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Cartan subalgebras of topological graph algebras and *k*-graph C*-algebras

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Abstract. In this paper, two sufficient and necessary conditions are given. The first one considers the boundary path groupoid of a topological graph without singular vertices, and it characterizes when the interior of its isotropy group bundle is closed. The second one concerns the path groupoid of a row-finite k-graph without sources, and it demonstrates when the interior of its isotropy is closed. It follows that the associated topological graph algebra and the associated k-graph C*-algebra have Cartan subalgebras due to a result of Brown–Nagy–Reznikoff–Sims–Williams.

1. Introduction

Renault in [17] defined the notion of Cartan subalgebras in the C*-algebras setting, as a C*-analog to Cartan subalgebras of von Neumann algebras studied by Feldman and Moore [3].

Definition 1.1 ([17, Def. 5.1])**.** Let B be an abelian C*-subalgebra of a C*-algebra A. Then B is called a *Cartan subalgebra* if the following hold:

(i) B contains an approximate identity of A;
(ii) B is a maximal abelian subalgebra of A;
(iii) there exists a faithful conditional expectation from A onto B;
(iv) $\{n \in A \mid nBn^*, n^*Bn \subset B\}$ generates A.

For an étale groupoid Γ (see Section 2.1), the inclusion of the interior of the isotropy $\mathrm{Iso}(\Gamma)^{\mathrm{o}}$ into Γ induces an inclusion of the C*-algebras

$$\iota_r : C_r^*(\mathrm{Iso}(\Gamma)^{\mathrm{o}}) \hookrightarrow C_r^*(\Gamma).$$

Brown, Nagy, Reznikoff, Sims, and Williams [2] characterized when this inclusion is Cartan.

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Theorem 1.2 ([2, Cor. 4.5])**.** *Let* Γ *be an étale groupoid. Suppose that* $\mathrm{Iso}(\Gamma)^{\mathrm{o}}$ *is abelian. Then* $\iota_r(C_r^*(\mathrm{Iso}(\Gamma)^{\mathrm{o}}))$ *is a Cartan subalgebra of* $C_r^*(\Gamma)$ *if and only if* $\mathrm{Iso}(\Gamma)^{\mathrm{o}}$ *is closed.*

This result has many applications. For example, it induces the Cuntz–Krieger uniqueness theorem [8] and the general Cuntz–Krieger uniqueness theorem [19].

The motivations for Brown et al. [2] originally came from graph algebras and their generalizations. Indeed, many C*-algebras coming from graphs have groupoid models, including:

- directed graphs [9],
- topological graphs [6], and
- k-graphs [7].

The difficulty of applying the full power of Theorem 1.2 to these situations, is that it is unclear when the interior of the isotropy is closed in the associated groupoid. Brown et al. showed in [2] using indirect means that the interior of the isotropy is always closed in the groupoid associated to directed graphs, but they also provided a counterexample [2, Ex. 4.7] in which the interior of the isotropy group bundle of the path groupoid of the k-graph is not closed. Yang provided some partial results for k-graph C*-algebras in [20], and proved the sufficiency of Theorem 1.2 for k-graph C*-algebras in [21]. However, there are currently no conditions intrinsic to a k-graph that show the associated groupoid satisfies the conditions of Theorem 1.2.

In this paper, we investigate examples of étale groupoids arising from directed graphs, topological graphs and k-graphs. Given a graph as we mentioned, we determine when the interior of the isotropy group bundle of the associated groupoid is closed. We begin with a brief review on the background of groupoid C*-algebras, graph algebras, topological graph algebras, and k-graph C*-algebras in Section 2.

Let E be a topological graph without singular vertices, $\Gamma(E^\infty,\sigma)$ the associated path groupoid, and $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ the interior of the isotropy of $\Gamma(E^\infty,\sigma)$. In Section 3 we study when $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is closed in $\Gamma(E^\infty,\sigma)$. In Example 3.1 we show that $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ need not be closed in general. Our main result is Theorem 3.8 which characterizes when $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is closed in $\Gamma(E^\infty,\sigma)$ using base points of cycles without entrances. More precisely, let E be a topological graph without singular vertices. We show that $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is closed in $\Gamma(E^\infty,\sigma)$ if and only if

$$V_n := \left\{ v \in E^0 \;\middle|\; \begin{array}{l} \text{there is an open neighborhood of } v \text{ consisting of} \\ \text{base points of cycles without entrances in } E^n \end{array} \right\}$$

is closed in E^0 for all $n \geq 1$.

Let Λ be a row-finite k-graph without sources, $\mathcal{G}_\Lambda$ the associated groupoid, and $\mathrm{Iso}(\mathcal{G}_\Lambda)^{\mathrm{o}}$ the interior of the isotropy of $\mathcal{G}_\Lambda$. In Section 4 we study when $\mathrm{Iso}(\mathcal{G}_\Lambda)^{\mathrm{o}}$ is closed in $\mathcal{G}_\Lambda$. By [2, Ex. 4.7] this does not hold in general. We then characterize when $\mathrm{Iso}(\mathcal{G}_\Lambda)^{\mathrm{o}}$ is closed in $\mathcal{G}_\Lambda$ using certain periodic paths in

Theorem 4.4. More precisely, for $p \neq q \in \mathbb{N}^k$, denote by $\Lambda^\infty_{p,q}$ the set consisting of $x \in \Lambda^\infty$ satisfying the following properties:

(i) $\sigma^p(x) = \sigma^q(x)$;

(ii) for any $p', q' \in \mathbb{N}^k$ with $p' - p = q' - q \in \mathbb{N}^k$, the pair $(x(0,p'), x(0,q'))$ is not cycline (see Definition 4.1), and $(x(0,p')\mu, x(0,q')\mu)$ is a cycline pair for some $\mu \in \Lambda$.

Then Theorem 4.4 says that $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ is closed if and only if $\Lambda^\infty_{p,q} = \varnothing$ for all $p \neq q \in \mathbb{N}^k$.

Finally, in Appendix A we present a short and direct proof that the interior of the isotropy group bundle of the boundary path groupoid of a row-finite directed graph without sources is always closed. This theorem can be inferred from Section 3, but we include it here because of its relative simplicity.

2. Preliminaries

Throughout this paper, all topological spaces are assumed to be second countable; all locally compact groupoids are assumed to be second-countable locally compact Hausdorff groupoids. By $\mathbb{N}$ (resp. $\mathbb{N}_+$), we denote the set of all nonnegative (resp. positive) integers.

2.1. **Groupoids.** In this subsection we recap the background of groupoid C*-algebras studied by Renault in [15, 18].

A groupoid is a small category (see [11]) where every morphism has an inverse. Let Γ be a groupoid. Denote by Γ^0 the set of objects in Γ which can be identified with the set of identity morphisms in Γ. If Γ is a groupoid then there exist maps $r, s : \Gamma \to \Gamma^0$ such that $r(\gamma) = \gamma\gamma^{-1}$ is the range of the morphism γ and $s(\gamma) = \gamma^{-1}\gamma$ is the source of the morphism γ. In this paper we deal exclusively with topological groupoids, that is, a groupoid with a topology in which composition and inversion are continuous. A locally compact groupoid is said to be *étale* if its range and source maps are both local homeomorphisms.

Notation 2.2. Let Γ be an étale groupoid. For $u \in \Gamma^0$, denote $\Gamma_u := s^{-1}(u)$, $\Gamma^u := r^{-1}(u)$, and by $\Gamma^u_u := \Gamma_u \cap \Gamma^u$ the *isotropy group* at u. Denote by $\mathrm{Iso}(\Gamma) := \bigcup_{u \in \Gamma^0} \Gamma^u_u$ the *isotropy group bundle*, which is closed in Γ. The interior of $\mathrm{Iso}(\Gamma)$, denoted by $\mathrm{Iso}(\Gamma)^\circ$, is open and is an étale subgroupoid of Γ.

Definition 2.3. Let Γ be an étale groupoid. Then Γ is said to be *essentially free* if the set of units whose isotropy groups are trivial is dense in Γ^0.

Definition 2.4. Let Γ be an étale groupoid. For $f, g \in C_c(\Gamma)$, $\gamma \in \Gamma$, define

$$f * g(\gamma) := \sum_{r(\beta)=r(\gamma)} f(\beta)g(\beta^{-1}\gamma) \quad \text{and} \quad f^*(\gamma) := \overline{f(\gamma^{-1})}.$$

Then $C_c(\Gamma)$ is a $*$-algebra. For $u \in \Gamma^0$, define $L^u : C_c(\Gamma) \to B(l^2(\Gamma_u))$ by

$$L^u(f)(\delta_\gamma) := \sum_{s(\beta)=r(\gamma)} f(\beta)\delta_{\beta\gamma} \quad \text{for all } f \in C_c(\Gamma),\ \gamma \in \Gamma.$$

L^u is a $*$-representation called the *left regular representation* at u. Define $L := \bigoplus_{u\in\Gamma^0} L^u$. For $f \in C_c(\Gamma)$, define $\|f\|_r := \|L(f)\|$. Then $\|\cdot\|_r$ is a C*-norm on $C_c(\Gamma)$ and the completion of $C_c(\Gamma)$ under the $\|\cdot\|_r$-norm is called the *reduced groupoid C*-algebra* of Γ which is denoted by $C^*_r(\Gamma)$ (it is the closure of $L(C_c(\Gamma))$).

Proposition 2.5 ([13, Prop. 1.9]). *Let Γ be a locally compact étale groupoid. Then the inclusion $\iota : C_c(\mathrm{Iso}(\Gamma)^o) \hookrightarrow C_c(\Gamma)$ induces an injective homomorphism $\iota_r : C^*_r(\mathrm{Iso}(\Gamma)^o) \hookrightarrow C^*_r(\Gamma)$.*

2.6. **Topological graphs.** Topological graphs were introduced in [4] which generalize directed graphs by adding topologies to the vertex and edge spaces. In this subsection, we review the background of topological graph algebras from [4].

Definition 2.7. A *topological graph* is a quadruple $E = (E^0, E^1, r, s)$ such that E^0, E^1 are locally compact Hausdorff spaces, $r : E^1 \to E^0$ is a continuous map, and $s : E^1 \to E^0$ is a local homeomorphism. In particular, a *directed graph* is a topological graph E where E^0 and E^1 are countable and discrete.

Let E be a topological graph. A subset U of E^1 is called an *s-section* if $s|_U : U \to s(U)$ is a homeomorphism with respect to the subspace topologies. Define the set of *finite receivers* E^0_{fin} consisting of all $v \in E^0$ which has an open neighborhood N such that $r^{-1}(\overline{N})$ is compact. Define the set of *sources* by $E^0_{\mathrm{sce}} := E^0 \setminus \overline{r(E^1)}$. Define the set of *regular vertices* by $E^0_{\mathrm{rg}} := E^0_{\mathrm{fin}} \setminus \overline{E^0_{\mathrm{sce}}}$. Moreover, define the set of *singular vertices* by $E^0_{\mathrm{sg}} := E^0 \setminus E^0_{\mathrm{rg}}$. Notice that the sets $E^0_{\mathrm{fin}}, E^0_{\mathrm{sce}}, E^0_{\mathrm{rg}}$ are all open, and the set E^0_{sg} is closed. In particular, a directed graph E is said to be *row-finite* if $E^0_{\mathrm{fin}} = F^0$; and E is said to be *without sources* if $E^0_{\mathrm{sce}} = \varnothing$. Notice that if $E_{\mathrm{rg}} = E^0$ then r is a proper map.

Let E be a topological graph. For $n \geq 2$, define

$$E^n := \Big\{\mu = (\mu_1, \ldots, \mu_n) \in \prod_{i=1}^{n} E^1 \mid s(\mu_i) = r(\mu_{i+1}),\, i = 1, \ldots, n-1\Big\}$$

endowed with the subspace topology of the product space $\prod_{i=1}^n E^1$. For convenience, for $\mu \in E^n$ we write $\mu = \mu_1 \cdots \mu_n$ instead of $(\mu_1, \ldots, \mu_n)$. We extend the range and source maps of E to E^n by $r^n(\mu_1 \cdots \mu_n) = r(\mu_1)$ and $s^n(\mu_1 \cdots \mu_n) = s(\mu_n)$, in this case we say μ *connects* $r^n(\mu)$ and $s^n(\mu)$. We call an open subset $O \subset E^n$ an *s^n-section* if $s^n|_O : O \to s^n(O)$ is a homeomorphism. Define the *finite path space* by $E^* := \coprod_{n=0}^{\infty} E^n$ with the disjoint union topology. We call elements of E^* *(finite) paths.*

Definition 2.8. Let E be a topological graph. For $n \geq 1$, a finite path $\mu \in E^n$ is called a *cycle* if $r(\mu_1) = s(\mu_n)$, and the vertex $r(\mu_1)$ is called the *base point* of the cycle. The cycle $\mu \in E^n$ is said to be *without entrances* if $r^{-1}(r(\mu_i)) = \{\mu_i\}$ for $i = 1, \ldots, n$. Furthermore, E is said to be *topologically free* if the set of base points of cycles without entrances has empty interior.

Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$. Define the *infinite path space* by

$$E^\infty := \Big\{x \in \prod_{i=1}^{\infty} E^1 \mid s(x_i) = r(x_{i+1}),\ i = 1, 2, \ldots\Big\}.$$

We call elements $x \in E^\infty$ *infinite paths* and for convenience write $x = x_1x_2\cdots$ instead of $x = (x_1, x_2, \ldots)$. Denote the length of a path $\mu \in E^* \amalg E^\infty$ by $|\mu|$. Endow E^∞ with the subspace topology inherited from the product space $\prod_{i=1}^{\infty} E^1$. A basis for this topology consists of $Z(U) := \{x \in E^\infty \mid x_1 \ldots x_n \in U\}$ where U is open in E^n for some $n \geq 1$. The product topology on E^∞ is locally compact Hausdorff by [6, Def. 4.7, Lem. 4.8]. The one-sided shift map $\sigma : E^\infty \to E^\infty$, $x_1x_2\cdots \mapsto x_2x_3\cdots$ is a local homeomorphism by [6, Lem. 7.1].

Definition 2.9 ([16, Def. 2.4])**.** Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$. Define the *boundary path groupoid* by

$$\Gamma(E^\infty, \sigma) := \big\{(x, k - \ell, y) \in E^\infty \times \mathbb{Z} \times E^\infty \mid \sigma^k(x) = \sigma^\ell(y)\big\}.$$

The range of (x, m, y) is x and its source is y, so

$$\mathrm{Iso}(\Gamma(E^\infty, \sigma)) = \{(x, k, x) \in \Gamma(E^\infty, \sigma)\}.$$

For $k, \ell \in \mathbb{N}$ and open subsets U, V of E^∞ such that σ^k is injective on U, and σ^ℓ is injective on V, denote

$$\mathcal{U}(U, V, k, \ell) := \big\{(x, k - \ell, y) \mid x \in U,\ y \in V,\ \sigma^k(x) = \sigma^\ell(y)\big\}.$$

The collection $\{\mathcal{U}(U, V, k, \ell)\}$ of subsets of $\Gamma(E^\infty, \sigma)$ as above forms an open base on $\Gamma(E^\infty, \sigma)$, and under this topology $\Gamma(E^\infty, \sigma)$ is an étale groupoid.

We give a characterization of convergent sequences in $\Gamma(E^\infty, \sigma)$. Fix a sequence $((x^t, n_t, y^t))_{t=1}^{\infty} \subset \Gamma(E^\infty, \sigma)$, and fix $(x, n, y) \in \Gamma(E^\infty, \sigma)$. Using the well-ordering principle, find $k, \ell \geq 0$ such that

(i) $n = k - \ell$, $\sigma^k(x) = \sigma^\ell(y)$;

(ii) for $k', \ell' \geq 0$ satisfying that $k' \leq k$, $\ell' \leq \ell$, $n = k' - \ell'$, $\sigma^{k'}(x) = \sigma^{\ell'}(y)$, we have $k' = k$, $\ell' = \ell$.

Then $(x^t, n_t, y^t) \to (x, n, y)$ if and only if $x^t \to x$, $y^t \to y$, and there exists $N \geq 1$ such that whenever $t \geq N$, we have $n_t = n$ and $\sigma^k(x) = \sigma^\ell(y)$.

Remark 2.10. Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$. In [4], Katsura defined the topological graph algebra $\mathcal{O}(E)$ by modifying Pimsner's construction in [14]. Recently, Kumjian and Li in [6] proved that

$$C^*_r(\Gamma(E^\infty, \sigma)) \cong \mathcal{O}(E).$$

2.11. ***k*-Graphs.** In this subsection, we recall the background of k-graph C*-algebras from [7].

Notation 2.12. Let $k \in \mathbb{N}_+$, $n, m \in \mathbb{N}^k$. Denote $n \vee m := (\max\{n_i, m_i\})_{i=1}^{k}$. For a category Λ we denote its objects by Λ^0 and the range and source maps from Λ to Λ^0 by r and s, respectively.

Definition 2.13 ([7, Defs. 1.1, 1.4])**.** Let $k \in \mathbb{N}_+$. A countable category Λ is called a *k-graph* if there exists a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *unique factorization property*, that is, for $\mu \in \Lambda$, $n, m \in \mathbb{N}^k$ with $d(\mu) = n + m$, there exist unique $\nu, \alpha \in \Lambda$ such that $d(\nu) = n$, $d(\alpha) = m$, $s(\nu) = r(\alpha)$, $\mu = \nu\alpha$. The functor d is called the *degree map* of Λ.

Let $(\Lambda_1, d_1), (\Lambda_2, d_2)$ be two k-graphs. A functor $f : \Lambda_1 \to \Lambda_2$ is called a *k-graph morphism* if $d_2 \circ f = d_1$.

For $n \in \mathbb{N}^k$ and $v \in \Lambda^0$ denote

$$v\Lambda := r^{-1}(v), \quad \Lambda^n := d^{-1}(n), \quad v\Lambda^n := v\Lambda \cap \Lambda^n.$$

Λ is *row-finite* if $|v\Lambda^n| < \infty$ for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$. Λ is *without sources* if $v\Lambda^n \neq \varnothing$ for all $v \in \Lambda^0$, $n \in \mathbb{N}^k$.

Example 2.14 ([7, Ex. 1.7 (ii)])**.** Let $k \in \mathbb{N}_+$. Define

$$\Omega_k := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k \mid p \leq q\},$$

$\Omega_k^0 := \mathbb{N}^k$, $r(p, q) := p$, $s(p, q) := q$, and $d(p, q) := q - p$. Then $(\Omega_k, \Omega_k^0, r, s)$ is a k-graph.

Definition 2.15 ([7, Defs. 2.1, 2.4])**.** Let $k \in \mathbb{N}_+$ and let Λ be a row-finite k-graph without sources. An *infinite path* is a k-graph morphism from Ω_k to Λ: denote by Λ^∞ the set of all infinite paths of Λ and extend r to Λ^∞ by $r(x) = x(0)$. Define $v\Lambda^\infty := \{x \in \Lambda^\infty \mid r(x) = v\}$. For $x \in \Lambda^\infty$ and $p \in \mathbb{N}^k$, denote by $\sigma^p(x)$ the unique element in Λ^∞ such that $x = x(0, p)\sigma^p(x)$. For $\mu \in \Lambda$, denote $Z(\mu) := \{\mu x \mid x \in \Lambda^\infty, s(\mu) = x(0)\}$. Endow Λ^∞ with the topology generated by the basic open sets $\{Z(\mu) \mid \mu \in \Lambda\}$.

Definition 2.16 ([7, Def. 2.7])**.** Let $k \in \mathbb{N}_+$ and let Λ be a row-finite k-graph without sources. Define the *path groupoid* by

$$\mathcal{G}_\Lambda := \{(x, p - q, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid p, q \in \mathbb{N}^k, \sigma^p(x) = \sigma^q(y)\}.$$

The range of (x, m, y) is x and its source is y. So $\mathrm{Iso}(\mathcal{G}_\Lambda) = \{(x, k, x) \in \mathcal{G}_\Lambda\}$.

For $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$, denote

$$Z(\mu, \nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) \mid x \in \Lambda^\infty, s(\mu) = x(0)\}.$$

Endow $\mathcal{G}_\Lambda$ with the topology generated by the basic open sets

$$\{Z(\mu, \nu) \mid \mu, \nu \in \Lambda, s(\mu) = s(\nu)\}.$$

By [7, Prop. 2.8], $\mathcal{G}_\Lambda$ is an étale groupoid and in particular each $Z(\mu, \nu)$ is a compact open bisection. Also Λ^∞ is a locally compact Hausdorff space and each $Z(\mu)$ is a compact open set.

We now give a characterization of convergent sequences in $\mathcal{G}_\Lambda$. Fix a sequence $((x^t, n, y^t))_{t=1}^\infty \subset \mathcal{G}_\Lambda$, and fix $(x, n, y) \in \mathcal{G}_\Lambda$. We have $(x^t, n, y^t) \to (x, n, y)$ if and only if for any $p, q \in \mathbb{N}^k$ satisfying that $p - q = n$ and $\sigma^p(x) = \sigma^q(y)$, there exists $N \geq 1$, such that $t \geq N$ implies

$$x^t(0, p) = x(0, p), \quad y^t(0, q) = y(0, q), \quad \sigma^p(x^t) = \sigma^q(y^t).$$

Remark 2.17. Let $k \in \mathbb{N}_+$ and let Λ be a row-finite k-graph without sources. Kumjian and Pask in [7] defined the k-graph C*-algebra $C^*(\Lambda)$ using the combinatorial method and they also showed that $C_r^*(\mathcal{G}_\Lambda) \cong C^*(\Lambda)$.

3. Cartan subalgebras of topological graph algebras

In this section, we give a complete characterization of when the interior of the isotropy group bundle of the boundary path groupoid for a topological graph E without singular vertices is closed.

The next example shows that for a topological graph E without singular vertices, $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is not closed in general.

Example 3.1. Let

$$E^0 = E^1 := \{0\} \cup \Big\{ \Big(\frac{1}{n}, 0\Big), \Big(-\frac{1}{n}, 0\Big) \mid n \geq 1 \Big\} \cup \Big\{ \Big(-\frac{1}{n}, \frac{1}{m}\Big) \mid m \geq n \geq 1 \Big\}$$

with the topology induced from $\mathbb{R}^2$. Define r to be the identity map. Define s to be the identity map on $\{0\} \cup \{(\frac{1}{n}, 0), (-\frac{1}{n}, 0)\}_{n=1}^\infty$. For $m \geq n \geq 1$, define

$$s\Big(-\frac{1}{n}, \frac{1}{m}\Big) := \Big(-\frac{1}{n}, \frac{1}{m+1}\Big).$$

Then E is a topological graph with $E^0_{\mathrm{sg}} = \varnothing$.

Denote $x := 000\cdots$. Then $(x, 1-0, x) \in \mathrm{Iso}(\Gamma(E^\infty,\sigma))$. For $n \geq 1$, denote

$$x^n := \Big(\frac{1}{n}, 0\Big)\Big(\frac{1}{n}, 0\Big)\Big(\frac{1}{n}, 0\Big)\cdots,$$
$$y^n := \Big(-\frac{1}{n}, \frac{1}{n}\Big)\Big(-\frac{1}{n}, \frac{1}{n+1}\Big)\Big(-\frac{1}{n}, \frac{1}{n+2}\Big)\cdots,$$
$$z^n := \Big(-\frac{1}{n}, \frac{1}{n+1}\Big)\Big(-\frac{1}{n}, \frac{1}{n+2}\Big)\Big(-\frac{1}{n}, \frac{1}{n+3}\Big)\cdots.$$

Notice that $(x^n, 1-0, x^n) \in \mathcal{U}(Z(\frac{1}{n},0), Z(\frac{1}{n},0), 1, 0) \subset \mathrm{Iso}(\Gamma(E^\infty,\sigma))$ and $(x^n, 1-0, x^n) \to (x, 1-0, x)$, which implies that $(x, 1-0, x) \in \overline{\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}}$. On the other hand, $(y^n, 1-0, z^n) \notin \mathrm{Iso}(\Gamma(E^\infty,\sigma))$ and $(y^n, 1-0, z^n) \to (x, 1-0, x)$. So $(x, 1-0, x) \notin \mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$. Therefore $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is not closed.

Notation 3.2. Let E be a topological graph. For $n \geq 1$, denote by C^n the set of cycles in E^n, which is a closed subset of E^n. For $k \geq 1$, $n \geq 1$, $\mu \in C^n$, and for an open neighborhood N of μ, denote

$$k\mu := \underbrace{\mu \cdots \mu}_{k}, \quad kN := \underbrace{N \times \cdots \times N}_{k}.$$

Denote by B^n the set of all cycles μ in C^n satisfying that there exist $k \geq 1$ and an open neighborhood N of μ such that

(i) for any distinct $\alpha, \beta \in kN$, there are no paths in N connecting $s(\alpha)$, $s(\beta)$; and
(ii) for any $\alpha \in kN$, there are no cycles in N with entrances and of base point $s(\alpha)$.

Remark 3.3. By condition (ii) of Notation 3.2, for each $n \geq 1$, each cycle in B^n has no entrances.

We show in Theorem 3.6 below that $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^\circ$ is closed in $\Gamma(E^\infty, \sigma)$ if and only if B^n is closed for all n.

Remark 3.4. If E is a directed graph and μ is a cycle in E, then we can take $N = \{\mu\}$; in this case condition (i) is vacuous and condition (ii) says that μ does not have an entrance. So for directed graphs, B^n consists of cycles of length n without entrances. Furthermore, since E is discrete, E^n is discrete and B^n is closed. Therefore, by Theorem 3.6 below, $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^\circ$ is always closed.

Lemma 3.5. *Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$, and let $(x, n, x) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))$. Pick $p, q \geq 0$ with $p - q = n$ and $\sigma^p(x) = \sigma^q(x)$.*

(i) *If $n > 0$, then $(x, n, x) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^\circ$ if and only if $x_{q+1} \cdots x_p \in B^n$.*
(ii) *If $n < 0$, then $(x, n, x) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^\circ$ if and only if $x_{p+1} \cdots x_q \in B^n$.*

Proof. We may assume that $n > 0$ and for the case $n < 0$ the argument is similar. Suppose that $(x, n, x) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^\circ$. Then there exist open subsets U, V of E^* such that $(x, n, x) \in \mathcal{U}(Z(U), Z(V), p, q) \subset \mathrm{Iso}(\Gamma(E^\infty, \sigma))$. Since $U \subset E^{n_U}$, $V \subset E^{n_V}$ for some n_U, n_V, picking $M > \max\{p, n_U, n_V\}$, we see that for $m \geq M$ there exists an open subset $W \subset E^m$ such that $(x, n, x) \in \mathcal{U}(Z(W), Z(W), p, q) \subset \mathcal{U}(Z(U), Z(V), p, q)$. Pick $m \geq M$. Write $m = q + kn + l$, where $k \geq 1$, $0 \leq l < n$. We may choose W such that

$$W = (W_0 \times k(W_1 \times \cdots \times W_n) \times (W_1 \times \cdots \times W_l)) \cap E^m,$$

where W_0 is an open neighborhood of $\mu_1 \cdots \mu_q$; $W_1 \times \cdots \times W_n$ is an open neighborhood of the cycle $\mu_{q+1} \cdots \mu_p$; $W_0, W_1, \ldots, W_n$ are open s-sections; and $r(W_1) \subset s(W_0)$. Let

$$W' = (W_0 \times (k+1)(W_1 \times \cdots \times W_n) \times (W_1 \times \cdots \times W_l)) \cap E^{m+n}.$$

Then $\mathcal{U}(Z(W'), Z(W), p, q) \subset \mathcal{U}(Z(W), Z(W), p, q)$.

We claim that $\mathcal{U}(Z(W'), Z(W), p, q) = \mathcal{U}(Z(W'), Z(W), m+n, m)$. First show

$$\mathcal{U}(Z(W'), Z(W), m+n, m) \subset \mathcal{U}(Z(W'), Z(W), p, q). \tag{1}$$

Fix $(y, n, z) \in \mathcal{U}(Z(W'), Z(W), m+n, m)$. Then

$$\sigma^{m-q}(\sigma^{q+n}(y)) = \sigma^{m+n}(y) = \sigma^m(z) = \sigma^{m-q}(\sigma^q(z)). \tag{2}$$

Since $\sigma^{q+n}(y), \sigma^q(z) \in Z(k(W_1 \times \cdots \times W_n) \times (W_1 \times \cdots \times W_l))$, by [6, Lem. 7.1] $\sigma^{q+n}(y) = \sigma^q(z)$. So $(y, n, z) \in \mathcal{U}(Z(W'), Z(W), p, q)$. Hence (1) holds. That $\mathcal{U}(Z(W'), Z(W), p, q) \subset \mathcal{U}(Z(W'), Z(W), m+n, m)$ follows from a permutation of equation (2).

Choose $m_0 = q + k_0 n \geq M$ for some $k_0 \geq 1$. Construct W_i, W and W' for this m_0 as above. We claim k_0 and $N := W_1 \times \cdots \times W_n$ satisfy conditions (i) and (ii) of Notation 3.2.

For condition (i), if $\alpha, \beta \in k_0N$ and $\gamma \in N$ with $s(\alpha\gamma) = s(\beta)$ pick $y \in E^\infty$ with $r(y) = s(\beta)$ and $\nu_\alpha, \nu_\beta \in W_0$ with $s(\nu_\omega) = r(\omega)$. Then

$$(\nu_\alpha\alpha\gamma y, n, \nu_\beta\beta y) \in \mathcal{U}(Z(W'), Z(W), m+n, m) \subset \mathrm{Iso}(\Gamma(E^\infty, \sigma)).$$

Therefore $\nu_\alpha\alpha\gamma y = \nu_\beta\beta y$ and so $\nu_\alpha = \nu_\beta$ and $\alpha = \beta$.

For condition (ii), suppose $\alpha \in k_0N$ and $\gamma \in s(\alpha)N$ is a cycle with an entrance. Write $\gamma = \gamma_1 \cdots \gamma_n$. Then there exists i_0 such that $r^{-1}(r(\gamma_{i_0})) - \{\gamma_{i_0}\} \neq \varnothing$. Pick $\delta \in r^{-1}(r(\gamma_{i_0})) - \{\gamma_{i_0}\}$. Take $\gamma' = \gamma_1 \cdots \gamma_{i_0-1}$ and pick $y \in Z(s(\delta))$ and $\nu \in W_0$. Consider $z = \gamma'\delta y$. Then

$$(\nu\alpha\gamma z, n, \nu\alpha z) \in \mathcal{U}(Z(W'), Z(W), m_0+n, m_0) \subset \mathrm{Iso}(\Gamma(E^\infty, \sigma)).$$

Therefore $\gamma\gamma'\delta = \gamma'\delta$ contradicting that $\delta \neq \gamma_{i_0}$.

Conversely, suppose that $x_{q+1} \cdots x_p \in B^n$. Then there exist $k \geq 1$ and an open neighborhood N of x satisfying conditions (i) and (ii) of Notation 3.2. Choose an arbitrary open s^q-section O containing $x_1 \cdots x_q$. Let

$$W := (O \times kN) \cap E^{q+kn}, \quad W' := (O \times (k+1)N) \cap E^{q+(k+1)n}.$$

Now

$$\begin{aligned}(x, n, x) &\in \mathcal{U}(Z(W'), Z(W), p, q)\\ &\subset \mathcal{U}(Z(W'), Z(W), p+kn, q+kn)\\ &\subset \mathrm{Iso}(\Gamma(E^\infty, \sigma)).\end{aligned}$$

So $(x, n, x) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$. □

Theorem 3.6. *Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$. Then $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$ is closed in $\Gamma(E^\infty, \sigma)$ if and only if B^n is closed in E^n for all $n \geq 1$.*

Proof. Suppose that $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$ is closed in $\Gamma(E^\infty, \sigma)$. Fix $n \geq 1$. Fix $(\mu^k)_{k=1}^\infty \subset B^n$ which is convergent to $\mu \in E^n$ (μ is a cycle). Then

$$(\mu^k\mu^k \cdots, n-0, \mu^k\mu^k \cdots) \to (\mu\mu \cdots, n-0, \mu\mu \cdots).$$

Fix $k \geq 1$. By the definition of B^n in Notation 3.2 and by Lemma 3.5, we have $(\mu^k\mu^k \cdots, n-0, \mu^k\mu^k \cdots) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$. Since $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$ is closed, $(\mu\mu \cdots, n-0, \mu\mu \cdots) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$. Again by the definition of B^n in Notation 3.2 and by Lemma 3.5, we have $\mu \in B^n$. So B^n is closed.

Conversely, suppose that B^n is closed in E^n for all $n \geq 1$. Fix a convergent net $(x^k, n_k, x^k) \subset \mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$ with the limit (x, n, x). We may assume that $n_k = n$ for all k and we take arbitrary $p, q \geq 0$ such that $p - q = n$, $\sigma^p(x^k) = \sigma^q(x^k)$, and $\sigma^p(x) = \sigma^q(x)$. We may further assume that $n > 0$ since the case $n < 0$ shares a symmetric proof. By Lemma 3.5, $x^k_{q+1} \cdots x^k_p \in B^n$. Since $x^k_{q+1} \cdots x^k_p \to x_{q+1} \cdots x_p$ and B^n is closed, we have $x_{q+1} \cdots x_p \in B^n$. Again by Lemma 3.5, $(x, n, x) \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$. □

Since B^n is closed for a directed graph, Theorem 3.6 gives Proposition A.2 (see Remark 3.4).

Notation 3.7. Let E be a topological graph. For $n \geq 1$, denote

$$V_n := \left\{ v \in E^0 \;\middle|\; \begin{array}{l} \text{there is an open neighborhood of } v \text{ consisting of} \\ \text{base points of cycles without entrances in } E^n \end{array} \right\}.$$

Notice that V_n is an open subset of E^0 for all $n \geq 1$.

Theorem 3.8. *Let E be a topological graph. Fix $n \geq 1$. Then $r^n(B^n) = V_n$ and $B^n = (r^n)^{-1}(V_n)$. Hence B^n is an open subset of E^n. Furthermore, suppose that $E^0_{\mathrm{sg}} = \varnothing$. Then B^n is closed if and only if V_n is closed. Therefore $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$ is closed in $\Gamma(E^\infty, \sigma)$ if and only if V_m is closed in E^0 for all $m \geq 1$.*

Proof. First of all, we show that $r^n(B^n) = V_n$. Fix $\mu \in B^n$. Then there exist $k \geq 1$ and an open neighborhood N of μ satisfying conditions (i) and (ii) of Notation 3.2. Then

$$W := s^{(k+1)n}((k+1)N \cap E^{(k+1)n})$$

is an open neighborhood of $r^n(\mu)$. For $v \in W$, there exists

$$\alpha = \alpha^{(1)} \cdots \alpha^{(k)} \alpha^{(k+1)} \in (k+1)N \cap E^{(k+1)n},$$

where $\alpha^{(i)} \in N$, such that $s^{(k+1)n}(\alpha) = v$. So $\alpha^{(k+1)}$ connects

$$s^{kn}(\alpha^{(1)} \cdots \alpha^{(k)}) = s^{kn}(\alpha^{(2)} \cdots \alpha^{(k+1)}).$$

By condition (i) of Notation 3.2, we have $\alpha^{(1)} \cdots \alpha^{(k)} = \alpha^{(2)} \cdots \alpha^{(k+1)}$. So $\alpha^{(1)} = \cdots = \alpha^{(k)} = \alpha^{(k+1)}$ is a cycle in E^n. By condition (ii) of Notation 3.2, $\alpha^{(1)} = \cdots = \alpha^{(k)} = \alpha^{(k+1)}$ has no entrances. So $r^n(\mu) \in V_n$ and $r^n(B^n) \subset V_n$.

Conversely, fix $v \in V_n$. Then there exists an open neighborhood W of v consisting of base points of cycles without entrances in E^n. So $(r^n)^{-1}(W)$ is an open neighborhood of $(r^n)^{-1}(v)$ consisting of cycles without entrances. It is straight-forward to see that $(r^n)^{-1}(v) \in (r^n)^{-1}(W) \subset B^n$. So $r^n(B^n) = V_n$. It follows immediately that $B^n = (r^n)^{-1}(V_n)$. Hence B^n is an open subset of E^n because V_n is open.

Finally, suppose $E^0_{\mathrm{sg}} = \varnothing$ and B^n is closed. Since $E^0_{\mathrm{sg}} = \varnothing$, r^n is proper, and so r^n is closed. Since $r^n(B^n) = V_n$, V_n is closed. Conversely, suppose that V_n is closed. Since $B^n = (r^n)^{-1}(V_n)$, B^n is closed. □

Corollary 3.9. *Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$. Then the following are equivalent.*

(i) *E is topologically free.*
(ii) *$V_n = \varnothing$ for all $n \geq 1$.*
(iii) *$B^n = \varnothing$ for all $n \geq 1$.*
(iv) *$\Gamma(E^\infty, \sigma)$ is essentially free.*

In these cases, $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^{\mathrm{o}}$ *is closed in* $\Gamma(E^\infty, \sigma)$.

Proof. (i) $\Leftrightarrow$ (ii). We prove the contrapositive. If $V_n \neq \varnothing$ for some n then E is not topologically free by definition. Now if E not topologically free then

[5, Prop. 6.12] gives a nonempty open subset $V \subset E^0$ and $n \in \mathbb{N}_+$ such that V consists of base points of cycles in E^n and thus $V_n \neq \varnothing$.

(ii) $\Leftrightarrow$ (iii) follows from Theorem 3.8.

(i) $\Leftrightarrow$ (iv) follows from [10, Prop. 3.7]. □

Corollary 3.10. *Let E be a topological graph such that $E^0_{\mathrm{sg}} = \varnothing$ and $\mathcal{O}(E)$ is simple. Then $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^\circ$ is closed in $\Gamma(E^\infty, \sigma)$.*

Proof. This follows from Corollary 3.9 and [5, Thm. 8.12]. □

4. Cartan subalgebras of k-graph algebras

In this section, we characterize when the interior of the isotropy of the path groupoid of a row-finite k-graph without sources is closed.

Definition 4.1 ([1, Def. 4.3])**.** Let $k \in \mathbb{N}_+$ and let Λ be a row-finite k-graph without sources. Then a pair $(\mu, \nu) \in \Lambda \times \Lambda$ is called a *cycline pair* if $s(\mu) = s(\nu)$ and $\mu x = \nu x$ for all $x \in s(\mu)\Lambda^\infty$.

The following lemma is stated without proof in [1, Rem. 4.11].

Lemma 4.2. *Let $k \geq 1$ and let Λ be a row-finite k-graph without sources. Then*

$$\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ = \big\{(x, p-q, x) \mid \sigma^p(x) = \sigma^q(x),\ (x(0,p), x(0,q)) \text{ is a cycline pair}\big\}.$$

Proof. First of all, fix $(x, n, x) \in \mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$. Then there exist $\mu, \nu \in \Lambda$ such that $(x, n, x) \in Z(\mu, \nu) \subset \mathrm{Iso}(\mathcal{G}_\Lambda)$. So

$$\sigma^{d(\mu)}(x) = \sigma^{d(\nu)}(x), \quad (x, n, x) = (\mu\sigma^{d(\mu)}(x), d(\mu) - d(\nu), \nu\sigma^{d(\nu)}(x)),$$

and (μ, ν) is a cycline pair.

Conversely, fix $(x, p-q, x) \in \mathcal{G}_\Lambda$ such that $\sigma^p(x) = \sigma^q(x)$ and $(x(0,p), x(0,q))$ is a cycline pair. Then $(x, p-q, x) \in Z(x(0,p), x(0,q)) \subset \mathrm{Iso}(\mathcal{G}_\Lambda)$. So we have $(x, p-q, x) \in \mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$. □

Notation 4.3. Let $k \geq 1$ and let Λ be a row-finite k-graph without sources. For $p \neq q \in \mathbb{N}^k$, denote by $\Lambda^\infty_{p,q}$ the set consisting of $x \in \Lambda^\infty$ satisfying the following properties:

(i) $\sigma^p(x) = \sigma^q(x)$;

(ii) for any $p', q' \in \mathbb{N}^k$ with $p' - p = q' - q \in \mathbb{N}^k$, the pair $(x(0,p'), x(0,q'))$ is not cycline, and $(x(0,p')\mu, x(0,q')\mu)$ is a cycline pair for some $\mu \in \Lambda$.

Theorem 4.4. *Let $k \geq 1$ and let Λ be a row-finite k-graph without sources. Then $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ is closed if and only if $\Lambda^\infty_{p,q} = \varnothing$ for all $p \neq q \in \mathbb{N}^k$.*

Proof. First of all, suppose that $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ is closed. Aiming at a contradiction, suppose that there exist $p \neq q \in \mathbb{N}^k$ such that $\Lambda^\infty_{p,q} \neq \varnothing$. Fix $x \in \Lambda^\infty_{p,q}$. For $n \geq 1$, let $p_n := p + (n, \ldots, n)$ and $q_n := q + (n, \ldots, n)$. By condition (ii) of Notation 4.3, for $n \geq 1$ there exist $y_n \in s(x(0,p_n))\Lambda^\infty$ and $\mu_n \in s(x(0,p_n))\Lambda$

such that $x(0,p_n)y_n \neq x(0,q_n)y_n$ and $(x(0,p_n)\mu_n, x(0,q_n)\mu_n)$ is a cycline pair. For $n \geq 1$, take an arbitrary $z_n \in s(\mu_n)\Lambda^\infty$. Then

$$(x(0,p_n)y_n, p-q, x(0,q_n)y_n) \to (x, p-q, x),$$
$$(x(0,p_n)\mu_n z_n, p-q, x(0,q_n)\mu_n z_n) \to (x, p-q, x).$$

However, $(x(0,p_n)y_n, p-q, x(0,q_n)y_n) \notin \mathrm{Iso}(\mathcal{G}_\Lambda)$; and by Lemma 4.2 one has $(x(0,p_n)\mu_n z_n, p-q, x(0,q_n)\mu_n z_n) \in \mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$. This is a contradiction. So $\Lambda^\infty_{p,q} = \varnothing$ for all $p \neq q \in \mathbb{N}^k$.

Conversely, suppose that $\Lambda^\infty_{p,q} = \varnothing$ for all $p \neq q \in \mathbb{N}^k$. Aiming at a contradiction assume that $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ is not closed. Then $\overline{\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ} \setminus \mathrm{Iso}(\mathcal{G}_\Lambda)^\circ \neq \varnothing$. Fix $(x, p-q, x) \in \overline{\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ} \setminus \mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$. We may assume that $p \neq q \in \mathbb{N}^k$ and $\sigma^p(x) = \sigma^q(x)$. Then there exist a sequence $(y_n, p-q, y_n)$ in $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ converging to $(x, p-q, x)$, and another sequence $(z_n, p-q, w_n)$ with $z_n \neq w_n$ for all $n \geq 1$ also converging to $(x, p-q, x)$. By Lemma 4.2 for $n \geq 1$ there exist $p_n, q_n \in \mathbb{N}^k$ with $p_n - q_n = p - q$ such that $\sigma^{p_n}(y_n) = \sigma^{q_n}(y_n)$ and $(y_n(0,p_n), y_n(0,q_n))$ is a cycline pair. Fix $p', q' \in \mathbb{N}^k$ with $p' - p = q' - q \in \mathbb{N}^k$. Then there exists $N \geq 1$ such that

$$y_N(0,p') = x(0,p'), \quad y_N(0,q') = x(0,q'), \quad \sigma^{p'}(y_N) = \sigma^{q'}(y_N),$$
$$z_N(0,p') = x(0,p'), \quad w_N(0,q') = x(0,q'), \quad \sigma^{p'}(z_N) = \sigma^{q'}(w_N).$$

Since $(y_N(0,p_N), y_N(0,q_N))$ is a cycline pair and $p_N \vee p' - q_N \vee q' = p - q$, we obtain a cycline pair

$$\begin{aligned}(y_N(0,p_N)&y_N(p_N, p_N \vee p'), y_N(0,q_N)y_N(q_N, q_N \vee q')) \\ &= (y_N(0,p')y_N(p', p_N \vee p'), y_N(0,q')y_N(q', q_N \vee q')) \\ &= (x(0,p')y_N(p', p_N \vee p'), x(0,q')y_N(q', q_N \vee q')).\end{aligned}$$

But we also have that $x(0,p')\sigma^{p'}(z_N) = z_N \neq w_N = x(0,q')\sigma^{q'}(w_N)$ which implies that $(x(0,p'), x(0,q'))$ is not a cycline pair. Hence $x \in \Lambda^\infty_{p,q}$ which is a contradiction. Therefore $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ is closed. □

Example 4.5. The following example is from [2, Ex. 4.7]. Consider the two-colored graph in Figure 1 where the factorization rules are given by

$$e_b\alpha_r = e_r\alpha_b, \quad e_b\beta_r = e_r\beta_b, \quad \alpha_b f_r = \alpha_r f_b, \quad f_b f_r = f_r f_b, \quad \beta_b g_r = \beta_r g_b,$$
$$\beta_b h_r = \beta_r h_b, \quad g_b g_r = g_r g_b, \quad g_b h_r = h_r g_b, \quad h_b g_r = g_r h_b, \quad h_b h_r = h_r h_b.$$

Define $x := e_b e_r e_b e_r e_b e_r \cdots$. It was shown in [2] that $\gamma := (x, (1,-1), x) \notin \mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ and $\mathrm{Iso}(\mathcal{G}_\Lambda)^\circ$ is not closed. We show that $x \in \Lambda^\infty_{(1,0),(0,1)}$ so that $\Lambda^\infty_{(1,0),(0,1)}$ is nonempty. Notice x satisfies condition (i) of Notation 4.3 by definition. For condition (ii) pick $p', q' \in \mathbb{N}^k$ with $p' \geq (1,0)$, $q' \geq (0,1)$ and $p' - q' = (1,-1)$. For $\mu = \alpha_b$, the pair $(x(0,p')\mu, x(0,q')\mu)$ is cycline. To show the pair $(x(0,p'), x(0,q'))$ is not cycline, define

$$y := \beta_b(g_b g_r h_b h_r)(g_b g_r h_b h_r)(g_b g_r h_b h_r)\cdots.$$

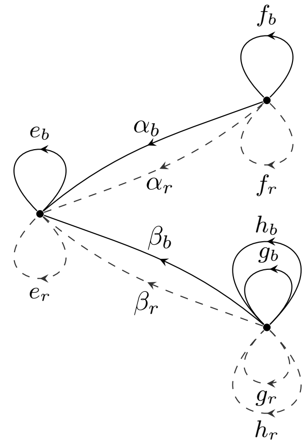

FIGURE 1. A 2-graph such that $\mathrm{Iso}(\mathcal{G}_\Lambda)^{\mathrm{o}}$ is not closed.

Then y is periodic, and we have $r(y) = s(x(0,p'))$ and $x(0,p')y \neq x(0,q')y$. Notice that in this example we used a periodic path to show that $(x(0,p'), x(0,q'))$ is not cycline.

APPENDIX A. CARTAN SUBALGEBRAS OF GRAPH ALGEBRAS

Let E be a directed graph: we consider E as a topological graph with the discrete topology. Suppose E is row-finite without sources, then combining [2, Cor. 4.5] and [12, Thm. 3.6], we get $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is closed in the boundary path groupoid $\Gamma(E^\infty,\sigma)$. In this appendix, we provide a direct proof of this result by investigating the boundary path groupoid of a directed graph. Given $\alpha,\beta \in E^*$ with $s(\alpha) = s(\beta)$, define

$$Z(\alpha,\beta) := \mathcal{U}(Z(\alpha), Z(\beta), |\alpha|, |\beta|).$$

The $Z(\alpha,\beta)$ form a basis for the topology on $\Gamma(E^\infty,\sigma)$.

Proposition A.1. *Let E be a row-finite directed graph without sources, let α,β with $|\alpha| \neq |\beta|$. Then $Z(\alpha,\beta) \cap \mathrm{Iso}(\Gamma(E^\infty,\sigma))$ is either empty or a singleton.*

Proof. If $Z(\alpha,\beta) \cap \mathrm{Iso}(\Gamma(E^\infty,\sigma))$ is empty, then we are done. Suppose that $Z(\alpha,\beta) \cap \mathrm{Iso}(\Gamma(E^\infty,\sigma))$ is nonempty. Since $|\alpha| \neq |\beta|$, without loss of generality, we assume that $|\alpha| < |\beta|$. Fix $(\alpha x, |\alpha| - |\beta|, \beta x) \in Z(\alpha,\beta) \cap \mathrm{Iso}(\Gamma(E^\infty,\sigma))$. Then $\alpha x = \beta x$. Since $|\alpha| < |\beta|$, we have $\beta = \alpha\gamma$, where γ is a cycle in $E^{|\beta|-|\alpha|}$. So $x = \gamma x$, which implies that $x = \gamma\gamma\cdots$. Therefore $Z(\alpha,\beta) \cap \mathrm{Iso}(\Gamma(E^\infty,\sigma))$ is a singleton. □

Proposition A.2. *Let E be a row-finite directed graph without sources and $\Gamma(E^\infty,\sigma)$ the associated groupoid. Then $\mathrm{Iso}(\Gamma(E^\infty,\sigma))^{\mathrm{o}}$ is closed.*

Proof. Suppose that $\gamma_i \to \gamma$ with $\gamma_i \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^o$. Since $\mathrm{Iso}(\Gamma(E^\infty, \sigma))$ is closed, we have $\gamma \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))$. If $\gamma \in \Gamma(E^\infty, \sigma)^{(0)}$ then $\gamma \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^o$ because $\Gamma(E^\infty, \sigma)$ is étale. Now suppose $\gamma \notin \Gamma(E^\infty, \sigma)^{(0)}$. Then there exist α, β with $|\alpha| \neq |\beta|$ such that $\gamma \in Z(\alpha, \beta)$. Since $\gamma_i \to \gamma$, there exists $i_0 \geq 1$ such that $\gamma_{i_0} \in Z(\alpha, \beta)$. By Proposition A.1, $Z(\alpha, \beta) \cap \mathrm{Iso}(\Gamma(E^\infty, \sigma))$ is a singleton. So $\gamma = \gamma_{i_0} \in \mathrm{Iso}(\Gamma(E^\infty, \sigma))^o$. Hence $\mathrm{Iso}(\Gamma(E^\infty, \sigma))^o$ is closed. □

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