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Localization techniques in circle-equivariant Kasparov theory

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Abstract. Let $\mathbb{T}$ be the circle and A be a $\mathbb{T}$-C*-algebra. Then the $\mathbb{T}$-equivariant K-theory $\mathrm{K}_*^{\mathbb{T}}(A)$ of A is a module over the representation ring $\mathrm{Rep}(\mathbb{T})$ of the circle. The latter is a Laurent polynomial ring. Using the support of the module as an invariant, and techniques of Atiyah, Bott and Segal, we deduce that there are examples of $\mathbb{T}$-C*-algebras A such that A and $A \rtimes \mathbb{T}$ are in the bootstrap category, but A is not $\mathrm{KK}^{\mathbb{T}}$-equivalent to any commutative $\mathbb{T}$-C*-algebra. We also assemble various results on $\mathbb{T}$-equivariant K-theory of smooth manifolds and deduce an equivariant version of the Lefschetz fixed-point formula for $\mathbb{T}$-equivariant geometric correspondences.

1. Introduction

This article has several purposes. The first is to show that many $\mathbb{T}$-C*-algebras are not $\mathrm{KK}^{\mathbb{T}}$-equivalent to any commutative $\mathbb{T}$-C*-algebra, even though both they and their cross products by $\mathbb{T}$ are in the boostrap category. These examples include the Cuntz–Krieger algebras O_A with their usual circle actions. To prove this statement we use a simple $\mathrm{K}_*^{\mathbb{T}}$-theoretic obstruction to commutativity based on ideas of Atiyah, Bott and Segal.

The phenomenon just described is in sharp contrast to the nonequivariant situation: every C*-algebra in the boostrap category is KK-equivalent to a commutative one.

Here and throughout this article, $\mathrm{K}_*^{\mathbb{T}}(A) := \mathrm{K}_0^{\mathbb{T}}(A) \oplus \mathrm{K}_1^{\mathbb{T}}(A)$ denotes equivariant K-theory with *complex coefficients*, i.e. is the integral K-theory tensored by $\mathbb{C}$. In particular $\mathrm{Rep}(\mathbb{T}) = \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[X, X^{-1}]$ is the ring of Laurent polynomials with complex coefficients.

Study of equivariant K-theory groups $\mathrm{K}_G^*(X)$ as modules over $\mathrm{Rep}(G)$ (G a compact group) began with a series of papers by Atiyah, Bott and Segal, written in the 60's (see [2, 3, 4, 23].)

A common strategy in these articles is first to prove results about the case $G = \mathbb{T}$, and then extend them to the case of connected groups G using Lie group theory (we will restrict entirely to $\mathbb{T}$ in this article.) The article [2] treats equivariant cohomology (for torus actions) and contains a lot of the essential ideas used by us here, except that we work in equivariant K-theory instead. Other good sources for equivariant K-theory are the articles [23] of Segal and Atiyah–Segal [3]. As we wish to reach a wider readership than only those who are familiar with these articles, we have explained supports and localization rather carefully in this article.

Any module over $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$, and in particular, the module $\mathrm{K}_*^{\mathbb{T}}(A)$ for a $\mathbb{T}$-C*-algebra A, yields a sheaf of modules over $\mathbb{C}^*$ defined by localizing the module to Zariski open sets. Such a sheaf has a support. The techniques of Atiyah and Segal are used in the first part of the article to check that, for any locally compact $\mathbb{T}$-space X, the support of $\mathrm{K}_*^{\mathbb{T}}\big(C_0(X)\big) = \mathrm{K}^*_{\mathbb{T}}(X)$ is always either contained in the unit circle or is all of $\mathbb{C}^*$. But, as we observe, for a Cuntz–Krieger algebra O_A with its standard circle action, the support of the sheaf $\mathrm{K}_*^{\mathbb{T}}(O_A)$ is the set of nonzero eigenvalues of the 0-1-valued matrix A.

Thus Cuntz–Krieger algebras have rather arbitrary algebraic integers as spectral points. In particular, they are not generally $\mathrm{KK}^{\mathbb{T}}$-equivalent to commutative $\mathbb{T}$-C*-algebras.

The second purpose of this paper is to strengthen several results on $\mathbb{T}$-equivariant K-theory of compact smooth manifolds due to Atiyah et al., for example, proving that after a suitable localization, $C(X)$ is $\mathrm{KK}^{\mathbb{T}}$-equivalent to $C(F)$ with $F \subset X$ the stationary set, and to describe $\mathbb{T}$-equivariant K-theory for smooth manifolds in terms of various geometric data. This discussion is mainly for purpose of proving the *Lefschetz theorem* in $\mathrm{KK}^{\mathbb{T}}$.

The equivariant Lefschetz theorem proved here generalizes the classical Lefschetz fixed-point formula. Recall that this formula equates a homological invariant of a smooth self-map $f : X \to X$ with a geometric invariant of the map. Our equivariant Lefschetz formula takes into account a $\mathbb{T}$-action for which the map is equivariant; moreover, it applies to more general morphisms in $\mathrm{KK}^{\mathbb{T}}(C(X), C(X))$ than just the ones induced from smooth maps: our techniques work just as well for geometric correspondences in the sense of [15].

Since $\mathbb{C}[X, X^{-1}]$ is a principal ideal domain, any finitely generated $\mathbb{C}[X, X^{-1}]$-module M decomposes uniquely into a torsion module and a free module $\cong \mathbb{C}[X, X^{-1}]^n$. Any module self-map of M thus has a $\mathbb{C}[X, X^{-1}]$-valued trace by compressing it to the free part of M. In particular, this applies to any element $f \in \mathrm{KK}_*^{\mathbb{T}}(C(X), C(X))$ where X is a $\mathbb{T}$-space, for f acts by a module map on $\mathrm{K}^*_{\mathbb{T}}(X)$. We denote by $\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(f_*) \in \mathbb{C}[X, X^{-1}]$ the *graded* module trace of f_* in this sense.

In the case of a morphism f represented by a smooth $\mathbb{T}$-equivariant geometric correspondence in the sense of [15], the Lefschetz theorem identifies the homological invariant $\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(f_*)$ with the Atiyah–Singer $\mathbb{T}$-index of a certain geometrically defined coincidence cycle constructed out of the correspondence: that is, we prove that

$$\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(f_*) = \mathrm{ind}_{\mathbb{T}}\big(\mathrm{Lef}(f)\big); \tag{1.1}$$

where $\mathrm{Lef}(f)$ is the class in $\mathrm{KK}^{\mathbb{T}}$ of a certain $\mathbb{T}$-equivariant Baum–Douglas cycle for X, depending geometrically on the correspondence representing f and $\mathrm{ind}_{\mathbb{T}}$ is the Atiyah–Singer $\mathbb{T}$-index.

In particular the right hand side is defined purely in terms of equivariant correspondences and geometric intersections, and hence is a local, topological invariant of the correspondence. The left hand side is of course homological and global in nature.

The equivariant Lefschetz theorem presented here is a special case of joint work with Ralf Meyer. See [9] for the more general version.

I would like to express my appreciation to Siegried Echterhoff and Ralf Meyer for their comments on the material here. The material in this note is related to joint work with both of them (independently.) I would also like to thank Nigel Higson for drawing my attention to the beautiful paper [2] of Atiyah and Bott on localization in equivariant cohomology.

Finally, the reader interested in further information on equivariant K-theory for compact group actions should see the important source [22], which deals extensively with the Universal Coefficient and Künneth theorems in the integral version of $\mathrm{KK}^{\mathbb{T}}$, and more generally, for Hodgkin groups. When one works integrally, the representation ring $\mathrm{Rep}(\mathbb{T})$ becomes $\mathbb{Z}[X, X^{-1}]$ which is no longer a principal ideal domain; this complicates some statements considerably.

2. The $\mathbb{T}$-spectrum of spaces

In the following, the reader should consider all $\mathbb{T}$-equivariant K-theory groups, e.g. $\mathrm{K}^*_{\mathbb{T}}(X)$ for a $\mathbb{T}$-space X, or $\mathrm{K}_*^{\mathbb{T}}(A)$ for a $\mathbb{T}$-C*-algebra A, as having complex coefficients. Thus, $\mathrm{K}_*^{\mathbb{T}}(A)$ denotes the usual integral equivariant K-theory of A tensored by the complex numbers.

Similarly, the symbol $\mathrm{Rep}(\mathbb{T})$ means the usual representation ring of the circle, tensored with the complex numbers, or, more conveniently for us, the ring $\mathbb{C}[X, X^{-1}]$ of Laurent polynomials in one variable, and complex coefficients. The isomorphism $\mathrm{Rep}(\mathbb{T}) \to \mathbb{C}[X, X^{-1}]$ is the character map.

This note makes crucial use of the fact that for any $\mathbb{T}$-C*-algebra A, the $\mathbb{T}$-equivariant K-theory $\mathrm{K}_*^{\mathbb{T}}(A)$ is a module over $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$. For unital, commutative $\mathbb{T}$-C*-algebras this is rather clear, since in this case $\mathrm{K}_*^{\mathbb{T}}(A)$ is a ring and the unital inclusion $\mathbb{C} \to A$ maps $\mathrm{Rep}(\mathbb{T})$ to a subring of $\mathrm{K}_*^{\mathbb{T}}(A)$. This induces the module structure. It is not hard to convince oneself that if even if A is not unital, and hence no ring embedding exists, the module structure still makes sense.

In the general case, we may point to the *external product* in equivariant Kasparov theory as a formal definition of the module structure: to translate to Kasparov language, $\mathrm{K}_*^{\mathbb{T}}(A) = \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A)$ and $\mathrm{Rep}(\mathbb{T}) = \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C})$ (tensored by the complex numbers.) So Kasparov external product gives grading-preserving maps

$$\mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A) \times \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) \to \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A)$$
$$\mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) \times \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, A) \to \mathrm{KK}_*^{\mathbb{T}}(\mathbb{C}, A)$$

These maps agree: external product is commutative.

More generally, $\mathrm{KK}^{\mathbb{T}}_*(A,B)$ is a graded $\mathrm{Rep}(\mathbb{T})$-module for any A, B.

For commutative A, i.e. for $\mathbb{T}$-spaces, the module structure of $\mathrm{K}^*_{\mathbb{T}}(X)$ over $\mathrm{Rep}(\mathbb{T})$ has been quite extensively studied by Atiyah and Segal in [1] and [2], and also by Atiyah and Bott in the context of equivariant cohomology in [2].

The following definition applies to arbitrary $\mathbb{C}[X,X^{-1}]$-modules, and indeed, to modules over more general polynomial rings.

Definition 2.1. Let M be a module over the ring $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X,X^{-1}]$. Its *annihilator* $\mathrm{ann}(M)$ is the ideal $\{f \in \mathbb{C}[X,X^{-1}] \mid fM = 0\}$. The *support* of M is defined by

$$\mathrm{supp}(M) := \bigcap_{f\in \mathrm{ann}(M)} Z_f$$

where $Z_f \subset \mathbb{C}^*$ is the zero set of f.

Thus a point z is not in the support of M if and only if there is a polynomial f such that $f(z) \neq 0$ but $fM = 0$. In particular, this can hold only if M has module torsion. Since a free module has no torsion, the support of a free module like $\mathbb{C}[X,X^{-1}]$ itself, is $\mathbb{C}^*$.

Under embeddings $M_1 \to M_2$ of $\mathbb{C}[X,X^{-1}]$-modules, supports can only increase as $\mathrm{ann}(M_2) \subset \mathrm{ann}(M_1)$ in this situation, which implies $\mathrm{supp}(M_1) \subset \mathrm{supp}(M_2)$. In particular, $\mathrm{supp}(M) = \mathbb{C}^*$ as soon as M contains a free submodule. If on the other hand one has a surjection $M_1 \to M_2$, then $\mathrm{ann}(M_1) \subset \mathrm{ann}(M_2)$ so that $\mathrm{supp}(M_2) \subset \mathrm{supp}(M_1)$ results.

The ring $\mathbb{C}[X,X^{-1}]$ is a principal ideal domain, i.e. any ideal is generated by a single polynomial f. This polynomial is unique up to multiplication by an invertible in $\mathbb{C}[X,X^{-1}]$, i.e. f can be replaced by fX^n for any integer n, and in particular f may always be taken to be a polynomial. Furthermore, any finitely generated module over a principal ideal domain decomposes uniquely into a direct sum of a free module and a torsion module. The torsion submodule is by definition $\{m \in M \mid fm = 0 \text{ for some } f \neq 0 \text{ in } \mathbb{C}[X,X^{-1}]\}$.

A finitely generated torsion module has a nonzero annihilator ideal because the annihilator ideal is the intersection of the annihilator ideals of the generators, this is an intersection of finitely many nonzero ideals and hence is nonzero. If the annihilator of the torsion module is generated by f, then the support of the torsion module is the zero set Z_f of f in $\mathbb{C}^*$, and in particular is a finite set of points of $\mathbb{C}^*$. If the module is not finitely generated, it may be torsion, but have a zero annihilator ideal, however. In this case, the support will be $\mathbb{C}^*$ (see below for an example.)

If a module has finite dimension as a vector space over $\mathbb{C}$ then of course it is torsion and finitely generated and the above discussion applies.

For any $\mathbb{C}[X,X^{-1}]$-module, ring multiplication by $X \in \mathbb{C}[X,X^{-1}]$ is an invertible, complex linear operator on the module, viewed just as a complex vector space. If M is torsion with nonzero annihilator ideal, then *the support is the set of eigenvalues of X and the generator f of the annihilator ideal is the minimal polynomial of X*. Indeed, factor $f(X) = (X-\lambda_1)^{k_1}\cdots(X-\lambda_n)^{k_n}$.

Each λ_i must be an eigenvalue of X since $\prod_{j\neq i}(X-\lambda_j)^{k_j}(X-\lambda_i)^{k_i-1}$ maps M into the kernel of $X-\lambda_i$. If the kernel of $X-\lambda_i$ is zero, we would have a polynomial of smaller degree annihilating M, false. So the kernel is nonzero. Furthermore, as $f(X)=0$ on M, $0=f(X)v=f(\lambda)v$ if v is any eigenvector of X with eigenvalue λ. Hence any eigenvalue of X is a root of f.

Remark 2.2. Finite generation is guaranteed for the $\mathbb{C}[X,X^{-1}]$-module $\mathrm{K}^*_{\mathbb{T}}(X)$ whenever X is a smooth, compact manifold and $\mathbb{T}$ acts smoothly (see [23]) or the discussion in Section 4 of this paper.

Definition 2.3. Let A be a $\mathbb{T}$-C*-algebra. The $\mathbb{T}$-*spectrum of* A is defined to be the support of $\mathrm{K}^{\mathbb{T}}_*(A)$ as an $\mathbb{C}[X,X^{-1}]$-module.

In the commutative case, we refer to the $\mathbb{T}$-spectrum of the corresponding space.

Remark 2.4. The definition of spectrum in terms of the module $\mathrm{K}^{\mathbb{T}}_*(A) := \mathrm{K}^{\mathbb{T}}_0(A)\oplus \mathrm{K}^{\mathbb{T}}_1(A)$ given above does not take into account the grading on $\mathbb{T}$-equivariant K-theory. A more natural invariant, in some ways, would take this into account, but we do not do this here because it is not necessary for our purposes. Note also that $\mathrm{K}^{\mathbb{T}}_*(A)\cong \mathrm{K}^{\mathbb{T}}_0\big(C(S^1)\otimes A\big)$ where the $\mathbb{T}$-action on the circle S^1 is trivial, which means that in computing module structures we can deal exclusively with $\mathbb{T}$-equivariant vector bundles.

In the case of the trivial $\mathbb{T}$-action on a point, $\mathrm{K}^*_{\mathbb{T}}(\cdot)=\mathrm{Rep}(\mathbb{T})$ and the module structure over $\mathrm{Rep}(\mathbb{T})$ is by ring multiplication. Hence the annihilator ideal is zero and $\mathbb{T}\text{-spec}(\cdot)=\mathbb{C}^*$.

If $A=C([0,\infty))$ with trivial $\mathbb{T}$-action, then $\mathrm{K}^{\mathbb{T}}_*(A)=0$ and hence $\mathbb{T}\text{-spec}(A)=\varnothing$ in this case.

Note that, although evaluation of Laurent polynomials at any $z\in\mathbb{C}^*$ yields a $\mathbb{C}[X,X^{-1}]$-module M such that $\mathrm{supp}(M)=\{z\}$, if this module is to arise from an equivariant K-theory module, then z must be an algebraic integer, at least if the module is finite dimensional over $\mathbb{C}$.

Proposition 2.5. *If* $\mathrm{K}^{\mathbb{T}}_*(A)$ *is finite dimensional over* $\mathbb{C}$*, then the* $\mathbb{T}$*-spectrum of* A *is a finite set of algebraic integers in* $\mathbb{C}^*$*.*

Proof. The spectrum in this case is the spectrum of X acting on $\mathrm{K}^{\mathbb{T}}_*(A)$. But X comes from an endomorphism of the underlying $\mathbb{T}$-equivariant K-theory with *integer coefficients* and therefore is represented in some basis for $\mathrm{K}^{\mathbb{T}}_*(A)$ by a matrix with integer coefficients, and $\mathbb{T}\text{-spec}(A)$ is its set of eigenvalues, so they are algebraic integers. □

Theorem 2.6. *If* $A=C_0(X)$ *is* any *commutative* $\mathbb{T}$-C**-algebra, then either* $\mathbb{T}\text{-spec}(A)=\mathbb{C}^*$ *or* $\mathbb{T}\text{-spec}(A)\subset\mathbb{T}$*. In the latter case, the spectrum is finite and each point of it is an* n*th root of unity where* n *is the order of some (finite) isotropy group of the action.*

If X *is compact, then* $\mathbb{T}\text{-spec}(X)=\mathbb{C}^*$ *if and only if* X *has a stationary point.*

The proof will occupy the rest of this section. We start by discussing stationary points. Suppose X has such a point. Then there is a $\mathbb{T}$-map from the one-point $\mathbb{T}$-space to X; it induces a module map $\mathrm{K}^*_{\mathbb{T}}(X) \to \mathrm{K}^*_{\mathbb{T}}(\cdot) = \mathrm{Rep}(\mathbb{T})$. If X is *compact* this map is surjective because the map from X to a point is proper in this case and gives a splitting. Hence $\mathbb{C}^* = \mathbb{T}\text{-spec}(\cdot) \subset \mathbb{T}\text{-spec}(X)$.

Thus, $\mathbb{T}\text{-spec}(X) = \mathbb{C}^*$ if X has a stationary point and is compact. This is rather common; for example, by the Hopf theorem any smooth $\mathbb{T}$-action on a smooth manifold of nonzero Euler characteristic has a stationary point. Hence having $\mathbb{T}$-spectrum $\mathbb{C}^*$ is rather generic for compact $\mathbb{T}$-spaces.

If X is not compact, it may have a stationary point without the spectrum being $\mathbb{C}^*$; for example $[0,\infty)$ with trivial $\mathbb{T}$-action has empty spectrum but many stationary points. The other implication also requires compactness in view of Example 2.8 below, where the spectrum is $\mathbb{C}^*$ but there is no stationary point.

To get an example of a space with non stationary point but with spectrum $\mathbb{C}^*$, observe first that for any collection $(M_\lambda)_{\lambda\in\Lambda}$ of nonzero $\mathbb{C}[X, X^{-1}]$-modules, the annihilator of the direct sum $M := \bigoplus M_i$ is, essentially tautologically, the intersection $\bigcap_i \mathrm{ann}(M_i)$ of the annihilators. But for the ring $\mathbb{C}[X, X^{-1}]$, there can only be finitely many ideals containing a given nonzero ideal, for if the given one is generated by (f) then any ideal containing (f) is generated by a divisor of f. Hence if there are infinitely many distinct ideals $\mathrm{ann}(M_i)$, then $\bigcap_i \mathrm{ann}(M_i)$ would have to be the zero ideal.

This shows the following.

Lemma 2.7. *If X is a $\mathbb{T}$-space which is a disjoint union $X = \bigsqcup_i X_i$ for a family of $\mathbb{T}$-spaces X_i. Then either the $\mathbb{T}$-spectrum of X is $\mathbb{C}^*$ or the sets* $\mathbb{T}\text{-spec}(X_i)$ *are all finite, there are only finitely many of them, and* $\mathbb{T}\text{-spec}(X)$ *is their union.*

Proof. $\mathrm{K}^*_{\mathbb{T}}(X) = \oplus_{i\in\Lambda}\mathrm{K}^*_{\mathbb{T}}(X_i)$ and the result follows from the preceding remarks. □

Example 2.8. Let $\mathbb{T}$ act on $X_n := \mathbb{T}$ with $t\cdot s := t^n s$. Let $\Omega_n \subset \mathbb{T}$ denote the subgroup of nth complex root of unity. Then $X_n \cong \mathbb{T}/\Omega_n$ with $\mathbb{T}$ acting by translation on the quotient. Thus $\mathrm{K}^*_{\mathbb{T}}(X_n) \cong \mathrm{Rep}(\Omega_n)$, and the $\mathbb{C}[X, X^{-1}] \cong \mathrm{Rep}(T)$ module structure is by restriction of representations, i.e. by restrictions of polynomials to $\Omega_n \subset \mathbb{C}^*$. The support is Ω_n, thus $\mathbb{T}\text{-spec}(X_n) = \Omega_n$.

Now let $X = \mathbb{T}\times\mathbb{N}$ with $\mathbb{T}$ acting as above in the nth copy of $\mathbb{T}$. By Lemma 2.7, the $\mathbb{T}$-spectrum of X is $\mathbb{C}^*$, although there is no stationary point.

The proof of Theorem 2.6 follows from the following two lemmas.

Lemma 2.9. *Let X be any $\mathbb{T}$-space and $Y \subset X$ be a closed $\mathbb{T}$-invariant subspace of X. Then*

$$\mathbb{T}\text{-spec}(X) \subset \mathbb{T}\text{-spec}(Y) \cup \mathbb{T}\text{-spec}(X - Y).$$

Proof. Consider the 6-term exact sequence of $\mathbb{T}$-equivariant K-theory groups associated to the exact sequence

$$0 \to C_0(X-Y) \xrightarrow{i} C(X) \xrightarrow{r} C_0(Y) \to 0.$$

Let $f \in \mathbb{C}[X, X^{-1}]$ annihilate $\mathrm{K}^*_{\mathbb{T}}(X-Y)$ and $\mathrm{K}^*_{\mathbb{T}}(Y)$. Then if $a \in \mathrm{K}^0_{\mathbb{T}}(X)$, $0 = f \cdot r_*(a) = r_*(f \cdot a)$ implies $f \cdot a = i_*(a')$ some $a' \in \mathrm{K}^0_{\mathbb{T}}(X-Y)$ and then $f^2 \cdot a = i_*(f \cdot a') = 0$ so f^2 annihilates $\mathrm{K}^0_{\mathbb{T}}(X)$. Similarly f^2 annihilates $\mathrm{K}^1_{\mathbb{T}}(X)$. Thus

$$f \in \mathrm{ann}\big(\mathrm{K}^*_{\mathbb{T}}(X-Y)\big) \cap \mathrm{ann}\big(\mathrm{K}^*_{\mathbb{T}}(Y)\big) \Rightarrow f^2 \in \mathrm{ann}\big(\mathrm{K}^*_{\mathbb{T}}(X)\big).$$

Hence $\mathrm{supp}\big(\mathrm{K}^*_{\mathbb{T}}(X)\big)$ is contained in $Z_{f^2} = Z_f$ for any $f \in \mathrm{ann}\big(\mathrm{K}^*_{\mathbb{T}}(X-Y)\big) \cap \mathrm{ann}\big(\mathrm{K}^*_{\mathbb{T}}(Y)\big)$. The result now follows. ☐

Lemma 2.10. *If $X := \mathbb{T} \times_H Y$ for some closed subgroup $H \subset \mathbb{T}$ and some H-space Y, then $\mathbb{T}\text{-spec}(X) \subset H$. In particular, if H is a proper subgroup, then the $\mathbb{T}$-spectrum of X consists of a set of nth roots of unity, where n is the cardinality of H.*

Proof. The $\mathrm{Rep}(\mathbb{T})$-module structure on $\mathrm{K}^*_{\mathbb{T}}(X) \cong \mathrm{K}^*_H(Y)$ factors through the restriction map $\mathrm{Rep}(\mathbb{T}) \to \mathrm{Rep}(H)$ and the $\mathrm{Rep}(H)$-module structure on $\mathrm{K}^*_H(Y)$. If f is a polynomial which vanishes on $H \subset \mathbb{T}$ then it restricts to zero in $\mathrm{Rep}(H)$ and hence acts by zero on $\mathrm{K}^*_H(Y) \cong \mathrm{K}^*_{\mathbb{T}}(X)$. Hence $\mathbb{T}\text{-spec}(X) \subset H$ as claimed. ☐

Remark 2.11. We remind the reader of two easy and well-known facts about induced spaces.

(i) Induced spaces $W = \mathbb{T} \times_H Y$ from a subgroup $H \subset \mathbb{T}$ are characterized among $\mathbb{T}$-spaces as those admitting a $\mathbb{T}$-map $\varphi : W \to \mathbb{T}/H$. We can recover Y from φ as the fiber over the identity coset in $\mathbb{T}/H$.

(ii) We often call induced spaces *slices*. Since we can always restrict a $\mathbb{T}$-map to a $\mathbb{T}$-invariant subspace, any $\mathbb{T}$-invariant subspace of a slice is a slice too.

(iii) A theorem of Palais (see [20]) asserts that any $\mathbb{T}$-space can be covered by *open* slices using stabilizer subgroups of the action. That is, if X is any $\mathbb{T}$-space and $x \in X$, then there exists an open subset $U \subset X$ with $x \in U$, and a $\mathbb{T}$-map $\varphi : U \to \mathbb{T}/H$ where $H := \mathbb{T}_x$ is the stabilizer of x. (This result holds more generally for actions of *Lie groups*.)

Note that if $\varphi : U \to \mathbb{T}/H$ is a slice with $H = \mathbb{T}_x$ for some $x \in U$, then $\mathbb{T}_y \subset \mathbb{T}_x$ for any $y \in U$.

Lemma 2.12. *Let X be a (locally compact) $\mathbb{T}$-space.*

(i) If X has no stationary points, then $\mathrm{K}^*_{\mathbb{T}}(X)$ is a torsion module and $\mathbb{T}\text{-spec}(Y)$ is a finite subset of $\mathbb{T}$ for every precompact $\mathbb{T}$-invariant subset $Y \subset X$. Furthermore, $\mathbb{T}\text{-spec}(Y) \subset \cup_{y \in \overline{Y}} \mathbb{T}_y$.

*(ii) If $\mathbb{T}\text{-spec}(X)$ is finite, $F \subset X$ is the stationary set, then $\mathrm{K}^*_{\mathbb{T}}(F) = 0$ and the $\mathbb{T}$-equivariant *-homomorphism $C_0(X-F) \to C_0(X)$ determines an isomorphism $\mathrm{K}^*_{\mathbb{T}}(X-F) \cong \mathrm{K}^*_{\mathbb{T}}(X)$ of $\mathbb{C}[X, X^{-1}]$-modules.*

Remark 2.13. The same arguments prove a stronger version of the second statement: that the $\mathbb{T}$-equivariant *-homomorphism $C_0(X-F) \to C_0(X)$ is invertible in $\mathrm{KK}^{\mathbb{T}}(C_0(X-F), C_0(X))$.

The condition of having finite $\mathbb{T}$-spectrum thus implies that the stationary set F is *homologically trivial*: that is, $\mathrm{K}^*_{\mathbb{T}}(F) = 0$. Compare the ray $[0,\infty)$ with the trivial action.

Proof. For the first statement, $\mathrm{K}^*_{\mathbb{T}}(X)$ is the inductive limit of the $\mathrm{K}^*_{\mathbb{T}}(Y)$, as $Y \subset X$ ranges over the precompact $\mathbb{T}$-invariant subsets of X. Therefore, if we can prove that the annihilator ideal of $\mathrm{K}^*_{\mathbb{T}}(Y)$ is nonzero for every precompact $\mathbb{T}$-invariant subset $Y \subset X$, we will be done. This is equivalent to showing that $\mathbb{T}\text{-spec}(Y)$ is finite for all such Y. If $Y \subset X$ is precompact, with closure $\overline{Y}$, then we can cover $\overline{Y}$ by finitely many $\mathbb{T}$-slices $\varphi : U_i \to \mathbb{T}/H_i$ using stabilizer subgroups H_i of the action on $\overline{Y}$. This gives a finite cover of Y itself by open slices (as in Remark 2.11, intersecting a slice with a $\mathbb{T}$-invariant subset always results in a slice.) Furthermore, since the H_i are stabilizer groups of points in $\overline{Y}$ and the action has no stationary points, all H_i are finite subgroups of $\mathbb{T}$.

Now prove the result by induction on the minimal number of slices required to cover Y, which we have just observed is finite. It can be covered by a single slice, then it is itself a slice, and the result follows from Lemma 2.10. If the result is true for precompact subsets of X that can be covered by $< n$ slices, and Y can be covered by n slices with domains $U_1, \dots, U_n$ and subgroups H_i, then the closed $\mathbb{T}$-invariant subspace $Y - U_n$ of Y can be covered by $n-1$ slices so by inductive hypothesis $\mathbb{T}\text{-spec}(Y - U_n) \subset \bigsqcup_{x \in Y} \mathbb{T}_x \subset \mathbb{T}$ is finite. The result for Y now follows from Lemma 2.9.

For the second statement, consider the exact sequence of $\mathbb{T}$-C*-algebras

$$0 \to C_0(X-F) \to C_0(X) \to C_0(F) \to 0.$$

This induces an exact sequence of $\mathrm{K}^*_{\mathbb{T}}$-groups. The restriction map $\mathrm{K}^*_{\mathbb{T}}(X) \to \mathrm{K}^*_{\mathbb{T}}(F)$ must vanish, because we have assumed that X has finite spectrum, (i.e. $\mathrm{K}^*_{\mathbb{T}}(X)$ is torsion) whereas $\mathrm{K}^*_{\mathbb{T}}(F)$ is free. This implies that we have a pair of short exact sequences

$$0 \to \mathrm{K}^{*+1}_{\mathbb{T}}(F) \to \mathrm{K}^*_{\mathbb{T}}(X-F) \to \mathrm{K}^*_{\mathbb{T}}(X) \to 0$$

for $* = 0,1$. But $X - F$ has no stationary points, so from the first part of this Lemma, $\mathrm{K}^*_{\mathbb{T}}(X-F)$ is torsion. Now a free $\mathbb{C}[X, X^{-1}]$-module $\mathrm{K}^{*+1}_{\mathbb{T}}(F)$ which injects into a torsion module $\mathrm{K}^*_{\mathbb{T}}(X-F)$ can only be the zero module. Hence $\mathrm{K}^*_{\mathbb{T}}(F) = 0$, $* = 0,1$ and $C_0(X-F) \to C(X)$ induces an isomorphism on $\mathrm{K}^*_{\mathbb{T}}$-theory. □

Proof of Theorem 2.6. Assume that $\mathbb{T}\text{-spec}(X) \neq \mathbb{C}^*$.

Then since X has finite $\mathbb{T}$-spectrum, $\mathrm{K}^*_{\mathbb{T}}(X) \cong \mathrm{K}^*_{\mathbb{T}}(X-F)$ as $\mathbb{C}[X, X^{-1}]$-modules, where $F \subset X$ is the stationary set, by Lemma 2.12. In particular, $\mathbb{T}\text{-spec}(X) = \mathbb{T}\text{-spec}(X-F)$ so by replacing X by $X-F$ we may assume that X itself has no stationary points.

Now by the preliminary discussion following Definition 2.1, since $\mathrm{K}^*_{\mathbb{T}}(X)$ has nonzero annihilator ideal, the support is the set of eigenvalues of X acting on $\mathrm{K}^*_{\mathbb{T}}(X)$. Suppose λ is an eigenvalue, $v \in \mathrm{K}^*_{\mathbb{T}}(X)$ an eigenvector for λ. Since $\mathrm{K}^*_{\mathbb{T}}(X)$ is the inductive limit of the $\mathrm{K}^*_{\mathbb{T}}(Y)$ as $Y \subset X$ ranges over the precompact $\mathbb{T}$-invariant subsets of X, there exists precompact Y and $w \in \mathrm{K}^*_{\mathbb{T}}(Y)$ mapping to v. By Lemma 2.12, since there are no stationary points, $\mathrm{K}^*_{\mathbb{T}}(Y)$ has a finite annihilator ideal, say generated by $g \in \mathbb{C}[X, X^{-1}]$, and moreover, the support of $\mathrm{K}^*_{\mathbb{T}}(Y)$ is contained in the unit circle. Since $gw = 0$, $gv = 0$, and as $gv = g(\lambda)v$, λ is a root of g, whence λ is contained in the unit circle as claimed.

This also proves that λ is an nth root of unity where n is the cardinality of some isotropy group of the action (on Y). □

3. The $\mathbb{T}$-spectra of C*-algebras

Let B be a C*-algebra equipped with an automorphism σ. Then $A := B \ltimes \mathbb{Z}$ is a $\mathbb{T}$-C*-algebra using the dual action

$$z\Big(\sum_{n\in\mathbb{Z}} b_n[n]\Big) := \sum_{n\in\mathbb{Z}} z^n b_n[n].$$

Hence it has a $\mathbb{T}$-spectrum. The Green–Julg theorem asserts that the $\mathbb{T}$-equivariant K-theory of A is isomorphic to the K-theory $\mathrm{K}_*(A \rtimes \mathbb{T})$ of the cross product. By Takai–Takesaki duality, this agrees with $\mathrm{K}_*(B)$.

Proposition 3.1. *Let B be a* C*-*algebra and* $\sigma \in \mathrm{Aut}(B)$. *Endow the cross product* $A := B \rtimes_\sigma \mathbb{Z}$ *with the dual action of* $\mathbb{T} \cong \widehat{\mathbb{Z}}$. *The automorphism induces an invertible linear map* $\sigma_* : \mathrm{K}_*(B) \to \mathrm{K}_*(B)$ *and hence a* $\mathbb{C}[X, X^{-1}]$ *module structure on* $\mathrm{K}_*(B)$. *This module is naturally isomorphic to* $\mathrm{K}^{\mathbb{T}}_*(A)$.

In particular, if $\mathrm{K}_*(B)$ is finite dimensional over $\mathbb{C}$, *then*

$$\mathbb{T}\text{-spec}(B \rtimes \mathbb{Z}) = \mathrm{Spec}(\sigma_*),$$

with $\mathrm{Spec}(\sigma_*)$ *the set of eigenvalues of the invertible linear map* $\sigma_* \in \mathrm{End}_{\mathbb{C}}\big(\mathrm{K}_*(B)\big)$.

Proof. This follows from Blackadar Proposition 11.8.3, which asserts that the isomorphism

$$\mathrm{K}_*(B) \cong \mathrm{K}_*(B \ltimes \mathbb{Z} \ltimes \mathbb{T}) = \mathrm{K}_*(A \rtimes \mathbb{T}) \cong \mathrm{K}^*_{\mathbb{T}}(A)$$

of Takai–Takesaki duality and the Green–Julg theorem, intertwines the group homomorphism σ_* and the group homomorphism of scalar multiplication by $X \in \mathbb{C}[X, X^{-1}] \cong \mathrm{Rep}(\mathbb{T})$.

Furthermore, if $\mathrm{K}^*_{\mathbb{T}}(A)$ has finite dimension, then $\mathbb{T}$-spec(A) has nonzero annihilator ideal and the support is the set of nonzero eigenvalues of the linear map X because it is the zero set of the minimal polynomial of the linear map X. □

Remark 3.2. Baaj–Skandalis duality (see [5]) is a functor $\mathrm{KK}^{\mathbb{T}} \to \mathrm{KK}^{\mathbb{Z}}$ which on objects sends a $\mathbb{T}$-C*-algebra B to the $\mathbb{Z}$-C*-algebra $B := A \rtimes \mathbb{T}$, with the dual action and sends a $\mathbb{T}$-equivariant *-homomorphism $A \to A'$ to the (obvious) induced $\mathbb{Z}$-equivariant map $B := A \rtimes \mathbb{T} \to B' := A' \rtimes \mathbb{T}$. Baaj and Skandalis extend this to a natural *isomorphism*

$$\mathrm{KK}^{\mathbb{T}}_*(A, A') \cong \mathrm{KK}^{\mathbb{Z}}_*(A \rtimes \mathbb{T}, A' \rtimes \mathbb{T})$$

of equivariant KK-groups. Note that this transformation maps an induced space $X = \mathbb{T} \times_H Y$ for some H-space Y and a closed subgroup H of $\mathbb{T}$ to the $\mathbb{Z}$-C*-algebra

$$C_0(X) \rtimes \mathbb{T} \cong C_0(Y) \rtimes H$$

with an appropriate dual action of $\mathbb{Z}$. The important point is that this $\mathbb{Z}$-action *factors through a periodic action*, i.e. factors through the homomorphism $\mathbb{Z} \to H \cong \mathbb{Z}/n$ for some n, and a $\mathbb{Z}/n$-action.

Under the Baaj–Skandalis transformation, the $\mathbb{T}$-spectrum of a $\mathbb{T}$-C*-algebra A corresponds, as we have observed above, to the spectrum, in the usual sense, of the endomorphism of $\mathrm{K}_*(A)$ by the generator $1 \in \mathbb{Z}$ of the $\mathbb{Z}$-action. Hence if the $\mathbb{Z}$-action is periodic, then the corresponding linear map has finite order, and hence its spectrum consists of roots of unity in the circle. This is, roughly, then, the counterpart of the situation in the first section, in the category $\mathrm{KK}^{\mathbb{Z}}$.

For instance let $A = \mathbb{C}$ with the trivial automorphism. Applying the proposition gives that the $\mathbb{T}$-spectrum of $C^*(\mathbb{Z})$ with its dual action of $\mathbb{T}$ is the single point $\{1\} \subset \mathbb{C}^*$.

Example 3.3. The $\mathbb{T}$-spectrum of the irrational rotation algebra $A_\theta := C(\mathbb{T}) \rtimes_{R_\theta} \mathbb{Z}$ with the dual action of $\mathbb{T}$ is also $\{1\}$ because σ_* is the identity map on $\mathrm{K}^*(\mathbb{T})$.

Example 3.4. Let A be an integer n-by-n matrix with entries either 0 or 1 and assume for simplicity that A is invertible over $\mathbb{C}$. Then the $\mathbb{T}$-spectrum of the associated Cuntz–Krieger algebra O_A is the set of eigenvalues of A. Indeed, $O_A \cong F_A \ltimes \mathbb{Z}$ where F_A is an appropriate AF-algebra, and $\cong$ means Morita equivalence. It is well-known and easily checked from the Bratteli diagram, that the K-theory of F_A is $\cong \mathbb{C}^n$, and the action of $\mathbb{Z}$ on it is by the matrix A. Hence the $\mathbb{T}$-spectrum of O_A is the spectrum of A.

Corollary 3.5. *The Cuntz–Krieger algebra O_A is not $\mathrm{KK}^{\mathbb{T}}$-equivalent to any commutative $\mathbb{T}$-C*-algebra as soon as the integer matrix A has some eigenvalue of modulus $\neq 1$.*

This happens for instance if $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

For the benefit of the reader (the result is well-known) we prove the following.

Lemma 3.6. *Both O_A and $O_A \rtimes \mathbb{T} \cong F_A$ are in the boostrap category $\mathcal{N}$.*

Proof. F_A is an AF algebra so is in $\mathcal{N}$. The Baum–Connes conjecture for $\mathbb{Z}$ is the statement that $C_0(\mathbb{R})$ with the $\mathbb{Z}$-action by translation is $\mathrm{KK}_1^{\mathbb{Z}}$-equivalent to $\mathbb{C}$. It follows from this that $O_A = F_A \rtimes \mathbb{Z}$ is KK-equivalent to $C_0(\mathbb{R}, F_A) \rtimes \mathbb{Z}$. There is an exact sequence

$$0 \to S \otimes F_A \otimes \mathbb{K} \to C_0(\mathbb{R}, F_A) \rtimes \mathbb{Z} \to F_A \otimes \mathbb{K} \to 0$$

of C*-algebras, obtained by evaluating functions on $\mathbb{R}$ at the integer points $\mathbb{Z} \subset \mathbb{R}$, a closed and $\mathbb{Z}$-invariant subset, and using $C_0(\mathbb{Z}) \rtimes \mathbb{Z} \cong \mathbb{K}$. Since $\mathbb{K}$ is KK-equivalent to $\mathbb{C}$ both ends are in the boostrap category. Hence $C_0(\mathbb{R}, F_A) \rtimes \mathbb{Z}$ is also. □

Remark 3.7. We have actually proved something stronger than Corollary 3.5, for we have shown that the $\mathbb{T}$-equivariant K-theory of O_A is not isomorphic in the category of $\mathbb{C}[X, X^{-1}]$-modules to the $\mathbb{T}$-equivariant K-theory of any locally compact Hausdorff $\mathbb{T}$-space.

We close this section with some further remarks on $\mathbb{T}$-equivariant K-theory of Cuntz–Krieger algebras, to see $\mathbb{T}$-spectra in a dynamical perspective.

Up to now we have considered $\mathrm{K}_*^{\mathbb{T}}(A) := \mathrm{K}_0^{\mathbb{T}}(A) \oplus \mathrm{K}_{\mathbb{T}}^1(A)$ as simply a $\mathbb{C}[X, X^{-1}]$-module without taking into consideration the grading. If we consider $\mathrm{K}_*^{\mathbb{T}}(A)$ as a $\mathbb{Z}/2$-graded $\mathbb{C}[X, X^{-1}]$-module, then an invariant of it—assuming it finite dimensional over $\mathbb{C}$—is the rational function

$$\mathrm{char}_A(t) := \frac{\det(1 - tX_+)}{\det(1 - tX_-)} \tag{3.1}$$

where $X_\pm$ denotes the action of the generator X on $\mathrm{K}_{0/1}^{\mathbb{T}}(A)$.

If A and B are $\mathrm{KK}^{\mathbb{T}}$-equivalent, they have the same rational function (3.1).

The following elementary result about (grading-preserving) linear transformations X on a $\mathbb{Z}/2$-graded vector space can be found in the appendices to Hartshorne's book [17]:

$$\mathrm{char}_A(t) = \exp\left(\sum_{n=1}^{\infty} \mathrm{trace}_s(X^n) \frac{t^n}{n}\right) \tag{3.2}$$

holds, where trace_s is the graded trace, the difference of the traces of X acting on $\mathrm{K}_1^{\mathbb{T}}(A)$ and $\mathrm{K}_0^{\mathbb{T}}(A)$.

We now specialize to the following situation: let $\phi_T : \mathbb{T}^n \to \mathbb{T}^n$ be a linear automorphism, where $T \in \mathrm{GL}_n(\mathbb{Z})$. We assume that T is self-adjoint, so it is diagonalizable over $\mathbb{C}$ with real, nonzero eigenvalues. We can form the cross product $A := C(\mathbb{T}^n) \rtimes_{\phi_T} \mathbb{Z}$, which is a $\mathbb{T}$-C*-algebra. By the Lefschetz fixed-point theorem,

$$\mathrm{trace}_s\big((\phi_T^*)^n\big) = (-1)^k\, P_n(\phi_T),$$

because the sign of $\det(1 - T)$ is $(-1)^k$ where k is the number (including multiplicities) of eigenvalues λ of T with $\lambda > 1$. Here $\mathrm{trace}_s(\phi_T)$ is the graded trace of the action of ϕ_T on $\mathrm{K}^*(\mathbb{T}^n)$, and $P_n(\phi_T)$ is the number of periodic points of order n.

Putting things together, we see that

$$\frac{\det(1-tX_+)}{\det(1-tX_-)} = \exp\left((-1)^k \cdot \sum_{n=1}^{\infty} P_n(\sigma)\frac{t^n}{n}\right).$$

The right hand side is called the *Artin-Mazur zeta function* of the map ϕ_T (see [1].)

To be explicit, if $n = 2$ and $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, so $k = 1$, $X_+ = \mathrm{Id}$ and X_- acts as T on $\mathrm{K}^1(\mathbb{T}^2) \cong \mathbb{C}^2$ and so $\mathrm{char}_{C(\mathbb{T})\rtimes_{\phi_T}\mathbb{Z}}(t) = t^2 - t - 1$ and

$$\mathbb{T}\text{-spec}\big(C(\mathbb{T}^2) \rtimes_{\phi_T} \mathbb{Z}\big) = \left\{1, \frac{1 \pm \sqrt{5}}{2}\right\}.$$

Note that this yields another example of a $\mathbb{T}$-C*-algebra not $\mathrm{KK}^{\mathbb{T}}$-equivalent to a commutative one.

4. $\mathbb{T}$-equivariant KK-theory of smooth manifolds and localization

If R is any commutative (unital) ring then any free, finitely generated R-module M has a well-defined *rank*, and any R-module self-map of M has a well-defined *trace*. We denote these invariants by $\mathrm{rank}_R(M)$ and $\mathrm{trace}_R(L)$ respectively, so that in particular $\mathrm{trace}_R(\mathrm{Id}) = \mathrm{rank}_R(M)$.

We will be mainly interested in the case where $R = \mathbb{C}[X, X^{-1}]$ or a localization of R.

Consider $\mathbb{C}[X, X^{-1}]$ as regular (rational) functions on $\mathbb{C}^*$. In algebraic geometry, if one wants to study the behavior of a variety near a point $z \in \mathbb{C}^*$, then one considers the set S of functions which are nonzero at z, and *localizes* $\mathbb{C}[X, X^{-1}]$ with respect to this multiplicative set (a subset of a ring is a multiplicative set if it includes the unit 1 and is closed under multiplication.)

This means that we invert all functions which are in S, i.e. invert functions which do not vanish at z. We therefore get all rational functions which are regular at z:

$$\mathbb{C}[X, X^{-1}]_z \cong \left\{ f \in \mathbb{C}(X) \;\middle|\; f = \frac{h}{g},\ g(z) \neq 0 \right\}.$$

This is a *local ring*: it has a unique maximal ideal, the ideal of $f \in \mathbb{C}[X, X^{-1}]_z$ such that $f(z) = 0$, and any $f \in \mathbb{C}[X, X^{-1}]$ such that $f(z) \neq 0$ is invertible in $\mathbb{C}[X, X^{-1}]_z$.

Note also that $\mathbb{C}[X, X^{-1}]$ embeds in its localization(s).

Localization can be defined for any commutative ring R with no zero divisors, at a multiplicative subset S (like the complement of a prime ideal) by considering the elements $\frac{r}{s}$ in the ring of fractions of R, such that $s \in S$. In this situation, R embeds in its localization.

For rings with zero divisors, localizations can still be defined, but the map from the original ring to its localization need not any longer be injective. Any element $r \in R$ such that there exists $s \in S$ so that $rs = 0$, is is killed by localization at S.

The prime ideals of the localization of a ring R at S correspond to the prime ideals of R which do not intersect S.

The "localizations" $\mathbb{C}[X, X^{-1}]_z$ just discussed, are the *stalks* of a sheaf of rings over $\mathbb{C}^*$ with the Zariski topology. For most of this paper, we will not use the stalks, but the values of the sheaf on Zariski open sets. To fix notation and terminology, we state the definition formally.

Definition 4.1. Let $f \in \mathbb{C}[X, X^{-1}]$. The *localization* of $\mathbb{C}[X, X^{-1}]$ at the Zariski open $U_f := \mathbb{C}^* - Z_f$ is the ring obtained from $\mathbb{C}[X, X^{-1}]$ by inverting all powers of f. We denote by $\mathbb{C}[X, X^{-1}]_f$ the localization of $\mathbb{C}[X, X^{-1}]$ at U_f. The assignment $U_f \mapsto \mathbb{C}[X, X^{-1}]_f$ defines a sheaf on $\mathbb{C}^*$ with the Zariski topology. The stalks of this sheaf are denoted $\mathbb{C}[X, X^{-1}]_z$ and are as discussed above.

Note that inverting f automatically inverts all divisors of f and hence inverts all polynomials which do not vanish on U_f, since the roots of such a polynomial are all roots of f, which implies it is a divisor of some positive power of f.

Hence $\mathbb{C}[X, X^{-1}]_f$ is simply the ring of *regular rational functions on U_f*.

Modules over a ring R can also be localized at multiplicative subsets of R, by setting

$$M_S := M \otimes_R R_S$$

where R_S is the localization of R at S. In the case of interest, where $R = \mathbb{C}[X, X^{-1}]$, we denote by M_f the localization of a $\mathbb{C}[X, X^{-1}]$-module at $S := \{1, f, f^2, \ldots\}$ (that is, at U_f.) The important point is that *localization of a module at U_f kills torsion supported in Z_f*. If M is a torsion module with finite support, then $M_f = 0$ if f vanishes on the support. More generally, of course, if the support of the torsion submodule of a finitely generated module M is Z_f then localizing M at U_f kills the torsion part, and the localization of the free part is free (over $\mathbb{C}[X, X^{-1}]_f$.) (Recall that since $\mathbb{C}[X, X^{-1}]$ is a principal ideal domain, every finitely generated $\mathbb{C}[X, X^{-1}]$-module splits uniquely into a torsion and a free module.)

We now consider the case where the module M has the form $M = \mathrm{K}^*_{\mathbb{T}}(X)$ where X is a $\mathbb{T}$-space. More generally, we may consider any $\mathrm{KK}^{\mathbb{T}}$-group, i.e. any $\mathrm{KK}^{\mathbb{T}}_*(A, B)$, for A and B $\mathbb{T}$-C*-algebras, with its $\mathbb{C}[X, X^{-1}]$-module structure. If $f \in \mathbb{C}[X, X^{-1}]$ we may localize any such module at f, yielding $\mathrm{KK}^{\mathbb{T}}_*(A, B)_f$. Localization is obviously compatible with the $\mathbb{Z}/2$-gradings, the intersection product (composition in $\mathrm{KK}^{\mathbb{T}}$) and the external product. In particular we may speak of $\mathrm{KK}^{\mathbb{T}}_f$-equivalence and so on.

Remark 4.2. Localization in K-theory is slightly different from localization in equivariant cohomology as in [2].

(i) The coefficient ring $\mathbb{C}[X, X^{-1}] = \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, \mathbb{C}) = \mathrm{KK}^{\mathbb{T}}_*(\mathbb{C}, \mathbb{C})$ we use is trivially graded, while the cohomological analogue $\mathrm{H}^*_{\mathbb{T}}(\mathrm{pnt}) := \mathrm{H}^*(B\mathbb{T}) \cong$

$\mathbb{C}[u]$ is $\mathbb{Z}$-graded with $\deg(u) = 2$. Atiyah's Completion Theorem relates the two rings: equivariant cohomology is the I-adic completion of $\mathbb{C}[X, X^{-1}]$ with respect to the ideal $I := \langle X - 1\rangle$ corresponding to $1 \in \mathbb{C}^*$. Supports of $\mathbb{C}[u]$-modules, like for example $H^*_{\mathbb{T}}(X) := H^*(E\mathbb{T} \times_{\mathbb{T}} X)$ for a $\mathbb{T}$-space X, are contained in $\mathbb{C}$ instead of $\mathbb{C}^*$. If the modules are graded, then their supports are always either all of $\mathbb{C}$ or are $\{0\}$, because they must be a cone (see [2]). Therefore the cohomological analogue of $\mathbb{T}$-spec is rather trivial: the support of the torsion submodule of $H^*_{\mathbb{T}}(X)$ must be $\{0\}$ and after localizing at $\mathbb{C}^* := \mathbb{C} - \{0\}$ we get a free module.

(ii) After localizing $\mathrm{H}^*_{\mathbb{T}}(X)$ by localizing, separately, its even and odd parts, the integer gradation on the module becomes lost; the $\mathbb{Z}/2$-grading is not lost, however.

Both of these facts would seem to support the idea that K-theory responds somewhat better to localization.

We are now going to refine some of our results from the first section about equivariant K-theory of spaces, using localization. See also [3], for some overlapping results.

We begin by discussing the issue of finite generation, which, importantly, implies that the torsion submodule of $\mathrm{K}^*_{\mathbb{T}}(X)$ has finite spectrum. Graeme Segal has proved the following.

Lemma 4.3 ([23, Prop. 5.4])**.** *If X is a smooth compact $\mathbb{T}$-manifold, then $\mathrm{K}^*_{\mathbb{T}}(X)$ is a finitely generated $\mathbb{C}[X, X^{-1}]$-module.*

The following is a useful geometric counterpart of Segal's lemma.

Lemma 4.4. *For a compact manifold X with smooth $\mathbb{T}$-action, there are only finitely many points $t \in \mathbb{T}$ which fix some point of $X - F$, where $F \subset X$ is the stationary set.*

Hence if $f \in \mathbb{C}[X, X^{-1}]$ is a polynomial which vanishes on these points, then $\mathrm{K}^*_{\mathbb{T}}\big(W \times (X - F)\big)_f = 0$ for any locally compact $\mathbb{T}$-space W.

Proof. For the first statement, since F is a smooth submanifold of X it has a normal bundle ν, which is a $\mathbb{T}$-equivariant real vector bundle. This may be identified with the orthogonal complement of $\mathrm{T}F$ in $\mathrm{T}X|_F$ with respect to any $\mathbb{T}$-invariant Riemannian metric. Since the fixed-point set of $t \in \mathbb{T}$ in T_xX (for $x \in F$) is exactly $\mathrm{T}F$, t fixes no nonzero vector in ν.

Let U be the corresponding $\mathbb{T}$-invariant open neighborhood of F. Since $\mathbb{T}$ acts freely on $\nu - 0$ it acts freely on $U - F$. We can cover the compact $X - U$ by finitely many open slices $W_i \subset X - U$, centered, say at points x_i, and if $x \in X - U$ is any point, then $\mathbb{T}_x \subset \mathbb{T}_{x_i}$ follows for $x \in W_i$. Since $\mathbb{T}_x = \{1\}$ for $x \in U$, $\bigcup_{x\in X-F} \mathbb{T}_x$ is a finite set as claimed.

If $(w, x) \in W \times X - F$ then of course $\mathbb{T}_{(w,x)} \subset \mathbb{T}_x$. It follows that if $f \in \mathbb{C}[X, X^{-1}]$ vanishes on $\bigcup_{x\in X-F} \mathbb{T}_x$ then it annihilates the image of $\mathrm{K}^*_{\mathbb{T}}(Y) \to \mathrm{K}^*_{\mathbb{T}}\big(W \times (X - F)\big)$ for any precompact $Y \subset W \times (X - F)$, cp. the arguments in the first paragraph of the proof of Lemma 2.12. Hence it annihilates $\mathrm{K}^*_{\mathbb{T}}\big(W \times (X - F)\big)$. □

Example 4.5. Consider $\mathbb{T} \times \mathbb{N}$ with the $\mathbb{T}$-action of Example 2.8. Let X be the one-point compactification of $X \times \mathbb{N}$, with $\mathbb{T}$-action the canonical extension of the action on $X \times \mathbb{N}$ (fixing the point at infinity). Then X is a compact space but there are infinitely many distinct points $t \in \mathbb{T}$ which fix some point of $X - F$. The equivariant K-theory is zero in dimension 1 and in dimension 0 is the $\mathbb{C}[X, X^{-1}]$-module

$$\mathrm{K}^0_{\mathbb{T}}(X) \cong \mathbb{C}[X, X^{-1}] \oplus \bigoplus_{n \in \mathbb{N}} \mathbb{C}[X, X^{-1}]/(f_n)$$

where $f_n(X) = \prod_{\omega \in \Omega_n} X - \omega$. The torsion submodule of $\mathrm{K}^0_{\mathbb{T}}(X)$ is not finitely generated and has support $\mathbb{C}^*$. Thus both Lemma 4.3 and Lemma 4.4 fail for X due to a lack of a "collaring" for the stationary set.

Lemma 4.6. *Let A, B and C be $\mathbb{C}[X, X^{-1}]$-modules, $\alpha : A \to B$ and $\beta : B \to C$ module maps, such that the sequence*

$$0 \longrightarrow \mathrm{im}(\alpha) \longrightarrow B \longrightarrow \ker(\beta) \longrightarrow 0$$

is exact and A and C are finitely generated. Then B is finitely generated.

Proof. This reduces immediately to whether $\ker(\beta)$ and $\mathrm{im}(\alpha)$ are finitely generated; the latter is obvious and the former follows from the fact that any submodule of a finitely generated $\mathbb{C}[X, X^{-1}]$-module is finitely generated, because $\mathbb{C}[X, X^{-1}]$ is noetherian. □

Corollary 4.7. $\mathrm{K}^*_{\mathbb{T}}(X - F)$ *is finitely generated for any smooth and compact $\mathbb{T}$-manifold X.*

Proof. By Lemma 4.3 $\mathrm{K}^*_{\mathbb{T}}(X)$ and $\mathrm{K}^*_{\mathbb{T}}(F)$ are finitely generated $\mathbb{C}[X, X^{-1}]$-modules (see Remark 4.8.) The result then follows from Lemma 4.6, for $\mathrm{K}^*_{\mathbb{T}}(X - F)$ fits into a 6-term exact sequence with the other terms $\mathrm{K}^*_{\mathbb{T}}(F)$ or $\mathrm{K}^*_{\mathbb{T}}(X)$. finitely generated. □

Remark 4.8. The stationary set of a smooth $\mathbb{T}$-action is smooth: a choice of a $\mathbb{T}$-invariant Riemannian metric yields, at every $x \in F$, an exponential map, $\mathrm{T}_x X \to X$, which is $\mathbb{T}$-equivariant and is a diffeomorphism in a small metric ball around the origin in $\mathrm{T}_x X$. Therefore $\exp_x$ intertwines (an open subset of) the stationary set of the *linear* action of $\mathbb{T}$ on $\mathrm{T}_x X$, to (an open subset) of the stationary set F. This yields a $\mathbb{T}$-equivariant smooth manifold chart around x in F.

It follows that $\mathrm{K}^*_{\mathbb{T}}(X - F)$ has a finite $\mathbb{T}$-spectrum, equivalently, a nonzero annihilator ideal, because it is finitely generated and torsion.

We will discuss $\mathbb{T}$-equivariant Poincaré duality for smooth manifolds in greater depth later; for now, the following statement is useful for proving certain things quickly.

Theorem 4.9. *Let X be a smooth and compact $\mathbb{T}$-manifold and let $D \in \mathrm{KK}^{\mathbb{T}}(C_0(\mathrm{T}X), \mathbb{C})$ the class of the Dirac operator on the almost-complex $\mathbb{T}$-manifold $\mathrm{T}X$. Then cup-cap product with D determines a natural family of isomorphisms*

$$\mathrm{KK}^{\mathbb{T}}_*(C(X) \otimes A, B) \cong \mathrm{KK}^{\mathbb{T}}_*(A, C_0(\mathrm{T}X) \otimes B)$$

for all $\mathbb{T}$-C^*-algebras A, B.

Theorem 4.9 is due to [8] in the nonequivariant setting. See also Kasparov [18] in the equivariant setting and his references. For a modern treatment of equivariant Poincaré duality see [14].

It follows from Poincaré duality that if W and Z are compact smooth $\mathbb{T}$-manifolds, then $\mathrm{KK}^{\mathbb{T}}_*(C(W), C(Z))$ is a finitely generated $\mathbb{C}[X, X^{-1}]$-module. Indeed, duality reduces us to proving that $\mathrm{K}^*_{\mathbb{T}}(\mathrm{T}W \times Z)$ is finitely generated, which follows from Lemma 4.10 below. From this, and consideration of the 6-term exact sequence associated to $F \subset X$, we deduce that the modules e.g. $\mathrm{KK}^{\mathbb{T}}_*(C_0(X - F), C(F))$ of morphisms in $\mathrm{KK}^{\mathbb{T}}$ between any two of $C(X), C(F)$ and $C_0(X - F)$, are finitely generated.

Lemma 4.10. *If X is a compact smooth $\mathbb{T}$-manifold and $V \to X$ is a real $\mathbb{T}$-equivariant vector bundle on X, then $\mathrm{K}^*_{\mathbb{T}}(V)$ is finitely generated. Moreover, if $\mathbb{T}$ acts freely on $V - 0$ then the restriction map*

$$\mathrm{K}^*_{\mathbb{T}}(V) \to \mathrm{K}^*_{\mathbb{T}}(X)$$

induces an isomorphism after localizing at $\mathbb{C}^* - \{1\}$.

In particular, $\mathbb{T}\text{-spec}(V) = \mathbb{T}\text{-spec}(X) \cup \{1\}$ if $\mathbb{T}$ acts freely on $V - 0$. A good example is $V = \mathrm{T}X$ for a compact smooth $\mathbb{T}$-manifold X where stationary points of the action are isolated, that is, where there is only a finite number of them. Fixing a $\mathbb{T}$-invariant Riemannian metric, any nonzero tangent vector which is fixed by the $\mathbb{T}$-action results in a geodesic which is point wise fixed by the action, contradicting that the stationary set consists of finitely many points. Thus the $\mathbb{T}$-action on nonzero tangent vectors is free.

Proof. Fix a $\mathbb{T}$-invariant metric on V and consider the exact sequence

$$0 \longrightarrow C_0(D_V) \longrightarrow C_0(\overline{D}_V) \longrightarrow C(S_V) \longrightarrow 0$$

where D_V is the open disk bundle, $\overline{D}_V$ the closed disk bundle, and S_V the sphere bundle. Since $\overline{D}_V$ is $\mathbb{T}$-equivariantly proper homotopy equivalent to X, which is a compact smooth manifold, and since S_V is also a compact smooth manifold, it follows from considering the associated 6-term exact sequence and Lemma 4.6 that $\mathrm{K}^*_{\mathbb{T}}(V) \cong \mathrm{K}^*_{\mathbb{T}}(D_V)$ is finitely generated.

If $\mathbb{T}$ acts freely on $V - 0$ then it acts freely on S_V and hence $\mathbb{T}\text{-spec}(S_V) \subset \{1\}$. Therefore, localizing at $\mathbb{C}^* - \{1\}$ kills $\mathrm{K}^*_{\mathbb{T}}(S_V)$ and the claim follows. □

Remark 4.11. Suppose that V carries a $\mathbb{T}$-equivariant K-orientation. The *Euler class* $e_V \in \mathrm{K}^{-\dim(V)}(X)$ of V can be defined as the restriction to X (the zero section in V) of the Thom class for V, in $\mathrm{K}^{-\dim(V)}_{\mathbb{T}}(V)$. The Thom class generates $\mathrm{K}^*_{\mathbb{T}}(V)$ as a free rank-one $\mathrm{K}^*_{\mathbb{T}}(X)$-module. It follows that the restriction map $\mathrm{K}^*_{\mathbb{T}}(V) \to \mathrm{K}^*_{\mathbb{T}}(X)$ identifies, under $\mathrm{K}^*_{\mathbb{T}}(V) \cong \mathrm{K}^*_{\mathbb{T}}(X)$, with the map

$$\mathrm{K}^*_{\mathbb{T}}(X) \to \mathrm{K}^*_{\mathbb{T}}(X), \quad \xi \mapsto \xi \cdot e_V.$$

It follows then from Lemma 4.10 that e_V becomes an invertible after we localize at $\mathbb{C}^* - \{1\}$, that is, e_V is an invertible in the ring $\mathrm{K}^*_{\mathbb{T}}(X)_f$ where $f(X) = X - 1$.

This fact is used frequently in connection with characteristic class computations in the work of Atiyah and Segal and in Atiyah and Bott's paper [2].

Theorem 4.12. *Let X be a compact smooth $\mathbb{T}$-manifold and $F \subset X$ the stationary set. Let*

$$\Omega := \{t \in \mathbb{T} \mid tx = x \text{ for some } x \in X - F\}.$$

Ω is finite. Let $f \in \mathbb{C}[X, X^{-1}]$ be a polynomial vanishing on Ω. Then $C_0(X - F)$ is $\mathrm{KK}^{\mathbb{T}}_f$-equivalent to the zero $\mathbb{T}$-C^*-algebra, and the localization $\rho_f \in \mathrm{KK}^{\mathbb{T}}(C(X), C(F))_f$ of the restriction morphism $\rho \in \mathrm{KK}^{\mathbb{T}}(C(X), C(F))$, is invertible (in $\mathrm{KK}^{\mathbb{T}}_f$).

This theorem is similar to [3, Prop. 1.5].

Proof. To prove that $C_0(X - F)$ is $\mathrm{KK}^{\mathbb{T}}_f$-equivalent to zero it suffices to prove that $\mathrm{KK}^{\mathbb{T}}_*(C_0(X - F), C_0(X - F))_f$ is the zero module over $\mathbb{C}[X, X^{-1}]_f$. By Lemma 4.4, if f vanishes on Ω then $\mathrm{K}^*_{\mathbb{T}}\big(\mathrm{T}F \times (X - F)\big)_f = 0 = \mathrm{K}^*_{\mathbb{T}}\big(\mathrm{T}X \times (X - F)\big)_f$ and by Poincaré duality for respectively F and X this implies that

$$\mathrm{KK}^{\mathbb{T}}_*(C(F), C_0(X - F))_f = 0, \quad \mathrm{KK}^{\mathbb{T}}_*(C(X), C_0(X - F))_f = 0. \tag{4.1}$$

Using the 6-term exact sequence applied to the first variable, we deduce that

$$\mathrm{KK}^{\mathbb{T}}_*(C_0(X - F), C_0(X - F))_f = 0,$$

too. Thus, $C_0(X - F)$ is $\mathrm{KK}^{\mathbb{T}}_f$-equivalent to the zero $\mathbb{T}$-C^*-algebra as claimed.

(We could not use Poincaré duality directly for $C_0(X - F)$ because it is noncompact, and duality works differently for noncompact spaces.)

Now from the 6-term exact sequence, and the fact just proved that $C_0(X - F)$ is $\mathrm{KK}^{\mathbb{T}}_f$-equivalent to zero, the map

$$\mathrm{KK}^{\mathbb{T}}_*(A, C(X))_f \xrightarrow{\cdot \otimes_{C(X)} \rho} \mathrm{KK}^{\mathbb{T}}_*(A, C(F))_f \tag{4.2}$$

induced by restriction to F is an isomorphism for any $\mathbb{T}$-C^*-algebra A. Now use the Yoneda lemma: set $A := C(F)$ and find a preimage $\alpha \in \mathrm{KK}^{\mathbb{T}}(C(F), C(X))_f$ of the identity morphism in $\mathrm{KK}^{\mathbb{T}}_*(C(F), C(F))_f$. Then the composition in $\mathrm{KK}^{\mathbb{T}}_f$

$$C(F) \xrightarrow{\alpha} C(X) \xrightarrow{\rho} C(F)$$

is the identity by the definitions, and the composition

$$C(X) \xrightarrow{\rho} C(F) \xrightarrow{\alpha} C(X)$$

is therefore multiplication by an idempotent $\gamma := \rho \otimes_{C(F)} \alpha \in \mathrm{KK}^{\mathbb{T}}(C(X), C(X))_f$. To show that $1 - \gamma = 0$ set $A := C(X)$, and observe that this is mapped to zero under composition with ρ, i.e. under the map (4.2). Since the latter is an isomorphism after localization, $1-\gamma = 0$. □

Remark 4.13. While a properly formulated version of Theorem 4.12 *should* be true without smoothness assumptions (cp. Theorem 2.6, which does not use such an assumption), we have not pursued it since we are mainly interested in smooth manifolds anyway, and because Example 4.5 shows that away from smooth manifolds, $\mathbb{T}$-spec$(X-F)$ may not be finite, which makes it more difficult to formulate a theorem.

Corollary 4.14. *Let D be a $\mathbb{T}$-C^*-algebra in the boostrap category, such that $D \rtimes \mathbb{T}$ is also in the boostrap category, let X be a smooth, compact $\mathbb{T}$-manifold, and f, Ω be as in Theorem 4.12. Then*

(i) $\mathrm{KK}^{\mathbb{T}}_*(C(X), D)_f \cong \mathrm{Hom}_{\mathbb{C}[X,X^{-1}]_f}\big(\mathrm{K}^*_{\mathbb{T}}(X)_f, \mathrm{K}^{\mathbb{T}}_*(D)_f\big)$,

(ii) $\mathrm{KK}^{\mathbb{T}}_*(\mathbb{C}, C(X)\otimes D)_f \cong \mathrm{K}^*_{\mathbb{T}}(X) \otimes_{\mathbb{C}[X,X^{-1}]_f} \mathrm{K}^{\mathbb{T}}_*(D)$.

Proof. The class of $\mathbb{T}$-spaces X for which both theorems hold (in $\mathrm{KK}^{\mathbb{T}}_f$) is closed under $\mathrm{KK}^{\mathbb{T}}_f$-equivalence so we may replace X by F by Theorem 4.12; since F is a trivial $\mathbb{T}$-space, $\mathrm{KK}^{\mathbb{T}}_*(C(F), D) \cong \mathrm{KK}_*(C(F), D\rtimes\mathbb{T})$ by the Green–Julg theorem, and by the UCT this is isomorphic to $\mathrm{Hom}_{\mathbb{C}}\big(\mathrm{K}^*(F), \mathrm{K}_*(D\rtimes\mathbb{T})\big) \cong \mathrm{Hom}_{\mathbb{C}}\big(\mathrm{K}^*(F), \mathrm{K}^{\mathbb{T}}_*(D)\big)$. This implies the corresponding isomorphisms after localization. Now $\mathrm{K}^*_{\mathbb{T}}(X)_f \cong \mathrm{K}^*_{\mathbb{T}}(F)_f \cong \big(\mathrm{K}^*(F)\otimes\mathbb{C}[X,X^{-1}]\big)_f \cong \mathrm{K}^*(F)\otimes \mathbb{C}[X,X^{-1}]_f$ and hence $\mathrm{Hom}_{\mathbb{C}[X,X^{-1}]_f}\big(\mathrm{K}^*_{\mathbb{T}}(X)_f, \mathrm{K}^{\mathbb{T}}_*(D)_f\big) \cong \mathrm{Hom}_{\mathbb{C}[X,X^{-1}]_f}\big(\mathrm{K}^*(F)\otimes\mathbb{C}[X,X^{-1}]_f, \mathrm{K}^*_{\mathbb{T}}(D)_f\big) \cong \mathrm{Hom}_{\mathbb{C}}\big(\mathrm{K}^*(F), \mathrm{K}^*_{\mathbb{T}}(D)_f\big)$ which proves the first statement. The second follows similarly (see the proof of Lemma 5.8.) □

We end this section with a fairly precise description of $\mathrm{K}^*_{\mathbb{T}}(X)$ for smooth $\mathbb{T}$-manifolds, starting with the following result, which uses ideas of Baum and Connes (see [6]).

Theorem 4.15. *Let X be a smooth, compact $\mathbb{T}$-manifold, $F\subset X$ the stationary set.*

For $\gamma\in\mathbb{T}$ we endow the $\mathbb{C}$-vector space $\mathrm{K}^*\big(\mathbb{T}\backslash(X^\gamma - F)\big)$ with the $\mathbb{C}[X,X^{-1}]$-module structure by evaluation $\mathbb{C}[X,X^{-1}]\to\mathbb{C}$ at γ. Then

$$\mathrm{K}^*_{\mathbb{T}}(X-F) \cong \oplus_{\gamma\in\mathbb{T}\text{-spec}(X-F)} \mathrm{K}^*\big(\mathbb{T}\backslash(X^\gamma - F)\big) \tag{4.3}$$

as $\mathbb{C}[X,X^{-1}]$-modules.

Remark 4.16. The usual geometric effect of localization of $\mathrm{K}^*_{\mathbb{T}}(X)$ at $\gamma\in\mathbb{T}$—it annihilates the contribution of $X - X^\gamma$, as we have seen—is obviously nil in the case where $\gamma = 1$. Thus Theorem 4.15 goes further in this case, informing us that the stalk at 1 of the sheaf determined by $\mathrm{K}^*_{\mathbb{T}}(X-F)$ is $\mathrm{K}^*\big(\mathbb{T}\backslash(X-F)\big)$ (with $\mathbb{C}[X,X^{-1}]$-module structure by evaluation at $1\in\mathbb{C}^*$.)

Proof. Set $\Omega := \{\gamma\in\mathbb{T} \mid \gamma x = x \text{ some } x\notin F\} \subset \mathbb{T}\text{-spec}(X-F)$. Ω is finite. We consider a theory defined on $\mathbb{T}$-spaces (like $X-F$) which can be covered by a finite number of open H-slices, where $H \subset \Omega$ is some subset. This class of spaces is clearly closed under passing to subspaces. If Z is such a space, let

$$\mathrm{F}(Z) := \bigoplus_{\gamma \in \Omega} \mathrm{K}^*(\mathbb{T}\backslash Z^\gamma)$$

with module structure evaluation of characters at γ in the corresponding summand. Observe that we may interpret this vector space as $\mathrm{K}^*(\widehat{Z})$ where

$$\widehat{Z} := \mathbb{T}\backslash\{(z,\gamma) \in Z \times \mathbb{T} \mid \gamma z = z\}.$$

Indeed, the space $\widehat{Z}$ fibers over Ω with fiber $\mathbb{T}\backslash X^\gamma$ over γ.

If $Y \subset Z$ is a closed $\mathbb{T}$-invariant subspace of Z in our class, then $\widehat{Y} \subset \widehat{Z}$ as a closed subspace, and $\widehat{Z} - \widehat{Y} = \widehat{Z - Y}$. Hence an inclusion of a closed $\mathbb{T}$-invariant subspace generates a corresponding 6-term exact sequence and the theory F is excisive. To show that it agrees with $\mathrm{K}^*_{\mathbb{T}}(\,\cdot\,)$ it is sufficient then to verify this for an induced space $U \cong \mathbb{T} \times_H Y$. In this case $\mathrm{K}^*_{\mathbb{T}}(U) \cong \mathrm{K}^*_H(Y)$ as $\mathbb{C}[X, X^{-1}]$-modules, where the $\mathbb{C}[X, X^{-1}]$-module action on $\mathrm{K}^*_H(Y)$ factors through the restriction $\mathbb{C}[X, X^{-1}] \to \mathrm{Rep}(H)$ and the $\mathrm{Rep}(H)$-module structure on $\mathrm{K}^*_H(Y)$. By a result of Baum and Connes for equivariant K-theory of finite group actions (see [6])

$$\mathrm{K}^*_H(Y) \cong \bigoplus_{h \in H} \mathrm{K}^*(H\backslash Y^h),$$

where the $\mathrm{Rep}(H) \cong \mathbb{C}[X, X^{-1}]/(f_H)$-module structure on the right hand side is by evaluation of characters at the points of H (here $f_H = \prod_{h \in H} X - h$ and (f_H) is the ideal of $\mathbb{C}[X, X^{-1}]$ generated by f_H.) We are using the fact that H is abelian, so that the centralizer of h in H is H. Localizing at $\gamma \in \Omega$ yields zero unless $\gamma \in H$, and in this case,

$$\begin{aligned}\mathrm{K}^*_H(Y)_\gamma &:= \mathrm{K}^*_H(Y) \otimes_{\mathbb{C}[X,X^{-1}]} \mathbb{C}[X, X^{-1}]_\gamma \\ &\cong \bigoplus_{h \in H} \left[\mathrm{K}^*(H\backslash Y^h) \otimes_{\mathbb{C}[X,X^{-1}]} \mathbb{C}[X, X^{-1}]_\gamma\right].\end{aligned}$$

Now for each term on the right hand side, the tensor product is over the evaluation map $\mathbb{C}[X, X^{-1}] \to \mathbb{C}$ at h. It follows that all terms in the sum on the right hand side vanish except for $h = \gamma$. The $\mathbb{C}[X, X^{-1}]_\gamma$-module structure on this term is evaluation of polynomials at γ. Thus,

$$\mathrm{K}^*_{\mathbb{T}}(U)_\gamma \cong \mathrm{K}^*(H\backslash Y^\gamma)_\gamma.$$

Given that $H\backslash Y^\gamma \cong \mathbb{T}\backslash U^\gamma$, the result follows. □

In particular, we now have an exact description of $\mathbb{T}$-spec(X) when X is a compact smooth manifold.

Corollary 4.17. *Let X be a compact smooth $\mathbb{T}$-manifold with no stationary points. Then*

$$\mathbb{T}\text{-spec}(X) = \{\gamma \in \mathbb{T} \mid \mathrm{K}^*(\mathbb{T} \setminus X^\gamma) \neq 0\}.$$

Before the proof, we use Theorem 4.15 determine the exact relation between the torsion submodule of $\mathrm{K}^*_{\mathbb{T}}(X)$ and the torsion module $\mathrm{K}^*_{\mathbb{T}}(X-F)$.

Let $\mathrm{Tors}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big)$ be the torsion part of $\mathrm{K}^i_{\mathbb{T}}(X)$ and $\mathrm{Free}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big)$ the free part. The 6-term exact sequence associated to the stationary set $F \subset X$ yields *surjections*

$$\mathrm{K}^i_{\mathbb{T}}(X-F) \to \mathrm{Tors}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big)$$

since the map $\mathrm{K}^i_{\mathbb{T}}(X) \to \mathrm{K}^i_{\mathbb{T}}(F)$ vanishes on the torsion part, since $\mathrm{K}^*_{\mathbb{T}}(F)$ is free, and *injections*

$$\mathrm{Free}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big) \to \mathrm{K}^i_{\mathbb{T}}(F) \cong \mathrm{K}^i(F) \otimes \mathbb{C}[X, X^{-1}],$$

since the map $\mathrm{K}^{i+1}_{\mathbb{T}}(X-F) \to \mathrm{K}^{i+1}_{\mathbb{T}}(X)$ has range in the torsion subgroup.

We have the boundary maps

$$\partial_i : \mathrm{K}^{i-1}(F) \otimes \mathbb{C}[X, X^{-1}] \longrightarrow \mathrm{K}^i_{\mathbb{T}}(X-F)\big) \tag{4.4}$$

and thus

$$\mathrm{coker}(\partial_i) \cong \mathrm{Tors}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big), \quad \ker(\partial_{i+1}) \cong \mathrm{Free}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big).$$

Theorem 4.15 and some geometric arguments (using smoothness) tells us more.

Corollary 4.18. *If X is a smooth compact $\mathbb{T}$-manifold, then the range of $\partial_i : \mathrm{K}^i_{\mathbb{T}}(F) \to \mathrm{K}^{i+1}_{\mathbb{T}}(X-F)$ is supported at $1 \in \mathbb{C}^*$. Hence ∂_i factors through a map*

$$\partial'_i : \mathrm{K}^i_{\mathbb{T}}(F) \to \mathrm{K}^{i+1}(\mathbb{T} \setminus X - F).$$

Thus $\mathrm{Tors}(\mathrm{K}^i_{\mathbb{T}}(X))_z \cong \mathrm{K}^i_{\mathbb{T}}(X-F)_z$ for all $z \in \mathbb{T} - \{1\}$, and for the component at $1 \in \mathbb{C}^*$ we have

$$\mathrm{Tors}(\mathrm{K}^i_{\mathbb{T}}(X)_1) \cong \mathrm{K}^i(\mathbb{T}\backslash X - F) \,/\, \mathrm{im}(\partial'_{i+1}).$$

Furthermore, the free modules $\mathrm{Free}\big(\mathrm{K}^*_{\mathbb{T}}(X)\big)$ and $\mathrm{K}^*_{\mathbb{T}}(F)$ have the same rank in each dimension.

Remark 4.19. The Lefschetz fixed-point theorem discussed below implies that the difference in ranks of the free part of $\mathrm{K}^0_{\mathbb{T}}(X)$ and the free part of $\mathrm{K}^1_{\mathbb{T}}(X)$ equals the difference of ranks of the $\mathbb{C}$-vector spaces $\mathrm{K}^0(F)$ and $\mathrm{K}^1(F)$. The above statement is stronger, since it holds before taking differences.

The boundary maps in the 6-term exact sequence of Theorem 4.15 can be computed fairly precisely if X is a smooth manifold with smooth $\mathbb{T}$-action, and this also proves the Corollary 4.18.

For the definition of correspondence, used below, see the discussion in Section 5.

Proof. (Of Corollary 4.18). F is a closed, smooth submanifold of X. Let ν be the normal bundle of the stationary set $F \subset X$; it can be endowed with a $\mathbb{T}$-action and invariant Riemannian metric. Let $\hat{\varphi} : \nu \to X$ the tubular neighborhood embedding.

Let $S\nu$ be the sphere bundle of ν and $\pi : \nu \to F$ the bundle projection. Let $j : S\nu \to X$ be its restriction to $S\nu$. Note that $j(S\nu)$ is disjoint from F and that j is a canonically $\mathbb{T}$-equivariantly K-oriented embedding with trivial normal bundle. To see this, define

$$\hat{f} : S\nu \times \mathbb{R} \cong U_F \subset X - F, \quad \hat{f}(x, \xi, s) := \hat{\varphi}(s\xi).$$

The restriction of $\hat{f}$ to the zero section $S\nu \times \{0\}$ is the embedding j.

The class in $\mathrm{KK}_1^{\mathbb{T}}(C(F), C_0(X - F))$ of the $\mathbb{T}$-equivariant extension

$$0 \longrightarrow C_0(X - F) \longrightarrow C(X) \longrightarrow C(F) \longrightarrow 0$$

is equal (see [8, Prop. 3.6]; the equivariant version goes through in the same way since we have a $\mathbb{T}$-equivariant normal bundle) to the class of the $\mathbb{T}$-equivariant correspondence

$$S\nu \xleftarrow{\pi_{S\nu}} (S\nu \times \mathbb{R}, \beta_{\mathbb{R}}) \xrightarrow{\hat{f}} X - F$$

where $\beta_{\mathbb{R}} \in \mathrm{K}^1_{\mathbb{T}}(\mathbb{R})$ is the Bott class (for the trivial $\mathbb{T}$-action on $\mathbb{R}$.)

Hence the class $\partial[V] \in \mathrm{K}^1_{\mathbb{T}}(X - F)$ is then represented by the smooth $\mathbb{T}$-equivariant correspondence $\mathrm{pnt} \leftarrow (S\nu, \pi^*_{S\nu}(V)) \xrightarrow{j} X \setminus F$, alternatively, as the class

$$\hat{f}_!\big(\pi^*_{S\nu}(V) \cdot \beta_{\mathbb{R}}\big) \in \mathrm{K}^1_{\mathbb{T}}(X - F)$$

of the Thom class of the (trivial) normal bundle, pushed forward to $X - F$ via $\hat{f}$.

Note that since $\mathbb{T}$ acts freely on $\nu - 0$, the open neighborhood U_F of $\hat{\varphi}(S\nu)$ may be assumed to meet none of the X^γ with $\gamma \in \mathbb{T} - \{1\}$). Hence localizing at $\gamma \neq 1$ kills the range of ∂_0, so its range is contained in the component of $\mathrm{K}^1_{\mathbb{T}}(X - F)$ over $1 \in \mathbb{T}$. Similarly for $i = 1$.

For the last statement, we know from the general discussion above that $\ker(\partial_{i+1}) \cong \mathrm{Free}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big)$ as $\mathbb{C}[X, X^{-1}]$-modules, which implies the corresponding statement after localization at $\mathbb{C}^* - \{1\}$. But we have just argued that ∂_{i+1} induces the zero map after localization at $\mathbb{C}^* - \{1\}$, so that its kernel after localization becomes $\mathrm{K}^i_{\mathbb{T}}(F)_f$ $(f(X) = X - 1)$. Hence the free $\mathbb{C}[X, X^{-1}]_f$-modules $\mathrm{Free}\big(\mathrm{K}^i_{\mathbb{T}}(X)_f\big)$ and $\mathrm{K}^i_{\mathbb{T}}(F)_f$ are isomorphic, so have the same rank, and it follows that $\mathrm{Free}\big(\mathrm{K}^i_{\mathbb{T}}(X)\big)$ and $\mathrm{K}^i_{\mathbb{T}}(F)$ have the same rank also, since localizing a free module does not change its rank. ☐

Remark 4.20. We make several remarks about the proof.

(i) We can describe the maps ∂_i' more precisely. In the proof of Corollary 4.18 we observed that there is a $\mathbb{T}$-equivariant correspondence

$$S\nu \xleftarrow{\pi_{S\nu}} (S\nu \times \mathbb{R}, \beta_{\mathbb{R}}) \xrightarrow{\hat{f}} X - F.$$

In fact by shrinking the neighborhood U_F of $S\nu$ if needed so that it is disjoint from F, we can factor $\hat{f}$ through an open embedding $\hat{f}' : S\nu \times \mathbb{R} \to U_F - F$ and the open embedding $U_F - F \to X - F$. The first yields a class in $\mathrm{KK}^1_{\mathbb{T}}(C(S\nu), C_0(U_F - F))$ but this group maps, using descent, to $\mathrm{KK}^1_{\mathbb{T}}(C(\mathbb{T}\backslash S\nu), C_0(U_F - F))$ since $\mathbb{T}$ acts freely on $S\nu$ and

$U_F - F$. Now the open embedding $U_F - F \to X - F$ induces an open embedding of quotient spaces $\mathbb{T}\backslash U_F - F \to \mathbb{T}\backslash X - F$ and an element $j! \in \mathrm{KK}(C_0(\mathbb{T}\backslash U_F - F), C_0(\mathbb{T}\backslash X - F))$. The map ∂_i' is the composition

(4.5)

$$\mathrm{K}^i_{\mathbb{T}}(F) \xrightarrow{\pi^*_{S\nu}} \mathrm{K}^i_{\mathbb{T}}(S\nu) \cong \mathrm{K}^i(\mathbb{T}\backslash S\nu) \xrightarrow{\hat{f}'} \mathrm{K}^{i+1}(U_F - F) \xrightarrow{j!} \mathrm{K}^{i+1}(\mathbb{T}\backslash X - F).$$

(ii) The boundary map $\partial_0 : \mathrm{K}^0(F) \otimes \mathbb{C}[X, X^{-1}] \to \mathrm{K}^1_{\mathbb{T}}(X - F)$ may be understood as giving an obstruction to extending a $\mathbb{T}$-equivariant vector bundle on F to a $\mathbb{T}$-equivariant vector bundle on X: this is possible for a given $[V]$ only if $\partial_0[V] = 0$, which is if and only if the class

$$\hat{f}_!\big(\pi^*_{S\nu}(V) \cdot \beta_{\mathbb{R}}\big) \in \mathrm{K}^1(\mathbb{T}\backslash X - F)$$

vanishes.

5. The Lefschetz theorem

Definition 5.1. Let X be a smooth, compact $\mathbb{T}$-manifold. Let

- $D \in \mathrm{KK}^{\mathbb{T}}_0(C_0(\mathrm{T}X), \mathbb{C})$ be the class of the $\mathbb{T}$-equivariant Dirac operator on the almost-complex manifold $\mathrm{T}X$.
- $\Theta \in \mathrm{KK}^{\mathbb{T}}_0\big(C_0(X), C_0(X \times \mathrm{T}X)\big)$ the class of the $\mathbb{T}$-equivariant K-oriented embedding $\rho : X \to X \times \mathrm{T}X$, $\rho(x) := \big(x, (x, 0)\big)$.
- s be the proper $\mathbb{T}$-map $\mathrm{T}X \to X \times \mathrm{T}X$, $s(x, \xi) := \big((x, \xi), x\big)$.

Then the *Lefschetz map* (see [14])

$$\mathrm{Lef} : \mathrm{KK}^{\mathbb{T}}_*\big(C(X), C(X)\big) \to \mathrm{KK}^{\mathbb{T}}_*(C(X), \mathbb{C})$$

is the composition

$$(5.1)\quad \mathrm{KK}^{\mathbb{T}}_*\big(C(X), C(X)\big) \xrightarrow{\otimes_{\mathbb{C}} 1_{\mathrm{T}X}} \mathrm{KK}^{\mathbb{T}}_*(C_0(X \times \mathrm{T}X), C_0(X \times \mathrm{T}X)) \xrightarrow{s^*} \mathrm{KK}^{\mathbb{T}}_*\big(C_0(X \times \mathrm{T}X), C_0(\mathrm{T}X)\big) \xrightarrow{\otimes_{C_0(\mathrm{T}X)} D} \mathrm{KK}^{\mathbb{T}}_*\big(C_0(X \times \mathrm{T}X), \mathbb{C}) \xrightarrow{\Theta \otimes_{C_0(X \times \mathrm{T}X)}} \mathrm{KK}^{\mathbb{T}}_*(C(X), \mathbb{C})$$

Thus the Lefschetz map associates to an equivariant morphism $X \to X$ in $\mathrm{KK}^{\mathbb{T}}$, an equivariant K-homology class for X. Such a class has an index in $\mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$.

Definition 5.2. The *Lefschetz index* $\mathrm{Ind}^L(\Lambda)$, where $f \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$ is the $\mathbb{T}$-equivariant index

$$\mathrm{Ind}^L(\Lambda) := (\mathrm{pnt})_* \, \mathrm{Lef}(\Lambda) \in \mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}],$$

where $\mathrm{pnt} : X \to \mathrm{pnt}$ is the map to a point.

In [15] and [16] we proved that $\mathbb{T}$-equivariant correspondences are cycles for a bivariant homology theory isomorphic to $\mathrm{KK}^{\mathbb{T}}$, with some restrictions on its arguments (e.g. to compact smooth $\mathbb{T}$-manifolds.)

Hence both the domain and codomain of the Lefschetz map can be described in terms of equivalence classes of correspondences; since we have defined the Lefschetz map itself in terms of correspondences, the Lefschetz map can be described in purely geometric terms. We give a brief summary.

Suppose the following data is given (see the original reference [8]), or [15].)

- M is a smooth $\mathbb{T}$-manifold (not necessarily compact).
- $b : M \to X$ is a smooth $\mathbb{T}$-map (not necessarily proper).
- $\xi \in \mathrm{RK}^*_{\mathbb{T},X}(M)$ is an equivariant K-theory class with compact support along the fibers of b.
- $f : M \to X$ is a $\mathbb{T}$-equivariant smooth K-oriented map.

This data is sometimes summarized by a diagram $X \xleftarrow{b} (M, \xi) \xrightarrow{f} X$. The quadruple (M, b, f, ξ) is a $\mathbb{T}$-*equivariant correspondence* from X to X.

It is convenient to assume that the correspondence—denote it Λ—also satisfies

- $f : M \to X$ is a submersion.
- The map $X \to X \times X$, $x \mapsto \big(f(x), b(x)\big)$ is transverse to the diagonal $X \to X \times X$.

These conditions imply that the *coincidence space*

$$\mathcal{C} := \{x \in M \mid f(x) = g(x)\}$$

has the structure of a smooth, equivariantly K-oriented $\mathbb{T}$-manifold (probably disconnected, but with only finitely many connected components, but each of the same dimension.)

Clearly it comes with a map $b|_{\mathcal{C}} : \mathcal{C} \to X$, so we obtain a Baum–Douglas cycle $(\mathcal{C}, b|_{\mathcal{C}}, \xi|_{\mathcal{C}})$ for X by restricting ξ to $\mathcal{C} \subset M$.

To a correspondence is associated a class, which by abuse of notation we also denote by Λ, in $\mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$. Here $* = \dim(M) - \dim(X) + \dim(\xi)$. See [15] for the details.

The following is a straightforward manipulation with correspondences.

Proposition 5.3. *If $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$ is represented by the $\mathbb{T}$-equivariant correspondence in general position in the sense described above, then $\mathrm{Lef}(\Lambda)$ is represented by the Baum–Douglas cycle $(\mathcal{C}, b|_{\mathcal{C}}, \xi|_{\mathcal{C}})$ for X. In particular,*

$$\mathrm{Ind}^L(\Lambda) = \mathrm{ind}_{\mathbb{T}}(D_{\mathcal{C}} \cdot \xi|_{\mathcal{C}}) \in \mathrm{Rep}(\mathbb{T}) \cong \mathbb{C}[X, X^{-1}]$$

holds; that is, the Lefschetz index of Λ equals the $\mathbb{T}$-index of the $\mathbb{T}$-equivariant Dirac operator on the coincidence manifold $\mathcal{C}$, twisted by $\xi|_{\mathcal{C}}$.

We will not prove this proposition; the proof can be found in [9] or the reader reasonably familiar with correspondences can prove it himself.

We aim to prove that $\mathrm{Ind}^L(\Lambda) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\Lambda_*)$ for where $\Lambda_* : \mathrm{K}^*_{\mathbb{T}}(X) \to \mathrm{K}^*_{\mathbb{T}}(X)$ is the action of Λ on equivariant K-theory; note that Λ_* is a $\mathbb{C}[X, X^{-1}]$-module map. Proving this statement has nothing to do with correspondences; it depends only on formal properties of $\mathrm{KK}^{\mathbb{T}}$.

The result provides a homological interpretation of the Lefschetz index along the lines of the classical theorem.

By the *trace* we mean the following. Firstly, since X is a smooth compact manifold, $\mathrm{K}^*_{\mathbb{T}}(X)$ is a finitely generated $\mathbb{C}[X, X^{-1}]$-module. Therefore (in each dimension $* = 0, 1$ it decomposes into a free part and a torsion part. Any $\mathbb{C}[X, X^{-1}]$-module self-map of $\mathrm{K}^*_{\mathbb{T}}(X)$ of even degree will induce a grading-preserving map on K-theory. We will define the trace of such a map to be the differences of the $\mathbb{C}[X, X^{-1}]$-valued module traces on $\mathrm{K}^0_{\mathbb{T}}(X)$ and $\mathrm{K}^1_{\mathbb{T}}(X)$. To define these individually, consider any $\mathbb{C}[X, X^{-1}]$ module, which we write as $M = T \oplus \mathbb{C}[X, X^{-1}]^k$ where T is torsion. Any self $\mathbb{C}[X, X^{-1}]$-module map of M sends T to itself and hence has an upper-triangular form $L = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ and we let $\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(L) := \mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(C)$. This is uniquely defined.

A $\mathbb{C}[X, X^{-1}]$-module self-map of $\mathrm{K}^*_{\mathbb{T}}(X)$ with odd degree will have trace zero, by definition.

Theorem 5.4. *(Lefschetz theorem in* $\mathrm{KK}^{\mathbb{T}}$*). Let X be a compact smooth $\mathbb{T}$-manifold and $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$. Then* $\mathrm{Ind}^L(\Lambda) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\Lambda_*)$.

Before proceeding, note that since Lef (and Ind^L) are both defined by basic $\mathrm{KK}^{\mathbb{T}}$-operations, both maps are compatible in the obvious sense with localization. For any A and B and any $\alpha \in \mathrm{KK}^{\mathbb{T}}_*(A, B)$, and any $f \in \mathbb{C}[X, X^{-1}]$, denote by $\alpha_f \in \mathrm{KK}^{\mathbb{T}}_*(A, B)_f$ the image of f under localization at U_f. Then compatibility means that the diagram

$$\begin{array}{ccccc} \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X)) & \xrightarrow{\mathrm{Lef}} & \mathrm{KK}^{\mathbb{T}}_*(C(X), \mathbb{C}) & \xrightarrow{\mathrm{ind}_{\mathbb{T}}} & \mathbb{C}[X, X^{-1}] \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))_f & \xrightarrow{\mathrm{Lef}} & \mathrm{KK}^{\mathbb{T}}_*(C(X), \mathbb{C})_f & \xrightarrow{\mathrm{ind}_{\mathbb{T}}} & \mathbb{C}[X, X^{-1}]_f \end{array} \tag{5.2}$$

commutes, where the lower row is the "localized" Lefschetz index map, defined using Kasparov products as on the top row, except with the localized classes D_f, Θ_f and so on.

Neither the first nor second vertical map need be injective, of course, but the third vertical map is injective because $\mathbb{C}[X, X^{-1}]$ is an integral domain. The diagram says that $\mathrm{Ind}^L(\Lambda)_f = \mathrm{Ind}^L_f(\Lambda_f)$ where Ind^L_f is the Lefschetz map in localized $\mathrm{KK}^{\mathbb{T}}$.

We define the localized module trace

$$\mathrm{trace}_{\mathbb{C}[X,X^{-1}]_f} : \mathrm{End}_{\mathbb{C}[X,X^{-1}]_f}(\mathrm{K}^*_{\mathbb{T}}(X)_f) \to \mathbb{C}[X, X^{-1}]_f$$

as with the nonlocalized version. Note that localization of a $\mathbb{C}[X, X^{-1}]$-module respects the decomposition into its torsion and free parts, so that

$$\mathrm{trace}_{\mathbb{C}[X,X^{-1}]_f}(L_f) = \big[\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(L)\big]_f \tag{5.3}$$

is clear, for any $\mathbb{C}[X, X^{-1}]$-module self-map of $\mathrm{K}^*_{\mathbb{T}}(X)$.

It will be sufficient to prove the following apparently weaker version of Theorem 5.4.

Lemma 5.5. *Let Ω be as in Theorem 4.12 and $f \in \mathbb{C}[X, X^{-1}]$ vanish on Ω. Then the Lefschetz theorem for X holds in $\mathrm{KK}^{\mathbb{T}}_f$. That is,*

$$\mathrm{Ind}^L_f(\Lambda_f) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]_f}\big((\Lambda_f)_*\big)$$

for any $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$.

Lemma 5.5 implies Theorem 5.4 because combining the diagram (5.2) and its algebraic analogue (5.3) gives

$$\begin{aligned}(5.4)\quad \mathrm{Ind}^L(\Lambda)_f = \mathrm{Ind}^L_f(\Lambda_f) &= \mathrm{trace}_{\mathbb{C}[X,X^{-1}]_f}\big(\Lambda_f)_*\big)\\ &= \big[\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\Lambda)\big]_f \in \mathbb{C}[X, X^{-1}]_f.\end{aligned}$$

By injectivity of $\mathbb{C}[X, X^{-1}] \to \mathbb{C}[X, X^{-1}]_f$, it follows that $\mathrm{Ind}^L(\Lambda) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\Lambda_*)$, yielding Theorem 5.4.

To prove Lemma 5.5 it is useful to use a slightly different formalism for the Lefschetz *indices* $\mathrm{Ind}^L(\,\cdot\,)$. This formalism is more general in the sense that it applies to noncommutative $\mathbb{T}$-C*-algebras as well, provided they have duals. (The Lefschetz *map* of Definition 5.1 exists in more generality than we have suggested, but does not work for noncommutative algebras because of the implicit use of the "diagonal map" $X \to X \times \mathrm{T}X$.)

As above, $s : \mathrm{T}X \to X \times \mathrm{T}X$ is the obvious section. Let $\Sigma : X \times \mathrm{T}X \to \mathrm{T}X \times X$ be the flip. Set

- $\Delta := \Sigma^* s_*(D) \in \mathrm{KK}^{\mathbb{T}}(C_0(\mathrm{T}X \times X), \mathbb{C})$,
- $\underline{\widehat{\Delta}} := (\mathrm{pnt})_*(\Theta) \in \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, C_0(X \times \mathrm{T}X))$,

We denote $A := C(X)$ and $B := C_0(\mathrm{T}X)$.

It is easily checked that Δ and $\underline{\widehat{\Delta}}$ satisfy the "zig-zag equations"

$$(5.5)\quad \big(\underline{\widehat{\Delta}} \otimes_{\mathbb{C}} 1_A\big) \otimes_{A\otimes B\otimes A} \big(1_A \otimes \Delta\big) = 1_A,\ \ \big(1_B \otimes_{\mathbb{C}} \underline{\widehat{\Delta}}\big) \otimes_{B\otimes A\otimes B} \big(\Delta \otimes_{\mathbb{C}} 1_B\big) = 1_B$$

and it follows that the map

$$\mathrm{KK}^{\mathbb{T}}_*(D_1, D_2 \otimes B) \to \mathrm{KK}^{\mathbb{T}}_*(D_1 \otimes, D_2),\ x \mapsto (x \otimes 1_A) \otimes_{B\otimes A} \Delta$$

is an isomorphism for every D_1, D_2 (cp. the briefly stated Theorem 4.9). The inverse map is defined similarly, using $\underline{\widehat{\Delta}}$. This is the kind of noncommutative Poincaré duality studied by the author in several papers, e.g. [10, 11, 14].

Set $\widehat{\Delta} := \Sigma_*(\underline{\widehat{\Delta}})$.

Lemma 5.6. *In the above notation: for any $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(A, A) := \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$,*

$$(5.6)\quad \mathrm{Ind}^L(\Lambda) = \big(\widehat{\Delta} \otimes_{B\otimes A} (1_B \otimes \Lambda)\big) \otimes_{B\otimes A} \Delta \in \mathrm{KK}^{\mathbb{T}}_*(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}[X, X^{-1}].$$

Similarly after localization.

Proof. Using the definitions

$$\begin{aligned}(5.7)\quad \mathrm{Ind}^L(\Lambda) &:= (\mathrm{pnt})_*\big(\mathrm{Lef}(\Lambda)\big)\\ &= (\mathrm{pnt})_*(\Theta) \otimes_{C_0(X\times \mathrm{T}X)} (\Lambda \otimes_{\mathbb{C}} 1_{C_0(\mathrm{T}X)}) \otimes_{C_0(X\times \mathrm{T}X)} [s^*] \otimes_{C_0(\mathrm{T}X)} D\\ &= \widehat{\underline{\Delta}} \otimes_{C_0(X\times \mathrm{T}X)} (\Lambda \otimes 1_{C_0(\mathrm{T}X)}) \otimes_{C_0(X\times \mathrm{T}X)} \Sigma^*(\Delta).\end{aligned}$$

where $[s^*] \in \mathrm{KK}^{\mathbb{T}}(C_0(X \times \mathrm{T}X), C_0(\mathrm{T}X))$ is the class of s. Carrying the flip across yields

$$(5.8)\qquad = \widehat{\Delta} \otimes_{C_0(\mathrm{T}X\times)} (1_{C_0(\mathrm{T}X)} \otimes_{\mathbb{C}} \Lambda) \otimes_{C_0(\mathrm{T}X\times X)} \Delta$$

as required. □

In particular, using the right hand side of (5.6), we can define the Lefschetz index of a morphism $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(A, A)$ for any $\mathbb{T}$-C*-algebra A for which there exists a triple $(B, \Delta, \widehat{\underline{\Delta}})$ satisfying (5.5). We call such A *dualizable.*

The author believes that A dualizable implies $\mathrm{K}^{\mathbb{T}}_*(A)$ is a finitely generated $\mathbb{C}[X, X^{-1}]$-module (see [14] for the nonequivariant proof) but does not have a reference. We are not interested in proving this here, since the A we consider obviously have finitely generated equivariant K-theory.

Suppose for such A there exists a C*-algebra A' and a $\mathrm{KK}^{\mathbb{T}}$-equivalence $\alpha \in \mathrm{KK}^{\mathbb{T}}(A, A')$. In this case, A' is also dualizable using $B' := B$,

$$(5.9)\qquad \Delta' := (1_B \otimes_{\mathbb{C}} \alpha^{-1}) \otimes_{B\otimes A} \Delta \in \mathrm{KK}^{\mathbb{T}}(B' \otimes A', \mathbb{C})$$

and

$$(5.10)\qquad \widehat{\underline{\Delta}}' := \widehat{\underline{\Delta}} \otimes_{A\otimes B} (\alpha \otimes 1_B) \in \mathrm{KK}^{\mathbb{T}}(\mathbb{C}, A' \otimes B').$$

Conjugation by α gives an isomorphism $\mathrm{KK}^{\mathbb{T}}_*(A, A) \cong \mathrm{KK}^{\mathbb{T}}(A', A')$ and it is easy to check that

Lemma 5.7.

$$(5.11)\qquad \mathrm{Ind}^L(\Lambda) = \mathrm{Ind}^L(\alpha \otimes_{A'} \Lambda \otimes_{A'} \alpha^{-1})$$

for any $\Lambda \in \mathrm{KK}_*(A', A')$*, and where the left hand side of this equation is defined using the dual* $(B', \Delta', \widehat{\underline{\Delta}}')$ *and the right hand side using* $(B, \Delta, \widehat{\underline{\Delta}})$*.*

This is of course what is to be expected if Ind^L is to agree with a $\mathbb{C}[X, X^{-1}]$-valued trace: the statement

$$\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\Lambda_*) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\alpha_*^{-1} \circ \Lambda_* \circ \alpha_*)$$

with $\Lambda_* : \mathrm{K}^{\mathbb{T}}_*(A') \to \mathrm{K}^*_{\mathbb{T}}(A')$, $\alpha_* : \mathrm{K}^{\mathbb{T}}_*(A) \to \mathrm{KK}^{\mathbb{T}}_*(A')$ the module maps induced by Λ and α, is obvious.

Lemma 5.7 also proves the independence of

$$\mathrm{Ind}^L : \mathrm{KK}^{\mathbb{T}}_*(A, A) \to \mathbb{C}[X, X^{-1}]$$

of the choice of dual $(B, \Delta, \widehat{\Delta})$, since any two duals for a fixed $\mathbb{T}$-C*-algebra A are related by a self-$\mathrm{KK}^{\mathbb{T}}$-equivalence of B as in (5.9) and (5.10).

This discussion has its obvious analogue in the localized category $\mathrm{KK}^{\mathbb{T}}_f$ (for any $f \in \mathbb{C}[X, X^{-1}]$). That is, we can speak of a C^*-algebra A being *dualizable* in $\mathrm{KK}^{\mathbb{T}}_f$, we may define the Lefschetz index map $\mathrm{Ind}^L_f : \mathrm{KK}^{\mathbb{T}}_*(A, A)_f \to \mathbb{C}[X, X^{-1}]_f$, and so on, cp. the discussion around (5.2) regarding the Lefschetz *map* for $A = C(X)$.

We now return to the case where $A = C(X)$ for a smooth, compact $\mathbb{T}$-manifold X. Let f be as in Theorem 4.12. Thus $\rho_f \in \mathrm{KK}^{\mathbb{T}}(C(X), C(F))_f$ is a $\mathrm{KK}^{\mathbb{T}}_f$-equivalence. Hence by the analogue in $\mathrm{KK}^{\mathbb{T}}_f$ of Lemma 5.7,

$$\mathrm{Ind}^L_f(\Lambda_f) = \mathrm{Ind}^L_f(\rho_f^{-1} \otimes_{C(X)} \Lambda_f \otimes_{C(F)} \rho_f).$$

Note that Ind^L_f is defined for the stationary set F because already

$$\mathrm{Ind}^L : \mathrm{KK}^{\mathbb{T}}_*(C(F), C(F)) \to \mathrm{KK}^{\mathbb{T}}_*(C(F), \mathbb{C})$$

is defined, because the stationary set F is a smooth $\mathbb{T}$-manifold (with the trivial action), and hence has a dual.

Our goal at this stage is therefore to prove that

$$\mathrm{Ind}^L_f(\mu) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]_f}(\mu_*) \tag{5.12}$$

for any $\mu \in \mathrm{KK}^{\mathbb{T}}(C(F), C(F))_f$. This will prove Lemma 5.5 and hence Theorem 5.4. But since F is a trivial $\mathbb{T}$-space, we can prove even the stronger statement

$$\mathrm{Ind}^L(\mu) = \mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\mu_*). \tag{5.13}$$

In fact this is simply a computation with bilinear forms, and applies to general groups.

Lemma 5.8. *Let G be a compact group, let A be a trivial G-C^*-algebra and B a G-C^*-algebra. Assume that as a C^*-algebra, A is in the boostrap category $\mathcal{N}$. Finally, assume that B and A are Poincaré dual, i.e. that there exist classes $\Delta \in \mathrm{KK}^G_0(B \otimes A, \mathbb{C})$ and $\underline{\widehat{\Delta}} \in \mathrm{KK}^G_0(\mathbb{C}, A \otimes B)$ such that* (5.5) *are satisfied. Let $\Lambda \in \mathrm{KK}_*(A, A)$ and $\widehat{\Delta} := \Sigma_*(\underline{\widehat{\Delta}}) \in \mathrm{KK}^G_0(\mathbb{C}, B \otimes A)$. Then*

$$\big(\widehat{\Delta} \otimes_{B\otimes A} (1_B \otimes \Lambda)\big) \otimes_{B\otimes A} \Delta = \mathrm{trace}_s(\Lambda_*)$$

holds, where the trace is that of the module map induced by Λ on the free, finitely generated $\mathrm{Rep}(G)$*-module* $\mathrm{K}^G_*(A) \cong \mathrm{K}_*(A) \otimes_{\mathbb{C}} \mathrm{Rep}(G)$.

In particular, Lemma 5.5, and hence Theorem 5.4 and (hence) all of its localized analogues (in particular (5.12)*) hold for trivial compact $\mathbb{T}$-manifolds X.*

Proof. Since A is a trivial G-C^*-algebra, $\mathrm{KK}^G_*(\mathbb{C}, B\otimes A) \cong \mathrm{KK}_*(\mathbb{C}, A\otimes B \rtimes G)$. The assumed equivariant duality implies nonequivariant duality and this implies (see [14] that $\mathrm{K}_*(A)$ is finite-dimensional. By the Green–Julg theorem $\mathrm{KK}^G_*(\mathbb{C}, B \otimes A) \cong \mathrm{KK}_*(\mathbb{C}, B \rtimes G \otimes A)$, and by the (nonequivariant) Künneth theorem ([7, Thm. 23.1.3]) this is $\cong \mathrm{KK}_*(\mathbb{C}, B \rtimes G) \otimes \mathrm{KK}_*(\mathbb{C}, A) \cong \mathrm{KK}^G_*(\mathbb{C}, B) \otimes \mathrm{KK}^G_*(\mathbb{C}, A)$; where the tensor product is in the category of $\mathrm{Rep}(G)$-modules.

Thus, external product

$$\mathrm{KK}^G_*(\mathbb{C}, A) \otimes \mathrm{KK}^G_*(\mathbb{C}, B) \to \mathrm{KK}^G_*(\mathbb{C}, A \otimes B)$$

is an isomorphism; for emphasis, the tensor product on the left hand side is in the category of $\mathrm{Rep}(G)$-modules.

We may find a finite basis $\{y_i^\epsilon\}$ for $\mathrm{K}^G_*(A)$ as an $\mathrm{Rep}(G)$-module with $y_i^\epsilon \in \mathrm{K}^G_\epsilon(A)$, and there exist $x_i^\epsilon \in \mathrm{K}^G_\epsilon(B)$ such that

$$\underline{\widehat{\Delta}} = \sum_i y_i^\epsilon \otimes_{\mathbb{C}} x_i^\epsilon \in \mathrm{K}^G_0(A \otimes B). \tag{5.14}$$

We have assumed ((5.5)) that

$$(\underline{\widehat{\Delta}} \otimes_{\mathbb{C}} 1_A) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta) = 1_A \in \mathrm{KK}_0(A, A). \tag{5.15}$$

Applying the functor from the category KK^G to the category of $\mathbb{Z}/2$-graded $\mathrm{Rep}(G)$-modules, we get that

$$y = y \otimes_A \big((\underline{\widehat{\Delta}} \otimes 1_A) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta) \big)$$

for all $y \in \mathrm{K}_*(A)$. Expanding the right hand side using (5.14) yields

$$\begin{aligned} y &= \sum_{i,\epsilon} \big(y_i^\epsilon \otimes_{\mathbb{C}} x_i^\epsilon \otimes_{\mathbb{C}} y \big) \otimes_{A \otimes B \otimes A} (1_A \otimes \Delta) \\ &= \sum_{i,\epsilon} y_i^\epsilon \otimes_{\mathbb{C}} \big((x_i^\epsilon \otimes_{\mathbb{C}} y) \otimes_{B \otimes A} \Delta \big) = \sum_{i,\epsilon} L_i^\epsilon(y) y_i^\epsilon \end{aligned} \tag{5.16}$$

where, as indicated, $L_i^\epsilon(y) = (x_i^\epsilon \otimes_{\mathbb{C}} y) \otimes_{B \otimes A} \Delta \in \mathrm{Rep}(G)$. Since the y_i^ϵ form a basis, we deduce by setting $y = y_j^\gamma$, that

$$L_i^\epsilon(y_j^\gamma) = \delta_{\epsilon,\gamma} \delta_{i,j}. \tag{5.17}$$

Now let $\Lambda \in \mathrm{KK}_0(A, A)$ (similar computations apply to odd morphisms.) We can write

$$\Lambda_*(y_i^\epsilon) = \sum_j \lambda_{ij}^\epsilon y_j^\epsilon. \tag{5.18}$$

Since the external product in KK^G is graded commutative,

$$\widehat{\Delta} := \Sigma_*(\underline{\widehat{\Delta}}) = \sum_{i,\epsilon} (-1)^\epsilon x_i^\epsilon \otimes_{\mathbb{C}} y_j^\epsilon.$$

We get, therefore,

$$\begin{aligned} &\big(\widehat{\Delta} \otimes_{B \otimes A} (1_B \otimes \Lambda) \big) \otimes_{B \otimes A} \Delta \\ &= \sum_{i,\epsilon} (-1)^\epsilon \big(x_i^\epsilon \otimes_{\mathbb{C}} y_i^\epsilon \big) \otimes_{B \otimes A} (1_B \otimes_{\mathbb{C}} \Lambda) \otimes_{B \otimes A} \Delta \\ &= \sum_{i,j,\epsilon} (-1)^\epsilon \lambda_{ij}^\epsilon (x_i^\epsilon \otimes_{\mathbb{C}} y_j^\epsilon) \otimes_{B \otimes A} \Delta = \sum_{i,\epsilon} \lambda_{ii}^\epsilon \end{aligned} \tag{5.19}$$

where the last step is using (5.17). This gives the graded trace of Λ_* acting on the free $\mathrm{Rep}(G)$-module $\mathrm{K}^G_*(A)$ as required.

The last statement follows from setting $A = C(X)$ as in the discussion around (5.5). □

We close with a brief discussion of equivariant Euler numbers, in order to illustrate the Lefschetz theorem.

Remark 5.9. The case of Euler numbers is the case where Λ is a "twist" of the identity correspondence, thus Λ has the form $X \xleftarrow{\mathrm{Id}} (X, \xi) \xrightarrow{\mathrm{Id}} X$ where $\xi \in \mathrm{K}^*_{\mathbb{T}}(X)$.

We first make a general observation about the Lefschetz map.

Lemma 5.10. *For any $\mathbb{T}$-C^*-algebra A and any $\mathbb{T}$-space X, $\mathrm{KK}^{\mathbb{T}}_*(C(X), A)$ is a module over $\mathrm{K}^*_{\mathbb{T}}(X)$. This module structure is "natural" with respect to A.*

Moreover, Lef $: \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X)) \to \mathrm{KK}^{\mathbb{T}}_*(C(X), \mathbb{C})$ *is a $\mathrm{K}^*_{\mathbb{T}}(X)$-module homomorphism.*

The $\mathrm{K}^*_{\mathbb{T}}(X)$-module structure on $\mathrm{K}_*^{\mathbb{T}}(X)$ corresponds to the process of twisting an elliptic operator by a vector bundle. Furthermore, it follows from the axiomatic definition of the Kasparov product that the Kasparov pairing $\mathrm{K}^*_{\mathbb{T}}(X) \times \mathrm{K}_*^{\mathbb{T}}(X) = \mathrm{KK}^{\mathbb{T}}_*(\mathbb{C}, C(X)) \times \mathrm{KK}^{\mathbb{T}}_*(C(X), \mathbb{C})$ maps (ξ, a) to $\mathrm{pnt}_*(a \cdot \xi)$, where the dot is the module structure, $\mathrm{pnt} : X \to \mathrm{pnt}$ is the map from X to a point.

We therefore have

$$\langle \xi, \mathrm{Lef}(\Lambda) \rangle = \mathrm{Ind}^L(\Lambda \cdot \xi) \in \mathbb{C}[X, X^{-1}]$$

for any $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$ and $\xi \in \mathrm{K}^*_{\mathbb{T}}(X)$, and, roughly, if we can realize $\mathrm{Lef}(\Lambda)$ as the class of a suitable elliptic operator, then this can be interpreted as the $\mathbb{T}$-index of that operator twisted by ξ. The module structure can also be easily described explicitly in topological terms, using correspondences.

The point is that the action of $\Lambda \cdot \xi \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$ on $\mathrm{K}^*_{\mathbb{T}}(X)$ is clearly the composition

$$\mathrm{K}^*_{\mathbb{T}}(X) \xrightarrow{\Lambda_*} \mathrm{K}^*_{\mathbb{T}}(X) \xrightarrow{\lambda_\xi} \mathrm{K}^*_{\mathbb{T}}(X)$$

where the map denoted λ_ξ is *ring multiplication by* ξ; this is clearly a $\mathbb{C}[X, X^{-1}]$-module map. Therefore we get a refinement of Theorem 5.4 involving the twisted Lefschetz numbers $\mathrm{Ind}^L(\Lambda \cdot \xi) = \langle \xi, \mathrm{Lef}(\Lambda) \rangle$.

Proposition 5.11. *In the above notation,*

$$\langle \xi, \mathrm{Lef}(\Lambda) \rangle = \mathrm{trace}_{\mathbb{C}[X, X^{-1}]}(\Lambda_* \circ \lambda_\xi) \in \mathbb{C}[X, X^{-1}]$$

for any $\Lambda \in \mathrm{KK}^{\mathbb{T}}_*(C(X), C(X))$ and $\xi \in \mathrm{K}^*_{\mathbb{T}}(X)$, where λ_ξ is the $\mathrm{Rep}(\mathbb{T})$*-module homomorphism of ring multiplication by ξ.*

We call the elements $e_X(\xi) := \mathrm{Ind}^L(\mathrm{Id} \cdot \xi)$ for $\xi \in \mathrm{K}^*_{\mathbb{T}}(X)$, the *twisted* $\mathbb{T}$-*equivariant Euler numbers of* X. Note that $e_X(\xi) = 0$ if ξ is an odd K-class.

We may interpret the Euler numbers in two different ways, given the above discussion:

- $e_X(\xi)$ is the $\mathbb{T}$-equivariant analytic index of the de Rham operator on X twisted by ξ.
- $e_X(\xi)$ is the module trace $\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(L_\xi)$ of ring multiplication by ξ on $\mathrm{K}^*_{\mathbb{T}}(X)$.

The first statement follows from the computation in [13], which proves the much stronger statement that $\mathrm{Lef}(\mathrm{Id}) = [D_{\mathrm{dR}}] \in \mathrm{K}_0^G(X)$, where D_{dR} is the de Rham (or "Euler") operator on X and G is *any* locally compact group acting properly and smoothly on X. For further information on the class of the de Rham operator and related issues, see [12] and [13], and the paper of Rosenberg and Lück [19] and of Rosenberg [21].

To compute the invariants in the first interpretation, let $g \in \mathbb{T}$ generate the circle topologically, so that $\mathrm{Fix}(g) = F$. Since $g : X \to X$ is $\mathbb{T}$-equivariantly homotopic to the identity, $\mathrm{Lef}(\mathrm{Id}) = \mathrm{Lef}([g^*])$. Now the computation of the Lefschetz map (for ordinary smooth self-maps) in [13] yields

$$\mathrm{Lef}([g^*]) = (i_F)_*([D^F_{\mathrm{dR}}])$$

where D^F_{dR} is the de Rham operator on F, $[D^F_{\mathrm{dR}}]$ its class in $\mathrm{KK}_0^{\mathbb{T}}(C(F), \mathbb{C})$, and $i_F : F \to X$ is the inclusion map. (The sign data in [13] vanishes because g is an isometry, which implies that the vector bundle map $\mathrm{Id} - Dg$ on the $\mathbb{T}$-equivariant normal bundle to F is homotopic to the identity bundle map.)

Thus, we see that $e_X(\xi) = e_F(\xi|_F)$ where $e_F(\xi|_F)$ denotes the equivariant Lefschetz number of the restriction of ξ to the smooth (trivial) $\mathbb{T}$-space F. By another application of the Lefschetz theorem, this time for the trivial $\mathbb{T}$-space F, yields that this equals the $\mathbb{T}$-index of the de Rham operator on F twisted by $\xi|_F$.

Since F is $\mathbb{T}$-fixed pointwise, we can further simplify this answer. Assume first that F is connected. The bundle $E|_F$ can be diagonalized into eigenspaces for the $\mathbb{T}$-action, $E|_F \cong \oplus_\lambda E_\lambda$ where $\mathbb{T}$ acts on E_λ by the character f_λ, some $f_\lambda \in \mathbb{C}[X, X^{-1}]$. Let $\xi_\lambda = [E_\lambda] \in \mathrm{K}^0(F)$. We see then that

$$e_F(\xi|_F) = \sum_\lambda e_F^{\mathrm{nonequ.}}(\xi_\lambda) f_\lambda$$

where in this formula $e_F^{\mathrm{nonequ.}}$ are the twisted, *nonequivariant* Euler numbers for the stationary manifold F.

Nonequivariant Euler numbers are straightforward to compute. The index of the de Rham operator on a connected compact manifold P, twisted by $\xi \in \mathrm{K}^0(P)$, is simply $\chi(P)\dim(\xi) \in \mathbb{Z}$, where χ is the numerical Euler characteristic.

We conclude that

$$e_X([E]) = \chi(F) \sum_\lambda \dim_{\mathbb{C}}(E_\lambda)\, f_\lambda.$$

If F has components $\{P\}$ then this formula becomes

$$\mathrm{trace}_{\mathbb{C}[X,X^{-1}]}(\lambda_\xi) = e_X(\xi) = \sum_P \chi(P) \sum_\lambda \dim_{\mathbb{C}}((E|_P)_\lambda)\, f_{\lambda,P}.$$

The right hand side is by and large easy to compute in specific situations. The case of isolated fixed-points is particularly transparent.

Proposition 5.12. *Let X be a smooth compact $\mathbb{T}$-manifold with a finite set of isolated stationary points. Then for any $\xi \in \mathrm{K}^0_{\mathbb{T}}(X)$,*

$$\operatorname{trace}_{\mathbb{C}[X,X^{-1}]}(\lambda_\xi) = \sum_{P \in F} \xi_P$$

where the ξ_P are the restrictions of $\xi \in \mathrm{K}^*_{\mathbb{T}}(X)$ to the points P, each such P yielding an element $\xi_P \in \mathrm{K}^*_{\mathbb{T}}(P) \cong \mathbb{C}[X, X^{-1}]$.

The following example illustrates the difference in computing the two invariants equated by the Lefschetz theorem.

Example 5.13. Let $X = \mathbb{CP}^1$ with the $\mathbb{T}$-action induced by the embedding $\mathbb{T} \to \mathrm{SU}_2(\mathbb{C}) \subset \mathrm{Aut}(\mathbb{C}^2)$, $z \mapsto \begin{bmatrix} z & 0 \\ 0 & \bar{z} \end{bmatrix}$. There are two stationary points, with homogeneous coordinates $[1,0]$ and $[0,1]$ respectively. Let H^* be the canonical line bundle on $\mathbb{CP}^1$, it is a $\mathbb{T}$-invariant subbundle of $\mathbb{CP}^1 \times \mathbb{C}^2$ so has a canonical structure of $\mathbb{T}$-equivariant vector bundle. Restricting H^* to the stationary points $[1,0]$ and $[0,1]$ yields respectively the characters X and X^{-1}, whence by the Lefschetz theorem

$$e_{\mathbb{CP}^1}([H^k]) = \operatorname{trace}_{\mathbb{C}[X,X^{-1}]}(\lambda^k_{[H]}) = X^k + X^{-k} \in \mathbb{C}[X, X^{-1}].$$

where H is the dual of H^*. Computation of the $\operatorname{trace}_{\mathbb{C}[X,X^{-1}]}(\lambda^k_{[H]})$ by homological methods requires computing $\mathrm{K}^*_{\mathbb{T}}(\mathbb{CP}^1)$ as both a ring and as a $\mathbb{C}[X,X^{-1}]$-module. By results of Atiyah and others, (see Segal's article [23] for a beautiful and concise proof) it is generated as a commutative unital ring by X and $[H]$ with the relations that X and $[H]$ are invertible and commute, and satisfy

$$([H] - X)([H] - X^{-1}) = 0.$$

Hence $[H]^2 = (X + X^{-1})[H] + 1$. This implies that as a $\mathbb{C}[X,X^{-1}]$-module, $\mathrm{K}^0_{\mathbb{T}}(\mathbb{CP}^1)$ is generated by the unit 1 of the ring, and the element $[H]$. This is a free basis, and with respect to it

$$\lambda_{[H]} = \begin{bmatrix} 0 & 1 \\ 1 & X + X^{-1} \end{bmatrix}.$$

The trace is $X + X^{-1}$. The formula for $\operatorname{trace}_{\mathbb{C}[X,X^{-1}]}(\lambda^k_{[H]})$ follows from induction, using the relation $\lambda^n_{[H]} = (X + X^{-1})\lambda^{n-1}_{[H]} + \lambda^{n-2}_{[H]}$, which comes from the relation given by the minimal polynomial $\lambda^2 - (X + X^{-1})\lambda - 1$ of $\lambda_{[H]}$.

References

[1] M. Artin and B. Mazur, On periodic points, Ann. of Math. (2) **81** (1965), 82–99. MR0176482 (31 #754)

[2] M. F. Atiyah and R. Bott, The moment map and equivariant cohomology, Topology **23** (1984), no. 1, 1–28. MR0721448 (85e:58041)

[3] M. F. Atiyah and G. B. Segal, The index of elliptic operators. II, Ann. of Math. (2) **87** (1968), 531–545. MR0236951 (38 #5244)

[4] M. Atiyah and G. Segal, On equivariant Euler characteristics, J. Geom. Phys. **6** (1989), no. 4, 671–677. MR1076708 (92c:19005)

[5] S. Baaj and G. Skandalis, C*-algèbres de Hopf et théorie de Kasparov équivariante, K-Theory **2** (1989), no. 6, 683–721. MR1010978 (90j:46061)

[6] P. Baum and A. Connes, K-theory for discrete groups, *Operator algebras and applications, Vol. 1*, 1–20, London Math. Soc. Lecture Note Ser., 135 Cambridge Univ. Press, Cambridge. MR0996437 (91i:46083)

[7] B. Blackadar, *K-theory for operator algebras*, second edition, Mathematical Sciences Research Institute Publications, **5**, Cambridge University Press, Cambridge, 1998. MR1656031 (99g:46104)

[8] A. Connes and G. Skandalis, The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183. MR0775126 (87h:58209)

[9] I. Dell'Ambrogio, H. Emerson, and R. Meyer, An equivariant Lefschetz fixed-point formula for correspondences. arXiv:1303.4777 [math.KT] (2013).

[10] H. Emerson, Noncommutative Poincaré duality for boundary actions of hyperbolic groups, J. Reine Angew. Math. **564** (2003), 1–33. MR2021032 (2005e:19007)

[11] H. Emerson, Lefschetz numbers for C*-algebras, Canad. Math. Bull. **54** (2011), no. 1, 82–99. MR2797970 (2012e:19008)

[12] H. Emerson and R. Meyer, Euler characteristics and Gysin sequences for group actions on boundaries, Math. Ann. **334** (2006), no. 4, 853–904. MR2209260 (2007b:19006)

[13] H. Emerson and R. Meyer, Equivariant Lefschetz maps for simplicial complexes and smooth manifolds, Math. Ann. **345** (2009), no. 3, 599–630. MR2534110 (2010h:19006)

[14] H. Emerson and R. Meyer, Dualities in equivariant Kasparov theory, New York J. Math. **16** (2010), 245–313. MR2740579 (2012b:19004)

[15] H. Emerson and R. Meyer, Bivariant K-theory via correspondences, Adv. Math. **225** (2010), no. 5, 2883–2919. MR2680187 (2012a:19013)

[16] H. Emerson and R. Meyer, Equivariant embedding theorems and topological index maps, Adv. Math. **225** (2010), no. 5, 2840–2882. MR2680186 (2011j:19009)

[17] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977. MR0463157 (57 #3116)

[18] G. G. Kasparov, Equivariant KK-theory and the Novikov conjecture, Invent. Math. **91** (1988), no. 1, 147–201. MR0918241 (88j:58123)

[19] W. Lück and J. Rosenberg, The equivariant Lefschetz fixed point theorem for proper cocompact G-manifolds, *High-dimensional manifold topology*, 322–361, World Sci. Publ., River Edge, NJ. MR2048727 (2005b:57064)

[20] R. S. Palais, On the existence of slices for actions of non-compact Lie groups, Ann. of Math. (2) **73** (1961), 295–323. MR0126506 (23 #A3802)

[21] J. Rosenberg, The K-homology class of the Euler characteristic operator is trivial, Proc. Amer. Math. Soc. **127** (1999), no. 12, 3467–3474. MR1610789 (2000d:58037)

[22] J. Rosenberg and C. Schochet, The Künneth theorem and the universal coefficient theorem for equivariant K-theory and KK-theory, Mem. Amer. Math. Soc. **62** (1986), no. 348. MR0849938 (87k:46147)

[23] G. Segal, Equivariant K-theory, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 129–151. MR0234452 (38 #2769)

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