Mathematik

Reductions, Resolutions and the Copolarity of Isometric Group Actions

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Introduction

Transformation groups play an important role in many parts of mathematics and theoretical physics. One reason is that they describe various kinds of symmetries of mathematical structures and physical systems. These symmetries in turn often lead to considerable reductions of degrees of freedom. For example, Riemannian manifolds with special curvature conditions (i.e. positive or non-negative curvature, Einstein manifolds) are much easier to understand if they have many symmetries, like homogenous or cohomogeneity-1-manifolds.

It sometimes occurs that an isometric action of a Lie group G on a Riemannian manifold M admits a *reduction*. By this we mean another Lie group W and some submanifold $\Sigma \subseteq M$, which satisfies certain conditions such that the action (W, Σ) is in a reasonable way related to the action (G, M). In many aspects the "best" situation, which can occur, and where we can give the vague notions above a precise meaning, is when Σ is a *section*. This is an embedded submanifold Σ, which intersects all G-orbits and which is perpendicular to the orbits in the intersection points. An isometric action that admits a section is called *polar*. Examples of polar actions occur, for instance, in Lie theory: The action by conjugation of a compact Lie group with a bi-invariant Riemannian metric on itself is polar. Every maximal torus is a section in this example. Also, every transitive isometric action is polar and a section is given by any point of the manifold in this case. It turns out that polar actions have a particularly nice structure theory. To begin with, a section is always totally geodesic and its dimension is equal to the cohomogeneity of the action. Furthermore, all sections are conjugate to each other and every section comes with a discrete group W, which acts on it and which is called the *generalized Weyl group*. The action of W on Σ has the following relation to the action of G on M. The G-orbits intersect Σ in a discrete set of points, which is parameterized by W. It follows that the orbit spaces are canonically isometric to each other and hence $G\backslash M$ actually has the structure of the orbifold $W\backslash\Sigma$. Furthermore, the smooth G-invariant functions on M can be canonically identified with the smooth W-invariant functions on Σ and integration of functions on M can be reduced to an integration along Σ and a principal orbit. The property of being polar is inherited from the G-action on M to the slice representation in every point of M and the orbit geometry of polar actions is also noteworthy. For instance, the principal orbits of polar representations are isoparametric submanifolds.

In the same way as one measures the non-transitivity of an isometric action by an integer, the cohomogeneity $\operatorname{cohom}(G, M)$, Gorodski, Olmos and Tojeiro introduced in [**GOT04**] an integer called *copolarity*, $\operatorname{copol}(G, M)$, which measures the non-polarity of an isometric action. Just as $\operatorname{cohom}(G, M) = 0$ means that the action is transitive, $\operatorname{copol}(G, M) = 0$ has the meaning that the action is polar. Actually, more interesting than the mere numeric value of the copolarity are the objects, which in [**GOT04**] are called *k-sections* and which we call *fat sections* in this thesis. These are connected totally geodesic submanifolds Σ of M, which intersect every G-orbit such that the normal space to every principal orbit in all intersection points is contained in the tangent space of Σ with codimension k. In addition, some regularity conditions have to be satisfied (Definition 1.1.1). The copolarity is now the minimal integer k such that a k-section, also called *minimal section*, exists.

For any fat section Σ we can form the *fat Weyl group* $W = W(\Sigma)$, which is the quotient of the normalizer of Σ in G by the centralizer of Σ in G. The pair (W, Σ) is then called a *reduction* of the action (G, M). Now the interesting point is that a reduction contains much information about the original action (G, M). Also, the structure theory of fat sections resembles very much the structure theory of sections of polar actions, which in turn can be viewed as a generalization of the structure theory of maximal tori in Lie theory. More precisely:

- The orbit spaces $G\backslash M$ and $W\backslash\Sigma$ are canonically isometric (Theorem 2.1.1).
- Any two minimal sections are conjugate.
- In every point q the isotropy group G_q acts transitively on the set of minimal sections passing through q (Corollary 2.1.4).
- The intersection of a G-orbit with a fat section is always a W-orbit and vice versa (Corollary 2.1.3).
- A fat section induces in each of its points a fat section of the slice representation. This implies that the copolarity of the slice representation cannot exceed the copolarity of (G, M) (Theorem 2.2.2).
- The G-regular points in Σ coincide with the W-regular points (Lemma 2.3.1).
- The copolarity of a reduction (W, Σ) is equal to the copolarity of (G, M) (Theorem 2.3.2).
- A minimal reduction contains the information on both the copolarity and cohomogeneity of (G, M): $\operatorname{copol}(G, M) = \dim W$ and
$$\operatorname{cohom}(G, M) = \dim \Sigma - \dim W \qquad \text{(Proposition 1.1.16).}$$
- The G-Killing fields decompose reductively and in a geometrically nice way along a minimal section (Theorem 2.5.5).

Some of these results have already been proved in [**GOT04**], however in the context of orthogonal representations. In this thesis we prove the above mentioned results for general isometric group actions and sometimes our proofs are entirely different from theirs. We furthermore prove a generalization of Weyl's classical integration formula for compact Lie groups to the case of an almost arbitrary isometric action (see Theorem 2.6.4 for details):

$$\int_M f(x)\,dx = \int_{G/N} \left(\int_\Sigma f(g\cdot s)\,\delta_{\mathcal{E}}(s)ds \right)\, d(gN).$$

Here N denotes the normalizer of the minimal section Σ and $\delta_{\mathcal{E}}$ (see Definition 2.6.2) is a special G-invariant function, which measures the volume of the orbits "outside" of Σ. At least when N is compact, this allows us to view an isometric group action as a generalized random matrix ensemble in the sense that M is the *integration manifold*, a minimal reduction (W, Σ) generalizes the *set of eigenvalues* and furthermore $\delta_{\mathcal{E}}$ generalizes the notion of a *joint density function* for classical random matrix ensembles. For polar actions this approach has been investigated in [**AWY06, AWY05**]. Another consequence of the integration formula is that, for compact G/N, we can identify the G-invariant integrable functions on M with the integrable W-invariant functions on Σ in a natural way (Theorem 2.6.4 (iii)). Actually, also the continuous invariant functions on M and Σ correspond to each other in a natural way (Corollary 2.1.2) and in

Section 2.7 we try to improve this correspondence to the case of invariant $\mathcal{C}^\infty$-functions and basic forms. However, we are only able to achieve this under restrictive additional assumptions (Theorem 2.7.3). For instance, these assumptions are met by the actions appearing in Theorem 5.1.4. Nevertheless, it seems natural to expect that the general result should also be true, without making any assumptions.

The main result of Section 2.4 shows that reductions can be used to study geometric features of actions: An isometric action is *variationally complete* if and only if a reduction has this property (Theorem 2.4.6). Is it true that a corresponding result holds for *taut* actions?

In [**GS00**] Grove and Searle investigate the notions *core* ${}_cM$, *core group* ${}_cG$ and *global resolution* ${}^rM$ of an isometric group action. More precisely, ${}_cM$ is defined as the union of those connected components of the fixed point set of a principal isotropy group H, which contain G-regular points. The core group is then defined as ${}_cG = N_G(H)/H$ and finally, the global resolution is the twisted product ${}^rM = G/H \times_{{}_cG} {}_cM$. A connected component of the core is called a *canonical fat section* in our thesis (Definition 1.1.11). These yield, whenever the principal isotropy groups of an effective action are non-trivial, examples of fat sections different from M. Hence, the copolarity is non-trivial in these cases as well, and often canonical fat sections are minimal sections. In Chapter 3 we show that the notion of the global resolution can be generalized to a *global resolution with respect to a fat section* Σ. This is denoted with M_Σ. In this way, we can show many of the results, which in [**GS00**] are stated for ${}^rM$, also for M_Σ. In particular, we obtain a construction of manifolds with non-negative curvature (Proposition 3.1.6). An advantage of minimal sections over cores is perhaps that minimal sections can also be defined for singular Riemannian foliations with locally closed leaves. This is explained in Chapter 4. It is therefore quite probable that many of the results of this thesis can be generalized to the case of singular Riemannian foliations with locally closed leaves. In Chapter 5 and 7 we explicitly determine the copolarity and minimal sections of special actions and representations. Finally, in Chapter 6, we show the surprising result that a certain infinite dimensional action, which is connected to the action in Chapter 5, is either polar or has copolarity equal to ∞.

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CHAPTER 1

Fat Sections, Fat Weyl Groups and the Copolarity of Isometric Actions

After fixing our notation, we define *fat sections* and the *copolarity* and give examples. We also recall some basic properties from [**GOT04**] and introduce *fat Weyl groups*.

An **isometric action** of a Lie group G on a (finite or infinite dimensional) Riemannian manifold M is a smooth homomorphism $\Phi : G \to \mathrm{Iso}(M)$, whose image is an embedded Lie subgroup of $\mathrm{Iso}(M)$[1]. We also denote the action by the associated map $\varphi : G \times M \to M, (g, q) \mapsto g \cdot q := \Phi(g)(q)$, or just by (G, M), if no confusion can arise. We consider regular points as points lying on principal orbits and all other points are called singular. Thus, points lying on exceptional orbits are also singular in our sense.

Now we come to the central notions of this thesis.

Definition 1.1.1. Let M be a complete Riemannian manifold and let (G, M) be an isometric action. A submanifold $\Sigma \subseteq M$ is called a **fat section** of (G, M) if:

(A) Σ is complete, connected, embedded and totally geodesic in M,
(B) Σ intersects every orbit of the G-action,
(C) for all G-regular $p \in \Sigma$ we have $\nu_p(G \cdot p) \subseteq T_p\Sigma$,
(D) for all G-regular $p \in \Sigma$ and $g \in G$ such that $g \cdot p \in \Sigma$ we have $g \cdot \Sigma = \Sigma$.

In this situation, following [**GOT04**], we also call Σ a k**-section**, where k denotes the codimension of $\nu_p(G \cdot p)$ in $T_p\Sigma$ for any regular point $p \in \Sigma$. The integer

$$\mathrm{copol}(G, M) := \min\{k \in \mathbf{N} \mid \text{there is a } k\text{-section } \Sigma \subseteq M\}$$

is called the **copolarity** of the G-action on M. If $\Sigma \subseteq M$ is a $\mathrm{copol}(G, M)$-section, then we say that Σ is **minimal**. If a submanifold $\Sigma \subseteq M$ satisfies only properties (A)-(C) above, then Σ is called a **pre-section**. Finally, if M is a minimal section of (G, M), we say that (G, M) has **trivial copolarity**.

Remark 1.1.2.

(i) The definitions are meaningful even if M and G are not necessarily finite dimensional Hilbert manifolds. The only difference is that one has to add the possibility that $\mathrm{copol}(G, M)$ may be equal to ∞.
(ii) If (G, M) is a polar action, then there exists a complete, connected and embedded submanifold Σ, called *section*, which intersects every orbit and such that in the intersection points the orbits are perpendicular to Σ. It follows that Σ is totally geodesic and satisfies property (D) in the definition above. Hence, we have $\mathrm{copol}(G, M) = 0$ and a section in the polar sense is a minimal section in the sense of Definition 1.1.1. Conversely, if an isometric action has

[1] Note that an isometric action defined in this way is proper. I.e. $G \times M \to M \times M, (g, q) \mapsto (g \cdot q, q)$ is a proper map. Conversely, at least in the finite dimensional case, every proper action $\Phi : G \to \mathrm{Iso}(M)$ is an isometric action, because $\mathrm{im}(\Phi)$ is closed in this case.

copolarity zero, the action is in fact polar and all minimal sections are sections in the polar sense. The copolarity therefore measures the failure of an isometric action to be polar.

(iii) For a given Riemannian manifold M, one can define the **copolarity of** M as the integer:

$$\operatorname{copol}(M) := \operatorname{copol}(\operatorname{Iso}(M), M).$$

Just like the symmetry rank, symmetry degree and the cohomogeneity of a Riemannian manifold (see for instance [**Wil06b**] for the definitions), the copolarity is also a measure for the amount of symmetry a Riemannian manifold carries. For instance, homogeneous spaces and cohomogeneity one manifolds are manifolds of copolarity zero.

Situations in which the copolarity of an action is nontrivial and not equal to zero and in which the minimal sections can be explicitly computed are described in Chapter 5 and 7. To give some flavor:

Example 1.1.3. The k-fold direct sum of the standard representation of $\mathbf{SO}(n)$ on $\mathbf{R}^n$ has nontrivial copolarity equal to $\frac{k(k-1)}{2}$ for $2 \leq k \leq n-1$ and a minimal section is given by $\mathbf{R}^{k^2}$, which is embedded into $\mathbf{R}^{kn}$ as block matrices with nonzero entries in the upper $(k \times k)$-block only.

Example 1.1.4. Consider the following action of $T^2 \times S(\mathbf{U}(1) \times \mathbf{U}(2))$ on $\mathbf{SU}(3)$. The first factor acts by matrix multiplication from the left and the second factor by matrix multiplication from the right by the inverted matrix. The copolarity in this case is equal to 1 and a minimal section is given by $\mathbf{SO}(3) \subset \mathbf{SU}(3)$.

Pre-sections can also be objects of independent interest:

Example 1.1.5. If G is a compact Lie group, which acts on itself via conjugation, then any connected subgroup H of maximal rank is a pre-section. In fact, it is well known that this action is polar and that a section is given by any maximal torus. Since H contains a maximal torus of G it follows that H is in fact a closed subgroup of G (see for instance [**Djo81**]). Therefore, H is a compact Lie group in its own right. It follows that for every G-regular point p in H the maximal torus through p, which a priori exists only in G, is in fact contained in H. This implies property (C) of Definition 1.1.1.

The following three lemmas are important in the study of fat sections and their properties. For instance, Lemma 1.1.6 is needed for the fact that the connected intersection of two fat sections is again a fat section (Proposition 1.1.9 (iii) and (iv)).

Lemma 1.1.6. *Let (G, M) be an isometric action and suppose that M is connected and finite dimensional. Let $p \in M$ be G-regular. Then $\exp_p(\nu_p(G \cdot p))$ intersects every G-orbit.*

Proof. Let $q \in M$ be an arbitrary point and let $r > 0$ be such that

$$N := \overline{B_r(p)} \cap G \cdot q \neq \emptyset.$$

The set N is compact. Therefore the continuous map $f : N \to \mathbf{R},\ f(x) := d(p, x)$ has a minimum in $x_0 \in N$. After enlarging r, if necessary, we may assume that $d(p, x_0) < r$. A distance minimizing geodesic γ from p to x_0 therefore minimizes the distance from p to N and also from p to $G \cdot q$. It follows that γ is perpendicular to $G \cdot q$ and therefore, γ is also perpendicular to $G \cdot p$. Hence, $\gamma \subseteq \exp_p(\nu_p(G \cdot p))$. □

The following two statements are Lemma 5.1 and Lemma 5.2 from [**GOT04**]. Note that we formulate the second lemma for general isometric actions, whereas in loc. cit. it is formulated for orthogonal representations. However, their proof works also in the general case.

Lemma 1.1.7. *Let (G, M) be an isometric action and let $q \in M$ be arbitrary. For $v \in \nu_q(G \cdot q)$ the following assertions are equivalent:*

(i) *v is G_q-regular.*
(ii) *There exists $\varepsilon > 0$ such that $\exp_q(tv)$ is G-regular for $0 < t < \varepsilon$.*
(iii) *$\exp_q(t_0 v)$ is G-regular for some $t_0 > 0$.*

Lemma 1.1.8. *Let Σ be a fat section of (G, M). For all $q \in \Sigma$ there is a G_q-regular $v \in T_q\Sigma \cap \nu_q(G \cdot q)$. Furthermore, v can be chosen such that $p = \exp_q v$ is G-regular and arbitrarily close to q.*

The following proposition lists several properties related to the copolarity of an isometric action. All of them are either observations already made in [**GOT04**] or immediate consequences of these observations and Definition 1.1.1.

Proposition 1.1.9. *Let M, N be finite dimensional Riemannian manifolds and G, H Lie groups which act smoothly and isometrically on M, resp. N. Let furthermore $p \in M$ be an arbitrary G-regular point.*

(i) *If (G, M) and (H, N) are orbit-equivalent (i.e. there is an isometry from M onto N, mapping G-orbits onto H-orbits), then $\mathrm{copol}(G, M) = \mathrm{copol}(H, N)$.*
(ii) *$\mathrm{copol}(G, M) = \mathrm{copol}(G^\circ, M)$, where G° denotes the identity component.*
(iii) *For any two fat sections Σ_1, Σ_2 containing p, the* ***connected intersection*** *(i.e. the connected component of p of the intersection $\Sigma_1 \cap \Sigma_2$) is again a fat section. Hence, a minimal section through p is unique.*
(iv) *The minimal section through p is the connected intersection of all fat sections containing p. It is also the connected intersection of all pre-sections through p.*
(v) *The G-translates of a given fat section Σ induce a foliation on M^{reg}, the set of regular points of (G, M).*
(vi) *Every minimal section arises from a given one by translation by an element of G. That is, G is transitive on the set of all minimal sections of (G, M).*
(vii) *The intersection of a principal orbit $G \cdot p$ with a fat section Σ is an embedded submanifold of M. In fact, it is homogeneous: If $N_G(\Sigma)$ denotes the normalizer of Σ in G, then*

$$\Sigma \cap (G \cdot p) = N_G(\Sigma) \cdot p, \text{ if } p \in \Sigma.$$

It may have several connected components.
(viii) *The set $\Sigma^{\mathrm{reg}} = \Sigma \cap M^{\mathrm{reg}}$ of G-regular points in a fat section Σ is open and dense in Σ.*

Clearly, M itself is always a fat section of (G, M) (hence, we speak of trivial copolarity if M is a minimal section). In many cases, the following proposition yields a more interesting fat section.

Proposition 1.1.10 ([**GOT04**, Section 3.2]). *If (G, M) is isometric and $p \in M^{\mathrm{reg}}$, then $\Sigma := \mathrm{Fix}(G_p, M)^\circ$, i.e. the connected component of the fixed point set of G_p containing p, is a k-section, where k is the dimension of the subspace of $T_p(G \cdot p)$ on which G_p acts trivially.*

Definition 1.1.11. We call the fat sections of Proposition 1.1.10 **canonical (fat) sections**. Furthermore, we say that a fat section is **sufficiently small** if it is contained in some canonical section. In particular, canonical sections and minimal sections are sufficiently small.

Remark 1.1.12 ([**GOT04**], Section 3.2). Canonical sections need not be minimal sections. For instance, if in Example 1.1.3 $k = 2$ and $n = 3$, then the principal isotropy groups are trivial, but a minimal section is strictly smaller than the representation space. Nevertheless, it often happens for an isometric action (G, M) that we may enlarge the group G to a group G', which also acts isometrically on M and which produces the same orbits as G, such that (G', M) has canonical minimal sections. By Proposition 1.1.9 (ii) both actions have the same copolarity and the same minimal sections. It is interesting to note that for every polar representation the sections can be obtained in this way ([**Str94**, Theorem 1.3]) and this is also the case for the representations we consider in Chapter 7.

We next introduce the notion of the fat Weyl group, which plays a central role throughout the whole paper.

Definition 1.1.13. Let Σ be a fat section of the isometric action (G, M). We put

$$W(\Sigma) := N_G(\Sigma)/Z_G(\Sigma)$$

and call it the **fat Weyl group** of Σ. The isometric action $(W(\Sigma), \Sigma)$ is called a **reduction** of (G, M) (induced by Σ), and if Σ is a minimal section, we call it the **minimal reduction** of (G, M).

Remark 1.1.14.

(i) It is possible to define fat sections without assuming them being embedded submanifolds. However, for our purposes it will be necessary that Σ is closed in M, because this turns $N_G(\Sigma)$ into a Lie subgroup of G and hence $W(\Sigma)$ also carries the structure of a Lie group.

(ii) Every compact Lie group appears as the fat Weyl group of some isometric action. This is a generalization of [**PT88**, Remark 5.6.20] and a construction will be described in greater detail in Chapter 3.

Example 1.1.15. Concerning Example 1.1.3, the fat Weyl group of the minimal section $\mathbf{R}^{k^2}$ is $\mathbf{O}(k)$, acting by multiplication from the left.

Proposition 1.1.16. *Let (G, M) be an isometric action and let $\Sigma \subseteq M$ be a fat section. We put $W = W(\Sigma)$. If Σ is sufficiently small, then $Z_G(\Sigma) = H$, where H is a principal isotropy group of (G, M). In particular, all principal isotropy groups along Σ coincide. It follows that W acts freely on Σ^{reg}, and if Σ is a minimal section, then $\operatorname{copol}(G, M) = \dim(W)$ and the following formula relates cohomogeneity and copolarity of (G, M):*

$$\dim \Sigma = \operatorname{cohom}(G, M) + \operatorname{copol}(G, M).$$

Proof. Every $h \in H$ fixes every element of Σ, because $\Sigma \subseteq \operatorname{Fix}(H, M)$. This shows $H \subseteq Z_G(\Sigma)$. Conversely, if $h \in Z_G(\Sigma)$ then for any point $p \in \Sigma$ with $H = G_p$ we have $h \cdot p = p$ and hence $h \in G_p = H$. □

CHAPTER 2

Structure Theory of Fat Sections and Reductions

2.1. Properties of Reductions

In this section we generalize several results of [**GOT04**, Section 5.2], where they have been stated in the case of orthogonal representations, to the case of an arbitrary isometric action. Interestingly, in comparison to loc. cit. we obtain the results in a reversed order. We start with a metric observation concerning orbit spaces[1], which is a much stronger result than [**GOT04**, Theorem 5.9].

In the following let (G, M) be an isometric group action and let Σ be a fat section. We put $W := W(\Sigma)$.

THEOREM 2.1.1. *The orbit spaces $W\backslash\Sigma$ and $G\backslash M$, both endowed with their respective orbital distance metric, are canonically isometric via the map*

$$\tilde{\iota} : W\backslash\Sigma \to G\backslash M,\ W \cdot q \mapsto G \cdot q.$$

PROOF. First of all, $\tilde{\iota}$ is a well defined map: If $q, q' \in \Sigma$ are such that $W \cdot q = W \cdot q'$, then there exists $n \in N_G(\Sigma) \subseteq G$ such that $n \cdot q = q'$. Hence $G \cdot q = G \cdot q'$. Since Σ intersects all G orbits, it is clear that $\tilde{\iota}$ is surjective. The following diagram is commutative:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\iota} & M \\ {\scriptstyle \pi_W}\downarrow & & \downarrow{\scriptstyle \pi_G} \\ W\backslash\Sigma & \xrightarrow[\tilde{\iota}]{} & G\backslash M. \end{array}$$

Since Σ is an embedded submanifold of M, ι is continuous and since the vertical maps are open and continuous, it follows that $\tilde{\iota}$ is a continuous map, too. The distance between two points $G \cdot q$ and $G \cdot q'$ in $G\backslash M$ is the length of a minimal geodesic segment γ in M connecting the orbits $G \cdot q$ and $G \cdot q'$. Each such segment is perpendicular to both orbits. If now q and q' are both G-regular, then by properties (A) and (C) of a fat section, γ is a segment in Σ. We may further assume that q and q' lie in Σ and thus, γ minimizes the distance between $W \cdot q$ and $W \cdot q'$. It follows that $\tilde{\iota}$ restricted to the open and dense subset of G-regular points in Σ (see Proposition 1.1.9 (viii)) is an isometry. By continuity and using that $W\backslash\Sigma$ and $G\backslash M$ are complete metric spaces, we see that $\tilde{\iota}$ is a surjective isometry. □

COROLLARY 2.1.2. *The map $\iota^* : \mathcal{C}^0(M)^G \to \mathcal{C}^0(\Sigma)^W$, $f \mapsto f|_\Sigma$ is an isomorphism of Banach algebras, where both spaces are equipped with the corresponding $\|.\|_\infty$-norm.*

[1]I would like to thank Burkhard Wilking for suggesting this metric approach to me.

Proof. Consider the following diagram of Banach algebras associated with the diagram from Theorem 2.1.1:

$$\begin{array}{ccc} \mathcal{C}^0(G\backslash M) & \xrightarrow{\tilde\iota^*} & \mathcal{C}^0(W\backslash\Sigma) \\ {\scriptstyle \pi_G^*}\downarrow & & \downarrow{\scriptstyle \pi_W^*} \\ \mathcal{C}^0(M)^G & \xrightarrow[\iota^*]{} & \mathcal{C}^0(\Sigma)^W. \end{array}$$

It is commutative and the top arrow is an isomorphism of Banach algebras since the spaces $G\backslash M$ and $W\backslash\Sigma$ are canonically homeomorphic under the map $\tilde\iota$ and the assignment $\tilde\iota^*(f) = f \circ \tilde\iota$ is clearly norm preserving. The vertical maps are Banach algebra isomorphisms by definition of the orbit space. Hence the bottom arrow is also an isomorphism of Banach algebras. □

The next result is a rephrasing of the injectivity of $\tilde\iota$ in Theorem 2.1.1 and it generalizes Proposition 1.1.9 (vii).

Corollary 2.1.3. *The fat Weyl group W parameterizes intersections of G-orbits with Σ: For all $q \in \Sigma$ we have $W \cdot q = (G \cdot q) \cap \Sigma$. In particular, $(G \cdot q) \cap \Sigma$ is an extrinsic homogenous submanifold of the spaces $G \cdot q$, Σ and M for every $q \in \Sigma$.*

Corollary 2.1.4. *For a given fat section Σ and every $q \in M$, the isotropy group G_q of q is transitive on the set of all G-translates of Σ that contain q. In particular, G_q is transitive on the set of minimal sections through q.*

Proof. Since Σ intersects every orbit we may assume that $q \in \Sigma$. Let $g \in G$ be such that $q \in g \cdot \Sigma$. We have to show that there is some $\tilde g \in G_q$ such that $\tilde g \cdot \Sigma = g \cdot \Sigma$ holds. Since we have $q \in g \cdot \Sigma$, it follows that $g^{-1} \cdot q \in \Sigma$. By Corollary 2.1.3 there is some $n \in N_G(\Sigma)$ such that $g^{-1} \cdot q = n \cdot q$ and it follows that $\tilde g := gn \in G_q$ and $\tilde g \cdot \Sigma = g\Sigma$. □

In every G-regular point p of a fat section Σ we have the orthogonal decomposition $T_p(G \cdot p) = (T_p(G \cdot p) \cap T_p\Sigma) \oplus \nu_p\Sigma$. This is a consequence of property (C) of a fat section. More generally we have:

Proposition 2.1.5. *In all points q of a fat section Σ the tangent space T_qM decomposes compatibly and orthogonally in two ways:*

$$T_qM = T_q\Sigma \oplus \nu_q\Sigma = T_q(G \cdot q) \oplus \nu_q(G \cdot q).$$

I.e. the following decompositions are orthogonal:

$$\begin{aligned} T_q\Sigma &= (T_q\Sigma \cap T_q(G \cdot q)) \oplus (T_q\Sigma \cap \nu_q(G \cdot q)), \\ \nu_q\Sigma &= (\nu_q\Sigma \cap T_q(G \cdot q)) \oplus (\nu_q\Sigma \cap \nu_q(G \cdot q)). \end{aligned}$$

The proof is basically the same as for [**GOT04**, Lemma 5.10].

Proof. We just have to prove the first equality. Clearly,

$$T_q\Sigma = T_q(W \cdot q) \oplus \nu_q^\Sigma(W \cdot q),$$

where $\nu_q^\Sigma(W \cdot q)$ denotes the orthogonal complement of $T_q(W \cdot q)$ in $T_q\Sigma$. By Corollary 2.1.3 we already have

$$T_q(W \cdot q) = T_q\Sigma \cap T_q(G \cdot q).$$

Furthermore, since $W \cdot q \subseteq G \cdot q$ we have

$$T_q\Sigma \cap \nu_q(G \cdot q) \subseteq \nu_q^\Sigma(W \cdot q).$$

For the converse inclusion, we first assume that $v \in \nu_q^\Sigma(W \cdot q)$ has the property that $p := \exp_q(v)$ is G-regular. Then the geodesic $\gamma(t) := \exp_q(tv)$ in Σ is perpendicular to the W-orbit through q and thus it is also perpendicular to the W-orbit through p. Since $\nu_p(G \cdot p) = \nu_p^\Sigma(W \cdot p)$ it follows that γ, as a geodesic in M, is perpendicular to the G-orbit through p. Again, γ is also perpendicular to the G-orbit through q, and it follows that

$$v \in T_q\Sigma \cap \nu_q(G \cdot q).$$

By Lemma 1.1.8, there always exists a G_q-regular $v \in T_q\Sigma \cap \nu_q(G \cdot q) \subseteq \nu_q^\Sigma(W \cdot q)$ such that $p = \exp_q(v)$ is G-regular and lies in a W-slice $S_q \subseteq \Sigma$ through q. Since the G-regular points form an open subset of M, there is an open neighborhood of p in S_q consisting of G-regular points. Under the exponential map this neighborhood gets mapped onto an open subset U of $\nu_q^\Sigma(W \cdot q)$ and by the above considerations, we have $U \subseteq T_q\Sigma \cap \nu_q(G \cdot q)$. An open subset of a vector space always contains a basis. It therefore follows that

$$\nu_q^\Sigma(W \cdot q) \subseteq T_q\Sigma \cap \nu_q(G \cdot q).$$

□

Definition 2.1.6. For a given fat section Σ and for every $q \in \Sigma$ we define

$$\begin{aligned} \mathcal{D}_q &:= T_q(G \cdot q) \cap T_q\Sigma \text{ and} \\ \mathcal{E}_q &:= T_q(G \cdot q) \cap \nu_q\Sigma. \end{aligned}$$

Following [**GOT04**] we extend $\mathcal{D}$ and $\mathcal{E}$ to G-invariant distributions on M^{reg} using property (D) of a fat section. This yields $T_p(G \cdot p) = \mathcal{D}_p \oplus \mathcal{E}_p$ for all $p \in M^{\mathrm{reg}}$.

Remark 2.1.7. Due to Proposition 2.1.5, $T_q(G \cdot q) = \mathcal{D}_q \oplus \mathcal{E}_q$ is an orthogonal decomposition for all $q \in \Sigma$ and both $\mathcal{D}$ and $\mathcal{E}$ are W-invariant (singular) distributions along Σ.

Theorem 2.1.8. *Let Σ be a fat section of (G, M). For every $q \in \Sigma$, the submanifold $W \cdot q \subseteq G \cdot q$ is totally geodesic in $G \cdot q$. Furthermore, for every $\eta \in \nu_q(G \cdot q) \cap T_q\Sigma$ the shape operator A_η of $G \cdot q$ leaves the decomposition $T_q(G \cdot q) = \mathcal{D}_q \oplus \mathcal{E}_q$ invariant.*

Proof. $W \cdot q$ is a submanifold of M, and by Proposition 2.1.5 we have

$$T_x(G \cdot q) = \mathcal{D}_x \oplus \mathcal{E}_x = (T_x(G \cdot x) \cap T_x\Sigma) \oplus (T_x(G \cdot x) \cap \nu_x\Sigma)$$

for all $x \in W \cdot q$. Therefore $T_x(G \cdot q)$ is invariant under the orthogonal reflection on $T_x\Sigma$. Now the claim follows from the next Lemma, which is [**BCO03**, Exercise 8.6.3]. □

Lemma 2.1.9. *Let Σ, N and $\Sigma \cap N$ be submanifolds of the Riemannian manifold M and suppose that Σ is totally geodesic. Suppose that T_pN is invariant under the orthogonal reflection at $T_p\Sigma$ for all $p \in \Sigma \cap N$, then $\Sigma \cap N$ is totally geodesic as a submanifold of N and A_η, the shape operator of N, leaves $T_p(\Sigma \cap N)$ invariant for all $p \in \Sigma \cap N$ and $\eta \in \nu_pN \cap T_p\Sigma$.*

Proof. We first fix some notation: the orthogonal reflection at $T_p\Sigma$ is the map

$$\sigma : T_pM = T_p\Sigma \oplus \nu_p\Sigma \to T_pM,\ u + w \mapsto u - w.$$

Let α, resp. $\tilde\alpha$ denote the second fundamental forms of $\Sigma \cap N$ in N, resp. of N in M and let $\nabla, \tilde\nabla$ resp. $\bar\nabla$ denote the Levi-Cività connections of $\Sigma \cap N, N$ resp. M. Then we have for all $X, Y \in \mathcal{X}(\Sigma \cap N)$:

$$\begin{aligned} \tilde\nabla_X Y &= \nabla_X Y + \alpha(X, Y), \\ \bar\nabla_X Y &= \tilde\nabla_X Y + \tilde\alpha(X, Y), \end{aligned}$$

from which we obtain the equation:

$$\underbrace{\bar{\nabla}_X Y}_{\in T\Sigma} - \underbrace{\tilde{\alpha}(X,Y)}_{\nu N} = \underbrace{\nabla_X Y}_{\in T\Sigma \cap TN} + \underbrace{\alpha(X,Y)}_{TN \cap \nu\Sigma}. \tag{1}$$

The assumption that T_pN is invariant under σ means, that

$$T_pN = (T_pN \cap T_p\Sigma) \oplus (T_pN \cap \nu_p\Sigma).$$

Accordingly, there is a similar decomposition for ν_pN. Hence, $\tilde{\alpha}$ decomposes as

$$\tilde{\alpha} = \tilde{\alpha}_1 + \tilde{\alpha}_2,$$

with $\tilde{\alpha}_1 \in \nu N \cap T\Sigma$ and $\tilde{\alpha}_2 \in \nu N \cap \nu\Sigma$. Now, applying σ to (1) we obtain

$$\bar{\nabla}_X Y - \tilde{\alpha}_1(X,Y) + \tilde{\alpha}_2(X,Y) = \nabla_X Y - \alpha(X,Y).$$

Subtracting (1) from this, we arrive at

$$2\underbrace{\tilde{\alpha}_2(X,Y)}_{\in \nu N \cap \nu\Sigma} = -2\underbrace{\alpha(X,Y)}_{TN \cap \nu\Sigma}.$$

Therefore, $\alpha(X,Y) = 0 = \tilde{\alpha}_2(X,Y)$. This shows that $\Sigma \cap N$ is totally geodesic in N. The shape operator $A_\eta v$ of N for $\eta \in \Gamma(\nu N)$ and $v \in T_pN$ is (up to a sign) the part of $\tilde{\nabla}_v \eta$ which is tangential to N. If η and v are also tangential to Σ, then $\nabla_v \eta \in T_p\Sigma$, since Σ is totally geodesic in M. Therefore, $A_\eta v \in T_p(N \cap \Sigma)$. □

Remark 2.1.10. In case that the action is polar, the orbits of the Weyl group consist of a discrete number of points and so, trivially, they are totally geodesic inside their ambient G-orbit. However, if the copolarity is positive and non-trivial, then the orbits of the fat Weyl group are proper positive-dimensional totally geodesic submanifolds of their ambient orbit. So one should expect that the theorem imposes certain restrictions on actions having non-trivial positive copolarity.

2.2. Copolarity and Reductions of the Slice Representation

We now generalize [**GOT04**, Theorem 5.6] from the case of representations to arbitrary isometric group actions, without making any further assumptions. Therefore, our proof follows a rather different approach than the one in loc. cit. We first need a lemma, which will also be frequently used in Section 2.5.

Lemma 2.2.1. *Let Σ be a totally geodesic submanifold of the Riemannian manifold M and let $\gamma \subseteq \Sigma$ be a geodesic. Then every Jacobi field J along γ splits uniquely into Jacobi fields Y and Z along γ such that Y is a Jacobi field in Σ and Z is perpendicular to Σ. Furthermore, every derivative of Z is perpendicular to Σ.*

Proof. Consider the orthogonal decomposition

$$J(t) = \underbrace{Y(t)}_{\in T_{\gamma(t)}\Sigma} + \underbrace{Z(t)}_{\in \nu_{\gamma(t)}\Sigma}$$

of J. Then Y and Z defined in this way are smooth vector fields along γ. Since J satisfies the Jacobi equation we have:

$$0 = J'' + R(J,\dot{\gamma},\dot{\gamma}) = Y'' + R(Y,\dot{\gamma},\dot{\gamma}) + Z'' + R(Z,\dot{\gamma},\dot{\gamma}). \tag{Δ}$$

Clearly, Y'' is tangential to Σ. Since Σ is totally geodesic, $R(Y,\dot{\gamma},\dot{\gamma})$ is also tangential to Σ. Since parallel transports of vectors normal to a totally geodesic submanifold stay perpendicular to the submanifold, it follows from the characterization of the covariant derivative by parallel transport that Z'' is perpendicular to Σ. Finally, the expression

$R(Z,\dot\gamma,\dot\gamma)$ is perpendicular to Σ, because for all $v \in T\Sigma$ we have, using the symmetry properties of the curvature tensor,

$$\langle R(Z,\dot\gamma,\dot\gamma),v\rangle = \langle \underbrace{R(v,\dot\gamma,\dot\gamma)}_{\in T\Sigma}, Z\rangle = 0.$$

Thus (Δ) implies that both Y and Z are Jacobi fields, and since Σ is totally geodesic we have that Y is a Jacobi field of Σ. □

THEOREM 2.2.2 (Slice Theorem). *If (G,M) is isometric, then for all $q \in M$:*

$$\operatorname{copol}(G_q,\nu_q(G\cdot q)) \leq \operatorname{copol}(G,M).$$

More generally, if Σ is a fat section of (G,M) and $q \in \Sigma$, then $V_q := \nu_q(G\cdot q)\cap T_q\Sigma$ is a fat section of $(G_q,\nu_q(G\cdot q))$. If W is the fat Weyl group of Σ, then W_q projects canonically onto the fat Weyl group of V_q.

PROOF. Let Σ be a fat section through q. Since V_q is a linear subspace of $\nu_q(G\cdot q)$, property (A) of a fat section is already satisfied. Property (B) will follow from property (C), which we will prove below. In fact, there exist G_q-regular points in V_q by Lemma 1.1.8. Then property (C) and Lemma 1.1.6 show that V_q intersects every G_q-orbit. We also have property (D): If $v \in V_q$ is G_q-regular, then, after scaling if necessary, we may assume that $p := \exp_q(v)$ lies in a slice S_q through q. Now assume that $g \in G_q$ satisfies $g\cdot v \in V_q$. This implies $g\cdot p \in \Sigma$. The G_q regular points in S_q are G-regular if we view them as points of M. Hence, p is also G-regular and therefore $g\cdot\Sigma = \Sigma$. It follows that

$$g\cdot V_q = T_q(g\cdot(\Sigma\cap S_q)) = T_q(\Sigma\cap S_q) = V_q.$$

We next show

$$V_q^\perp \subseteq T_v(G_q\cdot v).$$

Here $v \in V_q$ is an arbitrary G_q-regular point and $V_q^\perp$ is the orthogonal complement of V_q in $\nu_q(G\cdot q)$. This statement is equivalent to property (C) of a fat section. As in the proof of property (D) we may assume that $p = \exp_q(v)$ lies in a slice S_q through q. Since p is a G-regular point of Σ, property (C) of Σ now implies $\nu_p\Sigma \subseteq T_p(G\cdot p)$. Let $w \in V_q^\perp$ be arbitrary. Then $d\exp_q(v)(w) \in \nu_p\Sigma$. In fact, $d\exp_q(v)(w) = J(1)$, where J is the Jacobi field along $\gamma_v(t) = \exp_q(t\cdot v)$ with initial conditions $J(0) = 0$ and $J'(0) = w \in \nu_q\Sigma$. (see for instance [**Lan99**, Chapter IX, Theorem 3.1]). By Lemma 2.2.1, J is always orthogonal to Σ. In particular,

$$J(1) \in \nu_p\Sigma \subseteq T_p(G\cdot p).$$

Now let X be a G-Killing field such that $d\exp_q(v)(w) = X_p$. Since $d\exp_q(v)(w) \in T_pS_q$ and $(G\cdot p)\cap S_q = G_q\cdot p$ we may further assume that X is a G_q-Killing field. Therefore

$$d\exp_q(v)(w) = X_p \in T_p(G_q\cdot p).$$

Since $\exp_q$ intertwines the slice representation $(G_q,\nu_q(G\cdot q))$ with the G_q-action on S_q we have $G_q\cdot p = \exp_q(G_q\cdot v)$. This in turn implies

$$T_p(G_q\cdot p) = d\exp_q(v)(T_v(G_q\cdot v)),$$

and it follows that $w \in T_v(G_q\cdot v)$, because $d\exp_q(v)$ is a bijection.

We have therefore proved that V_q is a fat section of $(G_q,\nu_q(G\cdot q))$. Since (G,M) and $(G_q,\nu_q(G\cdot q)$ have the same cohomogeneity and $\dim V_q \leq \dim\Sigma$ it follows, by choosing Σ as a minimal section, that the copolarity of the slice representation is less than or equal to the copolarity of the G-action on M.

The fat Weyl group of V_q is given by

$$W(V_q) = N_{G_q}(V_q)/Z_{G_q}(V_q).$$

We first show that

$$N_{G_q}(V_q) = N_{G_q}(\Sigma) = (N_G(\Sigma))_q.$$

Let $g \in N_G(\Sigma) \cap G_q = N_{G_q}(\Sigma)$ be arbitrary. Then g leaves both $T_q\Sigma$ and $\nu_q(G \cdot q)$ invariant. It therefore also leaves $T_q\Sigma \cap \nu_q(G \cdot q) = V_q$ invariant and it follows that $g \in N_{G_q}(V_q)$. Conversely, for $g \in N_{G_q}(V_q)$, again as in the proof of property (D), it follows that $g \cdot \Sigma = \Sigma$ and hence $g \in N_{G_q}(\Sigma)$. Now it is easy to see that

$$Z_G(\Sigma) = Z_{G_q}(\Sigma) \subseteq Z_{G_q}(V_q).$$

From the following commuting diagram, we can thus read off that $W(\Sigma)_q$ projects canonically onto $W(V_q)$:

$$\begin{array}{ccc} N_{G_q}(\Sigma) & \xrightarrow{\text{id}} & N_{G_q}(V_q) \\ \downarrow{\scriptstyle \text{pr}} & & \downarrow{\scriptstyle \text{pr}} \\ W(\Sigma)_q & \twoheadrightarrow & W(V_q). \end{array}$$

□

From the proof we may further conclude:

COROLLARY 2.2.3. *If Σ is a pre-section of (G, M), then $V_q = \nu_q(G \cdot q) \cap T_q\Sigma$ is a pre-section of $(G_q, \nu_q(G \cdot q))$. If Σ is a sufficiently small section, then V_q is also sufficiently small and $W(V_q) = W_q$.*

REMARK 2.2.4. In the case that Σ is a minimal section, we do not know wether V_q is necessarily a minimal section of the slice representation $(G_q, \nu_q(G \cdot q))$, or not.

2.3. Stability of Copolarity under Reductions

We next show that the copolarity of a reduction (W, Σ) is equal to the copolarity of (G, M). We start with a Lemma, which may be interesting in its own right.

LEMMA 2.3.1. *If Σ is a fat section of an isometric action (G, M), then the G-regular points in Σ are $W(\Sigma)$-regular and viceversa.*

PROOF. According to Proposition 1.1.9 (viii) the set of G-regular points is open and dense in Σ. If we can show that the G-regular points in Σ all have the same $W(\Sigma)$-orbit type, then they must be $W(\Sigma)$-regular, because the $W(\Sigma)$-regular points form an open and dense subset of Σ, too. Let $p \in \Sigma$ be an arbitrary G-regular point. Then property (D) of a fat section implies $Z_G(\Sigma) \subseteq G_p \subseteq N_G(\Sigma)$ and therefore we have $(N_G(\Sigma))_p = G_p$. Let q be another G-regular point in Σ. If we connect q with $G \cdot p$ by a G-transversal geodesic γ, properties (A) and (C) of a fat section imply that γ is a geodesic of Σ. We may assume that $\gamma(0) = q$ and $\gamma(1) = g \cdot p$ for some $g \in G$. By property (D) again, we have that $g \in N_G(\Sigma)$. Since $G_q = G_{g \cdot p} = gG_pg^{-1}$ we have that p and q are of the same $W(\Sigma)$-orbit-type.

Conversely, let $q \in \Sigma$ be an arbitrary $W(\Sigma)$-regular point. By Theorem 2.2.2, V_q is a fat section of $(G_q, \nu_q(G \cdot q))$ and W_q projects canonically onto the fat Weyl group $W(V_q)$ of V_q. At the same time, Proposition 2.1.5 shows that V_q is also the representation space for the slice representation of $(W(\Sigma), \Sigma)$ in q. By assumption, W_q acts trivially on V_q. But since $W(V_q)$, by definition, acts effectively on V_q, this implies that $W(V_q)$ must be the trivial group. In particular, $(G_q, \nu_q(G \cdot q))$ is a polar representation with generalized

Weyl group $W(V_q)$. According to [**PT88**, Corollary 5.6.22] the latter is a Weyl group in the classical sense. However, a polar representation with trivial Weyl group must itself be trivial. So G_q acts trivially on $\nu_q(G \cdot q)$ which means that q is G-regular. □

THEOREM 2.3.2 (Stability theorem). *Let (G, M) be an isometric action and let Σ be an arbitrary fat section. Then a subset $\Sigma' \subseteq \Sigma$ is a fat section of (G, M) if and only if it is a fat section of $(W(\Sigma), \Sigma)$. It follows that*

$$\operatorname{copol}(G, M) = \operatorname{copol}(W(\Sigma), \Sigma).$$

If Σ is a minimal section, then the copolarity of $(W(\Sigma), \Sigma)$ is trivial.

PROOF. First of all, if Σ' is complete and connected, totally geodesic and embedded in Σ, then it also has these properties as a submanifold of M and viceversa. If Σ' intersects every G-orbit, then it also intersects every $W(\Sigma)$-orbit, because the latter are the intersections of G-orbits with Σ and we have $\Sigma' \subseteq \Sigma$ (Corollary 2.1.3). Conversely, if Σ' intersects every $W(\Sigma)$-orbit, then it also intersects every G-orbit, because every G-orbit contains a $W(\Sigma)$-orbit. Next, by Lemma 2.3.1, we need not distinguish between G-regular and $W(\Sigma)$-regular points in Σ'. We have for every regular $p \in \Sigma$:

$$\nu_p(G \cdot p) = \nu_p^{\Sigma}(W(\Sigma) \cdot p).$$

And therefore it follows for every regular $p \in \Sigma'$ that $\nu_p(G \cdot p) \subseteq T_p\Sigma'$ is equivalent to $\nu_p^{\Sigma}(W \cdot p) \subseteq T_p\Sigma'$. Finally, let $p \in \Sigma'$ be regular and let $g \in G$ be such that $g \cdot p \in \Sigma'$. Since $\Sigma' \subseteq \Sigma$, it follows that $g \in N_G(\Sigma)$. Now it is clear that Σ' has property (D) of a fat section with respect to (G, M) if and only if it it has this property with respect to $(W(\Sigma), \Sigma)$. □

2.4. A Remark on Variational Completeness and Co-Completeness

A central result of this section is that we show that variational completeness of an isometric action is inherited to every reduction of that action and conversely, variational completeness of a reduction extends to the variational completeness of the original action. As a slight excursion we will also generalize [**GOT04**, Theorem 4.1] in such a way that we do not require the considered fat section Σ to be flat but instead relax this condition to the case that Σ is free of conjugate points. This applies to a wider variety of situations, for instance if Σ has non-positive curvature. We first recall some definitions:

DEFINITION 2.4.1. Let N be a submanifold of M. An N**-geodesic** $\gamma : [0, \varepsilon) \to M$ is a geodesic of M which emanates perpendicularly from N. An N**-Jacobi field** J is a Jacobi field (along an N-geodesic γ) which is induced by a variation of N-geodesics.

One can show that if $\gamma(0) = p \in N$ and $v = \gamma'(0)$, then J is an N-Jacobi field if and only if it is a Jacobi field which satisfies $J(0) \in T_pN$ and $J'(0) + A_vJ(0) \in \nu_pN$, where A_v denotes the shape operator of N in the direction of v. Furthermore, the vector space $\mathcal{J}^N(\gamma)$ of all N-Jacobi fields along γ is isomorphic to $T_pM = T_pN \oplus \nu_pN$ via $J \mapsto J(0) + (J'(0) + A_vJ(0))$.

We fix a fat section Σ of (G, M) and let $N := G \cdot p$ denote a fixed principal orbit with $p \in \Sigma$. For $v \in \nu_pN$ let $\gamma_v(t) := \exp_p(tv)$.

The following Lemmas, together with their proofs, are Lemma 4.3 and Lemma 4.4 of [**GOT04**]. The second of these two characterizes under which conditions an N-Jacobi field is perpendicular to a given fat section whereas the first lemma shows that every N-Jacobi field, which is induced by a G-Killing field and which has the proper initial values, always satisfies this condition. We stress that no assumptions on the curvature of Σ are necessary at this point, because the beginning of section 4.1 in loc. cit. is perhaps a bit misleading in this regard (In fact, Corollary 4.6 of the same section shows that the authors are aware of this circumstance).

LEMMA 2.4.2 ([**GOT04**, Lemma 4.3]). *Suppose that J is an N-Jacobi field along γ_v such that $J(0) \in \mathcal{E}_p$. If J is the restriction along γ_v of a G-Killing field on M, then J satisfies $J'(0) + A_v J(0) = 0$.*

PROOF. Let X be a G-Killing field such that $J = X|_{\gamma_v}$. We then have

$$X_p = J(0) \in \mathcal{E}_p = \nu_p\Sigma \cap T_pN.$$

If ∇ denotes the Levi-Cività connection of M, then $J'(0) = \nabla_v X$. Let $w \in \nu_p N$ be arbitrary and let W be a vector field along γ_v which extends w in such a way that $W(t) \in \nu_{\gamma_v(t)}(G \cdot \gamma_v(t))$ for all t. Since $\gamma_v(t)$ is G-regular for small t, we have that W is tangent to Σ for small t and thus $\nabla_v W \in T_p\Sigma$. Now

$$\langle J'(0), w\rangle = \langle \nabla_v X, w\rangle = \frac{d}{dt}\underbrace{\langle X|_{\gamma_v}, W\rangle}_{=0} - \langle \underbrace{X_p}_{\in\nu_p\Sigma}, \underbrace{\nabla_v W}_{\in T_p\Sigma}\rangle = 0.$$

It follows that $J'(0) \in T_pN$, and since $A_v J(0) \in T_pN$, we get $J'(0) + A_v J(0) \in T_pN$. However, since J, like any N-Jacobi field, satisfies $J'(0) + A_v J(0) \in \nu_p N$, we conclude that $J'(0) + A_v J(0) = 0$. ☐

LEMMA 2.4.3 ([**GOT04**, Lemma 4.4]). *Let J be an N-Jacobi field along γ_v such that $J(0) \in \mathcal{E}_p$. Then J is always orthogonal to Σ if and only if $J'(0) + A_v J(0) = 0$.*

PROOF. If J is always orthogonal to Σ then $J'(0)$ is orthogonal to $T_p\Sigma$ because of Lemma 2.2.1. Also, by Theorem 2.1.8, A_v leaves $\mathcal{E}_p$ invariant, so $A_v J(0) \in \nu_p\Sigma$. Therefore, $J'(0) + A_v J(0) \in \nu_p\Sigma$ and in fact $J'(0) + A_v J(0) = 0$, since every N-Jacobi field satisfies $J'(0) + A_v J(0) \in \nu_p N$, and $\nu_p N \subseteq T_p\Sigma$.

Conversely, suppose that $J'(0) + A_v J(0) = 0$. Then $J'(0) = -A_v J(0) \in \mathcal{E}_p$, again by A_v-invariance of $\mathcal{E}_p$. Now J and J' are perpendicular to the totally geodesic submanifold Σ, and Lemma 2.2.1 tells us that J is then always orthogonal to Σ, because the initial conditions of the tangential Jacobi field Y associated with J are both zero. ☐

With the help of the lemmas we can give a refined decomposition of $\mathcal{J}^N(\gamma)$:

PROPOSITION 2.4.4. *Let $\tilde{N} := W(\Sigma) \cdot p$ and denote the $\tilde{N}$-Jacobi fields in Σ by $\mathcal{J}^{\tilde{N}}(\gamma)$. Then*

$$\mathcal{J}^N(\gamma) = \mathcal{J}^N_0(\gamma) \oplus \mathcal{J}^N_{\mathcal{D}}(\gamma) \oplus \mathcal{J}^N_{\mathcal{E}}(\gamma), \text{ where}$$

$$\begin{array}{rcll} \mathcal{J}^N_0(\gamma) & := & \{J \in \mathcal{J}^N(\gamma) \mid J(0) = 0, J'(0) \in \nu_p N\} & = \mathcal{J}^{\tilde{N}}_0(\gamma), \\ \mathcal{J}^N_{\mathcal{D}}(\gamma) & := & \{J \in \mathcal{J}^N(\gamma) \mid J(0) \in \mathcal{D}_p,\ J'(0) = -A_v J(0)\} & = \mathcal{J}^{\tilde{N}}_{\mathcal{D}}(\gamma), \\ \mathcal{J}^N_{\mathcal{E}}(\gamma) & := & \{J \in \mathcal{J}^N(\gamma) \mid J(0) \in \mathcal{E}_p,\ J'(0) = -A_v J(0)\} & \\ & = & \{X|_\gamma \mid X \text{ is a } G\text{-Killing field and } X_p \in \mathcal{E}_p\}. & \end{array}$$

In particular, if $J = J_0 + J_{\mathcal{D}} + J_{\mathcal{E}}$ is an N-Jacobi field represented with respect to the above decomposition, then, in view of Lemma 2.2.1, $J_0 + J_{\mathcal{D}}$ is the part of J which is everywhere tangential to Σ and $J_{\mathcal{E}}$ is part of J which is everywhere perpendicular to Σ.

PROOF. The decomposition follows from the isomorphism $\mathcal{J}^N(\gamma) \simeq T_pN \oplus \nu_pN$ and because of $T_p(G \cdot p) = \mathcal{D}_p \oplus \mathcal{E}_p$ (see Definition 2.1.6). Note that Theorem 2.1.8 implies that A_v leaves $\mathcal{D}_p$ invariant. This shows that every element $J_\mathcal{D}$ of $\mathcal{J}^N_\mathcal{D}(\gamma)$ is everywhere tangential to Σ, because $J_\mathcal{D}(0) \in \mathcal{D}_p$ and $J'_\mathcal{D}(0) = -A_v J_\mathcal{D}(0) \in \mathcal{D}_p$. It is also clear that every element J_0 of $\mathcal{J}^N_0(\gamma)$ is tangential to Σ, because of $J_0(0) = 0$ and $J'_0(0) \in \nu_pN$. We next show that J_0 and $J_\mathcal{D}$ are $\tilde{N}$-Jacobi fields. First of all, γ is a geodesic in M which starts in Σ and since $\gamma'(0) \in \nu_pN \subseteq T_p\Sigma$ we have that it is also tangential to Σ. Since Σ is totally geodesic in M, it follows that γ is a geodesic of Σ and further on, γ is a $\tilde{N}$-geodesic. Using Lemma 2.2.1 we see that J_0 and $J_\mathcal{D}$ are Jacobi fields on Σ. For J_0 we now have to show that $J'(0) \in \nu_p\tilde{N}$. But this is clear since we have $\nu_pN = \nu_p\tilde{N}$. Concerning $J_\mathcal{D}$, we have that $J_\mathcal{D}(0) \in \mathcal{D}_p = T_p\tilde{N}$ and if $\tilde{A}$ denotes the shape operator of $\tilde{N}$, then

$$J'_\mathcal{D}(0) + \tilde{A}_v J_\mathcal{D}(0) = J'_\mathcal{D}(0) + A_v J_\mathcal{D}(0) = 0,$$

where we have used that $\tilde{A}_v = A_v|_{\mathcal{D}_p}$, because $\tilde{N}$ is totally geodesic in N, by Theorem 2.1.8 again. By following the previous arguments backwards, we obtain that in fact the equalities $\mathcal{J}^N_0(\gamma) = \mathcal{J}^{\tilde{N}}_0(\gamma)$ and $\mathcal{J}^N_\mathcal{D}(\gamma) = \mathcal{J}^{\tilde{N}}_\mathcal{D}(\gamma)$ hold. The statements concerning $\mathcal{J}^N_\mathcal{E}(\gamma)$ are direct consequences of Lemma 2.4.2 and Lemma 2.4.3.

□

DEFINITION 2.4.5. An isometric action (G, M) is **variationally complete** if for every G-orbit N, every N-geodesic γ and every N-Jacobi field along γ, which vanishes for some $t_0 > 0$, is the restriction of a G-Killing field to γ.

It suffices to consider principal orbits only in order to show that an isometric action is variationally complete. This fact seems to be known in the literature. For instance, in [**GOT04**] this is implicitly assumed in the characterization of variational completeness via $\mathrm{covar}(G, M) = 0$ (see below). A proof can be found in [**LT07a**, Remark 5.5][2].

THEOREM 2.4.6. *An isometric action (G, M) is variationally complete if and only if a minimal reduction $(W(\Sigma), \Sigma)$ is variationally complete.*

PROOF. In the following let $p \in \Sigma$ be a regular point. Recall that due to Lemma 2.3.1 we do not have to distinguish between G-regular and W-regular points. Put

$$N := G \cdot p \text{ and } \tilde{N} := W(\Sigma) \cdot p$$

and let γ be an arbitrary $\tilde{N}$-geodesic starting in p.

Suppose that (G, M) is variationally complete. If $J \in \mathcal{J}^{\tilde{N}}(\gamma)$ satisfies $J(t_0) = 0$ for some $t_0 > 0$, then according to Proposition 2.4.4, we can view J as an N-Jacobi field along the N-geodesic γ. By variational completeness of (G, M), there is a G-Killing field X such that $J = X|_\gamma$. Let now $\mathrm{pr}_\Sigma X$ denote the orthogonal projection of X onto Σ, which by Theorem 2.5.5[3] is a W-Killing field on Σ (in this step we use that Σ is a minimal section). Since $X(\gamma(t)) = J(t) \in T_{\gamma(t)}\Sigma$ and therefore $J(t) = \mathrm{pr}_\Sigma X(\gamma(t))$, we may conclude that J is the restriction of a W-Killing field to γ.

For the converse direction, suppose now that $(W(\Sigma), \Sigma)$ is variationally complete. Let $p \in M$ be an arbitrary regular point and γ an N-geodesic starting in p. Without loss of generality, we may assume that $p \in \Sigma$ and that γ is an $\tilde{N}$-geodesic (a suitable translate $g \cdot \Sigma$ contains p and hence γ, and the minimal reduction $(W(g \cdot \Sigma), g \cdot \Sigma)$ is also variationally complete). We decompose an arbitrary N-Jacobi field J, which vanishes

[2]I would like to thank Alexander Lytchak for giving me this reference.

[3]Although we anticipate a result from section 2.5 we do not fall prey to circular reasoning.

for some $t_0 > 0$, according to Proposition 2.4.4 into the three parts $J = J_0 + J_{\mathcal{D}} + J_{\mathcal{E}}$. The Proposition tells us that $J_{\mathcal{E}}$ is already induced by a G-Killing field. From

$$0 = J(t_0) = \underbrace{J_0(t_0) + J_{\mathcal{D}}(t_0)}_{\in T_p\Sigma} + \underbrace{J_{\mathcal{E}}(t_0)}_{\in \nu_p\Sigma}$$

and the variational completeness of $(W(\Sigma), \Sigma)$ it follows that $J_0 + J_{\mathcal{D}}$ is induced by an $N(\Sigma)$-Killing field. But such a field is also a G-Killing field and it follows that J is the restriction of a G-Killing field to γ. □

Corollary 2.4.7. *An isometric action (G, M) is variationally complete if and only if some (and hence any) reduction $(W(\Sigma), \Sigma)$ is variationally complete.*

Proof. According to Theorem 2.3.2, (G, M) and $(W(\Sigma), \Sigma)$ have a common minimal reduction $(W(\Sigma'), \Sigma')$ with $\Sigma' \subseteq \Sigma$. Hence, we may apply Theorem 2.4.6 to (G, M) and $(W(\Sigma'), \Sigma')$ and then to $(W(\Sigma), \Sigma)$ and $(W(\Sigma'), \Sigma')$ and vice versa. □

Remark 2.4.8. As a matter of fact, Theorem 2.4.6 and Corollary 2.4.7 can also be deduced from [**LT07a**, Theorem 1.3] and our Theorem 2.1.1, because the first result states that variational completeness is a property, which depends solely on the metric properties of the orbit space of the action, and the second result states that the orbit space of an action is isometric to the orbit space of any reduction of that action.

Corollary 2.4.9. *If (G, M) is a polar and variationally complete action, then every section is free of conjugate points. In particular, if M is a Riemannian manifold of non-negative Ricci curvature or compact and of non-negative scalar curvature, then a variationally complete action on M is polar, if and only if it is hyperpolar.*

Proof. Polarity implies, that the generalized Weyl group $W(\Sigma)$ of any section Σ is finite. Furthermore, a Lie group acts variationally complete, if and only if its identity component does. However, if the trivial group acts variationally complete, this only means that every Jacobi field which vanishes in two different points, vanishes entirely. Hence, there are no conjugate points in Σ.

Note that a fat section Σ inherits the curvature conditions, which we assume on M, since Σ is totally geodesic. By a result of Mendonca and Zhou, [**MZ00**, Corollary 1], resp. Green [**Gre58**], we deduce from the above situation that Σ has to be flat. □

Remark 2.4.10. Conlon proved in [**Con72**] that hyperpolar actions are variationally complete. In general, the converse is false. Take, for instance, the action of the trivial group on a non-flat space of non-positive curvature. Then this action is variationally complete and polar, but not hyperpolar. However, Lytchak and Thorbergsson proved in [**LT07b**], that variationally complete actions on manifolds of non-negative curvature are hyperpolar.

We briefly recall the notion of variational co-completeness, which has been introduced in [**GOT04**]. Let $N = G \cdot p$ denote an arbitrary principal orbit and consider the isomorphism $\mathcal{J}^N(\gamma) \simeq T_pN \oplus \nu_pN$. For a subspace $U_p \subseteq T_pM$ consider the following condition:

(P) for every N-geodesic γ and every $J \in \mathcal{J}^N(\gamma)$ that vanishes for some $t_0 > 0$ and such that $(J(0), J'(0) + A_vJ(0)) \perp U_p$ it follows that J is the restriction of a G-Killing field to γ.

If U_p satisfies condition (P), then g_*U_p satisfies this condition in $g \cdot p$. Furthermore, we always have that $U_p = T_pM$ satisfies condition (P).

DEFINITION 2.4.11. We write $\operatorname{covar}_N(G, M) \leq \dim U_p$, if U_p satisfies condition (P). Then we say that the **variational co-completeness** of (G, M) is less than or equal to k, if $\operatorname{covar}_N(G, M) \leq k$ holds for all principal orbits N. We then also write $\operatorname{covar}(G, M) \leq k$.

Note that a canonical choice for U_p is always $T_p\Sigma = \nu_p N \oplus \mathcal{D}_p$, where Σ denotes a fat section through p. This is due to Proposition 2.4.4. In particular, we always have

$$\operatorname{covar}(G, M) \leq \operatorname{cohom}(G, M) + \operatorname{copol}(G, M).$$

This estimate can sometimes be considerably improved. The following result is a generalization of [**GOT04**, Theorem 4.1]. However, we do not hide the fact that all we did was to relax the curvature condition on the fat section appearing in the Theorem. The condition that Σ has to be flat in the proof of [**GOT04**, Lemma 4.2] can be weakened to the condition that Σ has no conjugate points. For example, this situation occurs whenever M has non-positive sectional curvature. The rest of the proof now just works as in Section 4.1 of loc. cit.

THEOREM 2.4.12. *Let (G, M) be an isometric action and $\Sigma \subseteq M$ a k-section. If Σ is free of conjugate points in the induced metric, then* $\operatorname{covar}(G, M) \leq k$. *In particular,*

$$\operatorname{covar}(G, M) \leq \operatorname{copol}(G, M).$$

We even obtain Corollary 4.5 of loc. cit. under these relaxed conditions:

COROLLARY 2.4.13. *Let (G, M) be an isometric action and let Σ be a pre-section wich we assume to have no conjugate points. Let N be a principal orbit and let $p \in N \cap \Sigma$. Then $\mathcal{D}_p = T_pN \cap T_p\Sigma$ has property (P).*

2.5. Decomposition of Killing Fields and Adapted Metrics

In this section we generalize parts of [**GOT04**, Section 5.4] and use the resulting decomposition of Killing fields along a minimal section to define certain adapted invariant Riemannian metrics on coset spaces G/H which will be useful later.

We consider the following relation on the set of G-regular points of M: We call two points $p, q \in M^{\text{reg}}$ **equivalent**, if they can be joined by a broken geodesic in M^{reg} whose segments are G-transversal geodesics. That is, there exists a finite sequence of G-transversal geodesics $\gamma_0, \ldots, \gamma_r : [0, 1] \to M^{\text{reg}}$ such that

$$\gamma_0(0) = p, \gamma_r(1) = q, \gamma_i(1) = \gamma_{i+1}(0), i = 0, \ldots, r-1.$$

It is easily checked that this relation is indeed an equivalence relation.

DEFINITION 2.5.1. We call the equivalence class containing the point $p \in M^{\text{reg}}$ the **G-network of** p and denote it by $\mathfrak{S}_p$[4]. We further denote by $\langle \mathfrak{S}_p \rangle$ the totally geodesic hull of $\mathfrak{S}_p$, i.e. the connected component of p of the intersection of all complete totally geodesic submanifolds containing $\mathfrak{S}_p$.

It is clear from this definition that $\mathfrak{S}_p \subseteq \langle \mathfrak{S}_p \rangle \subseteq \Sigma$ holds for every pre-section Σ through p. Furthermore, $\langle \mathfrak{S}_p \rangle$ intersects every G-orbit in M. In fact, the statement follows from $\exp_p(\nu_p(G \cdot p)) \subseteq \langle \mathfrak{S}_p \rangle$ and Lemma 1.1.6.

We next fix a fat section Σ of (G, M) and assume $p \in \Sigma$ to be G-regular. We show how one can construct a subset $\bar{\Sigma}_p$ of Σ which is again a fat section of (G, M). This

[4] $\mathfrak{S}_p$ is related to the notion of the *dual foliation* $\mathcal{F}^\perp$ to the singular Riemannian foliation $\mathcal{F}$ on M given by the G-orbits (see for instance [**Wil06a**]), but in general it differs from it.

generalizes the construction given in [**GOT04**, Section 5.4]. Our proofs are slightly different in some parts.

Definition 2.5.2. For a vector field Y on M let $\mathrm{Zero}(Y)$ denote the set of points where Y vanishes. If Y is a Killing field, then the connected components of $\mathrm{Zero}(Y)$ are closed and totally geodesic submanifolds of M (see for instance [**Kob58**]). Although we do not assume our isometric actions to be effective, we identify an element $X \in \mathfrak{g}$ with its induced G-Killing field on M. We denote the evaluation of the Killing field in p with X_p or $X(p)$. For any $X \in \mathfrak{g}$ we consider the orthogonal projection $\mathrm{pr}_\Sigma X$ of X to Σ, which by [**KN69**, Ch. VII, Theorem 8.9] is a Killing field of Σ. Then $\mathrm{Zero}(\mathrm{pr}_\Sigma X)$ consists of those points of Σ where X is perpendicular to Σ. Now put

$$\bar{\Sigma}_p := \big(\bigcap_{X \in I_p} \mathrm{Zero}(\mathrm{pr}_\Sigma X)\big)^\circ, \text{ where } I_p := \{X \in \mathfrak{g} \mid \mathrm{pr}_\Sigma X(p) = 0\}.$$

In words, $\bar{\Sigma}_p$ is the connected component of p of the common zero set of those projected G-Killing fields which vanish in p. It is a closed totally geodesic submanifold of Σ.

Proposition 2.5.3. *Let (G, M) be an isometric action, Σ a fat section and $p \in \Sigma$ be G-regular. Then*

$$\mathfrak{S}_p \subseteq \langle \mathfrak{S}_p \rangle \subseteq \bar{\Sigma}_p \subseteq \Sigma.$$

Furthermore, $\bar{\Sigma}_p$ is a fat section of (G, M).

Proof. Let $X \in I_p$ be arbitrary. Then X is a G-Killing field which satisfies $\mathrm{pr}_\Sigma X(p) = 0$. By definition, any $q \in \mathfrak{S}_p$ can be joined to p by a broken geodesic whose segments are G-transversal geodesics. Since we have $X_p \perp T_p\Sigma$, and thus $X_p \in \mathcal{E}_p$, we may apply the Lemmas 2.4.2 and 2.4.3 repeatedly along each segment and obtain that $X_q \perp T_q\Sigma$. Thus $q \in \mathrm{Zero}(\mathrm{pr}_\Sigma X)$ and we have shown that $\mathfrak{S}_p \subseteq \mathrm{Zero}(\mathrm{pr}_\Sigma X)$. Since $\mathrm{Zero}(\mathrm{pr}_\Sigma X)$ is complete and totally geodesic, we have $\langle \mathfrak{S}_p \rangle \subseteq \mathrm{Zero}(\mathrm{pr}_\Sigma X)$, and because X was arbitrary chosen from I_p it follows that

$$\langle \mathfrak{S}_p \rangle \subseteq \bar{\Sigma}_p.$$

Now, let us show that $\bar{\Sigma}_p$ is a fat section. Property (A) is obvious and property (B) follows from $\langle \mathfrak{S}_p \rangle \subseteq \bar{\Sigma}_p$ and the fact that $\langle \mathfrak{S}_p \rangle$ intersects every G-orbit. Concerning property (C), we first make the following observation: If $q \in \bar{\Sigma}_p$ is G-regular then $\bar{\Sigma}_q \subseteq \bar{\Sigma}_p$. In fact, we have $I_p \subseteq I_q$.

We have to show that $\nu_q(G \cdot q) \subseteq T_q\bar{\Sigma}_p$. Using the first part of the proof and the observation above, we conclude that $\mathfrak{S}_q \subseteq \bar{\Sigma}_q \subseteq \bar{\Sigma}_p$ and hence it follows that

$$\nu_q(G \cdot q) \subseteq T_q\bar{\Sigma}_q \subseteq T_q\bar{\Sigma}_p.$$

In order to show property (D) we first consider an arbitrary $g \in N_G(\Sigma)$ and claim that:

(1) $g \cdot \mathrm{Zero}(\mathrm{pr}_\Sigma X) = \mathrm{Zero}(\mathrm{pr}_\Sigma(\mathrm{Ad}_g X))$ and
(2) $g \cdot \bar{\Sigma}_p = \bar{\Sigma}_{g \cdot p}$.

Using that $\mathrm{Ad}_g X(p) = d\phi_g(p)(X_{g^{-1} \cdot p})$, we obtain (1) from the following computation:

$$\begin{aligned} g \cdot \mathrm{Zero}(\mathrm{pr}_\Sigma X) &= \{g \cdot p \in \Sigma \mid X_p \perp T_p\Sigma\} = \{p \in \Sigma \mid X_{g^{-1} \cdot p} \perp T_{g^{-1} \cdot p}\Sigma\} \\ &= \{p \in \Sigma \mid d\phi_{g^{-1}}(p)(\mathrm{Ad}_g X(p)) \perp d\phi_{g^{-1}}(T_p\Sigma)\} \\ &= \{p \in \Sigma \mid \mathrm{Ad}_g X(p) \perp T_p\Sigma\} = \mathrm{Zero}(\mathrm{pr}_\Sigma(\mathrm{Ad}_g X)). \end{aligned}$$

From (1) we conclude that

$$\mathrm{Ad}_{g^{-1}} X \in I_p \Leftrightarrow X \in I_{g \cdot p},$$

and now (2) follows from

$$\begin{aligned} g\cdot\bar{\Sigma}_p &= (\bigcap_{X\in I_p} g\cdot \mathrm{Zero}(\mathrm{pr}_\Sigma X))^\circ = (\bigcap_{X\in I_p} \mathrm{Zero}(\mathrm{pr}_\Sigma(\mathrm{Ad}_g X)))^\circ \\ &= (\bigcap_{\mathrm{Ad}_{g^{-1}}X\in I_p} \mathrm{Zero}(\mathrm{pr}_\Sigma X))^\circ = (\bigcap_{X\in I_{g\cdot p}} \mathrm{Zero}(\mathrm{pr}_\Sigma X))^\circ \\ &= \bar{\Sigma}_{g\cdot p}. \end{aligned}$$

If $g\in G$ satisfies $g\cdot p\in\bar{\Sigma}_p$ we have $g\in N_G(\Sigma)$, because $p\in\Sigma$ is G-regular. By (2) and the observation made in the proof of property (C) we then have

$$g\cdot\bar{\Sigma}_p = \bar{\Sigma}_{g\cdot p}\subseteq\bar{\Sigma}_p,$$

and since $\bar{\Sigma}_p$ is complete and connected it follows that $g\cdot\bar{\Sigma}_p=\bar{\Sigma}_p$.

For the general case let $q\in\bar{\Sigma}_p$ be an arbitrary G-regular point and let $g\in G$ satisfy $g\cdot q\in\bar{\Sigma}_p$. Again we have $g\in N_G(\Sigma)$. Let $\gamma:[0,1]\to M$ be a minimal geodesic from q to $G\cdot p$. Then $\gamma\subseteq\mathfrak{S}_q$ and there exists some $h\in G$ with $\gamma(1)=h\cdot p\in\bar{\Sigma}_p$. The previous arguments now imply that $h\cdot\bar{\Sigma}_p=\bar{\Sigma}_p$. Note that the arguments in the first two paragraphs of the proof show that for all $x\in\mathfrak{S}_y$ we have $\bar{\Sigma}_x=\bar{\Sigma}_y$. Thus it follows that

$$\bar{\Sigma}_q=\bar{\Sigma}_{h\cdot p}=h\cdot\bar{\Sigma}_p=\bar{\Sigma}_p.$$

Finally, if $g\cdot q\in\bar{\Sigma}_p$ then we have $g\cdot q\in\bar{\Sigma}_q$ and again by the previous arguments we see that

$$g\cdot\bar{\Sigma}_p=g\cdot\bar{\Sigma}_q=\bar{\Sigma}_q=\bar{\Sigma}_p.$$

□

COROLLARY 2.5.4. *If Σ is a minimal section of the isometric action (G,M) and $p\in\Sigma$ is G-regular, then for all $X\in\mathfrak{g}$ we have $X_p\perp T_p\Sigma$ if and only if $X_q\perp T_q\Sigma$ holds for all $q\in\Sigma$.*

PROOF. This follows from Proposition 2.5.3 since $\Sigma=\bar{\Sigma}_p$. □

Let $\Sigma\subseteq M$ be a minimal section and put $H:=Z_G(\Sigma)$ and $N:=N_G(\Sigma)$. As usual, we denote their corresponding Lie algebras by $\mathfrak{h}$, resp. $\mathfrak{n}$. We put $\mathfrak{m}:=\{X\in\mathfrak{g}\mid X\perp\Sigma\}$.

THEOREM 2.5.5. *The G-Killing fields decompose as*

$$\mathfrak{g}=\mathfrak{n}+\mathfrak{m},\ \ \textit{with}\ \mathfrak{h}=\mathfrak{n}\cap\mathfrak{m}.$$

In words: Every G-Killing field X decomposes uniquely, up to $Z_G(\Sigma)$-Killing fields, into an $N_G(\Sigma)$-Killing field X_1 and a G-Killing field X_2, which is perpendicular to Σ.

This decomposition is reductive in the sense that $[\mathfrak{n},\mathfrak{m}]\subseteq\mathfrak{m}$, and we even have for all $g\in N$ that $\mathrm{Ad}_g(\mathfrak{m})\subseteq\mathfrak{m}$. *Hence, $\mathfrak{g}/\mathfrak{h}=\mathfrak{n}/\mathfrak{h}\oplus\mathfrak{m}/\mathfrak{h}$ is $\mathrm{Ad}_G(N)$-invariant.*

PROOF. Let $p\in\Sigma$ be G-regular. Since X_p is normal to $\nu_p(G\cdot p)\subseteq T_p\Sigma$, we have that $(\mathrm{pr}_\Sigma X)(p)$ is tangent to $(G\cdot p)\cap\Sigma=N_G(\Sigma)\cdot p$ in p. Let X_1 denote an $N_G(\Sigma)$-Killing field which satisfies $X_1(p)=\mathrm{pr}_\Sigma X(p)$. Then $X_1|_\Sigma$ is tangent to Σ. Put $X_2:=X-X_1$. We have

$$X_2(p)=X(p)-\mathrm{pr}_\Sigma X(p)\in\nu_p(\Sigma),$$

hence, by Corollary 2.5.4, it follows that $X_2|_\Sigma$ is always perpendicular to Σ, and thus we also have $X_1|\Sigma=\mathrm{pr}_\Sigma X$. If $X=Y_1+Y_2$ is another decomposition of this type, then $X_1|_\Sigma=\mathrm{pr}_\Sigma X=Y_1|_\Sigma$. Hence, $0=Y_1|_\Sigma-X_1|_\Sigma$, and it follows that $Z:=Y_1-X_1$ is a

$Z_G(\Sigma)$-Killing field, because the latter are characterized as those G-Killing fields which vanish everywhere on Σ. Since

$$X = X_1 + X_2 = Y_1 + Y_2 = X_1 + Z + Y_2,$$

it follows that $Y_2 = X_2 - Z$ and clearly, Z is uniquely determined.

We next show the inclusion $\mathrm{Ad}_g(\mathfrak{m}) \in \mathfrak{m}$. Let $X \in \mathfrak{m}$, $g \in N$ and $p \in \Sigma^{\mathrm{reg}}$ be arbitrary elements. Then $g \cdot p \in \Sigma^{\mathrm{reg}}$ and it follows that

$$(\mathrm{Ad}_g X)(g \cdot p) = d\phi_g(p)(X_p) \perp d\phi_g(p)(T_p\Sigma) = T_{g\cdot p}\Sigma.$$

Thus $(\mathrm{Ad}_g X)(g \cdot p) \in \nu_{g\cdot p}\Sigma$ and by Corollary 2.5.4 we have $\mathrm{Ad}_g X \in \mathfrak{m}$. The inclusion $[\mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{m}$ follows easily from $\mathrm{Ad}_g(\mathfrak{m}) \subseteq \mathfrak{m}$. □

The following corollary is [**GOT04**, Lemma 5.3] but formulated for minimal sections of an arbitrary isometric action. The proof is entirely different as in loc. cit.

Corollary 2.5.6. *Let (G, M) be an isometric action and $\Sigma \subseteq M$ a minimal section. Let $q \in \Sigma$ be arbitrary and let $p \in S_q \cap \Sigma$ be G-regular, where S_q is a G-slice through q. Then we have an orthogonal decomposition:*

$$T_p(G_q \cdot p) = (T_p(G_q \cdot p) \cap T_p\Sigma) \oplus (T_p(G_q \cdot p) \cap \nu_p\Sigma).$$

Proof. We have $T_p(G_q \cdot q) = \{X_p \mid X \text{ is a } G_q\text{-Killing field}\}$. Let $u \in T_p(G \cdot q)$ be arbitrary and let X be a G_q-Killing field satisfying $X_p = u$. Consider a decomposition $X = X_1 + X_2$ with $X_1 \in \mathfrak{n}$ and $X_2 \in \mathfrak{m}$ as in Theorem 2.5.5. Then

$$0 = X_q = X_1(q) + X_2(q)$$

and therefore $X_1(q) = 0 = X_2(q)$. It follows that X_1 and X_2 are G_q-Killing fields and that X_1 is even an $N_{G_q}(\Sigma)$-Killing field. We thus have $u = X_p = X_1(p) + X_2(p)$ with $X_1(p) \in T_p(G_q \cdot p) \cap T_p\Sigma$ and $X_2(p) \in T_p(G_q \cdot p) \cap \nu_p\Sigma$, which proves the inclusion " $\subseteq$ ". Since the other inclusion is trivial, we have proved our claim. □

Corollary 2.5.7. *If G is connected and Σ is a minimal section of (G, M), then G is generated by $(N_G(\Sigma))^\circ$ and* $\exp(\mathfrak{m})$. *Furthermore, the subgroup $K \leq G$ generated by* $\exp(\mathfrak{m})$ *is a normal subgroup of G and G is a quotient of $N^\circ \ltimes K$.*

The decomposition of Theorem 2.5.5 can also be used to introduce a "nice" metric on G/H. A Riemannian metric on G/H is called **(G-W)-invariant** if it is left-G- and right-W-invariant (see Section 7).

Corollary 2.5.8. *For a minimal section Σ of (G, M) put $N = N_G(\Sigma)$, $H = Z_G(\Sigma)$ and $W = N/H$.*

- *(i) G/H admits a $(G$-$W)$-invariant Riemannian metric with $(\mathfrak{n}/\mathfrak{h})\perp(\mathfrak{m}/\mathfrak{h})$ if and only if $\mathfrak{m}/\mathfrak{h}$ carries an $\mathrm{Ad}_G(N)$-invariant scalar product and W is covered by the product of a compact Lie group and a vector group.*
- *(ii) If N is compact, then G/H admits a $(G$-$W)$-invariant Riemannian metric such that $\mathfrak{n}/\mathfrak{h}$ and $\mathfrak{m}/\mathfrak{h}$ are perpendicular.*

Proof. Clearly, the first statement implies the second one. By Theorem 2.5.5, the decomposition $\mathfrak{g}/\mathfrak{h} = \mathfrak{n}/\mathfrak{h} \oplus \mathfrak{m}/\mathfrak{h}$ is direct and $\mathrm{Ad}_G(N)$-invariant. Hence (i) follows from Proposition 8.1.2 (iii). □

Definition 2.5.9. Let Σ be a minimal section. Then we call a $(G$-$W)$-invariant Riemannian metric on G/H **adapted to** Σ if $(\mathfrak{n}/\mathfrak{h})\perp(\mathfrak{m}/\mathfrak{h})$ holds with respect to the $\mathrm{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$ induced by the metric.

The reason for this definition will become apparent in Section 2.6 where we prove a generalization of Weyl's integration formula. By Corollary 2.5.8 (ii) a metric adapted to Σ exists if N is compact. Also note that in the cases that the action is either polar (i.e. $\mathfrak{h} = \mathfrak{n}$) or has trivial copolarity (i.e. $\mathfrak{h} = \mathfrak{m}$) any left-invariant metric on G/H is adapted to Σ. Also note that it follows from Corollary 8.1.4 that, with respect to an adapted metric, W is totally geodesic in G/H, and if G/N carries the metric induced from $G/H \twoheadrightarrow G/N$, then $\mathfrak{m}/\mathfrak{h}$ is canonically isometric to $\mathfrak{g}/\mathfrak{n}$.

PROPOSITION 2.5.10. *For an isometric action (G, M) with minimal section Σ the normal bundle $\nu\Sigma^{\mathrm{reg}} \twoheadrightarrow \Sigma^{\mathrm{reg}}$ is trivial and a global trivialization can be given by G-Killing fields.*

PROOF. Let $p \in \Sigma^{\mathrm{reg}}$ be arbitrary. Let $X_1, \dots, X_r \in \mathfrak{m}$ such that $X_1(p), \dots, X_r(p)$ form a basis of $\nu_p\Sigma$. This can always be achieved, since $\nu_p\Sigma \subseteq T_p(G \cdot p)$. We claim that $X_1, \dots, X_r$ stay linearly independent along Σ^{reg}. This is easy to see, if one considers the orbit projection map $r : G \cdot p \to G \cdot q$, $g \cdot p \mapsto g \cdot q$, where $q \in \Sigma^{\mathrm{reg}}$ is an arbitrary point. Since both p and q are regular, this map is a diffeomorphism. Its differential is given by $dr(p)(u) = X_q$, where X is a G-Killing field such that $X_p = u \in T_p(G \cdot p)$. Since $dr(p)$ maps bases to bases and respects the decomposition $T_p(G \cdot p) = (T_p(G \cdot p) \cap T_p\Sigma) \oplus \nu_p\Sigma$, it follows that $X_1, \dots, X_r$ are everywhere linearly independent along Σ^{reg}. □

2.6. A Generalization of Weyl's Integration Formula

The aim of this section is to prove a generalization of Weyl's celebrated integration formula for compact Lie groups to the case of an almost arbitrary isometric action. We first formulate Fubini's Theorem for general submersions. We denote by μ_g the Riemannian measure with respect to the Riemannian metric g. Let $\pi : (M, g) \to (N, h)$ be a surjective submersion between two Riemannian manifolds. For $q \in N$ let δ_q denote the function

$$\delta_q : \pi^{-1}(q) \to \mathbf{R},\ p \mapsto \left|\det(d\pi(p)|_{\mathcal{H}_p})^{-1}\right|,$$

where $\mathcal{H}_p$ denotes the horizontal space to the fibre $\pi^{-1}(q)$ in p. (Recall that $|\det f|$ for a homomorphism $f : X \to Y$ of equal dimensional Euclidean vector spaces is defined via the usual determinant by $|\det f| := |\det A \circ f|$, where $A : Y \to X$ is an arbitrary auxiliary linear isometry.) If π is a Riemannian submersion, then $\delta_q \equiv 1$. In general this need not be the case. However, we always have $\delta_q > 0$; and using local frames, it is not difficult to see that δ_q is in fact smooth on $\pi^{-1}(q)$. For a function f on M we set $f_q := f|_{\pi^{-1}(q)}$ and denote the Riemannian measure on $\pi^{-1}(q)$ induced by g with μ_{g_q}. If f_q is integrable with respect to the weighted measure $\delta_q\mu_{g_q}$ on $\pi^{-1}(q)$, then we put

$$\bar{f}(q) := \int_{\pi^{-1}(q)} f_q \, \delta_q d\mu_{g_q}.$$

PROPOSITION 2.6.1 (Fubini's Theorem for submersions). *Let $\pi : (M, g) \twoheadrightarrow (N, h)$ be a surjective submersion between two Riemannian manifolds. If $f \in \mathcal{C}_c(M)$ (resp. f is integrable on M), then $\bar{f} \in \mathcal{C}_c(N)$ (resp. f_q is integrable for almost all $q \in N$ and $\bar{f}$ is integrable on N). Furthermore, we have:*

$$\int_M f \, d\mu_g = \int_N \bar{f} \, d\mu_h = \int_N \left(\int_{\pi^{-1}(q)} f_q \, \delta_q d\mu_{g_q} \right) d\mu_h.$$

PROOF. The proof is literally the same as in [**Sak96**, Chapter II, Theorem 5.6] (we also adopted the notation from there). The only modification which occurs is the factor δ_q, which enters in the way stated above. We leave the details to the reader. □

DEFINITION 2.6.2. Let Σ be a minimal section of (G, M). Consider for $s \in \Sigma$ the differential of the orbit map $\omega_s : G/H \to G \cdot s$ restricted to $\mathfrak{m}/\mathfrak{h}$.

$$d\omega_s(eH)|_{\mathfrak{m}/\mathfrak{h}} : \mathfrak{m}/\mathfrak{h} \to \nu_s\Sigma,\ X + \mathfrak{h} \mapsto X_s.$$

Here we denote, as in Theorem 2.5.5, by $\mathfrak{m}$ (resp. $\mathfrak{h}$) the elements of $\mathfrak{g}$ which induce Killing fields perpendicular to Σ (resp. which vanish on Σ). We assume that there is some inner product on $\mathfrak{m}/\mathfrak{h}$ and define

$$\delta_{\mathcal{E}} : \Sigma \to \mathbf{R},\ s \mapsto \left|\det(d\omega_s(eH)|_{\mathfrak{m}/\mathfrak{h}})\right|.$$

PROPOSITION 2.6.3. *Let $\varphi : G \times M \to M$ be an isometric action with minimal section Σ and assume that G/H carries a $(G\text{-}W)$-invariant metric adapted to Σ. Then $\delta_{\mathcal{E}} \in \mathcal{C}^0(\Sigma)^W$ and $\delta_{\mathcal{E}}|_{\Sigma^{\mathrm{reg}}} \in \mathcal{C}^\infty(\Sigma^{\mathrm{reg}})^W$. In particular, $\delta_{\mathcal{E}}$ extends to a G-invariant continuous function on M, which we also denote by $\delta_{\mathcal{E}}$. This function vanishes in the G-singular points of M. Furthermore, we have*

$$\delta_{\mathcal{E}}(s) = \left|\det((d\pi(s)|_{\nu_s\Sigma})^{-1})\right| \text{ for all } s \in \Sigma^{\mathrm{reg}},$$

where $\pi : M^{\mathrm{reg}} \twoheadrightarrow G/N$ is defined as in Theorem 3.1.2 (iv).

PROOF. Consider a fixed orthonormal basis $X_1 + \mathfrak{h}, \ldots, X_m + \mathfrak{h}$ of $\mathfrak{m}/\mathfrak{h}$ and a local orthonormal frame $Y_1, \ldots Y_m \in \Gamma(\Sigma, \nu(\Sigma))$ on a neighborhood of $s \in \Sigma$. We define the auxiliary linear isometry

$$A_s : \nu_s\Sigma \to \mathfrak{m}/\mathfrak{h}$$

by mapping $Y_i(s)$ to $X_i + \mathfrak{h}$. Then A_s varies smoothly in s. Also note that $d\omega_s(eH)$ depends smoothly on $s \in \Sigma$, because the action φ is assumed to be smooth. The usual determinant of square matrices is a polynomial in the matrix entries and thus $\det(A \circ d\omega_s(eH)|_{\mathfrak{m}/\mathfrak{h}})$ is smooth in s as it is a composition of smooth maps. If $s \in \Sigma^{\mathrm{reg}}$ we see that the former expression does not vanish on a small neighborhood of s. Hence its sign does not change there and we may conclude that

$$\delta_{\mathcal{E}}(s) = \left|\det(A \circ d\omega_s(eH)|_{\mathfrak{m}/\mathfrak{h}})\right|$$

is smooth in s. If s is not G-regular, then $\det(A \circ d\omega_s(eH)|_{\mathfrak{m}/\mathfrak{h}})$ may become zero in s and its sign may change in a neighborhood of s. Hence passing to the absolute value we see that $\delta_{\mathcal{E}}$ need no longer be smooth, but it still remains continuous.

If $s \in \Sigma^{\mathrm{reg}}$, then $d\pi(s)|_{\nu_s\Sigma} : \nu_s\Sigma \to \mathfrak{g}/\mathfrak{n}$ assigns to $v \in \nu_s\Sigma$ the unique element $X + \mathfrak{n} \in \mathfrak{g}/\mathfrak{n}$ which induces a Killing field on M, such that $X_s = v$. We thus have

$$d\omega_s(eH)|_{\mathfrak{m}/\mathfrak{h}} = (d\pi(s)|_{\nu_s\Sigma})^{-1},$$

where $T_{eH}(G/N) = \mathfrak{g}/\mathfrak{n}$ is identified with $\mathfrak{m}/\mathfrak{h}$ via the linear isometry

$$\mathfrak{m}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{n},\ X + \mathfrak{h} \mapsto X + \mathfrak{n}$$

(this is an isometry because of Corollary 8.1.4). Note that $\nu_s\Sigma$ is the horizontal space of the fibre Σ^{reg} of π in s. Thus the claimed formula follows for $\delta_{\mathcal{E}}(s)$ with $s \in \Sigma^{\mathrm{reg}}$. It remains to prove that $\delta_{\mathcal{E}}$ is W-invariant. For all $g \in N(\Sigma)$ we have

$$d\omega_{g\cdot s}(eH) = d\phi_g(s) \circ d\omega_s(eH).$$

Since $d\phi_g(s)$ is a linear isometry which leaves $\nu_s\Sigma$ invariant and since the determinant ignores linear isometries up to a sign, it follows that $\delta_{\mathcal{E}}$ is invariant under W. □

We are now in the position to formulate a generalization of Weyl's integration formula to the case of an arbitrary isometric group action.

THEOREM 2.6.4 (Weyl's integration formula). *Let $\varphi : G \times M \to M$ be an isometric action and let $\Sigma \subseteq M$ be a minimal section. We put $N = N_G(\Sigma)$ and $H = Z_G(\Sigma)$. Then $W = W(\Sigma) = N/H$. We assume that G/H carries a $(G\text{-}W)$-invariant metric adapted to Σ and that W and G/N are endowed with the metrics from Corollary 8.1.4. Furthermore all manifolds are equipped with their corresponding Riemannian measure. Then:*

(i) For every $f \in L^1(M)$

$$\int_M f(x)\,dx = \int_{G/N} \left(\int_\Sigma f(g\cdot s)\,\delta_{\mathcal{E}}(s)ds \right) d(gN).$$

Here $d(gN)$ denotes the Riemannian measure on G/N.

(ii) If G/N has finite volume, then for every $f \in L^1(M)^G$

$$\int_M f(x)\,dx = \mathrm{vol}(G/N) \int_\Sigma f(s)\,\delta_{\mathcal{E}}(s)ds.$$

(iii) If G/N is compact, the assignment

$$\Theta_p : L^p(M)^G \to L^p(\Sigma)^W,\ f \mapsto \sqrt[p]{\mathrm{vol}(G/N)\delta_{\mathcal{E}}}\, f|_\Sigma$$

is a surjective linear isometry for any $1 \le p < \infty$.

PROOF. In the following, we repeatedly use that Σ^{reg} and M^{reg} are dense in Σ, resp. M. We first consider the submersion $\pi : M^{\mathrm{reg}} \twoheadrightarrow G/N$ from Theorem 3.1.2 (iv). Its fibre in the point gN is given by $g\cdot\Sigma$. If we apply Fubini's Theorem to π, we get for any $f \in L^1(M)$:

$$\int_M f(x)\,dx = \int_{G/N} \bar{f}(gN)d(gN), \tag{1}$$

where

$$\bar{f}(gN) = \int_{g\cdot\Sigma} f(x)\,\delta(x)dx, \tag{2}$$

and $\delta(x)$ for $x = g\cdot s, s \in \Sigma^{\mathrm{reg}}$ is given by

$$\begin{aligned} \delta(g\cdot s) &= \left|\det((d\pi(g\cdot s)|_{\nu_{g\cdot s}(g\cdot\Sigma)})^{-1})\right| \\ &= \left|\det((dl_g(eN)\circ d\pi(s)|_{\nu_s\Sigma})^{-1})\right| \\ &= \delta(s) = \delta_{\mathcal{E}}(s). \end{aligned}$$

Here we used Proposition 2.6.3 in the last equation. Next, we apply the transformation formula to (2) with respect to the isometry ϕ_g. This yields:

$$\bar{f}(gN) = \int_\Sigma f(g^{-1}\cdot s)\,\delta_{\mathcal{E}}(s)ds. \tag{3}$$

Note that since ϕ_g is an isometry, no Jacobian appears in the formula. If we insert (3) into (1) and exchange $g^{-1}N$ by gN, we obtain formula (i).

Clearly, (ii) is a consequence of (i). For (iii) we proceed as follows. First, because of formula (ii), we have

$$\sqrt[p]{\mathrm{vol}(G/N)\delta_{\mathcal{E}}} f|_\Sigma \in L^p(\Sigma)^W,$$

and thus Θ is well defined. Next we observe that Θ is clearly a linear map. Due to formula (ii), it is an isometry and hence continuous and injective. It remains to prove that Θ is surjective. For this purpose let $\tilde{f} \in L^p(\Sigma)^W$ be arbitrary. Since Σ^{reg} is dense in Σ, we may approximate $\tilde{f}$ by a sequence $\tilde{f}_n \in \mathcal{C}^0_c(\Sigma^{\mathrm{reg}})^W$ with respect to the L_p-norm. According to Corollary 2.1.2, for each $\tilde{f}_n$ there is a function $h_n \in \mathcal{C}^0(M^{\mathrm{reg}})^G$ which restricts to $\tilde{f}_n$ on Σ^{reg}. In order to show that $\mathrm{supp}(h_n) = G \cdot \mathrm{supp}(\tilde{f}_n)$ is compact let $x_k = g_k \cdot y_k \in \mathrm{supp}(h_n)$ be an arbitrary sequence, where $n \in \mathbf{N}$ is fixed, $g_k \in G$ and $y_k \in \mathrm{supp}(\tilde{f}_n)$. Since G/N is compact, there is some $g \in G$ such that, after passing to some subsequence if necessary, $\lim_{k\to\infty}(g_k N) = gN$. Hence, we may assume that there is a sequence $m_k \in N$ such that $\lim_{k\to\infty}(g_k m_k) = g$. Since $\mathrm{supp}(\tilde{f}_n)$ is compact and N-invariant we may assume again that $\lim_{k\to\infty}(m_k^{-1} \cdot y_k) = y \in \mathrm{supp}(\tilde{f}_n)$. It hence follows that $x_k = (g_k m_k) \cdot (m_k^{-1} \cdot y_k)$ has a convergent subsequence. Therefore $\mathrm{supp}(h_n)$ is compact.

Now we define

$$f_n := \frac{h_n}{\sqrt[p]{\mathrm{vol}(G/N)\delta_{\mathcal{E}}}} \in \mathcal{C}_c^0(M^{\mathrm{reg}})^G.$$

Then $\Theta(f_n) = \tilde{f}_n$ and using formula (ii) again we see that f_n is a Cauchy sequence. Hence it converges to some element $f \in L^p(M)^G$. By continuity of Θ we may conclude that $\Theta(f) = \tilde{f}$ and we have proved the surjectivity of Θ. □

Corollary 2.6.5. *If, under the same assumptions as in Theorem 2.6.4, we further assume that M has finite volume, then:*

$$\mathrm{vol}(G/N) = \frac{\mathrm{vol}(M)}{\mathrm{vol}_{\mathcal{E}}(\Sigma)},$$

where $\mathrm{vol}_{\mathcal{E}}(\Sigma) = \int_\Sigma \delta_{\mathcal{E}}(s)ds$ is the weighted volume of Σ.

Proof. This follows immediately from Theorem 2.6.4 (ii). □

The next corollary has been investigated in [**Mag06**]. However, we later realized that over the years it has been independently discovered by several authors (see for instance [**AWY06, AWY05, FJ80, GT07**]). The authors of the first three articles do not explicitly mention the notion of a polar action.

Corollary 2.6.6. *With the same notation as in Theorem 2.6.4 assume that φ is polar and that W is finite. Then the following holds:*

(i) If G/H has finite volume, then $\delta_{\mathcal{E}}$ is a volume scaling function in the following sense:

$$\delta_{\mathcal{E}}(s) = \begin{cases} 0 & \text{if } s \text{ is singular,} \\ |G_s/H| \cdot \frac{\mathrm{vol}(G\cdot s)}{\mathrm{vol}(G/H)} & \text{if } s \text{ is regular or exceptional.} \end{cases}$$

(ii) For any $f \in L^1(M)$ we have the formula

$$\begin{aligned} \int_M f(x)\,dx &= \frac{1}{|W|}\int_{G/H}\left(\int_\Sigma f(g\cdot s)\,\delta_{\mathcal{E}}(s)ds\right)\,d(gH) \\ &= \frac{1}{|W|}\int_\Sigma\left(\int_{G/H} f(g\cdot s)\,d(gH)\right)\,\delta_{\mathcal{E}}(s)ds. \end{aligned}$$

(iii) If G/H has finite volume then for any $f \in L^1(M)^G$ we have

$$\int_M f(x)\,dx = \frac{\mathrm{vol}(G/H)}{|W|}\int_\Sigma f(s)\,\delta_{\mathcal{E}}(s)ds.$$

(iv) The assignment

$$\Psi : \mathcal{C}_c(M) \to \mathcal{C}_c(G/H \times \Sigma)^W,\ f \mapsto \{(gH, s) \mapsto f(g\cdot s)\delta_{\mathcal{E}}(s)\}$$

extends to a surjective linear isometry from $L^1(M)$ *onto* $L^1(G/H \times \Sigma)^W$. *Furthermore, if* G *is compact, then*

$$\Theta : L^p(M)^G \to L^p(\Sigma)^W,\ f \mapsto \sqrt[p]{\frac{\mathrm{vol}(G/H)}{|W|}\delta_{\mathcal{E}}}\, f|_\Sigma$$

is a surjective linear isometry.

Proof. For a polar action H is an open subgroup of N and thus $G/H \to G/N$ is a $|W|$-sheeted covering. The formulas in (ii) and (iii) now follow from Theorem 2.6.4 and Fubini's Theorem applied to the covering map. Also note that for fixed $s \in \Sigma$ and $f \in L^1(M)$ the map $gH \mapsto f(g \cdot s)$ is well defined and integrable on G/H. In contrast to this, $gN \mapsto f(g \cdot s)$ need not be well defined.

Since H is open in N we also have $\mathfrak{h} = \mathfrak{n}$ and $\mathfrak{m} = \mathfrak{g}$. The volume of $G \cdot s$ for non-singular $s \in \Sigma$ is defined as

$$\mathrm{vol}(G \cdot s) = \int_{G\cdot s} 1\, dy,$$

where dy is the Riemannian measure on $G \cdot s$ induced by the Riemannian metric on M. The orbit map $\omega_s : G/G_s \to G \cdot s$ is a diffeomorphism. If we transform the volume integral of $G \cdot s$ with respect to this diffeomorphism we obtain

$$\begin{aligned}\mathrm{vol}(G \cdot s) &= \int_{G/G_s} |\det d\omega_s(gG_s)|\, d(gG_s)\\ &= \frac{1}{|G_s/H|}\int_{G/H} |\det d\omega_s(gH)|\, d(gH)\\ &= \frac{1}{|G_s/H|}\int_{G/H} \delta_{\mathcal{E}}(s)\, d(gH) = \frac{\mathrm{vol}(G/H)}{|G_s/H|}\delta_{\mathcal{E}}(s).\end{aligned}$$

The second part of (iv) is a reformulation of (iii) in Theorem 2.6.4. The first part is proved as follows. First of all, Ψ is well defined. In fact let

$$F(gH, s) := f(g \cdot s)\delta_{\mathcal{E}}(s).$$

Then F is continuous with compact support. If $n \in N$ is arbitrary, then

$$F(gn^{-1}H, n \cdot s) = f(g \cdot s)\delta_{\mathcal{E}}(n \cdot s) = F(gH, s),$$

by W-invariance of $\delta_{\mathcal{E}}$. So F is W-invariant and this completes the proof that Ψ is well defined. Furthermore, Ψ is clearly linear. Using formula (i) we see that Ψ is an isometry which in turn implies that Ψ is injective and continuous. It follows that Ψ extends to a linear isometry from $L^1(M)$ to $L^1(G/H \times \Sigma)^W$. In order to show surjectivity of this map, first consider an arbitrary element F of $\mathcal{C}^0_c(G/H \times \Sigma^{\mathrm{reg}})^W$. From this we construct a function $f \in \mathcal{C}^0_c(M^{\mathrm{reg}})$ by

$$f := \frac{F \circ \varphi^{-1}}{\delta_{\mathcal{E}} \circ \mathrm{pr}_2 \circ \varphi^{-1}},$$

where $\mathrm{pr}_2 : G/H \times \Sigma^{\mathrm{reg}} \to \Sigma^{\mathrm{reg}}$ denotes the canonical projection. The numerator and the denominator are well defined functions, since F and $\delta_{\mathcal{E}}$ are W-invariant and the fibres of $\phi : G/H \times \Sigma^{\mathrm{reg}} \to M^{\mathrm{reg}}$ are precisely the W-orbits on $G/H \times \Sigma^{\mathrm{reg}}$. Continuity of f can be shown by considering local trivializations of the covering φ. Since the fibres of φ are finite, it follows that $F \circ \varphi^{-1}$ has compact support. Furthermore, $\delta_{\mathcal{E}} \circ \mathrm{pr}_2 \circ \phi^{-1}$ does not vanish on M^{reg} because $\delta_{\mathcal{E}}$ does not vanish on Σ^{reg}. So this f is well defined and satisfies $\Psi(f) = F$. We may now approximate an arbitrary element of $L^1(G/H \times \Sigma)^W$

by a sequence of elements in $\mathcal{C}_c^0(G/H \times \Sigma^{\mathrm{reg}})^W$. Using the above construction we can then show, similarly to the proof of Theorem 2.6.4 (iii), that Ψ is surjective. □

REMARK 2.6.7. In the special case that G is a compact Lie group acting on itself via conjugation, then a maximal torus T is a section and at the same time a principal isotropy group along T. Thus formula (ii) of Corollary 2.6.6 is the classical integration formula of Weyl in this case.

REMARK 2.6.8. Sometimes there is some kind of "root space decomposition" available which is adapted to the action. This in turn enables us to compute $\delta_{\mathcal{E}}$ explicitly in terms of the "roots". Below we list some examples of polar actions where this has been carried out in the literature (see for instance [**DK00**] or [**Hel84**]):

(i) Let G be a compact group acting on itself via conjugation: $(g, x) \mapsto gxg^{-1}$. Then a section is given by any maximal torus T of G and at the same time T is also the principal isotropy group along T. It is pretty straightforward to show that $\delta_{\mathcal{E}}(t) = \left|\det(\mathrm{id} - \mathrm{Ad}_t)|_{\mathfrak{g}/\mathfrak{t}}\right|$ for any $t \in T$. We recall some basic facts concerning the root space decomposition of $\mathfrak{g}$ which we will need:

Fix a maximal torus T of G and let $\mathfrak{t}$ denote its Lie algebra. An (infinitesimal) *root* α is an element of $(\mathfrak{t}^{\mathbf{C}})^*$, the dual space of $\mathfrak{t}^{\mathbf{C}}$, such that

$$\mathfrak{g}_\alpha := \{Y \in \mathfrak{g}^{\mathbf{C}} \mid \mathrm{ad}_X(Y) = \alpha(X) \cdot Y \text{ for all } X \in \mathfrak{t}\} \neq 0.$$

If α is a root then so is $-\alpha$. Let P be a choice of *positive roots*; that implies $0 \notin P$ and for each $\alpha \in P$ we have $-\alpha \notin P$. Now each L_α is defined by $L_\alpha := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{g}$. Note that since ad_X is a skew endomorphism with respect to some Ad-invariant inner product, we see that each α takes values in the imaginary numbers only. It is a fact that each nonzero $\mathfrak{g}_\alpha$ has complex dimension one, whereas the corresponding L_α has real dimension two. We then have that $\mathrm{Ad}_{\exp(X)}$ with $X \in \mathfrak{t}$ acts on each nonzero $\mathfrak{g}_\alpha$ as the multiplication operator $e^{\alpha(X)} \in U(1)$. Hence, the action of $\mathrm{Ad}_{\exp(X)}$ on L_α is the rotation through the angle $\alpha(X)/i$. The root space decomposition above is invariant under $(\mathrm{id} - \mathrm{Ad}_t)$. If we write an arbitrary $t \in T$ as $t = \exp(X)$ with $X \in \mathfrak{t}$, we obtain

$$\delta_{\mathcal{E}}(t) = 4^{|P|} \prod_{\alpha \in P} \sin^2\left(\frac{\alpha(X)}{2i}\right).$$

(ii) If we linearize the action in (i) in the neutral element of G, we obtain the adjoint representation Ad of G on $\mathfrak{g}$. In this case, a section is given by any maximal Abelian subspace $\mathfrak{t} \subseteq \mathfrak{g}$ which then corresponds to a maximal torus T of G via $T = \exp(\mathfrak{t})$. In this context, we have $\delta_{\mathcal{E}}(X) = |\det(\mathrm{ad}_X)|$ and ad_X acts on $\mathfrak{g}_\alpha$ as the multiplication operator $\alpha(X)$. Since $\alpha(X)$ is purely imaginary, the action of ad_X on L_α is a rotation of $\frac{\pi}{2}$ scaled by the factor $\frac{\alpha(X)}{i}$. Hence,

$$\delta_{\mathcal{E}}(X) = (-1)^{|P|} \prod_{\alpha \in P} (\alpha(X))^2 = \prod_{\alpha \in P} |\alpha(X)|^2.$$

(iii) A generalization of the polar action in (i) is present in the context of symmetric spaces. Let $M = G/K$ be a Riemannian symmetric space with $G = \mathrm{Iso}(M)^\circ$, the identity component of the isometry group of M, and $K = G_p$, the isotropy subgroup of G of some point $p \in M$. For simplicity reasons we will assume that M is either of compact type or noncompact type. It is then known that the action of K on $M = G/K$ by left translation is (hyper-)polar and a section is given by any maximal flat A of M through p. In order to compute $\delta_{\mathcal{E}}(a)$ for $a \in A$, we start with some preliminaries (see e.g. [**Hel01**, Chapter VII, §11] for the details). Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to K and assume the usual $\mathrm{Ad}_G(K)$-invariant inner product on $\mathfrak{g}$ derived from the Killing form; i.e. if B denotes the Killing form of $\mathfrak{g}$, then in the compact case, the inner product is $-B$, and in the noncompact case it is $-B(\cdot, \theta\cdot)$, where θ denotes the Cartan involution on $\mathfrak{g}$ corresponding to the symmetric subgroup K. We identify T_pM with $\mathfrak{p}$ and, under this identification, let $\mathfrak{a} := T_pA$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ is a maximal Abelian subspace of $\mathfrak{p}$. If $\pi : G \to G/K$ denotes the canonical projection and $d\pi(e) : \mathfrak{g} \to \mathfrak{p}$ the corresponding projection with respect to the Cartan decomposition and the above identification, then $\delta_{\mathcal{E}}(aK) = |\det(d\pi(e) \circ \mathrm{Ad}_{a^{-1}})|$. For further computations, we consider the decompositions

$$\mathfrak{p} = \mathfrak{a} \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{p}_\alpha \quad \text{and} \quad \mathfrak{k} = \mathfrak{z}_{\mathfrak{k}}(\mathfrak{a}) \oplus \sum_{\alpha\in\Sigma^+} \mathfrak{k}_\alpha,$$

where Σ denotes the set of restricted roots, Σ^+ a choice of positive roots,

$$\begin{aligned} \mathfrak{p}_\alpha &= \{Y \in \mathfrak{p} \mid (\mathrm{ad}_X)^2 Y = \alpha(X)^2 Y \text{ for all } X \in \mathfrak{a}\} \quad \text{and} \\ \mathfrak{k}_\alpha &= \{Y \in \mathfrak{k} \mid (\mathrm{ad}_X)^2 Y = \alpha(X)^2 Y \text{ for all } X \in \mathfrak{a}\}. \end{aligned}$$

We put $m_\alpha := \dim \mathfrak{p}_\alpha = \dim \mathfrak{k}_\alpha$. Now $d\pi(e) \circ \mathrm{Ad}_{a^{-1}}$ leaves the above decomposition of $\mathfrak{p}$ invariant and acts on each direct summand $\mathfrak{p}_\alpha$ as the operator $-\sinh\alpha(H) \cdot \frac{\mathrm{ad}_H}{\alpha(H)}$ in case that M is of noncompact type, and it acts as the operator $\sin\left(\frac{\alpha(H)}{i}\right) \cdot \frac{\mathrm{ad}_H}{\alpha(H)/i}$ in case that M is of compact type, where $a = \exp(H)$. In the noncompact case, we thus have:

$$\delta_{\mathcal{E}}(a) = \prod_{\alpha\in\Sigma^+} |\sinh\alpha(H)|^{m_\alpha},$$

whereas in the compact case we have:

$$\delta_{\mathcal{E}}(a) = \prod_{\alpha\in\Sigma^+} \left|\sin\left(\frac{\alpha(H)}{i}\right)\right|^{m_\alpha}.$$

(iv) As before, there is a Lie algebra version of (iii). The action of K on $\mathfrak{p}$ (using the same notation as in (iii)) given by $k \cdot X := \mathrm{Ad}_k(X)$ is (hyper-)polar. Every Cartan subspace $\mathfrak{a}$ of $\mathfrak{p}$ is a section. In view of the root space decomposition in (iii), we then obtain

$$\delta_{\mathcal{E}}(H) = \prod_{\alpha\in\Sigma^+} |\alpha(H)|^{m_\alpha}.$$

A further generalization of the above actions are the so called "Hermann actions". They constitute a quite well understood class of hyperpolar actions on symmetric spaces and their corresponding $\delta_{\mathcal{E}}$'s have been computed in [**GT07**].

EXAMPLE 2.6.9. Examples of non-polar actions where $\delta_{\mathcal{E}}$ can be explicitly calculated are stated in Proposition 7.1.4. For instance, in the case of the k-fold direct sum of the standard representation of $\mathbf{SO}(n)$ with $2 \le k \le n-1$ (cf. Example 1.1.3) we have

$$\delta_{\mathcal{E}}(p) = \frac{1}{\sqrt{2^{k(n-k)}}} |\det(B)|^{(n-k)},$$

where $p = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} \in \Sigma$ and $B \in \mathbf{R}^{k^2}$.

2.7. On a Generalization of Chevalley's Restriction Theorem

Recall that a smooth p-form $\omega \in \Omega(M)$ is called **G-invariant**, if for all $g \in G$ we have that $g^*\omega = \omega$. The set of all G-invariant p-forms on M will be denoted by $\Omega^p(M)^G$. A p-form ω is called **horizontal**, if for all $X \in \mathfrak{g}$ we have $\iota_X(\omega) = 0$. Here ι_X denotes contraction by the Killing field generated by X. The set of all G-invariant horizontal p-forms is denoted by $\Omega^p_{\text{hor}}(M)^G$. These forms are also called **basic** forms.

If Σ is a fat section with fat Weyl group W, then in view of Corollary 2.1.2 it is natural to ask whether the isomorphism ι^* also yields $\mathcal{C}^\infty(M)^G \simeq \mathcal{C}^\infty(\Sigma)^W$, or if we even have $\Omega^*_{\text{hor}}(M)^G \simeq \Omega^*_{\text{hor}}(\Sigma)^W$. In the polar case (i.e. $\operatorname{copol}(G, M) = 0$) the first statement has been proved by Palais and Terng in [**PT87**] and the second statement by Michor in [**Mic96, Mic97**]. In the general case we note the following:

Proposition 2.7.1. *The map $\iota^* : \mathcal{C}^\infty(M)^G \to \mathcal{C}^\infty(\Sigma)^W$, $f \mapsto f|_\Sigma$, is well defined and injective, and the G-invariant continuous extension $(\iota^*)^{-1}(f)$ of $f \in \mathcal{C}^\infty(\Sigma)^W$ to M is smooth on M^{reg}.*

Proof. Since Σ is an embedded submanifold of M, the restriction of f to Σ yields a smooth function on Σ. Thus $\iota^*|_{\mathcal{C}^\infty(M)^G}$ is well defined. Injectivity is trivial, because the map ι^* on $\mathcal{C}^0(M)^G$ is already injective by Corollary 2.1.2. Let now $f \in \mathcal{C}^\infty(\Sigma)^W$ be arbitrary and denote its G-invariant extension to M by F. Smoothness of F is a local condition. Thus, let $p \in M$ be an arbitrary point and let U be a tubular neighborhood of $G \cdot p$. Since F is G-invariant, we may assume that $p \in \Sigma$. Let furthermore S_p be a slice through p such that $U = G \cdot S_p$. It is known that $F|_U$ is smooth in p if and only if $F|_{S_p}$ is smooth in p. Since Σ is a fat section we have $S_p \subseteq \Sigma$ in the case that p is a G-regular point and S_q is also a slice with respect to the W-action on Σ. Hence $F|_{S_p} = f|_{S_p}$ is smooth in p. □

Proposition 2.7.2. *Let (G, M) be an isometric action and let Σ be a fat section with fat Weyl group $W = W(\Sigma)$. Then the mapping $i^* : \Omega^*_{\text{hor}}(M)^G \to \Omega^*_{\text{hor}}(\Sigma)^W$, which is obtained by restriction to Σ, is injective.*

Proof. The mapping i^* is well defined, since Σ is an embedded submanifold and due to Corollary 2.1.3. Suppose now that $i^*\omega = 0$ for some p-form $\omega \in \Omega^*_{\text{hor}}(M)^G$. Let $q \in \Sigma \cap M^{\text{reg}}$ be an arbitrary G-regular point in Σ. By property (C) of a fat section, we have a (not necessarily direct) decomposition of $T_qM = T_q\Sigma + T_q(G \cdot q)$. Let $X_1, \ldots, X_p$ be arbitrary vectors in T_qM. According to the above decomposition we can write $X_i = Y_i + Z_i$, where $Y_i \in T_q\Sigma$ and $Z_i \in T_q(G \cdot q)$ for all $i = 1, \ldots, p$. Now $\omega_q(X_1, \ldots, X_p)$ decomposes into a sum where each summand contains either Y_i or Z_i for all $i = 1, \ldots, p$. If a summand contains at least one Z_i, then it vanishes, since ω is horizontal. Otherwise, the summand is $\omega_q(Y_1, \ldots, Y_p)$ and vanishes because $i^*\omega = 0$. All in all we thus have that $\omega_q = 0$. Since ω is G-invariant, this holds along the whole orbit through q. Now $q \in M^{\text{reg}}$ was arbitrary, so ω vanishes on the G-regular set of M, and since the regular set is dense in M, we finally conclude that $\omega = 0$ on all of M. □

One would expect that i^* should also be surjective in general. However, we can show this only under the following strong assumptions:

Theorem 2.7.3. *Let (G, M) be an isometric action and let $\Sigma \subseteq M$ be a minimal section. Put $W = W(\Sigma)$. Suppose that the slice representation $(G_q, \nu_q(G \cdot q))$ is polar for* *every $q \in \Sigma$ and that $V_q = \nu_q(G \cdot q) \cap T_q\Sigma$ is a 0-section. Then $\Omega^*_{\text{hor}}(M)^G \simeq \Omega^*_{\text{hor}}(\Sigma)^W$. In particular, $\mathcal{C}^\infty(M)^G \simeq \mathcal{C}^\infty(\Sigma)^W$ and the isomorphism in both cases is given by the map i^* from proposition 2.7.2.*

Proof. All that is left to show is the surjectivity of i^*. The proof is basically the same as Michor's in [**Mic96**, 4.2]. The idea is the following: Given a form $\tilde{\omega} \in \Omega^p_{\text{hor}}(\Sigma)^W$, we have to construct a form $\omega \in \Omega^p_{\text{hor}}(M)^G$ with $i^*(\omega) = \tilde{\omega}$. In a first step, we construct ω locally, using that the slice representation is polar in every point in combination with Corollary 3.8 and Lemma 4.1 of [**Mic96**]. The corollary states that basic forms correspond to Weyl-invariant forms for polar representations, and the lemma states that basic forms on a slice can be extended to basic forms on the corresponding tube. Finally, we glue up the various local forms via a G-invariant partition of unity. To begin with, let $q \in M$ be an arbitrary point of M and let U denote a tube around $G \cdot q$ and S_q the corresponding slice through q. After a suitable translation, we may assume that $q \in \Sigma$. Let $B \subset T_qM$ be an open ball centered at 0, such that

$$\exp_q|_{(B\cap\nu_q(G\cdot q))} : (B \cap \nu_q(G \cdot q)) \to S_q$$

is a diffeomorphism onto S_q. Then

$$\exp_q|_{B\cap T_q\Sigma} : B \cap T_q\Sigma \to D$$

is a diffeomorphism onto an open neighborhood $D \subseteq \Sigma$ of q in Σ. By assumption, $V_q = T_q\Sigma \cap \nu_q(G \cdot q)$ is a section of the polar action of G_q on $\nu_q(G \cdot q)$. Furthermore, the generalized Weyl group of V_q is equal to the isotropy group W_q of W in q. Let $\omega' \in \Omega^p(B \cap T_q\Sigma)^{W_q}$ be the pullback of $\tilde{\omega}$ by $\exp_q|_{B\cap T_q\Sigma}$. By [**Mic96**, Corollary 3.8], there exists a unique form $\omega'' \in \Omega^p_{G_q-\text{hor}}(B \cap \nu_q(G \cdot q))^{W_q}$, which extends ω'. We may now push this form forward along $\exp_q|_{B\cap\nu_q(G\cdot q)}$ to S_q to obtain an element of $\Omega^p_{G_q-\text{hor}}(S_q)^{G_q}$. By Lemma 4.1 loc. cit. we finally obtain a form $\omega^q \in \Omega^p_{\text{hor}}(U)^G$ which satisfies $(i|_D)^*(\omega^q) = \tilde{\omega}|_D$.

The intersection $U \cap \Sigma$ consists of the disjoint union of all $w_j \cdot D$, where w_j is a representative of a coset of the quotient W/W_q. Choose for all j some element g_j of $N_G(\Sigma)$ which projects down to w_j. We then have

$$\begin{aligned}
(i|_{w_j\cdot D})^*(\omega^q) &= (l_{g_j} \circ i|_D \circ w_j^{-1})^*(\omega^q)\\
&= (w_j^{-1})^*(i|_D)^* l^*_{g_j}(\omega^q)\\
&= (w_j^{-1})^*(i|_D)^*(\omega^q) = (w_j^{-1})^*(\omega|_D) = \tilde{\omega}|_{w_j\cdot D}.
\end{aligned}$$

We therefore have $(i|_{U\cap\Sigma})^*(\omega)^q = \tilde{\omega}|U \cap \Sigma$ and, by choosing a suitable G-invariant partition of unity, we may glue the desired ω from the various ω^q, which then has the property $i^*(\omega) = \tilde{\omega}$. □

Remark 2.7.4.

(i) The assumption of polarity of the slice representation in the theorem above enters in a subtle way. Namely, in the step where [**Mic96**, Corollary 3.8] is invoked, a theorem enters, which states that the algebra of real G-invariant polynomials of a polar representation (G, V) with compact G is isomorphic to the algebra of real W-invariant polynomials on a section Σ via the restriction mapping $p \mapsto p|_\Sigma$. Crucial for this theorem in turn is that for any $x \in \Sigma$ the orthogonal projection of the orbit $G \cdot x$ onto Σ lies in the convex hull of $G \cdot x \cap \Sigma = W \cdot x$. This statement hinges on the fact, that $W \cdot x$ is a finite set in the polar case. We conjecture though, that a corresponding convexity theorem holds for minimal sections of arbitrary representations.

(ii) A situation, in which the seemingly restrictive assumptions of Theorem 2.7.3 are satisfied, is given in Theorem 5.1.4.

(iii) Forming the cohomology of the complex $\Omega^*_{\mathrm{hor}}(M)^G$ yields basic cohomology $H^*_{G-\mathrm{basic}}(M)$. Thus Theorem 2.7.3 implies that $H^*_{G-\mathrm{basic}}(M)$ is isomorphic to $H^*_{W-\mathrm{basic}}(\Sigma)$ under the assumptions stated. However, it was already observed by Koszul in [**Kos53**], that for compact M, the basic cohomology of (G, M) is isomorphic to the singular cohomology of the orbit space $G\backslash M$[5]. Hence, in view of Theorem 2.1.1, we obtain the isomorphism of basic cohomology already under the weaker assumption of M being compact, using that $G\backslash M$ is homeomorphic to $W\backslash\Sigma$.

[5]I thank Peter W. Michor for this information.

CHAPTER 3

Global Resolutions of Isometric Actions with Respect to Fat Sections

In this section we define the (global) resolution M_Σ of an isometric action (G, M) with respect to an arbitrary fat section Σ. The construction is related to the so called *core resolution* of Grove and Searle in [**GS00**], which we have already described in the introduction. The reason, why M_Σ is called a resolution, is that it is a G-space whose isotropy groups are smaller than that of M. So, roughly speaking, the G-orbits on M_Σ are less singular than the G-orbits on M, in a way which will be made precise later.

In the following, let (G, M) be an isometric action and let Σ be a fat section. Put $N = N_G(\Sigma)$ and $H = Z_G(\Sigma)$. Then $W = N/H$ is the fat Weyl group of Σ. Since Σ is a W-space, we may form the associated bundle $G/H \times_W \Sigma \twoheadrightarrow G/N$ with fibre Σ, where $G/H \times_W \Sigma$ is the orbit space of the W-action on $G/H \times \Sigma$ given by

$$nH \cdot (gH, s) := (gn^{-1}H, n \cdot s).$$

Its total space is a G-space with respect to the G-action $l \cdot [gH, s] := [lgH, s]$.

Definition 3.1.1. We call

$$M_\Sigma := G/H \times_W \Sigma$$

the **resolution** of (G, M) with respect to Σ. If Σ is a minimal section, we call M_Σ a **minimal resolution**.

We now list some features related to M_Σ (c.f. [**GS00**], Theorem 2.1):

Theorem 3.1.2. *Let $\varphi : G \times M \to M$ denote the group action (G, M). Then*

(i) The group action φ induces a smooth and surjective G-equivariant map:

$$\tilde{\varphi} : M_\Sigma \twoheadrightarrow M, [gH, s] \mapsto g \cdot s.$$

(ii) The isotropy group of the point $[eH, s] \in M_\Sigma = G/H \times_W \Sigma$ is given by:

$$G_{[eH,s]} = N(\Sigma) \cap G_s = N_{G_s}(\Sigma).$$

(iii) Σ is canonically N-equivariantly immersed into M_Σ via the map $s \mapsto [eH, s]$. The image $\tilde{\Sigma}$ is embedded into M_Σ because it is a fibre of $M_\Sigma \twoheadrightarrow G\backslash N$, and furthermore it intersects every G-orbit on M_Σ. It follows that $\tilde{\varphi}$ restricts to a W-equivariant diffeomorphism between $\tilde{\Sigma}$ and Σ.

(iv) The set of G-regular points $(M_\Sigma)^{\mathrm{reg}}$ can be identified with $G/H \times_W \Sigma^{\mathrm{reg}}$ and $\tilde{\varphi}$ restricts to a G-equivariant diffeomorphism from $(M_\Sigma)^{\mathrm{reg}}$ onto M^{reg}. It follows that we have a bundle

$$\pi : M^{\mathrm{reg}} \twoheadrightarrow G/N,\ g \cdot s \mapsto gN$$

with structure group W and whose fibres are the G-translates of Σ^{reg}. In particular, they are totally geodesic.

(v) The orbit spaces $G\backslash M_\Sigma$ and $G\backslash M$ are canonically homeomorphic.

(vi) $d\tilde{\varphi}_{[eH,s]} : T_{[eH,s]}M_\Sigma \to T_sM$ *is a linear isomorphism if and only if*

$$T_s(G \cdot s) + T_s\Sigma = T_sM. \qquad (*)$$

This is furthermore equivalent to $G_s \subseteq N$ *and also to* $(G_s)^\circ = (N \cap G_s)^\circ$. $\tilde{\varphi}$ *is a* G*-equivariant diffeomorphism if and only if* $(*)$ *is satisfied for all* $s \in \Sigma$.

(vii) The G*-translates of* $\tilde{\Sigma}$ *form a foliation of* M_Σ.

Proof. (i): If $[gH, s] = [\tilde{g}H, \tilde{s}] \in M_\Sigma$, then there is some $n \in N$ and $h \in H$ with

$$(\tilde{g}, \tilde{s}) = (gn^{-1}h, n \cdot s).$$

It follows that

$$\tilde{g} \cdot \tilde{s} = g \underbrace{n^{-1}hn}_{\in H} \cdot s = g \cdot s$$

and we have shown that $\tilde{\varphi}$ is well defined. Since Σ intersects every orbit, it follows that φ restricted to $G \times \Sigma$ maps onto M. Furthermore, H acts trivially on Σ and so the induced map

$$G/H \times \Sigma^{\mathrm{reg}} \to M^{\mathrm{reg}}, \ (gH, s) \mapsto g \cdot s$$

(by abuse of notation again denoted by φ) is still surjective. The following diagram is commutative:

$$\begin{array}{ccc} G/H \times \Sigma & & \\ {\scriptstyle \mathrm{pr}}\downarrow & \searrow^{\varphi} & \\ M_\Sigma & \xrightarrow[\tilde{\varphi}]{} & M. \end{array}$$

From this we can read off that $\tilde{\varphi}$ is also surjective and G-equivariant, and since the vertical map is a surjective submersion, it follows that $\tilde{\varphi}$ is a smooth map.

(ii): Let $g \in G_{[eH,s]}$ be arbitrary. Then there exists some $n \in N$ and $h \in H$ such that $(g, s) = (n^{-1}h, n \cdot s)$. This implies $n \in G_s$ and therefore $gh^{-1} \in G_s$. Since $H \subseteq G_s$, it follows that $g \in G_s \cap N$. If conversely $g \in G_s \cap N$, then

$$g \cdot [eH, s] = [gH, s] = [eH, g^{-1} \cdot s] = [eH, s],$$

showing that $g \in G_{[eH,s]}$. If $s \in \Sigma$ is G-regular, then $G_s \subseteq N$ and thus $G_{[eH,s]} = G_s$. From here it is not difficult to see that a G-regular $[eH, \tilde{s}]$ has an isotropy group which is conjugate to G_s.

(iii): This statement is easily verified.

(iv): The first part follows from (ii) and (iii). It remains to show that $\tilde{\varphi}|_{M_\Sigma}$ is injective and that it has a smooth inverse. Suppose that $g \cdot s = \tilde{g} \cdot \tilde{s}$ for $g, \tilde{g} \in G$ and $s, \tilde{s} \in \Sigma$. Then $\tilde{s} = \tilde{g}^{-1}g \cdot s$ and since s and $\tilde{s}$ are G-regular, it follows from property (D) of a fat section that $n := \tilde{g}^{-1}g \in N$. Hence,

$$[gH, s] = [gn^{-1}H, n \cdot s] = [\tilde{g}, \tilde{s}].$$

By property (C) of a fat section, the tangent space of Σ is in regular points transversal to the orbit. Using (vi) it follows that $\tilde{\varphi}|_{(M_\Sigma)^{\mathrm{reg}}}$ is a submersion and thus a diffeomorphism.

(v): We have the well defined map $f : G\backslash M_\Sigma \to G\backslash M, \ G \cdot [eH, s] \mapsto G \cdot s$, and the diagram

$$\begin{array}{ccc} M_\Sigma & \xrightarrow{\tilde{\varphi}} & M \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}} \\ G\backslash M_\Sigma & \xrightarrow[f]{} & G\backslash M \end{array}$$

is commutative. This shows that f is continuous and surjective. It is also easy to see that f is injective. To show that f^{-1} is continuous, we write it as a composition of continuous maps:

$$\begin{array}{ccccccc} G\backslash M & \overset{\tilde{\iota}^{-1}}{\to} & W\backslash\Sigma & \to & W\backslash\tilde{\Sigma} & \to & G\backslash M_\Sigma, \\ G\cdot s & \mapsto & W\cdot s & \mapsto & W\cdot[eH,s] & \mapsto & G\cdot[eH,s]. \end{array}$$

Here $\tilde{\iota}$ is the map from Theorem 2.1.1 and the other two maps are the continuous injections induced by the continuous maps $\Sigma \hookrightarrow \tilde{\Sigma}$, resp. $\tilde{\Sigma} \hookrightarrow M_\Sigma$, both of which appear in (iii).

(vi): From the diagram in the proof of (i) we see that

$$d\tilde{\varphi}_{[eH,s]} : T_{[eH,s]}M_\Sigma \to T_sM$$

is surjective if and only if

$$d\varphi_{(eH,s)} : T_{(eH,s)}G/H \times \Sigma \to T_sM$$

is surjective. The latter map is given by:

$$d\varphi_{(eH,s)}(X+\mathfrak{h}, v) = X_s + v,$$

where X_s is the value of the Killing field induced by $X + \mathfrak{h}$ on M in s. We have

$$\operatorname{im}(d\varphi_{(eH,s)}) = T_s(G\cdot s) + T_s\Sigma$$

and thus the map is surjective if and only if $T_sM = T_s(G\cdot s) + T_s\Sigma$ holds.

Using Proposition 2.1.5, we have

$$T_s(G\cdot s) + T_s\Sigma = T_s(G\cdot s) \oplus (T_s\Sigma \cap \nu_s(G\cdot s)). \qquad (**)$$

Note that the right hand side is an orthogonal decomposition. Thus $(*)$ holds if and only if

$$\nu_s(G\cdot s) \subseteq T_s\Sigma.$$

This in turn is equivalent to the statement that $G_s \subseteq N$. In fact, since the G_s-regular points in $\nu_s(G\cdot s)$ correspond to G-regular points in M under the exponential map, it follows from property (D) of a fat section that, if $\nu_s(G\cdot s) \subseteq T_s\Sigma$ holds, then $G_s \subseteq N$. Conversely, if $G_s \subseteq N$ then, according to the slice theorem (Theorem 2.2.2), $\nu_s^\Sigma(W\cdot s)$ is a G_s-invariant subspace of $\nu_s(G\cdot s)$, but this means that

$$\nu_s(G\cdot s) = \nu_s^\Sigma(W\cdot s) \subseteq T_s\Sigma.$$

Again by Proposition 2.1.5, we may rewrite the right hand side of $(*)$ as

$$T_s(G\cdot s) + T_s\Sigma = T_s\Sigma \oplus (T_s(G\cdot s) \cap \nu_s(\Sigma)).$$

Thus, $(*)$ is furthermore equivalent to

$$\dim M = \dim\Sigma + (\dim(G\cdot s) - \dim(W\cdot s)) \qquad (***)$$

Let $(.)_{\text{princ}}$ denote a principal isotropy group for the action in parentheses. Then

$$\dim\Sigma = \operatorname{cohom}(G,M) + \dim W - \underbrace{\dim(W,\Sigma)_{\text{princ}}}_{\dim(G,M)_{\text{princ}} - \dim H}$$

The right hand side of $(***)$ now computes to:

$$\begin{aligned} & \operatorname{cohom}(G,M) + \dim W - \dim(W,\Sigma)_{\text{princ}} + (\dim G - \dim G_s - \dim W + \dim W_s) \\ = \; & \dim M + \dim H - \dim G_s + \underbrace{\dim W_s}_{=\dim(G_s\cap N) - \dim H} \\ = \; & \dim M + \dim(G_s \cap N) - \dim G_s. \end{aligned}$$

It follows that $(***)$ is equivalent to $\dim(G_s \cap N) = \dim G_s$, which is the same as $(G_s)^\circ = (G_s \cap N)^\circ$.

Suppose that $\tilde{\varphi}$ is a local diffeomorphism. Since $\tilde{\varphi}$ restricted to $(M_\Sigma)^{\mathrm{reg}}$ is a diffeomorphism onto M^{reg} and since the regular points form an open and dense subset of their surrounding space, it follows that $\tilde{\varphi}$ is a diffeomorphism from M_Σ onto M.

(vii): Let $q := [gH, s] \in M_\Sigma$ be an arbitrary point. According to Corollary 2.1.4, $G_q = g(N(\Sigma) \cap G_s)g^{-1}$ is transitive on the set of G-translates of $\tilde{\Sigma}$ that contain q. Clearly, $g \cdot \tilde{\Sigma}$ contains q. For an arbitrary element $gng^{-1} \in G_q$, where $n \in N(\Sigma) \cap G_s$, we have

$$(gng^{-1}) \cdot (g \cdot \tilde{\Sigma}) = (gn) \cdot \tilde{\Sigma} = g \cdot \tilde{\Sigma}.$$

Therefore, only the G-translate $g \cdot \tilde{\Sigma}$ passes through q. □

Corollary 3.1.3 ([**GS00**, cf. Corollary 2.4]). *If (G, M) has only principal or exceptional orbits, then $M_\Sigma \simeq M$.*

Proof. Let $q \in \Sigma$ be arbitrary. According to Lemma 1.1.8 there is some G-regular point $p \in \Sigma$ in a slice around q. We thus have $G_p \subseteq G_q$ and by assumption $(G_q)^\circ = (G_p)^\circ$. Since $p \in \Sigma$ is G-regular, property (D) of a fat section implies that $G_p \subseteq N(\Sigma)$. Putting all this together yields:

$$(G_p)^\circ \subseteq (N(\Sigma) \cap G_q)^\circ \subseteq (G_q)^\circ = (G_p)^\circ.$$

Now the claim follows from Theorem 3.1.2 (vi). □

So far we have considered M_Σ only as a smooth manifold without any Riemannian metric on it. It is natural to demand that G should act isometrically on M_Σ. Furthermore, the Riemannian metric on M_Σ should be induced by a product metric on $G/H \times \Sigma$. Hence, we consider $(G\text{-}W)$-invariant metrics on G/H (cf. Section 7).

Proposition 3.1.4. *Suppose that G/H has a $(G\text{-}W)$-invariant Riemannian metric and Σ carries the Riemannian metric induced by M. Then M_Σ, endowed with the Riemannian metric submersed from $G/H \times \Sigma$, has the following properties:*

(i) (G, M_Σ) *is an isometric action.*
(ii) *If Σ is a k-section of (G, M), then $\tilde{\Sigma} = \{[eH, s] \mid s \in \Sigma\}$ is a k-section of (G, M_Σ) and $W(\tilde{\Sigma}) = W(\Sigma)$. In particular, the foliation of M_Σ given by the G-translates of $\tilde{\Sigma}$ has totally geodesic leaves.*
(iii) $(M_\Sigma)_{\tilde{\Sigma}} \simeq M_\Sigma$ *(G-equivalent).*
(iv) *If Σ is a minimal section of (G, M), then* $\operatorname{copol}(G, M_\Sigma) \leq \operatorname{copol}(G, M)$.

Proof. (i) is clear by the assumptions made on the metric on G/H.

(ii): By Theorem 3.1.2 (iii) we have that $\tilde{\Sigma}$ is complete, connected and embedded into M_Σ and intersects every G-orbit. Consider the principal bundle

$$\psi : G/H \times \Sigma \to M_\Sigma,\ (gH, s) \mapsto [gH, s],$$

which maps a point (gH, s) to its W-orbit $[gH, s] = \{(gn^{-1}H, n \cdot s) \mid nH \in W\}$. By our choice of metric, ψ is a Riemannian submersion.

We claim that $\tilde{\Sigma}$ is totally geodesic in M_Σ. In fact, $\psi^{-1}(\tilde{\Sigma}) = W \times \Sigma$ and since W is totally geodesic in G/H by Corollary 8.1.4, it follows that $W \times \Sigma$ is totally geodesic in $G/H \times \Sigma$. Thus $\tilde{\Sigma} = \psi(W \times \Sigma)$ is totally geodesic in M_Σ. This already gives us properties (A) and (B) of a fat section. The fibre of ψ over $[eH, s]$ is

$$\psi^{-1}([eH, s]) = \{(nH, n^{-1} \cdot s) \mid nH \in W\}.$$

In order to speak about metric relations in the tangent spaces of M_Σ we have to determine the vertical and horizontal distribution of ψ along $\{eH\} \times \Sigma$. They are defined as

$$\mathcal{V}_{(eH,s)} := T_{(eH,s)}\psi^{-1}([eH,s]) \text{ and } \mathcal{H}_{(eH,s)} := (\mathcal{V}_{(eH,s)})^\perp.$$

The definition of the fibre yields

$$\mathcal{V}_{(eH,s)} = \{(X + \mathfrak{h}, -X_s) \mid X + \mathfrak{h} \in \mathfrak{n}/\mathfrak{h}\} \subseteq \mathfrak{n}/\mathfrak{h} \times T_s(W \cdot s),$$

and a computation shows that

$$\mathcal{H}_{(eH,s)} = ((\mathfrak{n}/\mathfrak{h})^\perp \times \nu_s^\Sigma(W \cdot s)) \oplus A_s,$$

where $A_s := \mathcal{H}_{(eH,s)} \cap (\mathfrak{n}/\mathfrak{h} \times T_s(W \cdot s))$. In fact, A_s corresponds to the tangent space of the W-orbit through $[eH,s]$ (induced by the left action of G) and one can show that

$$A_s = \{(f_s(v), v) \mid v \in T_s(W \cdot s)\},$$

for some linear monomorphism $f_s : T_s(W \cdot s) \to \mathfrak{n}/\mathfrak{h}$, but we do not need this fact in the following. By our assumptions on the Riemannian metric on $G/H \times \Sigma$ and M_Σ, we have that ψ is a Riemannian submersion. Hence, we may identify subspaces of $T_{[eH,s]}M_\Sigma$ with certain subspaces of $\mathcal{H}_{(eH,s)}$. More precisely, we have

$$T_{[eH,s]}(G \cdot [eH,s]) \simeq \mathcal{H}_{(eH,s)} \cap (\underbrace{T_{(eH,s)}(G \cdot (eH,s)) + \mathcal{V}_{(eH,s)}}_{=\mathfrak{g}/\mathfrak{h} \times T_s(W \cdot s)}) = A_s \oplus ((\mathfrak{n}/\mathfrak{h})^\perp \times \{0\}),$$

and it follows that

$$\nu_{[eH,s]}(G \cdot [eH,s]) \simeq \{0\} \times \nu_s^\Sigma(W \cdot s) \subseteq (\{0\} \times \nu_s^\Sigma(W \cdot s)) \oplus A_s \simeq T_{[eH,s]}\tilde{\Sigma}.$$

We therefore have for *all* points $[eH,s] \in \tilde{\Sigma}$ (and not just the G-regular ones) that

$$\nu_{[eH,s]}(G \cdot [eH,s]) \subseteq T_{[eH,s]}\tilde{\Sigma}.$$

This shows property (C) of a fat section. We now come to property (D). If $[eH,s] \in \tilde{\Sigma}$ and $g \in G$ with $g \cdot [eH,s] = [gH,s] \in \tilde{\Sigma}$, it follows that $g \in N$ (again this holds not only in the G-regular points). We have therefore shown that $\tilde{\Sigma}$ is a k-section of (G, M_Σ) if Σ is a k-section of (G,M). That $W(\tilde{\Sigma}) = W(\Sigma)$ holds is also not difficult to show, in fact we even have $N_G(\tilde{\Sigma}) = N_G(\Sigma)$ and $Z_G(\tilde{\Sigma}) = Z_G(\Sigma)$.

(iii): Let $\tilde{\tilde{\varphi}} : (M_\Sigma)_{\tilde{\Sigma}} \to M_\Sigma$ denote the canonical G-equivariant surjection. That is

$$\tilde{\tilde{\varphi}} : G/H \times_W \tilde{\Sigma} \to M_\Sigma, \ [gH, [eH,s]] \mapsto [gH,s].$$

If $\tilde{\tilde{\varphi}}([gH,[eH,s]]) = \tilde{\tilde{\varphi}}([\tilde{g}H,[eH,\tilde{s}]])$, then $[gH,s] = [\tilde{g}H,\tilde{s}]$. Now $\tilde{g}H = gn^{-1}H$ and $\tilde{s} = n \cdot s$ for some $n \in N$. But this implies that

$$\begin{aligned} [\tilde{g}H,[eH,\tilde{s}]] &= [gn^{-1}H,[eH,n \cdot s]] \\ &= [gH, n \cdot [eH, n \cdot s]] \\ &= [gH,[nH,n \cdot s]] \\ &= [gH,[eH,s]]. \end{aligned}$$

This shows that $\tilde{\tilde{\varphi}}$ is injective. By Theorem 3.1.2 (ii) we have $G_{[eH,s]} \subseteq N$ for all $s \in \Sigma$ and then (vi) of the same Theorem implies that $\tilde{\tilde{\varphi}}$ is a submersion. It follows that the map is a G-equivariant diffeomorphism.

(iv) is an immediate consequence of (ii). □

REMARK 3.1.5.

(i) We do not know whether for a minimal section Σ of (G, M) it is actually possible that $\mathrm{copol}(G, M_\Sigma) < \mathrm{copol}(G, M)$.

(ii) According to Corollary 2.5.8 (ii), the assumptions made in the Proposition above are certainly satisfied if N is compact. We would also like to mention that there are other natural ways to endow M_Σ with a Riemannian metric such that $\tilde{\Sigma}$ is totally geodesic, see for instance [**Bes87**, Theorem 9.59]. We do not know however, if G then still acts isometrically on M_Σ.

The following proposition generalizes [**GS00**, Proposition 2.6]. Basically, the proof is the same as in loc. cit.

PROPOSITION 3.1.6. *Suppose that M is a Riemannian G-manifold with sectional curvature bounded from below by $k \leq 0$ and let G be a compact Lie group. For any fat section Σ the resolution M_Σ supports a G-invariant Riemannian metric whose sectional curvature is bounded from below by k.*

PROOF. As before, we write $N = N_G(\Sigma)$, $H = Z_G(\Sigma)$ and $W = N/Z$. Assume that G carries a bi-invariant Riemannian metric. Then the induced metric on G/H is $(G\text{-}W)$-invariant and has non-negative curvature. On Σ we consider the Riemannian metric induced from the ambient space. Since Σ is totally geodesic in M, it satisfies the same curvature bound as M. The same holds for $G/H \times \Sigma$ with the product metric. Since W acts isometrically on the product space $G/H \times \Sigma$, the projection onto M_Σ yields a Riemannian submersion. According to O'Neills Theorem ([**O'N66**, Corollary 1]), M_Σ now has the same curvature bound k from below as M. □

REMARK 3.1.7. As a concluding remark of this section, we show that for every triple $H \trianglelefteq N \leq G$, where G is a lie group, H and N are closed subgroups of G and such that N is compact, there exists some manifold Σ on which $W = N/H$ acts isometrically, with trivial principal isotropy group and such that $M := G/H \times_W \Sigma$ is a Riemannian G-manifold with fat section Σ and fat Weyl group W. This generalizes the construction in [**PT88**, 5.6.20]. In fact, since W is compact, there is some Euclidean vector space V on which W acts faithfully. Then W acts with trivial principal isotropy group on the k-fold inner direct sum $\Sigma := k \cdot V$ for some positive integer k. If now G/H is endowed with a $(G\text{-}W)$-invariant Riemannian metric, then $M := G/H \times_W \Sigma$ with the submersed metric from $G/H \times \Sigma$ is a G-manifold. Similarly as in the proof of Proposition 3.1.4 (ii) one can show that $\tilde{\Sigma} := \{[eH, s] \mid s \in \Sigma\}$ is a fat section with fat Weyl group W.

CHAPTER 4

Copolarity of Singular Riemannian Foliations

Since pre-sections are purely geometrical objects and since minimal sections can be expressed as connected components of the intersections of certain pre-sections (see Proposition 1.1.9 (iv)), there is a meaningful way to define these notions for singular Riemannian foliations. This also leads to the notion of copolarity for the latter. A reference for the following notions is [**Mol88**, Chapter 6]. A **transnormal system** $\mathcal{F}$ on a Riemannian manifold M is a partition of M into complete connected immersed submanifolds of M such that every geodesic perpendicular to one leaf is perpendicular to all other leaves it meets. A **singular Riemannian foliation (SRF)** is a transnormal system such that the module $\Xi_{\mathcal{F}}$ of all vector fields, which are tangent to all leaves in $\mathcal{F}$, spans for every $p \in M$ the tangent space T_pF of the leaf $F \in \mathcal{F}$ through p. A leaf F is called **regular** if it has maximal dimension, otherwise it is called **singular**.

The partition of a G-manifold M into the G-orbits is a transnormal system. Since the tangent space of every orbit is spanned by the G-Killing fields, this partition is also a singular Riemannian foliation. However, the principal and exceptional orbits are both considered as regular leaves for the singular foliation.

We can now define pre-sections for an SRF with locally closed leaves just as we did for a G-manifold:

Definition 4.1.1. Let M be a Riemannian manifold and let $\mathcal{F}$ be singular Riemannian foliation with locally closed leaves on M. A submanifold $\Sigma \subseteq M$ is called a **pre-section** for $\mathcal{F}$ if the following three conditions are satisfied:

(i) Σ is complete, connected, embedded and totally geodesic in M,
(ii) Σ intersects every leaf of $\mathcal{F}$,
(iii) for every regular leaf $F \in \mathcal{F}$ and all points $p \in \Sigma \cap F$ we have $\nu_p(F) \subseteq T_p\Sigma$.

If $p \in M$ is a point which lies on a regular leaf, then a pre-section of least dimension which contains p is called a **minimal section through** p.

The properties of a singular Riemannian foliation together with the assumption that the leaves are locally closed in M yield the following generalization of Lemma 1.1.6:

Lemma 4.1.2. *If $F \in \mathcal{F}$ is an arbitrary leaf, then for every $q \in F$ the set* $\exp_q(\nu_q(F))$ *intersects any other leaf of $\mathcal{F}$.*

Let p be a point in some regular leaf. Using the above lemma, it is easy to see that, if Σ_1 and Σ_2 are two pre-sections through p, then the connected component of $\Sigma_1 \cap \Sigma_2$ which contains p is again a pre-section. Hence, through every regular point p passes a unique minimal section. From here on it is seems quite natural to assume that all results on minimal sections of isometric group actions should carry over in one way or another to the case of minimal sections of SFRs with locally closed leaves.

However, a noteworthy point is that a corresponding definition of canonical fat sections (Definition 1.1.11) or cores ([**GS00**]) makes no sense for general singular Riemannian foliations with locally closed leaves. Hence, the minimal sections we defined above serve as a generalization of canonical fat sections.

CHAPTER 5

Copolarity of Actions induced by Polar Actions on Symmetric Spaces

In this section, our aim is to compute the copolarity of actions on compact lie groups which are associated to certain polar actions on symmetric spaces of compact type.

We first recall some notions for symmetric spaces in order to fix our notation (for the details we refer to Helgason's monograph [**Hel01**]). A *symmetric pair* (G, K) consists of a Lie group G and a closed subgroup K such that an involutive automorphism $\sigma : G \to G$ exists with $\mathrm{Fix}(\sigma)^\circ \subseteq K \subseteq \mathrm{Fix}(\sigma)$. If in addition $\mathrm{Ad}_G(K)$ is compact, then the pair is called *Riemannian*. The involution σ induces an involution of the Lie algebra $\mathfrak{g}$ of G (also denoted by σ). This yields the so called Cartan-decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is the $(+1)$- and $\mathfrak{p}$ the (-1)-eigenspace of σ. Note that $\mathfrak{k}$ is at the same time the Lie algebra of K. If $\pi : G \to G/K$ denotes the canonical projection, then $T_{eK}G/K$ is identified with $\mathfrak{p}$ via $d\pi(e)$.

It is well known that the complete connected totally geodesic submanifolds Σ of G/K correspond bijectively to the Lie triple systems $\mathfrak{m}$ of $\mathfrak{p}$. Furthermore, $\mathfrak{s} := [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is a Lie subalgebra of $\mathfrak{g}$ and its corresponding Lie subgroup S of G together with $L := S_{eK} = S \cap K$ form a Riemannian symmetric pair. We have $\Sigma = \pi(S) \simeq S/L$, and S is the smallest subgroup of G that acts transitively on Σ.

We could not find the following statement in the literature, although it seems natural to consider.

Lemma 5.1.1. *Let (G, K) be a Riemannian symmetric pair with G compact. Suppose that $\Sigma \subseteq G/K$ is a complete, connected and totally geodesic submanifold. Then Σ is embedded in G/K if and only if S is closed in G.*

Proof. If S is closed in G, then S acts isometrically on G/K. Therefore, its orbit $S \cdot eK = \Sigma$ is an embedded submanifold of G/K.

Conversely, $\mathfrak{s} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$ is a compact Lie algebra because $\mathfrak{g}$ is. Let $\mathfrak{s} = \mathfrak{z}(\mathfrak{s}) \oplus [\mathfrak{s}, \mathfrak{s}]$ denote the decomposition of $\mathfrak{s}$ into its center $\mathfrak{z}(\mathfrak{s})$ and its semisimple part $[\mathfrak{s}, \mathfrak{s}]$. It follows that $\exp([\mathfrak{s}, \mathfrak{s}])$ is closed in G ([**Mos50**], p. 615) and hence compact. The same holds for

$$\exp([\mathfrak{m}, \mathfrak{m}]) = (\exp([\mathfrak{s}, \mathfrak{s}]) \cap \mathrm{Fix}(\sigma))^\circ.$$

Since Σ is embedded in G/K, its image under $\phi : G/K \to G$, $gK \mapsto g\sigma(g)^{-1}$ yields the compact submanifold $\exp(\mathfrak{m})$ of G. Note that $\exp(\mathfrak{m})$ is closed under forming rational powers of elements. Applying σ to an element of $\exp(\mathfrak{m})$ has the same effect as forming its inverse. Clearly, $\exp(\mathfrak{m})$ projects onto Σ under π.

We next claim that every element $s \in S$ can be written as a product $s = xy$ where $x \in \exp(\mathfrak{m})$ and $y \in \exp([\mathfrak{m}, \mathfrak{m}])$. In fact, let $s \in S$ be arbitrary and let s_t be a path from e to s. Let then x_t be a path in $\exp(\mathfrak{m})$ which starts in e and satisfies

$$x_t^2 = s_t\sigma(s_t)^{-1} = \phi \circ \pi(s_t) \in \exp(\mathfrak{m})$$

for all t. We claim that $y_t := x_t^{-1} s_t$ is a path in S, which is fixed by σ. In fact,

$$\begin{aligned} \sigma(y_t) &= \sigma(x_t^{-1})\sigma(s_t) = x_t \sigma(s_t) s_t^{-1} s_t \\ &= x_t (x_t^2)^{-1} s_t = x_t^{-1} s_t = y_t. \end{aligned}$$

This shows $y \in \exp([\mathfrak{m}, \mathfrak{m}])$. It follows that $S = \exp(\mathfrak{m}) \exp([\mathfrak{m}, \mathfrak{m}])$ is closed in G. $\square$

Now let G be a compact Lie group equipped with a bi-invariant metric. Viewed as a symmetric space, G can be identified with $(G{\times}G)/\Delta(G)$, where $\Delta(G) = \{(g, g) \mid g \in G\}$. So $g \in G$ is identified with the coset $[g, e] = \{(gh, h) \mid h \in G\}$. Let $N \subseteq G$ be a totally geodesic submanifold of G. Then $\mathfrak{n} := T_e N$ is a Lie triple system of $\mathfrak{g} = L(G)$. As before, a transitive group of isometries of N can be realized as a subgroup of $G \times G$: Let $\tilde{\mathfrak{n}} := \{(X, -X) \mid X \in \mathfrak{n}\} \subset \mathfrak{g} \times \mathfrak{g}$. Obviously, $\tilde{\mathfrak{n}}$ is a Lie triple system, hence we may consider the Lie subalgebra

$$\mathfrak{s} := [\tilde{\mathfrak{n}}, \tilde{\mathfrak{n}}] \oplus \tilde{\mathfrak{n}} = \triangle([\mathfrak{n}, \mathfrak{n}]) \oplus \tilde{\mathfrak{n}} = \langle ([X, Y] + Z, [X, Y] - Z) \mid X, Y, Z \in \mathfrak{n} \rangle.$$

Lemma 5.1.2. *Let $S \subseteq G{\times}G$ be the connected Lie subgroup of $G{\times}G$ with $L(S) = \mathfrak{s}$. Then S is a group of isometries of N and we have for all $(g, h) \in S$: $g \cdot N \cdot h^{-1} = N$ and therefore*

$$T_{gh^{-1}} N = g \cdot T_e N \cdot h^{-1} = g \cdot \mathfrak{n} \cdot h^{-1}.$$

In particular, $(\exp(X), \exp(-X)) \in S$ for all $X \in \mathfrak{n}$, and hence

$$T_{\exp(2X)} N = \exp(X) \cdot \mathfrak{n} \cdot \exp(X).$$

Let now (G, K) be a Riemannian symmetric pair with G compact. The reason for all the preliminary work is the following: Whenever H is a subgroup of G, the action ψ of H on G/K by left translation lifts to an action φ of $H{\times}K$ on G in the following way: $(h, k) \cdot g := hgk^{-1}$. If $\mathrm{pr}_H : H \times K \to H$ denotes the projection onto the first factor, then the situation fits into the following commutative diagram:

$$\begin{array}{ccc} (H{\times}K) \times G & \xrightarrow{\ \varphi\ } & G \\ {\scriptstyle \mathrm{pr}_H \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ H \times G/K & \xrightarrow[\ \psi\ ]{} & G/K. \end{array}$$

The lift φ has certain distinctive features:

Proposition 5.1.3.

(i) π maps φ-orbits onto ψ-orbits: $\pi(HgK) = H \cdot (gK)$. The orbit spaces $(H{\times}K)\backslash G$ and $H\backslash G/K$ are canonically homeomorphic via $HgK \mapsto H \cdot (gK)$.

(ii) For the isotropy subgroups of both actions we have

$$\begin{aligned} (H{\times}K)_g &= \{(h, g^{-1}hg) \mid h \in H \cap gKg^{-1}\} \text{ and} \\ H_{gK} &= H \cap gKg^{-1}. \end{aligned}$$

Therefore, both groups are isomorphic via $\mathrm{pr}_H : (H{\times}K)_g \to H_{gK}$, $(h, k) \mapsto h$.

(iii) The actions ψ and φ have the same cohomogeneity. More precisely, the slice of φ through $g \in G$ is given by $\nu_g(HgK) = g \cdot (\mathrm{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{k}^\perp)$. The φ-orbits contain the fibres of π and since they are mapped onto the orbits of ψ, the slice through $g \cdot p$ is given by $\nu_{gK}(H \cdot (gK)) = d\pi(g)(\nu_g(HgK))$. Furthermore, the slice representation $((H{\times}K)_g, \nu_g(HgK))$ of φ is equivariantly isomorphic to the slice representation $(H_{gK}, \nu_{gK}(H \cdot (gK)))$ of ψ.

For the details we refer to [**GT02**].

THEOREM 5.1.4. *Let (G, K) be a Riemannian symmetric pair with compact G. Let H be a closed subgroup of G. If $(H, G/K)$ is polar and Σ is a section through eK with $\mathfrak{m} := T_{eK}\Sigma$, then*

$$\operatorname{copol}(H \times K, G) = \dim([\mathfrak{m}, \mathfrak{m}]).$$

A minimal section through e is given by the connected Lie subgroup S corresponding to the Lie subalgebra $\mathfrak{s} := [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m}$.

PROOF. We first show that S contains a minimal section. In a second step we show that each minimal section contains S. Without loss of generality we may assume that e is regular with respect to the $(H \times K)$-action.

Clearly, S is totally geodesic and complete as it is a Lie subgroup of G. Since Σ is embedded in G/K, Lemma 5.1.1 shows that S is embedded in G. Furthermore, since S maps under the projection $\pi : G \to G/K$ onto Σ, it intersects every orbit. Now suppose that $g \in S$ is regular with respect to the action φ. Then $\pi(g) = gK$ is regular with respect to ψ and the normal space $\nu_g(HgK)$ to the orbit HgK in g is given by

$$(\mathfrak{h}^\perp \cdot g) \cap (g \cdot \mathfrak{p}) = g \cdot (\operatorname{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{p}).$$

However, since the H-action on G/K is polar, we know that $\operatorname{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{p} = \mathfrak{m}$ (see [**Gor04a**, p. 195]). Since S is a Lie subalgebra of G, its tangent space in g is given by left translation of $\mathfrak{s}$ with g, i.e. $T_gS = g \cdot \mathfrak{s}$. Combining this with the above, we obtain:

$$\nu_g(HgK) = g \cdot (\operatorname{Ad}_{g^{-1}}(\mathfrak{h}^\perp) \cap \mathfrak{p}) = g \cdot \mathfrak{m} \subseteq g \cdot \mathfrak{s} = T_gS.$$

We have therefore established that any minimal section is contained in S.

Now assume that $N \subseteq S$ is a minimal section through e and write $\mathfrak{n} := T_eN$. In particular we have the inclusion $\nu_g(HgK) = g \cdot \mathfrak{m} \subseteq T_gN$ for all regular $g \in N$ and therefore $\mathfrak{m} \subseteq \mathfrak{n}$. Since the set of regular points of the $H \times K$-action on G is open and dense in G and e is assumed to be a regular point, there is a small $\varepsilon > 0$, such that for all $t \in (-\varepsilon, \varepsilon)$ and $X \in \mathfrak{m}$ with unit length, the value of $g^2 = \exp(t \cdot X)$ is regular. Applying the tangent space formula from lemma 5.1.2 it follows that $g^2 \cdot \mathfrak{m} \subseteq T_{g^2}N = g \cdot \mathfrak{n} \cdot g$, or in other words:

$$\operatorname{Ad}_g(\mathfrak{m}) = \operatorname{Ad}_{\exp(t/2 \cdot X)}(\mathfrak{m}) \subseteq \mathfrak{n}.$$

Since $\operatorname{Ad}_{\exp(X)} = e^{\operatorname{ad}_X}$, it follows for all $Y \in \mathfrak{m}$ and $t \in \mathbf{R}$:

$$\operatorname{Ad}_{\exp(t/2 \cdot X)}(Y) = e^{t/2 \cdot \operatorname{ad}_X}(Y) \in \mathfrak{n}.$$

Differentiating in $t = 0$ yields that $\operatorname{ad}_X Y = [X, Y] \in \mathfrak{n}$. By linearity of the Lie bracket we may thus conclude that $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{n}$ and therefore $\mathfrak{s} \subseteq \mathfrak{n}$ which in turn implies $S \subseteq N$. □

REMARK 5.1.5. The Lie group $S \subseteq G$ in the theorem above is a lift of the section $\Sigma \subseteq G/K$, which is minimal in the sense that it is a minimal section of the action of $(H \times K)$ on G. We have proved along the lines that, even if the action of H on G/K is not polar, we still have the following inequality:

$$\operatorname{copol}(H \times K, G) \leq \operatorname{copol}(H, G/K) + \dim([\mathfrak{m}, \mathfrak{m}]).$$

Here $\mathfrak{m}$ is the tangent space of a minimal section through eK. To be more precise, if $\Sigma \subseteq G/K$ denotes a minimal section with respect to the action ψ and $\mathfrak{m} = T_{eK}\Sigma$, then $S := \exp([\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{m})$ contains a minimal section of the action φ.

COROLLARY 5.1.6. *With the assumptions and notation as in Theorem 5.1.4:*

(i) *Assuming that* ψ *is polar, then* φ *is polar if and only if it is hyperpolar.*
(ii) *If* $H = \{e\}$*, then* $\operatorname{copol}(K, G) = \dim([\mathfrak{p}, \mathfrak{p}])$ *(the action is by right translation), and the copolarity is trivial.*

We can also describe the relation between the generalized Weyl group of Σ and the fat Weyl group of S:

PROPOSITION 5.1.7. *In addition to the assumptions of Theorem 5.1.4 let* e *be regular.*

(i) $N_{H\times K}(S) = \{(h, k) \in H\times K \mid hk^{-1} \in S\}$ *and* $Z_{H\times K}(S) = \triangle(H \cap K)$.
(ii) $\operatorname{pr}_H(N_{H\times K}(S)) = N_H(\Sigma)$ *and* $\operatorname{pr}_H(Z_{H\times K}(S)) = Z_H(\Sigma)$.
(iii) *The following diagram is commutative*

$$\begin{array}{ccc} N(S) & \xrightarrow{\operatorname{pr}_H} & N(\Sigma) \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ W(S) & \xrightarrow[\operatorname{pr}_W]{} & W(\Sigma), \end{array}$$

where pr_W *denotes the homomorphism induced by* $p_2 \circ \operatorname{pr}_H$*. Hence,* $W(S)$ *is mapped canonically onto* $W(\Sigma)$ *and has at least as many connected components as the latter.*
(iv) $N(\Sigma) \simeq N(S)/(\{e\}\times(K \cap S))$ *and* $W(\Sigma) \simeq W(S)/p_1(\{e\}\times(K \cap S))$.

PROOF. The description of the normalizer in (i) follows from property (D) of a fat section. The centralizer of a minimal section coincides with the isotropy group of any $(H \times K)$-regular point of S. Since e is a regular point, $Z_{H\times K}(S) = \Delta(H \cap K)$ follows from Proposition 5.1.3 (ii).

Let $(h, k) \in N_{H\times K}(S)$ be arbitrary. If we apply π to the equation $hSk^{-1} = S$ we obtain $h \cdot \Sigma = \Sigma$. This proves $h \in N_H(\Sigma)$. Conversely, assume that $h \in N_H(\Sigma)$ is an arbitrary element. In particular, $hK \in \Sigma$. Since $\pi(S) = \Sigma$, we can find an element $s \in S$ with $hK = sK$. It follows that $k := s^{-1}h \in K$, which we rewrite as $hk^{-1} = s \in S$. Since e is a regular point for the action φ, by assumption, we conclude that $(h, k) \in N_{H\times K}(S)$ by property (D) of a fat section. This completes the proof that pr_H maps $N_{H\times K}(S)$ onto $N_H(\Sigma)$.

The statement in (iii) is easily verified. The same is true in the case of (iv). In fact, the kernel of pr_H is given by

$$\ker(\operatorname{pr}_H) = \{(h, k) \in N(S) \mid h = 1,\ k \in S\} = \{e\}\times(K \cap S).$$

□

REMARK 5.1.8. With the assumptions made in Theorem 5.1.4, Proposition 5.1.3 (iii) shows that the assumptions made in Theorem 2.7.3 are satisfied. I.e. the basic forms on S and G are naturally isomorphic to each other. In particular, the smooth $(H \times K)$-invariant functions on G correspond to the smooth $N(S)$-invariant functions on S.

The polar, non-hyperpolar actions on compact rank one symmetric spaces yield interesting examples where Theorem 5.1.4 is applicable. These actions have been classified in [**PT99**]. We will now discuss an example in greater detail in order to illustrate our results so far.

Example: A torus action on $\mathbf{P}_2(\mathbf{C})$. In the following we identify the cyclic groups $\mathbf{Z}_n$ as canonical subgroups of $S^1 \subset \mathbf{C}$, that is $\mathbf{Z}_n = \{e^{\frac{2\pi k i}{n}} \mid k \in \mathbf{Z}\}$. Consider the complex projective 2-space

$$M = \mathbf{P}_2(\mathbf{C}) = \{[z_1 : z_2 : z_3] \mid (z_1, z_2, z_3) \in S^5 \subseteq \mathbf{C}^3\}.$$

Viewed as a symmetric space, M can be naturally identified with the quotient

$$G/K = \mathbf{SU}(3)/S(\mathbf{U}(1)\times\mathbf{U}(2)).$$

A point $[z] = [z_1 : z_2 : z_3] \in M$ is then mapped to the coset $gK \in G/K$ in the following manner: Let $v_1(z)$ and $v_2(z)$ be an orthonormal base of the orthocomplement of z with respect to the standard hermitian form on $\mathbf{C}^3$, such that the 3×3-matrix g, whose columns are given by $z, v_1(z)$ and $v_2(z)$ has determinant one. This process yields a well defined smooth map, independent of the choices for v_1 and v_2. The inverse map is given by assigning to the first column z of a representative g of the coset gK, the point $[z] \in \mathbf{P}_2(\mathbf{C})$. Now $H = T^2 = \{\mathrm{diag}(\lambda, \mu, \overline{\lambda\mu}) | \lambda, \mu \in S^1\}$ is a maximal torus in $\mathbf{SU}(3)$ and it acts on M by

$$\mathrm{diag}(\lambda, \mu, \overline{\lambda\mu}) \cdot [z_1 : z_2 : z_3] := [\lambda z_1 : \mu z_2 : \overline{\lambda\mu} z_3].$$

Since H is an Abelian group, conjugation is always the identity map. Hence, every isotropy type consists of a single group. There are the following inclusions amongst the isotropy subgroups:

$$\begin{array}{ccccccc}
 & & H_{[z_1:z_2:0]} & & & & \\
 & \nearrow & & \searrow & & & \\
H_{[z_1:z_2:z_3]} \simeq \mathbf{Z}_3 & \hookrightarrow & H_{[z_1:0:z_2]} & \hookrightarrow & H_{[1:0:0]} = H_{[0:1:0]} = H_{[0:0:1]} = H, \\
 & \searrow & & \nearrow & & & \\
 & & H_{[0:z_2:z_3]} & & & &
\end{array}$$

where each $z_1, z_2, z_3 \neq 0$. Every isotropy of a point with exactly one "0" is isomorphic to an S^1. The orbit space is homeomorphic to a closed triangle, whose vertices correspond to the three fixed points and whose edges correspond to the other sets of singular points. The interior points correspond to the regular points. Also note that the action is almost effective with kernel being equal to the principal isotropy group $\mathbf{Z}_3 \simeq \{(\lambda, \lambda, \lambda) | \lambda \in \mathbf{Z}_3\}$.

It is not difficult to see that the action ψ of T^2 on $\mathbf{P}_2(\mathbf{C})$ is polar. A section through $[1 : 1 : 1]$ is given by the real projective space $\Sigma := \mathbf{P}_2(\mathbf{R})$, sitting naturally in $\mathbf{P}_2(\mathbf{C})$. In particular, the cohomogeneity of ψ is 2. We now determine the generalized Weyl group of ψ. The normalizer of Σ is given by

$$N_H(\Sigma) = \{\mathrm{diag}(\lambda, \lambda, \bar{\lambda}^2) | \lambda \in \mathbf{Z}_6\} \cup \{\mathrm{diag}(\lambda, -\lambda, -\bar{\lambda}^2) | \lambda \in \mathbf{Z}_6\} \simeq \mathbf{Z}_2\times\mathbf{Z}_6.$$

Then the centralizer is

$$Z_H(\Sigma) = \{\mathrm{diag}(\lambda, \lambda, \lambda) | \lambda \in \mathbf{Z}_3\}.$$

Thus $Z_H(\Sigma)$ is entirely contained in the $\mathbf{Z}_6$-factor of the normalizer, which in turn implies that the generalized Weyl group $W(\Sigma)$ is isomorphic to $\mathbf{Z}_2\times\mathbf{Z}_2$. So the whole action of W on Σ is generated by the assignments

$$[x_1 : x_2 : x_3] \mapsto [-x_1 : -x_2 : x_3] \quad \text{and} \quad [x_1 : x_2 : x_3] \mapsto [x_1 : -x_2 : -x_3].$$

According to Theorem 5.1.4, the action φ of $H \times K$ on G has $S = \mathbf{SO}(3)$ as a minimal section and the copolarity of φ is 1. We write a general element (h,k) of $H\times K$ as a pair of matrices:

$$h = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \overline{\lambda\mu} \end{pmatrix} \text{ and } k = \begin{pmatrix} \tau^2 & 0 & 0 \\ 0 & \bar\tau a & \bar\tau b \\ 0 & -\overline{\tau b} & \overline{\tau a} \end{pmatrix},$$

where $\lambda, \mu, \tau \in S^1$ and $a, b \in \mathbf{C}$ with $|a|^2 + |b|^2 = 1$. A regular point of the $(H\times K)$action on G is given by

$$g = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & \sqrt{3} & 1 \end{pmatrix}$$

This follows from the recipe given above. Noting that $g \in S$, we may compute the normalizer $N(S)$ of S as the set of all pairs (h,k) such that $hgk^{-1} \in S$, which in our current situation is equivalent to the condition that hgk^{-1} is a real valued matrix. A computation shows

$$hgk^{-1} = \frac{1}{\sqrt{6}} \begin{pmatrix} \lambda\bar\tau^2\sqrt{2} & \lambda\tau(\bar b - \sqrt{3}\bar a) & \lambda\tau(a + \sqrt{3}b) \\ \mu\bar\tau^2\sqrt{2} & -2\mu\tau\bar b & -2\mu\tau a \\ \overline{\lambda\mu}\bar\tau^2\sqrt{2} & \overline{\lambda\mu}\tau(\bar b + \sqrt{3}\bar a) & \overline{\lambda\mu}\tau(a - \sqrt{3}b) \end{pmatrix}.$$

From this it follows that the normalizer is $N_{H\times K}(S) =$

$$\left\{ \left(\operatorname{diag}(\varepsilon_1\delta\tau^2, \varepsilon_2\delta\tau^2, \varepsilon_1\varepsilon_2\tau^2), \begin{pmatrix} \delta\tau^2 & & \\ & \tau^2\cos(\alpha) & \tau^2\sin(\alpha) \\ & -\delta\tau^2\sin(\alpha) & \delta\tau^2\cos(\alpha) \end{pmatrix} \right) \middle| \begin{array}{c} \varepsilon_1, \varepsilon_2, \delta \in \mathbf{Z}_2, \\ \tau \in \mathbf{Z}_3, \alpha \in \mathbf{R} \end{array} \right\},$$

whereas the centralizer is given by

$$Z_{H\times K}(S) = \{(\operatorname{diag}(\lambda,\lambda,\lambda), \operatorname{diag}(\lambda,\lambda,\lambda)) | \lambda \in \mathbf{Z}_3\}.$$

Therefore, $N_{H\times K}(S) \simeq \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_3 \times \mathbf{O}(2)$ via the isomorphism

$$(\varepsilon_1, \varepsilon_2, \tau, A) \mapsto \left(\operatorname{diag}(\varepsilon_1 \det(A)\tau^2, \varepsilon_2 \det(A)\tau^2, \varepsilon_1\varepsilon_2\tau^2), \begin{pmatrix} \det(A)\tau^2 & 0 \\ 0 & \bar\tau \cdot A \end{pmatrix} \right)$$

and $Z_{H\times K}(S) \simeq \mathbf{Z}_3$ is contained in the $\mathbf{Z}_3$-factor of the normalizer. This yields

$$W(S) \simeq \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{O}(2).$$

CHAPTER 6

An Infinite Dimensional Isometric Action

In [**GOT04**] it is shown that one may easily construct actions with prescribed fat sections in the following way: Take a polar action (G_1, M_1) with section Σ_1 and any action (G_2, M_2) whose principal orbit has dimension k. Then

$$(G, M) := (G_1 \times G_2, M_1 \times M_2)$$

has $\Sigma := \Sigma_1 \times M_2$ as a k-section. If (G_1, M_1) is an infinite dimensional isometric Fredholm action[1] and G_2 and M_2 are finite dimensional, then it follows that $\Sigma_1 \times M_2$ has finite dimension. Hence, $\mathrm{copol}(G, M)$ is also finite in this case. Besides these constructed examples, one might ask if there exist isometric Fredholm actions of infinite dimensional Lie groups on infinite dimensional manifolds with finite dimensional minimal sections. A natural candidate is the action by gauge transformation, which we describe in the following (see [**PT88, TT95**]). Let G be a compact Lie group with a bi-invariant Riemannian metric and let H and K be closed subgroups of G. The action by **gauge transformation** is defined as:

$$* : \mathcal{P}(G, H \times K) \times V \to V, \; (g, u) \mapsto \mathrm{Ad}_g(u) - g'g^{-1} = gug^{-1} - g'g^{-1}.$$

Here $\mathcal{P}(G, H \times K)$ denotes the Hilbert-Lie group of H^1 paths $g : I \to G$, emanating from $H \subseteq G$ and ending in $K \subseteq G$, and we let $V = H^0(I; \mathfrak{g}) = L^2(I; \mathfrak{g})$ denote the Hilbert space of L^2 integrable paths $u : I \to \mathfrak{g}$ in $\mathfrak{g} = L(G)$, equipped with the inner product

$$\langle u, v \rangle_0 := \int_0^1 \langle u(t), v(t) \rangle_1 \, dt \text{ with } \langle \cdot, \cdot \rangle_1 \ \mathrm{Ad}_G\text{-invariant}.$$

We give a short summary of several facts concerning the gauge transformation without proofs:

(i) $*$ is a smooth isometric Fredholm action by affine transformations.

(ii) The action of $\mathcal{P}(G, e \times G)$ on V is simply transitive. In other words, the orbit map $\alpha : \mathcal{P}(G, e \times G) \to V, \; g \mapsto g * \hat{0} = -g'g^{-1}$ is a diffeomorphism.

(iii) The map $\phi : V \to G, \; u \mapsto \alpha^{-1}(u)(1)$, obtained by mapping u into the group $\mathcal{P}(G, e \times G)$ and then evaluating the resulting path at $t = 1$, is a surjective Riemannian submersion.

(iv) The following diagram is commutative:

$$\begin{array}{ccc} \mathcal{P}(G, H \times K) \times V & \xrightarrow{\;\;*\;\;} & V \\ \Big\downarrow {\scriptstyle \pi \times \phi} & & \Big\downarrow {\scriptstyle \phi} \\ (H \times K) \times G & \xrightarrow[\;\;\varphi\;\;]{} & G, \end{array}$$

where π denotes the map $\pi : \mathcal{P}(G, H \times K) \to H \times K, \; g \mapsto (g(0), g(1))$. Thus, ϕ is equivariant with respect to π. Furthermore, the isotropy subgroups of both actions are isomorphic via π.

[1]A (proper) isometric action (G, M) is called **Fredholm** if $\mathrm{cohom}(G, M) < \infty$.

(v) For $u \in V$ we have that $\mathcal{P}(G, H\times K) * u = \phi^{-1}((H\times K)\cdot \phi(u))$.

(vi) The fibres of ϕ coincide with the orbits of $\Omega_e(G) = \mathcal{P}(G, e\times e)$. That is, for any $u \in V$, we have:

$$\phi^{-1}(\phi(u)) = \Omega_e(G) * u.$$

In particular, we have $\phi^{-1}(\exp(Y)) = \Omega_e(G) * \hat{Y}$ for all $Y \in \mathfrak{g}$.

(vii) For $u \in V$ let $\tilde{M} := \mathcal{P}(G, H\times K) * u$. The tangent space on $\tilde{M}$ in u is:

$$T_u(\tilde{M}) = \{[\xi, u] - \xi' \mid \xi \in H^1(I;\mathfrak{g}), \xi(0) \in \mathfrak{h}, \xi(1) \in \mathfrak{k}\}.$$

(viii) If $h \in \mathcal{P}(G, H\times K)$ with $u = h * \hat{0}$ and $x = \phi(u)$, then:

$$\nu_u(\tilde{M}) = \{hbx^{-1}h^{-1} \mid b \in \nu_x(HxK)\} = \mathrm{Ad}_{hx}(\mathrm{Ad}_x^{-1}(\mathfrak{h}^\perp) \cap \mathfrak{k}^\perp).$$

Hence, $\nu_{\hat{0}}(\tilde{M})$ is the set of constant paths in $\mathrm{Ad}_x^{-1}(\mathfrak{h}^\perp) \cap \mathfrak{k}^\perp = \nu_x(HxK)$.

The next lemma shows that Lemma 1.1.6 also holds for the action by gauge transformation.

Lemma 6.1.1. *$\nu_{\hat{0}}(\mathcal{P}(G, H\times K) * \hat{0})$ intersects all orbits of the $\mathcal{P}(G, H\times K)$-action on V.*

Proof. Let $\mathcal{P}(G, H\times K) * u$ be an arbitrary orbit and put $x := \phi(u)$. Now consider $X \in \mathrm{Ad}_x^{-1}(\mathfrak{h}^\perp) \cap \mathfrak{k}^\perp = \nu_x(HxK)$ such that $(H\times K)\cdot x = (H\times K)\cdot \exp(X)$. Such an X exists, because $\exp(\nu_x(HxK))$ intersect every $(H\times K)$-orbit on G. Using (v) above we obtain

$$\mathcal{P}(G, H\times K) * u = \phi^{-1}((H\times K)\cdot \phi(u)) = \phi^{-1}((H\times K)\cdot \exp(X)).$$

It follows, using (vi) above, that $\phi^{-1}(\exp(X)) = \Omega_e(G) * \hat{X} \subseteq \mathcal{P}(G, H\times K) * u$. □

In the following, we assume that (G, K) is a Riemannian symmetric pair with compact G and that $H \subseteq G$ is a closed subgroup. As usual, we identify $T_{eK}G/K$ with $\mathfrak{p}$ from the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Our aim is to show that if H acts polarly on G/K, then the action by gauge transformation is either polar (and hence hyperpolar), or it has infinite dimensional minimal sections and hence infinite copolarity. This gives a partial negative answer to the question we asked at the beginning of this section.

If $\mathfrak{l} \subseteq \mathfrak{g}$ is an arbitrary subset of $\mathfrak{g}$, we denote by $\hat{\mathfrak{l}} \subseteq V$ the set of constant paths in V with value in $\mathfrak{l}$. It is clear that if $\mathfrak{l}$ is a subspace (or subalgebra) of $\mathfrak{g}$, then $\hat{\mathfrak{l}}$ is a subspace (resp. subalgebra) of V which is canonically isomorphic to $\mathfrak{l}$. In particular, $\mathfrak{g}$ is embedded into V via $\hat{\mathfrak{g}}$.

Lemma 6.1.2. *Suppose that H acts polarly on G/K and let $\mathfrak{m} \subseteq \mathfrak{p}$ be the tangent space of a section through eK. If eK is H-regular, then every fat section $\mathcal{S} \subseteq V$ of the $\mathcal{P}(G, H\times K)$-action on V through $\hat{0}$ contains the linear subspace*

$$\mathrm{span}\{t \mapsto e^{(1-t)\cdot \mathrm{ad}_X}(Y) \mid X \in \mathfrak{m} \text{ regular},\ Y \in \mathfrak{m}\}.$$

*Here we call an element $X \in \mathfrak{g}$ **regular**, if $\exp(X) \in G$ is regular with respect to the $(H\times K)$-action on G.*

Proof. Since $\mathcal{S} \subseteq V$ is supposed to be a fat section through $\hat{0}$, it is complete, connected, and totally geodesic in V. Hence, $\mathcal{S}$ has to be a linear subspace of V.

From $\hat{\mathfrak{m}} = \nu_{\hat{0}}(\mathcal{P}(G, H\times K) * \hat{0}) \subseteq T_{\hat{0}}\mathcal{S} = \mathcal{S}$ and property (C) of a fat section we may conclude that for all regular $\hat{X} \in \hat{\mathfrak{m}}$ we have $\nu_{\hat{X}}(\mathcal{P}(G, H\times K) * \hat{X}) \subseteq \mathcal{S}$. Let $h \in \mathcal{P}(G, e\times G)$ be the path defined by $h(t) := \exp(-t\cdot X)$. Then $X = h * \hat{0}$ and

$\phi(\hat{X}) = h(1)^{-1} = \exp(X)$ is a regular element for the $(H\times K)$-action on G. Since the action of H on G/K is polar, it follows that $\mathrm{Ad}_{\exp(-X)}(\mathfrak{h}^\perp) \cap \mathfrak{p} = \mathfrak{m}$. From (viii) above we thus conclude that

$$\nu_{\hat{X}}(\mathcal{P}(G, H\times K) * \hat{X}) = \mathrm{Ad}_{h\exp(X)}(\mathrm{Ad}_{\exp(-X)}(\mathfrak{h}^\perp) \cap (\mathfrak{p})) = \mathrm{Ad}_{h\exp(X)}(\mathfrak{m}).$$

Since $h\exp(X) = \exp(-t\cdot X)\exp(X) = \exp((1-t)X)$ and $\mathrm{Ad}_{\exp(X)} = e^{\mathrm{ad}_X}$, we obtain $\mathrm{Ad}_{h\exp(X)}(\mathfrak{m}) = \{t \mapsto e^{(1-t)\cdot\mathrm{ad}_X}(Y) \mid Y \in \mathfrak{m}\}$. This fact together with $\mathcal{S}$ being linear completes the proof. □

THEOREM 6.1.3. *Let (G, K) be a Riemannian symmetric pair with compact G and let $H \subseteq G$ be a closed subgroup. Supposed that the action of H on G/K is polar, then the following are equivalent:*

(i) $\mathrm{copol}(\mathcal{P}(G, H\times K), V) < \infty$.
(ii) $\mathrm{copol}(\mathcal{P}(G, H\times K), V) = 0$.
(iii) The action of $\mathcal{P}(G, H\times K)$ on V is hyperpolar.
(iv) The action of $H\times K$ on G is hyperpolar.
(v) The action of H on G/K is hyperpolar.

PROOF. The equivalence of (iii), (iv) and (v) is well known. Furthermore, since sections in V are automatically flat and copol $= 0$ implies that an action is polar, (iii) is equivalent to (ii). Certainly, (ii) implies (i).

Let Σ be a section of $(H, G/K)$ through eK and assume that eK is H-regular. Then e is $H\times K$-regular and $\hat{0}$ is regular with respect to the action by gauge transformation. Put $\mathfrak{m} := T_{eK}\Sigma$. We now show that if $\mathrm{copol}(\mathcal{P}(G, H\times K), V) \neq 0$ then The copolarity must already be infinite. Let $X, Y \in \mathfrak{m}$ be elements with

$$(\mathrm{ad}_X)^2(Y) = -\delta Y \neq 0 \text{ and } \|Y\| = 1.$$

Such elements exist, since $\mathfrak{m}$ is a Lie triple system and $\mathfrak{m}$ is not Abelian. Otherwise, $\hat{\mathfrak{m}}$ would be a 0-section, which contradicts our assumption. Recalling that 0 is regular, there is a ball of regular elements around $0 \in \mathfrak{m}$. In fact, this is clear since $e \in G$ is regular and the set of regular points is open in G. We may thus further assume that $\varepsilon\cdot X$ is regular for all $\varepsilon \in [0, 1]$. By lemma 6.1.2, every minimal section $\mathcal{S}$ of $(\mathcal{P}(G, H\times K), V)$ contains the infinite family

$$\mathcal{M} := \{t \mapsto e^{(1-t)/p_n\cdot\mathrm{ad}_X}(Y) \mid n \in \mathbf{N}\} \subseteq V,$$

where p_n denotes the n-th odd prime number. We claim that $\mathcal{M}$ is linearly independent. Since every equivalence class in $L^2(I; \mathfrak{g})$ has at most one continuous representative, it suffices to show that the family $\mathcal{M}$ is linearly independent as a subset of $\mathcal{C}(I; \mathfrak{g})$. Furthermore, all members of $\mathcal{M}$ are analytic maps which can be extended analytically to $\mathbf{R}$ and so we only need to show that they are linearly independent when viewed as functions on $\mathbf{R}$.

Now assume there exist $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ such that

$$t \mapsto \sum_{k=1}^{n} \lambda_k e^{(1-t)/p_k\cdot\mathrm{ad}_X}(Y) = 0.$$

For any $s := (1-t) \in \mathbf{R}$, we then have:

$$\begin{aligned} 0 &= \left\langle \sum_{k=1}^{n} \lambda_k e^{s/p_k \cdot \mathrm{ad}_X}(Y), Y \right\rangle = \sum_{k=1}^{n} \lambda_k \langle e^{s/p_k \cdot \mathrm{ad}_X}(Y), Y\rangle \\ &= \sum_{k=1}^{n} \lambda_k \sum_{l=0}^{\infty} \frac{s^l}{p_k^l \cdot l!} \langle (\mathrm{ad}_X)^l (Y), Y\rangle \\ &= \sum_{k=1}^{n} \lambda_k \sum_{l=0}^{\infty} \frac{s^{2l}}{p_k^{2l} \cdot (2l)!} \delta^{2l} (-1)^l \\ &= \sum_{k=1}^{n} \lambda_k \cos\left(\frac{s\delta}{p_k}\right), \end{aligned}$$

where we made use of the continuity of the Ad-invariant inner product $\langle \cdot, \cdot \rangle$ and the fact that

$$\langle (\mathrm{ad}_X)^l (Y), Y \rangle = \begin{cases} 0 & \text{, for } l \text{ odd} \\ (-1)^{l/2} \delta^l & \text{, for } l \text{ even.} \end{cases}$$

By choosing

$$s_k := \pi \cdot (\prod_{l=1}^{n} p_l)/(2\delta p_k)$$

for $k = 1, \ldots, n$, we obtain that $\lambda_k = 0$. We have thus shown, that for any $n \in \mathbf{N}$ the first n members of $\mathcal{M}$ are linearly independent, which completes our proof. □

CHAPTER 7

Direct Sums of the Standard Representations ρ_n, μ_n, ν_n

In the following, let $\mathbf{K}$ denote one of the skew fields $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. The aim of this section is to compute the copolarity of the direct sums of the (real) standard representations ρ_n, μ_n, ν_n of $\mathbf{SO}(n), \mathbf{SU}(n), \mathbf{Sp}(n)$ on the corresponding $\mathbf{K}^n$. It turns out that in each case a minimal section is given by a canonical section (after enlarging the acting group if necessary).

LEMMA 7.1.1. *Let $n \geq 2$ be arbitrary and let $1 \leq k \leq n-1$. Then*

(i) The representation $k \cdot \rho_n$ of $\mathbf{SO}(n)$ on k copies of $\mathbf{R}^n$ has the same orbits as the corresponding representation of $\mathbf{O}(n)$ on k copies of $\mathbf{R}^n$.

(ii) The representation $k \cdot \mu_n$ of $\mathbf{SU}(n)$ on k copies of $\mathbf{C}^n$ has the same orbits as the the corresponding representation of $\mathbf{U}(n)$ on k copies of $\mathbf{C}^n$.

PROOF. Concerning (i), it is enough to show that for an element $J \in \mathbf{O}(n) - \mathbf{SO}(n)$ and every $v \in \mathbf{R}^{kn}$ we can always find some $A \in \mathbf{SO}(n)$ such that $A \cdot v = J \cdot v$. In fact, every element of $\mathbf{O}(n)$ can be written as a product $B \cdot J$ for some $B \in \mathbf{SO}(n)$. We consider the elements of R^{kn} as $(n \times k)$-matrices. Then the elements of $\mathbf{SO}(n)$ or $\mathbf{O}(n)$ act on R^{kn} by plain matrix multiplication from the left. Let

$$J := \operatorname{diag}(-1, 1, \ldots, 1) \in \mathbf{O}(n) \quad \text{and} \quad \tilde{J} := \operatorname{diag}(-1, 1, \ldots, 1, -1) \in \mathbf{SO}(n).$$

Next, we consider for an arbitrary element $v \in \mathbf{R}^{kn}$ its QR-decomposition. That is, $v = QR$ where Q is an element of $\mathbf{SO}(n)$ and R is a real upper triangular $n \times k$-matrix. Since $k \leq n-1$ we have in particular that the bottom row of R is always zero. The key observation now is, that because of this $JR = \tilde{J}R$. Let

$$A := JQJ\tilde{J}Q^t.$$

Then $A \in \mathbf{SO}(n)$, because $JQJ, \tilde{J}$ and Q are each elements of $\mathbf{SO}(n)$. We have:

$$A \cdot v = A \cdot QR = JQJ\tilde{J}R = JQR = J \cdot v.$$

The proof of (ii) works in the same manner. □

PROPOSITION 7.1.2. *The k-fold direct sum representations of the standard representations ρ_n, μ_n, ν_n have nontrivial copolarity if and only if $1 \leq k \leq n-1$. More precisely, besides the polar case $k = 1$ we have for $2 \leq k \leq n-1$:*

φ	H	$\operatorname{cohom}(G,V)$	Σ	$\operatorname{copol}(G,V)$	$W(\Sigma)$
$k \cdot \rho_n$	$\mathbf{SO}(n-k)$	$\frac{k(k+1)}{2}$	$\mathbf{R}^{k^2}$	$\frac{k(k-1)}{2}$	$\mathbf{O}(k)$
$k \cdot \mu_n$	$\mathbf{SU}(n-k)$	k^2	$\mathbf{R}^{2k^2}$	k^2	$\mathbf{U}(k)$
$k \cdot \nu_n$	$\mathbf{Sp}(n-k)$	$k(2k-1)$	$\mathbf{R}^{4k^2}$	$k(2k+1)$	$\mathbf{Sp}(k)$

Here H denotes a principal isotropy group along the minimal section Σ and we make use of the following conventions: we consider the representation space $V = \mathbf{K}^{n \times k}$ as the (real) space of rectangular $(n \times k)$-matrices with entries in $\mathbf{K}$, corresponding to the representations ρ_n, μ_n and ν_n. Then $G = \mathbf{SO}(n), \mathbf{SU}(n)$, resp. $\mathbf{Sp}(n)$ acts on V by left *multiplication. In each case, the elements of H are embedded into G as block matrices of the form,* $\begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}$, *with $A \in H$. Furthermore, $\Sigma \subseteq V$ is the space of block matrices of the form* $\begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}$, *where B is a $(k \times k)$-matrix with entries in $\mathbf{K}$.*

Proof. The first step of the proof, that the given Σ is indeed a minimal section, is to construct the G-network $\mathfrak{S}_v$ (see Definition 2.5.1) through a suitable G-regular point $v \in \Sigma$. We then show that Σ is the $\mathbf{R}$-linear span of $\mathfrak{S}_v$. The copolarity is of course the difference between the dimension of Σ and the cohomogeneity of the representation. For $2 \leq k \leq n-1$ let

$$v = (e_1, \ldots, e_k) = \begin{pmatrix} \mathbf{1} \\ \mathbf{0} \end{pmatrix}.$$

Then $v \in \Sigma$ and it is a G-regular point with isotropy group equal to H in the table above. From this we can compute the cohomogeneity of the representation φ. Let $\mathcal{A}(\mathbf{K})$ denote the set of all skew-symmetric, skew-hermitian, resp. skew-quaternionic-hermitian $k \times k$-matrices and let $\mathcal{S}(\mathbf{K})$ denote the set of all symmetric, hermitian, resp. quaternionic-hermitian $k \times k$-matrices. Then the tangent space at the orbit through v is described by:

$$\mathfrak{g} \cdot v = \left\{ \begin{pmatrix} A \\ B \end{pmatrix} \mid A \in \mathcal{A}(\mathbf{K}),\ B \in \mathbf{K}^{(n-k) \times k} \right\}.$$

The (real) inner product of two elements $x, y \in V$ is given by

$$\langle x | y \rangle = \operatorname{Re}(\operatorname{tr}(x^* y)),$$

where $*$ denotes transposition, complex-conjugate transposition or quaternionic-conjugate transposition. The normal space of the G-orbit through v is:

$$\nu_v(G \cdot v) = \left\{ \begin{pmatrix} C \\ \mathbf{0} \end{pmatrix} \mid C \in \mathcal{S}(\mathbf{K}) \right\}.$$

The point $w = (e_1, 2e_2, \ldots, ke_k)$ is G-regular and contained in $\nu_v(G \cdot v) \subseteq \Sigma$. In the case of $\mathbf{K} = \mathbf{R}$ we have

$$\mathfrak{g} \cdot v \cap \mathfrak{g} \cdot w = \left\{ \begin{pmatrix} \mathbf{0} \\ B \end{pmatrix} \mid B \in \mathbf{R}^{(n-k) \times k} \right\},$$

which coincides with the normal space of Σ in V. Therefore, in this case the formula

$$(U \cap W)^\perp = U^\perp + W^\perp$$

for subspaces $U, W \subseteq V$ already implies

$$\Sigma = \nu_v(G \cdot v) + \nu_w(G \cdot w)$$

and we are done with the proof. However, in the case of $\mathbf{K} = \mathbf{C}$ or $\mathbf{K} = \mathbf{H}$ we have

$$\mathfrak{g} \cdot v \cap \mathfrak{g} \cdot w = \left\{ \begin{pmatrix} D \\ B \end{pmatrix} \mid D \in \operatorname{Pu}_k(\mathbf{K}),\ B \in \mathbf{R}^{(n-k) \times k} \right\},$$

where $\operatorname{Pu}_k(\mathbf{K})$ is the set of $(k \times k)$-diagonal matrices with entries in the imaginary numbers, resp. pure quaternions. In both cases let $u \in \nu_v(G \cdot v)$ be the $(n \times k)$-matrix

$$u_{ij} := \begin{cases} 0 & \text{if } i = j \text{ or } i > k, \\ 1 & \text{else.} \end{cases}$$

It is clear that u is G-regular, since its rank is equal to k. We have

$$\mathfrak{g}\cdot v \cap \mathfrak{g}\cdot w \cap \mathfrak{g}\cdot u = \left\{ \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \\ & B & \end{pmatrix} \mid \lambda \in \mathrm{Pu}(\mathbf{K}),\ B \in \mathbf{R}^{(n-k)\times k} \right\}. \qquad (*)$$

In fact, let $a \in \mathfrak{g}$ be such that $a \cdot u \in \mathfrak{g}\cdot v \cap \mathfrak{g}\cdot w$. It then follows for all i, j:

$$(a\cdot u)_{ij} = \sum_{l=1}^{n} a_{il} u_{lj} = \sum_{\substack{l=1 \\ l\neq j}}^{k} a_{il}. \qquad (**)$$

For $i \neq j$ in $\{1, \dots, k\}$ this expression vanishes by assumption and it follows for a fixed $i \in \{1, \dots, k\}$:

$$0 = \sum_{\substack{l=1 \\ l\neq j}}^{k} a_{il} - \sum_{\substack{l=1 \\ l\neq m}}^{k} a_{il} = a_{im} - a_{ij}$$

for all $j, m \in \{1, \dots, k\} - \{i\}$. We thus have $(k-2)a_{ij} = -a_{ii} \in \mathrm{Pu}(\mathbf{K})$, which implies $a_{ij} = -\bar{a}_{ij}$ for all $i \neq j$ (in the case of $k = 2$ this follows from $(**)$ and $a\cdot u \in \mathfrak{g}\cdot v \cap \mathfrak{g}\cdot w$). Together with $a_{ij} = -\bar{a}_{ji}$ we conclude that $a_{ij} = a_{ji}$ holds for all $i \neq j$, and finally

$$(a\cdot u)_{ii} = \sum_{\substack{l=1 \\ l\neq i}}^{k} a_{il} = (n-1)a_{12} \text{ for all } i = 1, \dots, k,$$

which proves equation $(*)$.

The last element we consider is $\tilde{u} := (2e_2, e_1, 3e_3, \dots, ke_k) \in \nu_w(G\cdot w)$. This is again a G-regular point in the G-network through v and it is easily verified that

$$\mathfrak{g}\cdot v \cap \mathfrak{g}\cdot w \cap \mathfrak{g}\cdot u \cap \mathfrak{g}\cdot \tilde{u} = \left\{ \begin{pmatrix} \mathbf{0} \\ B \end{pmatrix} \mid B \in \mathbf{R}^{(n-k)\times k} \right\}.$$

As in the case of $\mathbf{K} = \mathbf{R}$ it now follows that

$$\Sigma = \nu_v(G\cdot v) + \nu_w(G\cdot w) + \nu_u(G\cdot u) + \nu_{\tilde{u}}(G\cdot \tilde{u}).$$

For $k \geq n$ the proof is basically the same as above. The fat Weyl group is computed as follows: the normalizer of Σ in G is given by

$$N(\Sigma) = S(\mathbf{O}(k)\times\mathbf{O}(n-k)),\ S(\mathbf{U}(k)\times\mathbf{U}(n-k)),\ \text{resp. } \mathbf{Sp}(k)\times\mathbf{Sp}(n-k).$$

Since the centralizer is

$$Z(\Sigma) = H = \{\mathbf{1}\}\times\mathbf{SO}(n-k),\ \{\mathbf{1}\}\times\mathbf{SU}(n-k),\ \text{resp. } \{\mathbf{1}\}\times\mathbf{Sp}(n-k),$$

we conclude that

$$W(\Sigma) = N(\Sigma)/Z(\Sigma) = \mathbf{O}(k),\ \mathbf{U}(k),\ \text{resp. } \mathbf{Sp}(k),$$

and it acts in an obvious fashion on Σ. □

REMARK 7.1.3. In [**GOT04**, Theorem 1.3] it is shown that an irreducible representation of a compact Lie group is taut if and only if it has copolarity ≤ 1. Now Gorodski showed in [**Gor04b**] that the (reducible) representations appearing in the table of Proposition 7.1.2 are taut. Hence, the characterization of tautness by copolarity ≤ 1 is not true in the case of reducible representations.

It is also possible to calculate the volume density function $\delta_{\mathcal{E}}$ of Definition 2.6.2 explicitly in the case of the direct sums of the standard representations.

PROPOSITION 7.1.4. *Using the same notation as in Proposition 7.1.2, let φ be the k-fold direct sum of one of the standard representation ρ_n, μ_n, ν_n and let $d := \dim_{\mathbf{R}}(\mathbf{K})$. If $p \in \Sigma$ is an arbitrary point, which we write as*

$$p = \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix}, \ B \in \mathbf{K}^{k^2},$$

then the density function of Definition 2.6.2 evaluated in p is equal to:

$$\delta_{\mathcal{E}}(p) = \frac{1}{\sqrt{2^{dk(n-k)}}} |\det(B^t)|^{d(n-k)}.$$

In the case that $\mathbf{K} = \mathbf{H}$, the determinant is to be understood in the sense of Dieudonné [**Die43**] *and only in this case does the transposition $(\cdot)^t$ matter.*

PROOF. As before, we consider the scalar product $\langle x|y\rangle = \mathrm{Re}(\mathrm{tr}(x^*y))$ on V. It furthermore defines a scalar product on $\mathfrak{g}$, which induces an adapted $(G\text{-}W)$-invariant Riemannian metric on G/H. The elements of $\mathfrak{n}$, resp. $\mathfrak{n}^\perp$ are block matrices of the form

$$\begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix}, \text{ resp. } \begin{pmatrix} \mathbf{0} & E^* \\ E & \mathbf{0} \end{pmatrix},$$

where $A \in \mathfrak{o}(k), \mathfrak{u}(k)$, resp. $\mathfrak{sp}(k)$, $B \in \mathfrak{o}(n-k), \mathfrak{u}(n-k)$, resp. $\mathfrak{sp}(n-k)$ and $\mathrm{tr}(A) + \mathrm{tr}(B) = 0$, and $E \in \mathbf{K}^{(n-k)k}$ is an arbitrary matrix. $\mathfrak{n}^\perp$ is isometric to $\mathfrak{m}/\mathfrak{h}$ and the orbit map $d\omega_p(e) : \mathfrak{n}^\perp \to \nu_p\Sigma$ is simply matrix multiplication of p by elements of $\mathfrak{n}^\perp$ from the left

$$d\omega_p(e) \begin{pmatrix} \mathbf{0} & E^* \\ E & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & E^* \\ E & \mathbf{0} \end{pmatrix} \begin{pmatrix} B \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ EB \end{pmatrix}.$$

For $i = k+1, \ldots, n$ and $j = 1, \ldots, k$ let $E_{ij} \in \mathbf{R}^{(n-k)k}$ denote the corresponding elementary matrix. Then

$$e_{ij} := \begin{pmatrix} \mathbf{0} & \frac{-1}{\sqrt{2}}(E_{ij})^* \\ \frac{1}{\sqrt{2}}E_{ij} & \mathbf{0} \end{pmatrix} \text{ and } f_{ij} := \begin{pmatrix} \mathbf{0} \\ E_{ij} \end{pmatrix}$$

form ON-bases of $\mathfrak{n}^\perp$, resp. $\nu_p\Sigma$, if we view both as $\mathbf{K}$-vector spaces. Identifying f_{ij} with e_{ij} yields a linear isometry between $\nu_p\Sigma$ and $\mathfrak{n}^\perp$. We arrange the e_{ij} as follows:

$$e_{k+1,1}, \ldots, e_{k+1,k}, e_{k+2,1}, \ldots, e_{k+2,k}, \ldots, e_{n,1}, \ldots, e_{n,k}.$$

With respect to this ordering, $d\omega_p(e)$ now has the following matrix representation:

$$\begin{pmatrix} \frac{1}{\sqrt{2}}B^t & & \\ & \ddots & \\ & & \frac{1}{\sqrt{2}}B^t \end{pmatrix}.$$

The block $\frac{1}{\sqrt{2}}B^t$ appears $(n-k)$-times. We are interested in the absolute value of the determinant of $d\omega_p(e)$ as a map between real vector spaces. In [**Asl96**] one can find the following relation for the determinant $\det(P)$ of a complex, resp. quaternionic matrix P and the determinant of its realification $\det_{\mathbf{R}}(P)$:

$$\det{}_{\mathbf{R}}(P) = |\det(P)|^d.$$

Using this and the fact, that the determinant is multiplicative even for $K = \mathbf{H}$ yields the claimed formula. □

Appendix

Invariant Metrics

In reminiscence of G-invariant metrics on G/H (cf. [**CE75**, Proposition 3.16]), we now investigate left-G-invariant metrics on a homogeneous space G/H which are also right-invariant under a certain group W. This concept is used in Sections 2.5, 2.6 and Chapter 3. Since we could not find a proper reference in the literature, we give full proofs of the statements made.

First recall that any triple $(H \trianglelefteq N \leq G)$, where G is a Lie group, H and N are closed subgroups of G and H is normal in N, gives rise to a W-principal bundle:

$$W \hookrightarrow G/H \twoheadrightarrow G/N,$$

where $W = N/H$. We have that G acts on G/H from the left and W acts properly and freely on G/H from the right by $(gH, nH) \mapsto gnH$. We are interested in the case that these actions are isometric and so we are lead to consider Riemannian metrics on G/H which are left-G- and right-W-invariant.

Definition 8.1.1. A Riemannian metric on G/H which is both left-G- and right-W-invariant is called **$(G$-$W)$-invariant**.

Proposition 8.1.2.

(i) *The $(G$-$W)$-invariant Riemannian metrics on G/H are in $1-1$ correspondence with the $\mathrm{Ad}_G(N)$-invariant scalar products on $\mathfrak{g}/\mathfrak{h}$.*

(ii) *If N is connected, then a scalar product $\langle\cdot|\cdot\rangle$ on $\mathfrak{g}/\mathfrak{h}$ is $\mathrm{Ad}_G(N)$-invariant if and only if $\mathrm{ad}_\mathfrak{n}$ is skew-symmetric with respect to $\langle\cdot|\cdot\rangle$.*

(iii) *If $\mathfrak{g}/\mathfrak{h}$ admits a decomposition $\mathfrak{g}/\mathfrak{h} = \mathfrak{n}/\mathfrak{h} \oplus \mathfrak{p}$ with $\mathrm{Ad}_G(N)(\mathfrak{p}) \subseteq \mathfrak{p}$, then the $\mathrm{Ad}_G(N)$-invariant scalar products on $\mathfrak{g}/\mathfrak{h}$, which satisfy $(\mathfrak{n}/\mathfrak{h})\perp\mathfrak{p}$, are in $1-1$ correspondence with pairs $(\langle\cdot|\cdot\rangle_{\mathfrak{n}/\mathfrak{h}}, \langle\cdot|\cdot\rangle_\mathfrak{p})$ of Ad_W-invariant scalar products on $\mathfrak{n}/\mathfrak{h}$, resp. $\mathrm{Ad}_G(N)$-invariant scalar products on $\mathfrak{p}$.*

Such a pair exists if and only if W is covered by a product of a compact Lie group with a vector group and if the image of N under $n \mapsto \mathrm{Ad}_n|_\mathfrak{p}$ in $\mathbf{GL}(\mathfrak{p})$ is relatively compact.

Conversely, if $\langle\cdot|\cdot\rangle$ on $\mathfrak{g}/\mathfrak{h}$ is $\mathrm{Ad}_G(N)$-invariant, then $\langle\cdot|\cdot\rangle|_{\mathfrak{n}/\mathfrak{h}}$ is Ad_W-invariant. If $\mathfrak{p} := (\mathfrak{n}/\mathfrak{h})^\perp$, then $\mathrm{Ad}_G(N)(\mathfrak{p}) \subseteq \mathfrak{p}$ and $\langle\cdot|\cdot\rangle|_\mathfrak{p}$ is $\mathrm{Ad}_G(N)$-invariant.

(iv) *If N is compact, then G/H admits a $(G$-$W)$-invariant Riemannian metric.*

Proof. (i): Let h be a Riemannian metric on G/H and put $\langle\cdot|\cdot\rangle := h_{eH}$. Then it is well known that h is left-G-invariant if and only if $\langle\cdot|\cdot\rangle$ is $\mathrm{Ad}_G(H)$-invariant. If additionally h is right-W-invariant, then $r_n^* h = h$ for all $n \in N$. Hence, we have for all $X, Y \in T_{gH}G/H$:

$$h_{gnH}(X \cdot n, Y \cdot n) = h_{gH}(X, Y).$$

Using the G-invariance we obtain

$$h_{eH}(n^{-1}g^{-1} \cdot X \cdot n, n^{-1}g^{-1} \cdot X \cdot n) = h_{eH}(g^{-1} \cdot X, g^{-1} \cdot Y).$$

Under the natural identification $T_{eH}G/H \simeq \mathfrak{g}/\mathfrak{h}$ this is equivalent to

$$\langle \mathrm{Ad}_n(X) \mid \mathrm{Ad}_n(Y)\rangle = \langle X \mid Y\rangle, \text{ for all } X, Y \in \mathfrak{g}/\mathfrak{h}.$$

Conversely, if we are given an $\mathrm{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$ then, in particular, it is $\mathrm{Ad}_G(H)$-invariant. It therefore gives rise to a left-G-invariant Riemannian metric on G/H. Furthermore, it is easy to see, that it is right-W-invariant.

(ii): This is a standard consideration.

(iii): If $\langle\cdot|\cdot\rangle$ is an $\mathrm{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}/\mathfrak{h}$ satisfying $(\mathfrak{n}/\mathfrak{h})\perp\mathfrak{p}$, then its restriction to $\mathfrak{n}/\mathfrak{h}$ resp. $\mathfrak{p}$ clearly yields the stated pair of invariant scalar products. Conversely, we may patch such a pair of invariant scalar products together to form an $\mathrm{Ad}_G(N)$-invariant scalar product $\langle\cdot|\cdot\rangle$ on $\mathfrak{g}/\mathfrak{h}$ by defining:

$$\langle X+Y|Z+W\rangle := \langle X|Z\rangle_{\mathfrak{n}/\mathfrak{h}} + \langle Y|W\rangle_{\mathfrak{p}}, \text{ for all } X, Z \in \mathfrak{n}/\mathfrak{h},\ Y, W \in \mathfrak{p}.$$

The Ad_W-invariance of $\langle\cdot|\cdot\rangle_{\mathfrak{n}/\mathfrak{h}}$ is equivalent to the existence of a bi-invariant Riemannian metric on W. Using [**CE75**, Proposition 3.34] yields that this is the case if and only if W is covered by the product of a compact Lie group and a vector group. Also, if $\langle\cdot|\cdot\rangle_{\mathfrak{p}}$ is $\mathrm{Ad}_G(N)$-invariant, then the image of N under $f : N \to \mathbf{GL}(\mathfrak{p})$, $n \mapsto \mathrm{Ad}_n|_{\mathfrak{p}}$ is contained in the compact group $\mathbf{O}(\mathfrak{p})$ and therefore relatively compact. Conversely, if $K := \overline{f(N)} \subseteq \mathbf{GL}(\mathfrak{p})$ is compact, then we may define by an averaging process a K-invariant scalar product on $\mathfrak{p}$, which in turn is $\mathrm{Ad}_G(N)$-invariant.

(iv) follows from (iii) and the fact that a representation of a compact Lie group is completely reducible. □

The following Proposition shows that the concept of a $(G\text{-}W)$-invariant metric on G/H is actually the same as that of a left-G-invariant metric on G/N. Nevertheless, the notion of a $(G\text{-}W)$-invariant metric is useful in the context of metrics adapted to a minimal section (Definition 2.5.9).

Proposition 8.1.3. *The $(G\text{-}W)$-invariant metrics on G/H correspond to the left-G-invariant metrics on G/N.*

Proof. If we are given a $(G\text{-}W)$-invariant metric on G/H, then the submersed metric on G/N under the canonical G-equivariant mapping $gH \mapsto gN$ is left-G-invariant. Conversely, if we start with a left-G-invariant metric on G/N, then G admits a left invariant metric which is right-N-invariant. This induces an $\mathrm{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}$ and since $\mathfrak{h}$ is $\mathrm{Ad}_G(N)$-invariant, the induced scalar product on $\mathfrak{g}/\mathfrak{h}$ is $\mathrm{Ad}_G(N)$-invariant. Using Proposition 8.1.2 (i), this yields a $(G\text{-}W)$-invariant Riemannian metric on G/H. □

The next result is basically [**Bes87**, Theorem 9.80].

Corollary 8.1.4. *If G/H carries a $(G\text{-}W)$-invariant Riemannian metric, then the principal fibre bundle $G/H \twoheadrightarrow G/N$ is a Riemannian submersion, where G/N is endowed with the quotient metric. Its fibres are totally geodesic. In particular W, viewed as a subset of G/H, is totally geodesic in G/H. Furthermore, the map*

$$(\mathfrak{n}/\mathfrak{h})^{\perp} \to \mathfrak{g}/\mathfrak{n},\ X+\mathfrak{h} \mapsto X+\mathfrak{n}$$

is a linear isometry.

Proof. By left-G-invariance, the fibres of the principal bundle $G/H \twoheadrightarrow G/N$ are all isometric to the fibre W over eN. Now W is the image of N under the canonical projection $G \to G/H$, which is a Riemannian submersion if G is endowed with a left-invariant metric that is right-N-invariant. Such a metric exists due to, [**CE75**, Proposition 3.16].

By the following lemma, N is a totally geodesic submanifold of G. Hence, its image W under the Riemannian submersion $G \to G/H$ is totally geodesic in G/H. $\square$

LEMMA 8.1.5. *Let G be a Lie group and $H \subseteq G$ a closed subgroup. If G carries a left-invariant metric which is right H-invariant, then the induced metric on H is bi-invariant and H is a totally geodesic submanifold of G.*

PROOF. Since the Riemannian metric on H is bi-invariant, it follows that the one-parameter subgroups of left-invariant vector fields on H are geodesics on H. Let $\langle\cdot|\cdot\rangle$ denote the $\mathrm{Ad}_G(N)$-invariant scalar product on $\mathfrak{g}$ induced by the metric on G. We have that ad_X is skew symmetric with respect to this scalar product for all $X \in \mathfrak{h}$. It follows (see for instance [**CE75**, Proposition 3.18]) that

$$\nabla_X Y = \frac{1}{2}[X, Y],$$

for all left invariant vector fields X, Y on G with $X_e, Y_e \in \mathfrak{h}$. In particular, for $Y = X$, it follows that the one-parameter subgroups of H are geodesics of G. $\square$

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