathcal{A} = (\mathcal{A}, \otimes, I)$. We also assume that $\varnothing$ is initial in $\mathcal{A}$, that is, there is exactly one morphism to a higher dimensional object. This is the case in all applications we have in mind.

Definition 3.1. Let $S = (P_1, P_2, \dots)$ be a sequence of $*$-ads and set

$$S_n = (P_1, \dots, P_n).$$

Let $\mathcal{A}(S_n)$ be the graded category whose objects are given by the following data:

(i) A pre σ-ad M_σ for each cell $\sigma \subset \{0, 1, \dots, n\}$ of Δ^n with the property

$$M_\sigma = \varnothing \quad \text{if } 0 \notin \sigma.$$

For $\sigma = \{0, 1, \dots, n\}$ we simply write M for the top pre ad.

(ii) An isomorphism of pre ads for each $i \notin \sigma$:

$$f_{\sigma,i} : \partial_i M_{(\sigma,i)} \xrightarrow{\cong} (-1)^{|\sigma,i|} M_\sigma \otimes P_i.$$

Here, ∂_i denotes the restriction to the face σ, (σ, i) means $\sigma \cup \{i\}$ and

$$|\sigma, i| = d(P_i) \sum_{s>i,\, s\in\sigma} d(P_s).$$

We demand for each object that $\partial_0 M_\sigma = \varnothing$ and that the diagram

$$\begin{array}{ccccc} \partial_j\partial_i M_{(\sigma,i,j)} & \longrightarrow & \partial_j M_{(\sigma,j)} \otimes P_i & \longrightarrow & M_\sigma \otimes P_j \otimes P_i \\ \big\downarrow{=} & & & & \big\downarrow{1\otimes\gamma} \\ \partial_i\partial_j M_{(\sigma,i,j)} & \longrightarrow & \partial_i M_{(\sigma,i)} \otimes P_j & \longrightarrow & M_\sigma \otimes P_i \otimes P_j \end{array}$$

commutes for all $i, j > 0$ (after taking the appropriate signs).

The dimension of an object $M.$ in $\mathcal{A}(S_n)$ is $d(M)-n$. Morphisms are morphisms of pre ads which commute with the isomorphisms $f_{\sigma,i}$.

Example 3.2. For $n = 0$ an object is determined by the value of the top cell of Δ^0 since $\varnothing$ is initial in $\mathcal{A}$. Hence we have

$$\mathcal{A}(\,) = \mathcal{A}.$$

For $n = 1$ an object is a Δ^1-pre ad $M = M_{\{0,1\}}$ and an object $N = M_{\{0\}}$ of $\mathcal{A}$ such that M has faces $\varnothing$ and $N \otimes P_1$.

Lemma 3.3. *For a ball complex L consider the graded category $\mathcal{B}$ of L-pre ads $\mathrm{pre}_{\mathcal{A}}(L)$ with values in $\mathcal{A}$. Then there is a natural equivalence of the form*

$$\mathrm{pre}_{\mathcal{A}}(K \times L) \cong \mathrm{pre}_{\mathcal{B}}(K).$$

Proof. We have a natural equivalence of categories over $\mathbb{Z}$:

$$\mathcal{C}\mathrm{ell}(K) \wedge_{\mathbb{Z}/2} \mathcal{C}\mathrm{ell}(L) \cong \mathcal{C}\mathrm{ell}(K \times L).$$

Here, the left-hand side has pairs $((\sigma, o), (\sigma', o'))$ of oriented cells as objects which are identified with $\varnothing$ if one of the cells is empty. In addition, each such pair is identified with the pair $((\sigma, -o), (\sigma', -o'))$. The claim follows from the obvious adjunction between products and functor sets. □

Proposition 3.4. *Let $\mathrm{ad}/S_n(K)$ be the set of pre K-ads in $\mathcal{A}(S_n)$ which give $(K \times \sigma)$-ads in $\mathcal{A}$ under the adjunction of Lemma 3.3 for each cell σ of Δ^n. Then ad/S_n defines an ad theory.*

Proof. The set ad/S_n clearly is pointed and full. Suppose that we are given a pre K-ad in $\mathcal{A}(S_n)$ which restricts to an adjoint of a $\tau \times \sigma$-ad for every $\tau \in K$. Then its adjoint restricts to a $K \times \sigma$-ad by the locality property of Definition 2.1 and hence it is an ad.

Next, we check the gluing property. A subdivision K' of K defines the subdivision $K' \times \sigma$ of $K \times \sigma$. Hence a $K' \times \sigma$-ad can be glued to a $K \times \sigma$-ad and the claim follows.

The cylinder J of the original ad theory takes a $K \times \sigma$-ad to a $K \times \sigma \times I$-ad and hence defines a cylinder for ad/S_n.

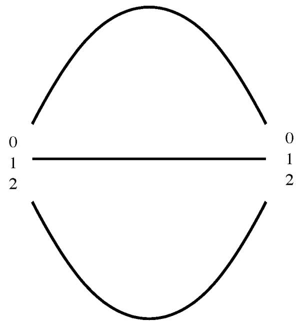

FIGURE 1

Finally, we have to show the stability axiom. An incidence number preserving k-isomorphism

$$\theta : \mathcal{C}\mathrm{ell}(K_0, L_0) \to \mathcal{C}\mathrm{ell}(K_1, L_1)$$

induces a k-isomorphism

$$\theta \times \mathrm{id} : \mathcal{C}\mathrm{ell}(K_0 \times \sigma, L_0 \times \sigma) \to \mathcal{C}\mathrm{ell}(K_1 \times \sigma, L_1 \times \sigma).$$

Hence, if M is a (K_0, L_0)-ad$/S_n$, then the ads induced by $(\theta \times \mathrm{id})^*$ of its adjoints assemble to $\theta^* M$. □

Example 3.5. Let ad be the ad theory of topological manifolds. Then the monoidal structure is the cartesian product. For $n = 0$ an object of ad$/S_0(*)$ is a manifold without boundary. For $n = 1$ and $S_1 = (*)$ we have a manifold with an arbitrary boundary. Figure 1 shows a manifold with a $\mathbb{Z}/3$-singularity, that is, an element of ad$/(P_1)(*)$ where P_1 consists of three points. (In other words, the local cone structure is $C\mathbb{Z}/3$.) In the notation of Example 3.2, the manifold N consists of two points.

Next we investigate how the ad theories ad$/S_n$ are related for different n. First observe that we have a map

$$\mu_{P_{n+1}} : \mathrm{ad}/S_n \to \mathrm{ad}/S_n$$

of degree $d = d(P_{n+1})$ which multiplies the ads by P_{n+1} from the right. Furthermore, there is a map of degree -1,

$$\pi : \mathrm{ad}/S_n \to \mathrm{ad}/S_{n+1},$$

which comes from considering an object of $\mathcal{A}(S_n)$ as an object of $\mathcal{A}(S_{n+1})$ by

$$\pi M_\sigma = \begin{cases} M_{\sigma \setminus \{n+1\}} & \text{if } (n+1) \in \sigma, \\ \varnothing & \text{otherwise.} \end{cases}$$

This certainly defines a pre K-ad $\pi(M)$ over S_{n+1} for each K-ad M over S_n.

Lemma 3.6. *$\pi(M)$ is an ad.*

Proof. We only check the top cell $\sigma = \{0, 1, \ldots, n\}$. The other cells are similar. The adjoint of M is a $K \times \Delta^n$-ad. The 1-isomorphism of graded categories

$$\mathcal{C}\mathrm{ell}(\Delta^{n+1}, \{n+1\} \cup \partial_{n+1}\Delta^{n+1}) \xrightarrow{\cong} \mathcal{C}\mathrm{ell}(\Delta^n)$$

can be multiplied with $\mathcal{C}\mathrm{ell}(K)$ and hence gives the desired ad by the stability axiom. □

Finally, we have a map

$$\delta : \mathrm{ad}/S_{n+1} \to \mathrm{ad}/S_n.$$

It takes a K-ad M over S_{n+1} to the K-ad over S_n given by the formula

$$\delta(M)(\sigma, o) = M(\sigma, o)_{|\{0,1,\ldots,n\}}.$$

Theorem 3.7. *Let $\Omega_*^{S_n}$ be the bordism group of the ad theory ad/S_n. Then the following sequence is exact:*

$$\cdots \xrightarrow{\delta_*} \Omega_*^{S_n} \xrightarrow{\mu_{P_{n+1}}{}_*} \Omega_{*+d}^{S_n} \xrightarrow{\pi_*} \Omega_{*+d-1}^{S_{n+1}} \xrightarrow{\delta_*} \Omega_{*-1}^{S_n} \xrightarrow{\mu_{P_{n+1}}{}_*} \cdots.$$

Proof. The proof is essentially the same as in [1, Thm. 3.2]. □

Example 3.8. Consider the sequence $S = (\varnothing, \varnothing, \ldots)$. Using the suspension axiom, it is not hard to see that ad/S_n consists of $n+1$ copies of the original Quinn spectrum. Hence the above exact sequence consists of short split exact sequences.

Example 3.9. Let R be a ring and suppose $x \in R$ is a nonzero divisor. Consider the ad theory of Example 2.2. Then the maps of spectra induced by μ_x, π and δ correspond to the Bockstein exact sequence in singular homology.

4. Example: Manifolds with singularities and assemblies

In this section we look at the ad theory of compact manifolds. For simplicity we restrict our attention to the unoriented topological case. It will then be clear how to do other cases of bordism theories.

We fix a sequence S of closed manifolds and write $Q[X]$ for the Quinn spectrum of $\mathrm{ad}[X]/S_n$. Here, X is a topological space and $\mathrm{ad}[X]$ is defined as in Example 2.3 with singular manifolds in X, that is, manifolds equipped with a continuous map to X. In the following, we call a simplicial set without the data of degeneracies a 'semi-simplicial set' (another name in the literature is 'Δ-set'.)

Proposition 4.1. *Suppose F is a functor from semi-simplicial sets to the category of symmetric spectra which sends homotopy equivalences to stable equivalences. Then there is a natural transformation in the homotopy category*

$$F[*] \wedge |X|_+ \to F[X]$$

which is the obvious equivalence if X is a point.

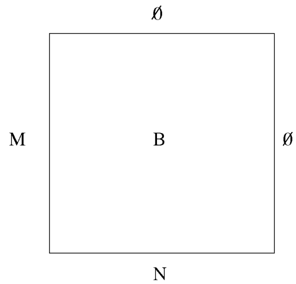

Figure 2

Proof. Some versions of the desired transformation are certainly known and run under the name assembly map (see for example the discussion in [11]). In this simple form it can be obtained as follows: for a semi-simplicial set X we have the natural homotopy equivalence

$$\operatorname*{hocolim}_{\Delta^n\to X} F[\Delta^n] \to \operatorname*{hocolim}_{\Delta^n\to X} F[*] \cong \operatorname*{colim}_{\Delta^n\to X} F[*]\wedge|\Delta^n|_+ \cong F[*]\wedge|X|_+$$

whose homotopy inverse can be composed with the map

$$\operatorname*{hocolim}_{\sigma:\Delta^n\to X} F[\Delta^n] \to \operatorname*{colim}_{\sigma:\Delta^n\to X} F[\Delta^n] \xrightarrow{(F[\sigma])} F[X]. \qquad \square$$

Theorem 4.2. *The spectrum $Q = Q[*]$ represents the homology theory of manifolds with singularities given in* [1].

Proof. We first show that the bordism groups of ad/S_n are naturally equivalent to the bordism groups of manifolds with singularities S_n. For that, recall that a $*$-ad in ad/S_n consists of σ-ads M_σ with $M_\sigma=\varnothing$ if $0\notin\sigma$ and a system of compatible isomorphisms

$$\partial_i M_{(\sigma,i)} \cong (-1)^{|\sigma,i|} M_\sigma \times P_i.$$

Hence, it defines a closed S_n-manifold, and a closed S_n-manifold gives a $*$-ad. A null bordism B is a family of $I\times\sigma$-ads with one end empty and the other being the bounding object M. The situation is illustrated for $n=1$ in Figure 2. Here, B and M are objects of $\mathcal{T}\mathrm{op}(P_1)$ and there are homeomorphisms

$$\begin{aligned}\partial_1 B &\cong N\times P_1,\\ \partial M &\cong \partial N\times P_1,\\ \partial B &\cong M\cup_{\partial N\times P_1} N\times P_1.\end{aligned}$$

For example in the case of the $\mathbb{Z}/3$-manifold M considered earlier, a null bordism is pictured in Figure 3. Here, N is a horizontal arc. This shows that the bordism groups coincide. The same argumentation shows that the bordism groups of the ad theory $\mathrm{ad}[X]/S_n$ coincide with the bordism groups of manifolds with singularities in X. This implies that Q is a homotopy invariant functor and hence the assembly map is well-defined by Proposition 4.1.

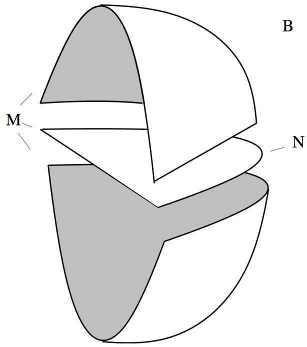

FIGURE 3

Finally, we have to show that the assembly map is a homotopy equivalence. Using the fact that bordism of singular S_n-manifolds defines a homology theory, we know that the functor

$$(X, Y) \mapsto \pi_*(Q[X], Q[Y])$$

together with the boundary operator defines a homology theory as well. Thus the assembly map defines a natural transformation between homology theories and is an isomorphism for a point. Thus the claim follows from the comparison theorem between homology theories. $\square$

5. A Bousfield–Kan spectral sequence

The exact sequences of the ad/S_n-bordism groups for different n are part of a spectral sequence of Bousfield–Kan type. In the case of classical complex bordism it first has been developed in [9].

Fix a commutative multiplicative ad theory ad over $\mathcal{A}$ and a sequence of $*$-ads $S = (P_1, P_2, \ldots)$. We will assume that $\varnothing$ is initial in $\mathcal{A}$. For a finite set $T = \{t_1, t_2, \ldots, t_n\}$ of natural numbers with $t_i \leq t_{i+1}$ for all i let

$$S_T = (P_{t_1}, \ldots, P_{t_n})$$

be the subsequence of S indexed by T.

For each T we are going to define the graded category $\mathcal{A}\langle S_T \rangle$. The notation is taken from the theory of manifolds with faces [6, §2.1] and should not be confused with $\mathcal{A}(S_T)$. The objects of $\mathcal{A}\langle S_T \rangle$ consist of the following data:

(i) A pre $*$-ad M_σ for each cell $\sigma \subset \{0, 1, \ldots, n\}$ of Δ^n with the property

$$M_\sigma = \varnothing \quad \text{if } 0 \notin \sigma.$$

(ii) Isomorphisms for each $i \notin \sigma$,

$$f_{\sigma,i} : M_{(\sigma,i)} \xrightarrow{\cong} (-1)^{|\sigma,i|} M_\sigma \otimes P_{t_i},$$

which are compatible with the face maps. Moreover, $\partial_0 M = \varnothing$ and for all $i, j > 0$ the diagram

$$\begin{array}{ccccc} M_{(\sigma,i,j)} & \longrightarrow & M_{(\sigma,j)} \otimes P_{t_i} & \longrightarrow & M_\sigma \otimes P_{t_j} \otimes P_{t_i} \\ \big\downarrow{=} & & & & \big\downarrow{1\otimes\gamma} \\ M_{(\sigma,i,j)} & \longrightarrow & M_{(\sigma,i)} \otimes P_{t_j} & \longrightarrow & M_\sigma \otimes P_{t_i} \otimes P_{t_j} \end{array}$$

commutes (with the appropriate signs).

Example 5.1. The category $\mathcal{A}\langle\{1\}\rangle$ consists of $*$-ads M_σ for $\sigma \in \Delta^1$ together with isomorphisms

$$M_{\{0,1\}} \cong M_{\{0\}} \otimes P_1$$

and $M_{\{1\}} = M_\varnothing = \varnothing$. An object of $\mathcal{A}\langle\{1,2\}\rangle$ is a collection of $*$-ads M_σ for $\sigma \in \Delta^2$ together with isomorphisms

$$\begin{aligned} M_{\{0,1,2\}} &\cong (-1)^{d(P_1)d(P_2)} M_{\{0,2\}} \otimes P_1, \\ M_{\{0,1,2\}} &\cong M_{\{0,1\}} \otimes P_2, \\ M_{\{0,1\}} &\cong M_{\{0\}} \otimes P_1, \\ M_{\{0,2\}} &\cong M_{\{0\}} \otimes P_2, \\ M_\sigma &= \varnothing \quad \text{if } 0 \notin \sigma \end{aligned}$$

which let the diagram above commute. In particular, we have a distinguished isomorphism

$$M \cong M_{\{0\}} \otimes P_1 \otimes P_2.$$

Let $\delta^k S_T$ be the sequence obtained from S_T with the kth entry omitted. Consider the face functors

$$\not\partial_k : \mathcal{A}\langle S_T\rangle \to \mathcal{A}\langle \delta^k S_T\rangle$$

given by

$$\not\partial_k (M.)_\sigma = (-1)^{|\sigma,k|^-} M_{(\sigma,k)}$$

with

$$|\sigma,k|^- = d(P_k) \sum_{0<s<k,\, s\in\sigma} d(P_s)$$

and isomorphisms $(-1)^{|\sigma,k|^-} f_{((\sigma,k),i)}$.

Example 5.2. Explicitly, the boundary functor

$$\not\partial_2 : \mathcal{A}\langle\{1,2\}\rangle \to \mathcal{A}\langle\{1\}\rangle$$

is given by

$$(\not\partial_2 M)_{\{0,1\}} = (-1)^{d(P_1)d(P_2)} M_{\{0,1,2\}} \cong M_{\{0,2\}} \otimes P_1 = (\not\partial_2 M)_{\{0\}} \otimes P_1.$$

For each set T there is an ad theory $\mathrm{ad}\langle S_T\rangle$ over $\mathcal{A}\langle S_T\rangle$ as follows: the K-ads are those pre ads M over $\mathcal{A}\langle S_T\rangle$ which are cell-wise K-ads in $\mathcal{A}$, that is, for each cell σ of Δ^n the functor M_σ from $\mathcal{C}\mathrm{ell}(K)$ to $\mathcal{A}$ is a K-ad.

Lemma 5.3. *For each T there is an equivalence of $\mathbb{Z}$-graded categories*

$$\psi : \mathcal{A} \to \mathcal{A}\langle S_T \rangle$$

which makes the diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\cdot P_{t_k}} & \mathcal{A} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ \mathcal{A}\langle S_T \rangle & \xrightarrow{\not\partial_k} & \mathcal{A}\langle \delta^k S_T \rangle \end{array}$$

strictly commutative. It induces a map of ad theories and hence an equivalence of Quinn spectra.

Proof. Define $\psi(M)$ by $(M_\sigma)_{\sigma\subset\Delta^n}$ with $M_\sigma = M \bigotimes_{i\in\sigma\setminus\{0\}} P_{t_i}$ and

$$\phi : \mathcal{A}\langle S_T \rangle \to \mathcal{A}, \quad M \mapsto M_{\{0\}}.$$

The composite $\phi\psi$ is the identity and $\psi\phi(M)$ is canonically isomorphic to M. By construction, these functors induce maps of ad theories and hence maps between Quinn spectra. Since isomorphic objects are bordant, the claim follows. For the commutativity of the diagram observe that the composite

$$\mathcal{A} \to \mathcal{A}\langle S_T \rangle \xrightarrow{\not\partial_{k\,*}} \mathcal{A}\langle \delta^k S_T \rangle \to \mathcal{A}$$

sends M to $M \otimes P_{t_k}$ by definition. □

The ad theories $\mathrm{ad}\langle S_T \rangle$ and the face functors

$$(-1)^{|T,k|} \not\partial_k : \mathrm{ad}\langle S_T \rangle \to \mathrm{ad}\langle \delta^k S_T \rangle$$

define a commutative cubical diagram of ad theories which will be denoted by $\mathrm{ad}\langle S \rangle$ in the sequel.

Lemma 5.4. *Let $Q(\mathrm{ad}\langle S_n \rangle)^+$ be the $n+1$-dimensional diagram indexed by subsets T of $\{0,1,\dots,n\}$ without $\varnothing$ given by*

$$Q(\mathrm{ad}\langle S_n \rangle)^+(T) = Q(\mathrm{ad}\langle S_T \rangle)$$

if $0 \in T$ and by $*$ otherwise. Then we have a natural homotopy equivalence

$$\operatorname{hocolim} Q(\mathrm{ad}\langle S_n \rangle)^+ \simeq Q(\mathrm{ad}/S_n).$$

Proof. The proof is an induction on the number of singularities. In the case of only one singularity the left-hand side is the cofiber of the map of Quinn spectra induced by $\not\partial_1$. This map coincides with the multiplication by P_1 on $\mathcal{A}$ by Lemma 5.3. The cofiber of the induced map of spectra has the same homotopy type as $Q(\mathrm{ad}/S)$ by the exact sequence of Theorem 3.7. Hence the induced map on cofibers is an isomorphism on bordism groups.

There are canonical isomorphisms

$$(\mathcal{A}\langle S_n \rangle)\langle P_{n+1} \rangle \cong \mathcal{A}\langle S_{n+1} \rangle,$$
$$(\mathcal{A}(S_n))(P_{n+1}) \cong \mathcal{A}(S_{n+1}).$$

We will describe the second one in more detail: an element of the left-hand side has the form

$$(M, N, \partial_1 M \xrightarrow{\cong} N \otimes P_{n+1})$$

with M a pre Δ^1-ad and N a pre $*$-ad with values in $\mathcal{A}(S_n)$. This is sent to the object L of $\mathcal{A}(S_{n+1})$ whose pre σ-ad for a cell $\tau \subset \sigma$ (with standard orientations) is

$$L_\sigma(\tau) = \begin{cases} M(\{0,1\})_{\sigma \setminus \{n+1\}}(\tau \setminus \{n+1\}) & \text{for } n+1 \in \tau, \\ M(\{0\})_{\sigma \setminus \{n+1\}}(\tau) & \text{for } n+1 \in \sigma \setminus \tau, \\ N_\sigma(\tau) & \text{for } n+1 \notin \sigma. \end{cases}$$

Note that the multiplication with $*$-ads is still defined (compare Corollary 6.1 below) even though the categories may not be monoidal.

The isomorphisms take ads to ads and hence provide the inductive step

$$\begin{aligned} Q(\mathrm{ad}/S_{n+1}) &\cong Q((\mathrm{ad}/S_n)/P_{n+1}) \\ &\cong \mathrm{hocofiber}\big(Q(\mathrm{ad}/S_n) \xrightarrow{\cdot P_{n+1}} Q(\mathrm{ad}/S_n)\big) \\ &\cong \mathrm{hocofiber}\big(Q(\mathrm{ad}/S_n)\langle P_{n+1}\rangle \xrightarrow{\not\partial} Q(\mathrm{ad}/S_n)\big) \\ &\cong \mathrm{hocofiber}\big(\mathrm{hocolim}\, Q(\mathrm{ad}\langle S_n\rangle^+\langle P_{n+1}\rangle) \xrightarrow{\not\partial} \mathrm{hocolim}\, Q(\mathrm{ad}\langle S_n\rangle^+)\big) \\ &\cong \mathrm{hocolim}\, Q(\mathrm{ad}\langle S_{n+1}\rangle)^+. \end{aligned}$$

The lemma is thus proved. □

The homotopy colimit identification of the spectra with singularities furnishes a spectral sequence of Bousfield–Kan type [3] with the homology of the chain complex

$$\cdots \to \bigoplus_{\#T=k} \pi_* Q(\mathrm{ad}\langle S_T\rangle) \xrightarrow{\not\partial} \bigoplus_{\#T=k-1} \pi_* Q(\mathrm{ad}\langle S_T\rangle) \to \cdots$$

with $\not\partial = \sum(-1)^k \not\partial_k$ being the E_2-term. This gives the following result.

Theorem 5.5. *There is a spectral sequence converging to the bordism groups of* ad/S_n *with* E_2*-term the homology of the Koszul complex* $K(P_1, \dots, P_n)$*, that is, the tensor product over* Ω_* *of the complexes*

$$0 \to \Omega_* \xrightarrow{\cdot P_k} \Omega_* \to 0.$$

6. Product structures

In this section we will investigate product structures on ad theories with singularities. We ask which multiplicative structures are inherited from an ad theory to its S_n-ad theory ad/S_n.

Suppose the ad theory is multiplicative. Then clearly we have an action

$$\mathrm{ad}(K) \times (\mathrm{ad}(L)/S_n) \to \mathrm{ad}(K \times L)/S_n,$$

and hence we obtain a module structure on the Quinn spectra.

Corollary 6.1. *The Quinn spectrum of the ad theory with singularities is a strict module spectrum over the original Quinn spectrum.*

Further product structures come from the following external product: we start with a multiplicative ad theory and finite sequences $P = (P_1, P_2, \ldots, P_n)$ and $Q = (Q_1, Q_2, \ldots, Q_m)$. Write (P, Q) for the sequence

$$(P_1, P_2, \ldots, P_n, Q_1, Q_2, \ldots, Q_m).$$

Proposition 6.2. *There is an external product*

$$\times : \mathrm{ad}/P(K) \times \mathrm{ad}/Q(L) \to \mathrm{ad}/(P, Q)(K \times L)$$

which is natural and associative.

Proof. Suppose M is a K-ad/(P) and N is an L-ad/(Q). For a subset ρ of $\{0, 1, \ldots, n + m\}$ set

$$\begin{aligned} \rho_0 &= \rho \cap \{0, 1, \ldots, n\}, \\ \rho_1 &= \big((\rho \cap \{n + 1, n + 2, \ldots, n + m\}) - n\big) \cup (\rho \cap \{0\}). \end{aligned}$$

Then the external product is given by

$$(M \times N)_\rho = (-1)^{\sum_{s\in\rho_0, t\in\rho_1} d(P_s)d(P_t)} M_{\rho_0} \times N_{\rho_1}.$$

The claimed properties are readily verified. □

Internal product structures are much harder to construct. In [9] an internal product on the level of homotopy groups was obtained with the help of a retraction map which reduces the singularity of type (P, P) to P under suitable hypothesis. In order to rigidify the products we will proceed differently. Instead of looking for a retraction map we construct a new ad theory which comes with an internal product and is homotopy equivalent to the old one in good cases.

Definition 6.3. Let $\tau \in \Sigma_n$ be a permutation and $P = (P_1, P_2, \ldots, P_n)$ a sequence. Let τP be the sequence $(P_{\tau_1}, \ldots, P_{\tau_n})$ and let

$$\tau^* : \mathcal{A}(P) \to \mathcal{A}(\tau P)$$

be the map which sends an object $M.$ to the object

$$(\tau^* M.)_\sigma = (-1)^{|\tau,\sigma|} M_{\tau\sigma}.$$

Here, $|\tau, \sigma|$ is the determinant of the matrix which is obtained by permuting the $d(P_i)$-dimensional blocks of a $\sum_{i\in\sigma} d(P_i)$-dimensional identity matrix through τ.

Lemma 6.4. *The functor τ^* induces an isomorphism of ad theories*

$$\tau^* : \mathrm{ad}/P \to \mathrm{ad}/(\tau P).$$

Proof. The proof is a consequence of the stability axiom. □

Consider the self-map

$$1 + (-1)^n \tau_{n,n} : \mathrm{ad}/(P,P) \to \mathrm{ad}/(P,P),$$

where $\tau_{n,n}$ is the permutation which twists the two blocks of length n. An ad in the image of this map has the property that for each oriented cell of the ball complex we have a Δ^{2n}-ad whose kth face for $k \leq n$ coincides with the $(k+n)$th face after permuting the two summands and applying $(-1)^n$. Moreover, the object on the top cell is twice the original object.

Definition 6.5. Let P be an arbitrary sequence. A K-ad of $\mathrm{ad}/(P,P)$ is said to be *close to* a K-ad/P if for each oriented cell of K the value M satisfies

$$\partial_k M = (-1)^n \partial_{k+n} M \quad \text{for all } 0 \leq k \leq n$$

and the same holds for the maps induced by the inclusions into the top cell. In other words, M is fixed under the action of $(-1)^n \tau_{n,n}$. We write $\mathrm{cl}(\mathrm{ad}/P)$ for all ads in $\mathrm{ad}/(P,P)$ which are isomorphic to those which are close to ad/P. We say that ad/P is *well behaved* if each of the theories $\mathrm{cl}(\mathrm{ad}/(P), \mathrm{cl}(\mathrm{ad}/(P,P)), \dots$ is an ad theory.

Definition 6.6. Let $\pi : \mathrm{ad}/P \to \mathrm{ad}/(P,P)$ be the inclusion map considered in Section 3. It comes from the map $\mathcal{A}(P) \to \mathcal{A}(P,P)$ which fills $\varnothing$ in the faces which do not contain the last n indices. Set

$$\rho_P = (1 + (-1)^n \tau_{n,n})\pi$$

and let $\mathrm{ad}//P$ be the colimit of the sequence

$$\mathrm{ad}/P \xrightarrow{\rho_P} \mathrm{cl}(\mathrm{ad}/P) \xrightarrow{\rho_{(P,P)}} \mathrm{cl}(\mathrm{ad}/(P,P)) \xrightarrow{\rho_{((P,P),(P,P))}} \cdots.$$

Theorem 6.7. *Let* ad/P *be well behaved.*

(i) $\mathrm{ad}//P$ *is a commutative multiplicative ad theory.*
(ii) *The canonical map from* ad *to* $\mathrm{ad}//P$ *respects the multiplication.*
(iii) *The canonical map from the spectrum* $Q(\mathrm{ad}/P)$ *to* $Q(\mathrm{ad}//P)$ *is a homotopy equivalence if* 2 *is inverted,* P *is regular and the cylinder of* P *admits an involution reversing isomorphism.*

Proof. The product of two ads M, N of the colimit, say in $\mathrm{cl}(\mathrm{ad}/(P,\dots,P))$, is given by their symmetrized exterior product $(1 + (-1)^n \tau_{n,n})(M \times N)$. This definition is independent of n by the hypothesis on $\mathcal{A}$. Clearly, the product is compatible with the map from ad.

The last assertion is more involved. It relies on arguments which are similar to the ones given in [9] for complex bordism. For simplicity we look at the case of only one singularity P of dimension m. We have short exact sequences

$$0 \to \Omega_{*-m} \xrightarrow{P} \Omega_* \to \Omega_*^P \to 0,$$
$$0 \to \Omega_*^P \to \Omega_*^{(P,P)} \to \Omega_{*-m-1}^P \to 0.$$

In particular, $\Omega_*^{(P,P)}$ is a free $\Omega_*^P \cong (\Omega/P)_*$-module on the generator 1 and a generator δ of dimension $m+1$. (We used here the fact that the obstruction for the vanishing of the multiplication by P map in a theory with singularities which contain P can be described by the bordism class of the mapping torus of P, see [5] for the classical case).

A convenient choice of δ is provided by the suspension of P, that is the cylinder of δ on the top cell and with P as first and second face. It maps to the unit of Ω^P_*. Since the cylinder of P admits an involution reversing isomorphism, we see that $-\tau_{1,1}\delta$ is isomorphic to $i\delta$. Hence, $1-\tau_{1,1}$ annihilates δ in the bordism group.

Hence, the map

$$Q(\mathrm{ad}/P) \to Q(\mathrm{cl}(\mathrm{ad}/P))$$

is a weak equivalence. An inverse on the level of homotopy groups is given by

$$\pi_* Q(\mathrm{cl}(\mathrm{ad}/P)) \to \pi_* Q(\mathrm{ad}/(P,P)) \to \pi_* Q(\mathrm{ad}/P),$$

the last map being induced by $(1-\tau_{1,1})/2$.

The same method applies to the other maps of the colimit. Note that the obstructions for the vanishing of the multiplication by P-map vanish and hence we get short exact sequences and can proceed as before. The general case for arbitrary many singularities is analogous. □

Corollary 6.8. *Under the conditions of Theorem 6.7*(iii) *the Quinn spectrum* $Q(\mathrm{ad}/P)$ *is homotopy equivalent to a commutative ring spectrum.*

Proof. This immediately follows from Theorem 6.7 and Theorem 2.5. □

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