Marco Schmidt

L^2-Betti Numbers of $\mathcal{R}$-Spaces and the Integral Foliated Simplicial Volume

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Marco Schmidt
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Dekan:	Prof. Dr. Klaus Hinrichs
Erster Gutachter:	Prof. Dr. Wolfgang Lück
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Introduction

The origin of this thesis is the following conjecture of Gromov [26, Section 8A, p. 232] revealing a connection between the L^2-Betti numbers $b_k^{(2)}(\widetilde{M})$and the simplicial volume $\|M\|$ of a closed oriented connected aspherical manifold M.

Conjecture. *Let M be a closed oriented connected aspherical manifold with $\|M\| = 0$. Then*

$$b_k^{(2)}(\widetilde{M}) = 0 \quad \textit{for all} \quad k \geq 0.$$

The first definition of L^2-Betti numbers for cocompact free proper G-manifolds with G-invariant Riemannian metric (due to Atiyah [2]) is given in terms of the heat kernel. We will briefly recall this original definition at the beginning of Chapter 1. Today, there is an algebraic and more general definition of L^2-Betti numbers which works for arbitrary G-spaces. Analogously to ordinary Betti numbers, they are given as the "rank" of certain homology modules. More precisely, the k-th L^2-Betti number $b_k^{(2)}(Z; \mathcal{N}G)$ of a G-space Z is the von Neumann dimension of the k-th twisted singular homology group of Z with coefficients in the group von Neumann algebra $\mathcal{N}G$. Here, von Neumann dimension means the dimension function developed by Lück [36],[37] for arbitrary modules (in the algebraic sense) over finite von Neumann algebras. The k-th L^2-Betti number $b_k^{(2)}(G)$ of a group G is defined as $b_k^{(2)}(EG; \mathcal{N}G)$, where $EG \to BG$ is the universal principal G-bundle. The notation $b_k^{(2)}(\widetilde{M})$ is short for $b_k^{(2)}\big(\widetilde{M}; \mathcal{N}\pi_1(M)\big)$. We will present the relevant definitions in Chapter 1. The standard reference for L^2-Betti numbers is Lück's extensive textbook [38].

The simplicial volume $\|M\|$ is a real valued homotopy invariant for closed oriented connected topological manifolds M. It measures the "complexity" of the fundamental homology class of M. Namely, $\|M\|$ is defined as the infimum of the ℓ^1-norms of real singular cycles representing the fundamental class. The simplicial volume was defined by Gromov in order to give volume estimates for Riemannian manifolds [24]. In particular, Gromov was interested in lower bounds for the minimal volume $\mathrm{minvol}(M)$, if M is in addition smooth. Definition and properties of the simplicial volume and the relation to the minimal volume are treated in Chapter 2. It is quite interesting that there are many connections to Riemannian geometry although the definition of simplicial volume only takes the topological structure of the manifold into account. For example, a manifold with vanishing simplicial volume cannot carry a Riemannian metric of negative sectional curvature [29].

At first sight the definitions of L^2-Betti numbers and the simplicial volume do not indicate a relationship between them, but in certain situations both invariants behave similarly. We will provide examples for this in Section 5.1. These similarities suggest a connection between L^2-Betti numbers and the simplicial volume. An immediate consequence of Gromov's Conjecture would be the fact that the Euler characteristic $\chi(M)$ vanishes if M is aspherical with $\|M\| = 0$. This fact could have been the motivation for Gromov to study relations between L^2-Betti numbers and simplicial volume. Gromov's Conjecture would also imply that the L^2-Betti numbers of the universal covering $\widetilde{M}$ of an aspherical manifold M vanish if M admits a selfmap $M \to M$ of degree $d \notin \{-1, 0, 1\}$. This was proved by Lück under the additional assumption that each normal subgroup of finite index of the fundamental group $\pi_1(M)$ is Hopfian [38, Theorem 14.40, p. 499]. A group G is called *Hopfian* if each surjective group homomorphism $G \to G$ is an isomorphism.

In this thesis, we will pick another approach to L^2-Betti numbers of G-spaces. We follow the philosophy that it is often fruitful to look at the same invariant from different points of view. A basic example for this is provided by the Betti numbers $b_k(Z)$ of finite CW-complexes Z: Looking only at the singular chain complex, it is not clear that they are finite. Looking only at the cellular chain complex, one does not see directly that they are independent of the CW-structure. But once one has shown that both chain complexes have isomorphic homology groups, it turns out that the $b_k(Z)$ are homotopy invariants with values in the nonnegative integers. We will see that L^2-Betti numbers are another good instance for this general principle. For some properties the original analytic approach of Atiyah is more convenient (e.g., the vanishing of L^2-Betti numbers of the universal covering of hyperbolic manifolds outside the middle dimension [16]), other properties are more easily proved with Lück's algebraic definition, such as the fact that L^2-Betti numbers of universal coverings of aspherical spaces with amenable fundamental group vanish [37, Section 5, p. 155 ff.]. In this thesis we will give another definition of L^2-Betti numbers which is adequate for analyzing Gromov's Conjecture.

We define L^2-Betti numbers of $\mathcal{R}$-spaces. These are spaces which are parametrized over a standard Borel probability space X and provided with an action of a standard equivalence relation $\mathcal{R} \subset X \times X$. The starting point of exploring standard equivalence relations is the work of Feldman and Moore [18]. The definition of $\mathcal{R}$-spaces is due to Connes. The basic example for standard equivalence relations is given by the orbit equivalence relation $\mathcal{R}_{G \curvearrowright X}$ of a standard action $G \curvearrowright X$. Each standard equivalence relation is an orbit equivalence relation of an appropriate measure preserving action $G \curvearrowright X$ [18], but one cannot assume that the action is essentially free. The definition of L^2-Betti numbers of $\mathcal{R}$-spaces makes use of the equivalence relation ring $\mathbb{Z}\mathcal{R}$ and the equivalence relation von Neumann algebra $\mathcal{N}\mathcal{R}$. With these two objects replacing the group ring and the group von Neumann algebra, the definition is analogous to the algebraic definition of L^2-Betti numbers of G-spaces. In fact, we first define a singular chain complex $C^X_\bullet(S; Z)$ for an $\mathcal{R}$-space S and consider then the homology of the complex $\mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C^X_\bullet(S; Z)$. The k-th L^2-Betti number of S is defined as

$$\dim_{\mathcal{NR}}\big(H_k(\mathcal{NR}\otimes_{\mathbb{Z}\mathcal{R}} C^X_\bullet(S;Z))\big).$$

If $G \curvearrowright X$ is a standard action, there is an induction functor

$$\mathrm{ind}\colon G\text{-Spaces} \to \mathcal{R}_{G\curvearrowright X}\text{-Spaces}$$

which sends a G-space Z to the $\mathcal{R}_{G\curvearrowright X}$-space $X \times Z$. In Chapter 4, we show the following result.

Theorem. *Let G be a countable group and $G \curvearrowright X$ a standard action on a standard Borel space X. Then for a countable free G-CW-complex Z one has*

$$b_k^{(2)}(X\times Z;\mathcal{N}\mathcal{R}_{G\curvearrowright X}) = b_k^{(2)}(Z;\mathcal{N}G) \quad \textit{for all} \quad k\geq 0.$$

This is a slight generalization of a result of Gaboriau who proved the same result for countable free simplicial complexes. He also defined L^2-Betti numbers $b_k^{(2)}(\mathcal{R})$ of a standard equivalence relation $\mathcal{R}$ and proved that $b_k^{(2)}(\mathcal{R}_{G\curvearrowright X}) = b_k^{(2)}(G)$ holds for a standard action $G \curvearrowright X$. Gaboriau used this result to show that L^2-Betti numbers of orbit equivalent groups coincide and those of measure equivalent groups coincide up to a non-zero multiplicative constant. Orbit equivalence will be introduced in Section 4.6, where we also give a new proof for the orbit equivalence invariance of L^2-Betti numbers. The definition of measure equivalence, which can be viewed as a measure theoretic analogue of quasi isometry, is due to Gromov and Zimmer. A lot of work on measure equivalence was done by Furman [19],[20]. It should be mentioned that Sauer [41] reproved Gaboriau's theorem using the dimension theory of Lück. More information about about measure equivalence can be found in [22],[41].

The first motivation for the approach to L^2-Betti numbers via the detour to $\mathcal{R}$-spaces was a remark of Gromov [27, p. 306f.], indicating how one could try to prove the conjecture relating L^2-Betti numbers and the simplicial volume of aspherical manifolds. The starting point for that is the upper bound

$$\sum_{j=0}^{n} b_j^{(2)}(\widetilde{M}) \leq 2^{n+1}\cdot \|M\|_{\mathbb{Z}},$$

where $\|M\|_{\mathbb{Z}}$ denotes the integral simplicial volumewhich is given as the minimum of ℓ^1-norms of *integral* fundamental cycles. This inequality is an easy application of the Poincaré duality theorem. Unfortunately, $\|M\|_{\mathbb{Z}}$ is only a very rough estimate for the simplicial volume $\|M\|$ (e.g., $\|M\|_{\mathbb{Z}} \geq 1$ holds for all manifolds M), and the Poincaré duality argument does not provide an upper bound for the sum of L^2-Betti numbers in terms of the coefficients of a *real* fundamental cycle (and therefore of the simplicial volume). Gromov's idea behind the *integral foliated simplicial volume* $\|M\|_{\mathcal{F},\mathbb{Z}}$ is to resolve this drawback by introducing *weighted* singular cycles to represent the fundamental class of M. The weight is given by the coefficients which are functions $f \in L^\infty(X;\mathbb{Z})$. One has the inequality $\|M\|_{\mathcal{F},\mathbb{Z}} \leq \|M\|_{\mathbb{Z}}$. Roughly speaking, the integral foliated simplicial volume is given as the infimum of the sum of ℓ^1-norms $\int_X |f|$ of the coefficient functions f of "weighted" fundamental cycles. This definition is only vaguely indicated by Gromov [27, p. 305f.]. The following theorem is posed there as an exercise (with 2^n instead of 2^{n+1} as constant factor):

Theorem. *Let M be a closed connected oriented manifold of dimension n. Then*

$$\sum_{j=0}^{n} b_j^{(2)}(\widetilde{M}) \leq 2^{n+1} \cdot \|M\|_{\mathcal{F},\mathbb{Z}}$$

holds.

In Chapter 5, we give a concise definition of $\|M\|_{\mathcal{F},\mathbb{Z}}$ and prove this theorem.

Note that there is no asphericity condition in the theorem. In order to prove Gromov's conjecture, one could try to prove that $\|M\| = 0$ implies $\|M\|_{\mathcal{F},\mathbb{Z}} = 0$ for aspherical manifolds M. Unfortunately, we are far away from such a result. The best outcome in this direction is $\|S^1 \times M\|_{\mathcal{F},\mathbb{Z}} = 0$.

Actually, the current definition of $\|M\|_{\mathcal{F},\mathbb{Z}}$ and the proof of the above theorem does not make use of $\mathcal{R}$-spaces. Hence the thesis somehow breaks up into two independent parts, one about $\mathcal{R}$-spaces and their L^2-Betti numbers and one dealing with Gromov's conjecture. There is a version of $\|M\|_{\mathcal{F},\mathbb{Z}}$, where $\mathcal{R}$-spaces and their L^2-Betti numbers occur, but this version needs the fact that the corresponding cohomology

$$H^\bullet\big(\hom_{\mathbb{Z}\mathcal{R}}\big(C_\bullet(S;\mathbb{Z}), \mathcal{N}\mathcal{R}\big)\big)$$

satisfies Eilenberg-Steenrod-type axioms, and we could not prove the excision axiom. Nevertheless, the definition of $\|M\|_{\mathcal{F},\mathbb{Z}}$ is motivated by the study of $\mathcal{R}$-spaces and their homology.

This thesis is organized as follows: The first two chapters consist of a survey about the concepts appearing in Gromov's Conjecture, namely L^2-Betti numbers and simplicial volume.

Chapter 1 deals with L^2-Betti numbers of G-spaces. We describe the algebraic approach to L^2-Betti numbers using Lück's dimension theory and collect the main properties without proof.

In Chapter 2, the simplicial volume of closed oriented connected manifolds is introduced. In addition to the definition and properties, we describe bounded cohomology as a main tool for exploring simplicial volume. A section about minimal volume is included since the study of this invariant was the main motivation for Gromov to analyze simplicial volume.

Chapters 3 and 4 contain the approach to L^2-Betti numbers via $\mathcal{R}$-spaces.

In Chapter 3, we present the definition of $\mathcal{R}$-spaces. We examine standard Borel spaces and assign to a standard Borel space X with probability measure the category of X-spaces. In analogy to ordinary singular homology we define singular homology of X-spaces. If $\mathcal{R} \subset X \times X$ is a standard equivalence relation, we describe the category of $\mathcal{R}$-spaces. These are special X-spaces, their additional feature is an action of $\mathcal{R}$. As a slogan, X-spaces should be thought of as an analog of ordinary topological spaces, and $\mathcal{R}$-spaces should be thought of as an analog of G-spaces.

The purpose of Chapter 4 is to define L^2-Betti numbers of $\mathcal{R}$-spaces and to compare them to ordinary L^2-Betti numbers in the situation of the orbit equivalence relation $\mathcal{R}_{G \curvearrowright X}$ and an induced $\mathcal{R}_{G \curvearrowright X}$-space $X \times Z$. We first define the von Neumann algebra $\mathcal{N}\mathcal{R}$ of a standard equivalence relation, following Feldman and Moore [18]. The definition of L^2-Betti numbers of $\mathcal{R}$-spaces is then analogous to the algebraic definition of ordinary L^2-Betti numbers. We show that the homology functor which showed up in the definition of L^2-Betti numbers of $\mathcal{R}$-spaces satisfies appropriate Eilenberg-Steenrod axioms up to dimension isomorphisms. From this, we deduce that the L^2-Betti numbers of a countable free G-CW-complex Z and those of the induced $\mathcal{R}_{G \curvearrowright X}$-space $X \times Z$ coincide. We define $\mathcal{R}$-CW-complexes and cellular homology. It is proved that the L^2-Betti numbers of an $\mathcal{R}$-CW-complex can be computed with the cellular complex. As an application of the theory of L^2-Betti numbers for $\mathcal{R}$-spaces, we provide another proof for the orbit equivalence invariance of L^2-Betti numbers.

Chapter 5 deals with Gromov's Conjecture. First, we give an overview of the conjecture. The rest of the chapter is devoted to the definition and properties of the *integral foliated simplicial volume* $\|M\|_{\mathcal{F},\mathbb{Z}}$. The main result is the upper bound

$$\sum_{j=0}^{n} b_j^{(2)}(\widetilde{M}) \leq 2^{n+1} \cdot \|M\|_{\mathcal{F},\mathbb{Z}}$$

for an n-dimensional manifold M. We prove that for simply connected M one has $\|M\|_{\mathbb{Z}} = \|M\|_{\mathcal{F},\mathbb{Z}}$. This is no surprise since then $b_0^{(2)}(\widetilde{M}) = b_0(M) = 1$ holds, and consequently $\|M\|_{\mathcal{F},\mathbb{Z}}$ cannot vanish. In contrast to that, we show $\|S^1\|_{\mathcal{F},\mathbb{Z}} = 0$. Furthermore, we prove that $\|M \times N\|_{\mathcal{F},\mathbb{Z}} \leq c \cdot \|M\|_{\mathcal{F},\mathbb{Z}} \cdot \|N\|_{\mathcal{F},\mathbb{Z}}$ holds, where the constant c depends only on $\dim(M) + \dim(N)$. We show that the integral foliated simplicial volume satisfies

$$\|M\| \leq \|M\|_{\mathcal{F},\mathbb{Z}} \leq \|M\|_{\mathbb{Z}}.$$

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Contents

1 L^2-Betti Numbers

The original definition of L^2-Betti numbers was given by Atiyah in 1976 [2]: The k-th L^2-Betti numbers of a cocompact free proper G-manifold M with G-invariant Riemannian metric is given by

$$b_k^{(2)}(M;G) = \lim_{t \to 0} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}}\big(e^{-t\cdot\Delta_k}(x,x)\big)\,\mathrm{vol}(x),$$

where $\mathcal{F} \subset M$ is a fundamental domain for the G-action. Furthermore, Δ_k is the Laplacian acting on L^2-integrable k-forms, and $e^{-t\cdot\Delta_k}(x,y)$ is the integral kernel (the *heat kernel*) of the operator $e^{-t\cdot\Delta_k}$ which is given by spectral calculus. In particular, for each $x \in M$ the map $e^{-t\cdot\Delta_k}(x,x)$ is an endomorphism of the finite dimensional vector space $\mathrm{Alt}^k(T_xM)$ of which we can take the ordinary trace in the sense of linear algebra.

Nowadays there is an algebraic and more general approach to L^2-Betti numbers in terms of homology due to Lück ([36],[37]). Using this approach L^2-Betti numbers can be defined for arbitrary G-spaces Z, but the main interest lies on the special case, where $Z = \widetilde{Y}$ is the universal covering of some space Y with the natural action of the fundamental group $G = \pi_1(Y)$. In that situation, L^2-Betti numbers can be viewed as a refinement of ordinary Betti numbers which takes the action of the fundamental group on the universal covering into account. In a first naive approach to such a refinement one would probably try to find a good notion of $\dim_{\mathbb{C}G}$ and then consider $\dim_{\mathbb{C}G}\big(H_k(\widetilde{Y};\mathbb{C})\big)$. The problem is that in general $\mathbb{C}G$ is not a “nice” ring and consequently such a notion would not have the desired properties. Therefore one passes from $\mathbb{C}G$ to the *group von Neumann algebra* $\mathcal{N}G$ which is a certain completion of $\mathbb{C}G$. For modules over $\mathcal{N}G$ there is a convenient dimension theory which will be used to define L^2-Betti numbers.

The standard reference for L^2-Betti numbers is Lück’s extensive textbook [38].

1.1 Basics about von Neumann Algebras

We will briefly recall what we need from the theory of finite von Neumann algebras. One can find this in any textbook on von Neumann algebras, e.g. in [14] or in [31] and [32].

Definition 1.1. A *von Neumann algebra* $\mathcal{N}$ is a weakly closed $*$-subalgebra of $L(H)$ which contains $\mathrm{id}_H\colon H \to H$, where H is a Hilbert space and $L(H)$ denotes the algebra of continuous linear operators on H.

The *weak topology* on $L(H)$ is generated by all seminorms $\eta_{x,y}(f) = |\langle f(x), y\rangle|$ with $x, y \in H$. The *strong topology* on $L(H)$ is generated by all seminorms $\eta_x(f) = \|f(x)\|$ with $x \in H$. In other words, the strong topology coincides with the topology of pointwise convergence on $L(H)$.

For the definition of L^2-Betti numbers we will need the following example of a von Neumann algebra.

Example 1.2. For a group G consider the Hilbert space $\ell^2(G)$ consisting of formal sums $\sum_{g\in G} \lambda_g \cdot g$ with complex coefficients $\lambda_g \in \mathbb{C}$ which are square summable, i.e. $\sum_{g\in G} |\lambda_g|^2 < \infty$. The group G acts on $\ell^2(G)$ from the left and from the right by translation. This induces embeddings

$$\rho_l\colon \mathbb{C}G \to L\big(\ell^2(G)\big)$$

and

$$\rho_r\colon \mathbb{C}G \to L\big(\ell^2(G)\big)$$

by linear continuation. The *group von Neumann algebra* $\mathcal{N}G$ of G is defined as the weak closure of $\rho_r(\mathbb{C}G)$ in $L\big(\ell^2(G)\big)$.

Definition 1.3. For a subset $A \subset L(H)$ its *commutant* is defined as

$$A' = \{f \in L(H) : af = fa \text{ for all } a \in A\}.$$

The following celebrated *double commutant theorem* of von Neumann gives rise to an alternative definition of von Neumann algebras. A proof can be found for example in [31, Theorem 5.3.1, p. 326].

Theorem 1.4. *For a $*$-closed subalgebra A of $L(H)$ the following statements are equivalent:*

(i) The algebra A is closed in the strong topology.

(ii) The algebra A is closed in the weak topology.

(iii) The equation $A = A''$ holds.

The elements of a von Neumann algebra $\mathcal{N} \subset L(H)$ can be partially ordered: For $a, b \in \mathcal{N}$ we define

$$a \leq b \iff b - a \text{ is positive.}$$

An element $m \in \mathcal{N}$ is called *positive* if it can be written as $m = ff^*$ for some $f \in \mathcal{N}$. This is equivalent to $\langle m(x), x\rangle \geq 0$ for all $x \in H$. An element $p \in \mathcal{N}$ is called a *projection* if $p^2 = p$ and $p = p^*$ hold. It immediately follows that projections are positive.

An important input for the definition of L^2-Betti numbers is the notion of a trace on a von Neumann algebra.

Definition 1.5. Let $\mathcal{N}$ be a von Neumann algebra. A *finite faithful normal trace* on $\mathcal{N}$ is a linear map $\mathrm{tr}\colon \mathcal{N} \to \mathbb{C}$ for which the following properties hold:

(i) The map tr satisfies the trace property, i.e. $\operatorname{tr}(ab) = \operatorname{tr}(ba)$ for all $a, b \in \mathcal{N}$.

(ii) For all positive $a \in \mathcal{N}$ one has $\operatorname{tr}(a) = 0$ if and only if $a = 0$.

(iii) The map tr satisfies
$$\operatorname{tr}(a) = \sup\{\operatorname{tr}(a_i) : i \in I\},$$
where $(a_i)_i$ is a monotone increasing net of positive elements $a_i \in \mathcal{N}$ and a is the supremum of the a_i.

A von Neumann algebra which possesses such a finite faithful normal trace is called *finite*.

Example 1.6. Let G be a group and $\mathcal{N}G$ its group von Neumann algebra. The map
$$\begin{aligned}\operatorname{tr}_{\mathcal{N}G} \colon \mathcal{N}G &\to \mathbb{C}\\ f &\mapsto \langle f(e), e\rangle_{\ell^2}\end{aligned}$$
defines a finite faithful normal trace on $\mathcal{N}G$, where $e \in G \subset \ell^2(G)$ is the unit element.

This is not hard to prove and follows for example from [38, Theorem 1.9, p. 18].

From now on we will simply say *trace* if we mean a finite faithful normal trace.

Definition 1.7. A $*$-homomorphism $f \colon \mathcal{N} \to \mathcal{M}$ between finite von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$ is called *trace preserving* if
$$\operatorname{tr}_{\mathcal{M}}\big(f(x)\big) = \operatorname{tr}_{\mathcal{N}}(x)$$
holds for every $x \in \mathcal{N}$.

If $f \colon \mathcal{N} \to \mathcal{M}$ is a $*$-homomorphism, then $\mathcal{M}$ gets an $\mathcal{N}$-module structure. The next lemma is proved in [41, Theorem 1.48].

Lemma 1.8. *Let $f \colon \mathcal{N} \to \mathcal{M}$ be a trace preserving $*$-homomorphism of finite von Neumann algebras. Then $\mathcal{M}$ is a flat $\mathcal{N}$-module.*

To a finite von Neumann algebra $\mathcal{N}$ with a given trace $\operatorname{tr}_{\mathcal{N}}$ we can assign the Hilbert space $\ell^2\mathcal{N}$. It is given as the Hilbert space completion of $\mathcal{N}$ with respect to the inner product $\langle x, y\rangle = \operatorname{tr}_{\mathcal{N}}(x^*y)$. Left multiplication with a given element $x \in \mathcal{N}$ is a linear map $\mathcal{N} \to \mathcal{N}$ which induces a continuous linear map $\ell^2\mathcal{N} \to \ell^2\mathcal{N}$. This construction yields a left $\mathcal{N}$-module structure on $\ell^2\mathcal{N}$. Analogously we get a right $\mathcal{N}$-module structure on $\ell^2\mathcal{N}$. There is the following result which is important for the exploration of finite von Neumann algebras.

Theorem 1.9. *Let $\mathcal{N}$ be a finite von Neumann algebra. The map*
$$\mathcal{N} \to L(\ell^2\mathcal{N})^{\mathcal{N}}$$
given by sending $x \in \mathcal{N}$ to right multiplication with x is an isometric $*$-antihomomorphism of $\mathbb{C}$-algebras.

This is proved for example in [14, Theorem 1 in I.5.2 and Theorem 2 in I.6.2]. The von Neumann algebra $\mathcal{N}$ can be viewed as a subalgebra of $L(\ell^2\mathcal{N})$, and hence $\ell^2\mathcal{N}$ serves as a Hilbert space on which $\mathcal{N}$ acts.

Example 1.10. Another type of examples of von Neumann algebras is given by $L^\infty(X,\mu)$, where X is a compact space and μ is a finite measure on its Borel σ-algebra. The associated Hilbert space is $L^2(X,\mu)$, where the action is given by the obvious embedding

$$L^\infty(X,\mu) \to L\big(L^2(X,\mu)\big)$$

which comes from pointwise multiplication.

It turns out that every separable abelian von Neumann algebra is of this form. More precisely, for each separable abelian von Neumann algebra $\mathcal{N}$ there is a compact space X with a finite measure μ on its Borel σ-algebra such that

$$\mathcal{N} \cong L^\infty(X,\mu),$$

for a proof see [14, Theorem 1 and 2 in I.7.3].

There is a trace $\mathrm{tr}_{L^\infty(X,\mu)}$ given by $\mathrm{tr}_{L^\infty(X,\mu)}(f) = \int_X f\, d\mu$.

1.2 Dimension Theory for Modules over Finite von Neumann Algebras

Lück developed a dimension theory for arbitrary modules over finite von Neumann algebras ([36],[37]). We will briefly recall the construction and the important properties. For a comprehensive presentation we refer to [38, Chapter 6]. There, everything is formulated for group von Neumann algebras, but the construction is exactly the same for arbitrary finite von Neumann algebras.

Note that *module over a von Neumann algebra* means module in the algebraic sense, i.e. the von Neumann algebra is just viewed as a ring. If not stated otherwise, all modules will be assumed to be *left* modules.

Definition 1.11. Let $\mathcal{N}$ be a finite von Neumann algebra with trace $\mathrm{tr}_\mathcal{N}$ and let P be a finitely generated (f. g.) projective $\mathcal{N}$-module. There exists a natural number n and a matrix $T \in M(n,n;\mathcal{N})$ such that $T^2 = T$ and $P \cong \mathrm{im}(r_T)$, where r_T is multiplication with T from the right. Now define

$$\dim_\mathcal{N}(P) = \mathrm{tr}_\mathcal{N}(T) = \sum_{j=1}^{n} \mathrm{tr}_\mathcal{N}(t_{jj}) \in [0,\infty). \tag{1.1}$$

This is independent of the choice of T. From the fact that T can be chosen such that $T = T^*$ it follows easily that $\dim_\mathcal{N}(P)$ is fact a nonnegative real number.

The generalization of Definition 1.11 to arbitrary $\mathcal{N}$-modules is the following one:

Definition 1.12. Let A be an $\mathcal{N}$-module. We define

$$\dim_\mathcal{N}(A) = \sup\big\{\dim_\mathcal{N}(P) : P \subset A \text{ is a f. g. projective submodule}\big\}. \tag{1.2}$$

From the definition it is not at all clear that this notion of dimension has any properties you would demand of a dimension function. Lück has shown that this notion has some nice "dimension properties". A few of them are listed in the following proposition.

Proposition 1.13. *The dimension function* $\dim_{\mathcal{N}}$ *satisfies the following properties.*

(i) Additivity.
Let A_0, A_1, A_2 *be* $\mathcal{N}$*-modules and let* $0 \to A_0 \to A_1 \to A_2 \to 0$ *be an exact sequence. Then*
$$\dim_{\mathcal{N}}(A_1) = \dim_{\mathcal{N}}(A_0) + \dim_{\mathcal{N}}(A_2).$$

(ii) Cofinality.
Let I *be a set and let* $A = \bigcup_{i \in I} A_i$ *be a directed union of* $\mathcal{N}$*-modules. Then one has*
$$\dim_{\mathcal{N}}(A) = \sup\{\dim_{\mathcal{N}}(A_i) : i \in I\}.$$

(iii) Let P *be a projective* $\mathcal{N}$*-module. Then*
$$\dim_{\mathcal{N}}(P) = 0 \iff P = 0.$$

For the next property of the dimension function we need some notation. Recall that the *dual module* M^* of an $\mathcal{N}$-module M is defined as $M^* = \hom_{\mathcal{N}}(M, \mathcal{N})$. One gets a right module structure on M^* given by $(f \cdot x)(m) = f(m) \cdot x$ for $f \in M^*$, $x \in \mathcal{N}$ and $m \in M$. The involution on $\mathcal{N}$ allows us to define a left module structure on M^* by $(x \cdot f)(m) = f(m) \cdot x^*$. A homomorphism $\varphi\colon M \to N$ of $\mathcal{N}$-modules induces a homomorphism $\varphi^*\colon N^* \to M^*$ given by $\varphi^*(f) = f \cdot \varphi$. The next definition can also be given for modules over arbitrary rings.

Definition 1.14. Let M be an $\mathcal{N}$-module and $K \subset M$ be a submodule. Then the *closure* $\overline{K}$ of K is defined as
$$\overline{K} = \{x \in M : \varphi(x) = 0 \text{ for all } \varphi \in M^* \text{ with } K \subset \ker(\varphi)\}.$$

The following result is taken from [38, Theorem 6.7 (2) and (4)(d)].

Lemma 1.15. *Let* M *be a finitely generated* $\mathcal{N}$*-module and let* $K \subset M$ *be a submodule.*

(i) The submodule $\overline{K} \subset M$ *is a direct summand, and* $M/\overline{K}$ *is a finitely generated projective* $\mathcal{N}$*-module.*

(ii) We get $\dim_{\mathcal{N}}(K) = \dim_{\mathcal{N}}(\overline{K})$.

Remark 1.16. The additivity of $\dim_{\mathcal{N}}$ with respect to short exact sequences implies that the subcategory $\mathcal{N}\text{-Mod}_0$ of all $\mathcal{N}$-modules of dimension 0 is a *Serre subcategory* of the abelian category $\mathcal{N}$-Mod of all $\mathcal{N}$-modules. Recall that a Serre subcategory of an abelian category is an abelian subcategory which is closed under subobjects, quotients and extensions. If $\mathcal{A}$ is a well powered abelian category and $\mathcal{B}$ is a Serre subcategory then there exists the *quotient category* $\mathcal{A}/\mathcal{B}$. It has the same objects as $\mathcal{A}$ and comes with an exact functor $\pi\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ with the property that for a morphism f in $\mathcal{A}$ the induced morphism $\pi(f)$ is an isomorphism if and only if $\ker(f)$ and $\operatorname{coker}(f)$ both belong to the objects of $\mathcal{B}$. Note that a category is called *well powered* if the family of subobjects of any object is small. Let us briefly recall what a *subobject* of an object in an arbitrary category is: If $\mathcal{A}$ is a category and a is an object of $\mathcal{A}$, then we consider monomorphisms $u\colon s \to a$. Two monomorphisms $u\colon s \to a$ and $v\colon t \to a$ satisfy $s \leq t$ by definition if there is an arrow $u'\colon s \to t$ with $u = v \circ u'$. We write $u \equiv v$ if $u \leq v$ and $v \leq u$ hold. Then the subobjects of a are defined to be the equivalence classes of this equivalence relation. It is clear that for a ring R, the subobjects of an object M of R-Mod are precisely the submodules of M in the usual sense. It follows immediately that $\mathcal{N}$-Mod is well powered. More details and a proof for the preceding claims can be found in [44, Section II, p. 40ff.].

Property (iii) of Proposition 1.13 is not true if one omits the restriction to projective modules: There are nontrivial $\mathcal{N}$-modules of dimension zero. We will see such examples later in this chapter.

The existence of nontrivial modules with trivial dimension leads to the following generalization of isomorphisms between $\mathcal{N}$-modules.

Definition 1.17. Let $f\colon A \to B$ be an $\mathcal{N}$-homomorphism of $\mathcal{N}$-modules. The map f is called a $\dim_{\mathcal{N}}$*-isomorphism* if $\dim_{\mathcal{N}}(\ker f) = \dim_{\mathcal{N}}(\operatorname{coker} f) = 0$ holds.

We will simply speak of a *dimension isomorphism* if it is clear from the context which von Neumann algebra is involved.

The fact that $\mathcal{N}$-Mod_0 is a Serre subcategory of $\mathcal{N}$-Mod has the consequence that dimension isomorphisms behave very much like isomorphisms. For example, there is a five-lemma for dimension isomorphisms.

The following lemma provides a useful local criterion to check whether an $\mathcal{N}$-module has dimension zero. It is taken from Sauer's thesis [41, Lemma 3.16].

Lemma 1.18. *Let A be an $\mathcal{N}$-module. Then* $\dim_{\mathcal{N}}(A) = 0$ *holds if and only if for every element $a \in A$ there is a sequence of projections $(p_n)_{n\in\mathbb{N}}$ in $\mathcal{N}$ such that* $\lim_{n\to\infty} \operatorname{tr}_{\mathcal{N}}(p_n) = 1$ *and $p_n \cdot a = 0$ for all $n \in \mathbb{N}$.*

There is also a connection between the vanishing of the dimension of a module and the dual module.

Lemma 1.19. *Let M be an $\mathcal{N}$-module. Then*

$$\dim_{\mathcal{N}}(M) = 0 \implies M^* = 0.$$

Proof. Suppose $\dim_{\mathcal{N}}(M) = 0$ and let $\varphi\colon M \to \mathcal{N}$ be an $\mathcal{N}$-homomorphism. From additivity of the dimension function we conclude $\dim_{\mathcal{N}}\big(\operatorname{im}(\varphi)\big) = 0$. Consequently, $\dim_{\mathcal{N}}\big(\overline{\operatorname{im}(\varphi)}\big) = 0$ by (ii) of Lemma 1.15. Part (i) of the same lemma implies that

$\overline{\mathrm{im}(\varphi)}$ is a direct summand of $\mathcal{N}$ and therefore of the form $\mathcal{N} \cdot p$ for a projection p. We get $\mathrm{tr}_{\mathcal{N}}(p) = 0$ from the definition of the dimension function and hence $p = 0$ from faithfulness of $\mathrm{tr}_{\mathcal{N}}$. Now $\mathrm{im}(\varphi) = 0$ follows. □

Remark 1.20. The converse of Lemma 1.19 is not true. A counterexample is constructed in [38, Exercise 6.5]. For finitely generated $\mathcal{N}$-modules M the vanishing of $\dim_{\mathcal{N}}(M)$ is in fact equivalent to the triviality of M^*.

The dimension function is compatible with induction by trace preserving $*$-homomorphisms:

Lemma 1.21. *Let $\mathcal{N} \to \mathcal{M}$ be a trace preserving $*$-homomorphism of finite von Neumann algebras. Then we have*

$$\dim_{\mathcal{N}}(N) = \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} N)$$

for any $\mathcal{N}$-module N.

A proof can be found in [41, Theorem 3.18]. The following notion, which is due to Sauer [41, Definition 3.24], generalizes the situation of Lemma 1.21.

Definition 1.22. Let $\mathcal{M}$ and $\mathcal{N}$ be two finite von Neumann algebras. An $\mathcal{M}$-$\mathcal{N}$-bimodule M is called *dimension compatible* if for each $\mathcal{N}$-module N the implication

$$\dim_{\mathcal{N}}(N) = 0 \implies \dim_{\mathcal{M}}(M \otimes_{\mathcal{N}} N) = 0$$

holds.

From Lemma 1.21 we can directly draw the following consequence.

Corollary 1.23. *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras. Then $\mathcal{M}$ is a dimension compatible $\mathcal{M}$-$\mathcal{N}$-bimodule.*

For later use in Chapter 4 we will state another result of Sauer [41, Lemma 3.28].

Lemma 1.24. *Let $\mathcal{N} \subset R \subset \mathcal{M}$ be an inclusion of rings, where $\mathcal{N}$ and $\mathcal{M}$ are finite von Neumann algebras. Let B be an $\mathcal{M}$-R-bimodule satisfying the following properties:*

(i) The ring R is dimension compatible as an $\mathcal{N}$-$\mathcal{N}$-bimodule.

(ii) The module B is dimension compatible as an $\mathcal{M}$-$\mathcal{N}$-bimodule.

(iii) The module B is flat as a right $\mathcal{N}$-module.

Then every R-homomorphism $M \to N$ of R-modules induces a $\dim_{\mathcal{M}}$-isomorphism

$$\mathrm{Tor}_\bullet^R(B, M) \to \mathrm{Tor}_\bullet^R(B, N).$$

1.3 Definition and Properties of L^2-Betti Numbers

In this section we will define L^2-Betti numbers of arbitrary spaces with group action.

If Z is a space with an action of a group G, then the group action $G \curvearrowright Z$ induces a $\mathbb{Z}G$-module structure on the singular chain complex $C_\bullet(Z;\mathbb{Z})$.

Definition 1.25. Let G be a group and let Z be a topological space with an action of G. Consider the chain complex of $\mathcal{N}G$-modules $\mathcal{N}G \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})$, where $\mathcal{N}G$ is equipped with the canonical $\mathcal{N}G$-$\mathbb{Z}G$-bimodule structure. The homology of $\mathcal{N}G \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})$ is called the *L^2-homology* of Z and is denoted by $H^G_\bullet(Z;\mathcal{N}G)$. The *k-th L^2-Betti number* of Z is defined as

$$b^{(2)}_k(Z;\mathcal{N}G) = \dim_{\mathcal{N}G}\big(H^G_k(Z;\mathcal{N}G)\big).$$

The k-th L^2-Betti number $b^{(2)}_k(G)$ of G is defined as

$$b^{(2)}_k(G) = b^{(2)}_k(EG;\mathcal{N}G),$$

where EG is the total space of the universal principal G-bundle $EG \to BG$.

Notation. We will mainly be interested in the L^2-Betti numbers $b^{(2)}_k\big(\widetilde{Z};\mathcal{N}\pi_1(Z)\big)$ of the universal covering $\widetilde{Z}$ of a topological space Z endowed with the canonical action of the fundamental group $\pi_1(Z)$. We will simply write $b^{(2)}_k(\widetilde{Z})$ in that case.

Remark 1.26. Later on, we will need the corresponding cohomology to $H^G_\bullet(Z;\mathcal{N}G)$, i.e. the cohomology of the cochain complex

$$\hom_{\mathbb{Z}G}\big(C_\bullet(Z;\mathbb{Z}),\mathcal{N}G\big).$$

It is called the *L^2-cohomology* of Z and is denoted by $H^\bullet_G(Z;\mathcal{N}G)$.

Remark 1.27. In the case of a cocompact free proper G-manifold M with G-invariant Riemannian metric the L^2-Betti numbers $b^{(2)}_k(M;\mathcal{N}G)$ as defined in Definition 1.25 coincide with the analytical L^2-Betti numbers defined at the beginning of this chapter. This follows from work of Dodziuk [15], a proof can be found in [38, 1.4.2, p. 58ff].

The following lemma due to Lück shows that one can compute L^2-Betti numbers of a G-CW-complex using its cellular chain complex.

Recall that a *G-CW-complex* Z is a G-space with a G-invariant filtration

$$\emptyset = Z_{[-1]} \subset Z_{[0]} \subset Z_{[1]} \subset \ldots \subset \bigcup_{n\in\mathbb{N}} Z_{[n]} = Z$$

such that the topology on Z is the colimit topology with respect to this filtration and each $Z_{[n]}$ is obtained from $Z_{[n-1]}$ by a G-pushout of the form

$$\begin{array}{ccc} \coprod_{i\in I_n} G/H_i \times S^{n-1} & \longrightarrow & Z_{[n-1]} \\ \big\downarrow & & \big\downarrow \\ \coprod_{i\in I_n} G/H_i \times D^n & \longrightarrow & Z_{[n]} \end{array}$$

Here, I_n is an index set and $H_i \subset G$ is a normal subgroup.

As in the non-equivariant case, the cellular complex $C_\bullet^{\text{cell}}(Z;\mathbb{Z})$ of Z has the relative singular homology group $H_n(Z_{[n]}, Z_{[n-1]};\mathbb{Z})$ as n-th chain group.

Lemma 1.28. *Let G be a group and let Z be a G-CW-complex . Then*

$$b_k^{(2)}(Z;\mathcal{N}G) = \dim_{\mathcal{N}G}\Big(H_k\big(\mathcal{N}G \otimes_{\mathbb{Z}G} C_\bullet^{\text{cell}}(Z;\mathbb{Z})\big)\Big).$$

This was proved by Lück [37, Lemma 4.2].

1.3.1 Basic Properties of L^2-Betti Numbers

We list some fundamental properties of L^2-Betti numbers whose proofs can be found in [38, Theorem 6.54].

Theorem 1.29. *The L^2-Betti numbers of G-spaces satisfy the following properties.*

(i) Homotopy invariance.
Let Z, Y be G-spaces and let $Z \to Y$ be a G-homotopy equivalence. Then

$$b_k^{(2)}(Z;\mathcal{N}G) = b_k^{(2)}(Y;\mathcal{N}G) \quad \text{for all } k \geq 0.$$

(ii) Künneth formula.
Let Z be a G-space and let Y be an H-space. Then $Z \times Y$ is a $G \times H$-space, and

$$b_k^{(2)}\big(Z \times Y;\mathcal{N}(G \times H)\big) = \sum_{i+j=k} b_i^{(2)}(Z;\mathcal{N}G) \cdot b_j^{(2)}(Y;\mathcal{N}H) \quad \text{for all } k \geq 0,$$

where the conventions $0 \cdot \infty = 0$, $r \cdot \infty = \infty$ for $r \in (0,\infty]$ and $r + \infty = \infty$ for $r \in [0,\infty]$ are used.

(iii) Restriction.
Let Z be a G-space and let $H \subset G$ be a subgroup of finite index $[G : H]$. Denote by $\text{res}_G^H(Z)$ the H-space obtained by restricting the G-action. Then

$$b_k^{(2)}\big(\text{res}_G^H(Z), \mathcal{N}H\big) = [G : H] \cdot b_k^{(2)}(Z;\mathcal{N}G) \quad \text{for all } k \geq 0,$$

where the convention $[G : H] \cdot \infty = \infty$ is used.

(iv) Induction.
Let $i\colon H \to G$ be an inclusion of groups and let Z be an H-space. Then $G \times_H Z$ is a G-space and

$$b_k^{(2)}(G \times_H Z;\mathcal{N}G) = b_k^{(2)}(Z;\mathcal{N}H) \quad \text{for all } k \geq 0.$$

(v) *Zeroth L^2-Betti number.*
Let Z be a path connected G-space. Then

$$b_0^{(2)}(Z;\mathcal{N}G) = \frac{1}{|G|},$$

where $\frac{1}{|G|}$ is assumed to be zero if the order $|G|$ of G is infinite.

As an immediate consequence of the restriction property in Theorem 1.29 we obtain the following result.

Corollary 1.30. *Let $Y \to Z$ be a d-sheeted covering with $d < \infty$. Then*

$$b_k^{(2)}(\widetilde{Y}) = d \cdot b_k^{(2)}(\widetilde{Z}) \quad \text{for all } k \geq 0.$$

In particular, the L^2-Betti numbers $b_k^{(2)}(\widetilde{Z})$ vanish if there exists a non trivial finite covering $Z \to Z$. Recall that ordinary Betti numbers are not multiplicative with respect to finite coverings. Also in general, L^2-Betti numbers of $\widetilde{Z}$ and ordinary Betti numbers of Z do not share a lot of common properties. One exception is given by the *Euler-Poincaré formula* which expresses the Euler characteristic of a finite CW-complex as the alternating sum of of its Betti numbers. The same is true for the L^2-Betti numbers of its universal covering [38, Theorem 1.35 (2)].

Lemma 1.31. *Let Z be a finite CW-complex . Then the equation*

$$\chi(Z) = \sum_{k=1}^{\dim(Z)} (-1)^k \cdot b_k(Z) = \sum_{k=1}^{\dim(Z)} (-1)^k \cdot b_k^{(2)}(\widetilde{Z})$$

holds.

There is another connection between L^2-Betti numbers and ordinary Betti numbers. Namely, in special situations one can approximate the L^2-Betti numbers of the universal covering of a space by the normalized Betti numbers of some finite covering. Recall the following definition. A group G is called *residually finite* if for any element $g \in G$ there is an epimorphism $p\colon G \to G'$ onto some finite group G' with $p(g) \neq e$, where $e \in G'$ denotes the unit element. For a countable group G this condition is equivalent to the fact that there is a sequence of subgroups

$$G = G_0 \supset G_1 \supset G_2 \supset \dots$$

of finite index $[G : G_n] < \infty$ for all $n \geq 1$ with $\bigcap_{n\geq 1} G_n = \{e\}$.

The following result was conjectured by Gromov [26] and proved by Lück [35].

Theorem 1.32. *Let Z be a finite connected CW-complex with residually finite fundamental group $G = \pi_1(Z)$ and let $G = G_0 \supset G_1 \supset G_2 \supset \dots$ be a sequence of subgroups of G of finite index with $\bigcap_{n\geq 1} G_n = \{e\}$. For $n \geq 1$ denote by Z_n the associated covering to $G_n \subset G$. Then we have*

$$\lim_{n\to\infty} \frac{b_k(Z_n)}{[G : G_n]} = b_k^{(2)}(\widetilde{Z}) \quad \text{for all} \quad k \geq 0.$$

1.3.2 L^2-Betti Numbers and Aspherical Spaces

A topological space Z is called *aspherical* if it is connected and possesses a universal covering $\widetilde{Z}$ which is contractible. For CW-complexes Z this is equivalent to

$$\pi_k(Z) = 0 \quad \text{for} \quad k \neq 1.$$

If Z is an aspherical CW-complex with fundamental group G, then Z is a model for the classifying space BG. Examples of aspherical spaces are closed Riemannian manifolds with non positive sectional curvature.

The following result is due to Cheeger and Gromov [10]. For a proof using the algebraic approach to L^2-Betti numbers see [38, Section 6.4].

Theorem 1.33. *Let Z be an aspherical CW-complex such that $\pi_1(Z)$ contains an infinite amenable normal subgroup. Then $b_k^{(2)}(\widetilde{Z}) = 0$ holds for all $k \geq 0$.*

A group G is called *amenable* if there is a G-invariant linear map $m\colon l^\infty(G, \mathbb{R}) \to \mathbb{R}$ such that

$$\inf\{f(g) : g \in G\} \leq m(f) \leq \sup\{f(g) : g \in G\}$$

holds for all $f \in l^\infty(G, \mathbb{R})$. There are many equivalent characterizations of amenability. Examples of amenable groups are finite groups and abelian groups. The class of amenable groups is closed with respect to subgroups, quotients, group extensions, and directed unions. The easiest example of a non amenable group is the free product $\mathbb{Z} * \mathbb{Z}$. A standard reference for amenability is [40].

The next result is taken from [38, Corollary 1.43].

Theorem 1.34. *Let M be a closed aspherical manifold with a non trivial S^1-action. Then the action has no fixed points, and the inclusion of any orbit into M induces a monomorphism on fundamental groups. Moreover, we get $b_k^{(2)}(\widetilde{M}) = 0$ for all $k \geq 0$.*

Note that the fact that the action is fixed point free has not to be assumed, it is a consequence of the assumptions.

1.3.3 L^2-Betti Numbers and Manifolds

As in the situation of classical Betti numbers, there is a Poincaré duality theorem for L^2-Betti numbers.

Theorem 1.35. *Let M be a closed manifold of dimension n. Then we have*

$$b_k^{(2)}(\widetilde{M}) = b_{n-k}^{(2)}(\widetilde{M}) \quad \textit{for all } k \in \mathbb{Z}.$$

For low dimensional manifolds the L^2-Betti numbers can be computed. For 1-dimensional closed manifolds the situation is clear. It follows directly from Corollary 1.30 that

$$b_k^{(2)}(\widetilde{S^1}) = 0 \quad \text{for all} \quad k \geq 0.$$

In dimension 2 the situation is as follows: Since S^2 is simply connected its L^2-Betti numbers coincide with its ordinary Betti numbers. In the case of the torus T^2 we can use Corollary 1.30 again to conclude

$$b_k^{(2)}(\widetilde{T^2}) = 0 \quad \text{for all} \quad k \geq 0.$$

For closed oriented surfaces F_g of genus $g \geq 1$ we get $b_k^{(2)}(\widetilde{F_g}) = 0$ for $k \neq 1$ and $b_1^{(2)}(\widetilde{F_g}) = 2 \cdot (g-1)$. This follows easily from Theorem 1.29 (v), Lemma 1.31 and Theorem 1.35.

For 3-dimensional closed manifolds the L^2-Betti numbers of their universal coverings can also be computed, we refer to [38, Chapter 4].

We mention the following proportionality principle which is proved in [38, Theorem 3.183]. It is based on work of Cheeger and Gromov [9, Proposition 6.4].

Theorem 1.36. *Let M and N be two Riemannian manifolds with isometric universal coverings. Then we get*

$$\frac{b_k^{(2)}(\widetilde{M})}{\mathrm{vol}(M)} = \frac{b_k^{(2)}(\widetilde{N})}{\mathrm{vol}(N)} \quad \textit{for all} \quad k \geq 0.$$

Note that the assumption of Theorem 1.36 is satisfied if M and N are both closed hyperbolic manifolds of the same dimension. A Riemannian manifold M is called *hyperbolic* if its sectional curvature satisfies $\sec(M) = -1$. This is equivalent to the fact that its universal covering $\widetilde{M}$ is isometric to the hyperbolic space $\mathbb{H}^n$, where n is the dimension of M. Hyperbolic manifolds are aspherical. The following result holds for L^2-Betti numbers of hyperbolic manifolds.

Theorem 1.37. *Let M be a closed hyperbolic manifold of dimension n. Then one has*

$$b_k^{(2)}(\widetilde{M}) = 0 \quad \textit{for} \quad 2 \cdot k \neq n$$

and

$$b_k^{(2)}(\widetilde{M}) > 0 \quad \textit{if} \quad 2 \cdot k = n.$$

This follows from work of Dodziuk [16], the idea of the proof is presented in [38, Theorem 1.62]. The first statement of Theorem 1.37 is a special case of the following *Singer Conjecture*:

Conjecture 1.38. *Let M be a closed aspherical manifold of dimension n. Then we get*

$$b_k^{(2)}(\widetilde{M}) = 0 \quad \textit{for} \quad 2 \cdot k \neq n.$$

2 Simplicial Volume

In this chapter we will discuss the simplicial volume of closed oriented connected manifolds. Although the definition is purely topological, the simplicial volume has many properties in terms of Riemannian geometry. This was in fact Gromov's main motivation for studying simplicial volume in [24]. We will review some of these relations in this chapter. In particular, we recall the definition of minimal volume, which was also introduced by Gromov [24], and discuss its connection to simplicial volume.

2.1 Definition and Properties of Simplicial Volume

First we define a norm on the real singular chain complex of a topological space. For a topological space X denote the *real singular chain complex* of X by $C_\bullet(X;\mathbb{R})$. In degree k it is the real vector space generated by all continuous maps $\sigma\colon \Delta^k \to X$ (which are called *singular simplices*), where $\Delta^k \subset \mathbb{R}^{k+1}$ is the k-dimensional standard simplex. By $\Sigma_k(X)$ we denote the set of all singular simplices in X of dimension k.

Definition 2.1. Let X be a topological space. Define the *ℓ^1-norm* on $C_k(X;\mathbb{R})$ by

$$\left\|\sum_{j=1}^{l} \lambda_j \cdot \sigma_j\right\|_1 = \sum_{j=1}^{l} |\lambda_j|.$$

Note that any norm on $C_k(X;\mathbb{R})$ induces a seminorm on $H_k(X;\mathbb{R})$. In particular, we obtain the following seminorm.

Definition 2.2. Let X be topological space. Define the *ℓ^1-seminorm* on $H_k(X;\mathbb{R})$ by

$$\|\alpha\|_1 = \inf\big\{\, \|z\| : z \text{ is a cycle in } C_k(X;\mathbb{R}) \text{ with } [z] = \alpha \big\}.$$

Remark 2.3. Note that in general this is not a norm since it is possible that $\|\alpha\|_1 = 0$ but $\alpha \neq 0$ in $H_k(X;\mathbb{R})$. We will see in a moment that this actually happens.

2.1.1 Definition of Simplicial Volume

Now we consider the case of a closed orientable connected manifold M of dimension n. Let $[M]$ be a fundamental class of M, i.e. a generator of $H_n(M;\mathbb{Z})$ and let $j\colon C_n(M;\mathbb{Z}) \to C_n(M;\mathbb{R})$ be the change of coefficients homomorphism. We will sometimes omit the j from the notation and regard elements of $C_n(M;\mathbb{Z})$ as elements of $C_n(M;\mathbb{R})$ without mentioning.

Definition 2.4. Let M be a closed oriented connected manifold. Define its *simplicial volume* as
$$\|M\| = \|j_*([M])\|_1.$$

2.1.2 First Properties of Simplicial Volume

Next we will collect some elementary properties of simplicial volume.

Lemma 2.5. *Let M be a closed oriented connected manifold which has a triangulation with k simplices in degree* $\dim(M)$*. Then $\|M\| \leq k$.*

Proof. Let n be the dimension of M. The sum of the n-dimensional simplices with suitable signs represents a generator of the simplicial homology $H_n^{\text{simpl}}(M;\mathbb{Z})$ in degree n. Its image under the canonical isomorphism $H_n^{\text{simpl}}(M;\mathbb{Z}) \xrightarrow{\cong} H_n(M;\mathbb{Z})$ is a generator $[M]$ of $H_n(M;\mathbb{Z})$. Hence $j([M])$ can be represented by a cycle with norm at most k. □

Lemma 2.6. *Let M and N be closed oriented connected manifolds of the same dimension and let $f\colon M \to N$ be a map of degree* $\deg(f) = d \in \mathbb{Z}$*. Then*
$$\|M\| \geq |d| \cdot \|N\|$$
holds.

Proof. Let $z \in C_n(M;\mathbb{R})$ represent $[M]$. It is clear that $\|f_\#(z)\|_1 \leq \|z\|_1$ holds, where $f_\#$ is the induced map on the chain complex. By definition of the degree, $f(z)$ represents $d \cdot [N]$. For $d \neq 0$ the claimed inequality follows from
$$\|N\| \leq \frac{1}{|d|} \cdot \|f(z)\|_1 \leq \frac{1}{|d|} \cdot \|z\|_1$$
and taking the infimum. In the case $d = 0$ the inequality is trivial. □

This shows immediately that simplicial volume is a homotopy invariant. As another corollary we get the following result.

Corollary 2.7. *Let M be a closed oriented connected manifold and let $f\colon M \to M$ be a selfmap with* $|\deg(f)| \geq 2$*. Then* $\|M\| = 0$ *holds.*

In particular, $\|S^n\| = \|T^n\| = 0$ for all $n \in \mathbb{N}$. Here S^n stands for the n-dimensional sphere, and T^n denotes the n-dimensional torus.

It is not hard to show that the inequality in Lemma 2.6 is an equality in the case of a covering map. A proof can be found in [43].

Lemma 2.8. *Let M and N be be closed oriented connected manifolds and $f\colon M \to N$ be a d-sheeted covering map. Then*
$$\|M\| = d \cdot \|N\|$$
holds.

One could ask what happens if one works with $H_n(M;\mathbb{Q})$ instead of $H_n(M;\mathbb{R})$ in the definition of simplicial volume. Denote the resulting number by $\|M\|_{\mathbb{Q}}$.

Lemma 2.9. *Let M be a closed oriented connected manifold. Then $\|M\| = \|M\|_{\mathbb{Q}}$ holds.*

Proof. Let $\varepsilon > 0$ and let $z \in C_n(M;\mathbb{R})$ be a real fundamental cycle. Moreover, let $\overline{z} \in C_n(M;\mathbb{Z})$ be an integral fundamental cycle. Then $z - \overline{z}$ is a boundary, i.e. there is a chain $c \in C_{n+1}(M;\mathbb{R})$ with $\partial_{n+1}(c) = z - \overline{z}$. Since $\mathbb{Q} \subset \mathbb{R}$ is dense there is a rational chain $c' \in C_{n+1}(M;\mathbb{Q})$ such that $\|c - c'\|_1 < \frac{\varepsilon}{n+2}$. The operator norm of the boundary operator $\partial_k \colon C_k(M;\mathbb{R}) \to C_{k-1}(M;\mathbb{R})$ satisfies $\|\partial_k\| \leq k+1$. Hence we get

$$\|z - (\overline{z} + \partial_{n+1}(c'))\|_1 = \|\partial_{n+1}(c - c')\|_1 < \varepsilon.$$

This proves the lemma since $\overline{z} + \partial_{n+1}(c')$ is a rational fundamental cycle. ☐

2.1.3 Bounded Cohomology

We now recall the notion of bounded cohomology as an important tool in the study of simplicial volume. Although the definition is not due to Gromov, his fundamental article [24] was the first systematic study of bounded cohomology.

For a topological space X denote by $C^\bullet(X;\mathbb{R})$ its *real singular cochain complex*. In degree k it is given by

$$C^k(X;\mathbb{R}) = \hom_{\mathbb{R}}\big(C_k(X;\mathbb{R}),\mathbb{R}\big) = \operatorname{map}\big(\Sigma_k(X),\mathbb{R}\big).$$

Definition 2.10. Let X be a topological space and let $C^\bullet(X;\mathbb{R})$ be its real singular cochain complex. A cochain $\varphi \in C^k(X;\mathbb{R})$ is called *bounded* if there is a constant $C > 0$ such that $\varphi(\sigma) \leq C$ holds for all $\sigma \in \Sigma_k(X)$. Define the *bounded cochain complex* $\widehat{C}^\bullet(X)$ of X as the subcomplex of $C^\bullet(X;\mathbb{R})$ generated by all bounded cochains. The *bounded cohomology* $\widehat{H}^\bullet(X)$ of X is defined to be the cohomology of $\widehat{C}^\bullet(X)$.

Remark 2.11. Bounded cohomology is a homotopy invariant and satisfies the dimension axiom, i.e. $\widehat{H}^k(\mathrm{pt}) = 0$ for $k \neq 0$ and $\widehat{H}^0(\mathrm{pt}) = \mathbb{R}$. But bounded cohomology is not a cohomology theory since the excision axiom fails. This fact makes bounded cohomology much harder to compute than singular cohomology. For example the second bounded cohomology group $\widehat{H}^2(S^1 \vee S^1)$ of the wedge of two circles is an infinitely generated $\mathbb{R}$-vector space although $S^1 \vee S^1$ is a one-dimensional CW-complex. For a proof see [23, Section 5A].

Dually to the seminorm on $H_\bullet(X;\mathbb{R})$, we will next define a seminorm on $\widehat{H}^\bullet(X)$.

Definition 2.12. For a bounded cochain $\varphi \in \widehat{C}^k(X)$ define its *ℓ^∞-norm* by

$$\|\varphi\|_\infty = \sup\big\{|\varphi(\sigma)| : \sigma \in \Sigma_k(X)\big\}.$$

The induced *ℓ^∞-seminorm* on $\widehat{H}^k(X)$ is given by

$$\|\psi\|_\infty = \inf\big\{\|\varphi\|_\infty : \varphi \text{ is a cocycle in } \widehat{C}^k(X) \text{ with } [\varphi] = \psi\big\}.$$

For a homology class $\alpha \in H_k(X;\mathbb{R})$ and a bounded cohomology class $\psi \in \widehat{H}^k(X)$ one can define the *Kronecker product* as $\langle \psi, \alpha \rangle = \varphi(z)$, where $\varphi \in \widehat{C}^k(X)$ is a cocycle representing ψ and $z \in C_k(X;\mathbb{R})$ is a cycle representing α. It is clear that this does not depend on the choice of the representing cycle and cocycle. We get the relation

$$|\langle \psi, \alpha \rangle| \le \|\psi\|_\infty \cdot \|\alpha\|_1.$$

The following lemma is the link between simplicial volume and bounded cohomology. Its proof is a simple application of the Hahn-Banach theorem [4].

Lemma 2.13. *(i) Let X be a topological space. For $\alpha \in H_k(X;\mathbb{R})$ it is $\|\alpha\|_1 = 0$ if and only if $\langle \psi, \alpha \rangle = 0$ holds for every $\psi \in \widehat{H}^k(X)$. If $\|\alpha\|_1 > 0$ then*

$$\frac{1}{\|\alpha\|_1} = \sup\bigl\{ \|\psi\|_\infty : \psi \in \widehat{H}^k(X) \text{ with } \langle \psi, \alpha \rangle = 1 \bigr\}.$$

(ii) Let M be a closed oriented connected manifold of dimension n. Then $\|M\| = 0$ holds if and only if the natural map $\widehat{H}^n(M) \to H^n(M;\mathbb{R})$ is the zero map. If $\|M\| > 0$ then

$$\frac{1}{\|M\|} = \|\psi\|_\infty,$$

where ψ is the image of a cohomological fundamental class under the canonical homomorphism $H^n(M;\mathbb{Z}) \to H^n(M,\mathbb{R})$ which is induced by inclusion of coefficients.

The next striking result is due to Gromov [24, Section 3.1]. An alternative proof was given by Ivanov [30, Theorem 4.3].

Theorem 2.14. *Let X and Y be path-connected topological spaces and let $f\colon X \to Y$ be continuous. If the induced homomorphism $\pi_1(f)\colon \pi_1(X) \to \pi_1(Y)$ is surjective and $\ker\bigl(\pi_1(f)\bigr)$ is amenable, then the induced homomorphism $\widehat{H}^\bullet(f)\colon \widehat{H}^\bullet(Y) \to \widehat{H}^\bullet(X)$ is an isometric isomorphism.*

We immediately get the following useful consequences.

Corollary 2.15. *(i) Let X be a path-connected topological space and let $f\colon X \to B\pi$ be the classifying map of its fundamental group $\pi = \pi_1(X)$. Then the induced map $\widehat{H}^\bullet(f)\colon \widehat{H}^\bullet(B\pi) \to \widehat{H}^\bullet(X)$ is an isometric isomorphism.*

(ii) Let X be a path-connected topological space with amenable fundamental group. Then

$$\widehat{H}^k(X) = 0 \quad \text{for all} \quad k \ge 1.$$

(iii) Let M be a closed oriented connected manifold with fundamental group $\pi = \pi_1(X)$ and classifying map $f\colon M \to B\pi$. Then we have $\|M\| = \|f_*([M])\|_1$.

(iv) Let M be a closed oriented connected manifold with amenable fundamental group. Then $\|M\| = 0$ holds.

2.1.4 Further Properties of Simplicial Volume

Up to now we have not seen a single example of a manifold with non vanishing simplicial volume. This changes if we turn our attention to hyperbolic manifolds. The next result is due to Gromov and Thurston [46], a detailed proof can be found in [4].

Theorem 2.16. *Let M be a closed oriented connected hyperbolic manifold of dimension n. Then we have*

$$\|M\| = \frac{\mathrm{vol}(M)}{v_n},$$

where v_n is the supremum of the volumes of all geodesic n-dimensional simplices in hyperbolic n-space $\mathbb{H}^n$.

In particular, we get $\|F_g\| = 4 \cdot (g-1)$, where F_g is the closed oriented surface of genus g.

For the definition of $\mathbb{H}^n$ see for example [4, Section A.1, p. 1f.].

For arbitrary negatively curved manifolds the method of proof of Theorem 2.16 can be used to conclude that their simplicial volume is non zero.

Theorem 2.17. *Let M be a closed oriented connected Riemannian manifold whose sectional curvature is everywhere negative. Then $\|M\| > 0$ holds.*

This is a result of Inoue and Yano [29].

Simplicial volume behaves well with respect to products and connected sums as the next results show. Both of them are due to Gromov [24], a proof of the first one can also be found in [4].

Theorem 2.18. *Let M and N be closed oriented connected manifolds of dimensions m and n. Then there is a constant $c(n+m) > 0$, not depending on M and N, such that*

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq c(n+m) \cdot \|M\| \cdot \|N\|.$$

Theorem 2.19. *Let M and N be closed oriented connected manifolds of the same dimension $n \geq 3$. Then we get*

$$\|M \# N\| = \|M\| + \|N\|.$$

Note that Theorem 2.19 is not true in dimension 2 since the simplicial volume of the connected sum of two tori is non zero.

The next result was proved by Gromov and Thurston [46],[24]. An extensive proof is given in [43].

Theorem 2.20. *Let M and N be closed oriented connected Riemannian manifolds with isometric universal coverings. Then we get*

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}.$$

The following Theorem was proved independently by Gromov [24] and Yano [49].

Theorem 2.21. *Let M be a closed oriented connected manifold. If there is a non trivial S^1-action on M then we get $\|M\| = 0$.*

2.2 Minimal Volume

There is the following finiteness theorem of Cheeger [8].

Theorem 2.22. *Let $n \geq 1$ be an integer. For any given $d > 0$ and $v > 0$ there is only a finite number of diffeomorphism classes of closed Riemannian manifolds M of dimension n such that $|\mathrm{sec}(M)| < 1$, $\mathrm{diam}(M) < d$ and $\mathrm{vol}(M) > v$ hold.*

This result motivates the question how small the volume of a given smooth manifold M can be if its sectional curvature satisfies $|\mathrm{sec}(M)| \leq 1$.

2.2.1 Definition of Minimal Volume

Definition 2.23. Let M be a closed oriented connected smooth manifold. We define its *minimal volume* $\mathrm{minvol}(M)$ as

$$\mathrm{minvol}(M) = \inf\bigl\{\mathrm{vol}(M, g) : g \text{ is a Riemannian metric on } M \text{ with } |\mathrm{sec}_g(M)| \leq 1\bigr\}.$$

Remark 2.24. The restriction to closed manifolds is not necessary. In the general situation one allows only *complete* Riemannian metrics g in the definition, a condition which is automatically satisfied for closed manifolds. Since we are only interested in closed manifolds, we will restrict our attention to that case.

It is clear that for the definition of the minimal volume we need a smooth structure on the manifold. Moreover, the next result of Bessières [5] shows that the minimal volume is not a topological invariant, it really depends on the smooth structure.

Theorem 2.25. *Let M be a closed connected hyperbolic stably parallelizable manifold of dimension $n \neq 4$ and let N be a closed connected smooth manifold of the same dimension. Then*

$$\mathrm{minvol}(M \# N) \geq \mathrm{minvol}(M)$$

with equality if and only if N is diffeomorphic to the n-sphere S^n.

It follows that if Σ is an exotic sphere one has $\mathrm{minvol}(M \# \Sigma) > \mathrm{minvol}(M)$ although $M \# \Sigma$ and M are homeomorphic.

2.2.2 Properties of Minimal Volume

It is obvious that the minimal volume of closed oriented connected smooth manifolds which admit a flat Riemannian metric vanishes.

Next we want to compute the minimal volume of closed oriented surfaces. Note that for a closed oriented surface F the Gauß-Bonnet theorem yields

$$\chi(F) = \frac{1}{2\pi} \cdot \int_F \mathrm{sec}(F) \ \mathrm{vol}\,.$$

Hence for any Riemannian metric on F whose sectional curvature satisfies $|\sec| \leq 1$, it follows that

$$\mathrm{vol}(F) \geq \int_F |\sec| \ \mathrm{vol} \geq \Big| \int_F \sec \ \mathrm{vol} \Big| = 2\pi \cdot |\chi(F)|.$$

Since on the closed oriented surface F_g of genus $g \geq 2$ there is a hyperbolic Riemannian metric, we have

$$\mathrm{minvol}(F_g) = 2\pi \cdot |\chi(F_g)| = 4\pi \cdot (g-1) = \pi \cdot \|F_g\|.$$

For arbitrary dimension, there is also a relation between the Euler characteristic and the minimal volume, namely we have the following result of Gromov [24, page 6].

Theorem 2.26. *There is a dimension constant $e_n > 0$ such that*

$$\chi(M) \leq e_n \cdot \mathrm{minvol}(M)$$

holds for all closed oriented connected smooth manifolds M of dimension n.

As a special case, we get that $\mathrm{minvol}(S^{2n}) \neq 0$ for all $n \geq 0$. This result combined with Theorem 2.25 shows that minimal volume is not additive with respect to connected sums.

There is a similar result as Theorem 2.21 for minimal volume. It is also due to Gromov [24, page 7].

Theorem 2.27. *Let M be a closed oriented connected smooth manifold. If there is a free S^1-action on M, then* $\mathrm{minvol}(M) = 0$ *holds.*

In particular, $\mathrm{minvol}(M \times S^1) = 0$ for all closed oriented connected smooth manifolds M, and also $\mathrm{minvol}(S^{2n+1}) = 0$ for all $n \geq 0$.

2.2.3 Relation between Simplicial and Minimal Volume

There is a relation between simplicial and minimal volume. It is based on the following theorem of Gromov [24, Main Inequality on page 12].

Theorem 2.28. *Let M be a closed oriented connected Riemannian manifold of dimension n whose Ricci curvature satisfies* $\mathrm{ric}(M) \geq \frac{-1}{n-1}$. *Then there is a constant c_n with* $0 < c_n < n!$ *such that*

$$\|M\| \leq c_n \cdot \mathrm{vol}(M).$$

Recall that the lower bound $\sec(M) \geq -\alpha^2$ for the sectional curvature implies the lower bound $\mathrm{ric}(M) \geq -(n-1) \cdot \alpha^2$ for the Ricci curvature. Hence the following corollary holds:

Corollary 2.29. *Let M be a closed oriented connected smooth manifold of dimension n. Then we have*

$$\|M\| \leq n! \cdot (n-1)^n \cdot \mathrm{minvol}(M).$$

3 $\mathcal{R}$-Spaces

In this chapter, $\mathcal{R}$-spaces are introduced. We will first collect some basic facts about standard Borel spaces which we need later to define and analyze $\mathcal{R}$-spaces. Standard Borel spaces are used since they form a category in which measure theory works well. Fundamental facts about measure theory can be found in detail for example in [11]. A good source for many results about standard Borel spaces is Kechris's monograph [33].

To a standard Borel space X with a probability measure μ we assign the category of X-spaces. Aside from some mild technical conditions, an X-space is simply a space fibered over X with measurable projection to X. Morphisms between X-spaces are measurable and fiber preserving maps which are fiberwise continuous. The natural example is the product $X \times Z$ for a topological space Z. We introduce singular homology $H_\bullet^X(S;\Lambda)$ with coefficients in a subring $\Lambda \subset \mathbb{C}$ for X-spaces S by a natural transfer of the definition of ordinary singular homology. We define standard equivalence relations $\mathcal{R} \subset X \times X$ on a standard Borel space in the sense of Feldman and Moore [18] and give some examples. Fundamental for our applications will be the orbit equivalence relation $\mathcal{R}_{G \curvearrowright X}$ of a standard action $G \curvearrowright X$. The definition of $\mathcal{R}$-spaces is given in 3.3.3. They arise as X-spaces provided with an "action" of $\mathcal{R}$. One should think of X-spaces as the analog of topological spaces, and of $\mathcal{R}$-spaces as the analog of G-spaces. We define an induction functor ind: G-Spaces $\to$ $\mathcal{R}_{G \curvearrowright X}$-Spaces which assigns to a G-space Z the $\mathcal{R}_{G \curvearrowright X}$-space $X \times Z$. The $\mathcal{R}_{G \curvearrowright X}$-action is induced by the diagonal action $G \curvearrowright X \times Z$.

3.1 Standard Borel Spaces

Let us recall some basic definitions and facts from measure theory. A *measurable space* is a set X together with a σ-algebra $\mathcal{A}$ of subsets of X. The elements of $\mathcal{A}$ are called *measurable* subsets of X. A morphism (or a *measurable map*) between measurable spaces $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ is a map $f\colon X_1 \to X_2$ such that $f^{-1}(A) \in \mathcal{A}_1$ holds for all $A \in \mathcal{A}_2$. Two measurable spaces $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ are called *isomorphic* if there is a bijective map $f\colon X_1 \to X_2$ such that f and f^{-1} are measurable. For a topological space X denote by $\mathcal{B}(X)$ its *Borel σ-algebra* generated by the open subsets of X.

A measurable space $(X, \mathcal{A})$ together with a measure μ on $\mathcal{A}$ is called a *measure space*. Later the case of a *probability measure* μ, i.e. $\mu(X) = 1$ will be important. A measure on $\mathcal{B}(X)$ is called a *Borel measure* on X. A measurable map $f\colon X_1 \to X_2$

between to measure spaces $(X_1, \mathcal{A}_1)$, $(X_2, \mathcal{A}_2)$ with measures μ_1 and μ_2 respectively is called *measure preserving* if $\mu_1\big(f^{-1}(A)\big) = \mu_2(A)$ holds for all $A \in \mathcal{A}_2$.

A *Polish space* is a separable topological space which is metrizable by a complete metric. A measurable space $(X, \mathcal{A})$ is called a *standard Borel space* if it is isomorphic to $\big(Y, \mathcal{B}(Y)\big)$ for some Polish space Y. Measurable subsets of standard Borel spaces will also be called *Borel sets*, measurable maps between standard Borel spaces will be called *Borel maps*. The σ-algebra $\mathcal{A}$ of a standard Borel space $(X, \mathcal{A})$ will also be called *Borel σ-algebra* although the Polish topology generating $\mathcal{A}$ is not part of the structure, see Remark 3.2 below.

In the next theorem we present some classical results about Polish and standard Borel spaces.

Theorem 3.1. *(i) Open and closed subsets of Polish spaces are Polish spaces.*

(ii) Countable disjoint unions of Polish spaces are Polish spaces.

(iii) Countable products of Polish spaces are Polish spaces.

(iv) Measurable subsets of standard Borel spaces are standard Borel spaces.

Remark 3.2. A standard Borel $(X, \mathcal{A})$ space can be given a topology by means of the isomorphism $(X, \mathcal{A}) \cong \big(Y, \mathcal{B}(Y)\big)$, where Y is Polish. Note that this topology is not part of the structure of $(X, \mathcal{A})$. For example, in general a measurable subset $Y \subset X$ with the induced topology of a chosen Polish topology on X does not turn Y into a Polish space. The point is that there exists a Polish topology on Y having the restricted Borel σ-algebra $\mathcal{B}(X)|_Y$ as its Borel σ-algebra.

The next useful theorem is due to Kuratowski. A proof can be found in [41, Theorem 1.3 on p. 15 f.].

Theorem 3.3. *Let $f\colon X \to Y$ be a measurable map between standard Borel spaces which is countable-to-1, i.e. for each $y \in Y$ the preimage $f^{-1}(y)$ is countable. Then the image $f(X) \subset Y$ is measurable, and there is a countable partition $(X_n)_{n\in\mathbb{N}}$ of X by measurable subsets X_n, such that $f|_{X_n}$ is injective and $f|_{X_n}\colon X_n \to f(X_n)$ is a Borel isomorphism for all $n \in \mathbb{N}$.*

If f is actually uniformly finite-to-1, i.e. there is $N \in \mathbb{N}$ such that $f^{-1}(y)$ has at most N elements, then there is a partition with the properties above consisting of at most N sets.

There is another helpful result about mappings between standard Borel spaces, a proof is given e.g. in [33, Corollary 15.2].

Theorem 3.4. *Let X and Y be standard Borel spaces and $f\colon X \to Y$ be a Borel map. If $f|_A$ is injective for a Borel subset $A \subset X$ then $f(A) \subset Y$ is a Borel subset and $f|_A\colon A \to f(A)$ is a Borel isomorphism.*

We get immediately the following corollary.

Corollary 3.5. *The graph $\mathrm{gr}(f) = \big\{(x, f(x)) : x \in X\big\}$ of a Borel map $f\colon X \to Y$ between standard Borel spaces X and Y is a Borel subset of $X \times Y$.*

Proof. Consider the injective Borel map

$$\overline{f}\colon X \to X \times Y, \quad x \mapsto \big(x, f(x)\big).$$

By Theorem 3.4 its image $\operatorname{im}(\overline{f}) = \operatorname{gr}(f)$ is a Borel subset of $X \times Y$. □

Remark 3.6. If not otherwise stated we will always use the *product σ-algebra* on the product of countably many measurable spaces $\big((X_n, \mathcal{A}_n)\big)_{n\in\mathbb{N}}$, which will be denoted by $\bigotimes_{n\in\mathbb{N}} \mathcal{A}_n$.

For later use we will state the following theorem due to Kunugui and Novikov, which is proved for example in [33, Theorem 28.7, p. 220].

Theorem 3.7. *Let X be a standard Borel space and Y be a Polish space with a basis $(V_n)_{n\in\mathbb{N}}$ for its topology. If $A \subset X \times Y$ is measurable such that each $A_x = \{y \in Y : (x, y) \in A\}$ is open in Y for all $x \in X$ then there is a family $(B_n)_{n\in\mathbb{N}}$ of Borel subsets $B_n \subset X$ such that*

$$A = \bigcup_{n\in\mathbb{N}} B_n \times V_n$$

holds.

We will need the following consequence of Theorem 3.7.

Lemma 3.8. *Let X be a standard Borel space and Y be a compact Polish space. If $A \subset X \times Y$ is a measurable set such that A_x is open in Y for all $x \in X$ then the set*

$$B_A = \{x \in X : A_x = Y\}$$

is measurable.

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be any basis for the topology of Y. By Theorem 3.7 we know that there is a family $(B_n)_{n\in\mathbb{N}}$ of measurable subsets of X with $A = \bigcup_{n\in\mathbb{N}} B_n \times V_n$. Now consider the set

$$\mathcal{V} = \Big\{I \subset \mathbb{N} : \bigcup_{i\in I} V_i = Y, I \text{ minimal with this property}\Big\}.$$

Since Y is compact $\mathcal{V}$ consists only of finite subsets and is therefore countable. The lemma now follows because of

$$B_A = \bigcup_{I\in\mathcal{V}} \bigcap_{i\in I} B_i.$$ □

3.2 X-Spaces and Singular Homology

In the following X stands for a standard Borel space which is σ-compact with respect to some Polish topology on X which generates the Borel σ-algebra. Note that any Polish topology on X is second countable. We assume that X is equipped with a probability measure μ.

Remark 3.9. Recall that a topological space is called *σ-compact* if it is the union of countably many compact subspaces.

Notation. In a measure space (X, μ) by the phrase "$\forall x \in X$" we always mean "for μ-almost all $x \in X$" unless otherwise stated.

3.2.1 Definition and Examples of X-Spaces

Roughly speaking an X-space is a collection of topological spaces (called the fibers) indexed by X such that the fibers depend measurably on X. Note that we will always work in the category of compactly generated spaces. We will collect the necessary background about compactly generated spaces in Section 3.4.

Definition 3.10. An *X-space* is a pair (S, p), where S is a σ-compact, second countable Hausdorff space and $p\colon S \to X$ is a measurable map to a standard Borel space X. For $x \in X$ we call $S_x = p^{-1}(x)$ the *fiber* over x. A morphism of X-spaces (S, p) and (T, q) is a measurable map $F\colon S \to T$ such that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{F} & T \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & X & \end{array}$$

and the restriction $F|_{S_x}\colon S_x \to T_x$ is continuous for each $x \in X$.

We will often omit the p from the notation and just write S for an X-space (S, p). If there is risk of confusion we will sometimes write p_S for the projection of an X-space S to X.

Remark 3.11. If not stated otherwise, the measurable structure on a topological space is always given by its Borel σ-algebra, which is generated by the open subsets of the space.

Example 3.12. The easiest example of an X-space is the product $X \times Z$ for a σ-compact, second countable Hausdorff space Z with the obvious projection. The topology on $X \times Z$ is given by the product topology, where we choose some σ-compact, second countable topology on X which generates the Borel σ-algebra. This choice does not affect the categorical properties of $X \times Z$ in the category of X-spaces since the measurable structure on $X \times Z$ and the topological structure in the fibers do not vary.

Definition 3.13. Let (S, p) be an X-space and let (T, q) be an X'-space. Then the $X \times X'$-space $(S \times T, p \times q)$ is called the *cartesian product* of (S, p) and (T, q).

3.2.2 Colimits of X-Spaces

There are several problems with limits of X-spaces due to lack of heredity properties of σ-compactness. For example, even for two X-spaces (S, p) and (T, q) it is not clear if their fibered product

$$S \times_X T = \{(s,t) \in S \times T : p(s) = q(t)\}$$

is again an X-space since $S \times_X T \subset S \times T$ is in general only measurable and not necessarily closed. Hence we can not draw σ-compactness of $S \times_X T$ from that of S and T.

The situation is much more convenient if we ask for colimits. Before we can show the existence of countable colimits in the category of X-spaces we need the following lemma which is of interest in its own right. There are many well known results in this direction, the proof for this particular case was communicated to me by Roman Sauer.

Lemma 3.14. *Let $f\colon Y \to Z$ be a continuous surjective map of σ-compact second countable Hausdorff spaces. Then f has a measurable section, i.e. there is a measurable map $s\colon Z \to Y$ (with respect to the Borel σ-algebras) such that $f \circ s = \mathrm{id}_Z$.*

Proof. First we consider the case that Y is compact. Since Y is second countable and Hausdorff it is metrizable. Hence there is a continuous surjective map $h\colon \mathcal{C} \to Y$ where $\mathcal{C} \subset [0,1]$ is the Cantor set [33, Theorem I.(4.18)]. We define a map $s'\colon Z \to \mathcal{C}$ by $z \mapsto \min\big((f \circ h)^{-1}(z)\big)$. The minimum exists since $f \circ h\colon \mathcal{C} \to Z$ is continuous and surjective and $\mathcal{C} \subset [0,1]$ is closed. Next we show that s' is measurable. It is

$$\begin{aligned} s'^{-1}\big(\mathcal{C} \cap [0,t]\big) &= \{y \in Y : \min\big((f \circ h)^{-1}(z)\big) \leq t\} \\ &= \{y \in Y : \exists t' \in \mathcal{C} \cap [0,t]\ f \circ h(t') = y\} \\ &= f \circ h\big(\mathcal{C} \cap [0,t]\big), \end{aligned}$$

and this set is compact and hence measurable. Since the sets of the form $\mathcal{C} \cap [0,t]$ generate the Borel σ-algebra of $\mathcal{C}$, measurability of s' follows. The map $s = h \circ s'$ is a measurable section of f.

In the general case pick an exhaustion $Y = \bigcup_{n=1}^{\infty} Y_n$ of Y by compact subspaces (recall that Y is σ-compact). Then $Z = \bigcup_{n=1}^{\infty} f(Y_n)$ is also an exhaustion by compact subspaces. One can construct a measurable section s_n of $f|_{Y_n}$ as above. Then the map $s\colon Z \to Y$ defined by $s(z) = s_n(z)$ for $z \in f(Y_n) - f(Y_{n-1})$ is a measurable section of f. □

Lemma 3.15. *Countable colimits exist in the category of X-spaces (Note that "countable colimit" means that there are also only countably many structure maps).*

Proof. Let $(S_i)_{i\in I}$ be a countable family of X-spaces and for each pair $(i,j) \in I^2$ let $(f^l_{i,j}\colon S_i \to S_j)_{l\in L_{i,j}}$ be a countable family of X-maps . It is easy to see that the disjoint union $\coprod_{i\in I} S_i$ is again σ-compact and second countable and therefore an honest X-space. We now build the quotient space

$$S = \operatorname*{colim}_{i\in I} S_i = \coprod_{i\in I} S_i / \sim,$$

where $s \sim f^l_{i,j}(s)$ for all $i,j \in I$, $l \in L_{i,j}$, $s \in S_i$. This is also an X-space since σ-compactness and second countability persist under continuous images. We will show that S is the colimit in the category of X-spaces. Let $q\colon \coprod_{i\in I} S_i \to S$ be the quotient map. First of all, there are natural maps $f_i\colon S_i \to S$ of X-spaces given by $f_i = q \circ j_i$, where $j_i\colon S_i \to \coprod_{i\in I} S_i$ is the inclusion. It is clear that $f_i = f_j \circ f^l_{i,j}$ holds, whenever there are maps $f^l_{i,j}$.

Let T be an X-space and let $F_i\colon S_i \to T$ be an X-map for each $i \in I$ such that $F_i = F_j \circ f^l_{i,j}$ for all suitable $i,j \in I$, $l \in L_{i,j}$. Obviously, there is a function $F\colon S \to T$ which satisfies $F \circ f_i = F_i$ for all $i \in I$ (since S is the colimit in the category of sets). We have to show that F is an X-map, i.e. that F is measurable and continuous on almost each fiber and that

$$\begin{array}{ccc} S & \xrightarrow{F} & T \\ & {\scriptstyle p_S}\searrow \quad \swarrow{\scriptstyle p_T} & \\ & X & \end{array} \tag{3.1}$$

commutes. The commutativity of (3.1) follows directly from the construction of the colimit and the map F. Measurability of F follows from Lemma 3.14 applied to the projection q. This yields a measurable section s of q, and now measurability of $F = \left(\coprod_{i\in I} F_i\right) \circ s$ follows.

Each F_i is an X-map and therefore continuous on $S_{i,x}$ for all $x \in X_i$, where $X_i \subset X$ is a subset of full measure. The subset $\bigcap_{i\in I} X_i \subset X$ also has full measure. One has

$$S_x = \operatorname*{colim}_{i\in I} S_{i,x}$$

for $x \in \bigcap_{i\in I} X_i$, and F_x is just the colimit of the $F_{i,x}$ in the category of topological spaces (which makes sense since all F_i are continuous on $S_{i,x}$). It follows that F is continuous on almost each fiber. □

3.2.3 Singular Homology for X-Spaces

We will develop a homology theory for X-spaces in analogy to singular homology for topological spaces.

Definition 3.16. A *singular X-simplex* of dimension n of an X-space (S,p) is an X-map

$$\sigma\colon X_\sigma \times \Delta^n \to S,$$

where $X_\sigma \subset X$ is a measurable subset. That means that σ is measurable and continuous on $\{x\} \times \Delta^n$ for almost all $x \in X_\sigma$. As usual, $\Delta^n \subset \mathbb{R}^{n+1}$ denotes the standard n-simplex. Define $\Sigma_n^X(S)$ as the set of n-dimensional X-simplices in S. We will use the notation σ_x for the ordinary singular simplex $\sigma|_{\{x\}\times\Delta^n}$.

In other words, σ is a measurable collection of singular simplices in the fibers of S. The following lemma about X-simplices will be useful later on.

Lemma 3.17. *Let S be an X-space and $T \subset S$ be a measurable subset which is either fiberwise open or fiberwise closed, i.e. the fiber $T_x = T \cap S_x \subset S_x$ is either open for almost all $x \in X$ or it is closed for almost all $x \in X$. Then for every X-simplex $\sigma\colon X_\sigma \times \Delta^n \to S$ the subset*

$$\left\{x \in X_\sigma : \operatorname{im}(\sigma_x) \subset T\right\}$$

is measurable.

Proof. In the case of a fiberwise open measurable subset $T \subset S$ we can apply Lemma 3.8 to the measurable subset $\sigma^{-1}(T) \subset X_\sigma \times \Delta^n$ to obtain measurability of $\left\{x \in X_\sigma : \operatorname{im}(\sigma_x) \subset T\right\}$.

If $T \subset S$ is measurable and fiberwise closed we use Theorem 3.7 to get

$$\sigma^{-1}(S - T) = \bigcup_{k\in\mathbb{N}} B_k \times V_k,$$

where $(V_k)_{k\in\mathbb{N}}$ is any basis for the topology of Δ^n and $(B_k)_{k\in\mathbb{N}}$ is a suitable family of Borel subsets of X_σ. We conclude

$$\left\{x \in X_\sigma : \operatorname{im}(\sigma_x) \subset T\right\} = X_\sigma - \bigcup_{k\in\mathbb{N}} B_k.$$

This proves the desired measurability. □

Now we will define the singular chain complex of S.

Definition 3.18. For a subring $\Gamma \subset \mathbb{C}$ we define the n-th *singular chain group* $C_n^X(S;\Gamma)$ of S with coefficients in Γ by

$$C_n^X(S;\Gamma) = L^\infty(X,\Gamma)[\Sigma_n^X(S)]/\sim,$$

where

$$\sum_\sigma f_\sigma \cdot \sigma \sim \sum_\sigma g_\sigma \cdot \sigma \iff \sum_\sigma f_\sigma(x) \cdot \sigma_x = \sum_\sigma g_\sigma(x) \cdot \sigma_x \quad \text{in } C_n(S_x)$$

for almost all $x \in X$. We will sometimes write $(\sum_\sigma f_\sigma \cdot \sigma)_x$ for $\sum_\sigma f_\sigma(x) \cdot \sigma_x$. For all $x \notin X_\sigma$ we regard σ_x as zero in $C_n(S_x;\Gamma)$.

If the coefficients are the integers $\mathbb{Z}$ themselves we will sometimes omit them from the notation and write $C_n^X(S)$ for $C_n^X(S;\mathbb{Z})$. Note that nevertheless $L^\infty(X)$ will always denote $L^\infty(X;\mathbb{C})$.

Remark 3.19. In the notation we will not always distinguish between an element of $L^\infty(X,\Gamma)[\Sigma_n^X(S)]$ and its class in $C_n^X(S;\Gamma)$ as one usually also does in the case of L^p-spaces (and as we did without mentioning it before with elements of $L^\infty(X)$).

The collection $\{C_n^X(S;\Gamma)\}_n$ becomes a chain complex of $L^\infty(X,\Gamma)$-modules with boundary operator $d_n\colon C_n^X(S;\Gamma) \to C_{n-1}^X(S;\Gamma)$ given by

$$d_n(\sigma) = \sum_{i=0}^{n} (-1)^i \cdot \sigma \circ (\mathrm{id}_{X_\sigma} \times \tau_i),$$

where $\tau_i\colon \Delta^{n-1} \to \Delta^n$ is the i-th face map.

For a pair $T \subset S$ of X-spaces we define the relative complex

$$C_\bullet^X(S,T;\Gamma) = \frac{C_\bullet^X(S;\Gamma)}{C_\bullet^X(T;\Gamma)}.$$

The boundary operator on $C_\bullet^X(S;\Gamma)$ induces a boundary operator on $C_\bullet^X(S,T;\Gamma)$.

Definition 3.20. Let $T \subset S$ be a pair of X-spaces. We define the *singular X-homology* of (S,T) as the homology of $C_\bullet^X(S,T;\Gamma)$ and denote it by $H_\bullet^X(S,T;\Gamma)$. The singular X-homology of S is defined as $H_\bullet^X(S,\emptyset;\Gamma)$ and denoted by $H_\bullet^X(S;\Gamma)$. Again, we will sometimes omit integral coefficients from the notation.

Remark 3.21. One could prove suitable variants of the Eilenberg-Steenrod axioms for $H_\bullet^X(\_,\_)$ up to $L^\infty(X)$-dimension isomorphisms. We will not do it here since we will prove equivariant versions of them later in this chapter. It will then be easy to establish the non equivariant case.

As a consequence we get the following result.

Theorem 3.22. *Let Z be a CW-complex . Then there is an $L^\infty(X)$-dimension isomorphism*

$$H_\bullet\big(Z;L^\infty(X)\big) \xrightarrow{\cong_{\dim}} H_\bullet^X(X \times Z;\mathbb{C}),$$

where $H_\bullet\big(Z;L^\infty(X)\big)$ is ordinary singular homology with $L^\infty(X)$-coefficients.

3.2.4 Reduced Form

The chain groups $C_n^X(S)$ are not free as $L^\infty(X)$-modules. For example, if σ_1 and σ_2 only agree on $A \times \Delta^n$, then $\chi_A \cdot \sigma_1 - \chi_A \cdot \sigma_2$ represents zero in $C_n^X(S)$. This fact causes some difficulties in the definition of maps on $C_n^X(S)$. We will therefore show that one can always find special representatives for X-chains.

Definition 3.23. For $\sum_{i=1}^k f_i \cdot \sigma_i \in L^\infty(X)[\Sigma_n^X(S)]$ let $\mathrm{supp}(f_i \cdot \sigma_i) = \mathrm{supp}(f_i) \cap X_{\sigma_i}$. By definition, $\sum_{i=1}^k f_i \cdot \sigma_i$ is in *reduced form* if for all $i,j \in \{1,\ldots,k\}$ with $i \neq j$ the set

$$A(i,j) = \mathrm{supp}(f_i \cdot \sigma_i) \cap \mathrm{supp}(f_j \cdot \sigma_j) \cap \{x \in X : \sigma_{i,x} = \sigma_{j,x}\}$$

has measure zero.

In other words $\sum_{i=1}^{k} f_i \cdot \sigma_i$ is in reduced form if and only if for almost all $x \in X$ the singular chain $\left(\sum_{i=1}^{k} f_i \cdot \sigma_i\right)_x$ is in standard form, i.e. each singular simplex occurs at most once in the sum.

Lemma 3.24. *Let $c \in C_n^X(S)$ be a singular X-chain. Then c can be represented in reduced form, i.e. there exists an element $\sum_{i=1}^{k} f_i \cdot \sigma_i \in L^\infty(X, \mathbb{Z})[\Sigma_n^X(S)]$ in reduced form such that $c = \left[\sum_{i=1}^{k} f_i \cdot \sigma_i\right]$ holds.*

Proof. Given a singular X-chain $c \in C_n^X(S)$ represented by $\sum_{l=1}^{k} g_l \cdot \sigma_l$ we will give a procedure which turns this representative into a reduced form element also representing c.

Order the pairs in $\left\{(i,j) \in \{1,\ldots,k\}^2 : i \neq j\right\}$ lexicographical, i.e.

$$(i,j) < (r,s) \iff (i < r) \quad \text{or} \quad (i = r \text{ and } j < s).$$

For each pair (i,j) with $i < j$ according to the lexicographical order do the following steps. If $A(i,j)$ has positive measure then change the coefficients:

- $g_j(x) = \begin{cases} g_i(x) + g_j(x), & x \in A(i,j) \\ g_j(x), & \text{otherwise} \end{cases}$
- $g_i(x) = \begin{cases} 0, & x \in A(i,j) \\ g_i(x), & \text{otherwise} \end{cases}$

One easily checks that each step in the algorithm does not change the represented element in $C_n^X(S)$. This algorithm clearly ends up in a reduced form element which represents c. □

Remark 3.25. The representation of chains by elements in reduced form is not unique. Also the output of the algorithm above depends on the enumeration of the simplices. Fortunately, we only need the existence of representatives in reduced form later on.

3.3 Definition and Examples of $\mathcal{R}$-Spaces

The relationship between X-spaces and $\mathcal{R}$-spaces can be seen as the one between ordinary topological spaces and topological spaces with the action of a group. We will define $\mathcal{R}$-spaces to be X-spaces equipped with the action of a suitable equivalence relation. Before we can make that precise we have to fix some notation.

3.3.1 Standard Equivalence Relations

First we have to define special equivalence relations on a standard Borel space X. From now on we will always assume that X is equipped with a probability measure μ.

Definition 3.26. A *standard equivalence relation* $\mathcal{R}$ on a standard Borel space X with probability measure μ is an equivalence relation $\mathcal{R} \subset X \times X$ such that the following conditions hold.

(i) The subset $\mathcal{R} \subset X \times X$ is measurable.

(ii) The equivalence classes of $\mathcal{R}$ are countable.

(iii) For each Borel isomorphism $\phi\colon A \to B$ between measurable subsets $A, B \subset X$ satisfying $\mathrm{gr}(\varphi) \subset \mathcal{R}$ one has $\mu(A) = \mu(B)$.

As usual, $\mathrm{gr}(\varphi) = \{(x, \varphi(x)) : x \in X\}$ denotes the graph of φ. A standard equivalence relation $\mathcal{R} \subset X \times X$ is called *ergodic* if any $\mathcal{R}$-invariant measurable subset $A \subset X$ satisfies $\mu(A) = 0$ or $\mu(A) = 1$.

By $[\mathcal{R}]$ we will denote the group of Borel automorphisms $\varphi\colon X \to X$ satisfying $\mathrm{gr}(\varphi) \subset \mathcal{R}$.

Remark 3.27. Note that a standard equivalence relation $\mathcal{R}$ is itself a standard Borel space. This follows directly from Theorem 3.1.

The next well known result shows that in some sense the group $[\mathcal{R}]$ is big. A proof is given e.g. in [21, Lemma 2.1].

Lemma 3.28. *Let $\mathcal{R} \subset X \times X$ be an ergodic standard equivalence relation. Then the group $[\mathcal{R}]$ acts transitively up to measure zero on the measurable subsets of X of a fixed measure.*

We will often use the following fact.

Lemma 3.29. *Let $\mathcal{R} \subset X \times X$ be a standard equivalence relation, and $A \subset \mathcal{R}$ be a measurable subset. Then there is a partition $A = \bigcup_{n\in\mathbb{N}} A_n$ into measurable subsets A_n such that both coordinate projections are injective on each A_n.*

Proof. Since the equivalence classes of $\mathcal{R}$ are countable the coordinate projections are countable-to-1 maps. Hence we can first apply Theorem 3.3 to the projection p_1 on the first factor to obtain a partition of A into measurable subsets A'_n such that p_1 is injective on each A'_n. Then we apply Theorem 3.3 again to the projection p_2 on the second factor restricted to each A'_n to get the desired partition. ☐

The most important example of a standard equivalence relation for us is provided by the following lemma.

Lemma 3.30. *Let X be a standard Borel space X with probability measure μ and G be a countable group with an action $G \times X \to X$ on X by measure preserving Borel automorphisms. Then the* orbit equivalence relation

$$\mathcal{R}_{G \curvearrowright X} = \{(x, g.x) : x \in X, g \in G\}$$

is a standard equivalence relation.

Proof. For fixed $g_0 \in G$ the subset $\{(x, g_0.x) : x \in X\}$ is a Borel subset and hence $\mathcal{R}_{G \curvearrowright X} \subset X \times X$ is Borel as a countable union of Borel subsets. The equivalence classes of $\mathcal{R}_{G \curvearrowright X}$ are countable since so is G. Now let $\varphi \colon A \to B$ a Borel isomorphism between Borel subsets $A, B \subset X$ with $\mathrm{gr}(\varphi) \subset \mathcal{R}_{G \curvearrowright X}$. Choose some enumeration $G = \{g_1, g_2, \ldots\}$. Then A is the disjoint countable union $A = \bigcup_{n=1}^{\infty} A_n$ with

$$A_n = \{a \in A : \varphi(a) = g_n.a \text{ and } \varphi(a) \neq g_j.a \text{ for all } j < n\}.$$

Since $G \curvearrowright X$ is measure preserving, we obtain $\mu\big(\varphi(A_n)\big) = \mu(g_n.A_n) = \mu(A_n)$ for all $n \in \mathbb{N}$ and therefore

$$\mu(B) = \sum_{n=1}^{\infty} \mu\big(\varphi(A_n)\big) = \sum_{n=1}^{\infty} \mu(A_n) = \mu(A)$$

since the union $B = \bigcup_{n=1}^{\infty} \varphi(A_n)$ is also disjoint due to bijectivity of φ. ☐

Somehow this is the only example of a standard equivalence relation since there is the following result of Feldman and Moore [18].

Theorem 3.31. *Let $\mathcal{R}$ be a standard equivalence relation on a standard Borel space X. Then there is a countable group G acting on X by measure preserving Borel automorphisms such that $\mathcal{R} = \mathcal{R}_{G \curvearrowright X}$ holds.*

For the proof of Lemma 3.30 we did not need the fact that the action is free. Also in Theorem 3.31 one can not assume freeness of the action. However we will need freeness of the action later, so we restrict our attention to this situation.

Definition 3.32. Let X be a standard Borel space with probability measure μ and G be a countable group. A group action $G \curvearrowright X$ is called a *standard action* if G acts by measure preserving Borel automorphisms and the action is essentially free, i.e. the stabilizer $G_x = \{g \in G : g.x = x\}$ is trivial for almost all $x \in X$. A standard action $G \curvearrowright X$ is called *ergodic* if any G-invariant measurable subset $A \subset X$ satisfies $\mu(A) = 0$ or $\mu(A) = 1$.

Remark 3.33. A standard action $G \curvearrowright X$ is ergodic if and only if the associated orbit equivalence relation $\mathcal{R}_{G \curvearrowright X}$ is ergodic.

Remark 3.34. If $G \curvearrowright X$ is an essentially free action on a standard Borel probability space X, we can easily switch to a free action on a standard Borel probability space X'. Namely, let

$$X_0 = \big\{x \in X : g.x = x \text{ for some } g \in G - \{e\}\big\}$$

and consider $X' = X - X_0$. The subset $X_0 \subset X$ is measurable since it is the countable union

$$X_0 = \bigcup_{g \in G - \{e\}} X(g), \quad X(g) = \{x \in X : g.x = x\}$$

and each $X(g)$ is measurable since it is the preimage of the (measurable) diagonal $\Delta_X = \{(x,x) : x \in X\}$ under the measurable map

$$\overline{l_g} \colon X \to X \times X, \quad x \mapsto (x, g.x).$$

Since the action $G \curvearrowright X$ was assumed to be essentially free the set X_0 has measure zero. As a measurable subspace of a standard Borel space X' is itself a standard Borel space, and the restricted measure on X' is again a probability measure.

Definition 3.35. A standard action $G \curvearrowright X$ is called *mixing* if for each pair of Borel subsets $A, B \subset X$ and each sequence $(g_n)_{n\in\mathbb{N}}$ with $g_n \xrightarrow{n\to\infty} \infty$ the condition

$$\lim_{n\to\infty} \mu(A \cap g_n.B) = \mu(A) \cdot \mu(B) \tag{3.2}$$

holds, where $g_n \xrightarrow{n\to\infty} \infty$ means that for each finite subset $F \subset G$ there exists $N_F \in \mathbb{N}$ with $g_n \notin F$ for all $n \geq N_F$.

Remark 3.36. It is immediately clear that each mixing action is ergodic. Furthermore, it suffices to check (3.2) for measurable subsets of a generating set of the σ-algebra on X.

Lemma 3.37. *Each countable group admits an ergodic standard action.*

Proof. Let G be a countable group. If G is finite then G itself with the normalized counting measure is a standard Borel probability space with a free measure preserving G-action given by the shift $(g,h) \mapsto gh$.

If G is countably infinite consider the standard Borel space $X = \{0,1\}^G$ equipped with the product measure of equipartition on $\{0,1\}$. The shift action

$$\big(g, (\lambda_h)_{h\in G}\big) \mapsto (\lambda_{gh})_{h\in G}$$

is easily seen to be measure preserving. It is not hard to show that the shift action is essentially free, this is done e.g. in [41].

To prove ergodicity of the action $G \curvearrowright \{0,1\}^G$, we show that it is mixing. For a finite subset $F \subset G$ and subsets $A_f \subset \{0,1\}$ for each $f \in F$, we define

$$X_F = \big\{(\lambda_g)_{g\in G} : \lambda_f \in A_f \text{ for all } f \in F\big\}.$$

The collection of those X_F generate the product σ-algebra. For sets of this form, one easily proves the mixing condition (3.2). □

Remark 3.38. The action $G \curvearrowright \{0,1\}^G$ is called *Bernoulli shift action*.

For later use we will now state the *Rohlin Lemma*, a classical and useful result from ergodic theory. A proof can be found e.g. in [1, Theorem 1.5.9, p. 47].

Theorem 3.39. *Let X be non-atomic probability space and $T \colon X \to X$ be a measure preserving ergodic Borel automorphism. Given $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a measurable subset $B \subset X$ such that the following conditions hold:*

(i) The subsets $B, T(B), \ldots, T^{n-1}(B)$ are pairwise disjoint.

(ii) It is $\mu\big(\bigcup_{j=0}^{n-1} T^j(B)\big) > 1 - \varepsilon$.

Remark 3.40. If $G \curvearrowright X$ is an ergodic standard action of an infinite group G on a standard Borel probability space X, then X is non-atomic.

3.3.2 A Measure on a Standard Equivalence Relation

Let $\mathcal{R}$ be a standard equivalence relation. We define a measure on $\mathcal{R}$, which is induced by the measure on X in a natural way. Since $p_1 \colon \mathcal{R} \to X$ is countable-to-1 by Lemma 3.29 there is a measurable partition $\mathcal{R} = \bigcup_{n\in\mathbb{N}} \mathcal{R}_n$ such that both coordinate projections are injective on each $\mathcal{R}_n$. It follows that for a measurable subset $B \in \mathcal{R}$ the function defined by

$$x \mapsto \big|B \cap p_1^{-1}(x)\big| = \sum_{n\in\mathbb{N}} \chi_{p_1(\mathcal{R}_n \cap B)}(x) \tag{3.3}$$

is measurable, where $|A|$ denotes the cardinality of A.

Definition 3.41. Let $\mathcal{R}$ be a standard equivalence relation. The measure ν on $\mathcal{R}$ is given by

$$\nu(B) = \int_X \big|B \cap p_1^{-1}(x)\big| \, d\mu(x)$$

for measurable subsets $B \subset \mathcal{R}$.

Remark 3.42. One could define the measure with respect to both coordinate projections. As a consequence of property (iii) of Definition 3.26 the measure is the same no matter which projection we take. For a proof of this fact see e.g. [41, Lemma 1.9].

The invariance of ν with respect to the coordinate projections will be very useful. It appears for example in the following lemma about products of standard equivalence relations.

Lemma 3.43. *Let $\mathcal{R} \subset X \times X$ and $\mathcal{R}' \subset X' \times X'$ be standard equivalence relations. Then their product*

$$\mathcal{R} \times \mathcal{R}' = \big\{ \big((x, x'), (y, y')\big) : (x, y) \in \mathcal{R},\ (x', y') \in \mathcal{R}' \big\} \tag{3.4}$$

is a standard equivalence relation on $X \times X'$.

Proof. Note that we have identified $\mathcal{R} \times \mathcal{R}' \subset X \times X \times X' \times X'$ with the set defined in (3.4) by the obvious Borel isomorphism which interchanges the second and third factor of the product. Hence it is clear that $\mathcal{R} \times \mathcal{R}'$ is a measurable subset.

For a pair $(x, x') \in X \times X'$ there are only countably many elements $y \in X$ with $(x, y) \in \mathcal{R}$ and only countably many elements $y' \in X'$ with $(x', y') \in \mathcal{R}'$. Consequently the equivalence classes of $\mathcal{R} \times \mathcal{R}'$ are countable.

Let $A, B \subset X \times X'$ be two subsets and $\varphi \colon A \to B$ be a Borel isomorphism such that $\mathrm{gr}(\varphi) \subset \mathcal{R} \times \mathcal{R}'$. The probability measures on X and X' will be denoted by μ and μ' respectively. First we define for each $(x, y) \in \mathcal{R}$ the sets

$$A_{x,y} = \{x' \in X' : (x, x') \in A,\ p_1(\varphi(x, x')) = y\}$$

and

$$B_{x,y} = \{y' \in X' : (y, y') \in B,\ p_1(\varphi^{-1}(y, y')) = x\}.$$

Clearly we have

$$A_x = \{x' \in X' : (x, x') \in A\} = \bigcup_{y \sim x} A_{x,y}$$

and

$$B_y = \{y' \in X' : (y, y') \in B\} = \bigcup_{x \sim y} B_{x,y},$$

where both unions are disjoint. One obtains a Borel isomorphism $\varphi_{x,y} \colon A_{x,y} \to B_{x,y}$ given by $x' \mapsto p_2(\varphi(x, x'))$. Because of $\mathrm{gr}(\varphi) \subset \mathcal{R} \times \mathcal{R}'$ we get $\mathrm{gr}(\varphi_{x,y}) \subset \mathcal{R}'$ and hence by measure invariance of standard equivalence relations $\mu'(A_{x,y}) = \mu'(B_{x,y})$. Now we can conclude

$$\begin{aligned}
\mu \times \mu'(A) &= \int_X \mu'(A_x)\, d\mu(x) \\
&= \int_X \sum_{y \sim x} \mu'(A_{x,y})\, d\mu(x) \\
&= \int_X \sum_{y \sim x} \mu'(B_{x,y})\, d\mu(x) \\
&= \int_X \sum_{x \sim y} \mu'(B_{x,y})\, d\mu(y) \\
&= \int_X \mu'(B_y)\, d\mu(y) \\
&= \mu \times \mu'(B)
\end{aligned}$$

In the step from line 3 to line 4 we used the invariance of the measure ν with respect to both coordinate projections for integrating the function $\mathcal{R} \to \mathbb{R}$ which is given by $(x, y) \mapsto \mu'(B_{x,y})$. □

3.3.3 Definition of $\mathcal{R}$-Spaces

Now we are prepared to give the definition of $\mathcal{R}$-spaces.

Definition 3.44. An $\mathcal{R}$-*space* is an X-space (S, p) together with a measurable map

$$\begin{aligned}
\rho \colon \mathcal{R} \times_X S &\to S \\
((x, y), s) &\mapsto (x, y).s = \rho((x, y), s),
\end{aligned}$$

where $\mathcal{R} \times_X S = \{((x,y),s) \in \mathcal{R} \times S : p(s) = y\}$, satisfying the following conditions:

(i) For all $(x,y) \in \mathcal{R}$, $s \in S_y$ we get $(x,y).s \in S_x$.

(ii) For all $(z,x), (x,y) \in \mathcal{R}$, $s \in S_y$ we get $(z,x).((x,y).s) = (z,y).s$.

(iii) For all $y \in X$, $s \in S_y$ we get $(y,y).s = s$.

The map ρ is called *$\mathcal{R}$-action*. An *$\mathcal{R}$-map* of two $\mathcal{R}$-spaces (S,p) and (T,q) is an X-map F satisfying $F((x,y).s) = (x,y).F(s)$ for all $(x,y) \in \mathcal{R}$ and $s \in S_y$.

The above definition was that of a *left* $\mathcal{R}$-space. One can define right $\mathcal{R}$-spaces as well, but we will deal with left $\mathcal{R}$-spaces only. Note that the $\mathcal{R}$-action on (S,p) induces a group action of $[\mathcal{R}]$ on S given by

$$\varphi.s = (\varphi(p(s)), p(s)).s.$$

Example 3.45. The first example of an $\mathcal{R}$-space is $\mathcal{R}$ itself, where the $\mathcal{R}$-action is given by $(x,y).(y,z) = (x,z)$. A little more general, for a Borel subset $A \subset X$ the *restricted equivalence relation*

$$\mathcal{R}|_A = \{(x,y) \in \mathcal{R} : y \in A\}$$

is an $\mathcal{R}$-space with the same action.

The next lemma provides another class of examples of $\mathcal{R}$-spaces which will be crucial for our applications.

Lemma 3.46. *Let G be a countable group. Then there is a standard action $G \curvearrowright X$ on a standard Borel space X and a functor*

$$\mathrm{ind}\colon G\text{-Spaces} \to \mathcal{R}_{G \curvearrowright X}\text{-Spaces},$$

where G-Spaces denotes the category of σ-compact, second countable Hausdorff spaces with an action of G and $\mathcal{R}_{G \curvearrowright X}$-Spaces denotes the category of $\mathcal{R}_{G \curvearrowright X}$-spaces.

Proof. By Lemma 3.37 there is a standard action of the group G on some standard Borel space X. We can use Remark 3.34 to obtain a free action.

Let Z be a G-space and define $\mathrm{ind}(Z) = X \times Z$. With the obvious projection and the $\mathcal{R}_{G \curvearrowright X}$-action defined by

$$(g.x, x).(x, m) = (g.x, g.m)$$

this is an $\mathcal{R}_{G \curvearrowright X}$ space. G-maps between G-spaces induce $\mathcal{R}_{G \curvearrowright X}$-maps between the corresponding $\mathcal{R}_{G \curvearrowright X}$-spaces in an obvious way. □

Remark 3.47. An example of an $\mathcal{R}$-map which is not induced by a G-map is provided in the proof of Theorem 4.36.

As in the case of X-spaces we can build products.

Definition 3.48. Let $\mathcal{R} \subset X \times X$ and $\mathcal{R}' \subset X' \times X'$ be two standard equivalence relations. If (S, p) is an $\mathcal{R}$-space and (T, q) is an $\mathcal{R}'$-space, then their *cartesian product* $(S \times T, p \times q)$ is an $\mathcal{R} \times \mathcal{R}'$-space, where the action is given by

$$\big((x, x'), (y, y')\big).(s, t) = \big((x, y).s, (x', y').t\big).$$

3.4 Compactly Generated Spaces

We will collect some facts which we need later on about compactly generated spaces, the category of topological spaces we are working in. We will see that this category is very convenient for a lot of constructions and will be useful for us in particular in Section 4.4 when we study $\mathcal{R}$-CW-complexes. The category of compactly generated spaces was introduced by Steenrod [42]. Most of the following facts are taken from [48, I.4] and [13, VI.6] which are good introductions to compactly generated spaces.

Definition 3.49. A topological space Z is called *compactly generated* if it is a Hausdorff space and each subset $A \subset Z$ with the property that $A \cap K \subset Z$ is closed for every compact subset $K \subset Z$ is itself closed in Z.

The category of compactly generated spaces and continuous maps will be denoted by $\mathcal{K}$. If we denote the category of Hausdorff spaces by $\mathcal{T}_2$, then there is a functor $k\colon \mathcal{T}_2 \to \mathcal{K}$ which does not change the underlying set of a Hausdorff space Z. A subset $A \subset Z$ is closed in $k(Z)$ if and only if $A \cap K$ is closed in Z for every compact subset $K \subset Z$. For a continuous map $f\colon Z_1 \to Z_2$, the map $k(f)$ is defined to be the same function considered as a map $k(Z_1) \to k(Z_2)$. One easily checks that $k(Z)$ is compactly generated and $k(f)$ is continuous. If Z is compactly generated, one has $k(Z) = Z$.

The functor k is right adjoint to the inclusion functor $\mathcal{K} \to \mathcal{T}_2$. In particular, for compactly generated Z_1 and continuous $f\colon Z_1 \to Z_2$ the map $k(f)\colon Z_1 \to k(Z_2)$ is also continuous. This implies that the singular chain complexes of Z and $k(Z)$ agree and therefore also their singular homology and cohomology.

In general, the cartesian product of two compactly generated spaces is not compactly generated. One resolves this defect by defining $Z_1 \times_{\mathcal{K}} Z_2 = k(Z_1 \times Z_2)$. It can be shown that $\times_{\mathcal{K}}$ is the categorical product in $\mathcal{K}$. We will need the following result about products of identifications which is proved e.g. in [13, Satz (6.13), p. 223]. Recall that an *identification* $f\colon Z_1 \to Z_2$ is a surjective map such that $A \subset Z_2$ is closed if and only if $f^{-1}(A) \subset Z_1$ is closed.

Lemma 3.50. *Let Z_1 and Z_1' be compactly generated and let $f\colon Z_1 \to Z_2$ and $f'\colon Z_1' \to Z_2'$ be identifications. Then the product $f \times f'\colon Z_1 \times_{\mathcal{K}} Z_1' \to Z_2 \times_{\mathcal{K}} Z_2'$ is an identification.*

We will now prove that the Borel σ-algebra of an X-space S is the same as that of $k(S)$. This is implied by the following lemma since an X-space is σ-compact and Hausdorff by part of its definition.

Lemma 3.51. *Let Z be σ-compact and Hausdorff. Then the Borel σ-algebra of Z and that of $k(Z)$ coincide.*

Proof. It suffices to show that a closed subset $A \subset k(Z)$ is contained in the Borel σ-algebra of Z. Since Z is σ-compact, there is a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets $K_n \subset Z$ with $Z = \bigcup_{n=1}^{\infty} K_n$. If $A \subset k(Z)$ is closed, then $A \cap K_n$ is closed in Z for all $n \in \mathbb{N}$ by definition. Consequently, $A = \bigcup_{n=1}^{\infty} A \cap K_n$ is an element of the Borel σ-algebra of Z since it is the countable union of closed subsets. □

A subspace $A \subset Z$ of a compactly generated space is not necessarily compactly generated. Again, one uses the functor k to overcome this problem and considers $k(A)$. One can show that $k(i)\colon k(A) \to Z$, where $i\colon A \to Z$ is the inclusion, has the formal properties of a subspace in $\mathcal{K}$, i.e. for a compactly generated space Z', a map $f\colon Z' \to k(A)$ is continuous if and only if $k(i) \circ f$ is continuous [13, Satz (6.7), p. 221].

The preceding arguments show that k becomes a functor from the category of X-spaces to the category of compactly generated X-spaces. First, $X = k(X)$ since metrizable spaces are compactly generated [13, Satz (6.1)(1), p. 218]. The projection $p\colon S \to X$ of an X-space (S, p) remains measurable since the σ-algebra of S does not change under k because of Lemma 3.51. By the same reason, an X-map $f\colon S \to T$ remains measurable if regarded as a map $k(S) \to k(T)$. For $x \in X$, the fiber over x is $k(S_x)$ by definition of the subspace topology in $\mathcal{K}$. Since k is a functor $\mathcal{T}_2 \to \mathcal{K}$ and for almost all $x \in X$ the map $f|_{S_x}\colon S_x \to T_x$ is continuous, the map $k(f)|_{k(S_x)}$ is also continuous for almost all $x \in X$.

Lemma 3.52. *Let S be an X-space. Then the singular X-simplices of S are precisely the X-simplices of $k(S)$.*

Proof. The domain $X_\sigma \times \Delta^n$ of an X-simplex σ is compactly generated, since the product of a compactly generated space and a locally compact space is compactly generated [13, Satz (6.8), p. 221] and X_σ is compactly generated as a metrizable space, whereas Δ^n is compact. Thus $k(X_\sigma \times \Delta^n) = X_\sigma \times \Delta^n$, and it follows that an X-map

$$\sigma\colon X_\sigma \times \Delta^n \to S$$

induces an X-map $X_\sigma \times \Delta^n \to k(S)$ by the arguments above. The other direction is clear since the topology of $k(S)$ is finer than that of S. □

4 L^2-Betti Numbers of $\mathcal{R}$-Spaces

In this chapter, we will define L^2-Betti numbers of $\mathcal{R}$-spaces and compare the L^2-Betti numbers of a G-space with those of the induced $\mathcal{R}_{G \curvearrowright X}$-space. First we need some algebraic ingredients analogous to the group ring and the group von Neumann algebra. We define the equivalence relation ring $\Lambda\mathcal{R}$ for subrings $\Lambda \subset \mathbb{C}$, closed under complex conjugation. In analogy to the group-situation, the equivalence relation von Neumann algebra $\mathcal{N}\mathcal{R}$ arises as a completion of $\mathbb{C}\mathcal{R}$. There is a trace $\mathrm{tr}_{\mathcal{N}\mathcal{R}} \colon \mathcal{N}\mathcal{R} \to \mathbb{C}$ turning $\mathcal{N}\mathcal{R}$ into a finite von Neumann algebra. The notion of the equivalence relation von Neumann algebra is due to Feldman and Moore [18]. The definition of L^2-Betti numbers of $\mathcal{R}$-spaces is based on Lück's algebraic definition of ordinary L^2-Betti numbers. We show that $\mathcal{R}$-homology (i.e. $H_n\big(\mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S;\mathbb{Z})\big)$) satisfies adapted Eilenberg-Steenrod axioms up to dimension isomorphisms. Furthermore, $\mathcal{R}$-CW-complexes will be considered as a natural analog of G-CW-complexes. We will prove that the L^2-Betti numbers of a countable free G-CW-complex Z and those of the induced $\mathcal{R}_{G \curvearrowright X}$-space $X \times Z$ coincide. As an application, we provide another proof for Gaboriau's theorem about the orbit equivalence invariance of L^2-Betti numbers.

4.1 The von Neumann Algebra of a Standard Equivalence Relation

The following construction should be thought of as an analog to the group ring.

4.1.1 The Equivalence Relation Ring

In the following, we will always tacitly use the measure ν on $\mathcal{R}$. This measure was introduced in Definition 3.41.

Definition 4.1. Let $\mathcal{R}$ be a standard equivalence relation and $\Lambda \subset \mathbb{C}$ be a subring, which is closed under complex conjugation. Then the *equivalence relation ring* $\Lambda\mathcal{R}$ is defined by

$$\begin{aligned}\Lambda\mathcal{R} = \big\{&\eta \in L^\infty(\mathcal{R},\Lambda) : \\ &\exists n \in \mathbb{N}\ \forall x \in X : |\{y : \eta(x,y) \neq 0\}|, |\{y : \eta(y,x) \neq 0\}| \leq n\big\}.\end{aligned}$$

The addition on $\Lambda\mathcal{R}$ is given by pointwise addition, whereas the multiplication is given by

$$\eta \cdot \rho(x,y) = \sum_{z\sim x} \eta(x,z) \cdot \rho(z,y).$$

Define an involution on $\Lambda\mathcal{R}$ by $\overline{\eta}(x,y) = \overline{\eta(y,x)}$.

Remark 4.2. There is an embedding of rings

$$j\colon L^\infty(X,\Lambda) \to \Lambda\mathcal{R} \tag{4.1}$$

given by restriction to the diagonal. This means that $j(f)$ maps (x,x) to $f(x)$ and vanishes outside the diagonal. We will usually omit the j from the notation and simply regard $f \in L^\infty(X,\Lambda)$ as an element of $\Lambda\mathcal{R}$.

Definition 4.3. The *augmentation morphism* $\varepsilon\colon \Lambda\mathcal{R} \to L^\infty(X,\Lambda)$ is defined by

$$\varepsilon(\eta)(x) = \sum_{y\sim x} \eta(x,y).$$

It is clear that $\varepsilon \circ j = \mathrm{id}_{L^\infty(X,\Lambda)}$ holds. Hence $L^\infty(X,\Lambda)$ is a direct summand of $\Lambda\mathcal{R}$. The augmentation morphism ε can be used to define an $\Lambda\mathcal{R}$-module structure on $L^\infty(X,\Lambda)$.

Definition 4.4. For $f \in L^\infty(X,\Lambda)$ and $\eta \in \Lambda\mathcal{R}$ we define

$$\eta \cdot f = \varepsilon\big(\eta \cdot f\big).$$

This defines a left $\Lambda\mathcal{R}$-module structure on $L^\infty(X,\Lambda)$.

Using the involution on $\Lambda\mathcal{R}$ we define a right $\Lambda\mathcal{R}$-module structure on $L^\infty(X,\Lambda)$ by $f \cdot \eta = \overline{\eta} \cdot f$.

The augmentation morphism becomes a $\Lambda\mathcal{R}$-morphism if $L^\infty(X,\Lambda)$ is endowed with the $\Lambda\mathcal{R}$-module structure of Definition 4.4.

Remark 4.5. Note that $L^\infty(X,\Lambda)$ is *not* an $L^\infty(X,\Lambda)$-$\Lambda\mathcal{R}$-bimodule since in general

$$(f \cdot g) \cdot \eta\ (x) = \sum_{y\sim x} \overline{\eta(y,x)} \cdot f(y) \cdot g(y) \neq \sum_{y\sim x} \overline{\eta(y,x)} \cdot f(x) \cdot g(y) = f \cdot (g \cdot \eta)\ (x).$$

Remark 4.6. There is an injective map $I\colon [\mathcal{R}] \to \mathbb{Z}\mathcal{R}$ given by

$$I(\varphi)(x,y) = \begin{cases} 1, & \varphi(y) = x \\ 0, & \text{otherwise.} \end{cases}$$

In other words, I is given by $\varphi \mapsto \chi_{\mathrm{gr}(\varphi^{-1})}$.

The following lemma about the elements of the equivalence relation ring will be quite useful later on.

Lemma 4.7. *For each $\eta \in \Lambda\mathcal{R}$ there are pairwise disjoint Borel subsets $E_1, \dots, E_k \subset \mathcal{R}$ and measurable maps $f_1, \dots, f_k \in L^\infty(X, \Lambda)$ with the property that both coordinate projections restricted to each E_j are injective and*

$$\eta = \sum_{j=1}^{k} f_j \cdot \chi_{E_j}$$

holds.

Proof. The coordinate projections $p_1, p_2\colon \operatorname{supp}(\eta) \to X$ are finite-to-1 maps between standard Borel spaces, since $\operatorname{supp}(\eta)$ is a measurable subset of the standard Borel space $\mathcal{R}$ and therefore itself standard Borel. By Lemma 3.29 one gets a finite partition of $\operatorname{supp}(\eta)$ into Borel sets $E_1, \dots, E_k$ such that p_1, p_2 restricted to each E_j are injective. For $x \in p_1(E_j)$ let $y_{x,j}$ be the unique point in X with $(x, y_{x,j}) \in E_j$ and define

$$f_j\colon X \to \Lambda, \qquad x \mapsto \begin{cases} \eta(x, y_{x,j}), & x \in p_1(E_j) \\ 0, & \text{otherwise.} \end{cases}$$

By direct computation one checks that $\eta = \sum_{j=1}^{k} f_j \cdot \chi_{E_j}$ holds. □

Remark 4.8. In the important special case $\mathcal{R} = \mathcal{R}_{G \curvearrowright X}$ there is a close relation between the group ring ΛG and the equivalence relation ring $\Lambda\mathcal{R}_{G \curvearrowright X}$, namely there is a ring monomorphism

$$\Lambda G \to \Lambda\mathcal{R}_{G \curvearrowright X}$$

which maps $\sum_{g \in G} \lambda_g \cdot g$ to the function $\mathcal{R}_{G \curvearrowright X} \to \Lambda$ given by

$$(g_0.x, x) \mapsto \lambda_{g_0}.$$

4.1.2 Definition of the Equivalence Relation von Neumann Algebra

In analogy to the group ring situation we can embed $\mathbb{C}\mathcal{R}$ into the bounded linear operators on $L^2(\mathcal{R})$ in two ways, i.e. there are linear maps $\rho_l\colon \mathbb{C}\mathcal{R} \to L\big(L^2(\mathcal{R})\big)$ given by

$$\rho_l(\phi)(\psi)(x,y) = \sum_{z \sim x} \phi(x,z) \cdot \psi(z,y)$$

and $\rho_r\colon \mathbb{C}\mathcal{R} \to L\big(L^2(\mathcal{R})\big)$ given by

$$\rho_r(\phi)(\psi)(x,y) = \sum_{z \sim x} \psi(x,z) \cdot \phi(z,y).$$

Definition 4.9. The von Neumann algebra $\mathcal{N}\mathcal{R}$ of a standard equivalence relation $\mathcal{R}$ is defined as the weak closure of $\rho_r(\mathbb{C}\mathcal{R})$ in $L\big(L^2(\mathcal{R})\big)$.

There is a trace $\operatorname{tr}_{\mathcal{N}\mathcal{R}}\colon \mathcal{N}\mathcal{R} \to \mathbb{C}$ on $\mathcal{N}\mathcal{R}$ defined by

$$\operatorname{tr}_{\mathcal{N}\mathcal{R}}(T) = \big\langle T(\chi_{\Delta_X}), \chi_{\Delta_X} \big\rangle_{L^2(\mathcal{R})}$$

where $\Delta_X \subset \mathcal{R}$ is the diagonal in $X \times X$ and χ_{Δ_X} is its characteristic function.

In [41, Theorem 1.46] it is proved that $\mathrm{tr}_{\mathcal{N}\mathcal{R}}$ is in fact a trace on $\mathcal{N}\mathcal{R}$. The following lemma is also proved in [41, Corollary 1.54].

Lemma 4.10. *The ring homomorphism $\mathbb{C}G \to \mathbb{C}\mathcal{R}_{G\curvearrowright X}$ defined in Remark 4.8 extends to a trace preserving $*$-homomorphism $\mathcal{N}G \to \mathcal{N}\mathcal{R}_{G\curvearrowright X}$.*

Since any trace preserving $*$-homomorphism of von Neumann algebras is a flat ring extension by Lemma 1.8, we get the following consequence.

Corollary 4.11. *The von Neumann algebra $\mathcal{N}\mathcal{R}_{G\curvearrowright X}$ is a flat $\mathcal{N}G$-module.*

4.2 L^2-Betti Numbers of $\mathcal{R}$-Spaces

Recall that in the case of an ordinary topological space X with the action of a group G we used the fact that the group action induces a $\mathbb{Z}G$-module structure on the singular chain complex $C_\bullet(X;\mathbb{Z})$ and considered the complex $\mathcal{N}G \otimes_{\mathbb{Z}G} C_\bullet(X;\mathbb{Z})$.

In order to proceed similarly in the situation of an $\mathcal{R}$-space (S,p) we first have to define a $\mathbb{Z}\mathcal{R}$-module structure on the singular chain complex $C_\bullet^X(S;\mathbb{Z})$.

4.2.1 A $\mathbb{Z}\mathcal{R}$-Module Structure on the Chain Complex of an $\mathcal{R}$-Space

The $\mathcal{R}$-action on (S,p) induces a group action of $[\mathcal{R}]$ on $\Sigma_n^X(S)$. Namely, for $\varphi \in [\mathcal{R}]$ and a singular X-simplex $\sigma\colon X_\sigma \times \Delta^n \to S$ we define an X-simplex by

$$\begin{aligned}\varphi.\sigma\colon \varphi(X_\sigma) \times \Delta^n &\to S \\ (x,t) &\mapsto \big(x,\varphi^{-1}(x)\big).\sigma\big(\varphi^{-1}(x),t\big).\end{aligned}$$

In the same way we can define an action of partial Borel isomorphisms $\psi\colon A \to B$ between Borel subsets $A,B \subset X$ with $\mathrm{gr}(\psi) \subset \mathcal{R}$ on $\sigma\colon X_\sigma \times \Delta^n \to S$. The domain of $\psi.\sigma$ is given by $B \cap \psi(X_\sigma \cap A)$, while the image of $(x,t) \in B \cap \psi(X_\sigma \cap A)$ is the same as above.

This action induces a $\mathbb{Z}\mathcal{R}$-module structure on $C_\bullet^X(S;\mathbb{Z})$ in the following way: Due to Lemma 4.7 each function $\eta \in \mathbb{Z}\mathcal{R}$ can be written as a finite sum $\sum_{j=1}^k f_j \cdot \chi_{E_j}$ with $f_j \in L^\infty(X,\mathbb{Z})$ and pairwise disjoint measurable subsets $E_j \subset \mathcal{R}$ such that both coordinate projections are injective on each E_j. Hence each E_j induces a Borel isomorphism $\psi_j\colon p_2(E_j) \to p_1(E_j)$ with $\mathrm{gr}(\phi_j) \subset \mathcal{R}$. We get $\eta = \sum_{j=1}^k f_j \cdot \chi_{\mathrm{gr}(\psi_j^{-1})}$.

For $\sigma \in \Sigma_n^X(S)$ define

$$\eta \cdot \sigma = \sum_{j=1}^k f_j \cdot (\psi_j.\sigma).$$

This is well defined since for almost all $x \in X$ one has

$$(\eta \cdot \sigma)_x = \sum_{\substack{y \in X_\sigma \\ y \sim x}} \eta(x,y) \cdot (x,y).\sigma_y.$$

Note that we obtain the equation $I(\eta) \cdot \sigma = \eta.\sigma$ (where $I\colon [\mathcal{R}] \to \mathbb{Z}\mathcal{R}$ was the embedding defined in Remark 4.6).

The $\mathbb{Z}\mathcal{R}$-module structure on $C_n^X(S)$ is now given by

$$\eta \cdot \sum_\sigma f_\sigma \cdot \sigma = \sum_\sigma (\eta \cdot f_\sigma) \cdot \sigma.$$

The assignment $(S, p) \mapsto C_n^X(S)$ becomes a functor

$$\mathcal{R}\text{-Spaces} \to \mathbb{Z}\mathcal{R}\text{-Mod},$$

where an $\mathcal{R}$-map $F\colon S \to T$ yields the $\mathbb{Z}\mathcal{R}$-module morphism $F_\#\colon C_n^X(S) \to C_n^X(T)$ induced by $\sigma \mapsto F \circ \sigma$.

This construction turns $\{C_n^X(S)\}_n$ into a chain complex of $\mathbb{Z}\mathcal{R}$-modules.

4.2.2 Definition of L^2-Betti Numbers of $\mathcal{R}$-Spaces

After the technical preparation we are now ready to define L^2-Betti numbers of $\mathcal{R}$-spaces.

Definition 4.12. Let S be an $\mathcal{R}$-space. The complex

$$\mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S; \mathbb{Z})$$

of $\mathcal{N}\mathcal{R}$-modules is called the *singular $\mathcal{R}$-complex* of S. Here $\mathcal{N}\mathcal{R}$ is equipped with the standard $\mathcal{N}\mathcal{R}$-$\mathbb{Z}\mathcal{R}$-bimodule structure. We will sometimes abbreviate

$$C_\bullet^{\mathcal{R}}(S; \mathcal{N}\mathcal{R}) = \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S; \mathbb{Z}).$$

The homology of $C_\bullet^{\mathcal{R}}(S; \mathcal{N}\mathcal{R})$ is called the *singular L^2-homology* of S and is denoted by $H_\bullet^{\mathcal{R}}(S; \mathcal{N}\mathcal{R})$. The *$n$-th L^2-Betti number* of S is defined as

$$b_n^{(2)}(S; \mathcal{N}\mathcal{R}) = \dim_{\mathcal{N}\mathcal{R}}\big(H_n^{\mathcal{R}}(S; \mathcal{N}\mathcal{R})\big).$$

The homology of $\mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S, T; \mathbb{Z})$ will be denoted by $H_\bullet^{\mathcal{R}}(S, T; \mathcal{N}\mathcal{R})$.

We will next establish some properties of L^2-homology of $\mathcal{R}$-spaces. Our main goal will be the equality

$$b_n^{(2)}(Z; \mathcal{N}G) = b_n^{(2)}(X \times Z; \mathcal{N}\mathcal{R}_{G \curvearrowright X}) \quad \text{for all} \quad n \geq 0$$

for G-spaces Z and standard Borel spaces X with a standard action $G \curvearrowright X$.

4.3 $\mathcal{R}$-Homology

In analogy to G-homology theorieswe will define $\mathcal{R}$-homology theories for $\mathcal{R}$-spaces and show that $H_\bullet^{\mathcal{R}}(.,.; \mathcal{N}\mathcal{R})$ is "nearly" such a theory. Let $\mathcal{R}$-Spaces^2 denote the category of pairs of $\mathcal{R}$-spaces.

4.3.1 $\mathcal{R}$-Homology Theories

The definition of $\mathcal{R}$-homology theories is motivated by the Eilenberg-Steenrod axioms for homology theories.

Definition 4.13. An *$\mathcal{R}$-homology theory* with values in an abelian category $\mathcal{A}$ is a collection of functors

$$\mathcal{H}_n \colon \mathcal{R}\text{-Spaces}^2 \to \mathcal{A}$$

such that the following conditions hold:

- *Long exact homology sequence of a pair*
  If $T \subset S$ is a pair of $\mathcal{R}$-spaces and T is fiberwise open or fiberwise closed in S, then there is a natural long exact sequence of the form

  $$\ldots \to \mathcal{H}_{n+1}(S,T) \to \mathcal{H}_n(T) \to \mathcal{H}_n(S) \to \mathcal{H}_n(S,T) \to \ldots .$$

- *Homotopy invariance*
  If $f, g \colon S \to T$ are $\mathcal{R}$-homotopic $\mathcal{R}$-maps, then $\mathcal{H}_\bullet(f) = \mathcal{H}_\bullet(g)$ holds.

- *Excision*
  If S is an $\mathcal{R}$-space and $A \subset B \subset S$ are $\mathcal{R}$-subspaces such that $\overline{A_x} \subset B_x^\circ$ holds for almost all $x \in X$, then the inclusion $(S - A, B - A) \hookrightarrow (S, B)$ induces an isomorphism

  $$\mathcal{H}_\bullet(S - A, B - A) \xrightarrow{\cong} \mathcal{H}_\bullet(S, B).$$

- *Countable Additivity*
  If I is a countable index set and S_α is an $\mathcal{R}$-space for each $\alpha \in I$, then the canonical map

  $$\bigoplus_{\alpha \in I} \mathcal{H}_\bullet(S_\alpha) \to \mathcal{H}_\bullet\big(\coprod_{\alpha \in I} S_\alpha\big)$$

  is an isomorphism.

The homology theory $\mathcal{H}_\bullet$ satisfies the *dimension axiom* if $\mathcal{H}_n(\mathcal{R}) = 0$ holds for each $n \neq 0$.

The following observation is obvious.

Lemma 4.14. *Let G be a group and $G \curvearrowright X$ be a standard action. Then an $\mathcal{R}$-homology theory $\mathcal{H}_\bullet$ with values in $\mathcal{A}$ yields a G-homology theory with values in $\mathcal{A}$ via*

$$(Z, Y) \mapsto \mathcal{H}_\bullet\big(\operatorname{ind}(Z), \operatorname{ind}(Y)\big),$$

where ind *is the induction functor defined in Lemma 3.46 and (Z, Y) is a pair of G-spaces.*

4.3.2 L^2-Homology is an $\mathcal{R}$-Homology Theory up to Dimension

It turns out that L^2-homology is not an $\mathcal{R}$-homology theory with values in $\mathcal{NR}$-Mod, but it is one modulo dimension. More precisely, L^2-homology becomes an $\mathcal{R}$-homology theory with values in the quotient category $\mathcal{NR}$-Mod$/\mathcal{NR}$-Mod$_0$ (compare Remark 1.16), where dimension isomorphisms become isomorphisms. We will give a proof of this fact now.

Theorem 4.15. *Singular L^2-homology $H^{\mathcal{R}}_{\bullet}(.,.;\mathcal{NR})$ is an $\mathcal{R}$-homology theory modulo dimension, i.e. the composite functor*

$$(S,T) \mapsto \pi\big(H^{\mathcal{R}}_{\bullet}(S,T;\mathcal{NR})\big)$$

is an $\mathcal{R}$-homology theory with values in $\mathcal{NR}$-Mod$/\mathcal{NR}$-Mod$_0$, where π is the exact functor defined in Remark 1.16. The dimension axiom holds with $H^{\mathcal{R}}_0(\mathcal{R};\mathcal{NR}) \cong_{\dim} \mathcal{NR}$.

Proof. First we prove the existence of the long exact homology sequence. By definition, the sequence of $\mathbb{Z}\mathcal{R}$-chain complexes

$$0 \to C^X_{\bullet}(T;\mathbb{Z}) \to C^X_{\bullet}(S;\mathbb{Z}) \to C^X_{\bullet}(S,T;\mathbb{Z}) \to 0$$

is exact. The first map does not split as a chain map, but it does split degreewise. Namely, for a chain in $C^X_n(S;\mathbb{Z})$ represented by $\sum_{j=1}^k f_j \cdot \sigma_j$ we consider the subsets

$$A_j = \big\{x \in X : \mathrm{im}(\sigma_{j,x}) \subset T\big\}. \tag{4.2}$$

Measurability of A_j follows from Lemma 3.17. The chain $\sum_{j=1}^k (\chi_{A_j} \cdot f_j) \cdot \sigma_j$ represents an element of $C^X_n(T;\mathbb{Z})$. This procedure yields a split of the inclusion map $C^X_n(T;\mathbb{Z}) \to C^X_n(S;\mathbb{Z})$. It follows that the first map in the sequence of $\mathcal{NR}$-chain complexes

$$0 \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C^X_{\bullet}(T;\mathbb{Z}) \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C^X_{\bullet}(S;\mathbb{Z}) \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C^X_{\bullet}(S,T;\mathbb{Z}) \to 0$$

is injective. Since the sequence is exact in the middle and on the right due to general homological algebra it is overall exact. Hence there is a long exact homology sequence for L^2-homology.

For the proof of homotopy invariance we want to imitate the proof in the classical situation (which can be found in any textbook on algebraic topology, e.g. in [6]). Let Y be a topological space. There is a natural map

$$\begin{aligned} D\colon C_n(Y;\mathbb{Z}) &\to C_{n+1}(Y \times I;\mathbb{Z}) \\ c &\mapsto c \times \mathrm{id}_{\Delta^1} \end{aligned}$$

satisfying $d \circ D + D \circ d = i_1 - i_0$, where (by abuse of notation) d denotes the boundary in $C_{\bullet}(Y;\mathbb{Z})$ and $C_{\bullet}(Y \times I;\mathbb{Z})$ as well and i_0, i_1 are induced by the obvious inclusions $Y \to Y \times I$. Moreover, there are integers $a_1, \ldots, a_k \in \mathbb{Z}$ and singular simplices $\tau_1, \ldots, \tau_k\colon \Delta^{n+1} \to \Delta^n \times I$ such that $D(\mathrm{id}_{\Delta^n}) = \sum_{j=1}^k a_j \cdot \tau_j$. By naturality of D we get

$$D(\tau) = \sum_{j=1}^{k} a_j \cdot (\tau \times \mathrm{id}_I) \circ \tau_j \tag{4.3}$$

for an arbitrary simplex $\tau\colon \Delta^n \to Y$. Now we use (4.3) to define the "X-version" of D. For an $\mathcal{R}$-space S the $\mathbb{Z}\mathcal{R}$-homomorphism $D^X\colon C_n^X(S;\mathbb{Z}) \to C_{n+1}^X(S\times I;\mathbb{Z})$ is given by

$$\sigma \mapsto \sum_{j=1}^{k} a_j \cdot (\sigma \times \mathrm{id}_I) \circ (\mathrm{id}_{X_\sigma} \times \tau_j),$$

for an X-simplex $\sigma\colon X_\sigma \times \Delta^n \to S$ in S and linear extension. Note that one has to be a little careful with linear extension in $C_n^X(S;\mathbb{Z})$ since it is not a free $L^\infty(X)$-module. To see that D^X is nevertheless well defined let $\sum_{l=1}^r f_l \cdot \sigma_l$ with $f_1,\dots,f_r \in L^\infty(X;\mathbb{Z})$ and singular X-simplices $\sigma_1,\dots,\sigma_r$ represent an element of $C_n^X(S;\mathbb{Z})$. Then we get for almost all $x\in X$ the equation

$$D^X\Big(\big[\sum_{l=1}^r f_l\cdot\sigma_l\big]\Big)_x = \sum_{l=1}^r\sum_{j=1}^k f_l(x)\cdot a_j\cdot\big((\sigma_l)_x\times\mathrm{id}_I\big)\circ\tau_j = D\Big(\sum_{l=1}^r f_l(x)\cdot(\sigma_l)_x\Big). \quad (4.4)$$

Hence D^X is well defined. Equation (4.4) also yields

$$d\circ D^X + D^X\circ d = i_1 - i_0,$$

where d now denotes the boundary operator of both $C_\bullet^X(S;\mathbb{Z})$ and $C_\bullet^X(S\times I;\mathbb{Z})$ and i_0, i_1 are induced by the inclusions $S\to S\times I$. Now homotopy invariance follows easily: Let $H\colon S\times I\to T$ be an $\mathcal{R}$-homotopy between $\mathcal{R}$-maps $f_0,f_1\colon S\to T$. Then we observe

$$d\circ(H_\Delta\circ D^X) + (H_\Delta\circ D^X)\circ d = (f_1)_\Delta - (f_0)_\Delta,$$

where the subscript Δ indicates the induced map on the singular chain complex. Hence $H_\Delta\circ D^X$ is a chain homotopy between $(f_0)_\Delta$ and $(f_1)_\Delta$. This property persists after tensoring with $\mathcal{N}\mathcal{R}$ and consequently f_0 and f_1 induce the same homomorphism in L^2-homology.

Next we are going to prove the excision axiom. Here we really have to work in the quotient category $\mathcal{N}\mathcal{R}$-Mod$/\mathcal{N}\mathcal{R}$-Mod$_0$ since the map which is supposed to be an isomorphism by the axioms will only be a $\dim_{\mathcal{N}\mathcal{R}}$-isomorphism, which becomes an isomorphism in the quotient category.

We first recall the proof in the classical situation since we are going to copy it: One defines a natural chain map $Y_\bullet\colon C_\bullet(Z;\mathbb{Z})\to C_\bullet(Z;\mathbb{Z})$ (the so called subdivision chain map) which is chain homotopic to the identity by a natural chain homotopy $F_\bullet\colon C_\bullet(Z;\mathbb{Z})\to C_{\bullet+1}(Z;\mathbb{Z})$. The maps Y and F have the property that for each singular simplex $\tau\colon\Delta^n\to Z$ the chains $Y_n(\tau)$ and $F_n(\tau)$ are built up by simplices with image in $\mathrm{im}(\tau)$. Furthermore, if Z is covered by a finite family $\mathcal{V}=\{V_1,\dots,V_l\}$ of open subsets, then there is a positive integer k such that $Y_n^k(\tau)\in C_n^{\mathcal{V}}(Z;\mathbb{Z})$, where $C_\bullet^{\mathcal{V}}(Z;\mathbb{Z})$ is the subcomplex of $C_\bullet(Z;\mathbb{Z})$ generated by all singular simplices with image in one of the elements of $\mathcal{V}$. By $F_\bullet^k$ we denote a natural chain homotopy between $Y_\bullet^k$ and $\mathrm{id}_{C_\bullet(Z;\mathbb{Z})}$.

To get the X-versions of $Y^k_\bullet$ and $F^k_\bullet$ let $Y^k_n(\mathrm{id}_{\Delta^n}) = \sum_{j=1}^r a_j \cdot \tau_j \in C_n(\Delta^n;\mathbb{Z})$ and define

$$Y_n^{X,k} \colon \sigma \mapsto \sum_{j=1}^r a_j \cdot \sigma \circ (\mathrm{id}_{X_\sigma} \times \tau_j)$$

for an X-simplex $\sigma\colon X_\sigma \times \Delta^n \to S$ in an $\mathcal{R}$-space S. One proceeds similarly to define $F_n^{X,k}\colon C_n^X(S;\mathbb{Z}) \to C_{n+1}^X(S;\mathbb{Z})$. Again it is easy to see that $Y_\bullet^{X,k}$ and $F_\bullet^{X,k}$ extend in a well defined way to $\mathbb{Z}\mathcal{R}$-homomorphisms on $C_\bullet^X(S;\mathbb{Z})$ by linearity and we get $Y_n^{X,k}(c)_x = Y_n^k(c_x)$ as well as $F_n^{X,k}(c) = F_n^k(c_x)$ for almost all $x \in X$ where c is an X-chain in $C_n^X(S;\mathbb{Z})$. The properties of the classical maps $Y^k_\bullet$ and $F^k_\bullet$ imply

$$Y_n^{X,k}(c) - c = F_{n-1}^{X,k} \circ d(c) + d \circ F_n^{X,k}(c).$$

For the proof of the excision axiom we first take a finite family $\mathcal{U} = \{U_1, \dots, U_r\}$ of $\mathcal{R}$-invariant subspaces of an $\mathcal{R}$-space S, such that $S_x = \bigcup_{j=1}^r \mathrm{int}(U_{j,x})$, where $\mathrm{int}(U_{j,x})$ is the topological interior of $U_{j,x}$. We denote by $C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z})$ the subcomplex of $C_\bullet^X(S;\mathbb{Z})$ generated by all singular X-simplices whose images are contained in one of the elements of $\mathcal{U}$. To show that the homomorphism

$$j\colon \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z}) \to \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S;\mathbb{Z}) \tag{4.5}$$

given by inclusion induces a $\dim_{\mathcal{N}\mathcal{R}}$-isomorphism in homology we proceed in two steps.

First consider the subset $\Sigma_{n,k}^X(S) \subset \Sigma_n^X(S)$ of those singular X-simplices σ satisfying $Y_n^{X,k}(\sigma) \in C_n^{X,\mathcal{U}}(S;\mathbb{Z})$. Now we define

$$C_{n,k}^X(S;\mathbb{Z}) = \big\{ \textstyle\sum_j f_j \cdot \sigma_j \in C_n^X(S;\mathbb{Z}) : \sigma_j \in \Sigma_{n,k}^X(S) \text{ for all } j \big\}. \tag{4.6}$$

Because of the $\mathcal{R}$-invariance of the subsets $U_j \subset S$, it follows that $C_{n,k}^X(S;\mathbb{Z})$ is a $\mathbb{Z}\mathcal{R}$-submodule of $C_n^X(S;\mathbb{Z})$. The inclusion $C_{n,k}^X(S;\mathbb{Z}) \hookrightarrow C_n^X(S;\mathbb{Z})$ splits for all $n \in \mathbb{N}$ by a similar construction as in (4.2). Hence

$$\mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_{n,k}^X(S;\mathbb{Z}) \to \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S;\mathbb{Z})$$

is injective. Since directed colimits commute with tensor products and preserve exactness, we conclude that

$$j_0\colon \operatorname*{colim}_{\overrightarrow{k}} \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_{\bullet,k}^X(S;\mathbb{Z}) \to \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S;\mathbb{Z})$$

is injective.

Next we want to show that j_0 is dimension surjective in each degree. We will use the local criterion (Lemma 1.18) to show that $\mathrm{coker}(j_0)$ has dimension zero in degree n. For this purpose let $\sum_{j=1}^k f_j \cdot \sigma_j$ represent a chain $c \in C_n^X(S;\mathbb{Z})$ and define

$$A_j(l) = (X - X_{\sigma_j}) \cup \big\{ x \in X_{\sigma_j} : \sigma_{j,x} \in C_{n,l}(S_x;\mathbb{Z}) \big\} \tag{4.7}$$

for $j = 1, \ldots, k$. Here, $C_{n,l}(S_x;\mathbb{Z})$ denotes the subcomplex of the ordinary chain complex $C_n(S_x;\mathbb{Z})$ of S_x generated by those singular simplices whose l-th subdivision is a sum of simplices which all have image in one of the covering sets. We will show measurability of $A_j(l)$. For one X-simplex $\sigma\colon X_\sigma \times \Delta^n \to S$ and one measurable set $U \subset S$ the subset

$$A_{\sigma,U} = (X - X_\sigma) \cup \big\{x \in X_\sigma : \mathrm{im}(\sigma_x) \subset \mathrm{int}(U_x)\big\}$$

is measurable by Lemma 3.17. Consequently, for a countable family Σ of X-simplices and a countable family $\mathcal{U}$ of measurable subsets of S

$$\begin{aligned} A_{\Sigma,\mathcal{U}} &= \big\{x \in X : \forall \sigma \in \Sigma \text{ with } x \in X_\sigma \ \exists U \in \mathcal{U} \text{ with } \mathrm{im}(\sigma)_x \subset \mathrm{int}(U_x)\big\} \\ &= \textstyle\bigcap_{\sigma\in\Sigma}\big(\bigcup_{U\in\mathcal{U}} A_{\sigma,U}\big) \end{aligned}$$

is measurable. This shows measurability of $A_j(l)$ since for

$$\Sigma = \big\{\text{all summands of } Y_n^{X,l}(\sigma_j)\big\} \quad \text{and} \quad \mathcal{U} = \{U_1, \ldots, U_r\}$$

the equation $A_j(l) = A_{\Sigma,\mathcal{U}}$ holds.

The properties of $Y_\bullet$ imply that for almost all $x \in X_{\sigma_j}$ there is an index $L \in \mathbb{N}$ such that $x \in A_j(l)$ for all $l \geq L$, in other words

$$\lim_{l\to\infty} \mu\big(A_j(l)\big) = 1. \tag{4.8}$$

By definition of $A_j(l)$ we get

$$1 \otimes \big[\textstyle\sum_{j=1}^{l} (\chi_{A_j(l)} \cdot f_j) \cdot \sigma_j\big] \in \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_{n,l}^X(S;\mathbb{Z}) \tag{4.9}$$

Define

$$A(l) = \bigcap_{j=1}^{k} A_j(l).$$

Then $\lim_{l\to\infty} \mu\big(A_j(l)\big) = 1$ for each $j = 1, \ldots, k$ implies

$$\lim_{l\to\infty} \mu\big(A(l)\big) = 1.$$

By (4.9) and the definition of $A(l)$, we obtain $1 \otimes \chi_{A(l)} \cdot \big[\sum_{j=1}^{k} f_j \cdot \sigma_j\big] = 0$ in $\mathrm{coker}(j_0)$. Since $(\chi_{A(l)})_{l\in\mathbb{N}}$ is a sequence of projections with $\mathrm{tr}_{\mathcal{N}\mathcal{R}}(\chi_{A(l)}) = \mu\big(A(l)\big) \xrightarrow{l\to\infty} 1$, the local criterion implies $\dim_{\mathcal{N}\mathcal{R}}\big(\mathrm{coker}(j_0)\big) = 0$. It follows that

$$H_\bullet\Big(\operatorname*{colim}_{\overrightarrow{k}} \mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_{n,k}^X(S;\mathbb{Z})\Big) \xrightarrow{(j_0)_*} H_\bullet\big(\mathcal{N}\mathcal{R} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S)\big) = H_\bullet^{\mathcal{R}}(S;\mathcal{N}\mathcal{R})$$

is a dimension isomorphism since it is induced by a chain map, which is a dimension isomorphism in each degree.

The second step in the proof that (4.5) is a homology isomorphism consists of showing that the chain map

$$j_1\colon \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z}) \to \underrightarrow{\operatorname{colim}}_k \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_{\bullet,k}^{X}(S;\mathbb{Z})$$

induces an isomorphism $(j_1)_*$ in homology. This can be shown as in the classical situation, compare [6]. For the sake of completeness we recall the argument. Note that by slight abuse of notation we will regard $Y_n^{X,k}$ and $F_n^{X,k}$ as maps on the tensor product $\mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S;\mathbb{Z})$.

Let $z \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^{X,\mathcal{U}}(S;\mathbb{Z})$ be a cycle with the property that for some $k \geq 0$ there is a chain $e \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_{n+1,k}^{X}(S;\mathbb{Z})$ with $d(e) = z$. For injectivity of $(j_1)_*$ we have to show that there is $e' \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_{n+1}^{X,\mathcal{U}}(S;\mathbb{Z})$ with $d(e') = z$. One gets

$$Y_{n+1}^{X,k}(e) - e = F_n^{X,k} \circ d(e) + d \circ F_{n+1}^{X,k}(e) = F_n^{X,k}(z) + d \circ F_{n+1}^{X,k}(e).$$

This implies

$$d \circ Y_{n+1}^{X,k}(e) - d(e) = d \circ F_n^{X,k}(z).$$

Now we can conclude

$$c = d(e) = d\big(Y_{n+1}^{X,k}(e) - F_n^{X,k}(z)\big).$$

If we define $e' = Y_{n+1}^{X,k}(e) - F_n^{X,k}(z)$ then it follows directly from the construction that $e' \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_{n+1}^{X,\mathcal{U}}(S;\mathbb{Z})$ holds. Hence $(j_1)_*$ is injective.

Next we show surjectivity of $(j_1)_*$. Let $z \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_{n,k}^{X}(S;\mathbb{Z})$ be a cycle. We have

$$Y_n^{X,k}(z) - z = F_{n-1}^{X,k} \circ d(z) + d \circ F_n^{X,k}(z) = d \circ F_n^{X,k}(z).$$

By construction, $Y_n^{X,k}(z) \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^{X,\mathcal{U}}(S;\mathbb{Z})$. This shows surjectivity of $(j_1)_*$.

The proof that the map $j = j_0 \circ j_1$ of (4.5) induces a $\dim_{\mathcal{NR}}$-isomorphism in homology is now complete.

The results obtained above enable us to complete the proof of the excision axiom. Let $A, B \subset S$ be $\mathcal{R}$-invariant subspaces with $\overline{A_x} \subset \operatorname{int}(B_x)$ for almost all $x \in X$. Then $\mathcal{U} = \{B, S - A\}$ is a system of $\mathcal{R}$-invariant subspaces with

$$S_x = \operatorname{int}(B_x) \cup \operatorname{int}\big((S-A)_x\big).$$

We have

$$\begin{aligned}\mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z}) &= \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \big(C_\bullet^X(B;\mathbb{Z}) + C_\bullet^X(S-A;\mathbb{Z})\big)\\ \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(B-A;\mathbb{Z}) &= \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \big(C_\bullet^X(B;\mathbb{Z}) \cap C_\bullet^X(S-A;\mathbb{Z})\big).\end{aligned}$$

Note that the sum in the first equation is not a direct sum. By one of the Noetherian isomorphisms we get an isomorphism

$$\mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^X(S-A;\mathbb{Z})}{C_\bullet^X(B-A;\mathbb{Z})} \xrightarrow{\cong} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z})}{C_\bullet^X(B;\mathbb{Z})}$$

induced by inclusion. This isomorphism fits into the following commutative diagram, with all maps induced by inclusion:

$$\begin{array}{ccc} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^X(S-A;\mathbb{Z})}{C_\bullet^X(B-A;\mathbb{Z})} & \xrightarrow{\cong} & \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z})}{C_\bullet^X(B;\mathbb{Z})} \\ & \searrow \qquad \swarrow & \\ & \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^X(S;\mathbb{Z})}{C_\bullet^X(B;\mathbb{Z})} & \end{array}$$

Since the map

$$\mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^{X,\mathcal{U}}(S;\mathbb{Z})}{C_\bullet^X(B;\mathbb{Z})} \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} \frac{C_\bullet^X(S;\mathbb{Z})}{C_\bullet^X(B;\mathbb{Z})}$$

induces a dimension isomorphism in homology by the results shown above and a five-lemma argument we conclude that

$$H_\bullet^{\mathcal{R}}(S-A, B-A; \mathcal{NR}) \to H_\bullet^{\mathcal{R}}(S, B; \mathcal{NR})$$

is also a dimension isomorphism.

To prove the countable additivity axiom we pick a countable index set I and a family of $\mathcal{R}$-spaces $\{S_\alpha\}_{\alpha\in I}$, and consider the canonical map

$$j\colon \bigoplus_{\alpha\in I} C_\bullet^X(S_\alpha;\mathbb{Z}) \to C_\bullet^X\big(\textstyle\coprod_{\alpha\in I} S_\alpha;\mathbb{Z}\big).$$

First we investigate the case where $I = \{1,\ldots,k\}$ is finite. If $\sigma\colon X_\sigma \times \Delta^n \to \coprod_{j=1}^k S_j$ is an X-simplex set $X_j = p_1\big(\sigma^{-1}(S_j)\big)$. If one defines

$$\sigma_j\colon X_j \times \Delta^n \to S$$

as the restriction of σ then

$$s(\sigma) = (\chi_{X_i} \cdot \sigma_i)_{i=1,\ldots,k}$$

extends to a split of j in each degree (but, as usual, not as a chain map). Hence $\mathrm{id}_{\mathcal{NR}} \otimes j$ is also split injective in each degree for finite I. Taking the direct colimit over the finite subsets of I therefore shows that

$$\mathrm{id}_{\mathcal{NR}} \otimes j\colon \bigoplus_{\alpha\in I} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X(S_\alpha;\mathbb{Z}) \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_\bullet^X\big(\textstyle\coprod_{\alpha\in I} S_\alpha;\mathbb{Z}\big)$$

is injective.

To show that $\mathrm{id}_{\mathcal{NR}} \otimes j$ is dimension surjective, consider the class

$$[1 \otimes c] \in \operatorname{coker}(\mathrm{id}_{\mathcal{NR}} \otimes j),$$

where $c \in C_n^X\big(\coprod_{\alpha\in I} S_\alpha;\mathbb{Z}\big)$ is represented by $\sum_{j=1}^k f_j \cdot \sigma_j$. We choose some enumeration $I = \{\alpha_0, \alpha_1, \ldots\}$ of I and define

$$A_j(l) = (X - X_{\sigma_j}) \cup \big\{x \in X_{\sigma_j} : \operatorname{im}(\sigma_{j,x}) \subset \textstyle\coprod_{i=0}^l S_{\alpha_i}\big\}$$

for $j = 1, \dots, k$. Measurability of $A_j(l)$ follows easily from Lemma 3.17 since

$$\coprod_{i=0}^{n} S_{\alpha_i} \subset \coprod_{\alpha \in I} S_\alpha$$

is open. The union $\bigcup_{n \geq 0} A_j(l)$ has measure 1, thus $\lim_{l\to\infty} \mu\big(A_j(l)\big) = 1$ follows for all $j = 1, \dots, k$. Each $\chi_{A_j(l)} \cdot \sigma_j$ is in the image of j, hence $\big[1 \otimes \sum_{j=1}^{k} (\chi_{A_j(l)} \cdot f_j) \cdot \sigma_j\big] = 0$ in $\operatorname{coker}(\mathrm{id}_{\mathcal{N}\mathcal{R}} \otimes j)$. If we define

$$A(l) = \bigcap_{j=1}^{k} A_j(l),$$

we obtain $\lim_{l\to\infty} \mu\big(A(l)\big) = 1$ and $\chi_{A(l)} \cdot [1 \otimes c] = 0$ in $\operatorname{coker}(\mathrm{id}_{\mathcal{N}\mathcal{R}} \otimes j)$. Now $\mu\big(A(l)\big) \xrightarrow{n\to\infty} 1$ implies $\mathrm{tr}_{\mathcal{N}\mathcal{R}}(\chi_{A(l)}) \xrightarrow{n\to\infty} 1$ and $\dim_{\mathcal{N}\mathcal{R}} \operatorname{coker}(\mathrm{id}_{\mathcal{N}\mathcal{R}} \otimes j) = 0$ follows from the local criterion 1.18.

Finally, we show the dimension axiom. We first observe that the equivalence classes of $\mathcal{R}$ are totally disconnected with respect to some Polish topology which induces the restricted Borel-σ-algebra on $\mathcal{R} \subset X \times X$. That follows from the general fact that connected sets in normal topological spaces are one-pointed or uncountable (see e.g. [17]) and that $\mathcal{R}$ with a Polish topology is normal. Hence a singular X-simplex $\sigma\colon X_\sigma \times \Delta^n \to \mathcal{R}$ can be identified with a map $f_\sigma\colon X_\sigma \to X$ with $\mathrm{gr}(f_\sigma) \subset \mathcal{R}$. We will show that therefore we can identify $C_n^X(\mathcal{R};\mathbb{C})$ with

$$\mathbb{C}_2\mathcal{R} = \big\{\eta \in L^\infty(\mathcal{R}) : \exists N \in \mathbb{N}\ \forall x \in X : |\{y : \eta(x,y) \neq 0\}| \leq N\big\}.$$

To an X-chain represented by $\sum_{j=1}^{k} f_j \cdot \sigma_j$ we define a function in $\mathbb{C}_2\mathcal{R}$ by

$$(x,y) \mapsto \sum_{\substack{j=1 \\ \sigma_j(x,t)=(x,y)}}^{k} f_j(x). \tag{4.10}$$

It is clear that this defines a monomorphism $C_n^X(\mathcal{R};\mathbb{C}) \to \mathbb{C}_2\mathcal{R}$. To see that this map is also surjective, pick some function $\eta \in \mathbb{C}_2\mathcal{R}$. By definition, the projection $\mathrm{pr}_1\colon \operatorname{supp}(\eta) \to X$ to the first coordinate is uniformly finite-to-1. Hence Theorem 3.3 implies that there is a partition $\operatorname{supp}(\eta) = \bigcup_{j=1}^{N} E_j$ such that $\mathrm{pr}_1|_{E_j}$ is injective for all $j = 1, \dots, N$. Now we define X-simplices $\sigma_j\colon \mathrm{pr}_1(E_j) \times \Delta^n \to \mathcal{R}$ by

$$(x,t) \mapsto (x, y_x^j)$$

and measurable maps $f_j\colon X \to \mathbb{C}$ by

$$x \mapsto \begin{cases} \eta(x, y_x^j), & x \in \mathrm{pr}_1(E_j) \\ 0, & \text{otherwise,} \end{cases}$$

where $y_j^x \in X$ is the unique element with $(x, y_x^j) \in \operatorname{supp}(\eta) \cap E_j$. Then the X-chain represented by $\sum_{j=1}^k f_j \cdot \sigma_j$ maps to η under the mapping defined in (4.10).

As in the classical case, the boundary operator $d_n \colon C_n^X(\mathcal{R};\mathbb{C}) \to C_{n-1}^X(\mathcal{R};\mathbb{C})$ is the identity for $n \geq 3$ odd and zero for $n \leq 1$ or n even. Now the dimension axiom follows, but we additionally want to compute $H_0^{\mathcal{R}}(\mathcal{R};\mathcal{N}\mathcal{R})$.

To do so, we first show that the inclusion $\mathbb{C}\mathcal{R} \hookrightarrow \mathbb{C}_2\mathcal{R}$ is a $\dim_{L^\infty(X)}$-isomorphism, i.e. $\dim_{L^\infty(X)}(\mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}) = 0$. Again, we want to apply the local criterion 1.18. For a given $\eta \in \mathbb{C}_2\mathcal{R}$ we consider the measurable subset

$$\operatorname{supp}(\eta) = \{(x,y) \in \mathcal{R} : \eta(x,y) \neq 0\} \subset \mathcal{R}.$$

The function given by

$$y \mapsto \big|\{x \in X : \eta(x,y) \neq 0\}\big|$$

is measurable (compare (3.3) on page 33). Hence

$$A_n = \big\{y \in X : |\{x \in X : \eta(x,y) \neq 0\}| \leq n\big\}$$

and

$$A_n' = \big\{y \in X : |\{x \in X : \eta(x,y) \neq 0\}| = n\big\}$$

are measurable subsets of X for all $n \in \mathbb{N}$. The invariance of the measure ν on $\mathcal{R}$ with respect to the choice of the coordinate projection gives us

$$\sum_{n=1}^{\infty} n \cdot \mu(A_n') = \nu\big(\operatorname{supp}(\eta)\big) \leq N$$

for some $N \in \mathbb{N}$. Hence for each $\varepsilon > 0$ we find some $k \in \mathbb{N}$ with $\sum_{n=k}^{\infty} n \cdot \mu(A_n') < \varepsilon$. It follows that

$$\nu\big(p_2^{-1}(X - A_k) \cap \operatorname{supp}(\eta)\big) = \int_{X - A_k} |\{x : \eta(x,y) \neq 0\}| \, d\mu(y) = \sum_{n=k+1}^{\infty} n \cdot \mu(A_n') < \varepsilon.$$

Let $B_n = \big\{x \in X : p_1^{-1}(x) \cap p_2^{-1}(X - A_n) \cap \operatorname{supp}(\eta) = \emptyset\big\}$. For each $n \geq k$ one gets

$$\mu(X - B_n) \leq \int_X |p_1^{-1}(x) \cap p_2^{-1}(X - A_n) \cap \operatorname{supp}(\eta)| \, d\mu(x) =$$

$$= \nu\big(p_2^{-1}(X - A_n) \cap \operatorname{supp}(\eta)\big) < \varepsilon$$

or, equivalently, $\mu(B_n) > 1 - \varepsilon$. For $p_n = \chi_{B_n}$ one has

$$p_n \cdot \eta(x,y) = \begin{cases} \eta(x,y), & x \in B_n \\ 0, & \text{otherwise.} \end{cases}$$

If $p_n \cdot \eta(x,y) \neq 0$ then we get $y \in A_n$. It follows that $p_n \cdot \eta \in \mathbb{C}\mathcal{R}$. By the local criterion, the inclusion $\mathbb{C}\mathcal{R} \hookrightarrow \mathbb{C}_2\mathcal{R}$ is a $\dim_{L^\infty(X)}$-isomorphism since we have

$$\operatorname{tr}_{L^\infty(X)}(p_n) = \mu(B_n) \xrightarrow{n\to\infty} 1.$$

We finally show that this implies that

$$\mathcal{N}\mathcal{R} = \mathcal{N}\mathcal{R} \otimes_{\mathbb{C}\mathcal{R}} \mathbb{C}\mathcal{R} \to \mathcal{N}\mathcal{R} \otimes_{\mathbb{C}\mathcal{R}} \mathbb{C}_2\mathcal{R} \tag{4.11}$$

is a $\dim_{\mathcal{N}\mathcal{R}}$-isomorphism.

Since $\mathbb{C}\mathcal{R} \hookrightarrow \mathbb{C}_2\mathcal{R}$ does not split as a $\mathbb{C}\mathcal{R}$-homomorphism we have to show injectivity of (4.11) by using another argument. From Lemma 1.24 we get

$$0 = \dim_{\mathcal{N}\mathcal{R}} \operatorname{Tor}_\bullet^{\mathbb{C}\mathcal{R}}(\mathcal{N}\mathcal{R}, 0) = \dim_{\mathcal{N}\mathcal{R}} \operatorname{Tor}_\bullet^{\mathbb{C}\mathcal{R}}(\mathcal{N}\mathcal{R}, \mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}).$$

It follows that $\dim_{\mathcal{N}\mathcal{R}}\big(\ker(\mathcal{N}\mathcal{R} \to \mathcal{N}\mathcal{R} \otimes_{\mathbb{C}\mathcal{R}} \mathbb{C}_2\mathcal{R})\big) = 0$ holds. By Corollary 1.23, $\mathcal{N}\mathcal{R}$ is dimension compatible as an $\mathcal{N}\mathcal{R}$-$L^\infty(X)$-bimodule. This implies

$$\dim_{L^\infty(X)}(\mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}) = 0 \;\Rightarrow\; \dim_{\mathcal{N}\mathcal{R}}(\mathcal{N}\mathcal{R} \otimes_{L^\infty(X)} \mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}) = 0.$$

Since $\mathcal{N}\mathcal{R} \otimes_{\mathbb{C}\mathcal{R}} \mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}$ is a quotient of $\mathcal{N}\mathcal{R} \otimes_{L^\infty(X)} \mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}$ it follows that

$$\dim_{\mathcal{N}\mathcal{R}}(\mathcal{N}\mathcal{R} \otimes_{\mathbb{C}\mathcal{R}} \mathbb{C}_2\mathcal{R}/\mathbb{C}\mathcal{R}) = 0.$$

We conclude $H_0^{\mathcal{R}}(\mathcal{R};\mathcal{N}\mathcal{R}) \cong \mathcal{N}\mathcal{R} \otimes_{\mathbb{C}\mathcal{R}} \mathbb{C}_2\mathcal{R} \cong_{\dim} \mathcal{N}\mathcal{R}$. □

Remark 4.16. It is natural to consider also singular $\mathcal{R}$-cohomology of an $\mathcal{R}$-space S, i.e. the cohomology of the cochain complex

$$\hom_{\mathbb{Z}\mathcal{R}}\big(C_\bullet^X(S;\mathbb{Z}), \mathcal{N}\mathcal{R}\big).$$

Unfortunately, we were not able to prove that it satisfies the excision axiom.

4.4 $\mathcal{R}$-CW-Complexes

In the $\mathcal{R}$-space setting, there is also an appropriate notion of CW-complexes which will be introduced in this section. It is an analog of free G-CW-complexes. As an additional feature, the cells of an $\mathcal{R}$-CW-complex are equipped with a weight in the unit interval $[0,1]$ given by $\mu(A)$ for a measurable subset $A \subset X$. We first investigate colimits in the category of $\mathcal{R}$-spaces based on the study of colimits in the category of X-spaces in 3.2.2.

4.4.1 Colimits of $\mathcal{R}$-Spaces

The next result is derived easily from Lemma 3.15.

Lemma 4.17. *Countable colimits exist in the category of $\mathcal{R}$-spaces.*

Proof. Let $(S_i)_{i\in I}$ be a countable family of $\mathcal{R}$-spaces together with a countable family $(f^l_{i,j}\colon S_i \to S_j)_{l\in L_{i,j}}$ of $\mathcal{R}$-maps for each $i, j \in I$. Since $\mathcal{R}$-spaces are special X-spaces, we can use Lemma 3.15 to get the existence of the colimit $\operatorname{colim}_{i\in I} S_i$ in the category of X-spaces. The induced $\mathcal{R}$-action on $\operatorname{colim}_{i\in I} S_i$ is well defined since the structure maps $f^l_{i,j}$ commute with the $\mathcal{R}$-action.

To see that $\operatorname{colim}_{i\in I} S_i$ is a colimit in the category of $\mathcal{R}$-spaces we have to show that the X-map $F\colon \operatorname{colim}_{i\in I} S_i \to S$ induced by a family of $\mathcal{R}$-maps $(F_i\colon S_i \to S)_{i\in I}$ which is compatible with the structure maps is $\mathcal{R}$-equivariant. This follows immediately from the equivariance of the maps F_i. □

4.4.2 Definition and Examples of $\mathcal{R}$-CW-Complexes

Being sure that pushouts exist in the category of $\mathcal{R}$-spaces we are now prepared to introduce $\mathcal{R}$-CW-complexes. Recall that for a Borel subset $A \subset X$ the restricted equivalence relation was defined as $\mathcal{R}|_A = \{(x,y) \in \mathcal{R} : y \in A\}$.

Definition 4.18. A relative $\mathcal{R}$-*CW-complex* is a pair (S,T) of $\mathcal{R}$-spaces together with a filtration

$$T = S_{[-1]} \subset S_{[0]} \subset S_{[1]} \subset \ldots \subset S = \bigcup_{n\geq 0} S_{[n]}$$

of $\mathcal{R}$-spaces such that the following conditions hold.

- The space S carries the weak topology with respect to the filtration, i.e. a subset $C \subset S$ is closed if and only if $C \cap S_{[n]}$ is closed in $S_{[n]}$ for all $n \geq 0$.
- For each $n \geq 0$ there is a countable index set I_n and a pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} \mathcal{R}|_{A_i} \times S^{n-1} & \longrightarrow & S_{[n-1]} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} \mathcal{R}|_{A_i} \times D^n & \longrightarrow & S_{[n]} \end{array}$$

  of $\mathcal{R}$-spaces, where each $A_i \subset X$ is a Borel subset.

The *weight* of the equivariant cell $\mathcal{R}|_{A_i} \times D^n$ is defined as $\mu(A_i)$. For each $n \geq -1$ the subspace $S_{[n]} \subset S$ is called the *n-skeleton* of S.

An $\mathcal{R}$-map $f\colon S \to S'$ between $\mathcal{R}$-CW-complexes is called *cellular* if $f(S_{[n]}) \subset S'_{[n]}$ holds for all $n \in \mathbb{N}$.

Remark 4.19. The unusual notation $S_{[n]}$ for the n-skeleton of S should avoid confusion of skeleta and fibers of S.

We already know that a G-space Z induces an $\mathcal{R}_{G\curvearrowright X}$-space $X \times Z$, where $G \curvearrowright X$ is a standard action. We will now show that the induction functor will map G-CW-complexes on $\mathcal{R}_{G\curvearrowright X}$-CW-complexes. We will use the following result. Note that in this context the category of G-spaces has as objects *σ-compact, second countable Hausdorff* G-spaces.

Lemma 4.20. *The functor*

$$\begin{array}{rcl} \mathrm{ind}\colon G\text{-}Spaces & \to & \mathcal{R}_{G\curvearrowright X}\text{-}Spaces \\ Z & \mapsto & X \times Z \end{array}$$

preserves countable colimits.

Proof. Let $(Z_i)_{i\in I}$ be a countable family of G-spaces together with a countable family $(f^l_{i,j}\colon Z_i \to Z_j)_{l\in L_{i,j}}$ of G-maps for each $i, j \in I$. Denote its colimit $\mathrm{colim}_{i\in I} Z_i$ by Z. We have to show that the $\mathcal{R}_{G\curvearrowright X}$-space $X \times Z$ satisfies the properties of a colimit in $\mathcal{R}_{G\curvearrowright X}$-Spaces.

If we are given an $\mathcal{R}_{G\curvearrowright X}$-space S together with $\mathcal{R}_{G\curvearrowright X}$-maps $g_i\colon X \times Z_i \to S$ for each $i \in I$ which commute with the structure maps, we define $g\colon X \times Z \to S$ by $g_x = \mathrm{colim}_{i\in I}(g_i)_x$ for all $x \in X$. Here, we regard the maps $(g_i)_x$ just as maps of sets without any further structure, but we will show in a moment that g indeed is an $\mathcal{R}_{G\curvearrowright X}$-map. The proof is similar to that of Lemma 3.15.

We will first show that g is measurable. By Lemma 3.14 there is a measurable section $s\colon \mathrm{colim}_{i\in I} Z_i \to \coprod_{i\in I} Z_i$ of the projection $q\colon \coprod_{i\in I} Z_i \to \mathrm{colim}_{i\in I} Z_i$. Hence, $g = \left(\coprod_{i\in I} g_i\right) \circ (\mathrm{id}_X \times s)$ is measurable.

Note that for each $i \in I$ there is a subset $X_i \subset X$ of full measure such that $(g_i)_x$ is continuous for all $x \in X_i$. Consequently, for $x \in \bigcap_{i\in I} X_i$ each $(g_i)_x$ is continuous. It follows that g_x is also continuous in all fibers over the full measure subset $\bigcap_{i\in I} X_i$. We therefore have shown that g is an X-map.

The $\mathcal{R}_{G\curvearrowright X}$-equivariance of g follows immediately from the $\mathcal{R}_{G\curvearrowright X}$-equivariance of the g_i. □

As a consequence, we will get a class of examples of $\mathcal{R}$-CW-complexes. Recall that we are working in the category of compactly generated spaces.

Corollary 4.21. *Let Z be a countable free G-CW-complex and $G \curvearrowright X$ be a free standard action. Then $X \times Z$ is an $\mathcal{R}_{G\curvearrowright X}$-CW-complex.*

Proof. For each $n \in \mathbb{N}$ there is a G-pushout

$$\begin{array}{ccc} \coprod_{i\in I_n} G \times S^{n-1} & \longrightarrow & Z_{[n-1]} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} G \times D^n & \longrightarrow & Z_{[n]} \end{array}.$$

By Lemma 4.20 the induced diagram

$$\begin{array}{ccc} \coprod_{i\in I_n} X \times G \times S^{n-1} & \longrightarrow & X \times Z_{[n-1]} \\ \downarrow & & \downarrow \\ \coprod_{i\in I_n} X \times G \times D^n & \longrightarrow & X \times Z_{[n]} \end{array}$$

is a pushout of $\mathcal{R}_{G\curvearrowright X}$-spaces.

The map $\varphi\colon X \times G \to \mathcal{R}_{G\curvearrowright X}$ given by

$$(x,g) \mapsto (x, g^{-1}.x)$$

is an isomorphism of $\mathcal{R}_{G\curvearrowright X}$-spaces. Hence we obtain an $\mathcal{R}_{G\curvearrowright X}$-pushout of the desired shape.

To see that $X \times Z$ carries the weak topology with respect to the subspaces $X \times Z_{[n]}$, we have to show that the maps

$$\coprod_{i\in I_n} \underbrace{\mathcal{R}_{G\curvearrowright X}}_{\cong X\times G} \times D^n \to X \times Z$$

which are induced by the lower horizontal map in the above pushout diagrams are identifications for all $n \in \mathbb{N}$. This follows from the fact that the maps

$$\coprod_{i\in I_n} G \times D^n \to Z$$

are identifications for all $n \in \mathbb{N}$ and products of identifications are identifications by Lemma 3.50. □

4.4.3 Cellular Homology

In analogy to the situation of ordinary CW-complexes or G-CW-complexes we will define the cellular chain complex of an $\mathcal{R}$-CW-complex S and show that its homology coincides with the singular homology of S.

Definition 4.22. We define the *cellular chain complex* $\big(C_\bullet^{\mathcal{R},\mathrm{cell}}(S;\mathcal{NR}), \beta_\bullet\big)$ of an $\mathcal{R}$-CW-complex S as follows: The n-th chain group is given by

$$C_n^{\mathcal{R},\mathrm{cell}}(S;\mathcal{NR}) = H_n^{\mathcal{R}}(S_{[n]}, S_{[n-1]};\mathcal{NR}).$$

The boundary operator $\beta_n\colon H_n^{\mathcal{R}}(S_{[n]}, S_{[n-1]};\mathcal{NR}) \to H_{n-1}^{\mathcal{R}}(S_{[n-1]}, S_{[n-2]};\mathcal{NR})$ is the composite

$$H_n^{\mathcal{R}}(S_{[n]}, S_{[n-1]};\mathcal{NR}) \xrightarrow{\partial_n} H_{n-1}^{\mathcal{R}}(S_{[n-1]};\mathcal{NR}) \xrightarrow{j_n} H_{n-1}^{\mathcal{R}}(S_{[n-1]}, S_{[n-2]};\mathcal{NR}),$$

where ∂_n is the boundary operator in the long exact homology sequence of the pair $(S_{[n]}, S_{[n-1]})$ and j_n is induced by inclusion. This is indeed a complex, and its homology is denoted by $H_\bullet^{\mathcal{R},\mathrm{cell}}(S;\mathcal{NR})$.

Next we compute the cellular homology of an $\mathcal{R}$-CW-complex. This is analogous to the situation of ordinary CW-complexes (or G-CW-complexes) for finite dimensional $\mathcal{R}$-CW-complexes and only a little more difficult for arbitrary $\mathcal{R}$-CW-complexes.

Proposition 4.23. *Let S be an $\mathcal{R}$-CW-complex. Then for each $n \in \mathbb{Z}$ there is a dimension isomorphism*

$$H_n^{\mathcal{R}}(S;\mathcal{NR}) \xrightarrow{\cong_{\dim}} H_n^{\mathcal{R},\mathrm{cell}}(S;\mathcal{NR}).$$

Proof. Using the axioms, one gets as in the case of an ordinary CW-complex

$$H_n^{\mathcal{R},\mathrm{cell}}(S;\mathcal{NR}) \cong_{\dim} H_n^{\mathcal{R}}(S_{[n+1]};\mathcal{NR}) \cong_{\dim} H_n^{\mathcal{R}}(S_{[n+2]};\mathcal{NR}) \cong_{\dim} \dots \tag{4.12}$$

(see for example [6, IV.10, pages 200 ff.]). This implies the result for finite dimensional $\mathcal{R}$-CW-complexes. For arbitrary $\mathcal{R}$-CW-complexes the direct transfer of the proof in the classical situation fails, because it is not clear that an $\mathcal{R}$-cycle has image in a finite subcomplex.

In the same manner as in the proof of excision of $\mathcal{R}$-homology on page 47 one shows that the map

$$\operatorname*{colim}_{\overrightarrow{l}} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S_{[l]};\mathbb{Z}) \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S;\mathbb{Z})$$

induced by inclusion is injective.

Let $z = \sum_{j=1}^k \psi_j \otimes \sigma_j$ be a cycle in $\mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S;\mathbb{Z})$. For almost all $x \in X$ the restricted simplices $\sigma_{j,x}$ have compact image and therefore lie in a finite subcomplex and hence in some $S_{[n(x)]}$ for an appropriate $n(x) \geq 0$. Let

$$A_j(l) = (X - X_{\sigma_j}) \cup \big\{x \in X_{\sigma_j} : \operatorname{im}(\sigma_{j,x}) \subset S_{[l]}\big\}.$$

Measurability of $A_j(l)$ follows from Lemma 3.17, because each skeleton $S_{[l]}$ is a measurable, fiberwise closed subset of S. We get $\mu\big(A_j(l)\big) \xrightarrow{l\to\infty} \mu(X) = 1$. Furthermore, we have $\sum_{j=1}^k (\chi_{A_j(l)} \cdot \psi_j) \otimes \sigma_j \in \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S_{[l]};\mathbb{Z})$. For the intersection

$$A_z(l) = \bigcup_{j=1}^k A_j(l)$$

we still have $\mu\big(A_z(l)\big) \xrightarrow{l\to\infty} \mu(X) = 1$. It is $\chi_{A_z(l)} \cdot z \in \operatorname*{colim}_{\overrightarrow{l}} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S_{[l]};\mathbb{Z})$ and $\lim_{l\to\infty} \operatorname{tr}_{\mathcal{NR}}(\chi_{A_z(l)}) = 1$. By the local criterion (Lemma 1.18),

$$\operatorname*{colim}_{\overrightarrow{l}} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S_{[l]};\mathbb{Z}) \to \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S;\mathbb{Z})$$

is $\mathcal{NR}$-dimension surjective, and consequently an $\mathcal{NR}$-dimension isomorphism. Hence we also get an $\mathcal{NR}$-dimension isomorphism in homology:

$$H_n\big(\bigcup_{l\geq 0} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S_{[l]};\mathbb{Z})\big) \cong_{\dim} H_n(\mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S;\mathbb{Z})) = H_n^{\mathcal{R}}(S;\mathcal{NR}).$$

Now with (4.12) we conclude

$$H_n^{\mathcal{R},\text{cell}}(S;\mathcal{NR}) \cong \underset{\overrightarrow{l}}{\text{colim}}\, H_n^{\mathcal{R}}(S_{[l]};\mathcal{NR}) \cong H_n\big(\bigcup_{l\geq 0} \mathcal{NR} \otimes_{\mathbb{Z}\mathcal{R}} C_n^X(S_{[l]};\mathbb{Z})\big),$$

and therefore $H_n^{\mathcal{R},\text{cell}}(S;\mathcal{NR}) \cong_{\dim} H_n^{\mathcal{R}}(S;\mathcal{NR})$ for all $n \in \mathbb{N}$. □

4.5 L^2-Betti Numbers Old and New

If Z is a G-CW-complex we can consider its L^2-Betti numbers $b_k^{(2)}(Z;\mathcal{N}G)$.For a standard action $G \curvearrowright X$ we can induce Z to the $\mathcal{R}_{G\curvearrowright X}$-CW-complex $X \times Z$ and consider its L^2-Betti numbers $b_k^{(2)}(X \times Z;\mathcal{NR}_{G\curvearrowright X})$. In this section we want to show that these two invariants coincide for finite free G-CW-complexes Z, i.e.

$$b_k^{(2)}(X \times Z;\mathcal{NR}_{G\curvearrowright X}) = b_k^{(2)}(Z;\mathcal{N}G) \quad \text{for all} \quad k \geq 0.$$

The proof is based on the following two lemmas.

Lemma 4.24. *Let G be a countable group and $G \curvearrowright X$ a standard action on a standard Borel space X. Let further Z be a countable free G-CW-complex. There is a natural dimension isomorphism*

$$H_k\big(\mathcal{NR}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})\big) \xrightarrow{\cong_{\dim}} H_k^{\mathcal{R}_{G\curvearrowright X}}(X \times Z;\mathcal{NR}_{G\curvearrowright X}). \tag{4.13}$$

It is induced by the assignment $\sigma \mapsto \text{id}_X \times \sigma$ for a singular simplex $\sigma\colon \Delta^k \to Z$.

Proof. By Lemma 4.14 and Theorem 4.15, the functor

$$(Z, Y) \mapsto H_\bullet^{\mathcal{R}_{G\curvearrowright X}}\big(\text{ind}(Z), \text{ind}(Y)\big)$$

yields a G-homology theory up to dimension, i.e. the composite functor

$$(Z, Y) \mapsto \pi \circ H_\bullet^{\mathcal{R}_{G\curvearrowright X}}\big(\text{ind}(Z), \text{ind}(Y)\big)$$

is an honest G-homology theory with values in $\mathcal{NR}_{G\curvearrowright X}$-Mod$/\mathcal{NR}_{G\curvearrowright X}$-Mod$_0$ (this quotient category was introduced in Remark 1.16).

Classical L^2-homology with $\mathcal{NR}_{G\curvearrowright X}$-coefficients, i.e. the functor

$$(Z, Y) \mapsto H_\bullet\Big(\mathcal{NR}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} \frac{C_\bullet(Z;\mathbb{Z})}{C_\bullet(Y;\mathbb{Z})}\Big)$$

is also a G-homology theory with values in $\mathcal{NR}_{G\curvearrowright X}$-Mod and becomes a G-homology theory with values in $\mathcal{NR}_{G\curvearrowright X}$-Mod$/\mathcal{NR}_{G\curvearrowright X}$-Mod$_0$ if one composes with the quotient functor π. We have

$$H_k^{\mathcal{R}_{G\curvearrowright X}}(X \times G;\mathcal{NR}_{G\curvearrowright X}) \cong_{\dim} H_k\big(\mathcal{NR}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(G;\mathbb{Z})\big) \cong \begin{cases} 0, & k > 0 \\ \mathcal{NR}_{G\curvearrowright X}, & k = 0, \end{cases}$$

and the dimension isomorphism

$$H_k\big(\mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(G;\mathbb{Z})\big) \to H_k^{\mathcal{R}_{G\curvearrowright X}}(X \times G; \mathcal{N}\mathcal{R}_{G\curvearrowright X})$$

is induced by $\sigma \mapsto \mathrm{id}_X \times \sigma$ for a singular simplex $\sigma\colon \Delta^k \to G$. Both homology theories are countably additive, satisfy the dimension axiom and coincide on the trivial free G-space G. As in the case of non equivariant homology theories (which is proved e.g. in [45, Theorem 7.55, p. 123]), it follows that

$$\pi \circ H_k^{\mathcal{R}_{G\curvearrowright X}}(X \times Z; \mathcal{N}\mathcal{R}_{G\curvearrowright X}) \cong \pi \circ H_k\big(\mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})\big)$$

holds for all countable free G-CW-complexes Z. This yields the desired dimension isomorphism (4.13). □

Lemma 4.25. *Let G be a countable group and $G \curvearrowright X$ a standard action on a standard Borel space X. Then for a G-space Z the equation*

$$b_k^{(2)}(Z;\mathcal{N}G) = \dim_{\mathcal{N}\mathcal{R}_{G\curvearrowright X}} H_k\big(\mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})\big). \tag{4.14}$$

holds.

Proof. For the proof of (4.14) note that

$$\mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z}) = \mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathcal{N}G} \mathcal{N}G \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z}).$$

In Corollary 4.11 we have shown that $\mathcal{N}\mathcal{R}_{G\curvearrowright X}$ is a flat $\mathcal{N}G$-module. We therefore obtain

$$H_k\big(\mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})\big) \cong \mathcal{N}\mathcal{R}_{G\curvearrowright X} \otimes_{\mathcal{N}G} H_k\big(\mathcal{N}G \otimes_{\mathbb{Z}G} C_\bullet(Z;\mathbb{Z})\big).$$

Now Lemma 1.21 implies equation (4.14). □

The following result is now a direct consequence of Lemma 4.24 and Lemma 4.25.

Theorem 4.26. *Let G be a countable group and $G \curvearrowright X$ a standard action on a standard Borel space X. Then for a countable free G-CW-complex Z one has*

$$b_k^{(2)}(X \times Z; \mathcal{N}\mathcal{R}_{G\curvearrowright X}) = b_k^{(2)}(Z;\mathcal{N}G) \quad \textit{for all} \quad k \geq 0.$$

4.6 Orbit Equivalence and L^2-Betti Numbers

We will apply the results of the preceding section to reprove that the L^2-Betti numbers of two infinite countable orbit equivalent groups coincide. First, we recall some facts about orbit equivalence.

4.6.1 Orbit Equivalence

Throughout this section, all groups appearing are assumed to be infinite.

Orbit equivalence is an equivalence relation of standard actions $G \curvearrowright X$. Two such actions are called orbit equivalent if they generate the same orbit equivalence relation. More precisely, we have:

Definition 4.27. Let $G \curvearrowright X$ and $H \curvearrowright Y$ be two standard actions. They are called *orbit equivalent* if there is a Borel isomorphism $\varphi\colon X \to Y$ such that $\varphi(G.x) = H.\varphi(x)$ holds for almost all $x \in X$.

Two countable groups G, H are called *orbit equivalent* if there exist orbit equivalent standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ on appropriate standard Borel probability spaces X and Y.

If G and H are two countable amenable groups, then any two ergodic standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are orbit equivalent [12]. On the other hand, no standard action of a non amenable group is orbit equivalent to a standard action of an amenable group [50, 4.3.3]. Hence, the orbit equivalence class of $\mathbb{Z}$ consists precisely of all infinite countable amenable groups.

Gaboriau proved that the L^2-Betti numbers of two countable orbit equivalent groups coincide [22, Théorèm 3.12]. This was also proved by Sauer [41, Theorem 3.38] using Lück's algebraic approach to L^2-Betti numbers. We will provide yet another proof for the orbit equivalence invariance of L^2-Betti numbers using classifying spaces and the singular homology for $\mathcal{R}$-spaces.

4.6.2 Classifying $\mathcal{R}$-Spaces

First, we define the homotopy relation for X-maps and $\mathcal{R}$-maps. This is just an obvious analog of homotopy and G-homotopy, and was tacitly used in the Eilenberg-Steenrod axioms for $\mathcal{R}$-homology.

Definition 4.28. Let S and T be two X-spaces. Two X-maps $f_0, f_1\colon S \to T$ are called *X-homotopic* if there exists an X-map $F\colon S \times [0,1] \to T$ such that $F(\_, j) = f_j$ holds for $j = 0, 1$. We will write $f_0 \simeq_X f_1$ in this case.

An X-map $f\colon S \to T$ is called an *X-homotopy equivalence* if there exists an X-map $g\colon T \to S$ such that $g \circ f \simeq_X \mathrm{id}_S$ and $f \circ g \simeq_X \mathrm{id}_T$ holds.

An X-space S is called *X-contractible* if the projection $p_S\colon S \to X$ is an X-homotopy equivalence.

We will do the same equivariantly.

Definition 4.29. Let S and T be two $\mathcal{R}$-spaces. Two $\mathcal{R}$-maps $f_0, f_1\colon S \to T$ are called *$\mathcal{R}$-homotopic* if there exists an $\mathcal{R}$-map $F\colon S \times [0,1] \to T$ such that $F(\_, j) = f_j$ holds for $j = 0, 1$. We will write $f_0 \simeq_{\mathcal{R}} f_1$ in this case.

An $\mathcal{R}$-map $f\colon S \to T$ is called an *$\mathcal{R}$-homotopy equivalence* if there exists an $\mathcal{R}$-map $g\colon T \to S$ such that $g \circ f \simeq_{\mathcal{R}} \mathrm{id}_S$ and $f \circ g \simeq_{\mathcal{R}} \mathrm{id}_T$ holds.

Remark 4.30. For an X-space S the product $S \times [0,1]$ is an X-space in a natural way. The analog is true for an $\mathcal{R}$-space S.

The following lemma will be useful later on in this section.

Lemma 4.31. *Let Z be a σ-compact second countable Hausdorff space and let T be an $\mathcal{R}$-space. Then for each measurable subset $A \subset X$ there is a bijection*

$$\mathrm{map}_{\mathcal{R}}(\mathcal{R}|_A \times Z, T) \to \mathrm{map}_X(A \times Z, T),$$

where on the right hand side, T is regarded as an X-space by forgetting the $\mathcal{R}$-action.

Proof. We define maps

$$\varphi\colon\ \mathrm{map}_{\mathcal{R}}(\mathcal{R}|_A \times Z, T) \to \mathrm{map}_X(A \times Z, T)$$

and

$$\psi\colon\ \mathrm{map}_X(A \times Z, T) \to \mathrm{map}_{\mathcal{R}}(\mathcal{R}|_A \times Z, T),$$

and show that they are mutually inverse. For an $\mathcal{R}$-map $f\colon \mathcal{R}|_A \times Z \to T$ define the X-map $\varphi(f)\colon A \times Z \to T$ by

$$\varphi(f)(x,z) = f(x,x,z).$$

For an X-map $g\colon A \times Z \to T$ the $\mathcal{R}$-map $\psi(g)\colon \mathcal{R}|_A \times Z \to T$ is defined by

$$\psi(g)\big((x,y),z\big) = (x,y).g(y,z).$$

By direct computation, one checks that $\psi(g)$ is $\mathcal{R}$-equivariant and that $\psi \circ \varphi(f) = f$ and $\varphi \circ \psi(g) = g$ hold. $\square$

The following definition is an analog of the classifying space EG for free G-spaces. The subsequent lemma shows that it has the corresponding property.

Definition 4.32. An $\mathcal{R}$-CW-complex T is called *classifying $\mathcal{R}$-space* if T is contractible as an X-space.

Lemma 4.33. *Let T be a classifying $\mathcal{R}$-space. Then for every countable $\mathcal{R}$-CW-complex S there is up to $\mathcal{R}$-homotopy exactly one $\mathcal{R}$-map $S \to T$.*

Proof. Since T is contractible as an X-space, there is an X-map $s\colon X \to T$ such that $s \circ p_T \simeq_X \mathrm{id}_T$. Hence, there is an X-map $F\colon T \times [0,1] \to T$ satisfying $F(\_,0) = s \circ p_T$ and $F(\_,1) = \mathrm{id}_T$. We will define the $\mathcal{R}$-map $f\colon S \to T$ by induction over the skeleta $S_{[n]}$. Thus, we first have to define f on the 0-skeleton

$$S_{[0]} = \coprod_{i \in I_0} \mathcal{R}|_{A_i}.$$

Define an X-map $A_i \to T$ by $x \mapsto s(x)$ for all $i \in I_0$. This gives rise to an $\mathcal{R}$-map $\mathcal{R}|_{A_i} \to T$ by Lemma 4.31. Hence, we have defined $f_{S_{[0]}}$.

Next, we extend $f|_{S_{[n-1]}}$ to $S_{[n]}$. By the pushout property and Lemma 4.31, it suffices to solve the extension problem

$$\begin{array}{ccc} A_i \times S^{n-1} & \xrightarrow{\varphi_i} & T \\ \downarrow & \nearrow & \\ A_i \times D^n. & & \end{array} \tag{4.15}$$

This is done using the following diagram:

$$\begin{array}{ccccc} A_i \times S^{n-1} \times \{0\} & \xrightarrow{q} & A_i \times \{0\} & & \\ \downarrow j & & \downarrow & \searrow^{s|_{A_i}} & \\ A_i \times S^{n-1} \times [0,1] & \longrightarrow & A_i \times D^n & & \\ & \searrow_{F\circ(\varphi_i \times \mathrm{id}_{[0,1]})} & & \searrow^{\Phi_i} & T \end{array}$$

Due to the pushout property and

$$\begin{aligned} F \circ (\varphi_i \times \mathrm{id}_{[0,1]})(x,r,0) &= F\big(\varphi_i(x,r),0\big) \\ &= s \circ p_T\big(\varphi_i(x,r)\big) \\ &= s(x) \\ &= s|_{A_i} \circ q(x,r,0) \end{aligned}$$

there exists the map Φ_i making the diagram commutative. Note that the lower horizontal map restricted to $A_i \times S^{n-1} \times \{1\}$ is induced by $S^{n-1} \hookrightarrow D^n$. Hence, the map Φ_i solves the extension problem (4.15). It follows that we can extend f to the n-skeleton $S_{[n]}$ of S. By induction (and a colimit argument if S is not finite dimensional), this yields an $\mathcal{R}$-map $f\colon S \to T$.

The proof that two $\mathcal{R}$-maps $f_0, f_1\colon S \to T$ are $\mathcal{R}$-homotopic is a simple application of the preceding arguments. In fact, to define a homotopy between f_0 and f_1 on the 0-skeleton $S_{[0]} = \coprod_{i \in I_0} \mathcal{R}|_{A_i}$, we have to solve the extension problem

$$\begin{array}{ccc} A_i \times \{0,1\} & \xrightarrow{f_0 \amalg f_1} & T \\ \downarrow & \nearrow & \\ A_i \times [0,1] & & \end{array}$$

and use Lemma 4.31 afterwards. But this extension problem is just (4.15) for $n = 1$ which was solved above. To extend the homotopy from $S_{[n-1]}$ to $S_{[n]}$, one has to complete the diagram

$$\begin{array}{ccc} A_i \times \big(S^{n-1} \times [0,1] \cup D^n \times \{0,1\}\big) & \longrightarrow & T \\ \downarrow & \nearrow & \\ A_i \times D^n \times [0,1]. & & \end{array}$$

But the vertical map is just the inclusion $A_i \times S^n \hookrightarrow A_i \times D^{n+1}$, so we get (4.15) again. □

Definition 4.34. The $\mathcal{R}$-homotopy type of an X-contractible $\mathcal{R}$-CW-complex is denoted by $E\mathcal{R}$. Any $\mathcal{R}$-CW-complex in that homotopy class is called a *model for* $E\mathcal{R}$.

As usual in the situation of classifying spaces for groups, we will sometimes write $E\mathcal{R}$ also for models of $E\mathcal{R}$. The following observation is an obvious consequence of Lemma 4.33.

Corollary 4.35. *Let G be a countable group and $G \curvearrowright X$ be a standard action. Then*

$$X \times EG \simeq_{\mathcal{R}} E\mathcal{R}_{G \curvearrowright X}.$$

4.6.3 Orbit Equivalence Invariance of L^2-Betti Numbers

We are now prepared to show Gaboriau's theorem that the L^2-Betti numbers of orbit equivalent groups coincide with help of our methods.

Theorem 4.36. *Let G and H be two infinite countable groups. If G and H are orbit equivalent, then*

$$b_k^{(2)}(G) = b_k^{(2)}(H) \quad \text{for all } k \geq 0$$

holds.

Proof. First, we can assume by means of the orbit equivalence $\varphi\colon X \to Y$ that G and H act on the same standard Borel space X. Moreover, we can assume that both actions are free by restricting to a full measure subset which will also be denoted by X (compare Remark 3.34). In this situation, we have $\mathcal{R}_{G \curvearrowright X} = \mathcal{R}_{H \curvearrowright X}$. Consequently, there is a map $\sigma\colon G \times X \to H$ which is defined by

$$g.x = \sigma(g,x).x.$$

The map σ is called a *cocycle* and satisfies the *cocycle condition*

$$\sigma(g'g,x) = \sigma(g',gx)\sigma(g,x).$$

Corollary 4.35 implies that $X \times EH$ is a model for $E\mathcal{R}_{H \curvearrowright X}$. Due to $\mathcal{R}_{G \curvearrowright X} = \mathcal{R}_{H \curvearrowright X}$ we know that $X \times EH$ is obviously also an $\mathcal{R}_{G \curvearrowright X}$-space. The $\mathcal{R}_{G \curvearrowright X}$-action can be written as

$$(gx,x).(x,z) = \big(gx, \sigma(g,x).z\big).$$

Hence, $X \times EH$ and $X \times EG$ are both models for $E\mathcal{R}_{G \curvearrowright X}$ and therefore Lemma 4.33 yields $X \times EH \simeq_{\mathcal{R}} X \times EG$. By $\mathcal{R}$-homotopy invariance of singular L^2-homology we obtain

$$b_k^{(2)}(X \times EG; \mathcal{N}\mathcal{R}_{G \curvearrowright X}) = b_k^{(2)}(X \times EH; \mathcal{N}\mathcal{R}_{H \curvearrowright X}).$$

The result follows since by Theorem 4.26

$$b_k^{(2)}(G) = b_k^{(2)}(EG; \mathcal{N}G) = b_k^{(2)}(X \times EG; \mathcal{N}\mathcal{R}_{G \curvearrowright X})$$

and
$$b_k^{(2)}(H) = b_k^{(2)}(EH;\mathcal{N}H) = b_k^{(2)}(X \times EH;\mathcal{N}\mathcal{R}_{H\curvearrowright X}). \qquad \square$$

Remark 4.37. The not explicitly given $\mathcal{R}$-homotopy equivalence $X \times EG \to X \times EH$ is an example for an $\mathcal{R}$-map which is not induced by a G-map.

5 Integral Foliated Simplicial Volume and L^2-Betti Numbers

The following conjecture is due to Gromov [26, p.232].

Conjecture 5.1. *Let M be a closed oriented connected aspherical manifold with $\|M\| = 0$. Then*

$$b_k^{(2)}(\widetilde{M}) = 0 \quad \textit{for all} \quad k \geq 0.$$

There is some evidence for this conjecture due to similar behavior of simplicial volume and L^2-Betti numbers in certain situations, although the definitions of the two invariants would not indicate a connection between them. We will review some results of the first two chapters under this aspect in Section 5.1.

In his influential and inspiring book [27] Gromov indicates how one could try to prove Conjecture 5.1. The idea is to define a new invariant, the *integral foliated simplicial volume* $\|M\|_{\mathcal{F},\mathbb{Z}}$ which bounds the sum of the L^2-Betti numbers of $\widetilde{M}$ up to a multiplicative constant. In this chapter, we give a concise definition of $\|M\|_{\mathcal{F},\mathbb{Z}}$ and show

$$\sum_{k=0}^{n} b_k^{(2)}(\widetilde{M}) \leq 2^{n+1} \cdot \|M\|_{\mathcal{F},\mathbb{Z}}.$$

This estimate appears as an exercise in [27, p. 307].

To prove Conjecture 5.1 one could try to show $\|M\| = \|M\|_{\mathcal{F},\mathbb{Z}}$ for aspherical manifolds M. Unfortunately, very little is known about the integral foliated simplicial volume $\|M\|_{\mathcal{F},\mathbb{Z}}$. In Section 5.2 we will define $\|M\|_{\mathcal{F},\mathbb{Z}}$ and derive some properties.

5.1 Overview of Gromov's Conjecture

Some evidence for Gromov's conjecture comes from similar behavior of the simplicial volume $\|M\|$ and the L^2-Betti numbers $b_k^{(2)}(\widetilde{M})$ in certain situations. For example, simplicial volume is multiplicative under finite coverings (cf. Lemma 2.8). The same is true for L^2-Betti numbers of the universal coverings, as was remarked in Corollary 1.30.

If the fundamental group $\pi_1(M)$ is amenable, then $\|M\| = 0$ by Corollary 2.15 (iv). For aspherical manifolds with amenable fundamental group we get $b_k^{(2)}(\widetilde{M}) = 0$ for all $k \geq 0$ as well by Theorem 1.33.

Recall that there is a proportionality principle for simplicial volume: If M and N are Riemannian manifolds with isometric universal covering $\widetilde{M} \cong \widetilde{N}$, then

$$\frac{\|M\|}{\mathrm{vol}(M)} = \frac{\|N\|}{\mathrm{vol}(N)}$$

by Theorem 2.20. The same proportionality principle holds for L^2-Betti numbers by Theorem 1.36.

Conjecture 5.1 holds for manifolds M with $\dim(M) \leq 3$: The only closed oriented connected 1-dimensional manifold is S^1 and we already know that $\|S^1\| = 0$ as well as $b_k^{(2)}(\widetilde{S^1}) = 0$ for all $k \geq 0$ by Corollary 2.7 and Corollary 1.30, respectively. In dimension 2, the torus $T^2 = S^1 \times S^1$ is the only closed oriented connected aspherical manifold with vanishing simplicial volume (note that oriented surfaces of genus $g \geq 2$ admit a hyperbolic Riemannian metric and have therefore positive simplicial volume by Theorem 2.16). We clearly have $b_k^{(2)}(\widetilde{T^2}) = 0$ for all $k \geq 0$, e.g. by Corollary 1.30. In dimension 3, it follows from the computation in [34] that the Singer Conjecture 1.38 holds [38, Section 11.1]. Of course, in odd dimensions n the Singer Conjecture is stronger than Conjecture 5.1 since it predicts that all L^2-Betti numbers vanish for the universal covering of a closed aspherical manifold of dimension n.

Gromov's conjecture becomes false if one drops the condition that the manifold in question is aspherical. Indeed, for any simply connected manifold M one has $\|M\| = 0$ by Corollary 2.15 (iv), but in this case the L^2-Betti numbers are just the ordinary Betti numbers, and hence $b_0^{(2)}(\widetilde{M}) \neq 0$.

One could ask if the converse of Gromov's conjecture holds. The answer is negative since for odd dimensional hyperbolic manifolds M (which are of course aspherical by the Cauchy-Hadamard theorem) one has $\|M\| > 0$ by Theorem 2.16, but $b_k^{(2)}(\widetilde{M}) = 0$ by Theorem 1.37.

5.2 The Integral Foliated Simplicial Volume

We will first establish a connection between the fundamental class of a closed connected oriented manifold M and the L^2-Betti numbers of $\widetilde{M}$. It is based on the equivariant Poincaré chain homotopy equivalence which is given by the cap product with a fundamental cycle. We obtain a bound for the sum of L^2-Betti numbers in terms of the fundamental cycle and will improve this bound later in this chapter using $\mathcal{R}$-homology.

5.2.1 A Bound for L^2-Betti Numbers in Terms of a Fundamental Cycle

Let us recall the definition of the cap product.

Definition 5.2. Let Z be a connected space with fundamental group $G = \pi_1(Z)$. Denote by $C_\bullet(\widetilde{Z};\mathbb{Z})$ the singular $\mathbb{Z}G$-chain complex of its universal covering $\widetilde{Z}$. We write $\hom_{\mathbb{Z}G}\big(C_\bullet(\widetilde{Z};\mathbb{Z}),\mathcal{N}G\big)$ for the $\mathcal{N}G$-cochain complex associated to the $\mathcal{N}G$-chain complex $\mathcal{N}G\otimes_{\mathbb{Z}G}C_\bullet(\widetilde{Z};\mathbb{Z})$. The *cap product* is defined as the map

$$\begin{aligned}\hom_{\mathbb{Z}G}\big(C_j(\widetilde{Z};\mathbb{Z}),\mathcal{N}G\big)\otimes_{\mathbb{Z}}\mathbb{Z}\otimes_{\mathbb{Z}G}C_n(\widetilde{Z},\mathbb{Z}) &\xrightarrow{\ \_\cap\_\ } \mathcal{N}G\otimes_{\mathbb{Z}G}C_{n-j}(\widetilde{Z};\mathbb{Z})\\ \varphi\otimes k\otimes\sigma &\longmapsto \overline{k\cdot\varphi(\sigma\rfloor_j)}\otimes{}_{n-j}\lfloor\sigma. \qquad (5.1)\end{aligned}$$

Remark 5.3. (i) By $\sigma\rfloor_j$ we denote the *front j-face* of a singular n-simplex σ. It is the j-simplex given by composing σ with the inclusion $\Delta^j\to\Delta^n$ induced by $e_i\mapsto e_i$ on the set of vertices. Analogously, the *back* $(n-j)$*-face* ${}_{n-j}\lfloor\sigma$ of σ is defined as the composition of σ with the inclusion $\Delta^{n-j}\to\Delta^n$ induced by $e_i\mapsto e_{j+i}$.

(ii) We have $\mathbb{Z}\otimes_{\mathbb{Z}G}C_n(\widetilde{Z},\mathbb{Z})\cong C_n(Z,\mathbb{Z})$.

(iii) If M is a closed connected oriented manifold of dimension n with fundamental group $G=\pi_1(M)$ and $\sum_{i=1}^k\lambda_i\otimes\sigma_i\in\mathbb{Z}\otimes_{\mathbb{Z}G}C_n(\widetilde{M},\mathbb{Z})$ is a fundamental cycle of M (i.e. a representative of the fundamental class $[M]$), then

$$\hom_{\mathbb{Z}G}\big(C_j(\widetilde{M};\mathbb{Z}),\mathcal{N}G\big)\xrightarrow{\ \_\cap\sum_{i=1}^k\lambda_i\otimes\sigma_i\ }\mathcal{N}G\otimes_{\mathbb{Z}G}C_{n-j}(\widetilde{M};\mathbb{Z})$$

is an $\mathcal{N}G$-homotopy equivalence. This equivariant version of Poincaré duality follows from [47, Theorem 2.1 on page 23].

Now we are ready to prove the following result.

Proposition 5.4. *Let M be a closed connected oriented manifold of dimension n and let further $\sum_{i=1}^k\lambda_i\cdot\sigma_i\in C_n(M;\mathbb{Z})$ be a fundamental cycle of M. Then*

$$b_j^{(2)}(\widetilde{M})\le\binom{n+1}{j}\cdot k$$

holds for all $j\ge 0$, and consequently

$$\sum_{j=0}^n b_j^{(2)}(\widetilde{M})\le 2^{n+1}\cdot k.$$

Proof. Consider the evaluation morphism of $\mathcal{N}G$-modules (where $G=\pi_1(M)$):

$$\begin{aligned}\mathrm{ev}_{n-j}\colon \hom_{\mathbb{Z}G}\big(C_{n-j}(\widetilde{M}),\mathcal{N}G\big) &\longrightarrow \bigoplus_{i=1}^k\ \bigoplus_{\substack{(n-j)\text{-faces}\\ \text{of }\sigma_i}}\mathcal{N}G\\ \varphi &\longmapsto \big(\varphi(\sigma_i^l)\big),\end{aligned}$$

where $\sigma_i^1,\dots,\sigma_i^{\binom{n+1}{j}}$ denote the $(n-j)$-faces of σ_i.

We define

$$H^\bullet_G(\widetilde{M};\mathcal{N}G) = H^\bullet\big(\hom_{\mathbb{Z}G}\big(C_\bullet(\widetilde{M}),\mathcal{N}G\big)\big).$$

Let $\psi \in H^{n-j}_G(\widetilde{M};\mathcal{N}G)$ be a cohomology class which can be represented by a cocycle $\varphi \in \hom_{\mathbb{Z}G}(C_{n-j}(\widetilde{M}),\mathcal{N}G)$ with $\varphi \in \ker(\mathrm{ev}_{n-j})$. In other words, we have

$$\varphi(\sigma_i^l) = 0 \quad \text{for all} \quad i = 1,\dots,k, \quad l = 1,\dots,\binom{n+1}{j}.$$

By definition of the cap product (cf. (5.1)), we get $\mathrm{ev}_{n-j}(\varphi) = 0$ and hence

$$\ker(\mathrm{ev}_{n-j}) \subset \ker\big(\_ \cap \textstyle\sum_{i=1}^k \lambda_i \otimes \sigma_i\big).$$

Since $\_ \cap [M]\colon H^{n-j}_G(\widetilde{M};\mathcal{N}G) \xrightarrow{\cong} H^G_j(\widetilde{M};\mathcal{N}G)$ is an isomorphism by the Poincaré duality Theorem (cf. Remark 5.3 (iii)), we conclude $\psi = 0$. Hence the projection $\mathrm{pr}\colon \ker(\delta_{n-j}) \to H^{n-j}_G(\widetilde{M};\mathcal{N}G)$ factorizes over $\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j})\cap\ker(\delta_{n-j})}$, i.e. we get a commutative diagram

$$\begin{array}{ccc}
\ker(\delta_{n-j}) & \xrightarrow{\quad\mathrm{pr}\quad} & H^{n-j}_G(\widetilde{M};\mathcal{N}G) \\
& \searrow \qquad\qquad \nearrow_{\overline{\mathrm{pr}}} & \\
& \dfrac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j})\cap\ker(\delta_{n-j})}. &
\end{array}$$

Here, $\delta_\bullet$ is the differential in the cochain complex $\hom_{\mathbb{Z}G}(C_\bullet(\widetilde{M}),\mathcal{N}G)$. Since $\overline{\mathrm{pr}}$ is surjective, additivity of the von Neumann dimension yields

$$\dim_{\mathcal{N}G}\big(H^{n-j}_G(\widetilde{M};\mathcal{N}G)\big) \leq \dim_{\mathcal{N}G}\Big(\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j})\cap\ker(\delta_{n-j})}\Big). \tag{5.2}$$

Moreover, we have the following composition of injective $\mathcal{N}G$-morphisms, where the right map is induced by ev_{n-j}:

$$\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j})\cap\ker(\delta_{n-j})} \to \frac{\hom_{\mathbb{Z}G}\big(C_{n-j}(\widetilde{M}),\mathcal{N}G\big)}{\ker(\mathrm{ev}_{n-j})} \xrightarrow{\overline{\mathrm{ev}_{n-j}}} \bigoplus_{\substack{(n-j)\text{-faces}\\ \text{of the } \sigma_i}} \mathcal{N}G.$$

Again by additivity of the von Neumann dimension, we get

$$\begin{aligned}
\dim_{\mathcal{N}G}\Big(\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j})\cap\ker(\delta_{n-j})}\Big) &\leq \dim_{\mathcal{N}G}\Big(\frac{\hom_{\mathbb{Z}G}\big(C_{n-j}(\widetilde{M}),\mathcal{N}G\big)}{\ker(\mathrm{ev}_{n-j})}\Big) \\
&\leq \dim_{\mathcal{N}G}\Big(\bigoplus_{i=1}^k \bigoplus_{\substack{(n-j)\text{-faces}\\ \text{of } \sigma_i}} \mathcal{N}G\Big) \\
&\leq \binom{n+1}{j}\cdot k.
\end{aligned} \tag{5.3}$$

Poincarè duality provides

$$b_j^{(2)}(\widetilde{M}) = \dim_{\mathcal{N}G}\big(H_G^{n-j}(\widetilde{M};\mathcal{N}G)\big),$$

and the result follows immediately by combining (5.2) and (5.3). □

Remark 5.5. The same argument applies to the usual Betti numbers, i.e. we get

$$b_j(M) \leq \binom{n+1}{j} \cdot k$$

for all $j \geq 0$, and hence

$$\sum_{j=0}^{n} b_j(M) \leq 2^{n+1} \cdot k.$$

If we define the *integral simplicial volume* $\|M\|_{\mathbb{Z}}$ as the minimum of $\sum_{i=1}^{k}|\lambda_i|$ for all fundamental cycles $\sum_{i=1}^{k} \lambda_i \cdot \sigma_i \in C_n(M;\mathbb{Z})$, Proposition 5.4 yields the following consequence.

Corollary 5.6. *Let M be a closed connected oriented manifold of dimension n. Then*

$$b_j^{(2)}(\widetilde{M}) \leq \binom{n+1}{j} \cdot \|M\|_{\mathbb{Z}},$$

for all $j \geq 0$, and therefore also

$$\sum_{j=0}^{n} b_j^{(2)}(\widetilde{M}) \leq 2^{n+1} \cdot \|M\|_{\mathbb{Z}}.$$

Unfortunately, $\|M\|_{\mathbb{Z}}$ is not closely related to the simplicial volume $\|M\|$. For example, one always has $\|M\|_{\mathbb{Z}} \geq 1$. Hence, the preceding results do not get us closer to Gromov's Conjecture. The problem is that the coefficients do not show up in the upper bound in Proposition 5.4, only the number of summands appears (and can be bounded by $\|M\|_{\mathbb{Z}}$ since the absolute value of any nonzero integer is at least one). In the following, we will try to resolve this defect by introducing the integral foliated simplicial volume.

5.2.2 A Generalized Cap Product

First, we have to generalize the definition of the cap product. In the following, X will denote a standard Borel probability space with a standard action of the fundamental group $G = \pi_1(M)$. We first introduce another description of the von Neumann algebra $\mathcal{NR}_{G \curvearrowright X}$ in terms of the crossed product ring $L^\infty(X) * G$ which is defined as follows.

Definition 5.7. Let G be a countable group and let $G \curvearrowright X$ be a standard action. The *crossed product ring* $L^\infty(X) * G$ has $L^\infty(X)[G]$ as underlying abelian group, and its multiplication is uniquely determined by the rule

$$g \cdot f = (f \circ l_{g-1}) \cdot g,$$

where $l_h \colon X \to X$ is given by $x \mapsto h.x$.

Remark 5.8. For the product of two elements $\sum_{g \in G} f_g \cdot g$ and $\sum_{g \in G} h_g \cdot g$ in the crossed product ring $L^\infty(X) * G$ we obtain

$$\big(\textstyle\sum_{g \in G} f_g \cdot g\big) \cdot \big(\textstyle\sum_{g \in G} h_g \cdot g\big) = \displaystyle\sum_{g \in G} \Big(\sum_{\substack{g_1, g_2 \in G \\ g_1 g_2 = g}} f_{g_1} \cdot (h_{g_2} \circ l_{g_1^{-1}}) \Big) \cdot g.$$

The map

$$\begin{aligned} L^\infty(X) * G &\to \mathbb{C}\mathcal{R}_{G \curvearrowright X} \\ \sum_{g \in G} f_g \cdot g &\mapsto \big((h.x, x) \mapsto f_h(h.x)\big) \end{aligned}$$

is an embedding of rings. Note that this map is well defined since the action $G \curvearrowright X$ is essentially free.

Definition 5.9. The von Neumann algebra $\mathcal{N}\big(L^\infty(X) * G\big)$ of the crossed product ring $L^\infty(X) * G$ is defined as the weak closure of $\rho_r\big(L^\infty(X) * G\big)$ in $L\big(L^2(\mathcal{R}_{G \curvearrowright X})\big)$.

The map ρ_r was defined in 4.1.2 on page 41. The following classical result yields another construction of $\mathcal{N}\mathcal{R}_{G \curvearrowright X}$ [39, IV, p. 192 ff.].

Lemma 5.10. *Let G be a countable group and let $G \curvearrowright X$ be a standard action. Then we have $\mathcal{N}\mathcal{R}_{G \curvearrowright X} = \mathcal{N}\big(L^\infty(X) * G\big)$.*

We will now describe the generalization of the classical cap product. We do this a little more general than we will actually use it since then the approach becomes more conceptional.

Let A be a ring with involution which contains $\mathbb{Z}G$ as a subring. Then A has a natural $\mathbb{Z}G$-bimodule structure. Let $B \subset A$ be a subring closed under involution such that for all $b \in B$ and $g \in G$ the product $g \cdot b \cdot g^{-1}$ (the product is built in A) is again in B. We will consider B as a right $\mathbb{Z}G$-module by $b \star g = g^{-1} \cdot b \cdot g$. We use the symbol $\star$ to avoid confusion with the right module structure on A.

Next, we explain other module structures we will use.

- As usual, we define a left $\mathbb{Z}G$-module structure on B by

$$g \star b = b \star g^{-1} = g \cdot b \cdot g^{-1}.$$

The corresponding left $\mathbb{Z}G$-module will be denoted by B^l. In the same way, we define a right $\mathbb{Z}G$-module structure on $C_n(\widetilde{M})$ by $\sigma \cdot g = g^{-1} \cdot \sigma$. The corresponding right $\mathbb{Z}G$-module will be denoted by $C_n^r(\widetilde{M})$.

- On $\hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big) \otimes_{\mathbb{Z}} C^r_n(\widetilde{M})$, a right $\mathbb{Z}G$ module structure is given by
$$(\varphi \otimes \sigma) \cdot g = \varphi \otimes g^{-1} \cdot \sigma.$$
Together with the left A-module structure given by $a \cdot (\varphi \otimes \sigma) = \varphi \cdot \overline{a} \otimes \sigma$, this turns $\hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big) \otimes_{\mathbb{Z}} C^r_n(\widetilde{M})$ into an A-$\mathbb{Z}G$-bimodule.

- On $A \otimes_Z C^r_{n-j}(\widetilde{M})$, a right $\mathbb{Z}G$-module structure is given by
$$(a \otimes \sigma) \cdot g = a \cdot g \otimes g^{-1} \cdot \sigma.$$
In fact, $A \otimes_Z C^r_{n-j}(\widetilde{M})$ becomes an A-$\mathbb{Z}G$-bimodule with the natural left A-module structure.

We will introduce some homomorphisms which will appear in the definition of the generalized cap product.

- The map
$$\begin{aligned} T\colon B \otimes_{\mathbb{Z}G} C_n(\widetilde{M}) &\to C^r_n(\widetilde{M}) \otimes_{\mathbb{Z}G} B^l \\ b \otimes \sigma &\mapsto \sigma \otimes b \end{aligned}$$
yields a well defined homomorphism of abelian groups.

- The cap product
$$\begin{aligned} \hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big) \otimes_{\mathbb{Z}} C^r_n(\widetilde{M}) &\xrightarrow{\ \cap\ } A \otimes_Z C^r_{n-j}(\widetilde{M}) \\ \varphi \otimes \sigma &\mapsto \overline{\varphi(\sigma\rfloor_j)} \otimes {}_{n-j}\lfloor\sigma \end{aligned}$$
yields a homomorphism of A-$\mathbb{Z}G$-bimodules.

- The map
$$\begin{aligned} m\colon A \otimes_{\mathbb{Z}} C^r_{n-j}(\widetilde{M}) \otimes_{\mathbb{Z}G} B^l &\to A \otimes_{\mathbb{Z}G} C_{n-j}(\widetilde{M}) \\ a \otimes \sigma \otimes b &\mapsto a \cdot \overline{b} \otimes \sigma \end{aligned}$$
yields a well defined homomorphism of A-modules.

These claims are easily checked by direct computation. To see that m is well defined, we calculate
$$\begin{aligned} m(a \otimes \sigma \otimes g \star b) &= m(a \otimes \sigma \otimes g \cdot b \cdot g^{-1}) \\ &= a \cdot \overline{g \cdot b \cdot g^{-1}} \otimes \sigma \\ &= a \cdot g \cdot \overline{b} \cdot g^{-1} \otimes \sigma \\ &= a \cdot g \cdot \overline{b} \otimes g^{-1} \cdot \sigma \\ &= m(a \cdot g \otimes g^{-1} \cdot \sigma \otimes b) \\ &= m\big((a \otimes \sigma) \cdot g \otimes b\big). \end{aligned}$$

Definition 5.11. The *generalized cap product* is given by the composition

$$\begin{array}{c} \hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big) \otimes_{\mathbb{Z}} B \otimes_{\mathbb{Z}G} C_n(\widetilde{M}) \\ \Big\downarrow {\scriptstyle \mathrm{id} \otimes T} \\ \hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big) \otimes_{\mathbb{Z}} C^r_n(\widetilde{M}) \otimes_{\mathbb{Z}G} B^l \\ \Big\downarrow {\scriptstyle \cap \otimes \mathrm{id}} \\ A \otimes_{\mathbb{Z}} C^r_{n-j}(\widetilde{M}) \otimes_{\mathbb{Z}G} B^l \\ \Big\downarrow {\scriptstyle m} \\ A \otimes_{\mathbb{Z}G} C_{n-j}(\widetilde{M}) \end{array}$$

of A-module homomorphisms.

By minor abuse of notation, we will write $\varphi \cap c$ for the generalized cap product of a cochain $\varphi \in \hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big)$ and a chain $c \in B \otimes_{\mathbb{Z}G} C_n(\widetilde{M})$.

The generalized cap product is then explicitly given by

$$\begin{aligned} \hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}), A\big) \otimes_{\mathbb{Z}} B \otimes_{\mathbb{Z}G} C_n(\widetilde{M}) &\xrightarrow{\ \cap\ } A \otimes_{\mathbb{Z}G} C_{n-j}(\widetilde{M}) \\ \varphi \otimes b \otimes \sigma &\mapsto \overline{\varphi(\sigma\rfloor_j)} \cdot \overline{b} \otimes {}_{n-j}\lfloor \sigma. \end{aligned}$$

Remark 5.12. One checks easily that

$$\partial_{n-j}\big(\varphi \cap c\big) = (-1)^j \cdot \big(\delta_{j-1}(\varphi) \cap c - \varphi \cap \partial_n(c)\big).$$

The computation works as in the classical situation [28, p. 240]. Consequently, the generalized cap product induces a map (also called cap product) in (co)homology.

From now on, we will only deal with the case

$$A = \mathcal{N}\big(L^\infty(X) * G\big) = \mathcal{N}\mathcal{R}_{G \curvearrowright X}$$

and

$$B = L^\infty(X, \mathbb{Z}).$$

Remark 5.13. In the case $B = L^\infty(X, \mathbb{Z})$, the right $\mathbb{Z}G$-module structure is given by

$$(f \star g)(x) = (g^{-1} \cdot f \cdot g)(x) = f(g.x).$$

Note that $L^\infty(X, \mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M})$ is *not* an $L^\infty(X, \mathbb{Z})$-module, because $L^\infty(X, \mathbb{Z})$ is not an $L^\infty(X, \mathbb{Z})$-$\mathbb{Z}G$-bimodule. In fact, for $f, f' \in L^\infty(X, \mathbb{Z})$ and $g \in G$ we have

$$(f \cdot f') \star g(x) = (f \cdot f')(g.x) = f(g.x) \cdot f'(g.x) \neq f(x) \cdot f'(g.x) = f \cdot \big(f' \star g\big)(x).$$

5.2.3 Generalized Poincaré Duality

We will now mildly extend the Poincaré duality theorem. Let M be an n-dimensional closed connected oriented manifold with fundamental group $G = \pi_1(M)$ and let X be a standard Borel space with probability measure μ provided with an essentially free action of G by measure preserving Borel isomorphisms.

In the following we will use the terms M, G, X and $\mathcal{R}$ as explained above without further mentioning.

Definition 5.14. Let

$$i_1 \colon \mathbb{Z} \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z}) \to L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$$

be the homomorphism of abelian groups given by

$$i_1(1 \otimes \sigma) = \mathrm{const}_1 \otimes \sigma,$$

where $\mathrm{const}_1 \colon X \to \mathbb{Z}$ is the constant function with value 1.

Remark 5.15. It is clear that i_1 is a chain map. If we want to emphasize the manifold M, we will write i_1^M.

With aid of the morphisms given in Definition 5.14, we can now link the classical cap product with the extended cap product. If no coefficients are given explicitly they are understood to be integral.

Lemma 5.16. *The diagram*

$$\begin{array}{ccc}
\hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}),\mathcal{NR}\big) \otimes_{\mathbb{Z}} \mathbb{Z} \otimes_{\mathbb{Z}G} C_n(\widetilde{M}) & \xrightarrow{\ _\cap_\ } & \mathcal{NR} \otimes_{\mathbb{Z}G} C_{n-j}(\widetilde{M}) \\
{=}\big\uparrow \qquad\qquad i_1\big\downarrow & & \big\downarrow{=} \\
\hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}),\mathcal{NR}\big) \otimes_{\mathbb{Z}} L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M}) & \xrightarrow{\ _\cap_\ } & \mathcal{NR} \otimes_{\mathbb{Z}G} C_{n-j}(\widetilde{M})
\end{array}$$

commutes, i.e. for a cochain $\varphi \in \hom_{\mathbb{Z}G}\big(C_j(\widetilde{M}),\mathcal{NR}\big)$ *and a chain* $c \in \mathbb{Z} \otimes_{\mathbb{Z}G} C_n(\widetilde{M})$ *one gets*

$$\varphi \cap c = \varphi \cap i_1(c).$$

Proof. It suffices to compute the terms in question for $c = 1 \otimes \sigma$, where $\sigma \colon \Delta^n \to \widetilde{M}$ is a singular simplex. We get

$$\begin{aligned}
\varphi \cap (1 \otimes \sigma) &= \overline{\varphi(\sigma\rfloor_j)} \otimes {}_{n-j}\lfloor\sigma \\
&= \overline{\varphi(\sigma\rfloor_j)} \cdot \chi_{\Delta_X} \otimes {}_{n-j}\lfloor\sigma \\
&= \overline{\chi_{\Delta_X} \cdot \varphi(\sigma\rfloor_j)} \otimes {}_{n-j}\lfloor\sigma \\
&= \overline{\mathrm{const}_1 \cdot \varphi(\sigma\rfloor_j)} \otimes {}_{n-j}\lfloor\sigma \\
&= \varphi \cap \big(\mathrm{const}_1 \otimes \sigma\big) \\
&= \varphi \cap i_1(1 \otimes \sigma)
\end{aligned}$$

since χ_{Δ_X}, the characteristic function of the diagonal $\Delta_X = \{(x,x) : x \in X\}$, is the identity in $\mathcal{NR}$ and moreover the image of const_1 under the obvious inclusion $L^\infty(X,\mathbb{Z}) \to \mathbb{Z}\mathcal{R}$. □

We immediately get the following Poincaré type theorem as a corollary. The notations H_j^G and H_G^j are analogs of those introduced in Definition 1.25 and Remark 1.26 respectively.

Corollary 5.17. *Let $[M] \in H_n(M;\mathbb{Z})$ be the fundamental class. Then the $\mathcal{NR}$-homomorphism*

$$\_ \cap (i_1)_*\big([M]\big) \colon H_G^j(\widetilde{M};\mathcal{NR}) \to H_{n-j}^G(\widetilde{M};\mathcal{NR})$$

is an isomorphism.

Definition 5.18. We will call $(i_1)_*\big([M]\big)$ the *measurable fundamental class* and its representatives *measurable fundamental cycles*.

5.2.4 A Better Upper Bound

We will now use the results of the previous subsection to improve the upper bound for L^2-Betti numbers in Proposition 5.4.

Theorem 5.19. *Let M be a closed connected oriented manifold of dimension n and let X be as above. Let further*

$$\sum_{i=1}^{k} f_i \otimes \sigma_i \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$$

be a measurable fundamental cycle. Then we have

$$b_j^{(2)}(\widetilde{M}) \leq \binom{n+1}{j} \cdot \sum_{i=1}^{k} \mu\big(\mathrm{supp}(f_i)\big) \tag{5.4}$$

for all $j \geq 0$, and consequently

$$\sum_{j=0}^{n} b_j^{(2)}(\widetilde{M}) \leq 2^{n+1} \cdot \sum_{i=1}^{k} \mu\big(\mathrm{supp}(f_i)\big). \tag{5.5}$$

Proof. The idea is to refine the proof of Proposition 5.4 in such a way that the measures of the supports of the coefficient functions f_i will enter the upper bound for the sum of the L^2-Betti numbers.

Consider the evaluation homomorphism

$$\begin{aligned} \mathrm{ev}_{n-j} \colon \hom_{\mathbb{Z}G}\big(C_{n-j}(\widetilde{M}),\mathcal{NR}\big) &\longrightarrow \bigoplus_{i=1}^{k} \bigoplus_{\substack{(n-j)\text{-faces}\\ \text{of } \sigma_i}} \mathcal{NR} \cdot f_i \\ \varphi &\longmapsto \overline{\big(f_i \cdot \varphi(\sigma_i^l)\big)}, \end{aligned}$$

where $\sigma_i^1, \ldots, \sigma_i^{\binom{n+1}{j+1}}$ denote the $(n-j)$-faces of σ_i. Note that

$$\overline{f_i \cdot \varphi(\sigma_i^l)} = \overline{\varphi(\sigma_i^l)} \cdot f_i,$$

because $f_i = \overline{f_i}$ holds. Hence, the image of ev_{n-j} is really contained in

$$\bigoplus_{i=1}^{k} \bigoplus_{\substack{(n-j)\text{-faces} \\ \text{of } \sigma_i}} \mathcal{NR} \cdot f_i.$$

Write $z = \sum_{i=1}^k f_i \otimes \sigma_i$ for short and let $\psi \in H_G^{n-j}(\widetilde{M}; \mathcal{NR})$ be a cohomology class which can be represented by a cocycle in $\ker(\mathrm{ev}_{n-j})$. By definition of the cap product, we obtain

$$\varphi \cap z = 0.$$

Hence we have shown $\ker(\mathrm{ev}_{n-j}) \subset \ker(\_ \cap i_1(z))$. Since $\_ \cap i_1(z)$ induces an isomorphism in homology, it follows that the projection

$$\ker(\delta_{n-j}) \to H_G^{n-j}(\widetilde{M}; \mathcal{NR})$$

factorizes over

$$\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j}) \cap \ker(\delta_{n-j})},$$

where $\delta_\bullet$ is the boundary of the cochain complex $\hom_{\mathbb{Z}G}\big(C_\bullet(\widetilde{M}), \mathcal{NR}\big)$. Hence we get the following commutative diagram:

$$\begin{array}{ccc} \ker(\delta_{n-j}) & \longrightarrow & H_G^{n-j}(\widetilde{M}; \mathcal{NR}) \\ & \searrow \qquad \nearrow_{\overline{\mathrm{pr}}} & \\ & \frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j}) \cap \ker(\delta_{n-j})} & \end{array}$$

Due to additivity of the von Neumann dimension, we conclude

$$\dim_{\mathcal{NR}}\big(H_G^{n-j}(\widetilde{M}; \mathcal{NR})\big) \leq \dim_{\mathcal{NR}}\Big(\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j}) \cap \ker(\delta_{n-j})}\Big). \tag{5.6}$$

Consider the composition of injective $\mathcal{NR}$-homomorphisms

$$\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j}) \cap \ker(\delta_{n-j})} \to \frac{\hom_{\mathbb{Z}G}\big(C_{n-j}(\widetilde{M}), \mathcal{NR}\big)}{\ker(\mathrm{ev}_{n-j})} \xrightarrow{\overline{\mathrm{ev}_{n-j}}} \bigoplus_{i=1}^{k} \bigoplus_{\substack{(n-j)\text{-faces} \\ \text{of } \sigma_i}} \mathcal{NR} \cdot f_i. \tag{5.7}$$

Furthermore, $f_i = f_i \cdot \chi_{\mathrm{supp}(f_i)}$ implies

$$\mathcal{NR} \cdot f_i \subset \mathcal{NR} \cdot \chi_{\mathrm{supp}(f_i)}.$$

Using additivity of the von Neumann dimension once more, we obtain

$$\dim_{\mathcal{NR}}\Big(\frac{\ker(\delta_{n-j})}{\ker(\mathrm{ev}_{n-j})\cap\ker(\delta_{n-j})}\Big) \le \dim_{\mathcal{NR}}\Big(\bigoplus_{i=1}^{k}\bigoplus_{\substack{(n-j)\text{-faces}\\ \text{of the }\sigma_i}} \mathcal{NR}\cdot\chi_{\mathrm{supp}(f_i)}\Big). \tag{5.8}$$

Directly from the definition of the von Neumann dimension, it follows

$$\dim_{\mathcal{NR}}\big(\mathcal{NR}\cdot\chi_{\mathrm{supp}(f_i)}\big) = \mathrm{tr}_{\mathcal{NR}}\big(\chi_{\mathrm{supp}(f_i)}\big) = \mu\big(\mathrm{supp}(f_i)\big). \tag{5.9}$$

Combining Corollary 5.17 with (5.6), (5.7), (5.8) and (5.9) yields

$$\dim_{\mathcal{NR}}\big(H_j^G(\widetilde{M};\mathcal{NR})\big) \le \binom{n+1}{j}\cdot\sum_{i=1}^{k}\mu\big(\mathrm{supp}(f_i)\big),$$

and now Theorem 4.26 completes the proof. □

5.2.5 Definition of the Integral Foliated Simplicial Volume

We will now imitate the definition of simplicial volume in the measurable context. More precisely, we first assign a "norm" to each measurable fundamental cycle in

$$L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$$

of a closed oriented connected manifold M and then take the infimum of the norms of measurable fundamental cycles. This value may depend on the choice of the standard Borel probability space X and will be denoted by $\|M\|^X_{\mathcal{F},\mathbb{Z}}$. Then we will again take the infimum, this time over G-isomorphism classes of standard Borel probability spaces X with standard G-action, where $G=\pi_1(M)$ is the fundamental group of M. The resulting value will be defined as the integral foliated simplicial volume $\|M\|_{\mathcal{F},\mathbb{Z}}$ of M.

Now we give the details of the definition.

Definition 5.20. For an element $\sum_{i=1}^k f_i\otimes\sigma_i \in L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$ we define its ℓ^1-*norm* by

$$\big\|\textstyle\sum_{i=1}^k f_i\otimes\sigma_i\big\|_1 = \displaystyle\sum_{i=1}^k\int_X |f_i|\,d\mu.$$

Usually, the phrase "norm" is used only on real or complex vector spaces. Nevertheless, we will use it in the present situation although $L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$ is just a $\mathbb{Z}$-module.

Remark 5.21. The ℓ^1-norm on $L^\infty(X,\mathbb{Z})\otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$ is well defined since

$$\int_X f(x)\,d\mu(x) = \int_X f(g.x)\,d\mu(x)$$

holds for all $f\in L^\infty(X)$ and $g\in G$ by measure preservation of the action $G\curvearrowright X$.

Definition 5.22. Let M be a closed oriented connected manifold of dimension n with fundamental group $G = \pi_1(M)$. Let further X be a standard Borel probability space with standard action $G \curvearrowright X$.The *integral X-foliated simplicial volume* of M is defined as

$$\|M\|^X_{\mathcal{F},\mathbb{Z}} = \inf\bigl\{ \|z\|_1 : z \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z}) \text{ is a measurable fundamental cycle of } M \bigr\}.$$

Remark 5.23. For a fundamental cycle $\sum_{i=1}^k \lambda_i \cdot \sigma_i \in \mathbb{Z} \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$ the induced cycle $\sum_{i=1}^k \mathrm{const}_{\lambda_i} \otimes \sigma_i$ is a measurable fundamental cycle by definition. We have

$$\int_X |\mathrm{const}_{\lambda_i}|\, d\mu = |\lambda_i|,$$

and consequently we obtain

$$\|M\|^X_{\mathcal{F},\mathbb{Z}} \leq \|M\|_{\mathbb{Z}}.$$

On the other hand, the proof of Theorem 5.35 will imply $\|M\| \leq \|M\|^X_{\mathcal{F},\mathbb{Z}}$ for all standard Borel probability space X with standard action $G \curvearrowright X$.

Next we want to get rid of the particular standard Borel G-probability space X by taking the infimum over isomorphism classes of such spaces. Before, we have to make precise, what we mean by *isomorphism* in that situation.

Definition 5.24. Let X and Y be standard Borel probability spaces with standard action of a countable group G. A map $X \to Y$ is called a *Borel G-isomorphism* if it is measurable, G-equivariant, measure preserving and bijective (measurability of the inverse map is then automatically given by Theorem 3.3).

Now we are prepared to give the definition of the integral foliated simplicial volume.

Definition 5.25. Let M be a closed oriented connected manifold with fundamental group $G = \pi_1(M)$. The *integral foliated simplicial volume* $\|M\|_{\mathcal{F},\mathbb{Z}}$ of M is defined as the infimum of $\|M\|^X_{\mathcal{F},\mathbb{Z}}$, where X runs through the set of Borel G-isomorphism classes of standard Borel probability spaces with standard G-action.

Remark 5.26. The Borel G-isomorphism classes of standard Borel probability G-spaces form indeed a set. Namely, there is a universal standard Borel G-space U_G with the property that any standard Borel G-space embeds measurably and G-equivariantly into U_G. This follows from [3, Theorem (2.6.1), p. 23]. By Theorem 3.4, the embedding is a Borel isomorphism onto its image.

Remark 5.27. The term *foliated* might be a bit confusing since there is no foliation around. It has its source in earlier work of Gromov [25, Section 2.4.B, p. 70] (based on unpublished work of Connes), where he defined an extension of the simplicial volume to foliations with transverse measure.

Note that the integral foliated simplicial volume defined above has nothing to do with the *foliated Gromov norm* in the sense of Calegari [7], which is defined on foliated manifolds by restricting the set of simplices allowed to represent the fundamental class to those which are transverse to the foliation.

5.2.6 Properties of the Integral Foliated Simplicial Volume

The intention of the definition of the foliated simplicial volume was to improve the upper bound for the sum of L^2-Betti numbers given in Corollary 5.6. Actually, Theorem 5.19 implies

Corollary 5.28. *Let M be a closed connected oriented manifold of dimension n. Then*

$$b_j^{(2)}(\widetilde{M}) \leq \binom{n+1}{j} \cdot \|M\|_{\mathcal{F},\mathbb{Z}},$$

and therefore also

$$\sum_{j=0}^{n} b_j^{(2)}(\widetilde{M}) \leq 2^{n+1} \cdot \|M\|_{\mathcal{F},\mathbb{Z}}.$$

Proof. Consider a measurable fundamental cycle

$$\sum_{j=1}^{k} f_j \otimes \sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z}).$$

For almost all $x \in \operatorname{supp}(f_j)$ we have $f_j(x) \geq 1$ since f_j is integer valued. We obtain

$$\sum_{j=1}^{k} \int_X |f_j(x)|\, d\mu(x) \geq \sum_{j=1}^{k} \mu\big(\operatorname{supp}(f_j)\big).$$

Now the result follows from Theorem 5.19. □

With regard to Conjecture 5.1 we would like to show that for all closed connected oriented aspherical manifolds M the vanishing of the simplicial volume $\|M\|$ implies the vanishing of the integral foliated simplicial volume $\|M\|_{\mathcal{F},\mathbb{Z}}$. Unfortunately, we can only show this in some very special cases.

Integral Foliated Simplicial Volume of Simply Connected Manifolds

In the opposite direction, one can ask for the integral foliated simplicial volume of simply connected manifolds. Note that by Corollary 2.15 their simplicial volume vanishes. In contrast, we have the following result for the foliated simplicial volume.

Proposition 5.29. *Let M be a closed connected oriented manifold with $\pi_1(M) = 0$. Then $\|M\|_{\mathcal{F},\mathbb{Z}} = \|M\|_{\mathbb{Z}} \geq 1$ holds.*

Proof. Let X be an arbitrary standard Borel probability space. Since $G = \pi_1(M)$ is the trivial group, we have

$$L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z}) \cong L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z}).$$

Any singular chain $c = \sum_{i=1}^k f_i \otimes \sigma_i \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z})$ induces singular chains $c_x = \sum_{i=1}^k f_i(x) \cdot \sigma_i \in C_n(M;\mathbb{Z})$ for almost all $x \in X$, and if two chains c, c' in $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z})$ are homologous, then so are their induced chains c_x, c'_x for almost all $x \in X$.

If $z \in C_n(M;\mathbb{Z})$ is a fundamental cycle, then $i_1(z)$ induces a fundamental cycle $i_1(z)_x$ for almost all $x \in X$. Hence, the same is true for any measurable fundamental cycle $\sum_{i=1}^k f_i \otimes \sigma_i \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}} C_n(M;\mathbb{Z})$. Thus, $\sum_{i=1}^k |f_i(x)| \geq \|M\|_{\mathbb{Z}}$ for almost all $x \in X$. It follows that

$$\begin{aligned} \left\|\sum\nolimits_{i=1}^k f_i \cdot \sigma_i\right\|_1 &= \sum_{i=1}^k \int_X |f_i(x)| \, d\mu(x) \\ &= \int_X \sum_{i=1}^k |f_i(x)| \, d\mu(x) \\ &\geq \int_X \|M\|_{\mathbb{Z}} \, d\mu(x) \\ &= \|M\|_{\mathbb{Z}}. \end{aligned}$$

This shows $\|M\|_{\mathcal{F},\mathbb{Z}} \geq \|M\|_{\mathbb{Z}}$, and the result follows with help of Remark 5.23. □

Integral Foliated Simplicial Volume of S^1

We will compute the integral foliated simplicial volume for the easiest closed manifold with positive dimension.

Proposition 5.30. *The equation $\|S^1\|^X_{\mathcal{F},\mathbb{Z}} = 0$ holds for every standard Borel probability space X with ergodic standard $\mathbb{Z}$-action. Consequently, $\|S^1\|_{\mathcal{F},\mathbb{Z}} = 0$ holds.*

Proof. Let X be a standard Borel probability space with ergodic standard $\mathbb{Z}$-action. Let further $n \in \mathbb{N}_{>0}$ be a positive integer and let $\varepsilon > 0$. Consider the obvious measurable fundamental cycle $\mathrm{const}_1 \otimes \sigma$ with

$$\begin{aligned} \sigma \colon I &\to \mathbb{R} \\ t &\mapsto t, \end{aligned}$$

where $I = [0,1]$ is identified with the standard 1-simplex Δ^1. Furthermore, for $j \in \mathbb{Z}$ we define $\sigma_j \colon I \to \mathbb{R}$ by $\sigma_j(t) = j + t$.

By the Rohlin Lemma (Theorem 3.39), there is a measurable subset $B \subset X$ such that $B, 1.B, \ldots, (n-1).B$ are pairwise disjoint and

$$\mu\Big(X - \bigcup\nolimits_{j=0}^{n-1} j.B\Big) < \varepsilon.$$

In the following, we will write $A = X - \bigcup_{j=0}^{n-1} j.B$. Obviously, the equation

$$\mathrm{const}_1 \otimes \sigma = \chi_A \otimes \sigma + \sum_{j=0}^{n-1} \chi_{j.B} \otimes \sigma$$

holds in $L^\infty(X;\mathbb{Z}) \otimes C_1(\mathbb{R};\mathbb{Z})$. For $j \in \mathbb{Z}$ we obtain $f \otimes \sigma_j = f(j.\_) \otimes \sigma$ by the tensor product relation in $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_1(\mathbb{R};\mathbb{Z})$. Consequently,

$$\chi_B \otimes \sigma_{-j} = \chi_{j.B} \otimes \sigma$$

in $L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_1(\mathbb{R};\mathbb{Z})$. We obtain

$$\begin{aligned} \mathrm{const}_1 \otimes \sigma &= \chi_A \otimes \sigma + \sum_{j=0}^{n-1} \chi_{j.B} \otimes \sigma \\ &= \chi_A \otimes \sigma + \sum_{j=0}^{n-1} \chi_B \otimes \sigma_{-j}. \end{aligned}$$

Next we observe that $\sum_{j=0}^{n-1} \chi_B \otimes \sigma_{-j}$ is homologous to $\chi_B \otimes \overline{\sigma}$, where $\overline{\sigma}$ is the singular simplex defined by

$$\begin{aligned} \overline{\sigma} \colon I &\to \mathbb{R} \\ t &\mapsto -(n-1) + n \cdot t. \end{aligned}$$

Hence $\chi_A \otimes \sigma + \chi_B \otimes \overline{\sigma}$ is a measurable fundamental cycle of S^1. We conclude

$$\|S^1\|^X_{\mathcal{F},\mathbb{Z}} \leq \mu(A) + \mu(B) < \varepsilon + \frac{1}{n}.$$

Now $\|S^1\|^X_{\mathcal{F},\mathbb{Z}} = 0$ follows since $n \in \mathbb{N}$ and $\varepsilon > 0$ can be chosen arbitrarily. □

Remark 5.31. The idea of the proof of Proposition 5.30 can be seen geometrically if one chooses a specific standard Borel space. Namely, let $X = S^1$ with the usual Lebesgue measure. The canonical measurable fundamental cycle is shown in Figure 5.1.

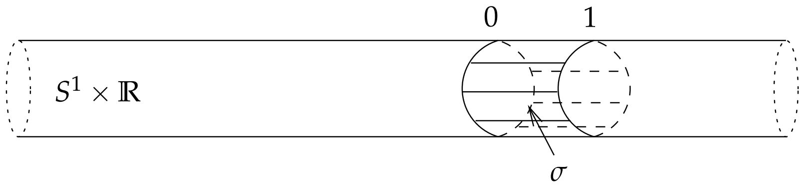

Figure 5.1: The canonical measurable fundamental cycle.

For given $n \in \mathbb{N}$ and $\varepsilon > 0$ let $\alpha \in [0, 1/n]$ be irrational with $\left|\frac{1}{n} - \alpha\right| < \frac{\varepsilon}{n}$. Let the $\mathbb{Z}$-action on X be induced by rotation around α. This action is ergodic since the

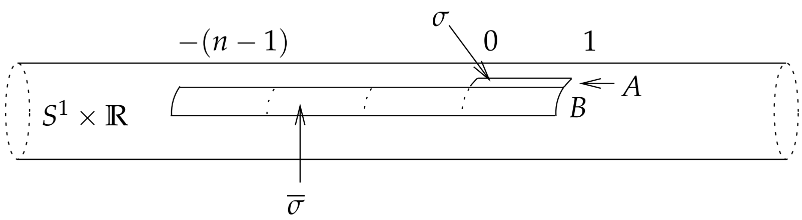

Figure 5.2: A measurable fundamental cycle of smaller weight.

rotation action of $\mathbb{Z}$ on S^1 is ergodic if and only if α is irrational (see [1, Proposition 1.2.5, p. 24] for a proof). We can define $B \subset X$ as $[0, \alpha) \subset S^1$. The measurable cycle $\chi_A \otimes \sigma + \chi_B \otimes \overline{\sigma}$ is shown in Figure 5.2.

Integral Foliated Simplicial Volume and Products

Recall that the simplicial volume of a product of two closed oriented manifolds M, N is related to the product of the simplicial volumes (Theorem 2.18). We will see that at least the upper bound

$$\|M \times N\| \leq c \cdot \|M\| \cdot \|N\|, \tag{5.10}$$

where c depends only on $\dim(M) + \dim(N)$, can be carried over to the foliated simplicial volume.

We will need the *cross product* map $C_n(M; \mathbb{Z}) \otimes_{\mathbb{Z}} C_m(N; \mathbb{Z}) \to C_{n+m}(M \times N; \mathbb{Z})$. Its definition uses the fact that $\Delta^n \times \Delta^m$ can be canonically triangulated. Denote by $\sum_{r=1}^{s} \lambda_r \cdot \alpha_r$ the corresponding chain in $C_{n+m}(\Delta^n \times \Delta^m; \mathbb{Z})$. For two singular simplices $\sigma\colon \Delta^n \to M$ and $\tau\colon \Delta^m \to N$ their cross product is defined as

$$\overline{\sigma \times \tau} = \sum_{j=r}^{s} \lambda_r \cdot (\sigma \times \tau) \circ \alpha_r.$$

The cross product is then defined by linear extension. We denote its value on a chain $c \otimes c' \in C_n(M; \mathbb{Z}) \otimes_{\mathbb{Z}} C_m(N; \mathbb{Z})$ is by $c \times c'$.

It is easy to see that the cross product induces a map

$$H_n(M; \mathbb{Z}) \otimes_{\mathbb{Z}} H_m(N; \mathbb{Z}) \to H_{n+m}(M \times N; \mathbb{Z})$$

in homology (which is also called cross product) and that $[M] \times [N] = [M \times N]$ holds.

Let $H = \pi_1(N)$ be the fundamental group of N and let Y be a standard Borel probability space with standard action $H \curvearrowright Y$. As before, $G = \pi_1(M)$ denotes the fundamental group of M and X denotes a standard Borel probability space with standard action $G \curvearrowright X$. Denote the orbit equivalence relation of $G \curvearrowright X$ by $\mathcal{R}$ and that of $H \curvearrowright Y$ by $\mathcal{R}'$. Consider two chains

$$\sum_{j=1}^{k} f_j \otimes \sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z}),$$
$$\sum_{i=1}^{l} g_i \otimes \tau_i \in L^\infty(Y,\mathbb{Z}) \otimes_{\mathbb{Z}H} C_m(\widetilde{N};\mathbb{Z}).$$

Definition 5.32. Define the *cross product* map

$$\big(L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})\big) \otimes_{\mathbb{Z}} \big(L^\infty(Y,\mathbb{Z}) \otimes_{\mathbb{Z}H} C_m(\widetilde{N};\mathbb{Z})\big) \\ \longrightarrow L^\infty(X \times Y;\mathbb{Z}) \otimes_{\mathbb{Z}(G\times H)} C_{n+m}(\widetilde{M} \times \widetilde{N};)$$

by

$$\big(\textstyle\sum_{j=1}^{k} f_j \otimes \sigma_j\big) \otimes \big(\sum_{i=1}^{l} g_i \otimes \tau_i\big) \mapsto \displaystyle\sum_{j=1}^{k}\sum_{i=1}^{l}\sum_{r=1}^{s} (f_j \times g_i) \otimes (\sigma_j \times \tau_i) \circ \alpha_r.$$

To see that the cross product is well defined, note that for an element $g \in G$ we obtain

$$\begin{aligned}(f \otimes g.\sigma) \times (f' \otimes \tau) &= \sum_{r=1}^{s} (f \times f') \otimes (g.\sigma \times \tau) \circ \alpha_r \\ &= \sum_{r=1}^{s} (f \times f') \otimes (g, e_H).(\sigma \times \tau) \circ \alpha_r \\ &= \sum_{r=1}^{s} \big(f(g.\_) \times f'\big) \otimes (\sigma \times \tau) \circ \alpha_r \\ &= \big(f(g.\_) \otimes \sigma\big) \times (f' \otimes \tau),\end{aligned}$$

where $e_H \in H$ denotes the unit element. Hence the definition of the cross product respects the tensor product relations.

For $c \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$ and $c' \in L^\infty(Y,\mathbb{Z}) \otimes_{\mathbb{Z}H} C_m(\widetilde{N};\mathbb{Z})$ we have

$$d(c \times c') = d(c) \times c' + (-1)^n \cdot c \times d(c'),$$

where, by slight abuse of notation, d denotes the boundary operators in the respective chain complexes. It follows that the cross product induces a map in homology.

Lemma 5.33. *The cross product of two measurable fundamental cycles of M and N respectively is a measurable fundamental cycle of $M \times N$.*

Proof. By definition, a measurable fundamental cycle $\sum_{j=1}^{k} f_j \otimes \sigma_j$ of M is homologous to $i_1^M(z)$ for a fundamental cycle $z \in \mathbb{Z} \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$. A measurable fundamental cycle $\sum_{i=1}^{l} g_i \otimes \tau_i$ of N is homologous to $i_1^N(z')$, where $z' \in \mathbb{Z} \otimes_{\mathbb{Z}H} C_m(\widetilde{N};\mathbb{Z})$

is a fundamental cycle. It follows directly from the definition of the cross product that $i_1^M(z) \times i_1^N(z') = i_1^{M\times N}(z \times z')$ holds. Hence the cross product

$$\left(\sum_{j=1}^{k} f_j \otimes \sigma_j\right) \times \left(\sum_{i=1}^{l} g_i \otimes \tau_i\right)$$

is a measurable fundamental cycle of $M \times N$. □

Now we are able to prove the product inequality for the integral foliated simplicial volume.

Theorem 5.34. *Let M and N be closed connected oriented manifolds of dimensions n and m respectively. Then*

$$\|M \times N\|_{\mathcal{F},\mathbb{Z}} \leq c \cdot \|M\|_{\mathcal{F},\mathbb{Z}} \cdot \|N\|_{\mathcal{F},\mathbb{Z}}$$

holds, where the constant c depends only on $n + m$.

Proof. Let X and Y be as above. Furthermore, let

$$\sum_{j=1}^{k} f_j \otimes \sigma_j \in L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z})$$

be a measurable fundamental cycle of M and let

$$\sum_{i=1}^{l} g_i \otimes \tau_i \in L^\infty(Y,\mathbb{Z}) \otimes_{\mathbb{Z}H} C_m(\widetilde{N};\mathbb{Z})$$

be a measurable fundamental cycle of N. By Lemma 5.33, their cross product is a measurable fundamental cycle of $M \times N$. If c denotes the maximum of $(n + m)$-simplices needed to triangulate $\Delta^{n+m-r} \times \Delta^r$ for $r = 0, \ldots, n + m$, we obtain

$$\begin{aligned}\left\|\sum_{j=1}^{k} f_j \otimes \sigma_j \times \sum_{i=1}^{l} g_i \otimes \tau_i\right\| &\leq c \cdot \sum_{j=1}^{k}\sum_{i=1}^{l} \int_{X\times Y} |f_j(x) \cdot g_i(y)|\, d(\mu \times \mu')(x,y) \\ &= c \cdot \sum_{j=1}^{k} \int_X |f_j(x)|\, d\mu(x) \cdot \sum_{i=1}^{l} \int_Y |g_i(y)|\, d\mu'(y).\end{aligned}$$

As a consequence we obtain

$$\|M \times N\|_{\mathcal{F},\mathbb{Z}}^{X\times Y} \leq c \cdot \|M\|_{\mathcal{F},\mathbb{Z}}^{X} \cdot \|N\|_{\mathcal{F},\mathbb{Z}}^{Y}.$$

The result follows by taking the infimum over the appropriate isomorphism classes of standard Borel spaces X and Y. □

Simplicial Volume and Integral Foliated Simplicial Volume

We will now prove that the integral foliated simplicial volume bounds the simplicial volume from above.

Theorem 5.35. *Let M be a closed oriented connected manifold. Then*

$$\|M\| \leq \|M\|_{\mathcal{F},\mathbb{Z}}$$

holds.

Proof. Let $p_1 \colon L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M};\mathbb{Z}) \to C_n(M;\mathbb{R})$ be the homomorphism given by $f \otimes \sigma \mapsto \big(\int_X f\, d\mu\big) \cdot p \circ \sigma$, where $p \colon \widetilde{M} \to M$ is the universal covering. Note that p_1 is well defined since $f \cdot g \otimes \sigma$ and $f \otimes g \cdot \sigma$ both map to $\big(\int_X f\, d\mu\big) \cdot p \circ \sigma$. The homomorphism p_1 fits into the commutative diagram

$$\begin{array}{ccc} C_n(M;\mathbb{Z}) & \xrightarrow{\;j_1\;} & L^\infty(X,\mathbb{Z}) \otimes_{\mathbb{Z}G} C_n(\widetilde{M},\mathbb{Z}) \\ \big\downarrow j & \swarrow p_1 & \\ C_n(M;\mathbb{R}), & & \end{array}$$

where j_1 is given by $\sigma \mapsto \mathrm{const}_1 \otimes \widetilde{\sigma}$ for some lift $\widetilde{\sigma} \colon \Delta^n \to \widetilde{M}$ of σ (whereas j_1 does not depend on the choice of the lift) and j is induced by inclusion of coefficients. If $\sum_{i=1}^k f_i \otimes \sigma_i$ is a measurable fundamental cycle, then it represents $(j_1)_*\big([M]\big)$ by definition. Consequently,

$$p_1\big(\textstyle\sum_{i=1}^k f_i \otimes \sigma_i\big) = \textstyle\sum_{i=1}^k \Big(\displaystyle\int_X f_i\, d\mu\Big) \cdot p \circ \sigma_i$$

represents $(p_1 \circ j_1)_*\big([M]\big) = j_*\big([M]\big)$. We obtain

$$\|M\| \leq \textstyle\sum_{i=1}^k \Big|\displaystyle\int_X f_i\, d\mu\Big| \leq \textstyle\sum_{i=1}^k \displaystyle\int_X |f_i|\, d\mu = \big\|\textstyle\sum_{i=1}^k f_i \otimes \sigma_i\big\|_1.$$

The result follows from taking infima over measurable fundamental cycles and then over G-isomorphism classes of standard Borel spaces with standard G-action. □

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Index

Lebenslauf

Marco Schmidt

Geburtsdatum/-ort	2.5.1974	Berlin
Familienstand		verheiratet
Eltern		Michael Schmidt und Marina Schmidt, geb. Bölke
Schulbildung	08/1981-07/1987	Nord-Grundschule Berlin
	08/1987-06/1994	Dreilinden-Oberschule (Gymnasium) Berlin
Hochschulreife	17.6.1994	Abitur an der Dreilinden-Oberschule (Gymnasium) Berlin
Zivildienst	07/1994-09/1995	Krankenhaus Heckeshorn Berlin
Studium	10/1995-06/2001	Diplomstudiengang Mathematik mit Nebenfach Informatik an der Freien Universität Berlin
Prüfungen	11.6.2001	Diplom im Fach Mathematik an der FU Berlin
Tätigkeiten	10/1997-09/2001	studentische Hilfskraft an der FU Berlin
	10/2003-02/2004	wissenschaftliche Hilfskraft an der WWU Münster
	03/2004-07/2004	wissenschaftlicher Mitarbeiter am Mathematischen Institut der WWU Münster
	10/2004-03/2005	wissenschaftliche Hilfskraft an der WWU Münster
Beginn der Promotion	10/2001	Mathematisches Institut der WWU Münster, Betreuer ist Prof. Dr. Wolfgang Lück