Local Shtukas and Divisible Local Anderson-Modules

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Abstract

We define the cotangent complex as it is defined in Abrashkin [1], Lichtenbaum; Schlessinger [13] and Messing [14] and then we compare all of them and prove that they are homotopically equivalent. Let ζ be an indeterminant over $\mathbb{F}_q$ and let $\mathbb{F}_q[\![\zeta]\!]$ be the ring of formal power series in ζ over $\mathbb{F}_q$. Let $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ be the category of $\mathbb{F}_q[\![\zeta]\!]$-schemes on which ζ is locally nilpotent. When $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ we prove the equivalence between the category of effective local shtukas over S and the category of z-divisible local Anderson-modules over S. The latter objects are analogues of BT-groups (also called p-divisible groups) in the equal characteristic case. Then we show how to associate a formal Lie group to any z-divisible local Anderson-modules over S. After this we treat a question of when a formal Lie group is a z-divisible local Anderson-module.

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Introduction

The z-divisible local Anderson modules are analogues of BT-groups (also called p-divisible groups) in equal characteristic case. The latter are objects in mixed characteristic over $\mathbb{Z}_p$. The z-divisible local Anderson-modules (called z-divisible groups in other text) were introduced in Hartl [9] but they already appeared in special case in work of Drinfeld [5], Genestier [8], Laumon [12], Taguchi [16] and Rosen [15]. In [9] our z-divisible local Anderson-modules were called "z-divisible groups" and it was claimed that their category is equivalent to the category of "Dieudonné $\mathbb{F}_q[\![z]\!]$-modules". The latter are called local shtukas in the present thesis and we provide the proofs for the statements claimed in [9]. In this thesis z-divisible local Anderson-modules over an arbitrary base scheme S (on which ζ is locally nilpotent) are studied. So we can think of these as nicely varying families of z-divisible local Anderson-modules parametrized by S. This thesis originates from the following question. Can we associate a formal Lie group to any z-divisible local Anderson-module? The answer is yes in the case when ζ is locally nilpotent.

Motivation: As we know BT-groups may arise from abelian variety, z-divisible local Anderson modules arise for example as the z-power torsion of Drinfeld modules or Anderson modules. On the other hand BT-groups are related via their Tate-module to p-adic Galois representation. This was one of the reason to study p-divisible groups.

Let us now give a more detailed summary of the various chapters. Let z be an indeterminant over $\mathbb{F}_q$ and suppose that S is a scheme over $\operatorname{Spec}\mathbb{F}_q[z]$. We denote the image of z in $\mathcal{O}_S$ by ζ. If G is a group on S which is also an $\mathbb{F}_q[z]$-module. We denote $\ker z^n$ by $G(n)$.

For a commutative group scheme G over a scheme S we define its co-Lie module ω_G as the $\mathcal{O}_S$-module of invariant differentials. It is canonically isomorphic to $e^*\Omega^1_{G/S}$ where $e: S \to G$ is the zero section i.e. $\omega_{G/S} = \mathcal{O}_S \otimes_{\mathcal{O}_G} \Omega^1_{G/S}$. We make the following definition of z-divisible local Anderson-module.

Definition. A z-divisible local Anderson module over S is a sheaf of $\mathbb{F}_q[\![z]\!]$-modules G on the big fppf-site of S such that for each integer $n \geq 1$:

1. G is of z-torsion i.e. $G = \varinjlim_n G(n)$,

2. G is z-divisible i.e. $z: \quad G \to G$ is an epimorphism,

3. The $\mathbb{F}_q$-modules $G(n)$ are representable by finite locally free group schemes over S and are strict.

4. Locally on S there exist a constant $d \in \mathbb{Z}_{\geq 0}$, such that $(z-\zeta)^d = 0$ on ω_G where $\omega_G = \varprojlim_n \omega_{G(n)}$ cf.2.3.4.

To associate a formal Lie group to any z-divisible local Anderson module over S in the case when ζ is locally nilpotent on S, we need to use the theory of relative cotangent complex. In chapter 1 we define the cotangent complex as it is defined in Abrashkin [1], Lichtenbaum; Schlessinger [13] and Messing [14] and then we compare all of them and prove that they are homotopically equivalent.

In section 1.4 and section 1.5 we define the cotangent complex in the sense Abrashkin and strict finite $\mathcal{O}$-module schemes. For that we fix some notations as p is a fixed prime number, $\mathcal{O}$ is a unitary commutative $\mathbb{F}_p$ -algebra and A is a commutaive unitary $\mathcal{O}$-algebra. In section 1.4 we define the cotangent complex in the sense of Abrashkin [1] in the following way.

For an augmented A-algebra B, we use the following notations: $\epsilon_B : B \to A$, the morphism of augmentation, and $\operatorname{Ker} \epsilon_B = I_B$, the augmentation ideal. If $A[\bar{X}] = A[X_1, ..., X_n]$, $n \geq 0$, is a polynomial ring we always assume that its augmentation ideal is $I_{A[\bar{X}]} = (X_1, ..., X_n)$ i.e. this is the ideal defining the zero section of $\operatorname{Spec}(A[\bar{X}])$. We define the category DAug_A are the triples $\mathcal{B} = (B, B^\flat, i_\mathcal{B})$ where B is a finite augmented A-algebra, $B^\flat$ is an augmented A-algebra and $i_\mathcal{B} : B^\flat \to B$ is an epimorphic map of augmented algebras such that locally on A there is a polynomial ring $A[\bar{X}] = [X_1, ..., X_n]$, $n \geq 0$, and an epimorphism of augmented A-algebras $j : A[\bar{X}] \to B^\flat$ satisfying the following properties:

• the ideal $I := \operatorname{Ker}(i_\mathcal{B} \circ j)$ is generated by elements of a regular sequence of length n in $A[\bar{X}]$;

• $\operatorname{Ker} j = I \cdot I_{A[\bar{X}]}$.

Abrashkin introduces the two A-modules $t^*_\mathcal{B} = I_{A[\bar{X}]}/I^2_{A[\bar{X}]}$ and $N_\mathcal{B} = I/(I \cdot I_{A[\bar{X}]})$. The cotangent complex in the sense of Abrashkin is defined as the complex

$$\mathcal{L}^{B/A}_{\cdot} \quad = \quad N_\mathcal{B} \to t^*_\mathcal{B} \tag{0.1}$$

and we show that $\mathcal{L}^{B/A}_{\cdot}$ is the fiber at the origin of the cotangent complex of B/A as have defined by Lichtenbaum; Schlessinger [13] and Messing [14]. In the above complex the map is the differential map.

In section 1.5 we define strict finite $\mathcal{O}$-module schemes. We denote by $\mathrm{DGr}(\mathcal{O})_A$ the category $\mathcal{O}$-module objects which satisfy the following definition of strictness.

Definition. Suppose $\mathcal{G} = \operatorname{Spec} \mathcal{B} = (\operatorname{Spec} B^\flat, \operatorname{Spec} B, \operatorname{Spec} i_\mathcal{B})$. A strict $\mathcal{O}$-action on $\mathcal{G}$ is a homomorphism $\mathcal{O} \to \operatorname{End}_{\mathrm{DGr}_A}(\mathcal{G})$ such that the induced action on $\mathcal{L}^{B/A}_{\cdot}$ is homotopic to the scalar multiplication via $\wp : \mathcal{O} \to A$.

Here $S = \mathrm{Spec}(A)$ is a scheme over $\mathcal{O}$ and $\wp : \mathcal{O} \to \Gamma(S, \mathcal{O}_S)$ is the structure morphism. Then we make the following definition strict finite $\mathcal{O}$-module schemes.

Definition. A strict $\mathcal{O}$-module scheme over S is a finite S-group scheme G together with a homomorphism $\mathcal{O} \to \mathrm{End}_S(G)$ which lifts to a homomorphism $\mathcal{O} \to \mathrm{End}_{\mathrm{DGr}_A}(\mathcal{G})$ such that the induced action on $\mathcal{L}^{B/A}_{\cdot}$ is homotopic to the scalar multiplication via $\wp : \mathcal{O} \to A$. We denote by $\mathrm{Gr}(\mathcal{O})_A$ the category of strict finite $\mathcal{O}$-module schemes.

Mainly $\mathrm{Gr}(\mathcal{O})_A$ is the quotient category of $\mathrm{DGr}(\mathcal{O})_A$ where the morphisms are the equivalence classes of morphisms $(G,\ G^\flat) \to (H,\ H^\flat)$ in the category $\mathrm{DGr}(\mathcal{O})_A$ which induce the same morphism $G \to H$.

In section 1.6 we fix p a prime number and q is a power of p. Let $S = \mathrm{Spec}(A)$ be a scheme over $\mathrm{Spec}\,\mathbb{F}_q$. We denote by $\sigma_q : S \to S$ its Frobenius endomorphism which acts as the identity on points and as the q-power map on the structure sheaf. In this section we explain that Drinfeld established a relation between finite shtukas over S and finite strict $\mathbb{F}_q$-module schemes over S. We first give the definition of finite $\mathbb{F}_q$-shtukas over S.

Definition. A *finite* $\mathbb{F}_q$*-shtuka* over S is a pair (M, F_M) consisting of a locally free $\mathcal{O}_S$-module M on S of finite rank and an $\mathcal{O}_S$-module homomorphism $F_M : \sigma_q^* M \to M$. Here $\sigma_q^* M = M \otimes_{\mathcal{O}_S, \sigma_q^*} \mathcal{O}_S$.

We denote by $\mathrm{Mod}(\mathbb{F}_q)_A$ the category of $\mathbb{F}_q$-shtukas over S.

After this we define the functor $\underline{M}_q : \mathrm{DGr}(\mathbb{F}_q)_A \to \mathrm{Mod}(\mathbb{F}_q)_A$ by setting for any $\mathcal{G} = (G, G^\flat) \in \mathrm{DGr}(\mathbb{F}_q)_A, \underline{M}_q(\mathcal{G}) = (M(\mathcal{G}), F_M)$ with

$$M(\mathcal{G}) = \{a \in A(G) \mid \Delta(a) = a \otimes 1 + 1 \otimes a,\ [\alpha](a) = \alpha a,\ \forall \alpha \in \mathbb{F}_q\},$$

where $F_M : \sigma_q^* M(\mathcal{G}) \to M(\mathcal{G})$ is induced by the q-th power map on $A(G)$.

Also we construct the inverse functor $\mathrm{Dr}_q : \mathrm{Mod}(\mathbb{F}_q)_A \to \mathrm{Gr}(\mathbb{F}_q)_A$ in a following way

If $(M, F_q) \in \mathrm{Mod}(\mathbb{F}_q)_A$, then $\mathrm{Dr}_q(M, F_q) = \mathrm{Spec}\,\mathcal{B}$ with $\mathcal{B} = (A(G), A(G)^\flat, i_\mathcal{B})$, defined by:

- $A(G) = \mathrm{Sym}_A M/I$ where the ideal I is generated by $\{m^q - F_q(m \otimes 1)|\ m \in M\}$, the comultiplication Δ is such that $\Delta(m) = m \otimes 1 + 1 \otimes m$ and the $\mathbb{F}_q$-action such that $[\alpha](m) = \alpha m$ for all $m \in M$ and $\alpha \in \mathbb{F}_q$;
- $A(G)^\flat = \mathrm{Sym}_A(M)/(I \cdot I_0)$ where the augmentation ideal I_0 is generated by all $m \in M$, the comultiplication $\Delta^\flat$ is such that $\Delta^\flat(m) = m \otimes 1 + 1 \otimes m$ and the $\mathbb{F}_q$-action such that $[\alpha]^\flat$ is given by the correspondence $m \mapsto \alpha m$ for all $m \in M$ and $\alpha \in \mathbb{F}_q$;
- $i_\mathcal{B}$ is the natural projection from $A(G)^\flat$ to $A(G)$.

After this we prove the following theorem

Theorem 1.6.4 (*Drinfeld* [*6*, 2.1], *Taguchi* [*17*, 1.7])

1. *The two contravariant functors* Dr_q *and* $\underline{M}_q$ *are mutually quasi-inverse anti-equivalences between the category of finite* $\mathbb{F}_q$*-shtukas over* S *and the category of finite strict* $\mathbb{F}_q$*-module schemes over* S*.*

2. *Both functors are* $\mathbb{F}_q$*-linear and map short exact sequences to short exact sequences. They preserve étale objects.*

Let (M, F) *be a finite* $\mathbb{F}_q$*-shtuka over* S*. Then*

3. *the* $\mathbb{F}_q$*-module scheme* $\mathrm{Dr}_q(M, F)$ *is radicial if and only if* F *is nilpotent locally on* S*.*

4. *the scheme* $\mathrm{Dr}_q(M, F)$ *is finite and locally free and the order of the* S*-group scheme* $\mathrm{Dr}_q(M, F)$ *is* $q^{\mathrm{rk}\, M}$*.*

5. *the* $\mathcal{O}_S$*-modules* $\omega_{\mathrm{Dr}_q(M,F)}$ *and* $\mathrm{coker}\, F$ *are canonically isomorphic.*

In section 1.7 we compare between cotangent complex of a finite locally free S-group scheme G and Frobenius map of finite $\mathbb{F}_q$-shtuka associated to it and we prove that they are isomorphic.

Let $\mathbb{F}_q[\![\zeta]\!]$ be the ring of formal power series in ζ over $\mathbb{F}_q$. Let $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ be the category of $\mathbb{F}_q[\![\zeta]\!]$-schemes on which ζ is locally nilpotent. In chapter 2 section 2.1, 2.2 and 2.3 the concept of formal Lie groups, local shtuka, z-divisible local Anderson module over S are defined, several sorites and several examples are given. This three sections consists of several definitions. We give the following definition of local shtuka and effective local shtuka.

Definition. A *local shtuka* of rank(or height) r over S is a pair (M, F_M) consisting of
• a sheaf M of $\mathcal{O}_S[\![z]\!]$-modules on S, which, Zariski-locally on S, is a free $\mathcal{O}_S[\![z]\!]$-module of rank r, and
• an isomorphism $F_M : \sigma^* M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$.

Definition. A local shtuka (M, F_M) over S is called *effective* if F_M is actually a morphism $F_M : \sigma^* M \to M$. Let (M, F_M) be effective. We say that

1. (M, F_M) has *dimension* d if $\mathrm{coker}\, F_M$ is locally free of rank d as an $\mathcal{O}_S$-module.

2. (M, F_M) is *étale* if $F_M : \sigma^* M \xrightarrow{\sim} M$ is an isomorphism.

3. F_M is *topologically nilpotent* if $\mathrm{im}\, F_M^n \subset zM$ locally on S for all large enough integers n.

In section 2.4 we extend Drinfeld's construction and the equivalence from Section 1.6 to an equivalence between the category of effective local shtukas over S and the category of z-divisible local Anderson-modules over S and we prove the following theorem

Theorem 2.4.3 *Let* $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$*.*

1. *The two contravariant functors Dr_q and $\underline{M}$ are mutually quasi-inverse anti-equivalences between the category of effective local shtukas over S and the category of z-divisible local Anderson-modules over S. Both functors are $\mathbb{F}_q[\![z]\!]$-linear.*

2. *Both functors preserve étale objects and map short exact sequences to short exact sequences.*

Let (M, F) be an effective local shtuka over S. Then

3. *the z-divisible local Anderson-module $Dr_q(M, F)$ is a formal Lie group if and only if F is topologically nilpotent.*

4. *the height and dimension of the z-divisible local Anderson-module $Dr_q(M, F)$ equal the rank and dimension of (M, F).*

5. *the $\mathcal{O}_S[\![z]\!]$-modules $\omega_{Dr_q(M,F)}$ and $\operatorname{coker} F$ are canonically isomorphic.*

In section 2.5 the relation between formal Lie groups and z-divisible local Anderson-modules is studied. First assuming $\zeta = 0$ on S and using the techniques of Messing [14], we show how to associate a formal Lie group to any z-divisible local Anderson-module G. In using the techniques of Messing [14], the main obstruction is getting the Verschiebung map $V_G : G^{(q)} \to G$ such that

$$\begin{aligned} F_G \circ V_G &= z^d|_{G^{(q)}}, \\ V_G \circ F_G &= z^d|_G \end{aligned}$$

which we get from the above equivalence. Let $G[n]$ will denote the kernel of the n^{th} iterate of q-Frobenius homomorphism:

$$G \xrightarrow{F_G} G^{(q)} \xrightarrow{F_{G^{(q)}}} G^{(q^2)} \ldots\ldots \to G^{(q^n)}$$

$\operatorname{Inf}^k_S(G)$ is the k-th infinitesimal neighbourhood of S in G in the sense of Messing [14] and we prove the following theorem.

Theorem 2.5.5 *When $\zeta = 0$ on S and G is a z-divisible local Anderson module over S, then $\varinjlim_n G[n]$ is a formal Lie group and is equal to $\bar{G} := \varinjlim_n \operatorname{Inf}^k_S(G)$.*

Via the use of the relative cotangent complex this result is extended to a z-divisible local Anderson-module G over S in the case ζ is locally nilpotent on S and we prove the following theorem.

Theorem 2.5.19 *Let ζ be locally nilpotent on S and let G be a z-divisible local Anderson module on S. Then $\bar{G} = \varinjlim_k \operatorname{Inf}^k(G)$ is a formal Lie group.*

Next we treat the question of when a formal Lie group which is also a $\mathbb{F}_q[z]$-module is a z-divisible local Anderson-module. Finally necessary and sufficient condition for a z-divisible local Anderson-module to be expressible as an extension of an ind-étale z-divisible local Anderson-module by a formal Lie group are given.

Proposition 2.5.27 *Let ζ be locally nilpotent on S and G be in z-divisible local Anderson modules over S. The following conditions are equivalent*

1. *$\bar{G}$ is a z-divisible local Anderson module.*
2. *G is an extension of an ind-étale z-divisible local Anderson module G'' by an ind-infinitesimal z-divisible local Anderson module G'.*

2' *G is an extension of an ind-étale z-divisible local Anderson module by a z-divisible formal Lie group.*

3. *For all n $G(n)$ is an extension of a finite étale group by a finite locally-free radiciel group.*

3' *$G(1)$ is an extension of a finite étale group by a finite locally-free radiciel group.*

4. *$s \mapsto$ separable rank $(G(1)_s)$ is locally constant function.*

In last section we compare of Tate-module of local shtuka and z-divisible local Anderson module and we prove the following theorem

Theorem 2.6.2 *There is a canonical $\mathbb{F}_q[\![z]\!]$-isomorphism $T_zG \xrightarrow{\sim} T_zM$ of $\mathrm{Gal}(K^{sep}/K)$-representations.*

where $S = \operatorname{Spec} \mathcal{O}_K$ where $\mathcal{O}_K$ are the integers in a local field K with $\zeta \in K^\times$ and K^{sep} is a fixed separable closure.

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Chapter 1

Cotangent complex and relation between strict finite group schemes and shtukas

Conventions. The conventions of commutative algebra that all the rings are with unit element unless it is not mentioned.

In this chapter we will define the cotangent complex as it is defined in Abrashkin [1], Lichtenbaum; Schlessinger [13] and Messing [14]. Then we will compare all of them and prove that they all are homotopically equivalent. In the end we will compare the cotangent complex with the Frobenius map of the associated finite shtuka and then we will prove that they are same.

At first we will define the cotangent complex in the way of Lichtenbaum; Schlessinger [13]. The following section has been taken from the paper Lichtenbaum; Schlessinger [13].

1.1 Cotangent complex in the sense of Lichtenbaum; Schlessinger

In the paper Lichtenbaum; Schlessinger [13] for a ring homomorphism $A \to B$ they have defined the cotangent complex of B over A in the following way. They start with an extension of B over A by which they mean an exact sequence

$$\mathcal{E}: \ 0 \longrightarrow E_2 \xrightarrow{e_2} E_1 \xrightarrow{e_1} R \xrightarrow{e_0} B \longrightarrow 0$$

where e_0 is a surjection of A-algebras, e_2 and e_1 are homomorphism of R-modules, and

$$e_1(x)y = e_1(y)x$$

for x, $y \in E_1$.

Note that the last condition is required to make E_2 a B-module. Indeed if $I = \operatorname{Ker} e_0$, if $a \in I$ and $x \in E_2$, choose $y \in E_1$ such that $e_1(y) = a$. Then $e_2(ax) = ae_2(x) = e_1(y)e_2(x) = e_1e_2(x).y = 0$, so $ax = 0$. This implies $IE_2 = 0$, so E_2 is a B-module.

Let A' be an A-algebra, B' be an A'-algebra, and $\mathcal{E}'$ an extension of B' over A'. By a homomorphism $\alpha : \mathcal{E} \to \mathcal{E}'$ we mean a collection $(b, \alpha_0, \alpha_1, \alpha_2)$ of maps which make the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E_2' & \xrightarrow{e_2'} & E_1' & \xrightarrow{e_1'} & R' & \xrightarrow{e_0'} & B' & \longrightarrow & 0 \\ & & \uparrow{\scriptstyle \alpha_2} & & \uparrow{\scriptstyle \alpha_1} & & \uparrow{\scriptstyle \alpha_0} & & \uparrow{\scriptstyle b} & & \\ 0 & \longrightarrow & E_2 & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & R & \xrightarrow{e_0} & B & \longrightarrow & 0 \end{array}$$

commutative. Here b and α_0 are homomorphisms of A-algebras, and α_1 and α_2 are homomorphisms of R-modules (where we regard R' and B' as A-algebras via $A \to A'$ and E_1' and E_2' as R-moules via α_0).

Definition 1.1.1. With each extension $\mathcal{E}$ of B over A we associate a three term complex $L^{\cdot}(\mathcal{E})$ of B-modules:

$$L^{\cdot}(\mathcal{E}) : \quad 0 \to E_2 \xrightarrow{d_2} E_1 \otimes_R B \xrightarrow{d_1} \Omega_{R/A} \otimes_R B \to 0 \,.$$

Here $\Omega_{R/A}$ is the module of Kähler differentials, d_2 is induced from e_2, and d_1 is defined as follows: let $I = \text{Ker } e_0$. Then $\text{Im } e_1 = I$ and we have $d : I/I^2 \to \Omega_{R/A} \otimes_R B$. Put $d_1 = d \circ (e_1 \otimes_R B)$.

$$\begin{array}{ccccc} E_1 \otimes_R B & & \xrightarrow{d \circ (e_1 \otimes_R B)} & & \Omega_{R/A} \otimes_R B \\ & {\scriptstyle e_1 \otimes_R B} \searrow & & \nearrow {\scriptstyle d} & \\ & & I \otimes_R B = I/I^2 & & \end{array}$$

Extensions of B over A are all obtained in the following way. Choose a surjection $e_0 : R \to B$ of A-algebras, and let $I = \text{Ker } e_0$. Then choose an exact sequence

$$0 \longrightarrow U \xrightarrow{i} F \xrightarrow{j} I \longrightarrow 0$$

of R modules and define $\phi : F \otimes F \to F$ by the formula

$$\phi(x \otimes y) = j(x)y - j(y)x.$$

$\text{Im}\,\phi \in U$, because $j(\phi(x \otimes y)) = j(x)j(y) - j(y)j(x) = 0$. Let $U_0 \subseteq U$ be the image of ϕ, and let $e_2 : U/U_0 \to F/U_0$ and $e_1 : F/U_0 \to R$ be the induced maps.

Definition 1.1.2. An extension of the form

$$0 \to U/U_0 \to F/U_0 \to R \to B \to 0$$

where R is a polynomial algebra over A and F is a free R-module is called a free extension of B over A.

It is clear that if

$$(*) \qquad \begin{array}{ccc} B & \xrightarrow{b} & B' \\ \uparrow & & \uparrow \\ A & \xrightarrow{a} & A' \end{array}$$

is a commutative diagram of ring homomorphisms, and $\mathcal{E}$ (resp $\mathcal{E}'$)is a free(resp. arbitrary) extension of B over A(resp of B' over A'), then there exist a homomorphism $\alpha : \mathcal{E} \to \mathcal{E}'$ extending b, and hence a homomorphism $\bar{\alpha} : L^{\cdot}(\mathcal{E}) \otimes_B B' \to L^{\cdot}(\mathcal{E}')$.

A complex of the form $L^{\cdot}(\mathcal{E})$, where $\mathcal{E}$ is a free extension of B over A is called a cotangent complex of B over A. In the paper Lichtenbaum; Schlessinger [13] we see that any two cotangent complexes of B over A are homotopic ([13, 2.1.9]).

An A-algebra B will be called finite if as an A-module it is locally free of finite rank i.e. flat and finitely presented. If these algebras are with a comultiplication map, then their spectrum will be a finite flat commutaive group schemes over A. When $\operatorname{Spec} B$ is a group scheme over A, it is known that B is a relative complete intersection [S.G.A.3, III 4.15], that is of the form $B = A[X_1, ..., X_n]/I$ with ideal I generated by a regular sequence which remains regular after all base changes.

Now we will apply the above results to find the cotangent complex of a group scheme G which is finite and locally free over $S = \operatorname{Spec} A$ i.e. represented by finitely presented flat A- algebra i.e. $G = \operatorname{Spec} B$, where B is a finite A-algebra, i.e. $B = A[\bar{X}]/I$, where $A[\bar{X}] = A[X_1, X_2..., X_n], n \geq 0$, and I is generated by elements of a regular sequence of length n in $A[\bar{X}]$.

To apply the results in our case we can take $R = A[\bar{X}]$ and say $I = (f_1, f_2..., f_n)$ and $F = A[\bar{X}]^{\oplus n}$. We begin with a lemma.

Lemma 1.1.3. *When the ideal $I = (f_1, f_2..., f_n)$ is generated by elements of a regular sequence. Then $U = U_0$ in Definition (1.1.2).*

Proof. To prove $U = U_0$ when the ideal $I = (f_1, f_2..., f_n)$ is generated by elements of a regular sequence we will prove first I is finitely presented over $A[\bar{X}]$.

Since $I = (f_1, ..., f_n)$ and $F = A[\bar{X}]^{\oplus n}$. To prove I is finitely presented over $A[\bar{X}]$ we will show that there is an exact sequence of $A[\bar{X}]$-modules

$$A[\bar{X}]^{\oplus \binom{n}{2}} \xrightarrow{i} A[\bar{X}]^{\oplus n} \xrightarrow{j} I \longrightarrow 0, \tag{1.1}$$

where the map j sends standard basis vectors e_i to f_i and the map i sends a basis g_{ij} of $A[\bar{X}]^{\oplus \binom{n}{2}}$ to $f_i e_j - f_j e_i$. From the definition of map j, it is surjective. To prove exactness in the middle let $\sum_{i=1}^{n} a_i e_i \in \ker j$, this implies $\sum_{i=1}^{n} a_i f_i = 0$. This implies,

$$\begin{aligned} a_n f_n &\equiv 0 \bmod (f_1, ...f_{n-1}) \text{ in } A[\bar{X}] \\ a_n f_n &= 0 \text{ in } A[\bar{X}]/(f_1, ...f_{n-1}) \end{aligned}$$

Since $f_n \in I$ is a non-zerodivisor in $A[\bar{X}]/(f_1, ...f_{n-1})$ we have $\bar{a}_n = 0$. This implies there exists $b_{nj} \in A[\bar{X}],\ 1 \leq j \leq n-1$ such that,

$$a_n = \sum_{j=1}^{n-1} b_{nj} f_j.$$

This implies

$$\sum_{i=1}^{n} a_i e_i = a_1 e_1 + ... + a_{n-1} e_{n-1} + (\sum_{j=1}^{n-1} b_{nj} f_j) e_n.$$

Take $\sum_{j=1}^{n-1} b_{nj} g_{nj} \in A[\bar{X}]^{\oplus \binom{n}{2}}$, this will give us

$$\sum_{i=1}^{n} a_i e_i = (a_1 + b_{n1} f_n) e_1 + ... + (a_{n-1} + b_{n,n-1} f_n) e_{n-1} \text{ (mod Im} i).$$

Continuing in this way we will get,

$$\sum_{i=1}^{n} a_i e_i = (a_1 + b_{n1} f_n + b_{n-1,1} f_{n-1} + ... + b_{2,1} f_2) e_1 \text{ (mod Im} i).$$

Hence $(a_1 + b_{n1} f_n + b_{n-1,1} f_{n-1} + ... + b_{2,1} f_2) f_1 = 0$. Since f_1 is non-zerodivisor in $A[\bar{X}]$ this implies we have

$$\sum_{i=1}^{n} a_i e_i = 0 \text{ (mod Im} i).$$

This proves that I is finitely presented over $A[\bar{X}]$. So we have an exact sequence

$$A[\bar{X}]^{\oplus \binom{n}{2}} \xrightarrow{i} A[\bar{X}]^{\oplus n} \xrightarrow{j} I \longrightarrow 0.$$

Let us take $U = \ker j$, so we have an exact sequence as $A[\bar{X}]$-modules ,

$$0 \rightarrow U \rightarrow A[\bar{X}]^{\oplus n} \rightarrow I \longrightarrow 0$$

and

$$\phi : A[\bar{X}]^{\oplus n} \otimes A[\bar{X}]^{\oplus n} \rightarrow A[\bar{X}]^{\oplus n}$$
$$e_i \otimes e_j \rightarrow f_i e_j - f_j e_i.$$

Since we have $U =< f_i e_j - f_j e_i >_{A[\bar{X}]} = \operatorname{Im} \phi = U_0$. This shows that $U = U_0$. □

So we have $U/U_0 = 0,\ F/U_0 \otimes_R B = I \otimes_R B = I/I^2$. Hence the cotangent complex

$$L^{\cdot}(\mathcal{E}) :\ \ 0 \longrightarrow U/U_0 \longrightarrow F/U_0 \otimes_R B \longrightarrow \Omega_{R/A} \otimes_R B \rightarrow 0;$$

is equal to

$$L^{\cdot}(\mathcal{E}) :\ \ 0 \longrightarrow 0 \longrightarrow I/I^2 \longrightarrow \Omega_{R/A} \otimes_R B \rightarrow 0\,.$$

Sometime we will denote this cotangent complex of B over A by $L^{B/A}_{\cdot}$ and we say this is the cotangent complex in the sense of Lichtenbaum and Schlessinger.

1.2 Cotangent complex in the sense of Messing

In this section we will define the cotangent complex in the sense of Messing [14] for the same finite and locally-free S-group scheme G. As we have assumed in the last section $S = \mathrm{Spec}(A)$ and $G = \mathrm{Spec}(B)$ where B is a finite and locally free A-algebra. To say G is a group amounts to saying B is a bi-algebra i.e., we have an algebra homomorphism $\Delta : B \to B \otimes B$ and $\epsilon : B \to A$ satisfying the usual identities. Thus $\check{B} = \mathrm{Hom}_A(B, A)$, the linear dual of B, is equipped with an algebra structure via the transpose of Δ and that of ϵ.

The hyperalgebra, $U(G)$, of G is by definition $\check{B}$ endowed with its algebra structure.

Remark 1.2.1. As is well known $\check{B}$ is in fact a bi-algebra and $\mathrm{Spec}(\check{B}) = G^*$, the Cartier dual of G.

Since $U(G)$ is a finite locally free $\mathcal{O}_S$-algebra we obtain a smooth group scheme $U(G)^\times$ whose points with values in the S-scheme T are by definition the invertible elements in the ring $\Gamma(T,\ U(G_T)) = \Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$. Also we have a natural monomorphism $G \hookrightarrow U(G)^\times$ which is defined by viewing a T-valued point of G as a homomorphism of $\mathcal{O}_T$ -algebras $B \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathcal{O}_T$ and hence as an element of $\Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$. The fact that such a homomorphism when viewed as an element of $\Gamma(T,\ U(G_T))$ is invertible follows from the commutativity of the following diagram

$$\begin{array}{ccc} G(T) \times G(T) & \xrightarrow{i \times i} & \Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T) \times \Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T) \\ \Big\downarrow \mathrm{Spec}\Delta & & \Big\downarrow \check{\Delta} \\ G(T) & \xrightarrow{\quad i \quad} & \Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T) \end{array}$$

The above diagram is commutative because both the left and right vertical arrow come from Δ. For every $f \in G(T)$ there exists $g \in G(T)$ such that $fg = 1$. Since $(i \times i)(f.g) = i(f).i(g) = i(fg) = i(1)$, it is enough to prove that $i(1) = 1$. $1 \in G(T)$ is equivalent to $\epsilon : B \otimes_{\mathcal{O}_S} \mathcal{O}_T \to \mathcal{O}_T$ which is equivalent to $(\check{\epsilon} : \mathcal{O}_T \to \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T) \equiv 1 \in \Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$. This shows $i(1) = 1$. Also it shows us that the morphism $G \hookrightarrow U(G)^\times$ is a homomorphism of group schemes.

Now we will prove that $U(G)^\times$ is affine over S. To prove this at first we will prove that the contravariant functor $U(G)$ sending an S-scheme T to the ring $\Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T)$ is representable. Since B is a finite and locally free $\mathcal{O}_S$-algebra, we can assume $B = \bigoplus_{i=1}^n \mathcal{O}_S X_i$ and $\check{B} = \bigoplus_{i=1}^n \mathcal{O}_S Y_i$. Then $\check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T = \bigoplus_{i=1}^n \mathcal{O}_T Y_i$ and $U(G)(T) = \Gamma(T,\ \check{B} \otimes_{\mathcal{O}_S} \mathcal{O}_T) = \bigoplus_{i=1}^n \Gamma(T,\ \mathcal{O}_T) Y_i$. We want $U(G)(T) = \mathrm{Hom}_{S-Sch}(T,\ U(G))$. Take $U(G) = \mathrm{Spec}\ \mathcal{O}_S[X_1, X_2, ..., X_n] = \mathbf{A}_S^n$.

Then $\mathrm{Hom}_{S-Sch}(T,\ \mathrm{Spec}\mathcal{O}_S[X_1, X_2, ..., X_n])$

$$\begin{aligned} &= \mathrm{Hom}_{\mathcal{O}_S-\mathrm{alg}}(\mathcal{O}_S[X_1, X_2, ..., X_n],\ \Gamma(T,\ \mathcal{O}_T)) \\ &= \mathrm{Hom}_{\mathcal{O}_S-\mathrm{mod}}(\bigoplus_{i=1}^{n} \mathcal{O}_S X_i,\ \Gamma(T,\ \mathcal{O}_T)) \\ &= \bigoplus_{i=1}^{n} \Gamma(T,\ \mathcal{O}_T) \cdot Y_i. \end{aligned}$$

The last equality comes from the map $f \in \mathrm{Hom}_{\mathcal{O}_S-\mathrm{mod}}(\bigoplus_{i=1}^{n} \mathcal{O}_S X_i, \Gamma(T,\ \mathcal{O}_T))$ sending to $\sum_{i=1}^{n} f(X_i) \cdot Y_i$. Hence we can take $U(G) = \mathrm{Spec}(\mathcal{O}_S[X_1, X_2, ..., X_n]) = \mathrm{Spec}(\mathrm{Sym}_{\mathcal{O}_S} B)$. Hence $U(G)$ is affine over S.

Now we will prove $U(G)^{\times} \hookrightarrow U(G) \times U(G)$ is a closed immersion. It will follow from the following fiber product diagram

$$\begin{array}{ccc} U(G)^{\times} = S \times_{U(G)} (U(G) \times U(G)) & \longrightarrow & U(G) \times U(G) \\ \downarrow & & \downarrow \text{Mult} \\ S & \xrightarrow{\text{unit section}} & U(G) \end{array}$$

Since $U(G)$ is affine, hence separated, the unit section is a closed immersion. Hence the upper horizontal row $U(G)^{\times} \hookrightarrow U(G) \times U(G)$ is a closed immersion. Therefore $U(G)^{\times}$ is affine over S. Since the unit section is a regular immersion the immersion $U(G)^{\times} \hookrightarrow U(G) \times U(G)$ is also regular.

Lemma 1.2.2. *The natural monomorphism $G \hookrightarrow U(G)^{\times}$ is a closed immersion.*

Proof. G is finite over S so proper and $U(G)^{\times}$ is affine over S(from above) so separated. Therefore $G \hookrightarrow U(G)^{\times}$ is proper. We know that every proper monomorphism is a closed immersion. □

Lemma 1.2.3. *The morphism $G \hookrightarrow U(G)^{\times}$ is a regular immersion.*

Proof. Since G is flat over S, so G is locally a complete intersection by [S.G.A.3, III 4.15]. Hence by [S.G.A., 6 VII 1.2, 1.4], we see $G \hookrightarrow U(G)^{\times}$ is a regular immersion. □

Let J be the ideal defining G in $U(G)^{\times}$; then the definition of [S.G.A., 6] for $L^{(G/S)}_{\cdot}$ which we adopt is:

Definition 1.2.4. The relative cotangent complex, $L^{(G/S)}_{\cdot}$, is the complex of $\mathcal{O}_G$-modules $J/J^2 \to \Omega_{U(G)^{\times}/S}|G$.

Since both terms in this complex are locally free we have $Le_G^*(L_\cdot^{(G/S)}) = e_G^*(L_\cdot^{(G/S)})$. Let $\pi : G \to S$ be the structural morphism. We have the following proposition from Messing [14].

Proposition 1.2.5. *Messing [14, Chap 2, Prop 3.2.9]*

$$\pi^* e_G^*(L_\cdot^{(G/S)}) \xrightarrow{\sim} L_\cdot^{(G/S)}. \quad (\textit{isomorphism as a complex})$$

Lemma 1.2.6. *The formation of $L_\cdot^{(G/S)}$ commutes with an arbitrary base change $S' \to S$.*

Proof. Messing [14, Chap 2, Lemma 3.2.11]. □

Recall the construction of $U(G)$ being functorial in G it follows that if $u : G \to H$ is a homomorphism of finite locally-free groups we have a commutative diagram:

$$\begin{array}{ccc} G & \hookrightarrow & U(G)^\times \\ {\scriptstyle u}\downarrow & & \downarrow \\ H & \hookrightarrow & U(H)^\times \end{array}$$

and hence deduce that there is a morphism $u^* L_\cdot^{(H/S)} \to L_\cdot^{(G/S)}$ corresponding to u.

Definition 1.2.7. The co-Lie complex, $\ell_\cdot^G$ is by definition $e_G^*(L_\cdot^{(G/S)})$. From the above we see that $G \to \ell_\cdot^G$ is a contravariant functor.

1.3 Comparison between cotangent complexes

In this section we compare the different cotangent complexes for a finite and locally free group scheme G over S i.e. $G = \operatorname{Spec} B, S = \operatorname{Spec} A$ and B is a finite A-algebra. At first we will compare the cotangent complexes in the sense of Messing [14] and Lichtenbaum, Schlessinger [13] and prove that both are homotopically equivalent.

For a finite A-algebra B, which is a quotient of polynomial ring over A in n variables by an ideal generated by elements of a regular sequence of length n i.e. $B = A[\bar{X}]/I$, where $A[\bar{X}] = A[X_1, ..., X_n]$, and $I = (f_1, ..., f_n)$ an ideal generated by a regular sequence. We have computed in section (1.1) that the cotangent complex in the sense of Lichtenbaum, Schlessinger of B over A is

$$L^\cdot(\mathcal{E}) : \;\; 0 \longrightarrow 0 \longrightarrow I/I^2 \longrightarrow \Omega_{R/A} \otimes_R B \to 0. \tag{1.2}$$

Here $R = A[\bar{X}]$.

In section (1.2) we have defined the cotangent complex in the sense of Messing for a finite locally free group scheme G over S is

$$L_{.}^{G/S} : 0 \to J/J^2 \to \Omega_{U(G)^{\times}/S}|G \to 0 \tag{1.3}$$

where J is the ideal defining G in $U(G)^{\times}$. We have seen that the immersion $G \to U(G)^{\times}$ is closed and also regular. Also we have proved that $U(G)^{\times} \hookrightarrow U(G) \times U(G)$ is a closed immersion which is regular. From [S.G.A., 6 Chap VII Prop 1.7] we know that a composition of regular immersions is regular so we have $G \hookrightarrow U(G) \times U(G)$ is a closed regular immersion. Since $U(G)$ is affine over S so $U(G) \times U(G)$.

To compare both complexes let us assume that $U(G) \times U(G) = \operatorname{Spec}(R)$, $U(G)^{\times} = \operatorname{Spec}(\bar{R})$, let I be the ideal defining G in $U(G) \times U(G)$ and let K be the ideal defining $U(G)^{\times}$ in $U(G) \times U(G)$. Now we have two closed immersions :

$$G \xrightarrow{i} U(G)^{\times} \xrightarrow{j} U(G) \times U(G),$$

so we have an exact sequence of R-modules

$$0 \to K \to I \to J \to 0 \ ;$$

after tensoring by $R/K = \bar{R}$, we have exact sequence :

$$K/K^2 \to I \otimes_R \bar{R} \to J \otimes_R \bar{R} \to 0 \ ;$$

by tensoring by $B = R/I = \bar{R}/J$, we obtain an exact sequence :

$$i^*(K/K^2) \to I \otimes_R B \to J \otimes_{\bar{R}} B \to 0 \ ;$$

Since i and j are regular immersions, from [S.G.A., 6 Chap VII Prop 1.7] we know that the following sequence

$$0 \to i^*(K/K^2) \to I \otimes_R B \to J \otimes_{\bar{R}} B \to 0 \ ;$$

is exact and is equal to

$$0 \to i^*(K/K^2) \to I/I^2 \xrightarrow{\alpha^0} J/J^2 \to 0.$$

We have denoted the third map by α^0.

Since the map $R \to \bar{R}$ is surjective and the kernel of this map is an ideal generated by a regular sequence we have an exact sequence of $\bar{R}$- modules

$$0 \to K/K^2 \to \Omega^1_{R/A} \otimes_R \bar{R} \to \Omega^1_{\bar{R}/A} \to 0.$$

Since in the above exact sequence all the terms are locally free, after tensoring with B we get an exact sequence

$$0 \to i^*(K/K^2) \to \Omega^1_{R/A} \otimes_R B \xrightarrow{\beta^0} \Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B \to 0.$$

We have denoted the third map by β^0.

Our aim was to prove that the cotangent complex in the sense of Lichtenbaum, Schlessinger and Messing are homotopically equivalent i.e. the following two complexes are homotopically equivalent

$$0 \longrightarrow I/I^2 \longrightarrow \Omega^1_{R/A} \otimes_R B \to 0$$
$$0 \to J/J^2 \to \Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B \to 0$$

are homotopically equivalent

To prove this we have to find maps $f^0,\ f^1,\ g^0,\ g^1$

$$\begin{array}{ccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_{R/A} \otimes_R B & \longrightarrow & 0 \\ & & f^1 \uparrow \downarrow g^1 & & f^0 \uparrow \downarrow g^0 & & \\ 0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B & \longrightarrow & 0 \end{array}$$

such that $fg-$ id and $gf-$ id are null homotopic.

This means there are homotopies $h_1,\ \tilde{h}_1$

$$\begin{array}{ccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_{R/A} \otimes_R B & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle f^1g^1-\mathrm{id}} & \swarrow {\scriptstyle h_1} & \downarrow {\scriptstyle f^0g^0-\mathrm{id}} & & \\ 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_{R/A} \otimes_R B & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B & \longrightarrow & 0 \\ & & \downarrow {\scriptstyle g^1f^1-\mathrm{id}} & \swarrow {\scriptstyle \tilde{h}_1} & \downarrow {\scriptstyle g^0f^0-\mathrm{id}} & & \\ 0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B & \longrightarrow & 0 \end{array}$$

satisfying

$$\begin{aligned} f^0g^0 - \mathrm{id} &= d_1h_1 \\ f^1g^1 - \mathrm{id} &= h_1d_1 \\ g^0f^0 - \mathrm{id} &= \tilde{d}_1\tilde{h}_1 \\ g^1f^1 - \mathrm{id} &= \tilde{h}_1\tilde{d}_1. \end{aligned}$$

We can take for $g^0 = \beta^0$, since $\beta^0 : \Omega^1_{R/A} \otimes_R B \to \Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B$ is surjective and $\Omega^1_{\bar{R}/A} \otimes_{\bar{R}} B$ is locally free, we can take f^0 as section of this map. Similarly for g^1 we take α^0, since $\alpha^0 : I/I^2 \to J/J^2$ is surjective and J/J^2 is locally free, then we take f^1 as section of the map α^0, so we have $g^0f^0 = \mathrm{id}$ and $g^1f^1 = \mathrm{id}$. Our need was $g^0f^0 - \mathrm{id} = \tilde{d}_1\tilde{h}_1,\ g^1f^1 - \mathrm{id} = \tilde{h}_1\tilde{d}_1$, so we can take $\tilde{h}_1 = 0$ and that will satisfy the last two conditions.

For the first two conditions let us see the following diagram

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & \overset{h_1}{\curvearrowleft} & \downarrow & & \\
 & & i^*(K/K^2) & = & i^*(K/K^2) & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_{R/A}\otimes_R B & \longrightarrow & 0 \\
 & & f^1 \uparrow\downarrow g^1 & & f^0 \uparrow\downarrow g^0 & & \\
0 & \longrightarrow & J/J^2 & \longrightarrow & \Omega^1_{\bar{R}/A}\otimes_{\bar{R}} B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & .
\end{array}
$$

Our aim is to find a homotopy h_1 in the above diagram such that $f^0g^0 - \mathrm{id} = d_1h_1$ and $f^1g^1 - \mathrm{id} = h_1d_1$. Since f^0 is a section of the map g^0, so $f^0g^0 - \mathrm{id} : \Omega^1_{R/A} \otimes_R B \to \ker(g^0) = i^*(K/K^2)$. Similarly $f^1g^1 - \mathrm{id} : I/I^2 \to i^*(K/K^2)$. Take $h_1 = \mathrm{id}_{i^*(K/K^2)} \circ (f^0g^0 - \mathrm{id})$. This h_1 satisfies the first two conditions. This follows from following commutative diagram

$$
\begin{array}{ccc}
i^*(K/K^2) & = & i^*(K/K^2) \\
{\scriptstyle f^1g^1 - \mathrm{id}}\uparrow & = & \uparrow{\scriptstyle f^0g^0 - \mathrm{id}} \\
I/I^2 & \xrightarrow{d_1} & \Omega^1_{\bar{R}/A}\otimes_{\bar{R}} B.
\end{array}
$$

From the above diagram we have,

$$
\begin{aligned}
f^1g^1 - \mathrm{id} &= h_1d_1 \\
\text{and } d_1h_1 &= d_1 \circ \mathrm{id}_{i^*(K/K^2)} \circ (f^0g^0 - \mathrm{id}) \\
&= f^0g^0 - \mathrm{id}.
\end{aligned}
$$

Hence we are done.

1.4 Cotangent complex in the sense of Abrashkin

Notations and Conventions. In the next two sections p is a fixed prime number, $\mathcal{O}$ is a unitary commutative $\mathbb{F}_p$ -algebra and A is a commutaive unitary $\mathcal{O}$-algebra.

In this section we will define the cotangent complex in the sense of Abrashkin [1]. For that we review the first section of Abrashkin's paper [1]. We start with deformations of augmented A-algebras.

Deformations of augmented A-algebras

For an augmented A-algebra B, we use the following notations: $\epsilon_B : B \to A$, the morphism of augmentation, and $\operatorname{Ker}\epsilon_B = I_B$, the augmentaion ideal. If $A[\bar{X}] = A[X_1, ..., X_n]$, $n \geq 0$, is a polynomial ring we always assume that its augmentation ideal is $I_{A[\bar{X}]} = (X_1, ..., X_n)$ i.e. this is the ideal defining the zero section of $\operatorname{Spec}(A[\bar{X}])$.

The objects of the category DAug_A are the triples $\mathcal{B} = (B, B^\flat, i_\mathcal{B})$ where B is a finite augmented A-algebra, $B^\flat$ is an augmented A-algebra and $i_\mathcal{B} : B^\flat \to B$ is an epimorphic map of augmented algebras such that locally on A there is a polynomial ring $A[\bar{X}] = [X_1, ..., X_n]$, $n \geq 0$, and an epimorphism of augmented A-algebras $j : A[\bar{X}] \to B^\flat$ satisfying the following properties:

• the ideal $I := \operatorname{Ker}(i_\mathcal{B} \circ j)$ is generated by elements of a regular sequence of length n in $A[\bar{X}]$;

• $\operatorname{Ker} j = I \cdot I_{A[\bar{X}]}$.

From above locally on A, $B = A[X_1, ..., X_n]/I$ with I generated by a regular sequence and $B^\flat = A[X_1, ..., X_n]/I \cdot I_{A[\bar{X}]}$. The above definition makes sense for any pointed A-scheme but we only use it for flat relative complete intersections.

A morphism $\bar{f} = (f,\ f^\flat) : \mathcal{B} \to \mathcal{C} = (C, C^\flat, i_\mathcal{C})$ in DAug_A is given by morphism of augmented A-algebras $f : B \to C$ and $f^\flat : B^\flat \to C^\flat$ such that $f \circ i_\mathcal{B} = f^\flat \circ i_\mathcal{C}$.

In the category DAug_A, $\mathcal{A} = (A, A, \mathrm{id}_A)$ is an initial object and any $\mathcal{B} = (B, B^\flat,\ i_\mathcal{B})$ has a natural augmentation to $\mathcal{A}$, $\epsilon_\mathcal{B} = (\epsilon_B, \epsilon_{B^\flat}) : \mathcal{B} \to \mathcal{A}$.

Now we will use the simpler notation $(B, B^\flat)$ instead of $(B, B^\flat, i_\mathcal{B})$ if it does not lead to any confusion.

Lemma 1.4.1. *If $(B, B^\flat) \in \mathrm{DAug}_A$, then $B^\flat$ is a finite A-algebra.*

Proof. We have an exact sequence

$$0 \to I/I \cdot I_{A[\bar{X}]} \to B^\flat \to B \to 0$$

and if we can prove $I/(I \cdot I_{A[\bar{X}]})$ is locally free A-module that will give us $B^\flat$ is a finite A-algebra. From the lemma (1.1.3) we know that I is a finitely presented $A[\bar{X}]$-module and have an exact sequence

$$A[\bar{X}]^{\oplus\binom{n}{2}} \xrightarrow{i} A[\bar{X}]^{\oplus n} \xrightarrow{j} I \longrightarrow 0.$$

After tensoring $\otimes_{A[\bar{X}]} A$ (Here we see A as $A[\bar{X}]$-module through the map $A[\bar{X}] \to A[\bar{X}]/(X_1, .., X_n) = A$) in the above exact sequence we get an exact sequence

$$A^{\oplus\binom{n}{2}} \xrightarrow{i\otimes 1} A^{\oplus n} \xrightarrow{j\otimes 1} I \otimes_{A[\bar{X}]} A \longrightarrow 0.$$

Since the map $i \otimes 1 = 0$, this implies $A^{\oplus n} \simeq I \otimes_{A[\bar{X}]} A = I/I \cdot I_{A[\bar{X}]}$. This gives us $I/I \cdot I_{A[\bar{X}]}$ is finitely presented moreover free A-module of finite rank $\Rightarrow$ $B^\flat$ is a finite A-algebra. □

Abrashkin introduces the two A-modules $t^*_{\mathcal{B}} = I_{A[\bar{X}]}/I^2_{A[\bar{X}]}$ and $N_{\mathcal{B}} = I/(I \cdot I_{A[\bar{X}]})$. They do not depend on the choice of the epimorphisms $j : A[\bar{X}] \to B^\flat$. Indeed, the first coincides with $I_{B^\flat}/I^2_{B^\flat}$ and the second with $\operatorname{Ker} i_{\mathcal{B}}$. Note also that both A-modules are locally free. This is obvious for the first module and for the second we have given a proof above.

We will denote $t^*_{\mathcal{B}}$ and $N_{\mathcal{B}}$ by t^*_G and N_G respectively if $G = \operatorname{Spec}(B)$.

The cotangent complex in the sense of Abrashkin is defined as the complex

$$\begin{aligned} \mathcal{L}^{B/A}_{\cdot} &= I/(I \cdot I_{A[\bar{X}]}) \to I_{A[\bar{X}]}/I^2_{A[\bar{X}]} \\ &= N_{\mathcal{B}} \to t^*_{\mathcal{B}} \end{aligned} \tag{1.4}$$

as concentrated in degrees -1 and 0 and we show that $\mathcal{L}^{B/A}_{\cdot}$ is the fiber at the origin of the cotangent complex of B/A that we have defined earlier. In the above complex the map is the differential map.

As earlier we have seen that the cotangent complex in the sense of Lichtenbaum, Schlessinger for a finite A-algebra B is the complex of B-modules

$$0 \to I/I^2 \to \Omega_{R/A} \otimes_R B \to 0$$

where $R = A[\bar{X}] = A[X_1, ..., X_n]$, and I is the kernel of the surjective map of A-algebras $R \to B$ generated by elements of a regular sequence. Our aim is to prove here that the cotangent complex in the sense of Abrashkin is the fiber at the origin of the cotangent complex in the sense of Lichtenbaum, Schlessinger i.e. we have to prove that the cotangent complex in the sense of Abrashkin is pull back(change of rings) of the cotangent complex in the sense of Lichtenbaum, Schlessinger that through the augmentation map $\epsilon_B : B \to A$. Now

$$\epsilon^*_B(I/I^2) = I/I^2 \otimes_{A[X_1,...,X_n]/I} A \cong I/I.I_{A[\bar{X}]}.$$

We get this isomorphism of A-modules by sending a basis $f_i \otimes 1$ to f_i and we get the inverse by sending f_i to $f_i \otimes 1$. It is easy to check both map are well defined and inverse of each other. And

$$\begin{aligned} \epsilon^*_B(\Omega^1_{R/A} \otimes_R B) &= (\Omega^1_{R/A} \otimes_R B) \otimes_{A[X_1,...,X_n]/I} A = \oplus_{i=1}^n B \mathrm{d}X_i \otimes_{A[X_1,...,X_n]/I} A \\ &= \oplus_{i=1}^n A \mathrm{d}X_i \cong I_{A[\bar{X}]}/{I_{A[\bar{X}]}}^2 = t^*_B. \end{aligned}$$

The above isomorphism comes from the surjective map of A -algebras $A[X_1, ..., X_n] \to A$ and then we take the conormal sequence. Since the maps in both complexes are the differential maps so compatible hence we are done.

Lemma 1.4.2. *If $\mathcal{B} = (B, B^\flat)$ and $\mathcal{C} = (C, C^\flat)$ are objects in $DAug_A$ and $f : B \to C$ is a morphism of augmented A-algebras then the set of all $f^\flat$ such that $(f, f^\flat) \in \operatorname{Hom}_{DAug_A}(\mathcal{B}, \mathcal{C})$ is not empty.*

Proof. Since $\mathcal{B} = (B, B^\flat) \in \mathrm{DAug}_A$ so there exists a polynomial ring $A[\bar{X}]$ and maps

$$A[\bar{X}] \xrightarrow{j_\mathcal{B}} B^\flat \xrightarrow{i_\mathcal{B}} B$$

with the ideal $I := \mathrm{Ker}(i_\mathcal{B} \circ j_\mathcal{B})$ is generated by elements of a regular sequence of length n in $A[\bar{X}]$ and $\mathrm{Ker}\, j_\mathcal{B} = I \cdot I_{A[\bar{X}]}$ i.e. $B = A[\bar{X}]/I$ and $B^\flat = A[\bar{X}]/(I \cdot I_{A[\bar{X}]})$. Similarly for $\mathcal{C} = (C, C^\flat)$ there exists polynomial ring $A[\bar{Y}]$ and maps $i_\mathcal{C},\ j_\mathcal{C}$. Let $\pi_\mathcal{B} : A \to B$ and $\pi_\mathcal{C} : A \to C$ be the structure morphisms of B and C respectively.

To prove the set $\{f^\flat : B^\flat \to C^\flat$ which lift the given $f : B \to C\}$ is non empty at first we will prove

$$\begin{array}{ccccc} A[\bar{X}] & \xrightarrow{j_\mathcal{B}} & B^\flat & \xrightarrow{i_\mathcal{B}} & B \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle f} \\ A[\bar{Y}] & \xrightarrow{j_\mathcal{C}} & C^\flat & \xrightarrow{i_\mathcal{C}} & C \end{array}$$

there exist F such that it makes the above diagram commutative. Given f, $F_{|A} = \mathrm{id}_A \implies F \circ \pi_\mathcal{B} = \pi_\mathcal{C}$. Choose $g_i \in A[\bar{Y}]$ with $i_\mathcal{C} \circ j_\mathcal{C}(g_i) = f \circ i_\mathcal{B} \circ j_\mathcal{B}(X_i)$. Set $F(X_i) := g_i$ implies there exist $F : A[\bar{X}] \to A[\bar{Y}]$ with $i_\mathcal{C} \circ j_\mathcal{C} \circ F = f \circ i_\mathcal{B} \circ j_\mathcal{B}$. This implies $\epsilon_C \circ i_\mathcal{C} \circ j_\mathcal{C}(g_i) = \epsilon_B \circ i_\mathcal{B} \circ j_\mathcal{B}(X_i) = 0 \implies g_i = F(X_i) \in I_{A[\bar{Y}]}$ implies $F(I_{A[\bar{X}]}) \subseteq I_{A[\bar{Y}]}$.

If $x \in I$ then $(i_\mathcal{C} \circ j_\mathcal{C})F(x) = f(i_\mathcal{B} \circ j_\mathcal{B}(x)) = 0$ implies $F(I) \subseteq J \implies F(I \cdot I_{A[\bar{X}]}) \subseteq J \cdot I_{A[\bar{Y}]}$. This implies there exists $f^\flat : B^\flat \to C^\flat$. Hence the set $\{f^\flat : B^\flat \to C^\flat$ which lift the given $f : B \to C\}$ is not empty. $\square$

From the above lemma the set $\{f^\flat : B^\flat \to C^\flat$ which lift the given $f : B \to C\}$ is non empty and it has a natural structure of a principal homogeneous space under homotopies $h : t^*_\mathcal{B} \to N_\mathcal{C}$ through the map

$$(h, f^\flat)(X_i) = f^\flat(X_i) + h(X_i) \text{ for all } i.$$

It is easy to check that the above maps are well defined. Thus the deformations $B^\flat$ is unique up to homotopy equivalence, which in turn is unique up to unique homotopy. Thus $\mathcal{L}^{B/A}_{\cdot}$ is independent of $B^\flat$ up to homotopy equivalence.

Let DAug^*_A be the following quotient category of DAug_A: it has the same objects but its morphisms are equivalence classes of morphisms from $\mathrm{Hom}_{\mathrm{DAug}_A}(\mathcal{B}, \mathcal{C})$ arising from the same A-algebra morphisms $f : B \to C$. Then the above description of morphisms in the category DAug_A implies that the forgetful functor $(B, B^\flat) \mapsto B$ is an equivalence of DAug^*_A and the category of augmented finite A-algebras. In this equivalence of categories full faithfulness comes from the definition of the functor and for a given B, there exist $B^\flat$ because B is a finite A-algebra, that is of the form $B = A[X_1, ..., X_n]/I$ with ideal I generated by a regular sequence so we can construct $B^\flat$ from here.

1.5 Strict finite $\mathcal{O}$-module schemes

In this section we will define strict finite $\mathcal{O}$-module schemes. We will start with deformations of affine group schemes, then we explain what an $\mathcal{O}$-module structure is on these deformations, and when this module structure is strict. We follow Abrashkin [1].

Deformation of affine group schemes

Let DSch_A be the dual category for DAug_A. Its objects appear in the form $\mathcal{H} = \operatorname{Spec}\mathcal{B} = (H, H^\flat, i_\mathcal{H})$ where $H = \operatorname{Spec} B$ and $H^\flat = \operatorname{Spec} B^\flat$ are finite flat pointed A-schemes, $\mathcal{B} = (B, B^\flat, i_\mathcal{B}) \in \mathrm{DAug}_A$, and $i_\mathcal{H} : H \to H^\flat$ is the closed embedding of pointed A-schemes induced by $i_\mathcal{B}$. We agree to use the simpler notation $(H, H^\flat)$ if there is no confusion. The category DSch_A has direct products: if for $i = 1, 2$, $\mathcal{B}_i = (B_i, B_i^\flat, i_{\mathcal{B}_i})$ with $B_i = A[\bar{X}_i]/I_i$, $B_i^\flat = A[\bar{X}_i]/(I_i \cdot I_{0i})$ (where $I_{0i} = I_{A[\bar{X}_i]}$), then the product $\operatorname{Spec}\mathcal{B}_1 \times \operatorname{Spec}\mathcal{B}_2$ is given by $\operatorname{Spec}(\mathcal{B}_1 \otimes \mathcal{B}_2)$, where $\mathcal{B}_1 \otimes \mathcal{B}_2 := (B_1 \otimes_A B_2, (B_1 \otimes_A B_2)^\flat, \kappa)$, $(B_1 \otimes_A B_2)^\flat$ is the quotient of $A[\bar{X}_1 \otimes 1,\ 1 \otimes \bar{X}_2]$ by the product of ideals $I_1 \otimes 1 + 1 \otimes I_2$ and $I_{01} \otimes 1 + 1 \otimes I_{02}$ and κ is the natural projection. Note that for $i = 1, 2$, the projections $\mathrm{pr}_i : \operatorname{Spec}(\mathcal{B}_1 \otimes \mathcal{B}_2) \to \operatorname{Spec}\mathcal{B}_i$ come from the natural embeddings of $A[\bar{X}_i]$ into $A[\bar{X}_1 \otimes 1,\ 1 \otimes \bar{X}_2]$.

Let DGr_A be the category of group objects in DSch_A. If $\mathcal{G} = \operatorname{Spec}\mathcal{B} \in \mathrm{DGr}_A$, then its group structure is given via the comultiplication $\bar{\Delta} = (\Delta, \Delta^\flat) : \mathcal{B} \to \mathcal{B} \otimes \mathcal{B}$, the counit $\bar{\epsilon} = (\epsilon, \epsilon^\flat) : \mathcal{B} \to \mathcal{A}$ and the coinversion $\bar{i} = (i, i^\flat) : \mathcal{B} \to \mathcal{B}$ morphisms, which satisfy the usual axioms. The morphisms in DGr_A are morphisms of group objects. Clearly, DGr_A is an additive category.

Note that:

(a) $G = \operatorname{Spec} B$ is a finite flat group scheme over A with the comultiplication Δ, the counit ϵ and the coinversion i;

(b) $\bar{\epsilon} = \bar{\epsilon}_\mathcal{B}$, where $\bar{\epsilon}_\mathcal{B}$ is the natural augmentation.in the category DAug_A;

(c) the counit axiom for $i = 1, 2,\ j = 1, 2$, that $\mathrm{pr}_i \circ \Delta_j^\flat = \mathrm{id}_{B^\flat}$ gives us the uniqueness of $\Delta^\flat$ as a lifting of Δ. This is proved as follows.

Since G is flat group scheme, denote $G^\flat = \operatorname{Spec}(B^\flat)$. The comultiplication map $B \to B \otimes_A B$ lifts to an $B^\flat \to (B \otimes_A B)^\flat$, or equivalently $(G \times G)^\flat \to G^\flat$. We will show that the axiom for $i = 1, 2$, that $\mathrm{pr}_i \circ \Delta^\flat = \mathrm{id}_{B^\flat}$ gives us the uniqueness of the lift. Here $\mathrm{pr}_i : B \otimes_A B \to B$, sends $b_1 \otimes b_2 \mapsto \epsilon(b_{3-i}) \cdot b_i$. This is implied by the following observation. Suppose that $\Delta_1^\flat$ and $\Delta_2^\flat$ are two lifts of Δ

$$\begin{array}{ccc} B^\flat & \longrightarrow & B \\ {\scriptstyle \Delta_1^\flat}\downdownarrows{\scriptstyle \Delta_2^\flat} & & \downarrow{\scriptstyle \Delta} \\ (B \otimes B)^\flat & \longrightarrow & B \otimes B \end{array}$$

where $B = R/I$, $B^\flat = R/I \cdot I_0$ with $R = A[\bar{X}]$, $I_0 = I_{A[\bar{X}]}$, so we have here $B \otimes B = (R \otimes R)/(1 \otimes I + I \otimes 1)$ and $(B \otimes B)^\flat = (R \otimes R)/(1 \otimes I_0 + I_0 \otimes 1)(1 \otimes I + I \otimes 1)$. Since $\Delta_1^\flat$ and $\Delta_2^\flat$ are lifts of Δ this implies for $r \in R$, $(\Delta_1^\flat - \Delta_2^\flat)(r) = 1 \otimes u + v \otimes 1$ for some

$u, v \in I$. The counit axiom for $i = 1, 2,\ \ j = 1, 2$, that is $\mathrm{pr}_i \circ \Delta_j^\flat = \mathrm{id}_{B^\flat}$, gives us $u = 0$ and $v = 0$ in $B^\flat$ and this gives the uniqueness of the lifting of Δ;

(d) if $\mathcal{B} = (B, B^\flat, i_\mathcal{B}) \in \mathrm{DAug}_A$ and $G = \mathrm{Spec}\, B$ is a finite flat group scheme over A, then there is a unique structure of a group object on $\mathrm{Spec}\,\mathcal{B}$, which is compatible with that of G;

$$\begin{array}{ccc} B^\flat & \longrightarrow & B \\ \Delta^\flat \downarrow & & \downarrow \Delta \\ (B\otimes B)^\flat & \longrightarrow & B \otimes B \end{array}$$

We take some arbitrary $\Delta^\flat$ (exists from lemma (1.4.2). Our aim is to find $\tilde{\Delta}^\flat$ such that for $i = 1, 2\ \ \mathrm{pr}_i \circ \tilde{\Delta}^\flat = \mathrm{id}_{B^\flat}$. If we write $B = A[\bar{X}]/I$, we know

$$\tilde{\Delta}^\flat(X_\nu) - \Delta^\flat(X_\nu) = h(X_\nu) \text{ for all } \nu$$

for some $h : I_{A[\bar{X}]}/I^2_{A[\bar{X}]} = t^*_\mathcal{B} \to N_{\mathcal{B}\otimes\mathcal{B}} = (I\otimes 1 + 1\otimes I)/(I\otimes 1 + 1\otimes I)(I_{A[\bar{X}]}\otimes 1 + 1\otimes I_{A[\bar{X}]})$. As we required

$$\mathrm{pr}_i \circ \tilde{\Delta}^\flat(X_\nu) = X_\nu \text{ for all } \nu,$$

i.e.

$$\mathrm{pr}_i \circ h(X_\nu) = X_\nu - \mathrm{pr}_i \circ \Delta^\flat(X_\nu).$$

So we take

$$h(X_\nu) = (X_\nu - \mathrm{pr}_1 \circ \Delta^\flat(X_\nu)) \otimes 1 + 1 \otimes (X_\nu - \mathrm{pr}_2 \circ \Delta^\flat(X_\nu)).$$

It is well defined in $N_{\mathcal{B}\otimes\mathcal{B}}$ because, if $a \in B^\flat$ then $a - \mathrm{pr}_i \circ \Delta^\flat(a) = 0$ in B i.e. $a - \mathrm{pr}_i \circ \Delta^\flat(a) \in I/I \cdot I_{A[\bar{X}]}$.

(e) if $f : G \to H$ is a morphism of group schemes and $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{DSch}_A}(\mathcal{G}, \mathcal{H})$, then $(f, f^\flat) \in \mathrm{Hom}_{\mathrm{DGr}_A}(\mathcal{G}, \mathcal{H})$.

To check this property we have to check that the following diagram

$$\begin{array}{ccc} (G\times G)^\flat & \longrightarrow & (H\times H)^\flat \\ \Delta_{G^\flat} \downarrow & & \downarrow \Delta_{H^\flat} \\ G^\flat & \xrightarrow{f^\flat} & H^\flat \end{array}$$

is commutative, where upper horizontal map $(G \times G)^\flat \to (H \times H)^\flat$ is $(f^\flat \times f^\flat)_{|(G\times G)^\flat}$ which we also denote by $f^\flat \times f^\flat$ i.e. we have to check

1. $(f^\flat \times f^\flat)((G \times G)^\flat) \subseteq (H \times H)^\flat$.

2. $\Delta_{H^\flat} \circ (f^\flat \times f^\flat) = f^\flat \circ \Delta_{G^\flat}$ on $(G \times G)^\flat$.

Equivalently if we write $G = \operatorname{Spec} B,\ H = \operatorname{Spec} C$, the following diagram

$$\begin{array}{ccc} C^\flat & \xrightarrow{f^\flat} & B^\flat \\ \Delta_{C^\flat}\downarrow & & \downarrow\Delta_{B^\flat} \\ (C\otimes C)^\flat & \xrightarrow{f^\flat\times f^\flat} & (B\otimes B)^\flat \end{array}$$

is commutative i.e. we have to check

1. there exists a map $(f^\flat \times f^\flat)_{|(C\otimes C)^\flat} : (C\otimes C)^\flat \to (B\otimes B)^\flat$.

2. $\Delta_{B^\flat} \circ f^\flat = (f^\flat \times f^\flat) \circ \Delta_{C^\flat}$.

To check first if we write $B = A[\bar{X}]/I,\ C = A[\bar{Y}]/J$, we have to check $(f^\flat \times f^\flat)((J\otimes 1 + 1\otimes J)(I_{A[\bar{Y}]}\otimes 1 + 1\otimes I_{A[\bar{Y}]})) \subseteq (I\otimes 1 + 1\otimes I)(I_{A[\bar{X}]}\otimes 1 + 1\otimes I_{A[\bar{X}]})$. We have seen in lemma (1.4.2) that $f^\flat(J) \subseteq I,\ f^\flat(I_{A[\bar{Y}]}) \subseteq I_{A[\bar{X}]}$, and this implies that $(f^\flat \times f^\flat)((J\otimes 1 + 1\otimes J)(I_{A[\bar{Y}]}\otimes 1 + 1\otimes I_{A[\bar{Y}]})) \subseteq (I\otimes 1 + 1\otimes I)(I_{A[\bar{X}]}\otimes 1 + 1\otimes I_{A[\bar{X}]})$ holds.

To check second let $F = \Delta_{B^\flat} \circ f^\flat - (f^\flat \times f^\flat) \circ \Delta_{C^\flat}$. Since $F : C^\flat \to (B\otimes B)^\flat$ is map which is after composing with the map $(B\otimes B)^\flat \to B\otimes B$ is zero. Hence F is a map from $C^\flat \to \operatorname{Ker}((B\otimes B)^\flat \to B\otimes B)$ i.e. if we write $B = A[\bar{X}]/I$, F is a map from

$$C^\flat \to (I\otimes 1 + 1\otimes I)/(I\otimes 1 + 1\otimes I)(I_{A[\bar{X}]}\otimes 1 + 1\otimes I_{A[\bar{X}]}).$$

Now we see that $\mathrm{pr}_{B^\flat,i} \circ F = \mathrm{pr}_i \circ (\Delta_{B^\flat} \circ f^\flat - (f^\flat \times f^\flat) \circ \Delta_{C^\flat}) = \mathrm{id}_{B^\flat} \circ f^\flat - f^\flat \circ \mathrm{pr}_i \circ \Delta_{C^\flat} = f^\flat - f^\flat \circ \mathrm{id}_{C^\flat} = 0$ for $i = 1, 2$ and $\mathrm{pr}_{B^\flat,1} = \mathrm{id} \otimes \epsilon_{B^\flat}$ and $\mathrm{pr}_{B^\flat,2} = \epsilon_{B^\flat} \otimes \mathrm{id}$. It will give us $\operatorname{im} F = 0$ so F is zero map hence we are done.

The above properties have the following interpretation. Define the quotient category DGr^*_A as the category consisting of the objects of the category DGr_A but where $\mathrm{Hom}_{\mathrm{DGr}^*_A}(\mathcal{G}, \mathcal{H})$ consists of equivalence classes of morphisms from the category DGr_A which induce the same morphisms of group schemes $G \to H$. Then the forgetful functor $\mathcal{G} \to G$ is an equivalence of categories.

The categories of strict $\mathcal{O}$-modules

Suppose that $\mathcal{G}$ is an $\mathcal{O}$-module object in the category DSch_A. Then $\mathcal{G}$ is an object of DGr_A and there is a map $\mathcal{O} \to \mathrm{End}_{\mathrm{DGr}_A}(\mathcal{G})$ satisfying the usual axioms from the definition of $\mathcal{O}$-modules. For $o \in \mathcal{O}$ and $\mathcal{G} = \operatorname{Spec} \mathcal{B}$, denote by $[\bar{o}] = ([o], [o]^\flat)$ the morphism of action of o on $\mathcal{B} = (B, B^\flat, i_{\mathcal{B}})$. Clearly, $G = \operatorname{Spec} B$ is an $\mathcal{O}$-module in the category of finite flat schemes over A. For any such G, the $\mathcal{O}$-module structure on the deformation $(G, G^\flat) \in \mathrm{DGr}_A$ is given by liftings $[o]^\flat : B^\flat \to B^\flat$ of morphisms $[o] : B \to B,\ o \in \mathcal{O}$. Note that $[o]^\flat$ are morphisms of augmented algebras. All such morphisms are automatically compatible with the group structure on this deformation, i.e. for any $o \in \mathcal{O}$, one has $\Delta^\flat \circ [o]^\flat = ([o]^\flat \otimes [o]^\flat) \circ \Delta^\flat$. So the above system of liftings $[o]^\flat,\ o \in O$, gives an $\mathcal{O}$-module structure if and only if for any $o_1, o_2 \in \mathcal{O}$,

$$[o_1 + o_2]^\flat = ([o_1] \otimes [o_2])^\flat \circ \Delta^\flat, \quad [o_1 o_2]^\flat = [o_2]^\flat \circ [o_1]^\flat \tag{1.5}$$

where $([o_1] \otimes [o_2])^\flat$ is induced by $[o_1]^\flat \otimes [o_2]^\flat$.

Note that since $\mathcal{O}$ is of characteristic p every $\mathcal{O}$-module scheme is p-torsion.

Let $S = \mathrm{Spec}(A)$ be a scheme over $\mathcal{O}$. We denote by $\wp : \mathcal{O} \to \Gamma(S, \mathcal{O}_S)$ the structure morphism.

We denote by $\mathrm{DGr}(\mathcal{O})_A$ the category of above $\mathcal{O}$-module objects which satisfy the following definition of strictness.

Definition 1.5.1. Suppose $\mathcal{G} = \mathrm{Spec}\,\mathcal{B}$. A strict $\mathcal{O}$-action on $\mathcal{G}$ is a homomorphism $\mathcal{O} \to \mathrm{End}_{\mathrm{DGr}_A}(\mathcal{G})$ such that the induced action on $\mathcal{L}^{B/A}_{\cdot}$ is homotopic to the scalar multiplication via $\wp : \mathcal{O} \to A$.

Since $\mathcal{L}^{B/A}_{\cdot}$ is independent of $B^\flat$ up to homotopy equivalence. Hence the above definition of strictness is independent of deformations $B^\flat$. Therefore we will change the notation for above category and denote by $\mathrm{Gr}(\mathcal{O})_A$ which will represent the category of strict $\mathcal{O}$-module schemes over S and have the following definition

Definition 1.5.2. A strict $\mathcal{O}$-module scheme over S is a finite S-group scheme G together with a homomorphism $\mathcal{O} \to \mathrm{End}_S(G)$ which lifts to a homomorphism $\mathcal{O} \to \mathrm{End}_{\mathrm{DGr}_A}(\mathcal{G})$ such that the induced action on $\mathcal{L}^{B/A}_{\cdot}$ is homotopic to the scalar multiplication via $\wp : \mathcal{O} \to A$.

Mainly $\mathrm{Gr}(\mathcal{O})_A$ is the quotient category of $\mathrm{DGr}(\mathcal{O})_A$ where the morphisms are the equivalence classes of morphisms $(G,\ G^\flat) \to (H,\ H^\flat)$ in the category $\mathrm{DGr}(\mathcal{O})_A$ which induce the same morphism $G \to H$.

Now we will compare this definition with Abrashkin's definition of strictness which is false for the case $\mathbb{F}_q,\ q \neq p$ and for that we will give a counter example. Abrashkin's definition says that the action of $\mathcal{O}$ on $\mathcal{G} = \mathrm{Spec}\,\mathcal{B} \in \mathrm{DGr}_A$ is strict if any $o \in \mathcal{O}$ acts on $t^*_{\mathcal{B}}$ and $N_{\mathcal{B}}$ via the scalar multiplication.

Let us consider a group $G = \mathbb{F}_{p^2}$ and take $\mathcal{O} = \mathbb{F}_{p^2} = \mathbb{F}_p[\lambda]$ with $\lambda \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$. We can give two different presentation of G as $\mathrm{Spec}\,B$ and $\mathrm{Spec}\,\tilde{B}$ where $B = A[X_1, X_2]/(X_1^p - X2, X_2^p - X_1)$ and $\tilde{B} = A[Y]/(Y^{p^2} - Y)$. In both presentation G is $\mathbb{F}_q$-module through the action $\lambda^*[X_1] = \lambda X_1,\ \lambda^*[X_2] = \lambda^p X_2$ and $\lambda^*[Y] = \lambda Y$. In the case $G = \mathrm{Spec}\,B = \mathrm{Spec}\,A[X_1, X_2]/(X_1^p - X2, X_2^p - X_1)$ we can take $R = A[X_1, X_2],\ I = (X_1^p - X_2, X_2^p - X_1)$ and $I_R = (X_1, X_2)$. We have $B^\flat = A[X_1, X_2]/(X_1^p - X_2, X_2^p - X_1)(X_1, X_2),\ t^*_{\mathcal{B}} = (X_1, X_2)/(X_1, X_2)^2 = AX_1 \oplus AX_2$ and $N_{\mathcal{B}} = (X_1^p - X_2, X_2^p - X_1)/(X_1^p - X2, X_2^p - X_1)(X_1, X_2)$. Now we have

$$\begin{aligned}\mathcal{L}^{B/A}_{\bullet} &= N_{\mathcal{B}} \longrightarrow t^*_{\mathcal{B}} \\ &= (X_1^p - X_2, X_2^p - X_1)/(X_1^p - X_2, X_2^p - X_1)(X_1, X_2) \to AX_1 \oplus AX_2.\end{aligned}$$

Since in the above complex the maps are differential maps, it sends $X_1^p - X_2 \to -X_2$ and $X_2^p - X_1 \to -X_1$. It is easy to check the action of λ on above complex is not equal to the scalar action but only homotopic to it. But in the second presentation the action of λ on cotangent complex is equal to the scalar action. So the Abrashkin' definition of strictness depends on presentation.

Remark 1.5.3. If $G = \mathrm{Spec}(B)$ is a smooth group scheme over $S = \mathrm{Spec}(A)$, then we have an exact sequence of B-modules

$$I/I^2 \to \Omega^1_{R/A} \otimes_R B \to \Omega^1_{G/S} \to 0$$

where R is an A-algebra such that $R \to B$ is surjective map of an A-algebra and I is the kernel of the map. Since G is smooth group scheme so $\Omega^1_{G/S}$ locally free. This implies that second map has a section so the cotangent complex $\mathcal{L}^{G/S}_{\cdot}$ is homotopically equivalent to the complex

$$0 \to 0 \to \Omega^1_{G/S} \to 0.$$

For the definition of $\Omega^1_{G/S}$ see definition (2.1.4).

Remark 1.5.4. If G is a finite smooth group scheme over S and an $\mathcal{O}$-module scheme then G is a strict $\mathcal{O}$-module because $\Omega^1_{G/S} = (0)$ and $\mathcal{L}^{G/S}_{\cdot}$ is homotopically equivalent to 0.

For example, if $q = p^n$ with $n \in \mathbb{N}$ and A is an $\mathbb{F}_q$-algebra and $\alpha \in \mathbb{F}_q$. It is easy to check that $\alpha_q = \mathrm{Spec}\, A[X]/(X^q)$ is an $\mathbb{F}_q$-module scheme, $\alpha \in \mathbb{F}_q$ acts via $\alpha[X] = \alpha X$. Now we will show that this module structure is strict. The cotangent complex in the sense of Abrashkin of α_q is

$$0 \to X^q \cdot A \to A \cdot dX \to 0,$$

if we write $B = A[X]/(X^q)$. It is easy to see that the action of $\alpha \in \mathbb{F}_q$ on the above cotangent complex is scalar multiple by α on the cotangent complex.

If we take for example $\alpha_p = \mathrm{Spec}\, A[X]/(X^p)$ and $p \neq q$. It is an $\mathbb{F}_q$-module scheme, $\alpha \in \mathbb{F}_q$ acts via $\alpha[X] = \alpha X$. Since $[\alpha](X^p) = \alpha^p X^p \neq \alpha X^p$, if $\alpha \in \mathbb{F}_q$ and $p \neq q$, then the $\mathbb{F}_q$-action on the cotangent complex is not a scalar multiple and since differential maps are zero maps so $\mathbb{F}_q$-action on the cotangent complex is not homotopic to a scalar multiple. Hence through this module structure $\alpha_p = \mathrm{Spec}\, A[X]/(X^p)$ is not a strict $\mathbb{F}_q$-module.

The constant etale group scheme $\mathbb{F}_q = \mathrm{Spec}\, \mathbb{F}_q[X]/(X^q - X)$ is a $\mathbb{F}_q$-module scheme, $\alpha \in \mathbb{F}_q$ acts via $\alpha[X] = \alpha X$ and through this module structure it is strict.

The constant multiplicative group $\mu_p = \mathrm{Spec}(\mathbb{F}_p[X]/(X^p - 1))$, $\Delta(X) = X \otimes X$, $e(X) = 1$ is a $\mathbb{F}_p$-module scheme, $\alpha \in \mathbb{F}_p$ acts via $\alpha[X] = X^\alpha$ but through this module structure is not strict, because the $\mathbb{F}_p$-action does not lift to $\mu_p^\flat$.

Example 1.5.5. If S is a scheme over $\mathrm{Spec}\, \mathbb{F}_q$, the additive group scheme $\mathbb{G}_{a,S}$ is an $\mathbb{F}_q$-module scheme over S. Likewise every S-group scheme which locally on S is isomorphic to $\mathbb{G}_{a,S}^d$ for some integer $d \geq 0$ is an $\mathbb{F}_q$-module scheme. Such a scheme is called an $\mathbb{F}_q$-*vector group scheme of dimension* d over S. For every $a \in \mathbb{F}_q$ the endomorphism $\mathrm{Lie}^* a$ on its co-Lie module equals the multiplication with a viewed as an element of $\Gamma(S, \mathcal{O}_S)$.

We will finish this section with some preliminary on commutative group scheme and a remark which we will need later on.

For a commutative group scheme G over a scheme S we define its co-Lie module ω_G as the $\mathcal{O}_S$-module of invariant differentials. It is canonically isomorphic to $e^*\Omega^1_{G/S}$ where $e : S \to G$ is the zero section i.e. $\omega_{G/S} = \mathcal{O}_S \otimes_{\mathcal{O}_G} \Omega^1_{G/S}$. Each element $a \in \mathcal{O}$ induces an endomorphism of ω_G which we denote by $\mathrm{Lie}^* a$. We have $\mathrm{Lie}\, G = \mathrm{Hom}_{\mathcal{O}_S}(\omega_G, \mathcal{O}_S)$ as $\mathcal{O}_S$-module.

Remark 1.5.6. If G is locally of finite presentation over S then $\omega_G = 0$ if and only if G is étale over S. Indeed, since $\Omega^1_{G/S}$ is a finitely generated $\mathcal{O}_G$-module, $\omega_G = 0$ implies by Nakayama that G is étale along the zero section. Being a group scheme it is étale everywhere.

1.6 Relation between finite shtukas and strict finite flat commutative group schemes

In this section we fix p a prime number and q is a power of p. Let $S = \mathrm{Spec}(A)$ be a scheme over $\mathrm{Spec}\, \mathbb{F}_q$. We denote by $\sigma_q : S \to S$ its Frobenius endomorphism which acts as the identity on points and as the q-power map on the structure sheaf. In this section we explain that Drinfeld established a relation between finite shtukas over S and finite strict $\mathbb{F}_q$-module schemes over S.

Definition 1.6.1. A *finite* $\mathbb{F}_q$-*shtuka* over S is a pair (M, F_M) consisting of a locally free $\mathcal{O}_S$-module M on S of finite rank and an $\mathcal{O}_S$-module homomorphism $F_M : \sigma_q^* M \to M$. A *morphism* $f : (M, F_M) \to (M', F_{M'})$ of finite shtukas is an $\mathcal{O}_S$-module homomorphism $f : M \to M'$ which makes the following diagram commutative

$$\begin{array}{ccc} \sigma_q^* M & \xrightarrow{\sigma_q^* f} & \sigma_q^* M' \\ \downarrow{\scriptstyle F_M} & & \downarrow{\scriptstyle F_{M'}} \\ M & \xrightarrow{f} & M'. \end{array}$$

Here $\sigma_q^* M = M \otimes_{\mathcal{O}_S, \sigma_q^*} \mathcal{O}_S$. A finite shtuka over S is called *étale* if F_M is an isomorphism.

The above objects were studied at various places in the literature. They were called "(finite) ϕ-sheaves" by Drinfeld [6], Taguchi and Wan [17, 18] and "Dieudonné $\mathbb{F}_q$-modules" by Laumon [12].

We denote by $\mathrm{Mod}(\mathbb{F}_q)_A$ the category of $\mathbb{F}_q$-shtukas over S.

For now suppose that A is an $\mathbb{F}_p$-algebra and study the category $\mathrm{Gr}(\mathbb{F}_p)_A$ of strict finite $\mathbb{F}_p$-modules G over A. At first we will give a complete description of the category $\mathrm{Gr}(\mathbb{F}_p)_A$ and we apply it to describe the category $\mathrm{Gr}(\mathbb{F}_q)_A$.

Suppose that G is a finite flat commutative group scheme over A. Consider $G^{(p)} = G \times_{(\mathrm{Spec}\, A, \sigma_p)} \mathrm{Spec}\, A$, where A is considered as an A-module via the map $\sigma_p^* : A \to A$ such that for any $a \in A$, $\sigma_p^*(a) = a^p$. Then $G^{(p)}$ has a natural structure of a finite flat commutative group scheme over A. Here $A(G)$ is a commutative A-algebra such that $G = \mathrm{Spec}\, A(G)$.

Let $F_G : G \to G^{(p)}$ be the relative Frobenius morphism of G over $\mathrm{Spec}\, A$. It is given by morphism of A-algebras $F_G^* : A(G^{(p)}) = A(G) \otimes_{(A,\sigma_p)} A \to A(G)$ such that for all $\alpha \in A(G)$ and $a \in A$, $F_G^*(\alpha \otimes a) = \alpha^p a$.

Let $V_G : G^{(p)} \to G$ be the p-Verschiebung morphism of group schemes. Recall from [G.A., Chap IV §3, 4] that it is given by the morphism of A-algebras $V_G^* : A(G) \to A(G^{(p)})$, which can be described as follows. We can proceed locally on $\mathrm{Spec}\, A$ and, therefore, can assume that $A(G)$ is a free A-module with a basis $a_1, \ldots\ldots, a_n$. Denote by $TS^p(A(G)) \subset A(G)^{\otimes p}$ the symmetric tensors of order p and $s : A(G)^{\otimes p} \to TS^p(A(G))$ the symmetrisation operator defined by

$$s(\alpha_1 \otimes \ldots\ldots \otimes \alpha_p) = \sum_{\sigma \in S_p} \alpha_{\sigma(1)} \otimes \ldots\ldots \otimes \alpha_{\sigma(p)} \quad \alpha_i \in A(G).$$

From the first lemma in [G.A., Chap IV §3, 4.1] we have a canonical bijection map $A(G^{(p)}) \to TS^p(A(G))/s(A(G)^{\otimes p})$, sending $\alpha \otimes a \mapsto \alpha^p a$, $\alpha \in A(G)$, $a \in A$, here α^p denotes the canonical image of $\alpha \otimes \ldots \otimes \alpha \in TS^p(A(G))$ in $TS^p(A(G))/s(A(G)^{\otimes p})$. That is we have a closed imbedding $G^{(p)} \hookrightarrow (G^p)^{S_p} = S^p G$. The comultiplication map

$$\Delta^{(p)} = \Delta \circ (\Delta \otimes \mathrm{id}) \circ \cdots \circ (\Delta \otimes \mathrm{id}^{\otimes p-2}) : A(G) \to A(G)^{\otimes p}$$

defines the multiplication map $G^p \to G$. Since it is invariant under S^p, it factorizes through $S^p G$. Since we have an embedding $G^{(p)} \hookrightarrow (G^p)^{S_p} = S^p G$ therefore we have induced a map $G^{(p)} \dashrightarrow G$, which is

$$\begin{array}{ccccc} G^p & \longrightarrow & S^p G & \hookleftarrow & G^{(p)} \\ & \searrow^{\text{mult}} & \downarrow & \swarrow & \\ & & G & & \end{array}$$

the required Verschiebung morphism of G. The way we have defined here the Verschiebung morphism of G, it is clear that it is the dual of Frobenius morphism. The above definition imply easily that $F_G \circ V_G = p\, \mathrm{id}_{G^{(p)}}$ and $V_G \circ F_G = p\, \mathrm{id}_G$. For more details see [G.A., Chap IV §3, 4].

Now we will explain the antiequivalence $\underline{M}$ of the category of finite flat commutative group schemes G over A with zero Verschiebung and the category $\mathrm{Mod}(\mathbb{F}_p)_A$ of $\mathbb{F}_p$-shtuka [G.A., Chap IV §3, 6]. Here $\underline{M}(G) = (M(G),\ F_{M(G)})$, where

$$M(G) := \mathrm{Hom}(G,\ \mathbb{G}_a) = \{a \in A(G) \mid \Delta(a) = a \otimes 1 + 1 \otimes a\}$$

is free A-module of finite rank and $F_{M(G)}$ is induced by the Frobenius F_G^*.

The inverse functor Dr_p can be described as follows. If $\mathcal{M} = (M,\ F_M) \in \mathrm{Mod}(\mathbb{F}_p)_A$ and $m_1, ..., m_n$ is an A-basis for M then for $1 \leq i \leq n$,

$$F_M(m_i \otimes 1) = \sum_{1 \leq j \leq n} r_{ij} m_j.$$

Then $\mathrm{Dr}_p(\mathcal{M}) = \mathrm{Spec}\, A(G)$, where $A(G) = A[X_1, ..., X_n]/I$ with the ideal I generated by the polynomials

$$X_i^p - \sum_{1 \leq j \leq n} r_{ij} X_j, \quad 1 \leq i \leq n, \tag{1.6}$$

with the group structure given via the comultiplication Δ such that $\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i$ and the counit e such that $e(X_i) = 0$ for $1 \leq i \leq n$.

We have a following theorem from Abrashkin [1] which gives us the description of the category $\mathrm{Gr}(\mathbb{F}_p)_A$.

Theorem 1.6.2. *[1, Thm 1] We have $G \in \mathrm{Gr}(\mathbb{F}_p)_A$ if and only if $V_G = 0$.*

Now assume that A is an $\mathbb{F}_q$-algebra, where $q = p^N$ and $N \in \mathbb{N}$, and now we will study the category $\mathrm{Gr}(\mathbb{F}_q)_A$ of strict finite $\mathbb{F}_q$-modules G over A as a full subcategory of $\mathrm{Gr}(\mathbb{F}_p)_A$.

For any A-moules M set $\sigma_q^* M = M \otimes_{(A, \sigma_q^*)} A$, where the A-module structure on the second component of this tensor product is given via the qth power map $\sigma_q^* : A \to A$. If $f : M \to N$ is morphism of A-modules, then we use the notation $\sigma_q^* f = f \otimes_{(A,\sigma_q^*)} A : \sigma_q^* M \to \sigma_q^* N$.

Consider the category $\mathrm{Mod}(\mathbb{F}_q)_A$ consisting of locally free of finite rank A-modules M with an A-linear morphism $F_M : \sigma_q^* M \to M$. If N is another object, then $\mathrm{Hom}_{\mathrm{Mod}(\mathbb{F}_q)_A}(M, N)$ consists of A-linear morphism $f : M \to N$ such that $F_N \circ f = \sigma_q^* f \circ F_M$.

Define the functor $\underline{M}_q : \mathrm{DGr}(\mathbb{F}_q)_A \to \mathrm{Mod}(\mathbb{F}_q)_A$ by setting for any $\mathcal{G} = (G, G^\flat) \in \mathrm{DGr}(\mathbb{F}_q)_A$, $\underline{M}_q(\mathcal{G}) = (M(\mathcal{G}), F_M)$ with

$$M(\mathcal{G}) = \{a \in A(G) \mid \Delta(a) = a \otimes 1 + 1 \otimes a,\ [\alpha](a) = \alpha a,\ \forall \alpha \in \mathbb{F}_q\},$$

where $F_M : \sigma_q^* M(\mathcal{G}) \to M(\mathcal{G})$ is induced by the qth power map on $A(G)$. If $\mathcal{H} \in \mathrm{DGr}(\mathbb{F}_q)_A$, $\underline{M}_q(\mathcal{H}) = (M(\mathcal{H}), F_q)$ and $(f,\ f^\flat) : \mathcal{G} \to \mathcal{H}$ is a morphism in the category $\mathrm{DGr}(\mathbb{F}_q)_A$, then $f(M(\mathcal{H})) \subset M(\mathcal{G})$ and $\underline{M}_q(f) = f|_{M(\mathcal{H})}$.

Now we construct the inverse functor $\mathrm{Dr}_q : \mathrm{Mod}(\mathbb{F}_q)_A \to \mathrm{Gr}(\mathbb{F}_q)_A$ in a similar way as we have done earlier.

If $(M, F_q) \in \mathrm{Mod}(\mathbb{F}_q)_A$, then $\mathrm{Dr}_q(M, F_q) = \mathrm{Spec}\,\mathcal{B}$ with $\mathcal{B} = (A(G), A(G)^\flat, i_{\mathcal{B}})$, defined by:

- $A(G) = \mathrm{Sym}_A M/I$ where the ideal I is generated by $\{m^q - F_q(m \otimes 1) |\ m \in M\}$, the comultiplication Δ is such that $\Delta(m) = m \otimes 1 + 1 \otimes m$ and the $\mathbb{F}_q$-action such that $[\alpha](m) = \alpha m$ for all $m \in M$ and $\alpha \in \mathbb{F}_q$;

• $A(G)^\flat = \mathrm{Sym}_A(M)/(I \cdot I_0)$ where the augmentation ideal I_0 is generated by all $m \in M$, the comultiplication $\Delta^\flat$ is such that $\Delta^\flat(m) = m \otimes 1 + 1 \otimes m$ and the $\mathbb{F}_q$-action such that $[\alpha]^\flat$ is given by the correspondence and $m \mapsto \alpha m$ for all $m \in M$ and $\alpha \in \mathbb{F}_q$;
• $i_{\mathcal{B}}$ is the natural projection from $A(G)^\flat$ to $A(G)$.

Clearly, $\mathcal{G} = \operatorname{Spec} \mathcal{B} \in \mathrm{DGr}(\mathbb{F}_q)_A$ and the correspondence $\mathrm{Dr}_q : (M, F_M) \mapsto \mathcal{G}$ can be naturally extended to the functor $\mathrm{Dr}_q : \mathrm{Mod}(\mathbb{F}_q)_A \to \mathrm{DGr}(\mathbb{F}_q)_A$. This functor is additive and faithful.

We have the following theorem from Abrashkin [1] which gives us the description of $\mathrm{DGr}(\mathbb{F}_q)_A$.

Theorem 1.6.3. *[1, Thm 2] The above defined functor* $\underline{M}_q : \mathrm{DGr}(\mathbb{F}_q)_A \to \mathrm{Mod}(\mathbb{F}_q)_A$ *induces an antiequivalence of the categories* $Gr(\mathbb{F}_q)_A$ *and* $\mathrm{Mod}(\mathbb{F}_q)_A$.

Note that any $\mathcal{G} \in \mathrm{DGr}(\mathbb{F}_q)_A$ can be identified in the category $\mathrm{Gr}(\mathbb{F}_q)_A$ with some $\mathrm{Dr}_q(M_0, F_q)$, where $(M_0, F_q) \in \mathrm{Mod}(\mathbb{F}_q)_A$.

Theorem 1.6.4. (*Drinfeld* [*6*, 2.1], *Taguchi* [*17*, 1.7])

1. *The two contravariant functors* Dr_q *and* $\underline{M}_q$ *are mutually quasi-inverse anti-equivalences between the category of finite* $\mathbb{F}_q$*-shtukas over* S *and the category of finite strict* $\mathbb{F}_q$*-module schemes over* S.

2. *Both functors are* $\mathbb{F}_q$*-linear and map short exact sequences to short exact sequences. They preserve étale objects.*

Let (M, F) *be a finite* $\mathbb{F}_q$*-shtuka over* S. *Then*

3. *the* $\mathbb{F}_q$*-module scheme* $\mathrm{Dr}_q(M, F)$ *is radicial if and only if* F *is nilpotent locally on* S.

4. *the scheme* $\mathrm{Dr}_q(M, F)$ *is finite and locally free and the order of the* S*-group scheme* $\mathrm{Dr}_q(M, F)$ *is* $q^{\mathrm{rk}\, M}$.

5. *the* $\mathcal{O}_S$*-modules* $\omega_{\mathrm{Dr}_q(M,F)}$ *and* $\operatorname{coker} F$ *are canonically isomorphic.*

Proof. Assertion 1 follows from the theorem (1.6.3). In this theorem both G and $\underline{M}(G)$ are étale if and only if the matrix T of Frobenius morphism $F_{M(G)} : \sigma_q^* M(G) \to M(G)$, $\in \mathrm{GL}_n(\mathcal{O}_S)$. Hence the second statement of 2 follows. Alternatively this is also a consequence of the fact that G is étale if and only if $Frob_G$ is an isomorphism. The $\mathbb{F}_q$-linearity is clear from the definitions. So let $0 \to G' \to G \to G'' \to 0$ be a short exact sequence of finite strict $\mathbb{F}_q$-module schemes. Then the exactness of $0 \to \underline{M}(G'') \to \underline{M}(G) \to \underline{M}(G')$ is obvious and the surjectivity of the last morphism also follows from the theorem (1.6.3). Conversely let $0 \to \underline{M}(G'') \to \underline{M}(G) \to \underline{M}(G') \to 0$ be a short exact sequence of finite shtukas. Then obviously $G' \to G$ is a closed immersion and the exactness of $0 \to G' \to G \to G'' \to 0$ follows from the identification

$$\underline{M}(G/G') \;=\; \ker\big(\underline{M}(G) \to \underline{M}(G')\big) \;=\; \underline{M}(G'').$$

If $M \simeq \mathcal{O}_S^n$ and $T = (t_{ij})$ is the matrix of the morphism $F : \sigma_S^* M \to M$ with respect to the basis $X_i,\ 1 \leq i \leq n$, then $\mathrm{Dr}_q(M, F)$ is the subscheme of $\mathbb{G}_{a,S}^n$, given by the system of equations

$$X_j^q - \sum_{i=1}^{n} t_{ij} X_i, \quad j = 1, ..., n. \tag{1.7}$$

It is easily seen that the quotient of $\mathcal{O}_S[X_1, ..., X_n]$ by the ideal generated by the polynomials of (1.7) is free $\mathcal{O}_S$-module with a basis $X_1^{m_1} \cdots X_n^{m_n},\ 0 \leq m_i \leq q$. Hence assertion 4 follows.

In the case where S is the spectrum of a field, 3 follows from the fact that G is connected if and only if $Frob_G$ is nilpotent. The general case is a direct consequence of this special case.

Finally assertion5 follows from the fact that both $\mathcal{O}_S$-modules are isomorphic to

$$\bigoplus_{i=1}^{n} \mathcal{O}_S\, X_i \,/\, (X_1, \ldots, X_n) \cdot T$$

from above description of $\mathrm{Dr}_q(M, F)$.

□

1.7 Comparison between cotangent complex and Frobenius map of finite $\mathbb{F}_q$-shtukas

In this section we will follow the same notation as in section (1.3) i.e. $G = \operatorname{Spec} B,\ S = \operatorname{Spec} A,\ R = A[X_1, ..., X_n]$ and S is a scheme over $\mathbb{F}_q$.

Let G be a finite flat commutative group scheme which is a strict $\mathbb{F}_q$-module scheme over S. Let $M(G)$ be the finite $\mathbb{F}_q$-shtuka associated to it and $F_{M(G)} : \sigma_q^* M(G) \to M(G)$ be its Frobenius map which is given by the matrix $T = (t_{ij})$ with respect to the basis $X_i,\ 1 \leq i \leq n$. From equivalence in section (1.6) we know that if

$$M(G) = \oplus_{i=1}^{n} \mathcal{O}_S X_i,$$

then $G = \operatorname{Spec}(\operatorname{Sym}_{\mathcal{O}_S} M(G)/I)$, where the ideal I is generated by the polynomials $f_i = X_i^q - (X_1, .., X_r) T e_i,\ 1 \leq i \leq n$. Since the A-algebra B is given by n equations above in $R = A[X_1, ..., X_n]$ and, therefore it is a relative complete intersection so $G \hookrightarrow \mathbb{G}_{a,S}^r$ is a regular immersion. The cotangent complex of G over S is

$$0 \to I/I^2 \to \Omega^1_{R/A} \otimes_R B \to 0. \tag{1.8}$$

Where the middle arrow is the differential map i.e. the map sending f_i to $df_i = -(dX_1, .., dX_r) \cdot T e_i$. Since the immersion $G \hookrightarrow \mathbb{G}_{a,S}^r$ is regular so I/I^2 is locally free and it is equal to $\oplus B f_i$. Also $\Omega^1_{R/A} \otimes_R B = \oplus B dX_i$.

Since we have the Frobenius map $F_{M(G)} : \sigma_S^* M(G) \to M(G)$ which is A-linear and which sends $\sigma_q^* X_i$ to $(X_1, .., X_r) T e_i$. After tensoring with $\otimes_A B$ we get a B-linear map $\sigma_S^* M(G) \otimes_A B \to M(G) \otimes_A B$ which we view as a complex

$$0 \to \sigma_S^* M(G) \otimes_A B \to M(G) \otimes_A B \to 0. \tag{1.9}$$

Now we can see that both complexes (1.8) and (1.9) are isomorphic via the following map,

$$\begin{array}{ccccccc} 0 & \longrightarrow & I/I^2 & \longrightarrow & \Omega^1_{R/A} \otimes_R B & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \sigma_S^* M(G) \otimes_A B & \longrightarrow & M(G) \otimes_A B & \longrightarrow & 0 \end{array} \tag{1.10}$$

if we send f_i to $-\sigma_S^* X_i$ in first vertical row and dX_i to X_i in the second vertical arrow. Note that all four elements in both complexes are locally free of same rank.

Chapter 2

Local shtukas and divisible local Anderson-modules

2.1 Formal Lie groups

Let S be a scheme, and X and Y with $Y \hookrightarrow X$ two sheaves on S for the fppf topology. Following Messing [14] we make the following definition.

Definition 2.1.1. $\mathrm{Inf}^k_Y(X)$ is the subsheaf of X whose sections over an S-scheme T are given as follows: $\Gamma(T,\ \mathrm{Inf}^k_Y(X)) = \{t \in \Gamma(T,\ X)|$ there is a covering $\{T_i \to T\}$ and for each T_i a closed subscheme $T_i{}'$ defined by an ideal whose $(k+1)^{st}$ power is (0) with the property that $t_{T_i{}'} \in \Gamma(T_i',\ X)$ is actually an element of $\Gamma(T_i',\ Y)\}$.

We call $\mathrm{Inf}^k_Y(X)$ the k^{th} infinitesimal neighborhood of Y in X.

If X and Y are schemes, we compare it to the k^{th} infinitesimal neighborhood of Y in X in the sense of [E.G.A., IV §16] when $i : Y \hookrightarrow X$ is an immersion and then prove that both are the same. Since $Y \hookrightarrow X$ is an immersion, we can assume that Y is a closed subscheme of X defined be a quasi-coherent $I \subseteq \mathcal{O}_X$ or to say $\mathcal{O}_Y$ is identified with sheaf of quotient rings $\mathcal{O}_X/I$. Then we can provide on $\mathcal{O}_X$ the I-adic filtration.

In the sense of [E.G.A.] the $\mathcal{O}_Y$-augmented sheaf $\mathcal{O}_X/I^{k+1}$ is called the k-th normal invariant of i; the ringed space $(Y, \mathcal{O}_X/I^{k+1})$ is called the k^{th} infinitesimal neighborhood of Y in X, and denoted by $Y^{(k)}$. The graded sheaf of rings associated to the filtered sheaf of rings $\mathcal{O}_X$

$$\mathrm{gr}_\bullet(i) = \oplus_{k\geq 0} I^k/I^{k+1}$$

is called the graded sheaf of rings associated to i. The sheaf $\mathrm{gr}_1(i) = I/I^2$ is called the conormal sheaf of i and we denote the sheaf $i^*(I/I^2)$ by ω_i. It is clear that the sheaf $\mathrm{gr}_\bullet(i)$ is a sheaf of graded algebras on the sheaf of rings $\mathcal{O}_Y = \mathrm{gr}_0(i)$ and $\mathrm{gr}_k(i)$ are $\mathcal{O}_Y$-modules.

Our aim was to prove that the sheaves $Y^{(k)}$ and $\mathrm{Inf}^k_Y(X)$ are the same when $i : Y \hookrightarrow X$ is an immersion. We certainly have a monomorphism $Y^{(k)} \to \mathrm{Inf}^k_Y(X)$. To show that this is an isomorphism we must show that for any affine scheme T over S the map $\Gamma(T,\ Y^{(k)}) \to \Gamma(T,\ \mathrm{Inf}^k_Y(X))$ is surjective. Thus we are reduce to showing that if $T = \mathrm{Spec}(A)$, $T' = \mathrm{Spec}(A')$ with $A \to A'$ faithfully flat, $J \subseteq A'$ is an ideal with $J^{k+1} = (0)$, and $\phi : T \to X$ is such that the composite map $\mathrm{Spec}(A'/J) \hookrightarrow T' \to T \to X$ factors through Y, then ϕ factors through $Y^{(k)}$. Let ψ denote the morphism $A \to A'$ corresponding to $T' \to T$. Then the hypothesis tells us that $I.A \subseteq \psi^{-1}(J)$. But $A \to A'$ being faithfully flat implies that it is injective, and hence $\psi^{-1}(J)$ has its $(k+1)^{\mathrm{st}}$ power equal to (0). But this certainly implies that $\phi : T \to X$ factors through $Y^{(k)}$.

It is easy to verify that formation of infinitesimal neighborhoods is compatible under base change i.e.

$$\mathrm{Inf}^k_{Y_{S'}}(X_{S'}) = (\mathrm{Inf}^k_Y(X))_{S'}.$$

Let X be a sheaf on S and $e_X : S \to X$ be a section. If it is understood that X is given together with a section, then we will write $\mathrm{Inf}^k(X)$ rather than $\mathrm{Inf}^k_S(X)$.

Definition 2.1.2. A pointed sheaf $(X,\ e_X)$ as above is ind-infinitesimal if

$$X = \varinjlim_k \mathrm{Inf}^k(X).$$

Remark 2.1.3. It follows immediately from the definition that $\mathrm{Inf}^k(X) =$
$\mathrm{Inf}^k(\mathrm{Inf}^{k+i}(X))$ for any $i \geq 0$.

Definition 2.1.4. Let $X \to S$ be a morphism of schemes; and for i take the diagonal morphism $\delta_{X/S} : X \to X \times_S X$. The quasi-coherent $\mathcal{O}_X$-module ω_i is then denoted by $\Omega^1_{X/S}$ and called the module of differntials of X over S.

If ω_{e_X} denotes the conormal sheaf of the immersion $e_X : S \hookrightarrow X$, we have $\omega_{e_X} \tilde{=} e^*_X(\Omega^1_{X/S})$ ([G.A., I §4,2.2]). From now we will denote ω_{e_X} by ω_X. In some text it has been denoted by $\mathrm{Lie}^*(X/S)$.

The following definition and its explanation is taken from Messing [14].

Definition 2.1.5. [14, Definition 1.1.4] A pointed sheaf $(X,\ e_X)$ on S is said to be a formal Lie variety if the following conditions are satisfied:

1. X is ind-infinitesimal and $\mathrm{Inf}^k(X)$ is representable for all $k \geq 0$.
2. $\omega_X = e^*_X(\Omega^1_{\mathrm{Inf}^1(X)/S}) = e^*_X(\Omega^1_{\mathrm{Inf}^k(X)/S})$ is locally free of finite type.
3. Denoting by $\mathrm{gr}^{\mathrm{inf}}(X)$ the unique graded $\mathcal{O}_S$-algebra, such that $\mathrm{gr}^{\mathrm{inf}}_i(X) = \mathrm{gr}_i(\mathrm{Inf}^i(X))$ holds for all $i \geq 0$, we have an isomorphism $\mathrm{Sym}(\omega_X) \tilde{\to} \mathrm{gr}^{\mathrm{inf}}(X)$ induced by the canonical mapping $\omega_X \to \mathrm{gr}^{\mathrm{inf}}_1(X)$

We proceed to translate this definition into more down to earth terms. First Condition 1 and Remark (2.1.3) tell us that for each k $\mathrm{Inf}^k(X)$ is an affine S-scheme and both S and $\mathrm{Inf}^k(X)$ have the same underlying topological space. $\mathrm{Inf}^k(X)$ is given by a quasi-coherent $\mathcal{O}_S$-algebra $\mathfrak{a}_k$ which is augmented:

$$0 \to I_k \to \mathfrak{a}_k \to \mathcal{O}_S \to 0$$

and furthermore $I_k^{k+1} = (0)$. We are also told that $\mathfrak{a}_{k+1}/I_{k+1}^{k+1} \cong \mathfrak{a}_k$, which makes obvious how we define the algebra $\mathrm{gr}^{\mathrm{inf}}(X)$. Condition 2 and 3 imply that locally on S we have

$$\mathfrak{a}_k \xleftarrow{\sim} \mathcal{O}_S[T_1, ..., T_N]/(T_1, ..., T_N)^{k+1},$$

these isomorphisms being compatible [4, Chap.III, §2, #8 Cor.3]. Hence locally on S, X is given by a power series ring $\mathcal{O}_S[[T_1, ..., T_N]]$ in the sense that for an S-scheme S', $\Gamma(S',\ X) \cong \mathrm{Nil}(\mathcal{O}_{S'}) \times ...... \times \mathrm{Nil}(\mathcal{O}_{S'})$ where $\mathrm{Nil}(\mathcal{O}_{S'})$ denotes the locally nilpotent section of $\mathcal{O}_{S'}$. These correspond exactly to the continuous homomorphisms

$$\mathcal{O}_S[[T_1, ..., T_N]] \longrightarrow \Gamma(S',\ \mathcal{O}_{S'})$$

when the latter is given the discrete topology.

Definition 2.1.6. A formal Lie group over S, $(e_G,$ G) is a group in the category of formal Lie varieties.

2.2 Local shtukas

Let ζ be an indeterminant over $\mathbb{F}_q$ and let $\mathbb{F}_q[\![\zeta]\!]$ be the ring of formal power series in ζ over $\mathbb{F}_q$. Let $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ be the category of $\mathbb{F}_q[\![\zeta]\!]$-schemes on which ζ is locally nilpotent. For S in $\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ we let $\mathcal{O}_S[\![z]\!]$ be the sheaf on S of formal power series in the indeterminant z. That is $\Gamma\big(U, \mathcal{O}_S[\![z]\!]\big) = \Gamma(U, \mathcal{O}_S)[\![z]\!]$ for open $U \subset S$ with the obvious restriction maps. This is a sheaf of $\mathcal{O}_S$-algebras on S since the global sections of a direct product of sheaves equal the direct product of their global sections. Moreover the topological space S endowed with the sheaf of rings $\mathcal{O}_S[\![z]\!]$ is a locally ringed space. This follows easily from the fact that z is contained in the Jacobson radical of $\Gamma\big(U, \mathcal{O}_S[\![z]\!]\big)$ for any U.

Lemma 2.2.1. *Let R be an $\mathbb{F}_q[\![\zeta]\!]$-algebra in which ζ is nilpotent. Then the sequence of $R[\![z]\!]$-modules*

$$0 \longrightarrow R[\![z]\!] \longrightarrow R[\![z]\!] \longrightarrow R \longrightarrow 0$$

$$1 \longmapsto z - \zeta \ \ , \ \ z \longmapsto \zeta$$

is exact. In particular $R[\![z]\!] \subset R[\![z]\!][\frac{1}{z-\zeta}]$.

Proof. If $\sum_i b_i z^i$ lies in the kernel of the first map then $b_i = \zeta b_{i+1} = \zeta^n b_{i+n}$ for all n. Since ζ is nilpotent all b_i are zero. Also due to the nilpotency of ζ the second map is well defined and surjective. For exactnes in the middle note that $\sum_i b_i(z^i - \zeta^i)$ is a multiple of $z - \zeta$. □

For $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ let $\mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}]$ be the sheaf associated to the presheaf

$$U \longmapsto \Gamma\big(U, \mathcal{O}_S[\![z]\!]\big)[\tfrac{1}{z-\zeta}].$$

Note that this presheaf is already a sheaf if S is quasi-compact. We denote by σ^* the endomorphism of $\mathcal{O}_S[\![z]\!]$ that acts as the identity on z and as $b \mapsto b^q$ on the elements $b \in \mathcal{O}_S$. For a sheaf M of $\mathcal{O}_S[\![z]\!]$-modules on S we let $\sigma^* M := M \otimes_{\mathcal{O}_S[\![z]\!],\sigma^*} \mathcal{O}_S[\![z]\!]$ and $M[\frac{1}{z-\zeta}] := M \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}]$ be the tensor product sheaves.

Definition 2.2.2. A *local shtuka* of rank(or height) r over S is a pair (M, F_M) consisting of

- a sheaf M of $\mathcal{O}_S[\![z]\!]$-modules on S, which, Zariski-locally on S, is a free $\mathcal{O}_S[\![z]\!]$-module of rank r, and
- an isomorphism $F_M : \sigma^* M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$.

A *morphism* of local shtukas $f : (M, F_M) \to (M', F_{M'})$ over S is a morphism of the underlying sheaves $f : M \to M'$ which satisfies $F_{M'} \circ \sigma^* f = f \circ F_M$.

Note that if (M, F_M) is a local shtuka, then $(M,\ (z-\zeta)^n \cdot F_M)$ is also local shtuka for every $n \in \mathbb{Z}$.

Lemma 2.2.3. *Let (M, F_M) be a local shtuka over S. Then locally on S there are $e, d, N' \in \mathbb{Z}$ s.t. $(z-\zeta)^d M \subseteq F_M(\sigma^* M) \subseteq (z-\zeta)^{-e} M$ and $z^{N'} M \subseteq F_M(\sigma^* M)$. For any such e the quotient $(z-\zeta)^{-e} M / F_M(\sigma^* M)$ is a locally free $\mathcal{O}_S$-module of finite rank.*

Proof. Locally on S, we can assume that $\sigma^* M$ and M are free $\mathcal{O}_S[\![z]\!]$-module of rank r for some r. Let $X_i,\ 1 \le i \le r$ be the basis of $\sigma^* M$ over $\mathcal{O}_S[\![z]\!]$. Since $F_M : \sigma^* M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$ is an isomorphism implies

$$F_M(X_i) = m_i/(z-\zeta)^{e_i},\ 1 \le i \le r$$

for some $m_i \in M$ and $e_i \in \mathbb{Z}$. Take $e = \max\{e_i\}$, then $F_M(\sigma^* M) \subseteq (z-\zeta)^{-e} M$. Let $Y_i,\ 1 \le i \le r$ be the basis of M over $\mathcal{O}_S[\![z]\!]$. Since $F_M : \sigma^* M[\frac{1}{z-\zeta}] \xrightarrow{\sim} M[\frac{1}{z-\zeta}]$ is an isomorphism implies Y_i is of the form $F_M(m'_i/(z-\zeta)^{d_i})$ for some $m'_i \in \sigma^* M$ and $d_i \in \mathbb{Z}$ i.e. $(z-\zeta)^{d_i} Y_i = F_M(m'_i)$. Take $d = \max\{d_i\}$, then $(z-\zeta)^d M \subseteq F_M(\sigma^* M)$. Since locally on S, ζ is nilpotent, take N' be the smallest integer which is power of p and greater than N and d, then $z^{N'} M \subseteq F_M(\sigma^* M)$.

From now the map F_M we mean $F_{M|\sigma^* M}$. Now we prove that $F_M : \sigma^* M \to (z-\zeta)^{-e} M$ is injective. The injectivity of F_M is equivalent to the injectivity of $(z-\zeta)^e F_M$. Now $(z-\zeta)^e F_M$ is injective by the lemma (2.2.1) and the following diagram.

$$\begin{array}{ccc} \sigma^* M & \hookrightarrow & \mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}] \otimes_{\mathcal{O}_S[\![z]\!]} \sigma^* M \\ {\scriptstyle (z-\zeta)^e F_{M|\sigma^* M}} \downarrow & & \downarrow \cong \\ M & \hookrightarrow & \mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}] \otimes_{\mathcal{O}_S[\![z]\!]} M \end{array}$$

In the above diagram the horizontal rows are injective because σ^*M and M are flat over $\mathcal{O}_S[\![z]\!]$ and the right vertical arrow is an isomorphism because (M, F_M) is local shtuka.

Now to prove $(z-\zeta)^{-e}M/F_M(\sigma^*M) = \operatorname{coker} F_M$ is locally free over S, we will prove first it is of finite presentation and flat over S. Since $z^{N'}M \subseteq F_M(\sigma^*M)$ i.e. $z^{N'} \operatorname{coker} F_M = 0$ implies

$$\operatorname{coker} F_M = \operatorname{coker}(F_M \bmod z^{N'} : \sigma^*M/z^{N'}\sigma^*M \to M/z^{N'}M)$$

i.e. $\operatorname{coker} F_M$ is of finite presentation over S.

Locally on S, we can assume that S is affine, say $S = \operatorname{Spec} R$. Let $\mathfrak{m} \subset R$ be a maximal ideal and $\kappa = R/\mathfrak{m}$. Now we consider the exact sequence

$$0 \to \operatorname{Tor}_1^{R[[z]]}(\kappa[[z]], \operatorname{coker} F_M) \to \sigma^*M/\mathfrak{m}\sigma^*M \to (z-\zeta)^e M/\mathfrak{m}(z-\zeta)^e M \\ \to \operatorname{coker} F_M \otimes_{R[[z]]} \kappa[[z]] \to 0.$$

As M is locally free of rank r we have $\sigma^*M/\mathfrak{m}\sigma^*M \cong \kappa[[z]]^{\oplus r} \cong (z-\zeta)^e M/\mathfrak{m}(z-\zeta)^e M$. Now $\operatorname{coker} F_M \otimes_{R[[z]]} \kappa[[z]]$ is torsion because $\operatorname{coker} F_M$ is killed by $z^{N'}$. So the map $\sigma^*M/\mathfrak{m}\sigma^*M \to (z-\zeta)^e M/\mathfrak{m}(z-\zeta)^e M$ is injective by the elementary divisor theorem. It gives us $\operatorname{Tor}_1^{R[[z]]}(\kappa[[z]], \operatorname{coker} F_M) = \operatorname{Tor}_1^{R[z]}(\kappa[z], \operatorname{coker} F_M) = \operatorname{Tor}_1^{R}(\kappa, \operatorname{coker} F_M) = 0$. Hence from Nakayama lemma $\operatorname{coker} F_M$ is flat over S. Hence $(z-\zeta)^{-e}M/F_M(\sigma^*M)$ is locally free over $\mathcal{O}_S$ of finite rank. □

Definition 2.2.4. A local shtuka (M, F_M) over S is called *effective* if F_M is actually a morphism $F_M : \sigma^*M \to M$. Let (M, F_M) be effective. We say that

1. (M, F_M) has *dimension* d if $\operatorname{coker} F_M$ is locally free of rank d as an $\mathcal{O}_S$-module.
2. (M, F_M) is *étale* if $F_M : \sigma^*M \xrightarrow{\sim} M$ is an isomorphism.
3. F_M is *topologically nilpotent* if $\operatorname{im} F_M^n \subset zM$ locally on S for all large enough integers n.

We define the *tensor product* of two local shtukas (M, F_M) and (N, F_N) over X as the local shtuka

$$\big(M \otimes_{\mathcal{O}_X[\![z]\!]} N\,,\, F_M \otimes F_N\big)\,.$$

The local shtuka $(\mathcal{O}_X[\![z]\!], F = \mathrm{id})$ is a *unit object* for the tensor product. Also there is a natural definition of *internal Hom's*. In particular the *dual* $(M^\vee, F_{M^\vee})$ of a local shtuka (M, F_M) over X is defined as the sheaf $M^\vee = \mathcal{H}om_{\mathcal{O}_X[\![z]\!]}\big(M, \mathcal{O}_X[\![z]\!]\big)$ together with

$$F_{M^\vee} = (\,.\circ F_M^{-1}) : \sigma^*M^\vee[\tfrac{1}{z-\zeta}] \xrightarrow{\sim} M^\vee[\tfrac{1}{z-\zeta}]\,.$$

Example 2.2.5. Every effective local shtuka (M, F) over S yields for every $n \in \mathbb{N}$ a finite shtuka $\big(M/z^nM, F \bmod z^n\big)$ and (M, F) equals the projective limit of these finite shtukas.

2.3 Divisible local Anderson-modules

Let z be an indeterminant over $\mathbb{F}_q$ and suppose that S is a scheme over $\operatorname{Spec}\mathbb{F}_q[z]$. We denote the image of z in $\mathcal{O}_S$ by ζ. If G is a group on S which is also an $\mathbb{F}_q[z]$-module. We denote $\ker z^n$ by $G(n)$.

For the next lemma suppose that $G = G(n)$.

Lemma 2.3.1. *The following conditions are equivalent*
(i) *G is a flat $\mathbb{F}_q[z]/z^n$-module*
(ii) $\operatorname{Ker}(z^{n-i}) = \operatorname{Im}(z^i)$ *for $i = 0, ..., n$.*

Proof. : First we show that (i) implies (ii). From (i) it follows that $\operatorname{gr}^\bullet(\mathbb{F}_q[z]/z^n) \otimes_{\mathbb{F}_q} \operatorname{gr}^0(G) \cong \operatorname{gr}^\bullet(G)$ (the associated graded group being taken with respect to the filtration defined by powers of z). Because of this we know that z^i induces an isomorphism from G/zG to $z^iG/z^{i+1}G$ for $i \leq n-1$. This gives us $\operatorname{Ker}(z^{n-1}) \subseteq \operatorname{Im}(z)$ i.e. $\operatorname{Ker}(z^{n-1}) = \operatorname{Im}(z)$ so we have the result for $i = 1$. Now $\operatorname{Ker}(z^{n-i}) \subseteq \operatorname{Ker}(z^{n-1}) \subseteq \operatorname{Im}(z)$ which implies that $\operatorname{Ker}(z^{n-i}) = z\operatorname{Ker}(z^{n-i+1}) = z \cdot z^{i-1}G = z^iG$ (by induction on i).

To prove that (ii) $\implies$ (i), we observe by taking $i = 1$ that $\operatorname{Im}(z) = \operatorname{Ker}(z^{n-i})$ and hence that z^{n-1} induces an isomorphism $G/zG \xrightarrow{\sim} z^{n-1}G$. Since this map factors through the epimorphisms $G/zG \to zG/z^2G \to \cdots \to z^{n-1}G$ we see that each of these maps is an isomorphism.Thus we have

$$\operatorname{gr}^\bullet(\mathbb{F}_q[z]/z^n) \otimes_{\mathbb{F}_q} \operatorname{gr}^0(G) \xrightarrow{\sim} \operatorname{gr}^\bullet(G). \tag{2.1}$$

For simlicity we will write $A = \mathbb{F}_q[z]/z^n$, $I = (z)$. Note that here the ideal $I = (z)$ is nilpotent. Since $A/I = \mathbb{F}_q$, so G/IG is flat over A/I. Equivalence of condition (i) and (iv) of Theorem 1 in [4, chap III, §5.2] implies that G is a flat $\mathbb{F}_q[z]/z^n$-module because we have 2.1 and G/IG is flat over A/I. □

Definition 2.3.2. Let $d \in \mathbb{N}_{>0}$. A truncated z-divisible local Anderson-module of level n and order of nilpotence d is an S-group such that:

1. If $n \geq 2d$ it is a finite $\mathbb{F}_q[z]/z^n$-module scheme G which is strict as $\mathbb{F}_q$-module with $(z-\zeta)^d$ homotopic to 0 on $L^{G/S}_{\cdot}$ and satisfies the equivalent condition of Lemma (2.3.1).
2. If $n < 2d$ it is of the form $\operatorname{Ker}(z^n : G \to G)$ for some truncated z-divisible local Anderson module G of level $2d$ with $(z-\zeta)^d$ homotopic to 0 on $L^{G/S}_{\cdot}$.

Definition 2.3.3. A z-divisible local Anderson module over S is a sheaf of $\mathbb{F}_q[\![z]\!]$-modules G on the big fppf-site of S such that for each integer $n \geq 1$:

1. G is of z-torsion i.e. $G = \varinjlim_n G(n)$,
2. G is z-divisible i.e. $z : \ G \to G$ is an epimorphism,

3. The $\mathbb{F}_q$-modules $G(n)$ are representable by finite locally free group schemes over S and are strict.

4. Locally on S there exist a constant $d \in \mathbb{Z}_{\geq 0}$, such that $(z-\zeta)^d = 0$ on ω_G where $\omega_G = \varprojlim_n \omega_{G(n)}$ cf.2.3.4.

Definition 2.3.4. Let G be a z-divisible local Anderson-module over S. We define the *co-Lie module of* G as

$$\omega_G \;:=\; \varprojlim_n \omega_{G(n)}.$$

We will see latter (Thm 2.4.3 or Thm 2.5.19) that ω_G is a finite locally free $\mathcal{O}_S$-module and we define the *dimension of* G as $\operatorname{rk} \omega_G$.

A z-divisible local Anderson module is called étale if $\omega_G = 0$. Since $\omega_G \twoheadrightarrow \omega_{G(n)}$, this is the case if and only if all $G(n)$ are étale. It is strict (here we strict mean $\mathbb{F}_q[\![z]\!]$-strict) if and only if z acts via scalar multiplication by ζ on ω_G.

Next we want to give an example for divisible local Anderson-modules.

Example 2.3.5. Let C be a smooth projective geometrically irreducible curve over $\operatorname{Spec} \mathbb{F}_q$ and let $\infty \in C$ be a closed point. Put $A := \Gamma(C \smallsetminus \{\infty\}, \mathcal{O}_C)$. Let $c : S \to \operatorname{Spec} A$ be a morphism of $\mathbb{F}_q$-schemes. A *Drinfeld-Anderson A-module of dimension d and rank r over S* is an A-module scheme E over S whose underlying $\mathbb{F}_q$-module scheme is a vector group scheme of dimension d such that the following conditions hold:

a) $(\operatorname{Lie}^* a - c^* a)^d = 0$ on ω_E for every $a \in A$,

b) the Zariski sheaf $\mathcal{H}om_{\mathbb{F}_q, S}(E, \mathbb{G}_{a,S})$ of $\mathbb{F}_q$-linear homomorphisms on S, equipped with the action of A by composition on the right is a locally free $A \otimes_{\mathbb{F}_q} \mathcal{O}_S$-module of finite rank r.

In case $d = 1$ this is also called a *Drinfeld module over S*.

Let $z \in A$. For every Drinfeld-Anderson A-module over S we can associate a z-divisible local Anderson-module as

$$G := \varinjlim_n E[z^n],$$

where $E[z^n] = \operatorname{Ker}(z^n : E \to E)$.

Its height equals the rank of the module E. If moreover $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ the dimension of G equals the dimension of E; cf.2.3.4

Remark 2.3.6 (on axiom 4)**.** Note the following difference to the theory of p-divisible groups. On a (finite) group scheme multiplication by p always induces multiplication with the scalar p on its co-Lie module. In the case of (finite) $\mathbb{F}_q[\![z]\!]$-module schemes, axiom 4 is the appropriate substitute for this fact, taking into account Example 2.3.5. It allows that $z - \zeta$ is nilpotent on $\omega_{G(n)}$.

Notation 2.3.7. Let G be a z-divisible local Anderson module. We denote by i_m the inclusion map $G(m) \hookrightarrow G(m+1)$ and by $i_{m,n} : G(m) \to G(m+n)$ the composite of the inclusions $i_{m+n-1} \circ \ldots \circ i_m$. We denote by $j_{m,n}$ the unique homomorphism $G(m+n) \to$

$G(n)$ which is induced by multiplication with z^m on $G(m+n)$ such that $i_{n,m} \circ j_{m,n} = z^m \operatorname{id}_{G(m+n)}$. Note here that $j_{m,n}$ is surjective because $z: \; G \to G$ is an epimorphism.

Remark 2.3.8. Let G be a z-divisible local Anderson module.

1. $G(n) = G(n+1)(n)$.

2. For any i such that $0 \leq i \leq n$, z^{n-i} induces an epimorphism $G(n) \to G(i)$. So we have exact sequences:

$$0 \to G(n-i) \xrightarrow{i_{n-i,i}} G(n) \xrightarrow{j_{n-i,i}} G(i) \to 0.$$

   Hence $G(n)$ satisfies the equivalent condition of lemma (2.3.1).

3. From the theory of finite group schemes over a field that the rank of the fiber of $G(1)$ at a point $s \in S$ is of the form $q^{h(s)}$ where h is a locally constant function on S. It also follows from remark 2) that the rank of the fiber of $G(n)$ at s is $q^{nh(s)}$ [G.A., Chap IV §3, 5], we say h is the height of the z-divisible local Anderson module.

4. Assume we have a system of groups $G(n)$ with $G(n)$ finite, locally free and strict $\mathbb{F}_q$ such that

   a) $G(n) = G(n+1)(n)$

   b) The rank of the fiber of $G(n)$ at s is $q^{nh(s)}$ where h is a locally constant function on S

   c) Locally on S there exist a constant $d \in \mathbb{Z}_{\geq 0}$, such that $(z-\zeta)^d = 0$ on ω_G where $\omega_G = \varprojlim_n \omega_{G(n)}$.

   We consider the exact sequence

$$0 \to G(n-i) \to G(n) \xrightarrow{z^{n-i}} G(i).$$

   By looking at each fiber and using the multiplicativity of the ranks $G(n)_s \to G(i)_s$ is faithfully flat. Therefore since $G(n)$ is flat over S, it follows that $G(n) \xrightarrow{z^{n-i}} G(i)$ is faithfully flat and hence an epimorphism. Thus we see that $G = \varinjlim_n G(n)$ is divisble local Anderson module.

Note that also we have to bound the order of nilpotency due to the following example which we do not want to consider a divisible local Anderson-module.

Example 2.3.9. Let S be the spectrum of a field L in which ζ is zero, and let $G(n)$ be the subgroup of $\mathbb{G}_a^n = \operatorname{Spec} L[x_1, \ldots, x_n]$ defined by the ideal $(x_1^q, \ldots, x_n^q)$. Make $G(n)$ into an $\mathbb{F}_q[\![z]\!]$-module scheme by letting z act through

$$z^*(x_1) = 0 \qquad \text{and} \qquad z^*(x_\nu) = x_{\nu-1} \quad \text{for } 1 < \nu \leq n\,.$$

Define $i_n : G(n) \to G(n+1)$ as the inclusion of the closed subgroup scheme defined by the ideal (x_{n+1}).

Then the inductive system $(G(n),\ i_n)$ satisfies axioms 1 to 3. But it does not satisfy 4. Therefore we can not drop the condition (c) in Remark 2.3.8/4.

2.4 The Local Equivalence

The category of z-divisible local Anderson-modules over S and the category of local shtukas over S are both $\mathbb{F}_q[\![z]\!]$-linear. Our next aim is to extend Drinfeld's construction and the equivalence from Section 1.6 to an equivalence between the category of effective local shtukas over S and the category of z-divisible local Anderson-modules over S.

For every effective local shtuka (M, F) over S we set

$$\mathrm{Dr}_q(M,F) \;:=\; \varinjlim_n \mathrm{Dr}_q\big(M/z^nM, F \bmod z^nM\big)$$

and for every z-divisible local Anderson-module $G = \varinjlim_n G(n)$ over S we set

$$\underline{M}(G) \;=\; \big(M(G), F_{M(G)}\big) \;:=\; \varprojlim_n \big(M(G(n)), F_{M(G(n))}\big)\,.$$

Multiplication with z on G gives $M(G)$ the structure of an $\mathcal{O}_S[\![z]\!]$-module.

Lemma 2.4.1. *Let $G = \varinjlim_n G(n)$ be a z-divisible local Anderson-module of height r over S then $M(G)$ is locally on S a free $\mathcal{O}_S[\![z]\!]$-module of rank r.*

Proof. Let $G = \varinjlim_n G(n)$ be a z-divisible local Anderson-module over S. So we have an exact sequence

$$0 \to G(n) \xrightarrow{i_n} G(n+1) \xrightarrow{z^n} G(n+1).$$

Then we get an exact sequences of $\mathcal{O}_S[\![z]\!]$-modules

$$M(G(n+1)) \;\xrightarrow{z^n}\; M(G(n+1)) \;\xrightarrow{M(i_n)}\; M(G(n)) \;\to\; 0\,.$$

We deduce from [4, Prop 3.2.11/14] that $M(G)$ is a finitely generated $\mathcal{O}_S[\![z]\!]$-module and the canonical map $M(G) \to M(G(n))$ identifies $M(G(n))$ with $M(G)/z^nM(G)$.

We claim that multiplication with z on $M(G)$ is injective. So let $\varprojlim(f_n)_n \in M(G)$, $f_n \in M(G(n))$ with $z \cdot f_n = 0$ in $M(G(n))$ for all n. To prove the claim consider the factorization

$$z \cdot \mathrm{id}_{M(G(n+1))} \;=\; M(j_{1,n}) \circ M(i_{n,1}) : \;\; M(G(n+1)) \;\longrightarrow\; M(G(n+1))$$

obtained from Notation (2.3.7). Since $M(j_{1,n})$ is injective, $f_n = M(i_{n,1})(f_{n+1})$ is zero as desired.

Locally on S the $\mathcal{O}_S$-module $M(G(1))$ is free. By Theorem 1.6.4 its rank is r. Let $m_1, \ldots, m_r$ be representatives in $M(G)$ of an $\mathcal{O}_S$-basis of $M(G(1))$ and consider the presentation

$$0 \to \ker \alpha \to \bigoplus_{i=1}^{r} \mathcal{O}_S[\![z]\!]\, m_i \xrightarrow{\alpha} M(G) \to 0\,.$$

Note that α is surjective by Nakayama's Lemma since z is contained in the radical of $\mathcal{O}_S[\![z]\!]$. Now the snake lemma implies that multiplication with z is surjective on $\ker \alpha$. But this can only be if $\ker \alpha$ is zero. Therefore $M(G)$ is locally on S a free $\mathcal{O}_S[\![z]\!]$-module of rank r. □

Recall the definition $\ell^G_{\cdot} = e^*_G(L^{(G/S)}_{\cdot})$ and in writing the cotangent complex $L^{G/S}_{\cdot}$ so that $\Omega^1_{U(G)^{\times}/S}|G$ is in degree 0 and I/I^2 is in degree -1, we define $\omega_G = \mathrm{H}_0(\ell^G_{\cdot})$, $n_G = \mathrm{H}_{-1}(\ell^G_{\cdot})$. Note that the use of the symbol ω_G is of course consistent with our previous notation. If G be a finite flat commutative group scheme which is a strict $\mathbb{F}_q$-module scheme over S and $\underline{M} = (M(G), F_{M(G)})$ be the finite $\mathbb{F}_q$-shtuka associated to it then $n_G \cong \operatorname{Ker} F_{M(G)}$ and $\omega_G \cong \operatorname{coker} F_{M(G)}$.

Lemma 2.4.2. *Let $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ and $G = \varinjlim_n G(n)$ be a z-divisible local Anderson-module of dimension d over S. Then*
(a) for all $n >> 0$, i_n induces an isomorphism $\omega_{G(n)} \xrightarrow{\sim} \omega_{G(n+1)}$.
(b) $n_{G(n)}$ satisfies the Mittag-Leffler condition.
(c) $\underline{M}(G)$ is an effective local shtuka over S.

Proof. Let $F_{M(G(n))} : \sigma^*(M(G(n))) \to M(G(n))$ is induced by the Frobenius $F^*_{G(n)}$. For the convenience we denote $M(G(n))$ by M_n and $F_{M(G(n))}$ by F_n. Since $i_n : G(n) \to G(n+1)$ induces a surjective map mod $z^n : M_{n+1} \to M_n$ and we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im} F_{n+1} & \longrightarrow & M_{n+1} & \longrightarrow & \operatorname{coker} F_{n+1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \bmod z^n} & & \downarrow{\scriptstyle \bmod z^n} & & \downarrow{\scriptstyle \bmod z^n} & & \\ 0 & \longrightarrow & \operatorname{im} F_n & \longrightarrow & M_n & \longrightarrow & \operatorname{coker} F_n & \longrightarrow & 0. \end{array}$$

The surjectivity of the map mod $z^n : M_{n+1} \to M_n$ induces surjectivity on $\operatorname{coker} F_{n+1} \to \operatorname{coker} F_n$. Let N be the order of nilpotency of ζ and N' be the smallest integer which is power of p and greater than N and d. Since $(z-\zeta)^d = 0$ on ω_G implies $z^{N'} = 0$ on ω_G implies $z^{N'} = 0$ on $\omega_{G(n)}$ for all n. Therefore $z^{N'} = 0$ on $\omega_{G(n)} \cong \operatorname{coker} F_n$ for all n. It implies if $n >> N'$, the map mod z^n : $\operatorname{coker} F_{n+1} \to \operatorname{coker} F_n$ is injective. Hence $\operatorname{coker} F_{n+1} \cong \operatorname{coker} F_n$, if $n >> 0$ i.e. for all $n >> 0$, i_n induces an isomorphism $\omega_{G(n)} \xrightarrow{\sim} \omega_{G(n+1)}$.

To prove (b) at first we fix n. From above lemma we have an exact sequence for all k

$$0 \to M_k \xrightarrow{z^n} M_{n+k} \xrightarrow{\bmod z^n} M_n \to 0.$$

Now for all k, we have commutative diagrams

$$\begin{array}{ccccccccc} 0 \to \ker F_{n+k} \to \sigma^* M_{n+k} \xrightarrow{F'_{n+k}} \operatorname{im} F_{n+k} \to 0 & & 0 \to \operatorname{im} F_{n+k} \xrightarrow{F''_{n+k}} M_{n+k} \to \operatorname{coker} F_{n+k} \to 0 \\ \downarrow \epsilon_k \qquad \downarrow \bmod z^n \qquad \downarrow \gamma_k & & \gamma_k \downarrow \qquad \downarrow \bmod z^n \qquad \downarrow \\ 0 \to \ker F_n \to \sigma^* M_n \xrightarrow[F'_n]{} \operatorname{im} F_n \to 0 & & 0 \to \operatorname{im} F_n \xrightarrow[F''_n]{} M_n \to \operatorname{coker} F_n \to 0. \end{array}$$

In the above commutative diagrams we have splitted $F_n : \sigma^* M_n \to M_n$ in two maps

$$\sigma^* M_n \xrightarrow{F'_n} \operatorname{im} F_n \xrightarrow{F''_n} M_n$$

i.e. $F_n = F''_n \circ F'_n$ and ϵ_k, γ_k are induced maps. In first commutative diagram, from snake lemma we have the following exact sequence

$$\sigma^* M_k \to \operatorname{Ker} \gamma_k \to \operatorname{coker} \epsilon_k \to 0.$$

We will denote the first map by α_k and the second by δ_k. If $n \geq N'$, where N' is as in (a), then we know from (a) that right vertical map in second commutative diagram is an isomorphism and in this case from snake lemma we have

$$\operatorname{Ker} \gamma_k \cong M_k.$$

We will denote this isomorphim by β_k. From above exact sequence and isomorphism we have following exact sequence

$$\sigma^* M_k \xrightarrow{\beta_k \alpha_k} M_k \xrightarrow{\delta_k \beta_k^{-1}} \operatorname{coker} \epsilon_k \to 0.$$

Since we have follwing commutative diagram from above commutative diagrams

$$\begin{array}{ccc} \sigma^* M_k & \xrightarrow{\beta_k \alpha_k} & M_k \\ z^n \big\downarrow & & \big\downarrow z^n \\ \sigma^* M_{n+k} & \xrightarrow{F_{n+k}} & M_k \end{array}$$

and $z^n_{|M_k}$ is injective, so we have $\beta_k \alpha_k = F_k$, hence $\operatorname{coker} F_k \cong \operatorname{coker} \epsilon_k$. Since for all $k \geq N'$, $\operatorname{coker} F_{k+1} \cong \operatorname{coker} \epsilon_k$. Hence for all $k \geq N'$, $\operatorname{coker} \epsilon_{k+1} \cong \operatorname{coker} \epsilon_k$. Since we have exact sequences

$$0 \to \operatorname{im} \epsilon_k \to \operatorname{Ker} F_n \to \operatorname{coker} \epsilon_k \to 0$$

and

$$0 \to \operatorname{im} \epsilon_{k+1} \to \operatorname{Ker} F_n \to \operatorname{coker} \epsilon_{k+1} \to 0.$$

Hence for all $k \geq N'$, $n \geq N'$, $\operatorname{im} \epsilon_k \cong \operatorname{im} \epsilon_{k+1}$. Hence $n_{G(n)} \cong \operatorname{Ker} F_n$ satisfies ML-condition.

To prove (c) we define $F_{M(G)} : \sigma^*(M(G)) \to M(G)$ by taking the limit of $F_{M(G(n))} : \sigma^*(M(G(n))) \to M(G(n))$. The maps $F_n : \sigma^* M_n \to M_n$ gives us two short exact sequences of projective systems

$$0 \to \ker F_n \to \sigma^*(M(G(n))) \to \operatorname{im} F_n \to 0$$

and

$$0 \to \operatorname{im} F_n \to M(G(n)) \to \operatorname{coker} F_n \to 0.$$

For first short exact sequence of projective systems after taking the projective limit we have an exact sequence

$$0 \to \varprojlim_n \ker F_n \to \sigma^*(M(G)) \to \varprojlim_n \operatorname{im} F_n \to 0$$

because $\ker F_n \cong n_{G(n)}$ satisfies the Mittag-Leffler condition. Since the projective system $\{M_n\}$ satisfies the Mittag-Leffler condition locally on S so does $\{\sigma^* M_n\}$ so does $\{\operatorname{im} F_n\}$. Hence after taking the projective limit of second short exact sequence of projective systems we have an exact sequence

$$0 \to \varprojlim_n \operatorname{im} F_n \to M(G) \to \varprojlim_n \operatorname{coker} F_n \to 0.$$

After combining both exact sequences we have an exact sequence

$$0 \to \varprojlim_n \ker F_n \to \sigma^*(M(G)) \to M(G) \to \varprojlim_n \operatorname{coker} F_n \to 0$$

It gives us that $\operatorname{coker} F_{M(G)} = \varprojlim_n \operatorname{coker} F_n$. Now $\omega_G = \varprojlim_n \omega_{G(n)} \cong \varprojlim_n \operatorname{coker} F_n$. So condition (4) of definition (2.3.3) implies that $(z-\zeta)^d$ annihilates $\varprojlim_n \operatorname{coker} F_n$. This will prove that the map

$$\sigma^*(M(G)) \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}] \longrightarrow M(G) \otimes_{\mathcal{O}_S[\![z]\!]} \mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}]$$

is surjective and in this map both modules are locally free over $\mathcal{O}_S[\![z]\!][\frac{1}{z-\zeta}]$ of the same rank so the above map is an isomorphism by Nakayama's Lemma. Hence $\underline{M}(G)$ is a local shtuka. □

Theorem 2.4.3. *Let $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$.*

1. *The two contravariant functors Dr_q and $\underline{M}$ are mutually quasi-inverse anti-equivalences between the category of effective local shtukas over S and the category of z-divisible local Anderson-modules over S. Both functors are $\mathbb{F}_q[\![z]\!]$-linear.*
2. *Both functors preserve étale objects and map short exact sequences to short exact sequences.*

Let (M, F) be an effective local shtuka over S. Then

3. *the z-divisible local Anderson-module $Dr_q(M, F)$ is a formal Lie group if and only if F is topologically nilpotent.*

4. *the height and dimension of the z-divisible local Anderson-module $Dr_q(M,F)$ equal the rank and dimension of (M,F).*

5. *the $\mathcal{O}_S[\![z]\!]$-modules $\omega_{Dr_q(M,F)}$ and $\operatorname{coker} F$ are canonically isomorphic.*

Proof. Let (M,F) is effective local shtuka over S and $G = \mathrm{Dr}_q(M,F) := \varinjlim_n \mathrm{Dr}_q\big(M/z^nM,\ F \bmod z^nM\big)$. For convenience of notation we denote M/z^nM by M_n and $\mathrm{Dr}_q\big(M/z^nM\big)$ by G_n . Since we have an exact sequence

$$M_{n+k} \xrightarrow{z^n} M_{n+k} \to M_n \to 0$$

induced by multiplication by z^n on M_{n+k} for all $n,k \in \mathbb{N}$. Then the functor Dr_q induce exact sequences of finite flat commutative group schemes

$$0 \to G_n \to G_{n+k} \xrightarrow{z^n} G_{n+k}$$

for all $n,k \in \mathbb{N}$. This implies that $G_n = \ker(z^n : G_{n+k} \to G_{n+k})$. Now we prove that $G_n = G(n)$ which will prove $G = \varinjlim_n G(n)$. Clearly $G_n \subseteq G(n)$. If $x \in G(n)(S)$ i.e. $x \in \varinjlim_n G_m(S)$ and $z^nx = 0$. Locally on S, we can assume that S is quasi compact, so we can find m such that $x \in G_m(S),\ z^nx = 0 \implies x \in G_n(S)$.

Now from the above exact sequence

$$M_{n+k} \xrightarrow{z^n} M_{n+k} \to M_n \to 0$$

we have induced injective map $z^n : M_k \to M_{n+k}$ implies we have a surjective $z^n : G_{n+k} \to G_k$ implies $z : G \to G$ is epimorphism.

Since $G(n) = \mathrm{Dr}_q\big(M/z^nM, F \bmod z^nM\big)$ is finite strict $\mathbb{F}_q$-module scheme over S by definiton of functor Dr_q.

By lemma 2.2.3 we know that locally on S there exist $d, N' \in \mathbb{N}$ such that $(z-\zeta)^d = 0$ on $\operatorname{coker} F_M$ and $z^{N'} = 0$ on $\operatorname{coker} F_M$. Now

$$\omega_G := \varprojlim_n \omega_{G_n} = \varprojlim_n \operatorname{coker}(F_M \bmod z^n) = \operatorname{coker}(F_M \bmod z^{N'}) = \operatorname{coker} F_M$$

implies $(z-\zeta)^d = 0$ on ω_G. The last equality comes from applying snake lemma to following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma^*M & \longrightarrow & M & \longrightarrow & \operatorname{coker} F_M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle z^{N'}} & & \downarrow{\scriptstyle z^{N'}} & & \downarrow{\scriptstyle z^{N'}} & & \\ 0 & \longrightarrow & \sigma^*M & \longrightarrow & M & \longrightarrow & \operatorname{coker} F_M & \longrightarrow & 0. \end{array}$$

where vertical arrows are the maps multiplication by $z^{N'}$. Hence $G = \varinjlim_n G_n$ is z-divisible local Anderson-module.

If G is z-divisible local Anderson module, from lemma (2.4.1) and lemma (2.4.2), $\underline{M}(G)$ is effective local shtuka. Both functors $\mathbb{F}_q[\![z]\!]$-linear by definition of functor. It proves 1).

From above when M is effective local shtuka we have prove that $\omega_G = \operatorname{coker} F_M$ which proves that Dr_q sends étale objects to étale objects and in proof of (c) in the lemma 2.4.2 we have prove that if G is z-divisible local Anderson module then $\operatorname{coker} F_M = \omega_G$ which proves that functor $\underline{M}$ sends étale objects to étale objects.

Now we prove (2). Let $0 \to M'' \to M \to M' \to 0$ be a short exact sequence of effective local shtukas. The exactness of $0 \to M'' \to M \to M' \to 0$ implies that exacness of $0 \to M''_n \to M_n \to M'_n \to 0$ which gives us exactness of $0 \to G'(n) \to G(n) \to G''(n) \to 0$, where $G = \mathrm{Dr}_q(M, F_M),\ G' = \mathrm{Dr}_q(M', F_{M'}),\ G' = \mathrm{Dr}_q(M', F_{M'}),\ G'' = \mathrm{Dr}_q(M'', F_{M''})$. Since direct limits in the category of modules is an exact functor implies $0 \to G' \to G \to G'' \to 0$ is exact. Let $0 \to G'' \to G \to G' \to 0$ be a short exact sequence z-divisible local Anderson module. The exactness of sequence $0 \to G'' \to G \to G' \to 0$ follows the exactness of $0 \to G''(n) \to G(n) \to G'(n) \to 0$ implies the exactness $0 \to M'_n \to M_n \to M''_n \to 0$, where $M = \underline{M}(G),\ M' = \underline{M}(G'),\ M'' = \underline{M}(G'')$. Since $\{M'_n\}$ satisfy the Mittag-Leffler condition assures the exacness of $0 \to M' \to M \to M'' \to 0$.

Now we prove 3). Let $G = \mathrm{Dr}_q(F, M)$. In proposition (2.5.23) we will see that G is formal lie group if and only if for all n $G(n)$ is radiciel and from theorem (1.6.4) $G(n)$ is radiciel if and only if $F_{G(n)}$ is nilpotent locally on S. If $x \in G(S)$, locally on S we can find $m \in \mathbb{N}$ such that $x \in G(m)(S)$ implies G is formal lie group if and only if F is nilpotent locally on S. The proof of 4) and 5) follows from the proof of lemmas (2.4.1) and (2.4.2). □

Lemma 2.4.4. *Let M be a effective local shtuka with $(z-\zeta)^d = 0$ on* $\operatorname{coker} F_M$. *Then $\exists V_M : M \to \sigma^* M$ with*

$$
\begin{aligned}
F_M \circ V_M &= (z-\zeta)^d|_M, \\
V_M \circ F_M &= (z-\zeta)^d|_{\sigma^* M}.
\end{aligned}
$$

.

Proof. Since F_M is injective and $(z-\zeta)^d = 0$ on $\operatorname{coker} F_M$, then the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \sigma^* M & \xrightarrow{F_M} & M & \longrightarrow & \operatorname{coker} F_M & \longrightarrow & 0 \\
 & & \downarrow {\scriptstyle (z-\zeta)^d} & \swarrow {\scriptstyle V_M} & \downarrow {\scriptstyle (z-\zeta)^d} & & \downarrow {\scriptstyle (z-\zeta)^d} & & \\
0 & \longrightarrow & \sigma^* M & \xrightarrow[F_M]{} & M & \longrightarrow & \operatorname{coker} F_M & \longrightarrow & 0
\end{array}
$$

implies that there exist $V_M : \ M \to \sigma^* M$ with

$$
\begin{aligned}
F_M \circ V_M &= (z-\zeta)^d|_M, \\
V_M \circ F_M &= (z-\zeta)^d|_{\sigma^* M}.
\end{aligned}
$$

□

Corollary 2.4.5. Let G be a z-divisible local Anderson module over S with $\zeta = 0$ on S such that $(z-\zeta)^d = 0$ on ω_G. Then $\exists V_G : G^{(q)} \to G$ with

$$\begin{aligned} F_G \circ V_G &= z^d|_{G^{(q)}}, \\ V_G \circ F_G &= z^d|_G. \end{aligned}$$

Proof. Since G is z-divisible local Anderson module over S, then $\underline{M}(G) = (M(G), F_{M(G)})$ is effective local shtuka over S. Since $(z-\zeta)^d = 0$ on ω_G and $\omega_G \cong \operatorname{coker} F_{M(G)}$, so from above lemma there exist $V_{M(G)} : \; M(G) \to \sigma^* M(G)$ with

$$\begin{aligned} F_{M(G)} \circ V_{M(G)} &= (z-\zeta)^d|_{M(G)}, \\ V_{M(G)} \circ F_{M(G)} &= (z-\zeta)^d|_{\sigma^* M(G)}. \end{aligned}$$

To have $V_G = \mathrm{Dr}_q(V_{M(G)}) : G^{(q)} \to G$ we need that $V_{M(G)}$ is a morphism $V_{M(G)} : (M(G), F_{M(G)}) \to (\sigma^* M(G), \sigma^* F_{M(G)})$ i.e. the following diagram is

$$\begin{array}{ccc} \sigma^* M(G) & \xrightarrow{\sigma^* V_{M(G)}} & (\sigma^*)^2 M(G) \\ \downarrow{\scriptstyle F_{M(G)}} & & \downarrow{\scriptstyle \sigma^* F_{M(G)}} \\ M(G) & \xrightarrow{V_{M(G)}} & \sigma^* M(G) \end{array}$$

commutative i.e. $V_{M(G)} \circ F_{M(G)} = \sigma^* F_{M(G)} \circ \sigma^* V_{M(G)}$ i.e. $(z-\zeta)^d|_{\sigma^* M(G)} = \sigma^*((z-\zeta)^d|_{M(G)}) = (z-\zeta^q)^d|_{\sigma^* M(G)}$, so we need $\zeta = \zeta^q = \zeta^{q^2} = \cdots = 0$. Hence we can define V_G because we have $\zeta = 0$ on S and in this case we have

$$\begin{aligned} F_G \circ V_G &= z^d|_{G^{(q)}}, \\ V_G \circ F_G &= z^d|_G. \end{aligned}$$

□

Remark 2.4.6. Every z-divisible local Anderson-module G gives a truncated z-divisible local Anderson module of level n as $G(n) := \operatorname{Ker}(z^n : G \to G)$.

Proof. If G is a z-divisible local Anderson-module then $G(n)$ ofcourse satisfy the equivalent condition of lemma (2.3.1). Let $M(G)$ be the effective local shtuka associated to G and $M(G(n))$ be the finite shtuka associated to $G(n)$. Since locally on S there exist $d \in \mathbb{Z}$ such that $(z-\zeta)^d = 0$ on $\operatorname{coker} F_{M(G)}$ implies there exist $V_{M(G)} : \; M(G) \to \sigma^* M(G)$ with

$$
\begin{aligned}
F_{M(G)} \circ V_{M(G)} &= (z-\zeta)^d|_{M(G)}, \\
V_{M(G)} \circ F_{M(G)} &= (z-\zeta)^d|_{\sigma^* M(G)}.
\end{aligned}
$$

If $n \geq 2d$ we can restrict $F_{M(G)}$ and $V_{M(G)}$ to $\sigma^* M(G(n))$ and $M(G(n))$ respectively. This implies we have maps $F_{M(G(n))}$ and $V_{M(G(n))}$ with

$$
\begin{aligned}
F_{M(G(n))} \circ V_{M(G(n))} &= (z-\zeta)^d|_{M(G(n))}, \\
V_{M(G(n))} \circ F_{M(G(n))} &= (z-\zeta)^d|_{\sigma^* M(G(n))}.
\end{aligned}
$$

Existence of $V_{M(G(n))}$ with above condition implies that $(z-\zeta)^d$ is homotopic to zero on $L_{\cdot}^{G(n)/S}$. If $n < 2d$, then there is nothing to prove.

□

Corollary 2.4.7. Let G be truncated z-divisible local Anderson module with order of nilpotence d over S with $\zeta = 0$ on S. Then $\exists V_G : G^{(q)} \to G$ with

$$
\begin{aligned}
F_G \circ V_G &= z^d|_{G^{(q)}}, \\
V_G \circ F_G &= z^d|_G.
\end{aligned}
$$

Proof. Let G be truncated z-divisible local Anderson module with order of nilpotence d with $\zeta = 0$ on S. Since z^d is homotopic to 0 on $L_{\cdot}^{G/S}$ implies there exists $V_G : G^{(q)} \to G$ with

$$
\begin{aligned}
F_G \circ V_G &= z^d|_{G^{(q)}}, \\
V_G \circ F_G &= z^d|_G.
\end{aligned}
$$

□

2.5 Relation between divisble local Anderson modules and formal Lie groups

In addition to the p-Verschiebung from chapter 1 section (1.6) we have proved that there exist the z^d-Verschiebung for divisible local Anderson modules in the case of $\zeta = 0$ on S i.e. there exist a map $V_G : G^q \to G$ such that

$$
\begin{aligned}
F_G \circ V_G &= z^d|_{G^{(q)}}, \\
V_G \circ F_G &= z^d|_G.
\end{aligned}
$$

Notation. $G[n]$ will denote the kernel of the n^{th} iterate of q-Frobenius homomorphism:

$$G \xrightarrow{F_G} G^{(q)} \xrightarrow{F_{G^{(q)}}} G^{(q^2)} ..... \to G^{(q^n)}$$

Definition 2.5.1. A sheaf of groups G on S is said to be of F-torsion if $G = \varinjlim_n G[n]$.

Definition 2.5.2. A sheaf of groups G on S is said to be F-divisible if $F_G : G \to G^{(q)}$ is an epimorphism.

To prove the next theorem we are going to use the theorem from Messing [14, 2.1.7] which explains equivalent conditions for a sheaf of groups G on S be a formal Lie group. The equivalent condition in that theorem for a sheaf of groups G on S to be a formal Lie group are

1. G is of F-torsion.
2. G is F-divisible.
3. The $G[n]$ are finite and locally free S-group schemes.

Lemma 2.5.3. *When $\zeta = 0$ on S and G is a z-divisible local Anderson module over S with $(z-\zeta)^d = 0$ on ω_G, then $G[n] \subseteq G(nd)$.*

Proof. When $\zeta = 0$ on S, we have $V_G^n \circ F_G^n = z^{nd}$. Hence $G[n] \subseteq G(nd)$. □

Lemma 2.5.4. *If G is finite and locally free then $f : G \to H$ is an epimorphism if and only if it is faithfully flat.*

Proof. Messing [14, Lemma 1.5 b] □

Theorem 2.5.5. *When $\zeta = 0$ on S and G is a z-divisible local Anderson module over S, then $\varinjlim_n G[n]$ is a formal Lie group and is equal to $\bar{G} := \varinjlim_n \mathrm{Inf}^k(G)$.*

Proof. From the above it suffices to show that $\varinjlim_n G[n]$ is of F-torsion, F-divisible and that the $G[n]$ are finite and locally free. By definition it is obvious that $\varinjlim_n G[n]$ is of F-torsion. Since by hypothesis $z^d : G^{(q)} \to G^{(q)}$ is surjective and since in our case we have a factorization of this morphism $F_G \circ V_G = z^d$

$$\begin{array}{ccc} G^{(q)} & \longrightarrow & G^{(q)} \\ & {\scriptstyle V_G} \searrow \quad \nearrow {\scriptstyle F_G} & \\ & G & \end{array}$$

it follows that $F_G : G \to G^{(q)}$ is surjective. Since $F_G^{-1}(G[n]^{(q)}) \subseteq G[n+1]$ it is clear that $F_G^{-1}(\varinjlim_n G[n]^{(q)}) \subseteq \varinjlim_n G[n+1] = \varinjlim_n G[n]$ and hence that $F_G : \varinjlim_n G[n] \to$

$\varinjlim_n G[n]^{(q)} = (\varinjlim_n G[n])^{(q)}$ is an epimorphism. Thus it remains to prove that the $G[n]$ are finite and locally free. Since we have the exact sequences

$$0 \to G[i] \to G[n] \xrightarrow{F_G^i} G[n-i]^{(q^i)} \to 0,$$

from descent theory we are reduced to showing that $G[1]$ is finite and locally free. Since we have $G[1] \subseteq G(d)$ and thus $G[1] = G(d)[1]$ it is certainly representable because it is a closed subscheme of $G(d)$. Now let us see the commuting diagram

$$\begin{array}{ccc} G(d) & \xrightarrow{\quad F_G \quad} & G(d)^q \\ & \searrow \quad \swarrow_{g} & \\ & S. & \end{array}$$

Since $g \circ F_G$ is finite and of finite presentation and g is affine so separated this implies F_G is finite and of finite presentation and hence it is the same for the morphism $G[1] \to S$. Thus to show $G[1]$ is finite and locally free it remains to show that it is flat over S. Since $F_G \circ V_G = z^d$ and

$$G(d)^{(q)} \xrightarrow{V_{G(d)}} G(d) \xrightarrow{F_G} G(d)^{(q)}$$

$z^d = 0$ on $G(d)^{(q)}$, the image of $V_{G(d)}$ is contained in $G[1]$, and we must have $V_G^{-1}(G[1]) \subseteq G(d)^{(q)}$ for the same reason. But just like F_G, V_G is also an epimorphism, and hence the morphism $G(d)^{(q)} \to G[1]$ induced by V_G is an epimorphism. Thus we have an exact sequence:

$$0 \to \ker V_{G(d)} \to G(d)^{(q)} \xrightarrow{V_{G(d)}} G[1] \to 0.$$

Passing to the fibers we see that for all $s \in S$, $V_{G(d),s}$ is faithully flat and hence as $G(d)^{(q)}$ is flat over S, it follows that $V_{G(d)} : G(d)^{(q)} \to G[1]$ is faithfully flat [E.G.A., IV 11.3.11]. This of course implies that $G[1]$ is flat over S and hence we have shown that $\varinjlim_n G[n]$ is a formal Lie group. To prove the last statement of the theorem we observe that $X = \varinjlim_n G[n]$ is ind-infinitesimal since this was part of the definition of a formal Lie group. As we want to prove $X = \varinjlim_k \mathrm{Inf}^k(G)$, it will be enough to prove that $\varinjlim_k \mathrm{Inf}^k(G) \subseteq X$ i.e. for every k there should exist some n such that $\mathrm{Inf}^k(G) \subseteq G[n]$. For that we will prove for any $n \geq 0$ we have $\mathrm{Inf}^{q^n-1}(G) \subseteq G[n]$. To see this we observe that for any S-scheme T, $F_G : \Gamma(T,\ G) \to \Gamma(T,\ G^{(q)})$ is simply the mapping sending ϕ to $\phi \circ F_T$ which comes from the following commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{F_T} & T \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S \end{array}$$

by applying the contravariant functor G to this above diagram. To take $\phi \in \mathrm{Inf}^{q^n-1}(G)$ (T) means there exist a covering $\{T_i \to T\}$ and for each T_i there is a closed immersion $J: T_i' \hookrightarrow T_i$ of order q^n, such that we have $\phi \circ J = 0$. If $J: T_i' \hookrightarrow T_i$ is a closed immersion by an ideal I with $I^{q^n} = 0$, then there exist $J': T_i \to T_i'$ with $J \circ J' = F_T^n$.

$$\begin{array}{ccc} T_i & \xrightarrow{F_T^n} & T_i \\ & {\scriptstyle J'}\searrow \quad \nearrow{\scriptstyle J} & \\ & T_i' & \end{array}$$

This implies $F_G^n(\phi) = \phi \circ F_T^n = \phi \circ J \circ J' = 0$ or $\phi \in G[n]$. Thus we have $X = \bar{X} = \varinjlim_k \mathrm{Inf}^k(X) \subseteq \varinjlim_k \mathrm{Inf}^k(G) \subseteq X$ which completes the proof. □

Now we will prove that when ζ is locally nilpotent on S i.e. $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ and G is a z-divisible local Anderson module over S, $\bar{G} := \varinjlim_n \mathrm{Inf}^k(G)$ is a formal Lie group. In order to do this we use the relative contangent complex $L_{\cdot}^{G(n)/S}$ that we studied in chapter 1. Also we use the following proposition from Messing [14, Prop 3.1.1] which explains us equivalent conditions for a sheaf of group G on S to be smooth along the section $e_G: G \to S$. Note that no hypothesis on S is necessary.

Proposition 2.5.6. *[14, Prop 3.1.1] Let $(G,\ e_G)$ be a pointed scheme, locally of finite presentation on S; i.e. e_G is a section of the structural morphism $\pi: G \to S$. Then the following two conditions are equivalent:*

1. *Locally (for the Zariski topology) on S, $\mathrm{Inf}_e^k(G)$ is isomorphic to a pointed scheme of the form $\mathrm{Spec}(\mathcal{O}_S[T_1, ..., T_n]/(T_1, ..., T_n)^{k+1})$; i.e., $\omega_G = e_G^*(\Omega^1_{G/S})$ is locally free of finite type, and $\mathrm{Sym}^i(\omega_G) \tilde{\to} \mathrm{gr}^i(G,\ e_G)$ for $i \leq k$.*
2. *For any affine scheme X_0 over S, an S-infinitesimal neighborhood X' of X_0 of order k, a sub-scheme X of X' containing X_0 and any S-morphism $f: X \to G$ such that $f|_{X_0}$ factors through $e_G: S \to G$, there is a prolongation of f to an $f': X' \to G$.*

$$\begin{array}{ccccc} X_0 & \overset{i}{\hookrightarrow} & X & \overset{j}{\hookrightarrow} & X' \\ \downarrow & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle f'} & \\ S & \overset{e_G}{\hookrightarrow} & G & & \end{array}$$

We say that G is smooth along the section e_G up to order k if G satisfies the equivalent conditions of proposition (2.5.6).

The following proposition from Messing [14, 3.3.1] gives us the necessary results in order to associate a formal Lie group to a z-divisible local Anderson-module. Recall the definition $\ell_{\cdot}^G = e_G^*(L_{\cdot}^{(G/S)})$.

Proposition 2.5.7. *[14, Prop 3.3.1] Let $G(n)$ be an inductive system of finite locally-free groups on an arbitrary scheme S. Let J be a coherent ideal of $\mathcal{O}_S$ such that $J^N = (0)$. Let $S_0 = \mathrm{Var}(J)$, $G_0(n) = G(n) \times_S S_0$ and in general let the subscript "0" denote the restriction to S_0. Assume we are given a mapping $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(n) \geq n$ for all n. Assume that whenever M is a quasi-coherent module on an affine open set $U_0 \subseteq S_0$, $\mathrm{Ext}^1_{\mathcal{O}_{U_0}}(\ell^{G_0(n)}_{\cdot}|_{U_0},\ M) \to \mathrm{Ext}^1_{\mathcal{O}_{U_0}}(\ell^{G_0(\phi(n))}_{\cdot}|_{U_0},\ M)$ is the zero map. (In the rest of the proposition we shall omit the $\mathcal{O}_{U_0}$ and $|_{U_0}$ and simply write S_0, or S depending on the context. This will not lead any confusion.) Let X' be an affine scheme over S and X be the subscheme defined by an ideal I such that $I^2 = (0)$. Assume we are given an S-morphism $f : X \to G(n)$. Then there is an S-morphism $f' : X' \to G(\phi^N(n))$ such that the following diagram commutes :*

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ G(n) & \longrightarrow & G(\phi^N(n)) \end{array}$$

Note - Under the hypothesis of proposition 2.5.7 the sheaf $G = \varinjlim_n G(n)$ is formally smooth, because

$$\Gamma(X,\ G) = \Gamma(X,\ \varinjlim_n G(n)) = \varinjlim_n \Gamma(X, G(n)),$$

since X is quasi-compact.

Let G be a z-divisible local Anderson module over S and $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$. In order to prove that $\bar{G} = \varinjlim_k \mathrm{Inf}^k(G)$ is a formal Lie group we will prove at first G satisfies the hypothesis of proposition (2.5.7).

Recall the definition $\omega_G = \mathrm{H}_0(\ell^G_{\cdot})$, $n_G = \mathrm{H}_{-1}(\ell^G_{\cdot})$.

We will need the following proposition and lemma and their corollaries from Messing [14] to proceed.

Proposition 2.5.8. *[14, Chap 2, Prop 3.3.4] Consider an exact sequence of finite locally-free S groups: $0 \to G' \to G \to G'' \to 0$. Then there is an exact triangle in the derived category, $D(S)$:*

$$\begin{array}{ccccc} & & \ell^{G'}_{\cdot} & & \\ & \swarrow & & \nwarrow & \\ \ell^{G''}_{\cdot} & & \longrightarrow & & \ell^{G}_{\cdot} \end{array}$$

giving rise to an exact sequence of $\mathcal{O}_S$-modules :

$$0 \to n_{G''} \to n_G \to n_{G'} \to \omega_{G''} \to \omega_G \to \omega_{G'} \to 0. \tag{2.2}$$

Lemma 2.5.9. *[14, Chap 2 Lemma 3.3.6] Suppose G is finite and locally-free on S. Then ω_G is locally free if and only if it is flat. If this is the case then n_G is also locally free $(of\ finite\ rank)$ and* $\mathrm{rank}(\omega_G) = \mathrm{rank}(n_G)$.

The following are corollaries of (2.5.8).

Corollary 2.5.10. Given $0 \to G' \to G \to G'' \to 0$ as in 2.5.8 then:
a) The following conditions are equivalent and they are implied by $G[1] \subseteq G'$

1. $\omega_G \to \omega_{G'}$ is an isomorphism
2. $\omega_{G''} \to \omega_{G'}$ is the zero map
3. $n_{G'} \to \omega_{G''}$ is surjective

b) Suppose $\omega_{G''}$ is locally free and $n_{G'}$ is of finite type and that $\mathrm{rank}_{k(s)}(n_{G'} \otimes k(s)) \leq \mathrm{rank}_{k(s)}(\omega_{G''} \otimes k(s))$ holds for all $s \in S$. Then the above conditions of *a)* are equivalent to

$3'$. $n_{G'} \to \omega_{G''}$ is an isomorphism and in this case $n_G \to n_{G'}$ is the zero map.

Proof. That the three conditions of a) are equivalent is immediate from (2.2). If $G[1] \subseteq G'$, then $G[1] = G'[1]$ and hence since $\mathrm{Inf}^1(G) \subseteq G[1]$ (at the end of the proof 2.5.5) we have $\omega_G = \omega_{G'}$ which is condition 1).

b) Assume the additional condition and that $n_{G'} \to \omega_{G''}$ is surjective. Then because $\omega_{G''}$ is locally free therefore locally on S we must have a splitting $n_{G'} = \mathrm{Ker} \times \omega_{G''}$ where Ker denotes the kernel of $n_{G'} \to \omega_{G''}$. By our assumption on the ranks we must have $\mathrm{Ker} \otimes_{\mathcal{O}_S} k(s) = 0$ for all $s \in S$. Ker is of finite type because $n_{G'}$ is. Therefore by Nakayama we must have each stalk of Ker is zero and therefore $\mathrm{Ker} = (0)$. This completes the proof since the last assertion is obvious. □

Remark 2.5.11. From (2.5.9) it follows that the inequality in the last corollary can be written $\mathrm{rank}(\omega_{G'_s}) \leq \mathrm{rank}(\omega_{G''_s})$ for all $s \in S$ if $\omega_{G'}$ is flat.

Corollary 2.5.12. [14, Chap 2, Cor 3.3.9] Assume given the exact sequence $0 \to G' \to G \to G'' \to 0$ as above and assume further that:

1. $\omega_G \to \omega_{G'}$ is an isomorphism
2. ω_G and $\omega_{G''}$ are locally free
3. $\mathrm{rank}(\omega_{G'_s}) \leq \mathrm{rank}(\omega_{G''_s})$ for all $s \in S$

Then for any affine open $U \subseteq S$ and for any quasi-coherent module M on U, the map

$$\mathrm{Ext}^1_{\mathcal{O}_U}(\ell^{G'}_{\cdot}|_U,\ M) \to \mathrm{Ext}^1_{\mathcal{O}_U}(\ell^{G}_{\cdot}|_U,\ M)$$

is the zero map. In particular for any quasi-coherent M on S the map $\mathrm{Ext}^1(\ell^{G'}_{\cdot},\ M) \to \mathrm{Ext}^1(\ell^{G}_{\cdot},\ M)$ is zero.

Corollary 2.5.13. Let S be a scheme such that $\zeta^{N+1} = 0$ on S, and let $G = G(nd)$ be a truncated z-divisble local Anderson module over S with order of nilpotence d and $n \geq N+1$. Then for any affine open subset U of S, the mapping (defined for any quasi-coherent M)
$\mathrm{Ext}^1(\ell^{G((n-N-1)d)}_{\cdot},\ M) \longrightarrow \mathrm{Ext}^1(\ell^{G(nd)}_{\cdot},\ M)$ is zero.

Proof. At first we prove that it is equivalent to prove for the case $N = 0$. i.e. we assume for $S_0 = V(\zeta) \subseteq S$ and for any affine open subset U_0 of S_0 and M be a quasi-coherent module on U_0 we have $\mathrm{Ext}^1(\ell^{G((n-1)d)}_{\cdot},\ M) \longrightarrow \mathrm{Ext}^1(\ell^{G(nd)}_{\cdot},\ M)$ is the zero map.

Now take the exact sequence

$$0 \to \zeta M \to M \to M/\zeta M \to 0.$$

and form the commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Ext}^1(\ell^{G((n-N-1)d)}_{\cdot},\ \zeta M) & \longrightarrow & \mathrm{Ext}^1(\ell^{G((n-N-1)d)}_{\cdot},\ M) & \longrightarrow & \mathrm{Ext}^1(\ell^{G((n-N-1)d)}_{\cdot},\ M/\zeta M) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Ext}^1(\ell^{G((n-N)d)}_{\cdot},\ \zeta M) & \longrightarrow & \mathrm{Ext}^1(\ell^{G((n-N)d)}_{\cdot},\ M) & \longrightarrow & \mathrm{Ext}^1(\ell^{G((n-N)d)}_{\cdot},\ M/\zeta M)
\end{array}$$

we see that the image of $\mathrm{Ext}^1(\ell^{G((n-N-1)d)}_{\cdot}, M)$ is contained in that of $\mathrm{Ext}^1(\ell^{G((n-N)d)}_{\cdot}$, $\zeta M)$ because the right vertical arrow is the zero map. This tells us that for any quasi-coherent module M on U the map $\mathrm{Ext}^1(\ell^{G((n-N-1)d)}_{\cdot},\ M) \to \mathrm{Ext}^1(\ell^{G(nd)}_{\cdot},\ M)$ is zero since by hypothesis $\zeta^{N+1} = 0$.

Now we prove the case $N = 0$. Then if $n = 1$, $G((n-N-1)d) = 0$ and there is nothing to prove. If $n \geq 2$ we must verify the conditions of the previous corollary for $G' = G((n-1)d)$ and $G = G(nd)$.

$$0 \longrightarrow G((n-1)d) \longrightarrow G(nd) \longrightarrow G(d) \longrightarrow 0$$

Since $G[1] \subseteq G(d)$ by lemma (2.5.3) so we have

$$G[1] = G((n-1)d)[1] = G(nd)[1] = G(d)[1].$$

It shows that $\omega_{G((n-1)d)} = \omega_{G(nd)} = \omega_{G(d)}$ and hence $\omega_G \to \omega_{G'}$ is an isomorphism. Condition 3) is also trivial as $G'' = G(d)$. To verify condition 2) holds we use statement 3) in the following proposition and appeal to Messing [14, Chap 2, §1, 2.1.4] to conclude the proof. □

Proposition 2.5.14. *Let S be a scheme such that $\zeta = 0$ on S and $G = G(nd)$ a truncated z-divisible local Anderson module of order of nilpotence d on S, in particular if $n = 1$ there is a truncated divisible local Anderson module $G(2d)$ giving rise to G. Then*

1. *For all i such that $0 \leq i \leq n$ we have $F_G^{n-i} : G[n] \to G[i]^{q^{n-i}}$ is an epimorphism.*
2. $\mathrm{Ker}\, F_G^n = \mathrm{im}\, V_G^n$ *and* $\mathrm{Ker}\, V_G^n = \mathrm{im}\, F_G^n$

3. $G[n]$ is finite and locally free on S

Proof. Condition 1) is trivial if $n = 1$. If $n \geq 2$ we have for each i the following commutative diagram:

$$\begin{array}{ccc} (z^{(n-i)d})^{-1}(G[i]^{(q^{n-i})}) & \xrightarrow{z^{n-i}\,\mathrm{id}_{G^{(q^{n-i})}}} & G[i]^{(q^{n-i})} \\ & {\scriptstyle V_G^{(n-i)d}} \searrow \quad \nearrow {\scriptstyle F_G^{n-i}} & \\ & G[n] & \end{array}$$

As the morphism $z^{(n-i)d}\,\mathrm{id}_{G^{(q^{n-i})}} : G^{(q^{n-i})} = G(nd)^{(q^{n-i})} \to G(id)^{(q^{n-i})}$ is surjective and $G[i]^{(q^{n-i})} \subseteq G(id)^{(q^{n-i})}$, it follows that F_G^{n-i} is an epimorphism. To prove 2) let us first assume that $G(nd)$ comes from a $G(2nd)$ which is a truncated divisible local Anderson module. But then we have an exact sequence:

$$0 \longrightarrow G(nd) \longrightarrow G(2nd) \xrightarrow{z^{nd}} G(nd) \to 0.$$

Therefore we have $G[n] \subseteq G(nd)$ and the map $(z^{nd})^{-1}(G[n]) \to G[n]$ is an epimorphism. But once again we can write $z^{nd} = V_G^n \circ F_G^n$ and since obviously $F_G^n : (z^{nd})^{-1}(G[n]) \to G(nd)^{(q^n)}$ we see $V_G^n : G(nd)^{(q^n)} \to G[n]$ is an epimorphism. That proves $\ker F_G^n = \operatorname{im} V_G^n$. The other case is of course handled in the same way. Thus we have proved statement 2 for the case $n = 1$. We proceed to prove it in general by induction on n. Consider the diagram:

$$\begin{array}{ccc} G(nd)^{(q^n)} & \xrightarrow{V_G^n} & G[n] \\ {\scriptstyle z^d}\downarrow & & \downarrow{\scriptstyle F_G} \\ G((n-1)d)^{(q^n)} & \xrightarrow[V_G^{n-1}]{} & G[n-1]^q \end{array}$$

By the induction hypothesis V_G^{n-1} is an epimorphism. Therefore $F_G \circ V_G^n$ is an epimorphism. Hence if we can show $G[1] \subseteq \operatorname{im} V_G^n$ it will follow that V_G^n is an epimorphism. But by the case $n = 1$ settled above

$$G[1] = V_G(G(d)^{(q)}) = V_G \circ z^{(n-1)d}(G(nd)^{(q)}) = V_G^n \circ F_G^{(n-1)}(G(nd)^{(q)}) \subseteq V_G^n(G(nd)^{(q^n)}).$$

The other case is of course handled in the same way. To prove 3) we observe that $G[n]$ is certainly finite and of finite presentation over S. Therefore to conclude it is locally free it suffices to show it is flat. This follows because we have a commutative diagram

$$\begin{array}{ccc} G(nd)^{q^n} & \xrightarrow{V_G^n} & G[n] \\ & \searrow \quad \swarrow & \\ & S & \end{array}$$

and V_G^n being an epimorphism is faithfully flat while $G(nd)^{q^n}$ is flat over S. □

Theorem 2.5.15. *If $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ and G is a Z-divisible local Anderson module over S, Then G is formally smooth.*

Proof. Let X' be an affine scheme over S and X a closed subscheme defined by an ideal of square zero. Let $f : X \to G$ be in $\Gamma(X,\ G)$. We must show f can be lifted to an $f' : X' \to G$.

$$\begin{array}{ccc} X & \hookrightarrow & X' \\ {\scriptstyle f}\big\downarrow & \swarrow {\scriptstyle f'} & \\ G & & \end{array}$$

As X is quasi-compact we have $\Gamma(X,\ G) = \varinjlim_n \Gamma(X,\ G(n)) = \varinjlim_n \Gamma(X,\ G(nd))$ and hence can assume $f : X \to G(nd)$ for some n. We cover X by a finite number of affine opens $U_i,\ i = 1, ..., m$ such that the image of U_i in S is contained in an affine open V_i. Since ζ is nilpotent on each V_i there is an integer N such that ζ^{N+1} is zero on $\cup V_i$. Replacing S by $S' = \cup V_i$ and G by $G_{S'}$ we are led to the case when ζ is nilpotent on S, ζ^{N+1} kills S. But now by 2.5.13 and 2.5.7 we see f can be lifted to an f' and the theorem is proved. □

Finally we shall prove that if $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ then $\bar{G} = \varinjlim_n \mathrm{Inf}^n(G)$ is a formal Lie group. We begin with a lemma.

Lemma 2.5.16. *Let G be a z-divisible local Anderson module on S with $(z-\zeta)^d = 0$ on $\omega_G = \varprojlim \omega_{G(n)}$ for some $d \in \mathbb{N}$. Assume we are given an S-scheme X' and a subscheme X defined by an ideal I such that $I^{k+1} = (0)$ and $\zeta^N \cdot I/I^2 = 0$ and suppose N' is the smallest integer which is power of p and greater than N and d. Then if $f' : X' \to G$ is such that $f = f'|_X : X \to G(n)$, we have $f' : X' \to G(n + kN')$.*

Proof. : The problem is local on X' and hence we can assume that X' is affine and thus quasi-compact. But then $f' \in \Gamma(X',\ G) = \varinjlim_m \Gamma(X',\ G(m))$ and hence we can assume that $f' : X' \to G(n')$ for some n'. Therefore we can assume that G is representable. We use induction on k. If we could show

$$\forall l : f'|_{\mathrm{V}(I^l)} : \mathrm{V}(I^l) \to G(n + (l-1)N'),$$

then by the case $l = k+1$ we would know $f' : X' = \mathrm{V}(I^{k+1}) \to G(n + kN')$. Thus it suffices to treat the case $l = 2$, i.e., $I^2 = 0$. Since $f : X \to G(n)$ we have $z^n \circ f = 0$ and $z^n f' \in G(X')$ has the property that its restriction to $G(X)$ is zero. Since $I^2 = 0$ and G is representable we know the group of sections of G over X' whose restriction to X is zero is isomorhic to the group $\mathrm{Hom}_{\mathcal{O}_X}(\omega_G \otimes_{\mathcal{O}_S} \mathcal{O}_X,\ I)$ [S.G.A 3 III 0.9]. This implies $z^n f' \mapsto h \in \mathrm{Hom}_{\mathcal{O}_X}(\omega_G \otimes_{\mathcal{O}_S} \mathcal{O}_X, I)$ and since ζ^N kills I

$$\begin{aligned} \Rightarrow \zeta^N \circ h &= 0 \\ \Rightarrow \zeta^{N'} \circ h &= 0 \quad \text{since } N' > N \end{aligned}$$

Since the action of $(z-\zeta)^d = 0$ on $\omega_G := \mathrm{Lie}^*(G/S)$

$$\begin{aligned} \Rightarrow\ (z-\zeta)^{N'} &= 0 \text{ on } \omega_G \text{ since } N' > d \\ \Rightarrow\ (\mathrm{Lie}^*\, z)^{N'} &= \zeta^{N'} \text{ on } \omega_G \text{ since } N' \text{ is multiple of } p, \end{aligned}$$

because z acts on ω_G through $\mathrm{Lie}^*\, z : \omega_G \to \omega_G$

$$\Rightarrow z^{N'} \circ (z^n f') \mapsto (\mathrm{Lie}^*\, z)^{N'} \circ h = \zeta^{N'} \circ h = 0$$

$$\Rightarrow\ f' \in G(n+N')(X').$$

□

Corollary 2.5.17. Let ζ^N kill S and let G be as in the lemma (2.5.16). Then the k-th infinitesimal neighborhood of $G(n)$ in G is the same as that of $G(n)$ in $G(n+kN')$. In particular $\mathrm{Inf}^k(G) = \mathrm{Inf}^k(G(kN'))$ and is therefore representable.

Proof. If $f : T' \to G$ belongs to the k-th infinitesimal neighborhood of $G(n)$ in G, then there is a covering family $\{T_i' \to T'\}$ and schemes T_i such that $T_i \hookrightarrow T_i'$ is a nilpotent immersion of order k and $f|_{T_i} : T_i \to G(n)$. But then by the lemma $f|_{T_i'} : T_i' \to G(n+kN')$ and hence $f \in \Gamma(T',\ G(n+kN'))$, which proves the corollary. □

Corollary 2.5.18. If ζ^N kills S and if $k < q^n$ we have $\mathrm{Inf}^k(G) \subseteq G(nd+(N-1)N')$ and hence $\mathrm{Inf}^k(G) = \mathrm{Inf}^k(G(nd+(N-1)N')$.

Proof. Let X' be an S-scheme and $X \hookrightarrow X'$ be a nilpotent immersion of order k. Denote with the subscript "0" the object obtained by reducing a given object modulo ζ. Given $f' : X' \to G$ whose restriction to X is zero, then we have $f_0' : X_0' \to G_0$ belong to $\mathrm{Inf}^k(G_0)$. By the reasoning at the end of the proof of (2.5.5) (where we show that $\mathrm{Inf}^{q^n-1} \subseteq G_0[n] \subseteq G_0(nd)$). This means that $f' \in G(X')$ has its restriction to $G(X_0') = G_0(X_0')$ belonging to $G(nd)(X_0')$. If we now apply lemma (2.5.16) with $I = \zeta\mathcal{O}_{X'}$, $k = N-1$, and $N = 1$, we find $f' \in G(nd+(N-1)N')(X')$. □

Theorem 2.5.19. *Let ζ be locally nilpotent on S and let G be a z-divisible local Anderson module on S. Then $\bar{G} = \varinjlim_k \mathrm{Inf}^k(G)$ is a formal Lie group.*

Proof. : As we know $\bar{G}$ is a subgroup of G we must show it is a formal Lie variety. By (2.5.17) $\mathrm{Inf}^k(G)$ is, locally on S, representable and therefore since it is a sheaf it is representable.

By theorem (2.5.15) we know G is formally smooth and obviously this implies that $\bar{G}$ is formally smooth. This tells us that $\mathrm{Inf}^k(G)$ satisfies the lifting condition 2) of (2.5.6) and hence, since locally on S $\mathrm{Inf}^k(G) = \mathrm{Inf}^k(G(m))$ for an appropriate m, It follows from (2.5.6) that locally on S $\mathrm{Inf}^k(G)$ satisfies condition 1) of that proposition. But now it is obvious that $\bar{G}$ satisfies condition 2) and 3) of definition (2.1.5) and hence is a formal Lie group. □

Remark 2.5.20. Since we know from theorem(2.4.3) that ω_G is locally free of finite rank but it also follows immediately from (2.5.19) that ω_G is locally free of finite rank and from (2.5.17) or, for a better estimate, (2.5.18) that locally on S $\omega_G = \omega_{G(m)}$ if m is sufficiently large. If ζ^N kills S, then $\omega_G = \omega_{G(N')}$ as follows from (2.5.17).

Now we study the relation between formal lie groups which is also $\mathbb{F}_q[z]$-module and z-divisible local Anderson modules on a scheme S, with ζ locally nilpotent on S.

Lemma 2.5.21. *Let B be a ring in which ζ is nilpotent, and I be a nilpotent ideal of B. Define a sequence of ideals $I_1 = \zeta I + I^2, ..., I_{n+1} = \zeta I_n + (I_n)^2$. Then for n sufficiently large $I_n = (0)$.*

Proof. Let $J = \zeta B + I$. Then it is easy to check that $I_n \subseteq J^{n+1}$. Since ζ and I both are nilpotent, so is the ideal J. This implies $I_n = 0$ for n sufficiently large. □

Lemma 2.5.22. *If ζ is locally nilpotent on S and G is formal lie group over S which is also an $\mathbb{F}_q[z]$-module such that locally on S there is an integer d for which* $(\mathrm{Lie}^* z - \zeta)^d = 0$ *on ω_G, then G is of z-torsion.*

Proof. We must show $G = \varinjlim_n G(n)$ and since this is statement about sheaves it suffices to check it locally on S. Thus we can assume $S = \operatorname{Spec} A$ with ζ nilpotent on A and G is given by a power series ring $A[\![X_1, ..., X_d]\!]$. If T is any affine S-scheme, say $T = \operatorname{Spec}(B)$, then an element of $G(T)$ will be an N-tuple $(b_1, ..., b_d)$ with each b_i nilpotent. Let I be the ideal generated by $\{b_1, ..., b_d\}$. Let N' be a power of q with $N' \geq d$. Then each component of $z^{N'} \cdot (b_1, ..., b_d)$ belongs to $\zeta^{N'} I + I^2$ and $z^{2N'} \cdot (b_1, ..., b_d)$ belongs to $\zeta^{N'} \cdot (\zeta^{N'} I + I^2) + (\zeta^{N'} I + I^2)^2, \cdots$. Thus by the previous lemma we see G is of z-torsion. □

Let S be the spectrum of an artin local ring with residue field of characteristic p, so $\zeta = 0$ in the residue field because ζ is nilpotent and let G be a z-divisible formal Lie group on S such that locally on S there is an integer d for which $(\mathrm{Lie}^* z - \zeta)^d = 0$ on ω_G for $d = \dim G$. From Proposition Messing [14, Chap II, 4.3] we know that $G(n)$ are finite and locally free and they are the kernel of an $\mathbb{F}_q$-linear homomorphism of formal Lie groups so they are strict. Hence G is a z-divisible local Anderson modules over S.

We have the following proposition for z-divisible local Anderson modules similar to the proposition for p-divisible group in Messing [14, Chap II, Prop 4.4] and also its proof follows similarly.

Proposition 2.5.23. *Let ζ be locally nilpotent on S and let G be a z-divisible local Anderson module over S. Then the following conditions are equivalent:*

1. *$G = \bar{G}$.*
2. *G is a formal Lie group.*
3. *For all n $G(n)$ is radiciel.*

4. $G(1)$ is radiciel.

Corollary 2.5.24. If ζ is locally nilpotent on S, there is an equivalence of categories between that of z-divisible local Anderson modules on S, with $G(1)$ radiciel, and the category of formal Lie groups G which are also $\mathbb{F}_q[z]$-modules with $z : G \to G$ an epimorphism, $G(1)$ finite and locally free and locally on S there is an integer d for which $(\mathrm{Lie}^* z - \zeta)^d = 0$ on ω_G.

Proof. By (2.5.22) and (2.5.23) both categories are identified with the same full subcategory of fppf sheaves of groupes on S which are also $\mathbb{F}_q[z]$-modules. □

Corollary 2.5.25. If S is artin, a z-divisible formal Lie group such that locally on S there is an integer d for which $(\mathrm{Lie}^* z - \zeta)^d = 0$ on ω_G is a z-divisible local Anderson module with $G(1)$ radiciel and conversely.

Proof. Follows from explanation above and (2.5.24) □

We have the following proposition for z-divisible local Anderson modules similar to the proposition for p-divisible group in Messing [14, Chap II, Prop 4.7] and also its proof follows similarly.

Proposition 2.5.26. *Let ζ be locally nilpotent on S and G be a z-divisible local Anderson module over S. In order that $\bar{G} = 0$ it is necessary and sufficient that G is ind-étale.*

We have the following proposition for z-divisible local Anderson modules similar to the proposition for p-divisible group in Messing [14, Chap II, Prop 4.9] and proof follows similarly using the lemma from Messing [14, Chap II, 4.8] which says if $X \xrightarrow{f} S$ be finite and locally free. Then the function $s \mapsto$ separable rank (X_s) is locally constant if and only if there are morphisms $i : X \to X'$, $f' : X' \to S$ which are finite and locally free with i radiciel and surjective, f' étale and $f = f' \circ i$ and the factorization is "unique" up to unique isomorphism and is functorial in X/S and the fact that whenever we have a finite $\mathcal{O}$-module scheme X over S, the canonical decomposition of X

$$0 \to X'' \to X \to X' \to 0$$

where X'' finite, locally free and radiciel, X' finite and étale are $\mathcal{O}$-invariant.

Proposition 2.5.27. *Let ζ be locally nilpotent on S and G be in z-divisible local Anderson modules over S. The following conditions are equivalent*

1. *$\bar{G}$ is a z-divisible local Anderson module.*

2. *G is an extension of an ind-étale z-divisible local Anderson module G'' by an ind-infinitesimal z-divisible local Anderson module G'.*

2' *G is an extension of an ind-étale z-divisible local Anderson module by a z-divisible formal Lie group.*

3. For all n $G(n)$ is an extension of a finite étale group by a finite locally-free radiciel group.

3' $G(1)$ is an extension of a finite étale group by a finite locally-free radiciel group.

4. $s \mapsto$ separable rank $(G(1)_s)$ is locally constant function.

We have the following lemma for z-divisible local Anderson modules on S similar to the lemma Messing [14, Chap II, Prop 4.11] and also its proof follows similarly.

Lemma 2.5.28. *Let ζ be locally nilpotent on S and let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of z-divisible local Anderson modules on S. Then $0 \to \bar{G}_1 \to \bar{G}_2 \to \bar{G}_3 \to 0$ is also exact.*

2.6 Comparison of Tate-modules

In this section we assume that $S \in \mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$. Let G be a z-divisible local Anderson module of rank r over S and $\underline{M}(G)$ be the local shtuka over S associated to G.

If $S = \operatorname{Spec} \mathcal{O}_K$ where $\mathcal{O}_K$ are the integers in a local field K with $\zeta \in K^\times$ and K^{sep} is a fixed separable closure. The z-adic Tate-module of G is defined as

$$T_z G := \varprojlim_n (G(n)(K^{\text{sep}}),\ z : G(n+1) \xrightarrow{\cdot z} G(n)).$$

So $T_z G$ is $\mathbb{F}_q[\![z]\!]$-module and

$$T_z G(K^{\text{sep}}) = \varprojlim_n (G(n)(K^{\text{sep}})) \cong \mathbb{F}_q[\![z]\!]^{\oplus \operatorname{rk} G}.$$

Since $\mathcal{O}_K \supseteq \mathbb{F}_q[\![\zeta]\!]$, so $K \supseteq \mathbb{F}_q((\zeta))$ and $\zeta \neq 0$ in K. It will be better to take $S \in$ Pro-$\mathcal{N}ilp_{\mathbb{F}_q[\![\zeta]\!]}$ i.e. $S = \varinjlim_n S_n = \varinjlim_n \operatorname{Spec} \mathcal{O}_K/\zeta^n = \operatorname{Spec}(\varprojlim_n \mathcal{O}_K/\zeta^n) = \operatorname{Spec} \mathcal{O}_K$.

For a local shtuka M we define the Tate module of M as

$$T_z M := ((M)^{\tau=1})^\vee = \operatorname{Hom}_{\mathbb{F}_q[\![z]\!]}((M \otimes K^{\text{sep}}[\![z]\!])^{\tau=1}, \mathbb{F}_q[\![z]\!]).$$

Here $M^{\tau=1} := \{m \in M | \tau(m) = m\}$, where the σ-linear map $\tau : M \to M$ is induced from the $\mathcal{O}_S$-linear map $F_M : \sigma^* M \to M$ i.e. $\tau(m) = F_M(m \otimes 1)$.

Definition 2.6.1. A pairing $\mathcal{G}_1 \times \mathcal{G}_2 \to \mathcal{H}$ of étale sheaves $\mathcal{G}_1, \mathcal{G}_2, \mathcal{H}$ of $\mathcal{A}$-modules on S is called perfect if it induces isomorphisms of $\mathcal{A}$-module sheaves $\mathcal{G}_i \cong \mathcal{H}om_{\mathcal{A}}(\mathcal{G}_{3-i}, \mathcal{H})$.

Theorem 2.6.2. *There is a canonical $\mathbb{F}_q[\![z]\!]$-isomorphism $T_z G \xrightarrow{\sim} T_z M$ of* Gal(K^{sep}/K)*-representations.*

Proof. From Böckle, Hartl [3, Lemma 3.4 and Thm 9.6] we have a perfect pairing

$$G(n)(K^{\text{sep}}) \times (M/z^n M)^{\tau=1}(K^{\text{sep}}) \longrightarrow \operatorname{Hom}_{\mathbb{F}_q\text{–Mod}}(\mathbb{F}_q[\![z]\!]/z^n, \mathbb{F}_q),$$

$$(p_n, m) \longmapsto (h_n : a \mapsto m(ap_n))$$

of $\mathbb{F}_q\llbracket z\rrbracket/z^n$-modules.

At first we will construct an isomorphism between $\mathrm{Hom}_{\mathbb{F}_q-\mathrm{Mod}}(\mathbb{F}_q\llbracket z\rrbracket/z^n, \mathbb{F}_q)$ and $z^{-n}\mathbb{F}_q\llbracket z\rrbracket/\mathbb{F}_q\llbracket z\rrbracket \cdot dz$. We define the map from

$$z^{-n}\mathbb{F}_q\llbracket z\rrbracket/\mathbb{F}_q\llbracket z\rrbracket \cdot dz \longrightarrow \mathrm{Hom}_{\mathbb{F}_q-\mathrm{Mod}}(\mathbb{F}_q\llbracket z\rrbracket/z^n, \mathbb{F}_q)$$

which sends $f \cdot dz \in z^{-n}\mathbb{F}_q\llbracket z\rrbracket/\mathbb{F}_q\llbracket z\rrbracket \cdot dz \longmapsto h_n : a \mapsto \mathrm{Res}_{z=0}(af)$ i.e. if $f = f_{-n}z^{-n} + \cdots + f_{-1}z^{-1}$ then $h_n(z^i) = f_{-(i+1)}$ and the inverse map is defined by

$$h_n \longmapsto \sum_{i=0}^{n-1} h_n(z^{n-i-1})z^{i-n} \cdot dz.$$

We compose the isomorphism $\mathrm{Hom}_{\mathbb{F}_q-\mathrm{Mod}}(\mathbb{F}_q\llbracket z\rrbracket/z^n, \mathbb{F}_q) \to z^{-n}\mathbb{F}_q\llbracket z\rrbracket/\mathbb{F}_q\llbracket z\rrbracket \cdot dz$ with the multiplication with z^n

$$z^n : z^{-n}\mathbb{F}_q\llbracket z\rrbracket/\mathbb{F}_q\llbracket z\rrbracket \cdot dz \tilde{\rightarrow} \mathbb{F}_q\llbracket z\rrbracket/z^n\mathbb{F}_q\llbracket z\rrbracket \cdot dz = \Omega_{\mathbb{F}_q\llbracket z\rrbracket/\mathbb{F}_q} \otimes_{\mathbb{F}_q\llbracket z\rrbracket} \mathbb{F}_q\llbracket z\rrbracket/z^n\mathbb{F}_q\llbracket z\rrbracket.$$

Then we have a perfect pairing

$$\varphi_n : G(n)(K^{\mathrm{sep}}) \times (M/z^nM)^{\tau=1}(K^{\mathrm{sep}}) \longrightarrow \mathbb{F}_q\llbracket z\rrbracket/z^n\mathbb{F}_q\llbracket z\rrbracket \cdot dz$$

$$(p_n, m) \longmapsto \sum_{i=0}^{n-1} m(z^{n-1-i} \cdot p_n) \cdot z^i dz$$

of $\mathbb{F}_q\llbracket z\rrbracket/z^n$-modules.

Now we have to check the compatibility of the map

$$\begin{array}{ccc} G(n)(K^{\mathrm{sep}}) \times (M/z^nM)^{\tau=1}(K^{\mathrm{sep}}) & \xrightarrow{\quad\varphi_n\quad} & \mathbb{F}_q\llbracket z\rrbracket/(z^n)\ dz \\ \uparrow{\scriptstyle \cdot z} \qquad\qquad \uparrow{\scriptstyle \mathrm{mod}\ z^n} & & \uparrow{\scriptstyle \mathrm{mod}\ z^n} \\ G(n+1)(K^{\mathrm{sep}}) \times (M/z^{n+1}M)^{\tau=1}(K^{\mathrm{sep}}) & \xrightarrow{\quad\varphi_{n+1}\quad} & \mathbb{F}_q\llbracket z\rrbracket/(z^{n+1})\ dz \end{array}$$

to get the perfect pairing

$$T_zG \times M^{\tau=1} \to \mathbb{F}_q\llbracket z\rrbracket \cdot dz.$$

Since $T_zG = \{(p_n)_n,\ n \in \mathbb{N} \mid p_n \in G(n)$ with $p_n = z \cdot p_{n+1}\}$. To check the commutativity of the above diagram we have to check $\varphi_{n+1}(p_{n+1}, m_{n+1}) \bmod z^n = \varphi_n(z \cdot p_{n+1},\ m_{n+1} \bmod z^n)$. The left hand side is equal to $\sum_{i=0}^{n} m_{n+1}(z^{n-i} \cdot p_{n+1}) \cdot z^i dz \bmod z^n = \sum_{i=0}^{n-1} m_{n+1}(z^{n-i} \cdot p_{n+1}) \cdot z^i dz$. The right side is equal to $\sum_{i=0}^{n-1}(m_{n+1} \bmod z^n)(z^{n-i-1} \cdot z \cdot p_{n+1}) \cdot z^i dz = \sum_{i=0}^{n-1} m_{n+1}(z^{n-i} \cdot p_{n+1}) \cdot z^i dz$. Hence we have perfect pairing of $\mathbb{F}_q\llbracket z\rrbracket$-modules

$$T_zG \times (M)^{\tau=1} \to \mathbb{F}_q\llbracket z\rrbracket \cdot dz.$$

so $T_zG \cong \mathrm{Hom}_{\mathbb{F}_q[\![z]\!]}((M)^{\tau=1}, \mathbb{F}_q[\![z]\!]) = (M^{\tau=1})^\vee = T_zM$.
If $\gamma \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ then

$$\begin{aligned}
\varphi_n(\gamma p_n, \gamma m) &= \sum_{i=0}^{n-1} \gamma m(z^{n-1-i} \cdot \gamma p_n) \cdot z^i dz \\
&= \sum_{i=0}^{n-1} m(z^{n-1-i} \gamma^{-1} \cdot \gamma p_n) \cdot z^i dz \\
&= \sum_{i=0}^{n-1} m(z^{n-1-i} p_n) \cdot z^i dz \\
&= \varphi_n(p_n, m)
\end{aligned}$$

Hence the above isomorphism is $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-equivariant.

□

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