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Cuntz-Li relations, inverse semigroups and groupoids

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Dedicated to Prof. V. S. Sunder on the occasion of his sixtieth birthday

Abstract. In this paper we show that the universal C^*-algebra satisfying the Cuntz-Li relations is generated by an inverse semigroup of partial isometries. We apply Exel's theory of tight representations to this inverse semigroup. We identify the universal C^*-algebra as the C^*-algebra of the tight groupoid associated to the inverse semigroup.

1. Introduction

Let R be an integral domain with only finite quotients. Assume that R is not a field and let K be its field of fractions. We denote the set of nonzero elements in R (resp. K) by $R^\times$ (resp. $K^\times$). In [3], Cuntz and Li studied the C^*-algebra, denoted $\mathfrak{A}_r[R]$, on $\ell^2(R)$ generated by the isometries induced by the multiplication and addition operations of the ring R. They showed that it is simple and purely infinite. It was also shown that this C^*-algebra is the universal C^*-algebra generated by isometries satisfying the relations reflecting the semigroup multiplication in $R \rtimes R^\times$ and one more important relation satisfied by the range projections. Also it was shown that $\mathfrak{A}_r[R]$ is Morita-equivalent to a crossed product of the form $C_0(\mathcal{R}) \rtimes (K \rtimes K^\times)$ where $\mathcal{R}$ is a locally compact Hausdorff space. For $R = \mathbb{Z}$, $\mathcal{R} = \mathbb{A}_f$ is the space of finite adeles. Alternate approaches to the algebra $\mathfrak{A}_r[R]$ were considered in [10], [2], and [19].

In [10], the situation in [3] was abstracted. Consider a semidirect product $N \rtimes H$ and a normal subgroup M of N. Let $P := \{a \in H \mid aMa^{-1} \subset M\}$. Then P is a semigroup. In [10], under certain hypotheses regarding the pair $(G = N \rtimes H, M)$, the crossed product algebra $C_0(\overline{N}) \rtimes G$ was considered. Here $\overline{N}$ is the profinite completion of N with respect to the group topology induced by the neighborhood base $\{aMa^{-1}\}_{a \in H}$ at the identity. Let $\overline{M}$ be the closure of M in $\overline{N}$. In [10], it was shown that the crossed product algebra

$C_0(\overline{N}) \rtimes G$ is Morita-equivalent to the C^*-algebra of the groupoid $\overline{N} \rtimes G|_{\overline{M}}$. In [10], it was shown that when H is abelian, $C^*(\overline{N} \rtimes G|_{\overline{M}})$ is the universal C^*-algebra generated by isometries satisfying the relations reflecting the semigroup multiplication in $M \rtimes P$ and one more important relation among the range projections. They also obtained sufficient conditions which will ensure that the reduced C^*-algebra $C^*_{red}(\overline{N} \rtimes G|_{\overline{M}})$ is simple and purely infinite.

Our objective in this paper is to weaken the hypothesis that H is abelian. Instead we assume $H = PP^{-1} = P^{-1}P$. This allows us to consider pairs like $(\mathbb{Q}^n \rtimes GL_n(\mathbb{Q}), \mathbb{Z}^n)$. Also we start with the universal C^*-algebra, denoted $\mathfrak{A}[N \rtimes H, M]$, generated by isometries satisfying the Cuntz-Li relations (see Def. 2.11). We show that $\mathfrak{A}[N \rtimes H, M]$ is generated by an inverse semigroup of partial isometries denoted by T. We show that $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_{tight}$, considered in [6], of the inverse semigroup T. We also identify the groupoid $\mathcal{G}_{tight}$ explicitly and show that $\mathcal{G}_{tight}$ is isomorphic to $\overline{N} \rtimes G|_{\overline{M}}$. The author had done a similar analysis for the Cuntz-Li algebra associated to the ring $\mathbb{Z}$ in [19]. At the end of this paper, we prove a duality result analogous to the duality result obtained in [4].

2. Semidirect products and the Cuntz-Li relations

Let $G = N \rtimes H$ be a semidirect product and let M be a normal subgroup of N. Let $P := \{a \in H \mid aMa^{-1} \subset M\}$. Then P is a semigroup containing the identity e. Assume that the following holds.

- (C1) The group $H = PP^{-1} = P^{-1}P$.
- (C2) For every $a \in P$, the subgroup aMa^{-1} is of finite index in M.
- (C3) The intersection $\bigcap_{a \in P} aMa^{-1} = \{e\}$ where e denotes the identity element of G.

Let $\mathcal{U} = \{aMa^{-1} \mid a \in H\}$. In [10], the following conditions were required to be satisfied. (Cp. [10, Sec. 2].)

- (E1) Given $U, V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \subset U \cap V$.
- (E2) If $U, V \in \mathcal{U}$ and $U \subset V$ then U is of finite index in V.
- (E3) The intersection $\bigcap_{U \in \mathcal{U}} U = \{e\}$.

We claim that (E1) is equivalent to the condition $H = PP^{-1}$. Assume (E1). Let $a \in H$ be given. Then there exists $c \in H$ such that $a^{-1}Ma \cap M \supseteq cMc^{-1}$. Then $c \in P$ and $ac \in P$. Note that $a = (ac)c^{-1} \in PP^{-1}$. Thus we have $H = PP^{-1}$.

Now suppose $H = PP^{-1}$. First note that for every $a, b \in P$, $aP \cap bP$ is nonempty. Now let $c, d \in H$ be given. Write $c = a_1a_2^{-1}$ and $d = b_1b_2^{-1}$ with $a_i, b_i \in P$. Choose $\alpha, \beta \in P$ such that $a_1\alpha = b_1\beta$. Let $a := a_1\alpha$. Then $c^{-1}a = a_2\alpha \in P$. Similarly $d^{-1}a \in P$. Hence $aMa^{-1} \subset cMc^{-1} \cap dMd^{-1}$. Thus (E1) holds.

Given (E1), note that (E3) is equivalent to (C3). For if $a \in H$, there exists $b \in P$ such that $aMa^{-1} \cap M \supseteq bMb^{-1}$. Thus for every $a \in H$,

$aMa^{-1} \supseteq \bigcap_{b\in P} bMb^{-1}$. Hence $\bigcap_{U\in\mathcal{U}} U = \bigcap_{a\in P} aMa^{-1}$. Thus given (E1), (E3) is equivalent to (C3). Clearly (E2) is equivalent to (C2).

Remark 2.1. In [10], the Cuntz-Li algebra associated to the pair $(N \rtimes H, M)$ (cp. Def. 2.11) was considered when H is abelian (cp. Hypothesis 9.2 and Theorem 9.11 in [10]). Here, we consider a slightly more general situation. We assume $H = P^{-1}P = PP^{-1}$.

Remark 2.2. The condition $H = P^{-1}P = PP^{-1}$ is equivalent to saying that P generates H and P is right and left reversible i.e. given $a, b \in P$, the intersections $Pa \cap Pb$ and $aP \cap bP$ are nonempty. Cancellative semigroups which are right (or left) reversible are called Ore semigroups. For more details on Ore semigroups, we refer to [5].

A semigroup P is called right reversible (left reversible) if $Pa\cap Pb$ (if $aP\cap bP$) is nonempty for every $a, b \in P$.

Throughout this article, whenever we write $G = N \rtimes H$ and M is a normal subgroup of N, we assume that conditions (C1), (C2) and (C3) hold. For $a \in P$, let $M_a = aMa^{-1}$. We will use this notation throughout.

Lemma 2.3. *Let $G = N \rtimes H$ and M be a normal subgroup of N. Let $N_0 := \bigcup_{a\in P} a^{-1}Ma$. Then N_0 is a subgroup of N and is invariant under conjugation by H.*

Proof. First observe that N_0 is closed under inversion. Let $a, b \in P$ be given. Choose an element c in the intersection $Pa \cap Pb$. Then $a^{-1}Ma \subset c^{-1}Mc$ and $b^{-1}Mb \subset c^{-1}Mc$. Now it follows that N_0 is closed under multiplication. Thus N_0 is a subgroup of N.

Obviously N_0 is invariant under conjugation by P^{-1}. Let $a, b \in P$ be given. Since P is right reversible, there exists $c, d \in P$ such that $ab^{-1} = c^{-1}d$. Now observe that $a(b^{-1}Mb)a^{-1} = c^{-1}(dMd^{-1})c \subset c^{-1}Mc$. Thus it follows that N_0 is closed under conjugation by P. This completes the proof. ☐

Remark 2.4. As a consequence of Lemma 2.3, we may very well assume as in [10] that $N = \bigcup_{a\in P} a^{-1}Ma$.

Let us consider a few examples which fits the setup that we are considering.

Example 2.5 ([3])**.** Let R be an integral domain such that for every nonzero $m \in R$, the ideal generated by m is of finite index in R. Assume that R is not a field. We denote the field of fractions of R by Q and the set of nonzero elements in Q by $Q^\times$. The multiplicative group $Q^\times$ acts on Q by multiplication. Now let $N := Q$, $H := Q^\times$ and $M := R$. Then $P = R^\times$ where $R^\times$ denotes the set of nonzero elements in R. Then conditions (C1)–(C3) hold for the pair $(N \rtimes H, M)$.

Example 2.6 ([10])**.** Let F be a finite group and consider the direct sum $N := \oplus_{\mathbb{Z}} F$. Then $H := \mathbb{Z}$ acts on N by shifting. Let $M := \oplus_{\mathbb{N}} F$ be the normal subgroup of N. Then it is easily verifiable that the pair $(N \rtimes H, M)$ satisfies the hypothesis (C1)–(C3).

In the following two examples, we think of elements of $\mathbb{Q}^n$ as column vectors.

Example 2.7. Let A be a $n \times n$ integer dilation matrix. In other words, A is an $n \times n$ matrix with integer entries such that every complex eigenvalue of A has absolute value greater than 1. Note that A is invertible over $\mathbb{Q}$ and $|\det(A)| > 1$. The matrix A acts on $\mathbb{Q}^n$ by matrix multiplication and thus induces an action of $\mathbb{Z}$ on $\mathbb{Q}^n$. We let the generator 1 of $\mathbb{Z}$ act on $\mathbb{Q}^n$ by $1.v = Av$ for $v \in \mathbb{Q}^n$. Let $N := \mathbb{Q}^n$, $H := \mathbb{Z}$ and $M := \mathbb{Z}^n$. Then $P = \mathbb{N}$. Let us verify the hypothesis (C1)–(C3).

- (C1) Note that H is abelian and $H = PP^{-1} = P^{-1}P$.
- (C2) For $r \geq 0$, the index of $A^r\mathbb{Z}^n$ is of finite index in $\mathbb{Z}^n$ and in fact its index is $|\det(A)|^r$.
- (C3) Lemma 4.1 of [8] implies that the operator norm $\|A^{-m}\|$ converges to 0 as m tends to infinity. Thus if $0 \neq v \in \bigcap_{r=0}^{\infty} A^r\mathbb{Z}^n$, then for every $m \geq 0$, $A^{-m}v \in \mathbb{Z}^n$. Thus we have $1 \leq \|A^{-m}v\| \leq \|A^{-m}\|\|v\|$ which is a contradiction. Thus (C3) holds.

The case $n = 1$ and $A = p$ where p is a prime number was discussed in [12]. In the previous example, we can consider integer matrices other than dilation matrices. It is possible that (C3) is satisfied for an integer matrix A such that $|\det(A)| > 1$ and $\bigcap_{r>0} A^r\mathbb{Z}^n = \{0\}$ without A being a dilation matrix. In fact we have the following nice characterization of condition (C3) when $n = 2$.

Lemma 2.8. *Let A be a 2×2 matrix with integer entries. Assume that $|\det(A)| > 1$. Then the following are equivalent.*

(1) The intersection $\bigcap_{r\geq 0} A^r\mathbb{Z}^2$ is trivial.
(2) Neither 1 nor -1 is an eigenvalue of A.

Proof. Suppose $\bigcap_{r\geq 0} A^r\mathbb{Z}^2 = \{0\}$. If 1 is an eigenvalue of A then there exists a nonzero $v \in \mathbb{Q}^2$ such that $Av = v$. By clearing denominators, we can assume that $v \in \mathbb{Z}^2$. Then clearly $v \in \bigcap_{r\geq 0} A^r\mathbb{Z}^2$. Thus we have shown that 1 is not an eigenvalue of A. Similarly we can show -1 is not an eigenvalue of A.

Now assume that neither 1 nor -1 is an eigenvalue of A. Let $\Gamma_r := A^r\mathbb{Z}^2$ and $\Gamma := \bigcap_{r\geq 0} \Gamma_r$. Since $\Gamma \subset \Gamma_r \subset \mathbb{Z}^2$, we have $[\mathbb{Z}^2 : \Gamma] \geq [\mathbb{Z}^2 : \Gamma_r] = |\det(A)|^r$. Hence Γ cannot be of finite index in $\mathbb{Z}^2$. This implies that Γ is of rank at most 1. If Γ is rank 1 then there exists a nonzero $v \in \mathbb{Z}^2$ such that $\Gamma = \mathbb{Z}v$. But $A : \Gamma \to \Gamma$ is a bijection. Thus it must either be multiplication by 1 or by -1. In other words, v is an eigenvector for A with eigenvalue 1 or -1. This is a contradiction. Thus Γ cannot be of rank 1 which in turn implies $\Gamma = \{0\}$. This completes the proof. $\square$

The matrix $A := \begin{bmatrix} 0 & 2 \\ 1 & -2 \end{bmatrix}$ has eigenvalues $\sqrt{3} - 1$ and $-\sqrt{3} - 1$. But A is not a dilation matrix but still (C3) holds for A.

Remark 2.9. It is not clear to the author whether (C3) can be characterized in terms of eigenvalues of the matrix in the higher dimensional case.

Let us now consider an example where H is nonabelian.

Example 2.10. Let $N = \mathbb{Q}^n$ and H be a subgroup of $GL_n(\mathbb{Q})$ containing the nonzero scalars. Just as in Example 2.7, H acts on N by matrix multiplication. Let $M = \mathbb{Z}^n$. Then P consists of elements of H whose entries are integers.

(C1) Let $A \in H$ be given. Then there exists a nonzero integer m such that $mA = Am \in P$. Hence $H = PP^{-1} = P^{-1}P$.
(C2) For $A \in P$, the subgroup $A\mathbb{Z}^n$ is of finite index and its index is $|\det(A)|$.
(C3) Since $\bigcap_{m\in\mathbb{Z}^\times} m\mathbb{Z}^n = \{0\}$, it follows that $\bigcap_{A\in P} A\mathbb{Z}^n = \{0\}$.

Definition 2.11. Let $G := N \rtimes H$ be a semidirect product and M be a normal subgroup of N such that (C1)–(C3) holds. We let $\mathfrak{A}[N \rtimes H, M]$ be the universal C^*-algebra generated by a set of isometries $\{s_a \mid a \in P\}$ and a set of unitaries $\{u(m) \mid m \in M\}$ satisfying the following relations.

$$s_a s_b = s_{ab}$$
$$u(m)u(n) = u(mn)$$
$$s_a u(m) = u(ama^{-1})s_a$$
$$\sum_{k\in M/M_a} u(k)e_a u(k)^{-1} = 1$$

where e_a denotes the final projection of s_a.

Note that $u(k)e_a u(k)^{-1}$ depends only on the coset $k(M_a)$. Moreover if k_1 and k_2 lie in different cosets of M_a then $u(k_1)e_a u(k_1)^{-1}$ and $u(k_2)e_a u(k_2)^{-1}$ are orthogonal.

For $a \in P$ and $m \in M$, consider the operators S_a and $U(m)$ on $\ell^2(M) \otimes \ell^2(H)$ defined as follows

$$S_a(\delta_n \otimes \delta_b) := \delta_{ana^{-1}} \otimes \delta_{ab}$$
$$U(m)(\delta_n \otimes \delta_b) := \delta_{mn} \otimes \delta_b.$$

Then $s_a \to S_a$ and $u(m) \to U(m)$ gives a representation of $\mathfrak{A}[N \rtimes H, M]$ on the Hilbert space $\ell^2(M) \otimes \ell^2(H)$. Let us call this representation the regular representation and denote its image by $\mathfrak{A}_r[N \rtimes H, M]$.

Remark 2.12. It should be noted that the regular representation for integral domains considered in [3] is different from ours.

3. An inverse semigroup for the Cuntz-Li relations

The main aim of this section is to show that the C^*-algebra $\mathfrak{A}[N \rtimes H, M]$ is generated by an inverse semigroup of partial isometries. We begin with a lemma similar to Lemma 1 of Section 3.1 in [3].

Lemma 3.1. *For every $a, b \in P$, one has*

$$e_a = \sum_{k\in M/M_b} u(aka^{-1})e_{ab}u(aka^{-1})^{-1}.$$

Proof. One has

$$\begin{aligned} e_a &= s_a s_a^* \\ &= s_a \left(\sum_{k \in M/M_b} u(k) e_b u(k)^{-1} \right) s_a^* \\ &= \sum_{k \in M/M_b} u(aka^{-1}) s_a e_b s_a^* u(aka^{-1})^{-1} \\ &= \sum_{k \in M/M_b} u(aka^{-1}) e_{ab} u(aka^{-1})^{-1}. \end{aligned}$$

This completes the proof. □

Let X be the linear span of $\{u(k)e_b u(k)^{-1} \mid b \in P, k \in M\}$. Denote the set of projections in X by F. By Lemma 3.1 and the left reversibility of P, it follows that $f \in F$ if and only if there exists $b \in P$ such that f is in the linear span of $\{u(k)e_b u(k)^{-1}\}$. The following lemma is an immediate corollary of Lemma 3.1 and the fact that P is left reversible.

Lemma 3.2. *The set F is a commutative semigroup of projections. Moreover F is invariant under the maps $x \to s_b x s_b^*$ for every $b \in P$ and $x \to u(m)xu(m)^{-1}$ for every $m \in M$.*

Now we show that F is also invariant under conjugation by s_a^* for every $a \in P$.

Lemma 3.3. *Let $a \in P$ be given. If $f \in F$, then $s_a^* f s_a \in F$. Moreover, $s_a^* u(m) e_b u(m)^{-1} s_a$ is in the linear span of $\{u(k)e_{a^{-1}c}u(k)^{-1}\}$ where c is any element in $aP \cap bP$.*

Proof. Let $a \in P$ and $f \in F$ be given. First observe that $s_a^* f s_a$ is selfadjoint. Also

$$\begin{aligned} (s_a^* f s_a)^2 &= s_a^* f s_a s_a^* f s_a \\ &= s_a^* f e_a f s_a \\ &= s_a^* e_a f s_a \text{ (since } F \text{ is commutative)} \\ &= s_a^* f s_a. \end{aligned}$$

Thus $s_a^* f s_a$ is a projection. Now to show that $s_a^* f s_a \in F$, it is enough to consider the case when $f = u(m)e_b u(m)^{-1}$. Now let $c \in aP \cap bP$ and write $c = a\alpha = b\beta$ with $\alpha, \beta \in P$.

Let $r_1, r_2, \cdots, r_n$ be distinct representatives of M/M_β. Then by Lemma 3.1, it follows that

$$\begin{aligned} s_a^* u(m) e_b u(m)^{-1} s_a &= \sum_{i=1}^{n} s_a^* u(mbr_i b^{-1}) e_{b\beta} u(mbr_i b^{-1})^{-1} s_a \\ &= \sum_{i=1}^{n} s_a^* u(mbr_i b^{-1}) e_{a\alpha} u(mbr_i b^{-1})^{-1} s_a. \end{aligned}$$

The term $s_a^*u(mbr_ib^{-1})e_{a\alpha}u(mbr_ib^{-1})^{-1}s_a$ survives if and only if $e_{a\alpha}u(mbr_i b^{-1})s_a \neq 0$ and that is if and only if $e_{a\alpha}u(mbr_ib^{-1})e_au(mbr_ib^{-1})^{-1} \neq 0$. But by Lemma 3.1 this happens precisely when there exists $t_i \in M/M_\alpha$ such that $mbr_ib^{-1} \equiv at_ia^{-1} mod M_{a\alpha}$.

Let

$$A := \{i \mid \text{ There exists } t_i \text{ such that } mbr_ib^{-1} \equiv at_ia^{-1} \bmod M_{a\alpha}\}.$$

For every $i \in A$, choose t_i such that $mbr_ib^{-1} \equiv at_ia^{-1} \bmod M_{a\alpha}$. Now we have

$$\begin{aligned} s_a^*u(m)e_bu(m)^{-1}s_a &= \sum_{i=1}^{n} s_a^*u(mbr_ib^{-1})e_{a\alpha}u(mbr_ib^{-1})^{-1}s_a \\ &= \sum_{i\in A} s_a^*u(mbr_ib^{-1})e_{a\alpha}u(mbr_ib^{-1})^{-1}s_a \\ &= \sum_{i\in A} s_a^*u(at_ia^{-1})e_{a\alpha}u(at_ia^{-1})^{-1}s_a \\ &= \sum_{i\in A} u(t_i)s_a^*e_{a\alpha}s_au(t_i)^{-1} \\ &= \sum_{i\in A} u(t_i)e_\alpha u(t_i)^{-1}. \end{aligned}$$

This completes the proof. □

Let us isolate the computation in the previous lemma in a remark. This will be used later.

Remark 3.4. Let $a, b \in P$ be given. Let $c \in aP \cap bP$. Choose α and β in P such that $c = a\alpha = b\beta$. Conjugation by a sends M_α to M_c. Thus we get a map denoted $\pi_\alpha^a : M/M_\alpha \to M/M_c$. Similarly conjugation by b gives a map $\pi_\beta^b : M/M_\beta \to M/M_c$. Note that both π_α^a and π_β^b are injective. Denote the quotient map $M \to M/M_c$ by q_c. For $m \in M$, define

$$A_m := \{r \in M/M_\beta \mid q_c(m)\pi_\beta^b(r) \in \pi_\alpha^a(M/M_\alpha)\}.$$

Then the computation in Lemma 3.3 can be restated as follows

$$\begin{aligned} &s_a^*u(m)e_bu(m)^{-1}s_a \\ &\qquad = \sum_{r\in A_m} u\Big(\big(\pi_\alpha^a\big)^{-1}(q_c(m)\pi_\beta^b(r))\Big)e_\alpha u\Big(\big(\pi_\alpha^a\big)^{-1}(q_c(m)\pi_\beta^b(r))\Big)^{-1}. \end{aligned}$$

Now we show that $\mathfrak{A}[N \rtimes H, M]$ is generated by an inverse semigroup of partial isometries.

Proposition 3.5. *Let* $T := \{s_a^*u(m)fu(m')s_{a'} \mid m, m' \in M, a, a' \in P,$ *and* $f \in F\}$. *Then* T *is an inverse semigroup of partial isometries containing* 0. *Moreover the set of projections in* T *coincides exactly with* F. *Also the linear span of* T *is a dense* $*$*-subalgebra of* $\mathfrak{A}[N \rtimes H, M]$.

Proof. The fact that T is closed under multiplication follows from the following calculation. Let $a_1, a_2, b_1, b_2 \in P$, $m_1, m_2, n_1, n_2 \in M$ and $e, f \in F$ be given. Choose $c \in Pb_1 \cap Pa_2$ and write c as $c = \beta b_1 = \alpha a_2$. Observe that

$$\begin{aligned}
&s_{a_1}^* u(m_1) e u(m_2) s_{a_2} s_{b_1}^* u(n_1) f u(n_2) s_{b_2} \\
&\quad = s_{a_1}^* u(m_1 m_2) u(m_2^{-1}) e u(m_2) s_\alpha^* s_\alpha s_{a_2} s_{b_1}^* s_\beta^* s_\beta u(n_1) f u(n_1^{-1}) u(n_1 n_2) s_{b_2} \\
&\quad = s_{a_1}^* u(m_1 m_2) u(m_2^{-1}) e u(m_2) s_\alpha^* s_{\alpha a_2} s_{\beta b_1}^* s_\beta u(n_1) f u(n_1^{-1}) u(n_1 n_2) s_{b_2} \\
&\quad = s_{a_1}^* u(m_1 m_2) s_\alpha^* s_\alpha u(m_2^{-1}) e u(m_2) s_\alpha^* s_c s_c^* s_\beta u(n_1) f u(n_1^{-1}) s_\beta^* s_\beta u(n_1 n_2) s_{b_2} \\
&\quad = s_{a_1}^* s_\alpha^* u(\alpha m_1 m_2 \alpha^{-1}) \big(s_\alpha u(m_2^{-1}) e u(m_2) s_\alpha^*\big) \\
&\qquad\qquad e_c \big(s_\beta u(n_1) f u(n_1^{-1}) s_\beta^*\big) u(\beta n_1 n_2 \beta^{-1}) s_\beta s_{b_2} \\
&\quad = s_{\alpha a_1}^* u(\alpha m_1 m_2 \alpha^{-1}) \big(s_\alpha u(m_2^{-1}) e u(m_2) s_\alpha^*\big) \\
&\qquad\qquad e_c \big(s_\beta u(n_1) f u(n_1^{-1}) s_\beta^*\big) u(\beta n_1 n_2 \beta^{-1}) s_{\beta b_2} \\
&\quad = s_{\alpha a_1}^* u(\alpha m_1 m_2 \alpha^{-1}) (s_\alpha \tilde{e} s_\alpha^*) e_c (s_\beta \tilde{f} s_\beta^*) u(\beta n_1 n_2 \beta^{-1}) s_{\beta b_2}
\end{aligned}$$

where $\tilde{e} = u(m_2^{-1}) e u(m_2)$ and $\tilde{f} = u(n_1) f u(n_1)^{-1}$. The above calculation together with Lemma 3.2 implies that T is closed under multiplication. Obviously T is closed under the involution $*$.

Now let us show that every element of T is a partial isometry. Let $v := s_a^* u(m) f u(m') s_{a'}$ be an element of T. Then

$$vv^* = s_a^* \Big(u(m)\big(f u(m') e_{a'} u(m')^{-1} f\big) u(m)^{-1}\Big) s_a.$$

Now Lemma 3.2 and Lemma 3.3 implies that $vv^* \in F$. Thus we have shown that every element of T is a partial isometry and the set of projections in T coincides with F. In other words T is an inverse semigroup.

Since T is closed under multiplication and involution, it follows that the linear span of T is a $*$-algebra. Moreover T contains $\{s_a \mid a \in P\}$ and $\{u(m) \mid m \in M\}$. Thus the linear span of T is dense in $\mathfrak{A}[N \rtimes H, M]$. This completes the proof. $\square$

The following equality will be used later. Let $a_1, a_2, b_1, b_2 \in P$ and $m_1, m_2 \in M$ be given. Choose $c \in Pb_1 \cap Pa_2$ and write c as $c = \beta b_1 = \alpha a_2$. Now the computation in Proposition 3.5 gives the following equality

$$s_{a_1}^* u(m_1) s_{b_1} s_{a_2}^* u(m_2) s_{b_2} = s_{\beta a_1}^* u(\beta m_1 \beta^{-1}) e_c u(\alpha m_2 \alpha^{-1}) s_{\alpha b_2} \tag{3.1}$$

Remark 3.6. We also need the following fact. If $v \in T$, let us denote its image in the regular representation by V. Observe that $v \neq 0$ if and only if $V \neq 0$. This is clear for projections in T. Now let $v \in T$ be a nonzero element. Then $vv^* \in F$ is nonzero. Thus $VV^* \neq 0$ which implies $V \neq 0$.

In the remainder of this article, we reserve the letter T to denote the inverse semigroup in Proposition 3.5 and F to denote the set of projections in T.

4. Tight representations of inverse semigroups

In this section, we show that the identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight in the sense of Exel and the C^*-algebra of the tight groupoid associated to T is isomorphic to $\mathfrak{A}[N \rtimes H, M]$. First let us recall the notion of tight characters and tight representations from [6].

Definition 4.1. Let S be an inverse semigroup with 0. Denote the set of projections in S by E. A *character for* E is a map $x : E \to \{0, 1\}$ such that

(1) the map x is a semigroup homomorphism, and
(2) $x(0) = 0$.

We denote the set of characters of E by $\widehat{E_0}$. We consider $\widehat{E_0}$ as a locally compact Hausdorff topological space where the topology on $\widehat{E_0}$ is the subspace topology induced from the product topology on $\{0, 1\}^E$.

For a character x of E, let $A_x := \{e \in E \mid x(e) = 1\}$. Then A_x is a nonempty set satisfying the following properties.

(1) The element $0 \notin A_x$.
(2) If $e \in A_x$ and $f \geq e$ then $f \in A_x$.
(3) If $e, f \in A_x$ then $ef \in A_x$.

Any nonempty subset A of E for which (1), (2) and (3) are satisfied is called a filter. Moreover if A is a filter then the indicator function 1_A is a character. Thus there is a bijective correspondence between the set of characters and filters. A filter is called an ultrafilter if it is maximal. We also call a character x maximal or an ultrafilter if its support A_x is maximal. The set of maximal characters is denoted by $\widehat{E_\infty}$ and its closure in $\widehat{E_0}$ is denoted by $\widehat{E_{tight}}$.

We refer to [19, Cor. 3.3] for the proof of the following lemma.

Lemma 4.2. *Let A be a unital C^*-algebra and $E \subset A$ be an inverse semigroup of projections containing $\{0, 1\}$. Suppose that E contains a finite set $\{e_1, e_2, \cdots, e_n\}$ of mutually orthogonal projections such that $\sum_{i=1}^n e_i = 1$. Then for every maximal character x of E, there exists a unique e_i for which $x(e_i) = 1$.*

Let us recall the notion of tight representations of semilattices from [6] and from [7]. The only semilattice we consider is that of an inverse semigroup of projections or in other words the idempotent semilattice of an inverse semigroup. Also our semilattice contains a maximal element 1. First let us recall the notion of a cover from [6].

Definition 4.3. Let E be an inverse semigroup of projections containing $\{0, 1\}$ and Z be a subset of E. A subset F of Z is called a *cover for* Z if given a nonzero element $z \in Z$ there exists an $f \in F$ such that $fz \neq 0$. A cover F of Z is called a finite cover if F is finite.

The following definition is actually Proposition 11.8 in [6].

Definition 4.4. Let E be an inverse semigroup of projections containing $\{0,1\}$. A representation $\sigma : E \to \mathcal{B}$ of the semilattice E in a Boolean algebra $\mathcal{B}$ is said to be *tight* if $\sigma(0) = 0$ and given $e \neq 0$ in E and for every finite cover F of the interval $[0,e] := \{x \in E \mid x \leq e\}$, one has $\sup_{f \in F} \sigma(f) = \sigma(e)$.

Let A be a unital C^*-algebra and S be an inverse semigroup containing $\{0,1\}$. Denote the set of projections in S by E. Let $\sigma : S \to A$ be a unital representation of S as partial isometries in A. Let $\sigma(C^*(E))$ be the C^*-subalgebra in A generated by $\sigma(E)$. Then $\sigma(C^*(E))$ is a unital, commutative C^*-algebra and hence the set of projections in it is a Boolean algebra which we denote by $\mathcal{B}_{\sigma(C^*(E))}$. We say the representation σ is *tight* if the representation $\sigma : E \to \mathcal{B}_{\sigma(C^*(E))}$ is *tight.* The proof of the following lemma can be found in [19, Lemma 3.6, p. 7].

Lemma 4.5. *Let X be a compact metric space and $E \subset C(X)$ be an inverse semigroup of projections containing $\{0,1\}$. Suppose that for every finite set of projections $\{f_1, f_2, \cdots, f_m\}$ in E, there exists a finite set of mutually orthogonal nonzero projections $\{e_1, e_2, \cdots, e_n\}$ in E and a matrix (a_{ij}) such that*

$$\sum_{i=1}^{n} e_i = 1,$$

$$f_i = \sum_j a_{ij} e_j.$$

Then the identity representation of E in $C(X)$ is tight.

As in [19], we prove that the identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight.

Proposition 4.6. *The identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight.*

Proof. We apply Lemma 4.5. Let $\{f_1, f_2, \cdots, f_n\}$ be a finite set of projections in T. By definition, given i there exists $a_i \in P$ such that f_i is in the linear span of $\{u(k)e_{a_i}u(k)^{-1}\}$. Let $c \in \bigcap_{i=1}^{n} a_i P$. By Lemma 3.1, it follows that for every i, f_i is in the linear span of $\{u(k)e_c u(k)^{-1} \mid k \in M/cMc^{-1}\}$. Appealing to Lemma 4.5, we can conclude that the identity representation of T in $\mathfrak{A}[N \rtimes H, M]$ is tight. This completes the proof. $\square$

Now we show that $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the C^*-algebra of the groupoid $\mathcal{G}_{tight}$ associated to T. For the convenience of the reader, we recall the construction of the groupoid $\mathcal{G}_{tight}$, considered in [6], associated to an inverse semigroup with 0.

Let S be an inverse semigroup with 0 and let E denote its set of projections. Note that S acts on $\widehat{E_0}$ partially. For $x \in \widehat{E_0}$ and $s \in S$, define $(x.s)(e) = x(ses^*)$. Then

- The map $x.s$ is a semigroup homomorphism, and
- $(x.s)(0) = 0$.

But $x.s$ is nonzero if and only if $x(ss^*) = 1$. For $s \in S$, define the domain and range of s as

$$D_s := \{x \in \widehat{E_0} \mid x(ss^*) = 1\},$$
$$R_s := \{x \in \widehat{E_0} \mid x(s^*s) = 1\}.$$

Note that both D_s and R_s are compact and open. Moreover s defines a homeomorphism from D_s to R_s with s^* as its inverse. Also observe that $\widehat{E_{tight}}$ is invariant under the action of S.

Consider the transformation groupoid $\Sigma := \{(x, s) \mid x \in D_s\}$ with the composition and the inversion being given by

$$(x, s)(y, t) := (x, st) \text{ if } y = x.s,$$
$$(x, s)^{-1} := (x.s, s^*).$$

Define an equivalence relation $\sim$ on Σ as $(x, s) \sim (y, t)$ if $x = y$ and if there exists an $e \in E$ such that $x \in D_e$ for which $es = et$. Let $\mathcal{G} = \Sigma/\sim$. Then $\mathcal{G}$ is a groupoid as the product and the inversion respects the equivalence relation $\sim$. Now we describe a topology on $\mathcal{G}$ which makes $\mathcal{G}$ into a topological groupoid.

For $s \in S$ and U an open subset of D_s, let $\theta(s, U) := \{[x, s] \mid x \in U\}$. We refer to [6] for the proof of the following proposition. We denote $\theta(s, D_s)$ by θ_s.

Proposition 4.7. *The collection $\{\theta(s, U) \mid s \in S, U$ open in $D_s\}$ forms a basis for a topology on $\mathcal{G}$. The groupoid $\mathcal{G}$ with this topology is a topological groupoid whose unit space can be identified with $\widehat{E_0}$. Also one has the following.*

(1) For $s, t \in S$, $\theta_s\theta_t = \theta_{st}$,
(2) for $s \in S$, $\theta_s^{-1} = \theta_{s^*}$,
(3) for $s \in S$, θ_s is compact, open and Hausdorff, and
(4) the set $\{1_{\theta_s} \mid s \in T\}$ generates the C^*-algebra $C^*(\mathcal{G})$.

We define the groupoid $\mathcal{G}_{tight}$ to be the reduction of the groupoid $\mathcal{G}$ to $\widehat{E_{tight}}$. In [6], it is shown that the representation $s \to 1_{\theta_s} \in C^*(\mathcal{G}_{tight})$ is tight and any tight representation of S factors through this universal one.

Proposition 4.8. *Let T be the inverse semigroup considered in Proposition 3.5. Denote the tight groupoid associated to T by $\mathcal{G}_{tight}$. Then $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to $C^*(\mathcal{G}_{tight})$.*

Proof. Let t_a and $v(m)$ be the images of s_a and $u(m)$ in $C^*(\mathcal{G}_{tight})$. By Proposition 4.6 and by the universal property of $\mathcal{G}_{tight}$, it follows that there exists a homomorphism $\rho : C^*(\mathcal{G}_{tight}) \to \mathfrak{A}[N \rtimes H, M]$ such that $\rho(t_a) = s_a$ and $\rho(v(m)) = u(m)$.

Given $a \in P$, the projections $\{u(k)e_a u(k)^{-1} \mid k \in M/M_a\}$ cover the projections in T. Since the representation of T in $C^*(\mathcal{G}_{tight})$ is tight, it follows that

$$\sum_{k \in M/M_a} v(k)(t_a t_a^*)v(k)^{-1} = 1.$$

Now the universal property of $\mathfrak{A}[N \rtimes H, M]$ implies that there exists a homomorphism $\sigma : \mathfrak{A}[N \rtimes H, M] \to C^*(\mathcal{G}_{tight})$ such that $\sigma(s_a) = t_a$ and $\sigma(u(m)) = v(m)$. It is then clear that σ and ρ are inverses of each other. This completes the proof. □

We identify the groupoid $\mathcal{G}_{tight}$ explicitly in the rest of the article.

5. Tight characters of the inverse semigroup T

In this section, we determine the tight characters of the inverse semigroup T defined in Proposition 3.5. Let

$$\overline{M} := \left\{ (r_a) \in \prod_{a \in P} M/M_a \;\middle|\; r_{ab} \equiv r_a \mod M_a \right\}.$$

We give $\overline{M}$ the subspace topology induced from the product topology on $\prod_{a\in P} M/M_a$. Here the finite group M/M_a is given the discrete topology. Then $\overline{M}$ is a compact, Hausdorff topological space. Moreover $\overline{M}$ is a topological group. Note that M embeds naturally into $\overline{M}$ via the imbedding $r \to (r_a := r)$. The map $r \to (r_a := r)$ is an imbedding since we have assumed that $\bigcap_{a \in P} M_a$ is trivial.

For $b \in P$ and $k \in M$, the set $U_{b,k} := \{(r_a) \in \overline{M} \mid r_b \equiv k \mod M_b\}$ is an open set. Moreover the collection $\{U_{b,k} \mid b \in P, k \in M\}$ forms a basis for $\overline{M}$. If $k \in M$ then clearly $k \in U_{b,k}$ for any $b \in P$. As a consequence, M is dense in $\overline{M}$.

For $r \in \overline{M}$, let

$$A_r := \{f \in F \mid f \geq u(r_a)e_a u(r_a)^{-1} \text{ for some } a \in P\}.$$

In the next lemma, we show that for every $r \in \overline{M}$, A_r is an ultrafilter and all ultrafilters are of this form.

Lemma 5.1. *For $r \in \overline{M}$, A_r is an ultrafilter. Moreover any ultrafilter is of the form A_r for some $r \in \overline{M}$.*

Proof. Let $r \in \overline{M}$ be given. First let us show that A_r is a filter. Clearly $0 \notin A_r$. Also if $f_1 \geq f_2$ and $f_2 \in A_r$ then $f_1 \in A_r$. Now suppose that $f_1, f_2 \in A_r$. Then there exists $a_1, a_2 \in P$ such that $f_i \geq u(r_{a_i})e_{a_i}u(r_{a_i})^{-1}$ for $i = 1, 2$. Choose $c \in a_1P \cap a_2P$. Then by Lemma 3.1, it follows that $e_c \leq e_{a_i}$ for $i = 1, 2$. Since $r \in \overline{M}$, it follows that $r_c \equiv r_{a_i} \mod M_{a_i}$ for $i = 1, 2$. Now observe that

$$\begin{aligned} f_1 f_2 &\geq u(r_{a_1})e_{a_1}u(r_{a_1})^{-1}u(r_{a_2})e_{a_2}u(r_{a_2})^{-1} \\ &= u(r_c)e_{a_1}u(r_c)^{-1}u(r_c)e_{a_2}u(r_c)^{-1} \\ &= u(r_c)e_{a_1}e_{a_2}u(r_c)^{-1} \\ &\geq u(r_c)e_c u(r_c)^{-1}. \end{aligned}$$

Thus $f_1 f_2 \in A_r$. Thus we have shown that A_r is a filter.

Now we show A_r is maximal. Let A be a filter which contains A_r. Consider an element $f \in A$. By definition there exists $a \in P$ and scalars $\alpha_k \in \{0,1\}$ such that

$$f = \sum_{k \in M/M_a} \alpha_k u(k) e_a u(k)^{-1}.$$

But both f and $u(r_a)e_a u(r_a)^{-1}$ belong to A and hence their product belongs to A. Thus the product $fu(r_a)e_a u(r_a)^{-1}$ is nonzero. This implies that $\alpha_{r_a} = 1$. Thus we have $f \geq u(r_a)e_a u(r_a)^{-1}$ or in other words $f \in A_r$. Hence $A = A_r$. This proves that A_r is maximal.

Let A be an ultrafilter. By Lemma 4.2, it follows that for every $a \in P$, there exists a unique $r_a \in M/M_a$ such that $u(r_a)e_a u(r_a)^{-1} \in A$. Let $r := (r_a)$. We claim that $r \in \overline{M}$. Let $a, b \in P$ be given. By Lemma 3.1, we have

$$u(r_a)e_a u(r_a)^{-1} = \sum_{k \in M/M_b} u(r_a aka^{-1}) e_{ab} u(r_a kak^{-1})^{-1}. \tag{5.2}$$

Since A is a filter containing $u(r_a)e_a u(r_a)^{-1}$ and $u(r_{ab})e_{ab}u(r_{ab})^{-1}$, it follows that their product is nonzero. This fact together with equation (5.2) implies that there exists $k \in M$, such that $r_{ab} \equiv r_a(aka^{-1}) \mod M_{ab}$. Thus $r_{ab} \equiv r_a \mod M_a$ for every $a, b \in P$. As a result, we have $r \in \overline{M}$. Since A is a filter it follows that $A_r \subset A$. We have already proved that A_r is maximal. Thus $A = A_r$. This completes the proof. □

The following proposition identifies the tight characters of T.

Proposition 5.2. *The map* $\overline{M} : r \to A_r \in \widehat{F_{tight}}$ *is a homeomorphism.*

Proof. It is clear from the definition that $r \to A_r$ is one-one. Let us denote this map by ϕ. We show ϕ is continuous. Consider a net r^α in $\overline{M}$ converging to r. We denote the indicator function of a set A by 1_A. Let $f \in F$ be given. Then there exists $a \in P$ and scalars α_k such that

$$f = \sum_k \alpha_k u(k) e_a u(k)^{-1}.$$

Then we have

$$1_{A_{r^\alpha}}(f) = \sum_k \alpha_k \delta_{r_a^\alpha, k}.$$

Since $r_a^\alpha = r_a$ eventually, it follows that $1_{A_{r^\alpha}}(f)$ converges to $1_{A_r}(f)$. This shows that $r \to A_r$ is continuous.

Now Lemma 5.1 implies that ϕ has range $\widehat{F_\infty}$. Since $\overline{M}$ is compact, it follows that $\widehat{F}_\infty$ is compact and hence closed. Thus $\widehat{F_\infty} = \widehat{F_{tight}}$. Thus $\phi : \overline{M} \to \widehat{F}_\infty$ is one-one, onto and continuous. Since $\overline{M}$ is compact, it follows that ϕ is in fact a homeomorphism. This completes the proof. □

From now on we will simply denote A_r by r and $1_{A_r}(f)$ by $r(f)$.

6. The groupoid $\mathcal{G}_{tight}$ of the inverse semigroup T

In this section, we will identify the tight groupoid $\mathcal{G}_{tight}$ associated to the inverse semigroup. Throughout this section, we assume $N = \bigcup_{a \in P} a^{-1}Ma$. By Remark 2.4, we can very well assume this. There is another natural groupoid which arises out of the following construction.

For every $a \in P$, the co-isometry s_a^* will give rise to an injection on $\overline{M}$ and the unitary $u(m)$ for $m \in M$ will act as a bijection on $\overline{M}$. Thus we get an action of the semigroup $M \rtimes P$, as injections, on $\overline{M}$. Now the space $\overline{M}$ can be enlarged to a space $\overline{N}$ and the action of $M \rtimes P$ can be dilated to get an action of $G = N \rtimes H$ on $\overline{N}$. We can then consider the transformation groupoid $\overline{N} \rtimes G$. But the unit space of $\mathcal{G}_{tight}$ is $\overline{M}$. Thus we restrict the transformation groupoid $\overline{N} \rtimes G$ to $\overline{M}$ and prove that it is isomorphic to $\mathcal{G}_{tight}$.

This dilation procedure has appeared in several works [See [11], [13]]. The basic principle goes back to [17].

First let us explain the action of $M \rtimes P$ on $\overline{M}$. The action of M on $\overline{M}$ is by left multiplication as M is a subgroup of $\overline{M}$. Let $a \in P$ and $r \in \overline{M}$ be given. For $b \in P$, choose $c \in aP \cap bP$ and write c as $c = a\alpha = b\beta$. We will use the notation as in Remark 3.4. Note that $M_c \subset M_b$ and we denote the induced quotient map $M/M_c \to M/M_b$ by $q_{b,c}$. Define $m_b = q_{b,c}(\pi_\alpha^a(r_\alpha))$. First let us show that m_b depends only on a and b and not on the choices made.

Suppose $c_1 = a\alpha_1 = b\beta_1$ and $c_2 = a\alpha_2 = b\beta_2$. Choose $\gamma_1, \gamma_2 \in P$ such that $\alpha_1\gamma_1 = \alpha_2\gamma_2$. Note that this implies $c_1\gamma_1 = c_2\gamma_2$. Now we have

$$\begin{aligned} q_{b,c_i}\pi_{\alpha_i}^a(r_{\alpha_i}) &= q_{b,c_i}\Big(\pi_{\alpha_i}^a\big(q_{\alpha_i,\alpha_i\gamma_i}(r_{\alpha_i\gamma_i})\big)\Big) \\ &= q_{b,c_i}\Big(q_{c_i,c_i\gamma_i}\big(\pi_{\alpha_i\gamma_i}^a(r_{\alpha_i\gamma_i})\big)\Big) \\ &= q_{b,c_i\gamma_i}\big(\pi_{\alpha_i\gamma_i}^a(r_{\alpha_i\gamma_i})\big). \end{aligned}$$

Note that the right hand side is constant for $i = 1, 2$. Thus we have

$$q_{b,c_1}(\pi_{\alpha_1}^a(r_{\alpha_1})) = q_{b,c_2}(\pi_{\alpha_2}^a(r_{\alpha_2})).$$

This shows that m_b is well defined. We leave it to the reader to check that $\tilde{m} = (m_b) \in \overline{M}$.

On M, the action of P is the usual conjugation. From now on, we denote the element $\tilde{m}$ by ara^{-1}. This way P acts on $\overline{M}$ injectively and continuously. This action of P together with the left multiplication action of M defines an action of $M \rtimes P$ on $\overline{M}$ (as injective,continuous transformations). We leave the details to the reader.

Lemma 6.1. *For $a \in P$, the kernel of the projection map $\overline{M} \ni (y_b) \to y_a \in M/M_a$ is $a\overline{M}a^{-1}$.*

Proof. By definition, it follows that $a\overline{M}a^{-1}$ is in the kernel of the ath projection. Now let $y = (y_b)$ be such that $y_a = 1$. Since M is dense in $\overline{M}$, there exists a sequence $y^n \in M$ such that $y^n \to y$ in $\overline{M}$. As M/M_a is finite, we can without loss of generality assume that $y^n \in M_a$ for every n. Thus there exists $x^n \in M$ such that $y^n = ax^na^{-1}$. But $\overline{M}$ is compact. Thus, by passing to a subsequence if necessary, we can assume that x^n converges to an element say $x \in \overline{M}$. Since conjugation by a is continuous, it follows that $y^n = ax^na^{-1}$ converges to axa^{-1}. But y^n converges to y. Thus $axa^{-1} = y$. This completes the proof. □

Now let us explain the dilation procedure that we promised at the beginning of this section. Consider the set $\overline{M} \times P$ and define a relation on $\overline{M} \times P$ by $(x, a) \sim (y, b)$ if there exists $\alpha, \beta \in P$ such that $\alpha a = \beta b$ and $\alpha x \alpha^{-1} = \beta y \beta^{-1}$. We leave the following routine checking to the reader.

(1) The relation $\sim$ is an equivalence relation. We denote the equivalence class containing (x, a) by $[(x, a)]$.
(2) Let $\overline{N} := \overline{M} \times P/\sim$. Then $\overline{N}$ is a group. The multiplication on $\overline{N}$ is defined as follows. For $a, b \in P$, choose α and β such that $\alpha a = \beta b$. Then

$$[(x, a)][(y, b)] = [(\alpha x \alpha^{-1} \beta y \beta^{-1}, \alpha a)].$$

The identity element of $\overline{N}$ is $[(e, e)]$ where (e, e) is the identity element of $\overline{M} \times P$ and the inverse of $[(x, a)]$ is $[(x^{-1}, a)]$.
(3) The group $\overline{N}$ is a locally compact Hausdorff topological group when $\overline{N}$ is given the quotient topology. Here P is given the discrete topology.
(4) The map $M \ni x \to [(x, e)] \in \overline{N}$ is a topological embedding. Thus $\overline{M}$ can be viewed as a subset of $\overline{N}$. Moreover $\overline{M}$ is a compact open subgroup of $\overline{N}$.
(5) The map $N \ni a^{-1}ma \to [(m, a)] \in \overline{N}$ is an embedding. When N is viewed as a subset of $\overline{N}$ via this embedding, N is dense in $\overline{N}$. Also $N \cap \overline{M} = M$.
(6) Let $a \in P$ be given. Define a map $\phi_a : \overline{N} \to \overline{N}$ as follows. Given $[(x, b)] \in \overline{N}$, choose $\alpha, \beta \in P$ such that $\alpha a = \beta b$. Define $\phi_a([(x, b)]) = [\beta x \beta^{-1}, \alpha)]$. One checks that ϕ_a is well defined. Moreover for $a \in P$, ϕ_a is a homeomorphism with ϕ_a^{-1} given by $\phi_a^{-1}[(x, b)] = [(x, ba)]$. Note that ϕ_a restricted to N is the usual conjugation. Also $\phi_a\phi_b = \phi_{ab}$ for $a, b \in P$. For $m \in M$, define $\psi_m : \overline{N} \to \overline{N}$ as $\psi_m([(x, a)]) = [(ama^{-1}x, a)]$. That is ψ_m is just left multiplication by m. One also has the following commutation relation. For $a \in P$ and $m \in M$,

$$\phi_a\psi_m = \psi_{ama^{-1}}\phi_a.$$

(7) Since we have assumed that $N = \bigcup_{a \in P} a^{-1}Ma$, it follows that any element of $g \in G = N \rtimes H$ can be written as $g = a^{-1}mb$ with $a, b \in P$ and $m \in M$. The map $a^{-1}mb \to \phi_a^{-1}\psi_m\phi_b$ is well defined and defines an action of G on $\overline{N}$. If $h = a^{-1}b \in H$ and $x \in \overline{N}$, we denote $\phi_a^{-1}\phi_b(x)$ as hxh^{-1}. If $n = a^{-1}ma$ and $x \in \overline{N}$, we denote $\phi_a^{-1}\psi_m\phi_a(x)$ as nx.
(8) Note that $\overline{N} = \bigcup_{a \in P} a^{-1}\overline{M}a$.
(9) *Universal Property:* Let L be a locally compact Hausdorff topological group on which H acts by group homomorphism. Suppose that K is a compact open subgroup of L which is invariant under P and $L = \bigcup_{a \in P} a^{-1}K$. If $\phi : \overline{M} \to K$ is a P-equivariant continuous bijection then the map

$\overline{N} \ni a^{-1}xa \to a^{-1}.\phi(x) \in L$ is a topological isomorphism and is H -equivariant.

Remark 6.2. It is not difficult to show by using (9) that $\overline{N}$ is the profinite completion of N when N is given the topology induced by the neighborhood base $\{aMa^{-1} \mid a \in H\}$ at the identity. In [10], the profinite completion model of $\overline{N}$ is used.

When considering transformation groupoids, we consider only right actions of groups and thus we change the above left action of G on $\overline{N}$ to a right action simply by defining $x.g = g^{-1}x$ for $x \in \overline{N}$ and $g \in G$. Now consider the transformation groupoid $\overline{N} \rtimes G$ and restrict it to $\overline{M}$. We show that the groupoid $\mathcal{G}_{tight}$ of the inverse semigroup T is isomorphic to the groupoid $\overline{N} \rtimes G|_{\overline{M}}$ i.e. to the transformation groupoid $\overline{N} \rtimes G$ restricted to the unit space $\overline{M}$. We will start with two lemmas which will be extremely useful to prove this.

Lemma 6.3. *If $a_1^{-1}m_1b_1 = a_2^{-1}m_2b_2$ then $s_{a_1}^*u(m_1)s_{b_1} = s_{a_2}^*u(m_2)s_{b_2}$.*

Proof. Suppose $a_1^{-1}m_1b_1 = a_2^{-1}m_2b_2$. Then $a_1^{-1}m_1a_1 = a_2^{-1}m_2a_2$ and $a_1^{-1}b_1 = a_2^{-1}b_2$. Choose $\beta_1, \beta_2 \in P$ such that $\beta_1 b_1 = \beta_2 b_2$. Then $a_1a_2^{-1} = \beta_1^{-1}\beta_2 = b_1b_2^{-1}$. Hence $\beta_1 m_1 \beta_1^{-1} = \beta_2 m_2 \beta_2^{-1}$. Now observe that

$$\begin{aligned}
s_{a_1}^*u(m_1)s_{b_1} &= s_{a_1}^*u(m_1)s_{\beta_1}^*s_{\beta_1}s_{b_1} \\
&= s_{a_1}^*s_{\beta_1}^*u(\beta_1m_1\beta_1^{-1})s_{\beta_1b_1} \\
&= s_{\beta_1a_1}^*u(\beta_1m_1\beta_1^{-1})s_{\beta_1b_1} \\
&= s_{\beta_2a_2}^*u(\beta_2m_2\beta_2^{-1})s_{\beta_2b_2} \\
&= s_{a_2}^*s_{\beta_2}^*u(\beta_2m_2\beta_2^{-1})s_{\beta_2b_2} \\
&= s_{a_2}^*u(m_2)s_{\beta_2}^*s_{\beta_2}s_{b_2} \\
&= s_{a_2}^*u(m_2)s_{b_2}.
\end{aligned}$$

This completes the proof. □

Lemma 6.4. *In $\mathcal{G}_{tight}$, $[(r, s_a^*u(m)fu(n)s_b)] = [(r, s_a^*u(mn)s_b)]$.*

Proof. First observe that $[(r, s_a^*)][r.s_a^*, u(m)fu(n)s_b] = [(r, s_a^*u(m)fu(n)s_b)]$. Thus it is enough to consider the case when a is the identity element of P. Now let $s = u(m)fu(n)s_b$, $t = u(mn)s_b$ and $e = u(m)fu(m)^{-1}$. Observe that $s = et$. Thus $ss^* = ett^*e$. Hence $r(ss^*) = 1$ implies $r(e) = 1$ and $r(tt^*) = 1$. Moreover $es = s = et$. Thus $[(r, s)] = [(r, t)]$. This completes the proof. □

Now we can state our main theorem.

Theorem 6.5. *Let $\phi : \overline{N} \rtimes G|_{\overline{M}} \to \mathcal{G}_{tight}$ be the map defined by*

$$\phi\big((x, a^{-1}mb)\big) = [(x, s_a^*u(m)s_b)].$$

Then ϕ is a topological groupoid isomorphism.

Proof. First let us show that ϕ is well defined. Let $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. Then by definition, there exists $y \in \overline{M}$ such that $m^{-1}axa^{-1} = byb^{-1}$. Choose α and β in P such that $c := a\alpha = b\beta$. By definition, this means that $\pi^a_\alpha(x_\alpha) \equiv q_c(m)\pi^b_\beta(y_\beta)$. Now Remark 3.4 implies that

$$s_a^* u(m) e_b u(m)^{-1} s_a \geq u(x_\alpha) e_\alpha u(x_\alpha)^{-1}.$$

Hence $x(s_a^* u(m) e_b u(m)^{-1} s_a) = 1$. Thus we have shown that ϕ is well-defined.

Before we show ϕ is a surjection, let us show that if $[(x, s_a^* u(m) s_b)] \in \mathcal{G}_{tight}$ then $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. To that effect, assume that $x(s_a^* u(m) e_b u(m)^{-1} s_a) = 1$. Choose $c \in aP \cap bP$ and write $c = a\alpha = b\beta$. By Remark 3.4, it follows that there exists $y \in M/M_\beta$ such that $q_c(m^{-1})\pi^a_\alpha(x_\alpha) = \pi^b_\beta(y)$. This implies that the bth coordinate of $m^{-1}axa^{-1}$ is 1, i.e. the identity element of M/M_b. Now Lemma 6.1 implies that there exists $z \in \overline{M}$ such that $m^{-1}axa^{-1} = bzb^{-1}$. Hence $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. Surjectivity is then an immediate consequence of Lemma 6.4.

Now we show ϕ is injective. Suppose $[(x, s_{a_1}^* u(m_1) s_{b_1})] = [(x, s_{a_2}^* u(m_2) s_{b_2})]$. Then there exists a projection $e \in F$ such that $0 \neq e(s_{a_1}^* u(m_1) s_{b_1}) = e(s_{a_2}^* u(m_2) s_{b_2})$. We can without loss of generality assume that $e = u(r_c) e_c u(r_c)^{-1}$. By Remark 3.6 and by reading the above equality in the regular representation, we immediately obtain $a_1^{-1} b_1 = a_2^{-1} b_2$ and $a_1^{-1} m_1 b_1 = a_2^{-1} m_2 b_2$. This implies that ϕ is injective.

Now let us show that ϕ is a groupoid morphism. First we show that ϕ preserves the range and source. By definition, ϕ preserves the range. Observe that ϕ is continuous and this is a direct consequence of Proposition 5.2. Let $\gamma = (x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$. Since M is dense in $\overline{M}$ there exists a sequence $x_n \in M$ such that x_n converges to x. Moreover the action of G on $\overline{N}$ is continuous and $\overline{M}$ is compact and open. Thus we can assume that $(x_n, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$ for every n. By definition, there exists $y \in \overline{M}$ such that $axa^{-1} = mbyb^{-1}$. Also let y_n be such that $ax_na^{-1} = mby_nb^{-1}$.

To keep things clear, if $z \in \overline{M}$, we denote the character determined by z as ξ_z. Let $v := s_a^* u(m) s_b$. Now if can show that $\xi_{x_n}.v = \xi_{y_n}$ then it will follow from continuity of ϕ that $\xi_x.v = \xi_y$. Thus we only need to show that $s(\phi(\gamma)) = \phi(s(\gamma))$ for $\gamma = (x, a^{-1}mb)$ with $x \in M$.

Now let $(x, a^{-1}mb) \in \overline{N} \rtimes G|_{\overline{M}}$ with $x \in M$. Then there exists $y \in M$ such that $axa^{-1} = mbyb^{-1}$. Let $v = s_a^* u(m) s_b$. To show $\xi_x.v = \xi_y$, as ξ_y is maximal, it is enough to show that the support of ξ_y is contained in $\xi_x.v$. Again it is enough to show that $u(y)e_c u(y)^{-1}$ is in the support of $\xi_x.v$. Choose α, β such that $a\alpha = bc\beta$. Note that

$$\begin{aligned} vu(y)e_c u(y)^{-1} v^* &= s_a^* u(m) s_b u(y) e_c u(y)^{-1} s_b^* u(m)^{-1} s_a \\ &= s_a^* u(mbyb^{-1}) s_b e_c s_b^* u(mbyb^{-1})^{-1} s_a \\ &= s_a^* u(axa^{-1}) e_{bc} u(axa^{-1})^{-1} s_a \\ &= u(x) s_a^* e_{bc} s_a u(x)^{-1} \end{aligned}$$

$$\geq u(x)s_a^* e_{bc\beta} s_a u(x)^{-1}$$
$$= u(x)s_a^* e_{a\alpha} s_a u(x)^{-1}$$
$$= u(x)e_\alpha u(x)^{-1} \in \mathrm{supp}(\xi_x).$$

Hence $u(y)e_c u(y)^{-1}$ is in the support of $\xi_x.v$. Thus we have shown that $\xi_x.v = \xi_y$. This proves that ϕ preserves the source.

Now we show ϕ preserves multiplication. Let $\gamma_1 = (x_1, a_1^{-1}m_1b_1)$ and $\gamma_2 = (x_2, a_2^{-1}m_2b_2)$. Since ϕ preserves the range and source, it follows that γ_1 and γ_2 are composable if and only if $\phi(\gamma_1)$ and $\phi(\gamma_2)$ are composable. Choose $\alpha, \beta \in P$ such that $\beta b_1 = \alpha a_2$. Now

$$\begin{aligned}\phi(\gamma_1)\phi(\gamma_2) &= [(x_1, s_{a_1}^* u(m_1)s_{b_1} s_{a_2}^* u(m_2)s_{b_2})] \\ &= [(x_1, s_{\beta a_1}^* u(\beta m_1 \beta^{-1})e_{\alpha a_2} u(\alpha m_2 \alpha^{-1})s_{\alpha b_2})] \text{ (by equation (3.1))} \\ &= [(x_1, s_{\beta a_1}^* u(\beta m_1 \beta^{-1} \alpha m_2 \alpha^{-1})s_{\alpha b_2})] \text{ (by Remark 6.4)} \\ &= \phi(\gamma_1\gamma_2).\end{aligned}$$

It is easily verifiable that ϕ preserves inversion.

For an open subset U of $\overline{M}$ and $g = a^{-1}mb$, consider the open set

$$\theta(U,g) := \{x \in \overline{M} \mid x.g \in \overline{M}\}.$$

The collection $\{\theta(U,g)\}$ forms a basis for $\overline{N} \rtimes G|_{\overline{M}}$. Moreover $\phi(\theta(U,g)) = \theta(U, s_a^* u(m)s_b)$. Thus ϕ is an open map. Thus we have shown that ϕ is a homeomorphism. This completes the proof. □

Corollary 6.6. *The algebra $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to $C^*(\overline{N} \rtimes G|_{\overline{M}})$.*

Proof. This follows from Theorem 6.5 and Proposition 4.8. □

7. Simplicity of $\mathfrak{A}_r[N \rtimes H, M]$

Let us recall a few definitions from [1]. Let $\mathcal{G}$ be an r-discrete groupoid and we denote its unit space by $\mathcal{G}^0$. The relation $\sim$ defined by $x \sim y$ if and only if there exists $\gamma \in \mathcal{G}$ such that $s(\gamma) = x$ and $r(\gamma) = y$ is an equivalence relation on $\mathcal{G}^0$. A subset $E \subset \mathcal{G}^0$ is said to be invariant if given $x \in E$ and $y \sim x$ then $y \in E$. For $x \in \mathcal{G}$, let $\mathcal{G}(x) := \{\gamma \in \mathcal{G} \mid s(\gamma) = r(\gamma) = x\}$ be the isotropy group of x.

A subset $S \subset \mathcal{G}$ is said to be a bisection if the range and source maps restricted to S are one-one. If S is a bisection, let $\alpha_S : r(S) \to s(S)$ be defined by $\alpha_S := s \circ r^{-1}$.

The groupoid $\mathcal{G}$ is said to be

- minimal if the only nonempty, open invariant subset of $\mathcal{G}^0$ is $\mathcal{G}^0$.
- topologically principal if the set of $x \in \mathcal{G}^0$ for which $\mathcal{G}(x) = \{x\}$ is dense in $\mathcal{G}^0$.
- locally contractive if for every nonempty open subset U of $\mathcal{G}^0$, there exists an open subset $V \subset U$ and an open bisection S with $\overline{V} \subset s(S)$ and $\alpha_{S^{-1}}(\overline{V})$ not contained in V.

Conjugation by P on M gives rise to a semigroup homomorphism from P to the semigroup of injective maps on M. In [10], the action of P on M is called an effective action if the above semigroup homomorphism is injective i.e. given $h \in H$ with $h \neq 1$, then there exists $s \in M$ such that $hsh^{-1} \neq s$. In [10], the following facts were proved about the transformation groupoid $\overline{N} \rtimes G$.

(1) The groupoid $\overline{N} \rtimes G$ is minimal and locally contractive.
(2) The groupoid $\overline{N} \rtimes G$ is topologically principal if and only if P acts effectively on M.
(3) Thus the reduced C^*-algebra $C^*_{red}(\overline{N} \rtimes G)$ is simple and purely infinite if P acts effectively on M. [Refer to [1].]

Analogous statements hold for the groupoid $\mathcal{G}_{tight}$ associated to the inverse semigroup T.

Remark 7.1. In [10], only the if part (in (2)) was proved. But then the other direction, i.e. if $\overline{N} \rtimes G$ is topologically principal then P acts effectively on M, is easy to verify.

Also note that $\overline{M}$ is a closed subset of $\overline{N}$ which meets each G orbit of $\overline{N}$. Moreover $\overline{M}$ is open as well. Hence by appealing to Example 2.7 in [16] , we conclude that $C^*(\overline{N} \rtimes G)$ and $C^*(\overline{N} \rtimes G|_{\overline{M}})$ are Morita-equivalent.

We end this section by showing that $\mathfrak{A}_r[N \rtimes H, M]$ is isomorphic to the reduced C^*-algebra $C^*_{red}(\mathcal{G}_{tight})$.

Proposition 7.2. *Let $\mathcal{G} := \overline{N} \rtimes G|_{\overline{M}}$. Then the reduced C^*-algebra of the groupoid $\mathcal{G}$ is isomorphic to $\mathfrak{A}_r[N \rtimes H, M]$.*

Proof. Let e be the identity element of $\overline{M}$. Define $\mathcal{G}^e := \{\gamma \in \mathcal{G} \mid r(\gamma) = e\}$. Then $\mathcal{G}^e := \{(e, hm) \mid m \in M, h \in H\}$. Thus $L^2(\mathcal{G}^e)$ can be identified with $\ell^2(M) \otimes \ell^2(H)$. Consider the representation π_e of $C^*_{red}(\mathcal{G})$ on $L^2(\mathcal{G}^e)$ defined as follows. For $f \in C_c(\mathcal{G})$, define $\pi_e(f)$ by the following formula

$$(\pi_e(f)(\xi))(\gamma) := \sum_{\gamma_1 \in \mathcal{G}^e} f(\gamma^{-1}\gamma_1)\xi(\gamma_1).$$

Since M is dense in $\overline{M}$, it follows that the largest open invariant set not containing e is the empty set. Hence π_e is faithful.

For $a \in P$ and $m \in M$, we let S_a and $U(m)$ be the images of s_a and $u(m)$ in $C^*_{red}(\mathcal{G})$. Let $\{\delta_m \otimes \delta_b \mid m \in M, b \in H\}$ be the canonical basis of $\ell^2(M) \otimes \ell^2(H)$. Consider the unitary operator V on $\ell^2(M) \otimes \ell^2(H)$ defined by

$$V(\delta_m \otimes \delta_b) := \delta_{m^{-1}} \otimes \delta_{b^{-1}}.$$

For $a \in P$ and $k \in M$, we leave it to the reader to check the following equality

$$V\pi_e(S_a)V^*(\delta_m \otimes \delta_b) = \delta_{ama^{-1}} \otimes \delta_{ab},$$
$$V\pi_e(U(k))V^*(\delta_m \otimes \delta_b) = \delta_{km} \otimes \delta_b.$$

Since $\{S_a \mid a \in P\}$ and $\{U(k) \mid k \in M\}$ generate $C^*_{red}(\mathcal{G})$, it follows that $C^*_{red}(\mathcal{G})$ is isomorphic to $\mathfrak{A}_r[N \rtimes H, M]$. This completes the proof. □

We now show that Corollary 6.6 and Proposition 7.2 can also be expressed in terms of crossed products as in [10]. We need to digress a bit before we do this.

Let $\mathcal{G}$ be an r-discrete, locally compact and Hausdorff groupoid. Let $Y \subset \mathcal{G}^0$ be a compact open subset of the unit space. Assume that Y meets each orbit of $\mathcal{G}^0$. Let

$$\mathcal{G}^Y := \{\gamma \in \mathcal{G} \mid s(\gamma) \in Y\},$$
$$\mathcal{G}_Y^Y := \{\gamma \in \mathcal{G} \mid s(\gamma), r(\gamma) \in Y\}.$$

Since Y is clopen, it follows that $\mathcal{G}^Y$ and $\mathcal{G}_Y^Y$ are clopen. Thus if $f \in C_c(G^Y)$, then f can be extended to an element in $C_c(\mathcal{G})$ by declaring its value to be zero outside $\mathcal{G}^Y$. Thus we have the inclusion $C_c(\mathcal{G}^Y) \subset C_c(\mathcal{G})$. Similarly, we have the inclusion $C_c(\mathcal{G}_Y^Y) \subset C_c(\mathcal{G}^Y)$. The algebra $C_c(\mathcal{G}_Y^Y)$ is a $*$-subalgebra of $C_c(\mathcal{G})$.

The space $C_c(\mathcal{G}^Y)$ is a pre-Hilbert $C_c(\mathcal{G}_Y^Y) \subset C^*(\mathcal{G}_Y^Y)$ module with the inner product and the right multiplication given by

$$\langle f_1, f_2\rangle(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} \overline{f_1(\gamma_1^{-1})} f_2(\gamma_2) \ \text{ for } \ \gamma \in \mathcal{G}_Y^Y,\ f_1, f_2 \in C_c(\mathcal{G}^Y),$$
$$(f.g)(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)g(\gamma_2) \ \text{ for } \ \gamma \in \mathcal{G}^Y,\ f \in C_c(\mathcal{G}^Y),\ g \in C_c(\mathcal{G}_Y^Y).$$

Moreover there is left action of $C_c(\mathcal{G})$ on $C_c(\mathcal{G}^Y)$ and it is given by

$$\begin{aligned}(f.\phi)(\gamma) &= (f * \phi)(\gamma) \\ &= \sum_{\gamma_1\gamma_2=\gamma} f(\gamma_1)\phi(\gamma_2)\end{aligned}$$

for $\gamma \in \mathcal{G}^Y$, $f \in C_c(\mathcal{G})$ and $\phi \in C_c(\mathcal{G}^Y)$.

Now Theorem 2.8 and Example 2.7 of [16] implies the following. The "completion" of $C_c(\mathcal{G})$-$C_c(\mathcal{G}_Y^Y)$ bimodule $C_c(\mathcal{G}^Y)$ is a $C^*(\mathcal{G})$-$C^*(\mathcal{G}_Y^Y)$ imprimitivity bimodule implementing a strong Morita equivalence between $C^*(\mathcal{G})$ and $C^*(\mathcal{G}_Y^Y)$.

Let us denote the completion of $C_c(\mathcal{G}^Y)$ by $\mathcal{E}$. For $x, y \in \mathcal{E}$, let $\theta_{x,y}$ be the compact operator on $\mathcal{E}$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle$. For $x \in \mathcal{E}$, the operator norm of $\theta_{x,x}$ is $\|x\|^2$.

The following proposition has also appeared in [14]. (See [14, Lemma 5.18].) The proof is exactly as in [14]. We include the proof for the sake of completeness.

Proposition 7.3. *The inclusion $C_c(\mathcal{G}_Y^Y) \subset C_c(\mathcal{G})$ extends to an isometric embedding from $C^*(\mathcal{G}_Y^Y)$ to $C^*(\mathcal{G})$. Also the inclusion $C_c(\mathcal{G}_Y^Y) \subset C_c(\mathcal{G})$ extends to an isometric embedding from $C^*_{red}(\mathcal{G}_Y^Y)$ to $C^*_{red}(\mathcal{G})$.*

Proof. Let $f \in C_c(\mathcal{G}_Y^Y)$ be given. Consider f as an element of $C_c(\mathcal{G}^Y) \subset \mathcal{E}$. Then $\theta_{f,f}$ restricted to $C_c(\mathcal{G}^Y)$ is just multiplication by $f * f^*$. Since $\mathcal{E}$ is a

$C^*(\mathcal{G})$-$C^*(\mathcal{G}_Y^Y)$ imprimitivity bimodule, it follows that

$$\begin{aligned}\|f\|_{C^*(\mathcal{G})}^2 &= \|f * f^*\|_{C^*(\mathcal{G})}\\ &= \|\theta_{f,f}\|\\ &= \|f\|_{\mathcal{E}}^2\\ &= \|f^* * f\|_{C^*(\mathcal{G}_Y^Y)}\\ &= \|f\|_{C^*(\mathcal{G}_Y^Y)}^2.\end{aligned}$$

For $x \in \mathcal{G}^0$, let $\mathcal{G}^{(x)} := r^{-1}(x)$. Consider $\ell^2(\mathcal{G}^{(x)})$ and let $\{\delta_\gamma : \gamma \in \mathcal{G}^{(x)}\}$ be the standard orthonormal basis. Consider the representation π_x of $C_c(\mathcal{G})$ on $\ell^2(\mathcal{G}^{(x)})$ defined by

$$\pi_x(f)(\delta_\gamma) = \sum_{\alpha \in \mathcal{G}^{(x)}} f(\alpha^{-1}\gamma)\delta_\alpha. \tag{7.3}$$

The reduced C^*-algebra $C^*_{red}(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ under the norm $\|.\|$ given by $|f|_{red} = \sup_{x \in \mathcal{G}^0} \|\pi_x(f)\|$. (We refer the reader to [18].)

Let $\mathcal{G}_Y^{(x)} := \{\gamma \in \mathcal{G}^{(x)} \mid s(\gamma) \in Y\}$. If $x \in Y$, let π_x^Y be the representation of $C_c(\mathcal{G}_Y^Y)$ on $\ell^2(\mathcal{G}_Y^{(x)})$ defined by the same formula as in equation (7.3). Now observe the following.

(1) Let $\gamma_0 \in \mathcal{G}$ be such that $s(\gamma_0) = x$ and $r(\gamma_0) = y$. Then $U : \ell^2(\mathcal{G}^{(x)}) \to \ell^2(\mathcal{G}^{(y)})$ defined by $U(\delta_\gamma) = \delta_{\gamma_0\gamma}$ is a unitary. Moreover $U\pi_x(.)U^* = \pi_y(.)$.
(2) Since Y meets each orbit of $\mathcal{G}^0$, it follows from (1) that for $f \in C_c(\mathcal{G})$, $\|f\|_{red} = \sup_{x\in Y} \|\pi_x(f)\|$.
(3) If $x \in Y$, then write $\ell^2(\mathcal{G}^{(x)})$ as $\ell^2(\mathcal{G}^{(x)}) = \ell^2(\mathcal{G}_Y^{(x)}) \oplus (\ell^2(\mathcal{G}_Y^{(x)}))^\perp$. With this decomposition, for $f \in C_c(\mathcal{G}_Y^Y)$, we have $\pi_x(f) = \pi_x^Y(f) \oplus 0$.

Now the above three observations imply that for $f \in C_c(\mathcal{G}_Y^Y)$, $\|f\|_{C^*_{red}(\mathcal{G}_Y^Y)} = \|f\|_{C^*_{red}(\mathcal{G})}$. This completes the proof. □

Remark 7.4. The representations used to define the regular representation in [18] is different from what we have used. But the inversion map of the groupoid intertwines our representations with those used in [18].

The C^*-algebra of the groupoid $\overline{N} \rtimes G$ is naturally isomorphic to $C_0(\overline{N}) \rtimes G$. Let $\Phi : C_c(\overline{N}) \rtimes G \to C_c(\overline{N} \rtimes G)$ be the map defined by

$$\Phi(fU_g)(x,h) := \begin{cases} f(x), & \text{if } g = h,\\ 0, & \text{otherwise,}\end{cases} \tag{7.4}$$

for $f \in C_c(\overline{N})$ and $g \in G$. Here $\{U_g \mid g \in G\}$ denotes the canonical unitaries (corresponding to the group elements) in the multiplier algebra of $C_0(\overline{N}) \rtimes G$. Then Φ extends to an isomorphism from $C_0(\overline{N}) \rtimes G$ onto $C^*(\overline{N} \rtimes G)$ (Cp. [18, Cor. 2.3.19, p. 34]).

Let $p := 1_{\overline{M}} \in C_c(\overline{N}) \subset C_0(\overline{N}) \rtimes G$ where $1_{\overline{M}}$ is the characteristic function associated to the compact open subset $\overline{M}$. Note that $\Phi(1_{\overline{M}}) = 1_{\overline{M}\times\{e\}}$.

Proposition 7.5. *The full corner $p(C_0(\overline{N}) \rtimes G)p$ is isomorphic to $\mathfrak{A}[N \rtimes H, M]$. Here the projection p is given by $p = 1_{\overline{M}}$.*

Proof. Let $i : C_c(\overline{N} \rtimes G|_{\overline{M}}) \to C_c(\overline{N} \rtimes G)$ be the natural inclusion. It is easy to verify that the image of i is $1_{\overline{M}\times\{e\}} C_c(\overline{N} \rtimes G) 1_{\overline{M}\times\{e\}}$. Now from Proposition 7.3, it follows that $C^*(\overline{N} \rtimes G|_{\overline{M}})$ is isomorphic to $1_{\overline{M}\times\{e\}} C^*(\overline{N} \rtimes G) 1_{\overline{M}\times\{e\}}$. But we have the isomorphism $\Phi : C_0(\overline{N}) \rtimes G \to C^*(\overline{N} \rtimes G)$ with $\Phi(1_{\overline{M}}) = 1_{\overline{M}\times\{e\}}$. Hence $\mathfrak{A}[N \rtimes H, M]$ is isomorphic to the corner $1_{\overline{M}}(C_0(\overline{N}) \rtimes G)1_{\overline{M}}$.

Let $A = C_0(\overline{N}) \rtimes G$. Then ApA is an ideal in A containing $p = 1_{\overline{M}}$. Note that for every $g \in G$, $x_g := U_g 1_{\overline{M}} 1_{\overline{M}} \in ApA$. Hence $1_{g\overline{M}} = U_g 1_{\overline{M}} U_g^* = x_g x_g^* \in ApA$. Hence for every $g \in G$, $1_{g.\overline{M}} \in ApA$. Thus $1_{a^{-1}\overline{M}a} \in ApA$ for every $a \in P$. Thus we have $C_c(\overline{N}) \subset ApA$ (See Remark 7.6) and hence $C_0(\overline{N}) \subset ApA$. As a consequence we have $ApA = C_0(\overline{N}) \rtimes G$. Thus the projection p is full. This completes the proof. □

Remark 7.6. If $K \subset \overline{N}$ is compact then there exists $b \in P$ such that $K \subset b^{-1}\overline{M}b$. For $\{a^{-1}\overline{M}a \mid a \in P\}$ is an open cover of $\overline{N}$. Thus there exists $a_1, a_2, \cdots, a_n \in P$ such that $K \subset \bigcup_{i=1}^{n} a_i^{-1}\overline{M}a_i$. Choose $b \in \bigcap_{i=1}^{n} Pa_i$. Then for every i, $a_i^{-1}\overline{M}a_i \subset b^{-1}\overline{M}b$. (Reason: M is dense in $\overline{M}$ and $ba_i^{-1} \in P$.) Hence $K \subset b^{-1}\overline{M}b$.

Remark 7.7. Using the second half of Proposition 7.3, it can be shown that the C^*-algebra $\mathfrak{A}_{red}[N \rtimes H, M]$ is isomorphic to the full corner $1_{\overline{M}}(C_0(\overline{N}) \rtimes_{red} G)1_{\overline{M}}$. We leave the details to the reader.

8. Cuntz-Li Duality theorem

The purpose of this section is to establish a duality result for the C^*-algebra associated to Examples 2.7 and 2.10. This is analogous to the duality result obtained in [4] for the ring C^*-algebra associated to the ring of integers in a number field. The proof is really a step by step adaptation of the arguments in [4] to our situation.

Let $\Gamma \subset GL_n(\mathbb{Q})$ be a subgroup and let $\Gamma_+ := \{\gamma \in \Gamma \mid \gamma \in M_n(\mathbb{Z})\}$. Assume that the following holds.

(1) The group $\Gamma = \Gamma_+\Gamma_+^{-1} = \Gamma_+^{-1}\Gamma_+$.
(2) The intersections $\bigcap_{\gamma\in\Gamma_+} \gamma\mathbb{Z}^n = \bigcap_{\gamma\in\Gamma_+} \gamma^t\mathbb{Z}^n = \{0\}$.

Let $\Gamma^{op} := \{\gamma^t \mid \gamma \in \Gamma\}$. Then Γ^{op} is a subgroup of $GL_n(\mathbb{Q})$. Also Γ satisfies (1) and (2) if and only if Γ^{op} satisfies (1) and (2). If Γ contains the nonzero scalars then (1) and (2) are satisfied.

For the rest of this section, we let Γ be a subgroup of $GL_n(\mathbb{Q})$ which satisfies (1) and (2). The group Γ acts on $\mathbb{Q}^n$ by left multiplication. Let $N_\Gamma := \bigcup_{\gamma\in\Gamma_+} \gamma^{-1}\mathbb{Z}^n$. Then by Lemma 2.3, it follows that N_Γ is a subgroup of $\mathbb{Q}^n$ and Γ leaves N_Γ invariant. Consider the semidirect product $N_\Gamma \rtimes \Gamma$. Then the pair $(N_\Gamma \rtimes \Gamma, \mathbb{Z}^n)$ satisfies the hypotheses (C1), (C2) and (C3). Let us denote the C^*-algebra $\mathfrak{A}[N_\Gamma \rtimes \Gamma, \mathbb{Z}^n]$ by $\mathfrak{A}_\Gamma$.

Note that $N_\Gamma \rtimes \Gamma$ acts on $\mathbb{R}^n$ on the right as follows. For $\xi \in \mathbb{R}^n$ and $(v,\gamma) \in N_\Gamma \rtimes \Gamma$, let $\xi.(v,\gamma) = \gamma^{-1}(\xi - v)$. This right action of $N_\Gamma \rtimes \Gamma$ on $\mathbb{R}^n$ gives rise to a left action of $N_\Gamma \rtimes \Gamma$ on $C_0(\mathbb{R}^n)$ as follows. For $g \in N_\Gamma \rtimes \Gamma$ and $f \in C_0(\mathbb{R}^n)$, let $(g.f)(x) = f(x.g)$.

The main theorem of this section is the following.

Theorem 8.1. *The C^*-algebras $\mathfrak{A}_{\Gamma^{op}}$ and $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$ are Morita-equivalent.*

To prove this we need a bit of preparation. If $\gamma \in \Gamma_+$, then γ leaves $\mathbb{Z}^n$ invariant and induces a map on the quotient $\frac{N_\Gamma}{\mathbb{Z}^n}$ which we still denote by γ. Let

$$\overline{N_\Gamma} := \left\{ (z_\gamma)_{\gamma\in\Gamma_+} \in \prod_{\gamma\in\Gamma_+} \frac{N_\Gamma}{\mathbb{Z}^n} \;\middle|\; \delta z_{\gamma\delta} = z_\gamma \text{ for every } \gamma,\delta \in \Gamma_+ \right\}.$$

We give $\frac{N_\Gamma}{\mathbb{Z}^n}$ the discrete topology. The abelian group $\overline{N_\Gamma}$ is given the subspace topology inherited from the product topology on $\prod_{\gamma\in\Gamma_+} \frac{N_\Gamma}{\mathbb{Z}^n}$. The topological group $\overline{N_\Gamma}$ is Hausdorff.

Now we describe the action of Γ_+ on $\overline{N_\Gamma}$. Let $\gamma \in \Gamma_+$ and $z \in \overline{N_\Gamma}$ be given. For $\delta \in \Gamma_+$, choose $\alpha, \beta \in \Gamma_+$ such that $\gamma\alpha = \delta\beta$. Let $(\gamma.z)_\delta = \beta z_\alpha$. It is easily verifiable that γ is a homeomorphism. The inverse of γ is given by $(\gamma^{-1}z)_\delta = z_{\gamma\delta}$. This way Γ_+ acts on $\overline{N_\Gamma}$ and induces an action of Γ on $\overline{N_\Gamma}$.

Proposition 8.2. *We have the following.*

(1) The map $N_\Gamma \ni v \to (\gamma^{-1}v)_{\gamma\in\Gamma_+} \in \overline{N_\Gamma}$ is injective and is Γ-equivariant. Moreover, when N_Γ is viewed as a subset of $\overline{N_\Gamma}$ via this embedding, N_Γ is dense in $\overline{N_\Gamma}$.

(2) Let $\overline{M_\Gamma} := \{z \in \overline{N_\Gamma} \mid z_e = 0\}$ is a compact open subgroup of $\overline{N_\Gamma}$. Also the intersection $\overline{M_\Gamma} \cap N_\Gamma = \mathbb{Z}^n$. Hence $\mathbb{Z}^n$ is dense in $\overline{M_\Gamma}$.

(3) Also $\overline{N_\Gamma} = \bigcup_{\gamma\in\Gamma_+} \gamma^{-1}\overline{M_\Gamma}$. As a consequence, $\overline{N_\Gamma}$ is locally compact.

Proof. The fact that $v \to (\gamma^{-1}v)_\gamma$ is injective follows from the assumption that $\bigcap_{\gamma\in\Gamma_+} \gamma\mathbb{Z}^n = \{0\}$. Let $\gamma \in \Gamma_+$ and $v \in N_\Gamma$ be given. Let us denote the image of v in $\overline{N_\Gamma}$ by $\tilde{v}$. We need to show that for $\delta \in \Gamma_+$, the δth coordinate of $\gamma.\tilde{v}$ is $\delta^{-1}\gamma v$. Choose α and β in Γ_+ such that $\gamma\alpha = \delta\beta$. Then by definition $(\gamma.\tilde{v})_\delta = \beta\alpha^{-1}v = \delta^{-1}\gamma v$. Thus we have shown that the embedding $N_\Gamma \ni v \to (\gamma^{-1}v)_{\gamma\in\Gamma_+} \in \overline{N_\Gamma}$ is Γ_+ -equivariant and consequently is Γ -equivariant.

For $\gamma \in \Gamma_+$ and $v \in N_\Gamma$, let

$$U_{\gamma,v} := \{z \in \overline{N_\Gamma} \mid z_\gamma \equiv v \mod \mathbb{Z}^n\}.$$

Clearly the collection $\{U_{\gamma,v} \mid \gamma \in \Gamma_+, v \in N_\Gamma\}$ forms a basis for $\overline{N_\Gamma}$. Note that $\gamma.v \in U_{\gamma,v}$. Thus N_Γ is dense in $\overline{N_\Gamma}$.

For $\gamma \in \Gamma_+$, let $N_\gamma := \gamma^{-1}\mathbb{Z}^n$. Note that for $\gamma \in \Gamma_+$, $\frac{N_\gamma}{\mathbb{Z}^n}$ is finite. Now observe that $\overline{M_\Gamma} = \overline{N_\Gamma} \cap \prod_\gamma \frac{N_\gamma}{\mathbb{Z}^n}$. Thus $\overline{M_\Gamma}$ is compact. Since the projection onto the eth coordinate is a continuous homomorphism, it follows that $\overline{M_\Gamma}$ is an open subgroup. The equality $\overline{M_\Gamma} \cap N_\Gamma = \mathbb{Z}^n$ is obvious.

Let $z \in \overline{N_\Gamma}$ be given. Since $N_\Gamma = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\mathbb{Z}^n$, it follows that there exists $\gamma \in \Gamma_+$ such that $\gamma z_e = 0$. Then $\gamma.z \in \overline{M_\Gamma}$. Thus $\overline{N_\Gamma} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M_\Gamma}$. As $\overline{N_\Gamma}$ is a union of compact open subsets, it follows that $\overline{N_\Gamma}$ is locally compact. This completes the proof. □

Let $\overline{N'}$ and $\overline{M'}$ be the groups considered in Section 6 applied to the pair $(N_\Gamma \rtimes \Gamma, \mathbb{Z}^n)$. Let us now convince ourselves that the pair $(\overline{N'}, \overline{M'})$ is Γ-equivariantly isomorphic to the pair $(\overline{N_\Gamma}, \overline{M_\Gamma})$. Let $\gamma, \delta \in \Gamma_+$ be given.

Denote the quotient map $\mathbb{Z}^n \to \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n}$ by q_γ. Then q_γ descends to a map $\frac{\mathbb{Z}^n}{\gamma\delta\mathbb{Z}^n} \to \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n}$ which we denote by $q_{\gamma,\delta}$. Multiplication by γ^{-1} maps $\mathbb{Z}^n$ injectively onto $\gamma^{-1}\mathbb{Z}^n$ and takes $\gamma\mathbb{Z}^n$ onto $\mathbb{Z}^n$. We denote the resulting isomorphism from $\frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n} \to \frac{\gamma^{-1}\mathbb{Z}^n}{\mathbb{Z}^n}$ again by γ^{-1}. Then we have the following commutative diagram where the vertical arrows are isomorphisms.

$$\begin{array}{ccc} \frac{\mathbb{Z}^n}{\gamma\delta\mathbb{Z}^n} & \xrightarrow{q_{\gamma,\delta}} & \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n} \\ \Big\downarrow{\scriptstyle (\gamma\delta)^{-1}} & & \Big\downarrow{\scriptstyle \gamma^{-1}} \\ \frac{(\gamma\delta)^{-1}\mathbb{Z}^n}{\mathbb{Z}^n} & \xrightarrow[\delta]{} & \frac{\gamma^{-1}\mathbb{Z}^n}{\mathbb{Z}^n} \end{array} \tag{8.5}$$

Recall that

$$\overline{M'} = \left\{ (z_\gamma)_{\gamma \in \Gamma_+} \in \prod_{\gamma \in \Gamma_+} \frac{\mathbb{Z}^n}{\gamma\mathbb{Z}^n} \;\middle|\; q_{\gamma,\delta}(z_{\gamma\delta}) = z_\gamma \right\},$$

$$\overline{M_\Gamma} = \left\{ (z_\gamma)_{\gamma \in \Gamma_+} \in \prod_{\gamma \in \Gamma_+} \frac{\gamma^{-1}\mathbb{Z}^n}{\mathbb{Z}^n} \;\middle|\; \delta z_{\gamma\delta} = z_\gamma \right\}.$$

Let $i : \mathbb{Z}^n \to \overline{M'}$ be the embedding given by $i(v) = (v)_{\gamma \in \Gamma_+}$ and $j : \mathbb{Z}^n \to \overline{M_\Gamma}$ be the embedding described in Proposition 8.2. Then $j(v) = (\gamma^{-1}v)_{\gamma \in \Gamma_+}$ for $v \in \mathbb{Z}^n$. Now the commutative diagram 8.5 implies that the map $\varphi : \overline{M'} \to \overline{M_\Gamma}$ given by $\varphi((z_\gamma)) = (\gamma^{-1}z_\gamma)$ is an isomorphism and $\varphi(i(v)) = j(v)$ for $v \in \mathbb{Z}^n$. It is also clear that φ is a homeomorphism.

Claim: φ is Γ_+-equivariant.

Proof. First the embeddings i and j are Γ_+-equivariant. Since $\varphi \circ i = j$, it follows that $\varphi(\gamma.i(v)) = \gamma.\varphi(i(v))$ if $\gamma \in \Gamma_+$ and $v \in \mathbb{Z}^n$. Since $i(\mathbb{Z}^n)$ is dense in $\overline{M'}$ (and the maps involved are continuous), it follows that $\varphi(\gamma.x) = \gamma.\varphi(x)$ for $x \in \overline{M'}$ and $\gamma \in \Gamma_+$.

Now since $\overline{N_\Gamma} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M_\Gamma}$ and $\overline{N'} = \bigcup_{\gamma \in \Gamma_+} \gamma^{-1}\overline{M'}$, it follows from the universal property, as explained in Section 6 (item (9)), that the map $\gamma^{-1}x \to \gamma^{-1}\varphi(x)$ (with $x \in \overline{M'}$) extends to a Γ-equivariant isomorphism from $\overline{N'} \to \overline{N_\Gamma}$. □

Now we describe the Pontrjagin dual of the discrete group N_Γ. For $x, \xi \in \mathbb{R}^n$, let $\langle x, \xi\rangle := x^t\xi$. If $x, \xi \in \mathbb{R}^n$, we let $\chi_\xi(x) = e^{2\pi i\langle x,\xi\rangle}$. We identity $\mathbb{R}^n$ with $\widehat{\mathbb{R}^n}$ via the map $\xi \to \chi_\xi$. If $\xi \in \mathbb{R}^n$, restricting χ_ξ to N_Γ gives a character of N_Γ. Moreover the map $\mathbb{R}^n \ni \xi \to \chi_\xi \in \widehat{N_\Gamma}$ is continuous.

Let $z \in \overline{N}_{\Gamma^{op}}$ be given. Let $\chi_z : N_\Gamma \to \mathbb{T}$ be defined as follows. For $x \in \gamma^{-1}\mathbb{Z}^n$ for some $\gamma \in \Gamma_+$, let $\chi_z(x) = e^{2\pi i\langle \gamma x, z_\gamma\rangle} = e^{2\pi i\langle x, \gamma^t z_\gamma\rangle}$. It is easy to verify that χ_z is well defined and χ_z is a character of N_Γ. Clearly $\overline{N_{\Gamma^{op}}} \ni z \to \chi_z \in \widehat{N_\Gamma}$ is continuous. Note that if $z \in N_{\Gamma^{op}}$ and $x \in N_\Gamma$ then $\chi_z(x) = e^{2\pi i\langle x,z\rangle}$.

Proposition 8.3. *The map $\Psi : \mathbb{R}^n \times \overline{N}_{\Gamma^{op}} \to \widehat{N_\Gamma}$ defined by*

$$\Psi(\xi, z) = \chi_\xi\chi_{-z}$$

is a surjective homomorphism with kernel $\Delta = \{(x, x) \mid x \in N_{\Gamma^{op}}\}$. The induced map $\widetilde{\Psi} : \frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta} \to \widehat{N_\Gamma}$ is a topological isomorphism.

Proof. Clearly Ψ is a continuous group homomorphism and $\Psi(\Delta) = \{1\}$. Now let us show that the kernel of Ψ is Δ. Let (ξ, z) be such that $\Psi(\xi, z) = 1$. Then for every $\gamma \in \Gamma_+$ and $x \in \mathbb{Z}^n$, we have

$$\begin{aligned} 1 &= \chi_\xi(\gamma^{-1}x)\chi_{-z}(\gamma^{-1}x) \\ &= e^{2\pi i\langle x,(\gamma^t)^{-1}\xi\rangle}e^{-2\pi i\langle x,z_\gamma\rangle} \\ &= e^{2\pi i\langle x,(\gamma^t)^{-1}\xi - z_\gamma\rangle}. \end{aligned}$$

Thus for every $\gamma \in \Gamma_+$, we have $z_\gamma - (\gamma^t)^{-1}\xi \in \mathbb{Z}^n$. In other words, we have $\xi \in N_{\Gamma^{op}}$ and $z = \xi$ in $\overline{N}_{\Gamma^{op}}$. Hence $(\xi, z) \in \Delta$. Thus we have shown that the kernel of Ψ is Δ which implies that $\widetilde{\Psi}$ is one-one.

Next we claim $\frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$ is compact. Let $\lambda : \mathbb{R}^n \times \overline{N}_{\Gamma^{op}} \to \frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$ be the quotient map. We also write $\lambda(\xi, z)$ as $[(\xi, z)]$. We claim that $\lambda([0, 1]^n \times \overline{M}_{\Gamma^{op}}) = \frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$. This will prove that $\frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$ is compact.

Let $[(\xi, z)]$ be an element in the quotient $\frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$. Choose $v \in \mathbb{Z}^n$ and $\gamma \in \Gamma_+$ such that $z_e \equiv (\gamma^t)^{-1}v$. Then $[(\xi, z)] = [(\xi - (\gamma^t)^{-1}v, z - (\gamma^t)^{-1}v)]$. Choose $w \in \mathbb{Z}^n$ such that $\xi - (\gamma^t)^{-1}v - w \in [0, 1]^n$. Let $\xi' = \xi - (\gamma^t)^{-1}v - w$ and $z' = z - (\gamma^t)^{-1}v - w$. Then $\xi' \in [0, 1]^n$ and $z' \in \overline{M}_{\Gamma^{op}}$. Moreover $\lambda(\xi, z) = \lambda(\xi', z')$. Thus the image of $[0, 1]^n \times \overline{M}_{\Gamma^{op}}$ under λ is $\frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$.

The image of $\widetilde{\Psi}$ is a compact subgroup of $\widehat{N_\Gamma}$ and it separates points of N_Γ. (The image of $\mathbb{R}^n \times \{0\}$ under Ψ separates points of N_Γ.) Hence $\widetilde{\Psi}$ is onto. Since $\frac{\mathbb{R}^n\times\overline{N}_{\Gamma^{op}}}{\Delta}$ is compact, it follows that $\widetilde{\Psi}$ is a topological isomorphism. This completes the proof. □

Consider the semidirect product $\mathbb{R}^n \rtimes \Gamma^{op}$ where Γ^{op} acts on $\mathbb{R}^n$ by left multiplication. The semidirect product $\mathbb{R}^n \rtimes \Gamma^{op}$ acts on $\widehat{N_\Gamma} = \frac{\mathbb{R}^n \times \overline{N_{\Gamma^{op}}}}{\Delta}$ on the right as follows. For $[(\xi, z)] \in \widehat{N_\Gamma}$ and $(v, \gamma) \in \mathbb{R}^n \rtimes \Gamma^{op}$, let $[(\xi, z)].(v, \gamma) = [(\gamma^{-1}(\xi + v), \gamma^{-1}z)]$. This right action of $\mathbb{R}^n \rtimes \Gamma^{op}$ on $\widehat{N_\Gamma}$ induces a left action of $\mathbb{R}^n \rtimes \Gamma^{op}$ on $C^*(N_\Gamma) \cong C(\widehat{N_\Gamma})$.

The crossed product $C^*(N_\Gamma) \rtimes (\mathbb{R}^n \rtimes \Gamma^{op})$ is isomorphic to the iterated crossed product $(C^*(N_\Gamma) \rtimes \mathbb{R}^n) \rtimes \Gamma^{op}$. (Cp. [20, Prop. 3.11, p. 87].) But then the map $\Gamma \ni \gamma \to (\gamma^t)^{-1} \in \Gamma^{op}$ is an isomorphism. Thus the crossed product $(C^*(N_\Gamma) \rtimes \mathbb{R}^n) \rtimes \Gamma^{op} \cong (C^*(N_\Gamma) \rtimes \mathbb{R}^n) \rtimes \Gamma$.

Let us fix notations. Let τ be the action of $\mathbb{R}^n$ on $C^*(N_\Gamma)$. Let β be the action of Γ on $C^*(N_\Gamma) \cong C(\widehat{N_\Gamma})$, induced by the action of Γ^{op} and the identification $\Gamma \cong \Gamma^{op}$. For $v \in N_\Gamma$, $\xi \in \mathbb{R}^n$ and $\gamma \in \Gamma$, it is easy to verify the following,

$$\tau_\xi(\delta_v) = e^{-2\pi i\langle \xi, v\rangle}\delta_v,$$
$$\beta_\gamma(\delta_v) = \delta_{\gamma v},$$

where $\{\delta_v \mid v \in N_\Gamma\}$ denotes the canonical unitaries of $C^*(N_\Gamma)$. The action of Γ^{op} on $C^*(N_\Gamma) \rtimes \mathbb{R}^n$, induces an action of Γ (via the identification $\Gamma \ni \gamma \to (\gamma^t)^{-1}$) and let us denote it by $\widetilde{\beta}$. For $\gamma \in \Gamma$, and $f \in C_c(\mathbb{R}^n, C^*(N_\Gamma))$, we have

$$\widetilde{\beta_\gamma}(f)(x) = |\det(\gamma)|\beta_\gamma(f(\gamma^t x)).$$

Now consider the crossed product $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma) \cong C^*(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$. Let us denote the action of N_Γ and Γ on $C^*(\mathbb{R}^n)$ by σ and α. For $v \in N_\Gamma$, $\gamma \in \Gamma$ and $f \in C_c(\mathbb{R}^n)$, we have

$$(\sigma_v f)(\xi) = e^{2\pi i\langle \xi, v\rangle} f(\xi),$$
$$(\alpha_\gamma f)(\xi) = |\det(\gamma)| f(\gamma^t \xi).$$

Denote the action of Γ on $C^*(\mathbb{R}^n) \rtimes N_\Gamma$ by $\widetilde{\alpha}$. For $\gamma \in \Gamma$, $v \in N_\Gamma$ and $f \in C^*(\mathbb{R}^n)$, one has

$$\widetilde{\alpha_\gamma}(f\delta_v) = \alpha_\gamma(f)\delta_{\gamma v}.$$

Let us recall the following lemma which is Lemma 4.3 in [4].

Lemma 8.4 ([4])**.** *Let G be a locally compact abelian group and H be a subgroup of the Pontrjagin dual $\widehat{G}$. Endow H with the discrete topology. Let σ be the action of H on $C^*(G)$ and τ be the action of G on $C^*(H)$ given by $\sigma_h(f) = [g \to h(g)f(g)]$ and $\tau_g(\tilde{f}) = [h \to h(-g)\tilde{f}(h)]$. Then the map $\phi : C_c(H, C_c(G)) \to C_c(G, C_c(H))$ defined by $\phi(f)(g)(h) = h(-g)f(h)(g)$ extends to an isomorphism between $C^*(G) \rtimes_\sigma H$ and $C^*(H) \rtimes_\tau G$.*

We are now ready to prove the following proposition.

Proposition 8.5. *The crossed products $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$ and $C(\widehat{N_\Gamma}) \rtimes (\mathbb{R}^n \rtimes \Gamma^{op})$ are isomorphic.*

Proof. It is enough to show that the crossed products $(C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma) \rtimes_{\widetilde{\alpha}} \Gamma$ and $(C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n) \rtimes_{\widetilde{\beta}} \Gamma$ are isomorphic. We show that $C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma$ and $C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n$ are Γ-equivariantly isomorphic. Then the isomorphism between the crossed products will follow.

Identify $\mathbb{R}^n$ with $\widehat{\mathbb{R}^n}$ via the map $\xi \to \chi_\xi$. (Recall that χ_ξ is the character given by $\chi_\xi(x) = e^{2\pi i <x,\xi>}$.) Consider N_Γ as a subgroup of $\widehat{\mathbb{R}^n}$ via the natural inclusion $N_\Gamma \subset \mathbb{R}^n$. Note that the action σ of N_Γ on $C^*(\mathbb{R}^n)$ and τ of $\mathbb{R}^n$ on $C^*(N_\Gamma)$ are exactly as in Lemma 8.4.

Thus Lemma 8.4 implies that $C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma \cong C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n$. Let $\phi : C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma \to C^*(N_\Gamma) \rtimes_\tau \mathbb{R}^n$ be the isomorphism prescribed by Lemma 8.4. We claim ϕ is Γ-equivariant. First note that $\phi(f\delta_v)(\xi) = e^{-2\pi i\langle \xi,v\rangle} f(\xi)\delta_v$ for $f \in C_c(\mathbb{R}^n)$ and $v \in N_\Gamma$.

Let $\gamma \in \Gamma$ be given. Now observe that

$$\begin{aligned}\widetilde{\beta_\gamma}(\phi(f\delta_v))(\xi) &= |\det(\gamma)|\beta_\gamma(\phi(f\delta_v)(\gamma^t\xi)) \\ &= |\det(\gamma)|e^{-2\pi i\langle \gamma^t\xi,v\rangle} f(\gamma^t\xi)\delta_{\gamma v} \\ &= |\det(\gamma)|e^{-2\pi i\langle \xi,\gamma v\rangle} f(\gamma^t\xi)\delta_{\gamma v}.\end{aligned}$$

On the other hand, observe that

$$\begin{aligned}\phi(\widetilde{\alpha_\gamma}(f\delta_v))(\xi) &= \phi(\alpha_\gamma(f)\delta_{\gamma v})(\xi) \\ &= e^{-2\pi i\langle \xi,\gamma v\rangle}\alpha_\gamma(f)(\xi)\delta_{\gamma v} \\ &= e^{-2\pi i\langle \xi,\gamma v\rangle}|\det(\gamma)|f(\gamma^t\xi)\delta_{\gamma v}.\end{aligned}$$

Hence for every $\gamma \in \Gamma$, $\widetilde{\beta_\gamma}\phi(f\delta_v) = \phi\widetilde{\alpha_\gamma}(f\delta_v)$. Since $\{f\delta_v \mid f \in C_c(\mathbb{R}^n), v \in N_\Gamma\}$ is total in $C^*(\mathbb{R}^n) \rtimes_\sigma N_\Gamma$, it follows that for every γ, $\widetilde{\beta_\gamma}\phi = \phi\widetilde{\alpha_\gamma}$. In other words, ϕ is Γ-equivariant. This completes the proof. ☐

Proof of Theorem 8.1. By Corollary 6.6, it follows that $\mathfrak{A}_{\Gamma^{op}}$ is isomorphic to the C^*-algebra of the groupoid $\widetilde{\mathcal{G}} := \overline{N}_{\Gamma^{op}} \rtimes (N_{\Gamma^{op}} \rtimes \Gamma^{op})|_{\overline{M}_{\Gamma^{op}}}$. By Proposition 8.5, it follows that $C_0(\mathbb{R}^n) \rtimes (N_\Gamma \rtimes \Gamma)$ is isomorphic to the C^*-algebra of the groupoid $\mathcal{G} := \widehat{N_\Gamma} \rtimes (\mathbb{R}^n \rtimes \Gamma^{op})$. We will show that $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are equivalent in the sense of [16].

By Proposition 8.3, $\widehat{N}_\Gamma = \frac{\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}}{\Delta}$ where $\Delta := \{(x,x) \mid x \in N_{\Gamma^{op}}\}$. Denote the quotient map $\mathbb{R}^n \times \overline{N}_{\Gamma^{op}} \to \frac{\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}}{\Delta}$ by λ. Let $X := \lambda(\{0\} \times \overline{M}_{\Gamma^{op}})$. Then X is a closed subset of $\mathcal{G}^0$ and it is easy to verify that X meets each orbit of $\mathcal{G}^0$. Let

$$\mathcal{G}_X := \{\alpha \in \mathcal{G} \mid s(\alpha) \in X\} = s^{-1}(X).$$

We claim that the (restricted) source map $s : \mathcal{G}_X \to X$ and the range map $r : \mathcal{G}_X \to \mathcal{G}^0$ are open. Let $U \subset \mathcal{G}$ be an open subset. Then $s(U \cap \mathcal{G}_X) = s(U) \cap X$. Since $s : \mathcal{G} \to \mathcal{G}^0$ is open, it follows that $s : \mathcal{G}_X \to X$ is open.

Now we prove that $r : \mathcal{G}_X \to \mathcal{G}^0$ is open. It is enough to show that $r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$ is open whenever $U \subset \frac{\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}}{\Delta}$ and $V \subset \mathbb{R}^n$ are open and

$\gamma \in \Gamma^{op}$. We claim that

$$r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X) = U \cap \lambda(-V \times \gamma \overline{M}_{\Gamma^{op}}).$$

Let $[(\xi, z)] \in r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$. Then there exists $([(\eta, y)], v, \gamma) \in U \times V \times \{\gamma\}$ such that $[(\eta, y)].(v, \gamma) \in X$ and $[(\xi, z)] = [(\eta, y)]$. Thus there exists $u \in N_{\Gamma^{op}}$ such that $\gamma^{-1}(\xi + v) = u$ and $\gamma^{-1}z - u = x$ for some $x \in \overline{M}_{\Gamma^{op}}$. Hence $[(\xi, z)] = [(-v, \gamma x)]$. Clearly $[(\xi, z)] \in U$. Hence $[(\xi, z)] \in U \cap \lambda(-V \times \gamma \overline{M}_{\Gamma^{op}})$. Thus we have shown that

$$r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X) \subset U \cap \lambda(-V \times \gamma \overline{M}_{\Gamma^{op}}).$$

Now let $[(\xi, z)] \in U \cap \lambda(-V \times \gamma \overline{M}_{\Gamma^{op}})$. Then there exists $(v, x) \in V \times \overline{M}_{\Gamma^{op}}$ such that $[(\xi, z)] = [(-v, \gamma x)]$. This is equivalent to saying that $[(\xi, z)].(v, \gamma) \in X$. Thus $([(\xi, z)], v, \gamma) \in (U \times V \times \{\gamma\}) \cap \mathcal{G}_X$ and $r([(\xi, z)], v, \gamma) = (\xi, z)]$. This proves that $U \cap \lambda(-V \times \gamma \overline{M}_{\Gamma^{op}}) \subset r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$.

This proves the claim that $r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X) = U \cap \lambda(-V \times \gamma \overline{M}_{\Gamma^{op}})$. Now since λ is open and $\overline{M}_{\Gamma^{op}}$ is open, it follows that $r((U \times V \times \{\gamma\}) \cap \mathcal{G}_X)$ is open. Thus we have shown that $r : \mathcal{G}_X \to \mathcal{G}^0$ is open.

Now by Example 2.7 of [16], it follows that $\mathcal{G}$ and $\mathcal{G}_X^X := \{\alpha \in \mathcal{G}_X \mid r(\alpha) \in X\}$ are equivalent. Recall that $\widetilde{\mathcal{G}} = \overline{N}_{\Gamma^{op}} \rtimes (N_{\Gamma^{op}} \rtimes \Gamma^{op})|_{\overline{M}_{\Gamma^{op}}}$ The right action of $N_{\Gamma^{op}} \rtimes \Gamma^{op}$ on $\overline{N}_{\Gamma^{op}}$ is given by $x.(v, \gamma) = \gamma^{-1}(x - v)$. Let $\Phi : \widetilde{\mathcal{G}} \to \mathcal{G}_X^X$ be defined by $\Phi(x, v, \gamma) = ([(0, x)], v, \gamma)$. It is easy to check that Φ is a groupoid isomorphism and it is continuous. Now we prove that Φ is a topological isomorphism.

Let (x_n, v_n, γ) be a sequence in $\widetilde{\mathcal{G}}$ such that $\Phi(x_n, v_n, \gamma)$ converges to $([(0, x)], v, \gamma))$. First note that $x \to [(0, x)]$ is a topological embedding of $\overline{M}_{\Gamma^{op}}$ into $\widehat{N_\Gamma}$. Thus, it follows that x_n converges to x in $\overline{M}_{\Gamma^{op}}$. Now $\Phi(x_n, v_n, \gamma)$ converges to $[(0, x)], v, \gamma)$ implies that v_n tends to v in $\mathbb{R}^n$ and $\gamma^{-1}(x - v_n)$ tends to $\gamma^{-1}(x - v)$ in $\overline{M}_{\Gamma^{op}}$. Hence v_n converges to v in $\overline{N}_{\Gamma^{op}}$. Thus $(v_n, v_n) \to (v, v)$ in $\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}$. But Δ is a discrete subgroup of $\mathbb{R}^n \times \overline{N}_{\Gamma^{op}}$. Hence $v_n = v$ eventually. Therefore, $(x_n, v_n, \gamma) \to (x, v, \gamma)$ in $\widetilde{\mathcal{G}}$. So, Φ is a topological isomorphism.

Since $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are equivalent in the sense of [16], it follows from Theorem 2.8 in [16] that $C^*(\mathcal{G})$ and $C^*(\widetilde{\mathcal{G}})$ are Morita-equivalent. This completes the proof. □

8.1. **Examples.** We end this article by considering two examples.

Example 8.6. First we show that the duality result for the ring C^*-algebra associated to number fields obtained in [4] can be derived from Theorem 8.1.

Consider a number field K of degree n. Denote the ring of integers in K by O_K. Let $\{w_1, w_2, \cdots, w_n\}$ be a $\mathbb{Z}$-basis for O_K. Then $\{w_1, w_2, \cdots, w_n\}$ is a $\mathbb{Q}$-basis for K. Identify K with $\mathbb{Q}^n$ via the map $\beta : \mathbb{Q}^n \ni (x_1, x_2, \cdots, x_n)^t \to \sum_{i=1}^n x_i w_i \in K$. By definition, $\beta(\mathbb{Z}^n) = O_K$.

If $a \in K$, then a acts on K by left multiplication and is $\mathbb{Q}$-linear. Thus a gives rise to a matrix with respect to the basis $\{w_1, w_2, \cdots, w_n\}$ which we denote by $\alpha(a)$. Explicitly, for $1 \leq j \leq n$, let

$$aw_j := \sum_{i=1}^{n} \alpha_{ij}(a)w_i. \tag{8.6}$$

Let $\alpha(a) := (\alpha_{ij}(a))$. Then $\alpha : K \to M_n(\mathbb{Q})$ is an injective ring homomorphism. We also have the following equivariance. For $a \in K$ and $x \in \mathbb{Q}^n$, $\beta(\alpha(a)x) = a\beta(x)$.

Let $\Gamma := \alpha(K^\times)$. Then Γ is a subgroup of $GL_n(\mathbb{Q})$. Now the pair $(K \rtimes K^\times, O_K)$ is isomorphic to $(\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n)$. Thus the ring C^*-algebra associated to O_K is nothing but $\mathfrak{A}[\mathbb{Q}^n \rtimes \Gamma, \mathbb{Z}^n]$. Hence Theorem 8.1 applies. The only thing that one needs to verify is $\bigcap_{a \in O_K} \alpha(a)^t\mathbb{Z}^n$ is trivial. Since $\bigcap_{a\in O_K} aO_K = \{0\}$, it follows that $\bigcap_{a\in O_K} \alpha(a)\mathbb{Z}^n = \{0\}$. We produce a matrix X with rational entries whose determinant is nonzero and $X\alpha(a)X^{-1} = \alpha(a)^t$ for every $a \in O_K$. Then it will follow that $\bigcap_{a\in O_K} \alpha(a)^t\mathbb{Z}^n = \{0\}$. (See also Lemma 8.10.)

Let $\mathrm{Tr} : M_n(\mathbb{Q}) \to \mathbb{Q}$ be the usual trace and let $\mathrm{tr} := \mathrm{Tr} \circ \alpha$. Denote the $n \times n$ matrix whose (i, j)th entry is $\mathrm{tr}(w_iw_j)$ by X. Then X has determinant nonzero and its determinant is called the discriminant of the number field K.

Lemma 8.7. *For every $a \in K$, $X\alpha(a)X^{-1} = \alpha(a)^t$.*

Proof. Fix $a \in K$. Let $Y = (\mathrm{tr}(aw_iw_j))$. Multiplying equation (8.6) by w_k and taking trace, we get

$$Y_{jk} = \sum_{i=1}^{n} \alpha_{ij}(a)X_{ik}.$$

In other words, we have $Y = \alpha(a)^tX$. But Y and X are symmetric. Thus taking transpose, we get $Y = X\alpha(a)$. Hence $X\alpha(a) = \alpha(a)^tX$. This completes the proof. □

Let $\mathbb{A}_\infty$ denote the ring of infinite adeles associated to K.

Theorem 8.8 ([4])**.** *For a number field K, the ring C^*-algebra $\mathfrak{A}[K \rtimes K^\times, O_K]$ is Morita-equivalent to $C_0(\mathbb{A}_\infty) \rtimes (K \rtimes K^\times)$.*

Proof. Note that for $\Gamma = \alpha(K^\times)$, $N_\Gamma = \mathbb{Q}^n$ and $N_{\Gamma^{op}} = \mathbb{Q}^n$ (since Γ contains the diagonal matrices with rational entries). Thus Lemma 8.7 implies that the matrix $X = (\mathrm{tr}(w_iw_j))$ implements an isomorphism between the dynamical systems $(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma)$ and $(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma^{op})$. The map

$$(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma) \ni (\xi, (v, \gamma)) \to (X\xi, (Xv, \gamma^t)) \in (\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma^{op})$$

is the required isomorphism. (Note that Γ is commutative.)

Consider the map $\delta : \mathbb{R}^n \ni (x_1, x_2, \cdots, x_n) \to \sum_{i=1} x_iw_i \in \mathbb{A}_\infty$. Then from standard number theoretic arguments, (for example, using Theorem 13.5 (p. 70) and Theorem 4.4 (p. 110) in [9]), it follows that δ (together with identifications α and β) implements an isomorphism between $(\mathbb{A}_\infty, K \times K^\rtimes)$ and $(\mathbb{R}^n, \mathbb{Q}^n \rtimes \Gamma)$. Now Theorem 8.1 yields the required result. This completes the proof. □

Example 8.9. Let A be an $n\times n$ matrix with integer entries such that $\det(A) \neq 0$ and $\bigcap_{r=0}^{\infty} A^r\mathbb{Z}^n = \{0\}$. Let $\Gamma := \{A^r \mid r \in \mathbb{Z}\} \cong \mathbb{Z}$. Denote the subgroup N_Γ by N_A and the Cuntz-Li algebra $\mathfrak{A}[N_\Gamma \rtimes \Gamma, \mathbb{Z}^n]$ by $\mathfrak{A}_A$. Denote the transpose A^t by B. Then $\Gamma^{op} = \{B^r \mid r \in \mathbb{Z}\} \cong \mathbb{Z}$.

We claim that the duality result is applicable to this example. The only thing that needs verification is $\bigcap_{r=0}^{\infty} B^r\mathbb{Z}^n = \{0\}$. This follows from the following lemma.

Lemma 8.10. *Let A be a $n \times n$ matrix with integer entries and denote A^t by B. Then $\bigcap_{r=0}^{\infty} A^r\mathbb{Z}^n = \{0\}$ if and only if $\bigcap_{r=0}^{\infty} B^r\mathbb{Z}^n = \{0\}$.*

Proof. Since A and B are similar over $\mathbb{Q}$, it follows that there exists $Y \in GL_n(\mathbb{Q})$ such that $YAY^{-1} = B$. Choose a nonzero integer m such that $X = mY \in M_n(\mathbb{Z})$. One has $XA = BX$. By induction, it follows that $XA^r = B^rX$ for every $r \geq 0$. First note that it is enough to show that $\bigcap_{r=0}^{\infty} A^r\mathbb{Z}^n \neq \{0\}$ implies $\bigcap_{r=0}^{\infty} B^r\mathbb{Z}^n \neq \{0\}$.

Suppose v is a nonzero element in $\bigcap_{r=0}^{\infty} A^r\mathbb{Z}^n$. Then

$$
\begin{aligned}
Xv &\in \bigcap_{r=0}^{\infty} XA^r\mathbb{Z}^n \\
&= \bigcap_{r=0}^{\infty} B^rX\mathbb{Z}^n \subset \bigcap_{r=0}^{\infty} B^r\mathbb{Z}^n.
\end{aligned}
$$

Since X is invertible over $\mathbb{Q}$, it follows that Xv is a nonzero element in $\bigcap_{r=0}^{\infty} B^r\mathbb{Z}^n$. Thus if $\bigcap_{r=0}^{\infty} A^r\mathbb{Z}^n \neq \{0\}$ then $\bigcap_{r=0}^{\infty} B^r\mathbb{Z}^n \neq \{0\}$. This completes the proof. □

Now Theorem 8.1 and Proposition 8.5 implies the following proposition.

Proposition 8.11. *The C^*-algebra $\mathfrak{A}_{A^t}$ is Morita-equivalent to $C_0(\mathbb{R}^n) \rtimes (N_A \rtimes \mathbb{Z})$. Also $\mathfrak{A}_{A^t}$ is Morita-equivalent to $(C^*(N_A) \rtimes \mathbb{R}^n) \rtimes \mathbb{Z}$.*

Proposition 8.11 for the case when $n = 1$ and $A = (2)$ was proved in [12]. In this case, the C^*-algebra $\mathfrak{A}_{A^t} = \mathfrak{A}_A$ is the C^*-algebra $\mathcal{Q}_2$ considered in [12]. The subgroup $\bigcup_{r=0} 2^{-r}\mathbb{Z}$ is denoted $\mathbb{Z}[\frac{1}{2}]$ in [12]. The Morita equivalence between $\mathcal{Q}_2$ and $C_0(\mathbb{R}) \rtimes (\mathbb{Z}[\frac{1}{2}] \rtimes (2))$ is called the 2-adic duality theorem in [12]. (Cp. Corollary 5.5 and Theorem 7.5 in [12].)

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