Analytic filling of totally real tori

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Abstract. We prove that any embedded Maslov index two analytic disc attached to a totally real torus in the complex two-dimensional affine space extends to an analytic filling provided that the torus is contained in a regular level set of a strictly plurisubharmonic function.

1. Introduction

We consider an embedded two-dimensional torus T in the affine space $\mathbb{C}^2$. We assume that T is *totally real* in the sense that no complex line is tangent to the torus T. We abbreviate

$$S^1 = \mathbb{R}/2\pi\mathbb{Z} \quad \text{and} \quad \mathbb{D} = \{|z| \leq 1\} \subset \mathbb{C}.$$

The totally real torus T admits an *analytic filling* provided there exists an embedding F of the solid torus $S^1 \times \mathbb{D}$ into $\mathbb{C}^2$ such that

(F_1) the boundary $S^1 \times \partial\mathbb{D}$ is mapped onto T, and
(F_2) for all $t \in S^1$ the restriction of $u_t := F(t, \cdot)$ to the interior of $\mathbb{D}$ defines a holomorphic map, i.e., u_t solves the Cauchy–Riemann equation

$$\partial_x u_t + \mathrm{i}\partial_y u_t = 0$$

on $\mathrm{Int}\,\mathbb{D}$, where $z = x + \mathrm{i}y$.

If a thickened disc $(-\varepsilon, \varepsilon) \times \mathbb{D}$ for some $\varepsilon > 0$ is embedded instead, then the map F is called a *local filling*. The aim of this note is to prove the following extension result.

Theorem 1.1. *If T is contained in a regular level set of a strictly plurisubharmonic function on $\mathbb{C}^2$, then any local filling of T extends after restriction to a global analytic filling of T.*

The *Clifford torus* is the embedded totally real torus given by the product of unit circles $\partial\mathbb{D} \times \partial\mathbb{D}$ inside $\mathbb{C} \times \mathbb{C}$ so that the solid tori $\mathbb{D} \times \partial\mathbb{D}$ and $\partial\mathbb{D} \times \mathbb{D}$ induce an analytic filling each. Notice that the Clifford torus is contained

in the expanded 3-sphere $\sqrt{2} \cdot S^3$ of all vectors having length $\sqrt{2}$. A small perturbation of the Clifford torus inside $\sqrt{2} \cdot S^3$ that fixes a neighborhood of $\{1\} \times \partial\mathbb{D}$ or $\partial\mathbb{D} \times \{1\}$ for example yields totally real tori that admit local fillings but might not be foliated by circles of the Hopf fibration. In view of Theorem 1.1 the perturbed Clifford tori still admit global analytic fillings. This can be obtained by the perturbation results of Alexander [2] and Bedford [4], alternatively.

More generally, we consider the 2-sphere S^2 in $S^3 \subset \mathbb{C} \times \mathbb{C}$ obtained by intersecting with the real hyperplane $\mathbb{C} \times \mathbb{R}$. The intersections with the complex lines $\mathbb{C} \times \{s\}$, $s \in (-1, 1)$, define an analytic filling of S^2, cp. [15]. The filling collapses at the singular points $(0, \pm 1)$, at which S^2 is tangent to the complex lines $\mathbb{C} \times \{\pm 1\}$. Attach an embedded 1-handle to S^2 inside S^3 that is obtained from a small tubular neighborhood of an embedded path that connects $(0, \pm 1)$ and is everywhere transverse to the field of complex lines $TS^3 \cap iTS^3$, cp. [13, §3.3.2]. The construction results in a possibly knotted totally real torus inside S^3, cp. [20, §4.5, §5.3]. The totally real torus admits a local and, hence, with Theorem 1.1 a global analytic filling. A general existence result for analytic fillings of totally real tori that are unknotted in S^3 is obtained by Duval–Gayet [10].

We remark that the example of attaching a 1-handle generalizes to small perturbations of embedded 2-spheres in regular level sets of strictly plurisubharmonic functions invoking the Giroux elimination lemma [18] and local Bishop discs [7] or global fillings obtained by Bedford–Gaveau [5], Gromov [19], Eliashberg [11], Bedford–Klingenberg [6], Kruzhilin [22, 23], and Ye [29].

1.2. **Totally real isotopies.** By the results of Borrelli [8, 9] there are infinitely many isotopy classes of embedded totally real tori in $\mathbb{C}^2$. Precisely one class admits analytic fillings, see Proposition 6.1 below. In Section 6 we will prove:

Corollary 1.3. *If T is contained in a regular level set of a strictly plurisubharmonic function on $\mathbb{C}^2$ and admits a local filling, then T is isotopic to the Clifford torus through totally real tori.*

1.4. **Non-trapped characteristics.** The complex lines tangent to a regular level set of a strictly plurisubharmonic function constitute a field of real two-dimensional planes, which turns out to be a contact structure, see Section 2.1 and Section 5. The foliation on T cut out by the contact structure is the so-called *characteristic foliation*, which is transverse to the foliation obtained by the boundary circles of an analytic filling. In Section 4.2 we will show:

Corollary 1.5. *In the context of Theorem 1.1 cylinders in T cut out by boundary circles of holomorphic discs that belong to an analytic filling do not admit trapped characteristics.*

In view of [14] we call a characteristic which does not connect the two boundary components of the cylinder to be *trapped*. The corollary says that any boundary circle induced by an analytic filling is a global circle of section

for the characteristic foliation of T. In particular, any integrating and non-vanishing vector field defines a Poincaré section map on all boundary circles. Hence, the characteristic foliation is *homotopically trivial*, i.e., any integrating characteristic line field is homotopic to the kernel of a non-singular closed 1-form.

2. Recollections

2.1. **Pseudo-convexity.** A real-valued function H on $\mathbb{C}^2$ is called *strictly plurisubharmonic* if the 2-form $-\mathrm{d}(\mathrm{d}H \circ \mathrm{i})$ is positive on complex lines, i.e.,

$$-\mathrm{d}(\mathrm{d}H \circ \mathrm{i})(v, \mathrm{i}v) > 0$$

for all non vanishing tangent vectors v of $\mathbb{C}^2$. This is equivalent to say that $H \circ u$ is strictly subharmonic for any holomorphic map $u : D \to \mathbb{C}^2$ defined on an open domain D of $\mathbb{C}^2$, cp. [16, §3.1].

We assume that T is contained in the regular level set $M := H^{-1}(0)$ and write $W = H^{-1}((-\infty, 0])$. Then $\partial W = M$ as oriented manifolds. The restriction of $-\mathrm{d}H \circ \mathrm{i}$ to the tangent bundle of M defines a positive contact form on M whose kernel ξ equals $TM \cap \mathrm{i}TM$. Therefore, ξ is invariant under the multiplication by i. The *characteristic foliation* T_ξ of T is the intersection of TT with ξ. Because T is a totally real torus, the intersection is transverse so that T_ξ is indeed a one-dimensional foliation on T. Choosing a co-orientation of T in M, the complex orientation of ξ orients T_ξ, i.e., the leaves of T_ξ are the so-called *characteristic leaves* of T.

The strong *maximum principle* of Eberhard Hopf applied to $H \circ u$ yields that any non-constant holomorphic map $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ sends the interior of $\mathbb{D}$ into the interior of W such that the restriction of u to $\partial\mathbb{D}$ is an immersion positively transverse to ξ, see [15, Prop. 4.2]. Therefore, we can choose an orientation of T such that the holomorphic discs given by the local filling of T intersect the leaves of T_ξ positively.

2.2. **Factorizability.** One consequence of the maximum principle is that all non-constant holomorphic maps $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ do not have any mixed self-intersection points, i.e., u does not map an interior point of $\mathbb{D}$ to $u(\partial\mathbb{D})$. In view of Lazzarini's work [24, 25] this implies that u either is *simple* or *multiply covered*. This means that there exist

- a holomorphic map $\pi : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{D}, \partial\mathbb{D})$ that is continuous up to the boundary and satisfies $\pi^{-1}(\partial\mathbb{D}) = \partial\mathbb{D}$, and
- a holomorphic map $v : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ with a dense set of points $z \in \mathbb{D}$ satisfying $T_z v \neq 0$ and $v^{-1}(v(z)) = \{z\}$

such that $u = v \circ \pi$. Being simple corresponds to π having mapping degree 1 as it is satisfied for v.

2.3. **Topological index.** Denote by A a relative homotopy class of continuous maps $u : (\mathbb{D}, \partial\mathbb{D}) \to (W, T)$. The *Maslov index* $\mu(A)$ of A is defined to be the Maslov index of the bundle pair (u^*TW, u^*TT) for any disc map u

representing A, see [27, §C.3]. Notice, that the Maslov index $\mu([u])$ does not change if the map u is perturbed through homotopies relative T that even take values outside $W \subset \mathbb{C}^2$. In other words, $\mu(A)$ is uniquely determined by the image $\partial_* A$ under the boundary homomorphism ∂_* that maps $\pi_2(\mathbb{C}^2, T)$ isomorphically onto $\pi_1(T)$.

We remark that the Maslov index $\mu(A)$ is an *even* integer for all classes A because T is orientable. This is because the forgetful map from the space of *oriented* real planes through $0 \in \mathbb{C}^2$ onto the space of *all* real planes through $0 \in \mathbb{C}^2$ has degree two, cp. [26, pp. 52–53], [3, Appendix] and [27, p. 554].

2.4. Intersection product.

A smooth map $(\mathbb{D}, \partial\mathbb{D}) \to (W, T)$ is called *admissible* if it sends $\operatorname{Int}\mathbb{D}$ into $\operatorname{Int} W$, restricts to an immersion on $\partial\mathbb{D}$, and is transverse to M. Any non-constant holomorphic map $(\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ is admissible, see Section 2.1. For admissible maps $u_1, u_2 : (\mathbb{D}, \partial\mathbb{D}) \to (W, T)$ an *intersection number* $u_1 \bullet u_2$ is defined provided u_1 and u_2 intersect in only finitely many points and the span of the tangent spaces to $u_1(\mathbb{D})$ and $u_2(\mathbb{D})$ at all boundary intersection points is not three-dimensional, see [15, §8]. We will say that u_1 and u_2 intersect *nicely*.

To each interior intersection point a local intersection multiplicity is assigned. For boundary intersection points one takes local intersection multiplicities of extensions of $u_1(\mathbb{D})$ and $u_2(\mathbb{D})$ by local Schwarz reflections, see [15, pp. 569–571]. By definition the intersection number $u_1 \bullet u_2$ is the total sum of local intersection multiplicities, where interior intersections are counted twice, see [15, Def. 8.8]. It is shown in [15, Prop. 8.9] that $u_1 \bullet u_2$ only depends on the relative homotopy classes $[u_1]$ and $[u_2]$ in $\pi_2(W, T)$. By [15, Prop. 8.2] the condition on u_1 and u_2 to intersect nicely is generic. Hence, the intersection number defines an *intersection product* $\bullet$ on $\pi_2(W, T)$, see [15, Rem. 8.11].

2.5. Non-negativity of intersections.

By [15, Prop. 9.1] any two distinct holomorphic discs $u_1, u_2 : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ intersect nicely. The intersection number $u_1 \bullet u_2$ is nonnegative and equals zero if and only if u_1 and u_2 have disjoint images, see [15, Thm. 9.2].

2.6. Relative adjunction inequality.

The *embedding defect* of a relative homotopy class A in $\pi_2(W, T)$ is defined to be

$$D(A) = A \bullet A - \mu(A) + 2.$$

For all simple holomorphic maps $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ the embedding defect $D([u])$ is nonnegative and vanishes if and only if u is an embedding, see [15, Thm. 9.4].

Proposition 2.7. *Let F be a local filling of T. Then the Maslov index of $F(t, \cdot)$ equals 2 for all t.*

Proof. Set $u_t = F(t, \cdot)$. The embedding defect $D([u_0])$ vanishes because u_0 is an embedding. The intersection product $[u_0] \bullet [u_0]$ is equal to the intersection number $u_0 \bullet u_t$ for $t > 0$ small, which is zero as $u_0(\mathbb{D})$ and $u_t(\mathbb{D})$ are disjoint. Therefore, $\mu([u_0]) = 2$. □

Example 2.8. Consider the holomorphic embedding $u(z) = (z, z)$, $z \in \mathbb{D}$, that takes boundary values on the Clifford torus $\partial\mathbb{D} \times \partial\mathbb{D}$ inside $\sqrt{2} \cdot S^3$. As u has Maslov index 4, there is no filling of the Clifford torus that extends u, which of course can be verified directly.

Remark 2.9. In [22], Kruzhilin showed that any totally real torus that is contained in a regular level set of a strictly plurisubharmonic function admits a family of Maslov index 2 holomorphic discs that are attached to the torus passing through all its points. Therefore, the total obstruction to extend to an analytic filling lies in the vanishing of the self-intersection number of the represented relative homology class.

2.10. **Automatic transversallity.** We formulate the converse of Proposition 2.7. For that consider a holomorphic embedding $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$. The collection of three pairwise disjoint local paths in T that intersect $u(\partial\mathbb{D})$ transversally in a single point will be referred to as the *transverse constraints*.

Proposition 2.11. *Let $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$ be a holomorphic embedding of Maslov index 2. Then there exists a local filling F of T that extends $u = F(0, \cdot)$ such that the curves $t \mapsto F(t, \mathrm{i}^k)$, $k = 0, 1, 2$, locally parametrize the transverse constraints. Moreover, the filling is unique up to re-parametrizations in the time variable t and shrinking the time interval.*

Proof. This is worked out in [20, §2] and [21, §3.2]. The necessary modifications in view of the transverse constraints can be achieved similarly to [15, Lem. 7.5 and Prop. 7.6] by a choice of a Riemannian metric that turns T and the three transverse constraints into totally geodesic submanifolds. $\square$

3. A moduli space

In this section we assume the situation of Theorem 1.1 and denote the local filling of T by F.

3.1. **Definition.** Set $u_0 = F(0, \cdot)$. Provide the space of all holomorphic maps $u : (\mathbb{D}, \partial\mathbb{D}) \to (\mathbb{C}^2, T)$, which by Section 2.1 take values in W, with the C^∞-topology. The subspace consisting of all u that are homologous to u_0 in W relative T is denoted by $\widetilde{\mathcal{M}}$. Notice, that all holomorphic discs $u \in \widetilde{\mathcal{M}}$ are simple. To see this, factor $u = v \circ \pi$ as described in Section 2.2. Because $\mu([v])$ is even and using Proposition 2.7, we see that the degree of the holomorphic map π must be one. In particular, the group G of conformal automorphisms of $(\mathbb{D}, \mathrm{i})$ acts without fixed points via re-parametrizations. The *moduli space* $\mathcal{M}$ is defined to be the quotient $\widetilde{\mathcal{M}}/G$.

Because $D([u_0])$ vanishes, all holomorphic maps in $\widetilde{\mathcal{M}}$ are embeddings with Maslov index 2, cp. Proposition 2.7. By Section 2.5 the images of holomorphic maps in $\widetilde{\mathcal{M}}$ that have distinct images, i.e., represent distinct classes in $\mathcal{M}$, are in fact disjoint because $[u_0] \bullet [u_0] = 0$. With the arguments from Proposition 2.11 $\widetilde{\mathcal{M}}$ is a four-dimensional, and hence $\mathcal{M}$ a one-dimensional, smooth manifold.

3.2. **Geometric bounds.** Denote by B a ball in $\mathbb{C}^2$ that contains the torus T. Because B is a sub-level set of the strictly plurisubharmonic function $\mathbf{z} \mapsto \frac{1}{4}|\mathbf{z}|^2$, all holomorphic discs $u(\mathbb{D})$ with $u(\partial\mathbb{D}) \subset T$ are contained in B. This again follows from the maximum principle.

3.3. **Cutting the torus.** The image of $u_0|_{\partial\mathbb{D}}$ is an oriented knot in T, which is transverse to the characteristic leaves of T, see Section 2.1. Therefore, $u_0(\partial\mathbb{D})$ can not bound a disc inside T. In view of [28, p. 25] the complement $T \setminus u_0(\partial\mathbb{D})$ is diffeomorphic to $S^1 \times (0,1)$. We may assume that the boundary circles of the holomorphic discs that belong to the local filling F coincide with the slices $S^1 \times \{t\}$ for $t \in (0,\varepsilon) \cup (1-\varepsilon, 1)$ so that we can add two copies of $u_0(\partial\mathbb{D})$ to the cylinder to get $S^1 \times [0,1]$. According to the orientation convention in Section 2.1 the leaves of T_ξ point inwards $S^1 \times [0,1]$ along $S^1 \times \{0\}$.

3.4. **Energy bounds.** Denote by ω the symplectic form $d\mathbf{x} \wedge d\mathbf{y}$ of $\mathbb{C}^2 \equiv \mathbb{R}^4$. The energy $E(u)$ of a holomorphic disc u is defined by $\int_{\mathbb{D}} u^*\omega$. Because i and ω are compatible, $E(u)$ is equal to the Dirichlet energy of u, see [27, §2.2]. Both descriptions imply that the energy is invariant under conformal re-parametrizations.

Proposition 3.5. *The energy function $[u] \mapsto E(u)$ on $\mathcal{M}$ is bounded from above.*

Proof. We consider a holomorphic map $u \in \widetilde{\mathcal{M}}$ that is not a conformal re-parametrization of u_0. Then $u(\partial\mathbb{D})$ divides $T \setminus u_0(\partial\mathbb{D})$, which is diffeomorphic to $S^1 \times (0,1)$, into two nonempty cylindrical components, because the knots $u_0(\partial\mathbb{D})$ and $u(\partial\mathbb{D})$ are homotopic in $S^1 \times [0,1]$. We denote the cylinder for which $u(\partial\mathbb{D})$ is an oriented boundary component by C. Therefore, by Stokes theorem,

$$\int_C \omega = \int_{u(\partial\mathbb{D})} \lambda - \int_{u_0(\partial\mathbb{D})} \lambda = E(u) - E(u_0)$$

writing λ instead of $\mathbf{x}d\mathbf{y}$. Choose an area form σ on T so that $\omega = f\sigma$ for a smooth function f on the 2-torus T. Therefore,

$$\int_C \omega \leq \max_C |f| \cdot \sigma(C),$$

where we denote the total area of a subset $U \subset T$ by $\sigma(U)$. Combining both expressions, we obtain

$$E(u) \leq E(u_0) + \max_C |f| \cdot \sigma(C).$$

Repeating the argument with C replaced by $T \setminus C$, we eventually obtain

$$E(u) \leq E(u_0) + \tfrac{1}{2}\max_T |f| \cdot \sigma(T)$$

because the minimum of two real numbers is smaller than the arithmetic mean of the two. $\square$

3.6. **Compactness.** Any sequence in $\widetilde{\mathcal{M}}$ has a subsequence that Gromov converges to a stable holomorphic disc $\mathbf{u}$, which represents the relative homotopy class $[u_0]$, see [12, Thm. 1.1]. By Liouville's theorem, $\mathbf{u}$ has no spherical components.

Proposition 3.7. $\mathcal{M}$ *is compact.*

Proof. We consider a sequence in $\widetilde{\mathcal{M}}$. We can assume that the boundary circles of the holomorphic discs stay in the complement of $F((-\varepsilon/2, \varepsilon/2) \times \partial \mathbb{D})$. Denote by $u^1, \ldots, u^N$ the components of a limiting stable holomorphic disc of a Gromov converging subsequence. We conclude that the circles $u^j(\partial \mathbb{D})$ are contained in the complement of $u_0(\partial \mathbb{D})$, i.e., in $S^1 \times (0, 1)$. We will show that $N = 1$ so that the chosen subsequence descends to a converging sequence in the moduli space $\mathcal{M}$.

As described in Section 2.2 each of the holomorphic discs u^j factors through a simple holomorphic disc v^j via a branched covering map of degree $m_j \geq 1$. Therefore, $[u_0]$ equals

$$[u_0] = \sum_{j=1}^{N} m_j [v^j]$$

in $\pi_2(W, T)$. We can assume that the images of the v^j's are pairwise distinct. If not we combine any pair of classes that represent the v^j's with common image to a single class weighted with the sum of the multiplicities. This procedure shrinks N and enlarges the m_j's.

We will utilize the argument from [17, pp. 549–550] and assume by contradiction that $N \geq 2$. Because $[u_0] \bullet [u_0] = 0$, we get with Section 2.5 that $[u_0] \bullet [v^j] = 0$ for all j. Substituting the above expression for $[u_0]$ once more, we get for all j,

$$0 = \sum_{k=1}^{N} m_k [v^k] \bullet [v^j]$$

or equivalently

$$-m_j [v^j] \bullet [v^j] = \sum_{k \neq j} m_k [v^k] \bullet [v^j].$$

At least two of the v^j's, which originate from the bubble tree of the limiting stable holomorphic disc, must intersect. Section 2.5 yields that the right-hand side is positive. Therefore, $[v^j] \bullet [v^j] \leq -1$ for all j.

Because $\mu([u_0]) = 2$, we get furthermore

$$2 = \sum_{j=1}^{N} m_j \mu([v^j]).$$

Moreover, by Section 2.6 the embedding defect

$$0 \leq D([v^j]) = [v^j] \bullet [v^j] - \mu([v^j]) + 2$$

of the $[v^j]$'s is nonnegative. Combining both yields

$$2 \leq \sum_{j=1}^{N} m_j \big([v^j] \bullet [v^j] + 2\big).$$

We conclude that at least one of the v^j's, v^1 say, has self-intersection number equal to $[v^1] \bullet [v^1] = -1$.

We can approximate v^1 by a smooth admissible map $w : (\mathbb{D}, \partial\mathbb{D}) \to (W, T)$ that represents $[v^1]$ and intersects v^1 transversely in a finite number of points, see [15, Rem. 8.6] and Section 2.4. Hence, $v^1 \bullet w = -1$. Because interior intersections are weighted with factor two, the contribution from interior intersection points to the intersection number $v^1 \bullet w$ is even. Therefore, $v^1(\partial\mathbb{D})$ and $w(\partial\mathbb{D})$ intersect in an *odd* number of points, which are transverse intersections by construction. This implies that the ordinary intersection product

$$[v^1|_{\partial\mathbb{D}}] \cdot [v^1|_{\partial\mathbb{D}}] = [v^1|_{\partial\mathbb{D}}] \cdot [w|_{\partial\mathbb{D}}] \neq 0$$

does not vanish on the first homology of $T \setminus u_0(\partial\mathbb{D})$. In other words $[v^1|_{\partial\mathbb{D}}] \neq 0$, so that the classes of $v^1|_{\partial\mathbb{D}}$ and $u_0|_{\partial\mathbb{D}}$ are nontrivially co-linear in the $\mathbb{Z}$-module $H_1(T \setminus u_0(\partial\mathbb{D}))$, the latter being identified with $\mathbb{Z}[u_0|_{\partial\mathbb{D}}]$. We infer that

$$[u_0|_{\partial\mathbb{D}}] \cdot [u_0|_{\partial\mathbb{D}}] \neq 0.$$

This is a contradiction as $u_0(\partial\mathbb{D})$ and $F(\varepsilon/2, \partial\mathbb{D})$ are disjoint. Hence, we get $N = 1$. $\square$

4. EXTENSIONS OF LOCAL FILLINGS

4.1. **Holomorphic discs with one boundary marked point.** Consider the quotient space

$$\mathcal{M}^1 = \widetilde{\mathcal{M}} \times_G \partial\mathbb{D}$$

by the action

$$g * (u, z) = (u \circ g, g^{-1}(z)).$$

By Section 3.1 and Proposition 3.7 the moduli space $\mathcal{M}^1$ is a closed surface. Charts can be obtained via local fillings as described in Proposition 2.11. For any $u \in \widetilde{\mathcal{M}}$ and transverse constraints c_0, c_1, c_2 that intersect $u(\partial\mathbb{D})$ in $u(1), u(\mathrm{i}), u(-1)$, respectively, there exists a local filling $F : (-\varepsilon, \varepsilon) \times \mathbb{D} \to \mathbb{C}^2$ that extends $u = F(0, \cdot)$. The local filling is uniquely determined up to reparametrizations in time. The map $(-\varepsilon, \varepsilon) \times \partial\mathbb{D} \to \mathcal{M}^1$, $(t, z) \mapsto [F(t, \cdot), z]$ is a local parametrization near (u, z). In order to obtain coordinate changes, consider local fillings F_1 and F_2 that extend u and $u \circ h$, $h \in G$, respectively. By shrinking the time intervals, we can assume that the images coincide. Therefore, we obtain a diffeomorphism

$$F_2^{-1} \circ F_1 = (f, g) : (-\varepsilon_1, \varepsilon_1) \times \mathbb{D} \to (-\varepsilon_2, \varepsilon_2') \times \mathbb{D},$$

where $f : (-\varepsilon_1, \varepsilon_1) \to (-\varepsilon_2, \varepsilon_2')$ is a smooth strictly increasing function and $g = g_t$, $t \in (-\varepsilon_1, \varepsilon_1)$, a smooth 1-parameter family of conformal automorphisms in G such that $f(0) = 0$ and $g_0 = h^{-1}$, see [20, Thm. 18] and [21, Prop. 3.12].

The desired coordinate change $(-\varepsilon_1, \varepsilon_1) \times \partial\mathbb{D} \to (-\varepsilon_2, \varepsilon_2') \times \partial\mathbb{D}$ according to the parametrizations F_1 and F_2 spells out as $(t, z) \mapsto (f(t), g_t(z))$. In particular, the evaluation map

$$\mathrm{ev} : \mathcal{M}^1 \to T, \quad [u, z] \mapsto u(z)$$

equals $(t, z) \mapsto F(t, z)$ in the chart obtained by F. Because ev is injective by Sections 2.5 and 2.6, the evaluation map ev is a diffeomorphism.

4.2. No trapped characteristics on the cut-off torus.

We identify the complement $T \setminus u_0(\partial\mathbb{D})$ with $S^1 \times (0, 1)$ according to the conventions made in Section 3.3.

Proposition 4.3. *Each characteristic leaf of $S^1 \times (0, 1)$ connects $S^1 \times \{0\}$ with $S^1 \times \{1\}$.*

Proof. Choose a characteristic leaf ℓ on $S^1 \times (0, 1)$ that intersects $S^1 \times \{0\}$. For each point $p \in \ell$ there exists a local filling F_p such that $F_p(0, \partial\mathbb{D})$ intersects p, see Section 4.1. Denote by

$$U = \bigcup_{p \in \ell} F_p\big((-\varepsilon_p, \varepsilon_p) \times \partial\mathbb{D}\big)$$

the union of the images of all local fillings F_p, $p \in \ell$, in $S^1 \times (0, 1)$. Denote by s_0 the supremum of

$$\mathrm{proj}_2\big(U \cap (\{0\} \times (0, 1))\big)$$

in the unit interval $(0, 1)$. Then there exists a local filling F_0 such that $F_0(0, \partial\mathbb{D})$ intersects the point $(0, s_0)$ in $S^1 \times (0, 1)$. The image of F_0 in $S^1 \times (0, 1)$ overlaps with U. Because ℓ intersects boundary circles of holomorphic discs transversally, each boundary circle of each holomorphic disc $F_0(t, \cdot)$ intersects ℓ nontrivially. Hence, $s_0 = 1$ and $F_0(0, \partial\mathbb{D})$ coincides with $S^1 \times \{1\}$ so that ℓ connects $S^1 \times \{0\}$ with $S^1 \times \{1\}$. $\qquad\square$

4.4. Gluing local fillings.

We consider a local filling $F : (-\varepsilon, \varepsilon) \times \mathbb{D} \to \mathbb{C}^2$ of T and set $u_0 = F(0, \cdot)$. In view of Proposition 4.3 we choose three disjoint smooth knots K_0, K_1, K_2 in T such that each coincides with a connected characteristic leaf on $S^1 \times [\varepsilon, 1 - \varepsilon]$ and is transverse to the boundary circles of $F(t, \cdot)$, whose images correspond to the slices in $S^1 \times ((0, \varepsilon) \cup (1 - \varepsilon, 1))$. Therefore, each holomorphic disc $u \in \widetilde{\mathcal{M}}$ intersects each knot K_k transversely. We can assume that $u_0(\mathrm{i}^k) \in K_k$ for $k = 0, 1, 2$.

Let $\mathcal{M}_{1,\mathrm{i},-1}$ be the moduli space of all $u \in \widetilde{\mathcal{M}}$ such that $u(1) \in K_0$, $u(\mathrm{i}) \in K_1$, $u(-1) \in K_2$. Requiring the three marked points to lie on the respective knots, which play the role of *global* transverse constraints, is the same as to build the abstract quotient $\mathcal{M}$. A variant of the considerations in Section 4.1 that ignores the marked point shows that

$$\mathcal{M}_{1,\mathrm{i},-1} \to \mathcal{M}, \quad u \mapsto [u]$$

is a diffeomorphism. Therefore, $\mathcal{M}_{1,\mathrm{i},-1}$ is a circle, which of course can be seen directly with Proposition 2.11 and [15, Prop. 7.1].

The evaluation map

$$\mathrm{ev} : \mathcal{M}_{1,\mathrm{i},-1} \times \partial \mathbb{D} \to T, \quad (u,z) \mapsto u(z)$$

is a diffeomorphism, as it factors through the diffeomorphism

$$\mathcal{M}_{1,\mathrm{i},-1} \times \partial \mathbb{D} \to \mathcal{M}^1, \quad (u,z) \mapsto [u,z],$$

and the evaluation map we considered earlier, see Section 4.1. By construction $\mathrm{ev}_1 = \mathrm{ev}(\cdot,1)$ maps $\mathcal{M}_{1,\mathrm{i},-1}$ diffeomorphically onto the knot K_0. Therefore, choosing a regular parametrization $c : S^1 \to K_0$, the map

$$S^1 \times \mathbb{D} \to \mathbb{C}^2, \quad (t,z) \mapsto \big(\mathrm{ev}_1^{-1}(c(t))\big)(z)$$

turns out to be an analytic filling of T, cp. [15, Prop. 5.2].

Proof of Theorem 1.1. Denote by F_1 the global analytic filling just obtained. The given local filling is denoted by F_2. Then $F_2^{-1} \circ F_1$ has the form (f,g) as indicated in Section 4.1. This time f sends (a,b) to $(-\varepsilon,\varepsilon)$, where we can assume that $a \in (-\pi,0)$ and $b \in (0,\pi)$. The path $g = g_t$ in G is parametrized by $t \in (a,b)$. Replace f by a smooth strictly increasing function that is the identity near 0 and coincides with the old near $\pm\varepsilon$. Similarly, replace the path g_t by a path in G that is constantly equal to the identity near 0 and coincides with the old path near $\pm\varepsilon$. This is possible because the group G of Möbius transformations on $\mathbb{D}$ is diffeomorphic to $S^1 \times \mathbb{R}$ and, hence, connected. The analytic filling that equals $F_2 \circ (f,g)$ on $(a,b) \times \mathbb{D}$ and F_1 elsewhere on $S^1 \setminus (a,b) \times \mathbb{D}$ is an extension after restriction of F_2. □

5. GENERALIZATIONS

Theorem 1.1 remains valid if we replace $\mathbb{C}^2$ by any *Stein surface*, which is a complex two-dimensional complex manifold that admits a proper holomorphic embedding into a complex affine space, cp. [13, pp. 283–284]. In fact, the integrability of the almost complex structure is not used in the proof. The theorem therefore can be phrased in the following form:

Let (M,ξ) be a closed oriented contact 3-manifold with a *positive contact form* α, i.e., $\alpha \wedge d\alpha > 0$. Let (W,ω) be a *weak filling* of (M,ξ), i.e., a compact symplectic manifold, which is oriented by ω^2, such that $\partial W = M$ as oriented manifolds and the restriction of ω to ξ is positive. Assume that the symplectic manifold admits an almost complex structure J that turns $T \subset M$ into a totally real torus, leaves ξ and its symplectic orthogonal invariant, and is *tamed* by ω, i.e., ω is positive on J-complex lines. In particular, the boundary M is *J-convex*, i.e., the complex tangencies ξ define a positive contact structure, see [17, p. 538] and [15, Rem. 4.3].

Assume that T admits a local filling F. Then F extends to a global filling of T after restriction, either if $H_2 W$ has no spherical classes A with $\omega(A) > 0$ and $A \cdot A = -1$, or J is perturbed away from $F([-\varepsilon/2,\varepsilon/2] \times \mathbb{D})$ to be generic for precisely those classes and (W,ω) is *minimal*, i.e., does not admit any embedded symplectic sphere of self-intersection equal to -1. Indeed, if J turns all simple stable holomorphic discs that have an *exceptional sphere* component

as described into a regular moduli space problem, then no further assumptions on W are necessary.

The proof is a variation of the one we gave. In order to get uniform energy bounds, replace the primitive λ in Proposition 3.5 by a 3-chain that has boundary $u(\mathbb{D}) + C - u_0(\mathbb{D})$, which exists by assumption. Moreover, in order to prove compactness, combine the intersection argument from Proposition 3.7 with the one from [17, pp. 549–550], and from [16, p. 276] in the generic case described at last.

6. From filling to isotopy

We consider a totally real torus $T \subset \mathbb{C}^2$ which is provided with an analytic filling $F : S^1 \times \mathbb{D} \to \mathbb{C}^2$. We write u_t instead of $F(t, \cdot)$ for all $t \in S^1$ and parametrize the Clifford torus by

$$S^1 \times \partial\mathbb{D} \to \mathbb{C}^2, \quad (t, \mathrm{e}^{\mathrm{i}\theta}) \mapsto (\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\theta}).$$

Corollary 1.3 will follow from the next proposition.

Proposition 6.1. *There exists an isotopy Φ of embeddings $S^1 \times \partial\mathbb{D} \to \mathbb{C}^2$ that connects $\Phi_0 = F$ with the chosen parametrization Φ_1 of the Clifford torus such that the image tori $\Phi_s(S^1 \times \partial\mathbb{D})$ are totally real for all $s \in [0,1]$.*

Proof. The embeddings $F(\cdot, 0)$ and $t \mapsto (\mathrm{e}^{\mathrm{i}t}, 0)$ of S^1 into $\mathbb{C}^2$ are isotopic by the unknotting theorem [1, Cor. 7.2]. We denote the isotopy by

$$\varphi : [0,1] \times S^1 \to \mathbb{C}^2, \quad (s,t) \mapsto \varphi_s(t).$$

The isotopy defines a trivial complex plane bundle $E = \varphi^* T\mathbb{C}^2$, which has a preferred non-vanishing section X defined by $X_{(s,t)} = T_t \varphi_s(\partial_t)$. The complex line bundle generated by X is denoted by E_1. Let E_2 be a complementary complex subbundle of E_1 in E such that the fiber of E_2 over $(0,t)$ equals the tangent plane of the disc $u_t(\mathbb{D})$ at $u_t(0)$ and the fiber of E_2 over $(1,t)$ is equal to $\{\mathrm{e}^{\mathrm{i}t}\} \times \mathbb{C}$. Choose a complex trivialization of E that preserves the splitting $E_1 \oplus E_2$. Together with the normal exponential map of the Euclidean metric we obtain an isotopy Φ of embeddings

$$(S^1 \times (-\varepsilon, \varepsilon)) \times D_\varepsilon^2 \to \mathbb{C}^2$$

for some small $\varepsilon > 0$. The isotopy Φ restricts to φ on $[0,1] \times S^1 \times \{0\} \times \{0\}$ such that $\Phi_1(t, r; z) = (\mathrm{e}^{\mathrm{i}t}, z)$ and that $\Phi_s^* \mathrm{i}$ leaves the splitting $(S^1 \times (-\varepsilon, \varepsilon)) \times D_\varepsilon^2$ invariant.

The desired isotopy will be a combination of isotopies. In a first step, we shrink T via

$$(t, \mathrm{e}^{\mathrm{i}\theta}) \mapsto F(t, r\mathrm{e}^{\mathrm{i}\theta})$$

into the image of Φ_0, which is a neighborhood of $F(S^1, 0)$, by making r small. The neighborhood contains a second family of embedded totally real tori that are parametrized by $\Phi_0(t, 0; r\mathrm{e}^{\mathrm{i}\theta})$. The tangent spaces to the tori for t and θ fixed converge in both families to the real plane spanned by the vectors $X_{(0,t)}$ and $T_{(t,0;0)}\Phi_0(0, \mathrm{i}\mathrm{e}^{\mathrm{i}\theta})$ as r tends to zero. Hence, for r sufficiently small the

angle between the respective tangent planes is less than $\pi/2$ so that the convex combination of $F(t, re^{i\theta})$ and $\Phi_0(t, 0; re^{i\theta})$ induces an isotopy of embeddings. Shrinking r again if necessary, the isotopy will be through totally real tori because the space of all real planes in the Grassmannian of all two-dimensional subspaces of $\mathbb{C}^2 \equiv \mathbb{R}^4$ is open. The third isotopy is given by

$$(t, e^{i\theta}) \mapsto \Phi_s(t, 0; re^{i\theta})$$

sending s to 1 and the last brings $\partial D_1^2 \times \partial D_r^2$ to the Clifford torus by radially expanding the second circle factor. $\qquad\square$

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