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Deligne-Lusztig Characters associated with Galois Representations and their Reductions mod p

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Abstract

Let p be a prime and L a finite extension of the field of p-adic numbers $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. With any smooth n-dimensional irreducible representation of the absolute Galois group of L with coefficients in $\overline{\mathbb{F}}_q$ we associate a Deligne-Lusztig character of $\mathrm{GL}_\mathrm{n}(\mathbb{F}_q)$ over an algebraically closed field of characteristic 0. We reduce such a character to a virtual representation of $\mathrm{GL}_n(\mathbb{F}_q)$ over $\overline{\mathbb{F}}_q$. From this virtual representation we can construct a virtual module over the pro-p Iwahori Hecke algebra $\mathcal{H}^{(1)}$ of $\mathrm{GL}_n(L)$ over $\overline{\mathbb{F}}_q$. If $q = p$ and $n = 2$, this establishes a bijection between isomorphism classes of smooth irreducible 2-dimensional Galois representations and irreducible supersingular 2-dimensional modules over $\mathcal{H}^{(1)}$. We will also compare our construction to Große-Klönne's functor for general n when $L = \mathbb{Q}_p$.

Introduction

Let p be a prime, L a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers and $\mathcal{O}_L$ the corresponding valuation ring with prime element π_L and residue class field $\mathbb{F}_q$. Its absolute Galois group $G_L := \mathrm{Gal}(\overline{L}/L)$ is a very important object in number theory. It is the aim of the Langlands program to understand this group via its representations. More specifically, for each positive integer n, one aims to establish a connection between G_L-representations and representations of the group $\mathrm{GL}_n(L)$. In this thesis we will focus on those representations on $\overline{\mathbb{F}}_p$-vector spaces. The word "representation" will always be used for a smooth representation over $\overline{\mathbb{F}}_q$ unless otherwise stated. In the case of $L = \mathbb{Q}_p$ and $n = 2$ Breuil has established a bijective correspondence between isomorphism classes of supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ and isomorphism classes of irreducible 2-dimensional representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ in [Bre03].

However the situation becomes more complicated when $n > 2$ or $L \neq \mathbb{Q}_p$. Instead of representations of $\mathrm{GL}_n(L)$ one can consider modules over appropriate algebras, the so-called pro-p Iwahori Hecke algebras. Let $I^{(1)}$ be the standard pro-p Iwahori subgroup of $\mathrm{GL}_n(L)$, i.e. the subgroup consisting of all matrices in $\mathrm{GL}_n(\mathcal{O}_L)$ which have unipotent upper triangular reduction mod π_L. With each representation M of $\mathrm{GL}_n(L)$ one can associate its $I^{(1)}$-invariants $M^{I^{(1)}}$ which have a natural structure of a right module over the pro-p Iwahori Hecke algebra $\mathcal{H}^{(1)} := \mathrm{End}_{\overline{\mathbb{F}}_p[G(L)]}\overline{\mathbb{F}}_p[I^{(1)}\backslash G(L)]$. However, the link provided by the functor of $I^{(1)}$-invariants between $\mathcal{H}^{(1)}$-right modules and $\mathrm{GL}_n(L)$-representations with nonzero $I^{(1)}$-invariants is not as tight as the one afforded by the functor of invariants under the Iwahori subgroup in characteristic 0. For example, if $n = 2$, it is not an equivalence of categories if $q \neq p$ by [Oll09].

We will disregard that problem in this thesis and aim to establish a connection between irreducible n-dimensional G_L-representations and supersingular simple modules over the pro-p Iwahori Hecke algebra $\mathcal{H}^{(1)}$ of $\mathrm{GL}_n(L)$. In the last few years correspondences between both sides have been set up. The first such correspondence was of numerical nature: Vignéras and Ollivier have shown that the number of n-dimensional simple supersingular modules (with fixed action of the double coset of π_L considered as an element of $\mathrm{GL}_n(L)$) over $\mathcal{H}^{(1)}$ is equal to the number of smooth irreducible n-dimensional representations of G_L (with fixed determinant of a Frobenius). The arguments rely on combinatorics, explicitly computing the order of both sides in [Vig05] and another characterization of supersingular modules by means of the restriction of such a module to the affine subalgebra $\mathcal{H}^{(1)}_{aff}$ of $\mathcal{H}^{(1)}$ in [Oll10]. In [GK13] Große-Klönne constructs a contravariant functor from the category of $\mathcal{H}^{(1)}$-

modules of finite length to étale (φ, Γ)-modules over Fontaine's ring $\mathcal{O}_{\mathcal{E}}$ for $L = \mathbb{Q}_p$ using the Bruhat-Tits tree. Composing with Fontaine's equivalence of categories we obtain a finite dimensional representation of $G_{\mathbb{Q}_p}$. This functor induces a bijection between the set of simple supersingular n-dimensional $\mathcal{H}^{(1)}$-modules and the set of isomorphism classes of irreducible n-dimensional $G_{\mathbb{Q}_p}$-representations.

In this thesis we use another approach. We proceed as follows: The purpose of **Chapter 1** is to collect general facts on reductive groups with an emphasis on reductive groups over finite fields. Our main example will be the reductive group GL_n for which we will introduce some notation that will be used throughout this thesis.

In **Chapter 2** we will consider representations of G_L. We start by introducing the Weil group W_L and its topology and will then show that the categories of finite representations of G_L resp. W_L are equivalent. This allows us to work with W_L representations from now on. Let $I_L \subseteq W_L$ be the inertia subgroup and P_L its unique pro-p Sylow subgroup. As irreducible representations of W_L are trivial on P_L, we may restrict to considering representations of the factor group W_L/P_L. The concept is now to express a representation of W_L/P_L in terms of algebraic groups. We will more generally consider a split reductive group G over $\mathbb{F}_q$ with dual group G^* and a continuous homomorphism $\rho : W_L/P_L \to G^*$. It turns out that the image of the restriction of such a homomorphism to I_L/P_L is a semisimple cyclic subgroup of G^*, hence we may assume that it is contained in a fixed split maximal torus T^*. This is well-defined up to Weyl group conjugation, i.e. with ρ we can associate a Frobenius-stable Weyl group orbit of continuous homomorphisms from I_L/P_L to T^* which we will be able to reinterpret as a Frobenius-stable Weyl group orbit of homomorphisms from $Y(T) \otimes_{\mathbb{Z}} Z$ to $\overline{\mathbb{F}}_q^\times$ where $Y(T)$ is the cocharacter group of the split maximal torus T dual to T^* and $Z := \varprojlim_{(p,m)=1} \mu_m(\overline{\mathbb{F}}_q)$. This allows us to construct a $G(\mathbb{F}_q)$-conjugacy class of an in general non-split maximal torus $T_w \subseteq G$ and a character θ of $T_w(\mathbb{F}_q)$. In the remainder of chapter 2 we will explicitly compute T_w and θ for the case that ρ is an irreducible representation of GL_n, and consider the case $G = \mathrm{SL}_n$.

In **Chapter 3** we will use the $G(\mathbb{F}_q)$-conjugacy class of the pair (T_w, θ) constructed in chapter 2 to construct a virtual representation of $G(\mathbb{F}_q)$. Deligne and Lusztig have assigned a virtual representation $R^\theta(w)$ over an algebraically closed field in characteristic 0 to such a conjugacy class. We can view this virtual representation as a virtual representation over $\overline{L} = \overline{\mathbb{Q}}_p$ in a canonical way which allows us to reduce it to a virtual representation over $\overline{\mathbb{F}}_q$ in a well-defined way. Jantzen has given a formula how to compute these reductions in terms of restrictions of Weyl modules which we shall make explicit for lower dimensional cases.

If ρ is an irreducible representation, we will see that $\pm R^\theta(w)$ is an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ over an algebraically closed field of characteristic 0. For future calculations we will need a classification of the irreducible representations of the finite group $\mathrm{GL}_n(\mathbb{F}_q)$. We state two such classifications, one in terms of the structure of GL_n as an algebraic group, the other in terms of the structure of $\mathrm{GL}_n(\mathbb{F}_q)$ as a group with BN-pair. Then we will illuminate the connection between these two classifications. This will be helpful because the first one naturally arises in the reduction of $R^\theta(w)$ to $\overline{\mathbb{F}}_q$ and the second one is natural when we will establish a connection to modules over Hecke algebras later.

We will start **Chapter 4** by reviewing the definitions and presentations for the finite Hecke algebras and pro-p Iwahori Hecke algebras. For us the three most important Hecke algebras are the finite Hecke algebra $\mathcal{H}_0^{(1)}$ of the group $G(\mathbb{F}_q)$ with respect to the unipotent radical of a Borel subgroup, the affine pro-p Iwahori Hecke algebra $\mathcal{H}_{aff}^{(1)}$ and the pro-p Iwahori Hecke algebra $\mathcal{H}^{(1)}$. Among them we have the inclusions $\mathcal{H}_0^{(1)} \subseteq \mathcal{H}_{aff}^{(1)} \subseteq \mathcal{H}^{(1)}$. We will give presentations of these algebras in terms of appropriate Weyl groups and finite tori. With these presentations we can examine the behavior of these algebras for varying fields L. For any finite extension of L we will construct injective algebra homomorphisms between the corresponding Hecke algebras.

The remainder of chapter 4 will be devoted to the study of supersingular modules: We start by reviewing a central theorem by Ollivier on supersingular modules which states that simple supersingular modules are the same as simple modules which contain "nontrivial" characters for the affine pro-p Iwahori Hecke algebra. This allows us to carry on our construction: In chapter 3 we have constructed a virtual representation of $\mathrm{GL}_n(\mathbb{F}_q)$ from an irreducible n-dimensional Galois representation. The irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ are in bijection with the simple right modules over the finite Hecke algebra $\mathcal{H}_0^{(1)}$ by taking invariants under the unipotent radical of the standard Borel subgroup. In particular, we obtain an isomorphism of Grothendieck groups $\mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_n(\mathbb{F}_q)]) \cong \mathrm{G}_0(\mathcal{H}_0^{(1)})$. Then we construct a map from $\mathrm{G}_0(\mathcal{H}_0^{(1)})$ to $\mathrm{G}_0(\mathcal{H}_{aff}^{(1)})$. Further we can give a homomorphism from $\mathrm{G}_0(\mathcal{H}_{aff}^{(1)})$ to $\mathrm{G}_0(\mathcal{H}^{(1)})$ which uses another parameter, namely a unit of $\overline{\mathbb{F}}_q$. For this we will take the determinant of the image of a Frobenius under ρ. Combining these constructions we have associated a virtual module of $\mathcal{H}^{(1)}$ with our irreducible Galois representation.

Finally, in **Chapter 5** we examine the constructions of chapters 2, 3 and 4 by explicitly computing the image of ρ in $\mathrm{G}_0(\mathcal{H}^{(1)})$. We will begin by discussing the case $G = \mathrm{GL}_2$ and $q = p$. However, it turns out that our construction is not even near being a correspondence between irreducible 2-dimensional Galois representations and simple supersingular 2-dimensional $\mathcal{H}^{(1)}$-modules. But with a slight modification it will be: We introduce a "shift" map on $\mathrm{G}_0(\mathrm{GL}_n(\mathbb{F}_q))$ and include it as an intermediate step of the construction. This way we obtain another virtual module over $\mathcal{H}^{(1)}$. We show that there exists a simple supersingular 2-dimensional $\mathcal{H}^{(1)}$-module $M(\rho)$ such that the virtual module associated with ρ by this construction is either equal to $M(\rho)$ or $2M(\rho)$. Further, we get:

Theorem. Assume that $L/\mathbb{Q}_p$ is totally ramified, i.e. $q = p$. The assignment $\rho \mapsto M(\rho)$ is a bijection between irreducible 2-dimensional W_L-representations and irreducible 2-dimensional supersingular simple $\mathcal{H}^{(1)}$-modules. If $L = \mathbb{Q}_p$, this bijection coincides with the one afforded by Große-Klönne's functor.

For $G = \mathrm{GL}_3$ we can imitate this behavior: In contrast to the GL_2-case we have to introduce 2 different shift maps depending on ρ. We will find (slight modifications) of the simple supersingular modules given by Große-Klönne's functor appearing as summands in the virtual $\mathcal{H}^{(1)}$-module given by our construction and compute their multiplicities which are always equal to 1, 2 or 3. Finally, we propose a set of n different shift maps for general n and examine their behavior generically for GL_4.

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Contents

Chapter 1

Reductive Groups

1.1 Notations and basic Definitions

Let p be a prime number. We fix some finite extension L of $\mathbb{Q}_p$ with algebraic closure $\overline{L}$. Denote by $\mathcal{O}_L$ and $\mathcal{O}_{\overline{L}}$ the corresponding rings of integers and by $\mathfrak{m}_L$, $\mathfrak{m}_{\overline{L}}$ their maximal ideals. We will identify the residue fields $\mathcal{O}_L/\mathfrak{m}_L$ and $\mathcal{O}_{\overline{L}}/\mathfrak{m}_{\overline{L}}$ with $\mathbb{F}_q$, where $q = p^r$ is a power of p, resp. $\overline{\mathbb{F}}_q$.

In this section, we review the results which we will need from the theory of algebraic groups and of finite groups of Lie type. For proofs and more details, see [Spr09] for the general case or [Car93] for the case of linear algebraic groups over $\overline{\mathbb{F}}_q$.

Algebraic Groups

For the moment, let K be an algebraically closed field of arbitrary characteristic and let G be an algebraic group over K, i.e. an algebraic variety over K, which is a group such that the multiplication map $G \to G, (x, y) \mapsto xy$ and the formation of inverse elements $G \to G, g \mapsto g^{-1}$ are morphisms of varieties. We will usually omit the coordinate ring of the underlying algebraic variety from our notations.

An algebraic group is endowed with a topology, the Zariski topology. Hence, we may speak of open, closed and connected subsets of G. A homomorphism of algebraic groups is a morphism of varieties between two algebraic groups, which is also a group homomorphism. If the underlying variety is affine, we call G a linear algebraic group. A closed subgroup of an algebraic group has the structure of an algebraic group such that the inclusion map is a morphism of algebraic groups. If G is a linear algebraic group, then so is every closed subgroup.

Any linear algebraic group over K is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$ for some positive integer n. As $\mathrm{GL}_n(K)$ is an algebraic group itself, linear algebraic groups are, up to isomorphism, exactly the closed subgroups of the groups $\mathrm{GL}_n(K)$. This justifies the name *linear* algebraic group.

Unipotent and Semisimple Elements

In the following, assume G to be a closed subgroup of $\mathrm{GL}_n(K)$ by a chosen embedding. We call an element $g \in G$ semisimple, if it is diagonalizable (or, if $K = \overline{\mathbb{F}}_q$, equivalently, the order of g is prime to p) and unipotent if the matrix $g-1 \in M_n(K)$ is nilpotent (or, if $K = \overline{\mathbb{F}}_q$, equivalently, if the order of g is a power of p). These definitions do not depend on the chosen embedding $G \to \mathrm{GL}_n(K)$. We call a linear algebraic group unipotent, if all its elements are unipotent and for any linear algebraic group G, let G_u be the subset of unipotent elements in G. In general, this is not a subgroup. Morphisms of algebraic groups respect semisimple and unipotent elements. Each element in G has a Jordan decomposition, i.e. there exist unique elements $g_s \in G$ and $g_u \in G$, such that g_s is semisimple, g_u is unipotent and $g = g_s g_u = g_u g_s$.

Borel Subgroups, Parabolic Subgroups and Tori

A subgroup B of G is called a Borel subgroup if it is closed, connected, solvable and maximal with these properties. There always exists a Borel subgroup and two Borel subgroups are conjugate in G. Additionally, every element of G lies in some Borel subgroup. A closed subgroup P of G containing a Borel subgroup is called parabolic.

From now on, let G be a connected linear algebraic group. Then the set of closed connected solvable normal subgroups of G has a unique maximal element. We will call this subgroup the radical of G and denote it by $R(G)$. Analogously, the set of closed connected unipotent normal subgroups has a unique maximal element $R_u(G)$, called the unipotent radical of G. We have $R(G)_u = R_u(G)$. G is called semisimple, if $R(G) = \{1\}$ and reductive, if $R_u(G) = \{1\}$.

Denote by $\mathbb{G}_m$ the algebraic group $K^\times$. Then a torus is an algebraic group T which is isomorphic to $\mathbb{G}_m^n$ for some $n \geq 1$. Hence, a torus consists only of semisimple elements. n is called the rank of T. A subtorus T of G is a closed subgroup, which is a torus. We call it maximal, if it is not properly contained in any subtorus. Two maximal subtori in G are conjugate and every semisimple element lies in a maximal torus. Further, any maximal torus is contained in a Borel subgroup. The rank of G is by definition the rank of a maximal torus in G.

A character of G is a homomorphism from G to $\mathbb{G}_m$ and dually, a cocharacter of G is a homomorphism from $\mathbb{G}_m$ to G. We write $X(G) = \mathrm{Hom}(G, \mathbb{G}_m)$ and $Y(G) = \mathrm{Hom}(\mathbb{G}_m, G)$. Pointwise multiplication makes $X(G)$ an abelian group and, if G is abelian, it makes $Y(G)$ an abelian group. If $G = T$ is a torus we will write the group law additively because of the isomorphisms

$$X(T) = \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{G}_m^n, \mathbb{G}_m) \cong \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)^n \cong \mathbb{Z}^n$$

and

$$Y(T) = \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m^n) \cong \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)^n \cong \mathbb{Z}^n,$$

where n is the rank of T.

If $\chi \in X(T)$ and $\gamma \in Y(T)$, $\chi \circ \gamma \in \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)$ and hence there exists a unique $n_{\chi,\gamma} \in \mathbb{Z}$ such that $\chi(\gamma(x)) = x^{n_{\chi,\gamma}}$ for all $x \in \mathbb{G}_m$. This allows us to define a pairing $\langle\cdot,\cdot\rangle : X(T) \times Y(T) \to \mathbb{Z}$ by $(\chi, \gamma) \mapsto n_{\chi,\gamma}$. This is a perfect pairing, i.e. the homomorphisms $X(T) \to \mathrm{Hom}(Y(T), \mathbb{Z}), \chi \mapsto \langle\chi, \cdot\rangle$ and $Y(T) \to \mathrm{Hom}(X(T), \mathbb{Z}), \gamma \mapsto \langle\cdot, \gamma\rangle$ are isomorphisms.

We return to the situation of a general connected linear algebraic group G. Fix a maximal torus $T \subseteq G$, denote by $N(T)$ its normalizer and by $C(T)$ its centralizer in G. The quotient $W_0 := N(T)/C(T)$ is finite and we call it the finite Weyl group of G with respect to T. If G is reductive we have $C(T) = T$ and hence $W_0 = N(T)/T$. W_0 acts from the left on T by setting ${}^w t = \dot{w} t \dot{w}^{-1}$, where $\dot{w} \in N(T)$ is any lift of w. This does not depend on the choice of $\dot{w}$. The action on T induces left-actions of W_0 by $\mathbb{Z}$-linear automorphisms on $X(T)$ and $Y(T)$ by setting ${}^w\chi(t) = \chi({}^{w^{-1}}t)$ for all $t \in T$ and ${}^w\gamma(x) = {}^w(\gamma(x))$ for all $\chi \in X(T)$, $\gamma \in Y(T)$, $w \in W_0$, $t \in T$ and $x \in \mathbb{G}_m$. The pairing $\langle\cdot,\cdot\rangle$ is W_0-invariant, i.e., we have $\langle {}^w\chi, {}^w\gamma\rangle = \langle\chi, \gamma\rangle$ for all $\chi \in X(T)$, $\gamma \in Y(T)$ and $w \in W_0$.

1.2 Root Data and Weyl Groups

Abstract Root Data

Here, we will summarize the theory of abstract root data. For proofs and a more detailed treatment, see [Bor09, Kapitel 1].

Definition 1.2.1. A root datum is a quadruple $(X, Y, \Phi, \check{\Phi})$ with the following properties:

(i) X and Y are free abelian groups of the same finite rank.

(ii) Φ and $\check{\Phi}$ are nonempty subsets of X resp. Y such that there exists a bijection $\alpha \mapsto \check{\alpha}$ from Φ to $\check{\Phi}$.

(iii) There exists a perfect pairing $\langle\cdot,\cdot\rangle : X \times Y \to \mathbb{Z}$ such that $\langle\alpha, \check{\alpha}\rangle = 2$ for all $\alpha \in \Phi$.

(iv) For the endomorphisms $s_\alpha \in \mathrm{End}_{\mathbb{Z}}(X)$ and $s_{\check{\alpha}} \in \mathrm{End}_{\mathbb{Z}}(Y)$ given by

$$s_\alpha(x) = x - \langle x, \check{\alpha}\rangle\alpha \text{ for all } x \in X$$

and

$$s_{\check{\alpha}}(y) = y - \langle\alpha, y\rangle\check{\alpha} \text{ for all } y \in Y$$

we have $s_\alpha(\Phi) = \Phi$ and $s_{\check{\alpha}}(\check{\Phi}) = \check{\Phi}$.

In the following we will also consider the pairing $\langle\cdot,\cdot\rangle$ as part of the given datum. A based root datum is a quintuple $(X, Y, \Phi, \check{\Phi}, \Delta)$ such that $(X, Y, \Phi, \check{\Phi})$ is a root datum and

(v) Δ is a basis of the underlying root system, i.e. $\Delta \subseteq \Phi$, Δ is linearly independent and for each $\alpha \in \Phi$ there exist unique $n_\beta \in \mathbb{Z}$ such that $\alpha = \sum_{\beta\in\Delta} n_\beta \beta$. The n_β are either all nonnegative or all nonpositive.

The elements of Φ resp. $\check{\Phi}$ are called the roots resp. coroots and those belonging to Δ are the simple roots. Note that, if α is a root, $-\alpha = s_\alpha(\alpha)$ is also a root. It can be shown that for every root datum $(X, Y, \Phi, \check{\Phi})$ there exists a subset $\Delta \subseteq \Phi$ such that $(X, Y, \Phi, \check{\Phi}, \Delta)$ is a based root datum. However, we will not need this fact.

Given a based root datum, we can define the positive roots as all those roots which are linear combination of simple roots with nonnegative coefficients and denote this set by Φ^+. On the other hand, if we have a system of positive roots defined by Δ, we can reconstruct Δ as the set of all positive roots, which are not the sum of two or more other roots. Additionally, we have the monoid

$$X_+ = \{x \in X : \langle x, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \Delta\} = \{x \in X : \langle x, \check{\alpha} \rangle \geq 0 \text{ for all } \alpha \in \Phi^+\}.$$

This is a finitely generated submonoid of X. Its elements are called the dominant weights.

A root datum is called irreducible if there are no nonempty subsets $\Phi_1, \Phi_2 \subseteq \Phi$ such that $\langle \Phi_1, \check{\Phi}_2 \rangle = 0$ and $\Phi_1 \cup \Phi_2 = \Phi$.

With every root datum, we can associate a dual given by $(Y, X, \check{\Phi}, \Phi, \check{\Delta})$, where $\check{\Delta} = \{\check{\alpha} : \alpha \in \Delta\}$. This is again a based root datum.

Definition 1.2.2. Two based root data $(X_1, Y_1, \Phi_1, \check{\Phi}_1, \Delta_1)$ and $(X_2, Y_2, \Phi_2, \check{\Phi}_2, \Delta_2)$ are called isomorphic if there exist isomorphisms of $\mathbb{Z}$-modules $\delta : X_1 \to X_2$ and $\epsilon : Y_1 \to Y_2$ such that the following conditions are satisfied:

(i) $\langle \delta(x), \epsilon(y) \rangle = \langle x, y \rangle$ for all $x \in X_1$ and $y \in Y_1$,

(ii) $\delta(\Phi_1) = \Phi_2$ and $\epsilon(\check{\Phi}_1) = \check{\Phi}_2$,

(iii) $\delta(\alpha)\check{} = \epsilon(\check{\alpha})$ for all $\alpha \in \Phi_1$,

(iv) $\delta(\Delta_1) = \Delta_2$.

Two root data are said to be dual if each of them is isomorphic to the dual root datum of the other.

Another way to construct new root data from a given one is the following: We can define the root lattice $Q = \sum_{\alpha \in \Phi} \mathbb{Z}\alpha = \bigoplus_{\alpha \in \Delta} \mathbb{Z}\alpha \subseteq X$ and the coroot lattice $\check{Q} = \sum_{\check{\alpha} \in \check{\Phi}} \mathbb{Z}\check{\alpha} = \bigoplus_{\check{\alpha} \in \check{\Delta}} \mathbb{Z}\check{\alpha} \subseteq Y$. These induce $\mathbb{Q}$-vector spaces $V_Q = Q \otimes_{\mathbb{Z}} \mathbb{Q}$ and $V_{\check{Q}} = \check{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ so that we can extend $\langle \cdot, \cdot \rangle$ to a perfect pairing from $V_Q \times V_{\check{Q}}$ to $\mathbb{Q}$. This allows us to define the weight lattice

$$P = \{x \in V_Q : \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \check{Q}\}$$

and the coweight lattice

$$\check{P} = \{y \in V_{\check{Q}} : \langle x, y \rangle \in \mathbb{Z} \text{ for all } x \in Q\}.$$

By definition, $\langle \cdot, \cdot \rangle$ induces perfect pairings $Q \times \check{P} \to \mathbb{Z}$ and $P \times \check{Q} \to \mathbb{Z}$. Hence, we obtain new based root data $(Q, \check{P}, \Phi, \check{\Phi}, \Delta)$ and $(P, \check{Q}, \Phi, \check{\Phi}, \Delta)$. We say that a root datum is simply connected if $X = P$, and of adjoint type if $X = Q$.

On X, we can introduce a partial order given by

$$x_1 \leq x_2 \text{ if and only if } x_2 - x_1 \in \sum_{\alpha \in \Phi^+} \mathbb{N}_0 \alpha.$$

We denote the set of minimal elements of Φ with respect to $\leq$ by Φ_m. A root datum is irreducible if and only if Φ_m consists of exactly one element.

Root Data of Reductive Groups

For this subsection, let G be a connected reductive group and fix a maximal torus $T \subseteq G$ and a Borel subgroup B containing T. In the following, we will construct a root datum as before from these data.

B decomposes as the semidirect product $B = TU$, where $U = R_u(B)$ is its unipotent radical, a subgroup of B. There exists a unique Borel subgroup $B^- \subseteq G$ containing T such that $B \cap B^- = T$. We call B and B^- opposite Borel subgroups of G. Denote by $U^- = R_u(B^-)$ its unipotent radical. We have $U \cap U^- = \{1\}$.

Now, let X be a nontrivial subgroup of U or U^- normalized by T which is minimal with these properties. Then X is isomorphic to the additive group $\mathbb{G}_a = K$. Hence, conjugation induces a homomorphism $T \to \mathrm{Aut}(\mathbb{G}_a) = \mathbb{G}_m$, i.e. character of T. This character does not depend on the choice of the isomorphism between X and $\mathbb{G}_a$. We call these characters the roots of (G, T) and denote by Φ the set of roots. Although a Borel subgroup was chosen to define the notion of a root, this definition does not depend on the choice of this Borel subgroup. If $\alpha \in \Phi$, we denote by U_α the corresponding minimal subgroup and fix an isomorphism $u_\alpha : \mathbb{G}_a \to U_\alpha$. The U_α are called the root subgroups of G with respect to T. We will call the roots corresponding to subgroups of U (resp. subgroups of U^-) positive (resp. negative) and denote the set of of those roots by Φ^+ (resp. Φ^-). Φ is the disjoint union of Φ^+ and Φ^- and we have $\Phi^+ = -\Phi^-$. Let Δ be the set of all positive roots, which are not sum of two or more positive roots.

Let $\alpha \in \Phi$ be a root. Then, we have the subgroup $\langle U_\alpha, U_{-\alpha} \rangle$ of G. After possibly replacing $u_{-\alpha}$ by another isomorphism of that sort, there exists a unique homomorphism $\phi_\alpha : \mathrm{SL}_2(K) \to \langle U_\alpha, U_{-\alpha} \rangle$, such that

$$\phi_\alpha(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) = u_\alpha(x) \text{ and } \phi_\alpha(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}) = u_{-\alpha}(x)$$

for all $x \in K$. We can now define a cocharacter $\check{\alpha} \in Y(T)$ by

$$\check{\alpha}(x) = \phi_\alpha(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix})$$

for all $t \in K^\times$. We call the elements obtained this way the coroots and denote the set of coroots by $\check{\Phi}$. The quintuple $(X(T), Y(T), \Phi, \check{\Phi}, \Delta)$ is a based root datum as defined before. Up to isomorphism, each connected reductive group is uniquely determined by its root datum.

We can generalize the U_α by

$$U_w := \prod_{\alpha \in \Phi^+, w^{-1}(\alpha) \in \Phi^-} U_\alpha$$

for $w \in W_0$. This implies $U_\alpha = U_{s_\alpha}$ for any simple α. With this, we obtain the Bruhat decomposition:

Proposition 1.2.3. *(i) G is the disjoint union of the double cosets $\{BwB\}_{w\in W_0}$*

(ii) For any lift n of $w \in W_0$, the map

$$U_{w^{-1}} \times B \to BwB, (u,b) \mapsto unb$$

is an isomorphism of varieties.

Proof. See [Spr09][8.3.6 (ii) and 8.3.8]. □

The Finite Weyl Group

Given a based root datum $(X, Y, \Phi, \check{\Phi}, \Delta)$, we can define its finite Weyl group W_0 to be the subgroup of $\mathrm{GL}(X)$ generated by $\{s_\alpha\}_{\alpha\in\Phi}$. This is canonically isomorphic to the finite Weyl group of the dual root datum, i.e. the subgroup of $\mathrm{GL}(Y)$ generated by all $s_{\check{\alpha}}$. This isomorphism sends s_α to $s_{\check{\alpha}}$ for all $\alpha \in \Phi$. We will identify these groups and write s_α instead of $s_{\check{\alpha}}$ in this case.

If a root datum arises from a reductive group, the Weyl group defined this way is isomorphic to $N(T)/T$ by identifying s_α with the image of $\phi_\alpha(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})$ in W_0. Thus, we may identify both these Weyl groups and it will cause no confusion to call them both W_0.

Let $S_0 = \{s_\alpha\}_{\alpha\in\Delta}$. We will call the elements of S_0 simple reflections. It follows from the general theory of root data that (W_0, S_0) is a Coxeter system, i.e. we have $s^2 = 1$ for all $s \in S_0$ and W_0 has the presentation

$$W_0 = \langle S_0 : (ss')^{m_{s,s'}} = 1 \rangle,$$

where $m_{s,s'}$ is the order of ss' in W_0 and s and s' run through the elements of S_0.

For each $w \in W_0$, there exists a minimal $n \in \mathbb{N}_0$ such that there exist $s_1, \dots, s_n \in S_0$ such that $w = s_1 \dots s_n$. An expression of this kind with minimal n is called reduced and n is called the length of w. This defines a length function $l : W_0 \to \mathbb{N}_0$. It can also be described by $l(w) = \#(\Phi^+ \cap w^{-1}\Phi^-)$ and the set $\Phi^+ \cap w^{-1}\Phi^-$ determines w uniquely. We have $l(w) = l(w^{-1})$ for all $w \in W_0$.

With respect to this length function, there exists a unique element of maximal length in W_0 which we will denote by w_0. As w_0^{-1} is also an element of the same length, we get $w_0 = w_0^{-1}$. For any $\Delta' \subseteq \Phi$, such that $(X, Y, \Phi, \Phi^+, \Delta')$ is a based root datum, and any numbering $\Delta' = \{s_1, \dots, s_d\}$, the element $v = s_1 \cdot \dots \cdot s_d$ is called a Coxeter element of W_0. Obviously v depends on these choices, but it can be shown that two Coxeter elements in W_0 are conjugate (see [Hum90] Proposition 3.16). The converse is obvious and hence the set of Coxeter elements is a full conjugacy class in W_0.

In addition to the natural action of W_0 on X, there exists another twisted action, the so-called dot-action. To define this, let $\rho \in \frac{1}{2}\sum_{\alpha\in\Phi^+} \alpha + (X \otimes \mathbb{Q})^{W_0} \subseteq X \otimes \mathbb{Q}$. Using such an element, we can define

$$w \cdot x = w(x + \rho) - \rho$$

for $w \in W_0$ and $x \in X$. It can easily be verified that this defines an action of W_0 on the set X which does not depend on the choice of ρ.

Affine Weyl Groups

For this subsection, we are still given a based root datum $(X, Y, \Phi, \check{\Phi}, \Delta)$. Recall that W_0 identifies with the subgroup of $\mathrm{GL}(Y)$ generated by all $s_{\check{\alpha}}$. It operates on Y and $\check{Q}$ which allows us to define the groups

$$W_{aff} = W_0 \ltimes \check{Q}$$

and

$$W = W_0 \ltimes Y.$$

W_{aff} is a normal subgroup of W. We call W_{aff} the affine Weyl group and W the extended affine Weyl group or simply the Weyl group. For an element (w, y) of either of these groups we shall write we^y. This exponential notation has the advantage that we can write the group law in Y resp. $\check{Q}$ additively but multiplicatively in W_{aff} resp. W. For example, we have $e^{y_1} \cdot e^{y_2} = e^{y_1+y_2}$. By definition, we have

$$we^y = e^{w(y)}w \text{ for all } y \in Y, w \in W_0.$$

As in the case of the finite Weyl group it turns out that W_{aff} is a Coxeter group. More precisely, setting $S_{aff} = S_0 \cup \{s_\alpha e^{\check{\alpha}} : \alpha \in \Phi_m\}$, the tuple (W_{aff}, S_{aff}) is a Coxeter system. We will call the elements of S_{aff} simple affine reflections. As in the finite case, we obtain a length function on W_{aff}. As this restricts to the length function of W_0, there is no harm in denoting both by l.

Although the extended affine Weyl group W is not a Coxeter group in general, it is not very far away from being one. Let Ω be the normalizer of S_{aff} in W. Then we can write W as a semidirect product

$$W = W_{aff} \rtimes \Omega.$$

This way we get a well defined extension of l to W by setting $l(uw) = l(w)$ for $u \in \Omega$ and $w \in W_{aff}$. We also have $l(wu) = l(w)$ for $u \in \Omega$ and $w \in W_{aff}$ and $\Omega = \{u \in W : l(u) = 0\}$. Note that $\Omega \cong Y/\check{Q}$ is abelian.

The length function on W and W_{aff} can be expressed by affine roots. For this let

$$\Phi_{aff} := \Phi \times \mathbb{Z} \subseteq X \times \mathbb{Z}$$

and denote by

$$\Phi^+_{aff} := \Phi \times \mathbb{Z}_{>0} \cup \Phi^+ \times \{0\}$$

resp.

$$\Phi^-_{aff} := \Phi \times \mathbb{Z}_{<0} \cup \Phi^- \times \{0\}$$

the subsets of positive resp. negative affine roots. W and W_{aff} operate on the set of affine roots by

$$we^y(\alpha, k) := (w(\alpha), k - \langle \alpha, y \rangle).$$

As for the finite Weyl group, we have for $w \in W$:

$$l(w) = \#\Phi^+_{aff} \cap w^{-1}\Phi^-_{aff}.$$

1.3 Rationality

k-Group Functors

Now we will generalize the concept of algebraic groups by loosening the assumption that K is an algebraically closed field. We follow [Jan87][I.1 and I.2]. Let k be any commutative ring with unit. A k-functor is a functor from the category of commutative k-algebras to the category of sets. Similarly a k-group functor is a functor from the category of commutative k-algebras to the category of groups. We may regard every k-group functor as a k-functor by composing with the forgetful functor from the category of groups to the category of sets. If G and H are two k-functors (resp. k-group functors) we will denote by $\mathrm{Mor}(G, H)$ (resp. $\mathrm{Hom}(G, H)$) the set of all natural transformations from G to H considered as k-functors (resp. k-group functors). Hence, if $f \in \mathrm{Mor}(G, H)$, f lies in $\mathrm{Hom}(G, H)$ if and only if $f(A)$ is a group homomorphism for every commutative k-algebra A.

Now let k' be any k-algebra. Then each commutative k'-algebra has a natural structure as a k-algebra. Hence, every k-functor X defines a k'-functor $X_{k'}$ given by $X_{k'}(A) = X(A)$ for any commutative k'-algebra A. This way we obtain a functor from the category of k-functors to the category of k'-functors which we will call the base chance from k to k'. We will say that a k'-functor X' is defined over k if there exists a k-functor X such that $X = X'_{k'}$.

For any k-algebra R we can define a k-functor Sp_kR by $(Sp_kR)(A) := \mathrm{Hom}(R, A)$, where $\mathrm{Hom}(R, A)$ denotes the set of k-algebra homomorphism from R to A. For $\varphi \in \mathrm{Hom}(A, A')$ we define

$$(Sp_kR)(\varphi) : \mathrm{Hom}(R, A) \to \mathrm{Hom}(R, A'), \alpha \mapsto \varphi \circ \alpha.$$

A k-functor will be called an affine scheme over k if it is isomorphic to some Sp_kR. For $n \geq 1$, we set $\mathbb{A}^n := Sp_k(k[T_1, \ldots, T_n])$, where $k[T_1, \ldots, T_n]$ denotes the polynomial ring in n variables over k. This allows us to recover R form Sp_kR: Yoneda's Lemma yields the bijection $\mathrm{Mor}(Sp_kR, X) \xrightarrow{\cong} X(R)$ given by $f \mapsto f(R)(id_R)$ for every k-functor X. In particular, we get for $X = \mathbb{A}^1$

$$\mathrm{Mor}(Sp_kR, \mathbb{A}^1) \xrightarrow{\cong} \mathbb{A}^1(R) = \mathrm{Hom}(k[T], R) \xrightarrow{\cong} R.$$

For any affine k-functor, the set $k[X] := \mathrm{Mor}(X, \mathbb{A}^1)$ has a natural structure as a k-algebra: For example, we can define f_1+f_2 by $(f_1+f_2)(A)(x) = f_1(A)(x)+f_2(A)(x)$ for $f_1, f_2 \in k[X]$, any commutative k-algebra A and $x \in X(A)$. Multiplication and scalar multiplication can be defined similarly. With this definition we get that $k[Sp_kR] \cong R$. We will call an affine scheme X algebraic if $k[X]$ is isomorphic to a k-algebra of the form $k[T_1, \ldots, T_n]/I$ for some $n \in \mathbb{N}$ and a finitely generated ideal $I \subseteq k[T_1, \ldots, T_n]$. We call X reduced if the ring $k[X]$ is reduced.

A k-group scheme is a k-group which is an affine scheme when considered as a k-group functor. We call an algebraic k-group scheme if it is algebraic considered as an affine scheme. Similarly, a reduced algebraic k-group scheme is an algebraic k-group scheme which is reduced. If k is an algebraically closed field the category of linear algebraic groups identifies with the subcategory of all reduced algebraic

k-groups. Hence, the concept of k-groups generalizes the concept of linear algebraic groups presented before.

Frobenius maps

Now we turn to the special situation that $k = \mathbb{F}_q$ is a finite field and $K = \overline{\mathbb{F}}_q$ is its algebraic closure. Assume we are given an linear algebraic group G over K, which is defined over k, i.e. $G = G'_K$ for some reduced algebraic k-group G'. For each commutative k-algebra A we have the k-linear Frobenius endomorphism ϕ_A of A given by $a \mapsto a^q$. Hence $A \mapsto G'(\phi_A)$ is an endomorphism of G' which we shall denote by $F_{G'}$, the Frobenius endomorphism of G'. By restriction this becomes an endomorphism F of G, which we may view as a linear algebraic group. It corresponds to the homomorphism

$$K[G] = k[G] \otimes_k K \to K[G] = k[G] \otimes_k K, f \otimes a \mapsto f^q \otimes a.$$

When we speak of a group with a Frobenius map we mean a linear algebraic group over $\overline{\mathbb{F}}_q$ which is defined over $\mathbb{F}_q$ so the the corresponding reduced algebraic $\mathbb{F}_q$-group functor induces this Frobenius map. In this situation, the $\mathbb{F}_q$-rational points $G(\mathbb{F}_q)$ are given by the Frobenius fixed points G^F. Note that if X is a closed subvariety of some affine space K^n, then F is the map which raises each component to its q-th power. A subvariety of G is defined over $\mathbb{F}_q$ if and only if it is stable under the action of F.

Let $\Lambda : G \to G, g \mapsto g^{-1}F(g)$. The following theorem by Lang and Steinberg is very useful for studying linear algebraic groups with Frobenius maps:

Theorem 1.3.1. *If G is a connected linear algebraic group with Frobenius map, the map Λ is surjective.*

Proof. See [Spr09, Thm. 4.4.17]. □

For example, this implies that a connected group G with a Frobenius map is quasi-split, i.e. there exists a Borel subgroup in G which is defined over $\mathbb{F}_q$. To see this, let $B \subseteq G$ be any Borel subgroup. The two Borel subgroups B and $F(B)$ are conjugate in G, hence there exists an $h \in G$ such that ${}^hF(B) = B$. By the Lang-Steinberg Theorem, we find some $g \in G$ with $g^{-1}F(g) = \Lambda(g) = h$ and obtain

$$F({}^gB) = {}^{F(g)}F(B) = {}^{gh}F(B) = {}^gB,$$

hence ${}^gB$ is defined over $\mathbb{F}_q$.

Now, choosing a maximal torus T inside a Borel subgroup B defined over $\mathbb{F}_q$ and applying the Lang-Steinberg Theorem to the groups $T \subseteq B$ as in the above situation, one sees that each Borel defined over $\mathbb{F}_q$ contains a maximal torus which is defined over $\mathbb{F}_q$.

Let G be a group with a Frobenius map F and a maximal torus T contained in an F-stable Borel subgroup. Then the action of F on T induces actions on the character and cocharacter groups which we will denote by F again. Explicitly, we have

$$(F(\chi))(t) = \chi(F(t)) \text{ for all } \chi \in X(T)$$

and

$$(F(\gamma))(x) = F(\gamma(x)) \text{ for all } x \in \mathbb{G}_m.$$

This obviously implies that $\langle F(\chi), \gamma\rangle = \langle \chi, F(\gamma)\rangle$ for $\chi \in X(T)$ and $\gamma \in Y(T)$. Note that F also acts on the finite Weyl group: With T its normalizer $N(T)$ is F-stable and hence F acts on W by $F(nT) = F(n)T$ for $n \in N(T)$. We have $W^F = N^F/T^F$.

Now we can introduce the notion of duality for groups with Frobenius map. Let G resp. G^* be groups with Frobenius maps F resp. F^* and F-stable maximal tori T resp. T^* which are contained in F-stable Borel subgroups. Denote their root data by $(X(T), Y(T); \Phi, \check{\Phi}, \Delta)$ resp. $(X(T^*), Y(T^*); \Phi^*, \check{\Phi}^*, \Delta^*)$. We say that G and G^* are in duality if the root datum of G is isomorphic to the dual root datum of G^* by isomorphisms $\delta : X(T) \to Y(T^*)$ and $\epsilon : Y(T) \to X(T^*)$ compatible with F, i.e.

$$\delta(F(\chi)) = F^*(\delta(\chi)) \text{ for all } \chi \in X(T^*)$$

and similarly for ϵ.

If W_0 resp. W_0^* are the corresponding finite Weyl groups, there exists an isomorphism between W_0 and W_0^* which we will also denote by δ. It is determined by $\delta(s_\alpha) = s_{\delta(\alpha)\check{}}$ for all $\alpha \in \Phi$. This isomorphism is induced by the isomorphism $\mathrm{GL}(X(T)) \cong \mathrm{GL}(Y(T^*))$ defined by δ. Induction on the word length shows that we have

$$\delta(w(\chi)) = \delta(w)(\delta(\chi))$$

for all $w \in W_0$. Note that the formula looks a little different than the one in [Car93, Prop. 4.2.3] where w has to be inverted on one side of the equation. This is explained by the fact that we haven chosen to let W_0 operate from the left on the cocharacters rather than from the right.

Rationality for tori

Assume we are given a group G defined over k. A torus T defined over k is called (k-)split if it is isomorphic to some $\mathbb{G}_m^n$ over k. We will usually only speak of split tori if the corresponding field is clear from the context. We call G (k-)split if it contains a maximal torus which is (k-)split.

Now assume again that we are given a group G with Frobenius map F and a maximal torus T_0 contained in an F-stable Borel subgroup in G. Each other maximal torus in G is conjugate to T_0, hence of the form ${}^gT_0$ for some $g \in G$. This torus is defined over $\mathbb{F}_q$ if and only if it is invariant under the Frobenius map hence if and only if

$${}^gT_0 = F({}^gT_0) = {}^{F(g)}F(T_0) = {}^{F(g)}T_0,$$

or equivalently if and only if $\Lambda(g) = g^{-1}F(g)$ lies in the normalizer of T_0. The reduction of $\Lambda(g)$ modulo T_0 defines an element of the finite Weyl group. We get the following classification of $G(\mathbb{F}_q)$-conjugacy classes of maximal tori in G.

Proposition 1.3.2. *The map ${}^gT_0 \mapsto \Lambda(g) \mod T_0$ induces a well-defined bijection from the set of G^F-conjugacy classes of maximal tori in G to the F-conjugacy classes in W.*

Proof. See [Car93, prop. 3.3.2, prop. 3.3.3]. □

1.4 Example: GL_n

In this section we will illuminate all the constructions discussed before for the linear algebraic group $\mathrm{GL}_n(K)$, i.e. the group of all invertible $n \times n$ matrices over K, for an algebraically closed field K. It can be realized as the closed subset

$$\{(a_{11}, \ldots, a_{nn}, a) \in K^{n^2+1} : det((a_{ij})_{i,j})a = 1\}$$

of K^{n^2+1}. Hence the corresponding affine algebra is $K[T_{i,j}, \det(T_{i,j})^{-1}]$.

Let $B \subseteq \mathrm{GL}_n(K)$ be the subgroup of invertible upper triangular matrices and let T be the subgroup of invertible diagonal matrices. Then T is contained in B, T is a maximal torus and B is a Borel subgroup of $\mathrm{GL}_n(K)$. The corresponding opposite Borel subgroup B^- is the subgroup of lower triangular matrices. The radical of $\mathrm{GL}_n(K)$ is its center, i.e. the set of nonzero scalar multiples of the identity matrix. In particular, $\mathrm{GL}_n(K)$ is reductive and not semisimple.

For $1 \leq i \leq n$ denote by χ_i the character of T given by

$$\begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix} \mapsto x_i$$

and by γ_i the cocharacter of T given by

$$x \mapsto \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & x & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix},$$

where the i-th diagonal entry is x and all other diagonal entries are 1. The χ_i resp. γ_i are a basis of $X(T)$ resp. $Y(T)$, which we will both identify with $\mathbb{Z}^n$ by means of these bases and write $(x_1, \ldots, x_n)$ for the element $\sum_{i=1}^n x_i\chi_i \in X(T)$ and $(y_1, \ldots, y_n)$ for the element $\sum_{i=1}^n y_i\gamma_i \in Y(T)$. $\{\chi_i\}_{1\leq i\leq n}$ is the dual basis of $\{\gamma_i\}_{1\leq i\leq n}$ with respect to the pairing $\langle\cdot,\cdot\rangle$.

The roots of G with respect to T are of the form $\alpha_{i,j} = \chi_i - \chi_j$ with $1 \leq i \neq j \leq n$. The associated coroots are $\check{\alpha}_{i,j} = \gamma_i - \gamma_j$. The choice of our Borel subgroup marks the roots $\alpha_{i,j}$ for $i < j$ as positive and the roots $\alpha_i := \alpha_{i,i+1}$ for $1 \leq i \leq n-1$ as simple. Hence, $S_0 = \{s_{\alpha_1}, \ldots, s_{\alpha_{n-1}}\}$. The root subgroup $U_{\alpha_{i,j}}$ consists of those matrices which have ones on their diagonal and all other entries except for possibly the one in the i-th row and the j-th column are zero. We have the isomorphism

$$u_{\alpha_{i,j}} : \mathbb{G}_a \to U_{\alpha_{i,j}}, x \mapsto E_n + xE_{i,j},$$

where E_n is the $n \times n$ identity matrix and and $E_{i,j}$ is the matrix with 1 in the (i,j)-position and zeros elsewhere.

For $1 \leq i \neq j \leq n$, we have $s_{\alpha_{i,j}}(\chi_i) = \chi_j$. Thus, for each $w \in W_0$ there exists a unique $\sigma \in S_n$ such that $w(\chi_i) = \chi_{\sigma(i)}$. The group homomorphism $W_0 \to S_n$ defined this way is surjective, because S_n is generated by the transpositions and injective because the χ_i generate $X(T)$. We will identify W_0 with S_n.

For each $\sigma \in S_n$, we can define a permutation matrix $P_\sigma = (p_{ij})_{i,j}$ by $p_{ij} = \delta_{i,\sigma(j)}$. We obtain a group homomorphism from W_0 to $\mathrm{GL}_n(K)$ given by $\sigma \mapsto P_\sigma$. Its image is obviously contained in the normalizer of T and it is a splitting of the group homomorphism $N(T) \to N(T)/T \cong W_0$. We emphasize the fact that the existence of such a splitting is specific for GL_n and not true for general reductive groups. For example in $\mathrm{SL}_2(K)$, there exists no element of order 2 congruent to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ modulo the torus of diagonal matrices with determinant 1.

Via the isomorphism $W_0 \cong S_n$, the simple reflection $s_i := s_{\alpha_i}$ identifies with the transposition $(i, i+1)$ of S_n for $1 \leq i \leq n-1$. The longest element w_0 is given by $w_0(\chi_i) = \chi_{n+1-i}$ for $1 \leq i \leq n$ and hence we have $w_0(\alpha_i) = -\alpha_{n-i}$ for $1 \leq i \leq n-1$. Our choice and numbering of Δ defines the Coxeter element $v = s_1 \ldots s_{n-1}$. This identifies with the n-cycle $(1, 2, \ldots, n) \in S_n$, i.e.

$$v(\chi_i) = \begin{cases} \chi_{i+1} & \text{if } 1 \leq i \leq n-1, \\ \chi_1 & \text{if } i = n \end{cases}$$

and we get the same statement with γ instead of χ. The set of Coxeter elements in S_n for different choices of Δ and numberings of the simple roots consists precisely of the n-cycles.

Via the identifications $X(T) \cong \mathbb{Z}^n$ resp. $Y(T) \cong \mathbb{Z}^n$, the root- and coroot-lattices correspond to those elements $(x_1, \ldots, x_n) \in \mathbb{Z}^n$ (resp. $(y_1, \ldots, y_n) \in \mathbb{Z}^n$) with $\sum x_i = 0$ (resp. $\sum y_i = 0$).

The unique minimal root is $\alpha_0 = \alpha_{n,1} = -\alpha_{1,n} = -\chi_1 + \chi_n$. This means that the set of Coxeter generators S_{aff} for W_{aff} is given by $S_0 \cup \{s_{\alpha_0} e^{\check{\alpha}_0}\}$, where

$$s_{\alpha_0}(\chi_i) = \begin{cases} \chi_n & \text{if } i = 1 \\ \chi_i & \text{if } 1 < i < n \\ \chi_1 & \text{if } i = n. \end{cases}$$

We set $s_0 =: s_{\alpha_0} e^{\check{\alpha}_0}$.

The group Ω is of a very simple nature in this case: Setting $u_0 = e^{\gamma_1} v$, we have $\Omega = \langle u_0 \rangle \cong \mathbb{Z}$. Additionally, the element u_0 permutes the simple affine reflections transitively. More precisely, we have $u_0 s_i u_0^{-1} = s_{i+1}$ for $0 \leq i \leq n-2$ and $u_0 s_{n-1} u_0^{-1} = s_0$. Of course, this behavior is specific to the case of GL_n: For example $\Omega \cong Y(T)/\check{Q}$ is trivial for any simply connected semisimple linear algebraic group, such as SL_n.

Let us fix some $\alpha = \alpha_{i,j} \in \Phi$. Then, SL_2 becomes a subgroup of GL_n via

$$\phi_\alpha : \mathrm{SL}_2 \to \mathrm{GL}_n, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \sum_{k \neq i,j} E_{k,k} + aE_{i,i} + bE_{i,j} + cE_{j,i} + dE_{j,j}$$

$$= \begin{pmatrix} 1 &&&&&&&&& \\ & \ddots &&&&&&&& \\ && 1 &&&&&&& \\ &&& a &&&& b && \\ &&&& 1 &&&&& \\ &&&&& \ddots &&&& \\ &&&&&& 1 &&& \\ &&& c &&&& d && \\ &&&&&&&& 1 & \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{pmatrix}.$$

This yields the isomorphisms $u_\alpha : \mathbb{G}_a \to U_\alpha, x \mapsto E_n + xE_{i,j}$.

Now we turn to questions of rationality for $K = \overline{\mathbb{F}}_q$. The $\overline{\mathbb{F}}_q$-group functor GL_n is obviously defined over $\mathbb{F}_q$ and the corresponding Frobenius map raises each matrix entry to its q-th power, i.e. $F((a_{i,j})_{i,j}) = (a_{i,j}^q)_{i,j}$. As expected, the $\mathbb{F}_q$-rational points coincide with the invertible $n \times n$ matrixes with entries in $\mathbb{F}_q$.

In the following, we will examine the $\mathrm{GL}_n(\mathbb{F}_q)$-conjugacy classes of maximal tori defined over $\mathbb{F}_q$ in $\mathrm{GL}_n(K)$. Recall from Proposition 1.3.2 that the $\mathrm{GL}_n(\mathbb{F}_q)$-conjugacy classes are in bijection to the F-conjugacy classes of W_0. It is easily seen that F acts trivially on W and thus we can replace F-conjugacy classes by regular conjugacy classes. For following applications, we will be particularly interested in the torus, which corresponds to the conjugacy class of the Coxeter elements. One representative is given by the n-cycle v defined above. Choosing $g_v \in \mathrm{GL}_n(K)$ with $\Lambda(g_v)T = v$, one representative for the corresponding $\mathrm{GL}_n(\mathbb{F}_q)$-conjugacy class of tori is given by ${}^{g_v}T$. We could explicitly determine such an element g but this will not be necessary in the following. A matrix ${}^{g_v}diag(t_1, \ldots, t_n) \in {}^{g_v}T$ is an $\mathbb{F}_q$-rational point if and only if

$$\begin{aligned} {}^{g_v}diag(t_1, \ldots, t_n) &= F({}^{g_v}diag(t_1, \ldots, t_n)) = {}^{F(g_v)}diag(t_1^q, \ldots t_n^q) \\ &= {}^{g_v \Lambda(g_v)}diag(t_1^q, \ldots t_n^q) = {}^{g_v v}diag(t_1^q, \ldots t_n^q) \\ &= {}^{g_v}diag(t_n^q, t_1^q, \ldots, t_{n-1}^q). \end{aligned}$$

It follows that the $\mathbb{F}_q$-rational points are given by the conditions

$$t_1 = t_n^q = t_{n-1}^{q^2} = \ldots = t_2^{q^{n-1}} = t_1^{q^n}.$$

From this, it is clear that ${}^{g_v}T(\mathbb{F}_q) \cong \mathbb{F}_{q^n}^\times$ (in a non-unique way). In particular, the torus ${}^{g_v}T$ is not split. More precisely, the center is a maximal split subtorus of ${}^{g_v}T$. Hence, ${}^{g_v}T$ is "as non-split as possible".

1.5 Groups with BN-Pair

For a moment, let G be any group.

Definition 1.5.1. We say that (G, B, N, S) is a BN-pair, if the following conditions are satisfied

(i) G is generated by the two subgroups B and N.

(ii) $B \cap N$ is a normal subgroup of N.

(iii) $N/(B \cap N)$ is generated by a subset S consisting of elements of order 2.

(iv) For each $s \in S$ and $w \in N/(B \cap N)$, we have $sBw \subseteq BwB \cup BswB$.

(v) For all $s \in S$, $sBs \not\subseteq B$.

In points (iv) and (v) of the preceding definition, we view elements of $N/(B \cap N)$ as cosets modulo $B \cap N$, hence subsets of G. Hence, expressions like sBw make sense as subsets of G. Sometimes, BN-pairs are also called Tits systems.

Now assume we are given a BN-pair (G, B, N, S). Then denote by W_0 the quotient $N/(B \cap N)$ and by T the intersection $B \cap N$. From now on we will write S_0 instead of S for better compatibility with our notations in the case of reductive groups. The tuple (W_0, S_0) is a Coxeter system (cf. [Bou02, Thm. 2 in §2.4]). We can define a root system by means of (W_0, S_0) in the following way: Let $\Delta = \{\delta_s : s \in S_0\}$ and let V be the $\mathbb{R}$-vector space with basis Δ equipped with the symmetric bilinear form given by $\langle \delta, \delta' \rangle := -\cos(\frac{\pi}{s_\delta s_{\delta'}})$. Then W_0 acts faithfully on V by the morphism which sends s_δ to the reflection at the hyperplane orthogonal to δ. We set $\Phi := W_0(\Delta)$. Clearly this returns the root system if W_0 is defined by a root datum as introduced before. Each element of Φ is an integral linear combination of elements in Δ with either only nonnegative or only nonpositive coefficients. This way, we can define positive resp. negative roots Φ^+ resp. Φ^-.

Now assume that W_0 is finite. There exists a unique $w_0 \in W_0$ of maximal length. Further, let $I \subseteq S$ be any subset and let W_I be the subgroup generated by I. The tuple (W_I, I) is again a Coxeter System. Let w_I denote its longest element.

For each $w \in W_0$, we may write $B^w := w^{-1}Bw$, because B contains T and $U^w := w^{-1}Uw$, because T normalizes U and T is abelian. For $w \in W_0$, we set $B_w = B \cap B^{w_0 w}$ and $U_w = U \cap U^{w_0 w}$.

Definition 1.5.2. Let (G, B, N, S) be a BN-pair. We call it split of characteristic p, if G is finite and the following conditions are satisfied:

1. $B \cap B^{w_0} = T$.

2. There exists a normal subgroup U of B such that B decomposes as a semidirect product $B = UT$.

3. U is a p-group and T is of order prime to p.

We call (G, B, N, S) strongly split, if additionally for each subset $I \subseteq S$, $U_I := U \cap U^{w_I}$ is normal in U.

Now assume we are given a split BN-pair (G, B, N, T) of characteristic p with finite W_0. If α is a simple root, we write B_α for B_{s_α}. Denote by X_α the set of p-elements in B_α. For $w \in W_0$ and $\alpha \in \Delta$, the subgroup ${}^wX_\alpha$ only depends on $w\alpha$ and will be denoted by $X_{w\alpha}$. Because each root is conjugate to a simple root by W_0, we obtain subgroups X_α for all $\alpha \in \Phi$. Let G_α be the subgroup generated by X_α and $X_{-\alpha}$ and let $T_\alpha = T \cap G_\alpha$.

In our applications, we will always be in the strongly split case of characteristic p:

Lemma 1.5.3. *Let G be a connected reductive group defined and split over $\mathbb{F}_q$ with split maximal torus T defined over $\mathbb{F}_q$ contained in a Borel subgroup B defined over $\mathbb{F}_q$. Further, let N be the normalizer of T in G and let S_0 be the system of simple reflections defined by the choice of B. Then $(G(\mathbb{F}_q), B(\mathbb{F}_q), N(\mathbb{F}_q), S_0)$ is a strongly split BN-pair of characteristic p. Moreover, we have $X_\alpha = U_\alpha(\mathbb{F}_q)$ for each $\alpha \in \Phi$.*

Proof. This is basically contained in [Car93, 1.18]: With T, also N is defined over $\mathbb{F}_q$, hence the statement makes sense. The only thing not shown there is the fact that $U(\mathbb{F}_q) \cap U(\mathbb{F}_q)^{w_I}$ is normal in $U(\mathbb{F}_q)$ for each subset $I \subseteq S_0$, where $B = TU$. So, let $I \subseteq S_0$, Φ_I be the subroot system of Φ corresponding to I with corresponding positive roots $\Phi_I^+ = \Phi_I \cap \Phi^+$.

By [CE04, Thm. 2.21 (iii)], there exists a sequence $\alpha_1, \ldots, \alpha_N$ consisting of all positive roots such that for any $u \in U(\mathbb{F}_q)$, there exist unique $x_{\alpha_i} \in X_{\alpha_i}$ with

$$u = x_{\alpha_1} \cdot \ldots \cdot x_{\alpha_N}.$$

For $w \in W_0$, such u is contained in $U(\mathbb{F}_q)^w$ if and only if $\alpha_i \in w^{-1}R^+$ for each i with $x_{\alpha_i} \neq 1$ by [CE04, Lemma 2.22]. Writing

$$R^+ \cap {w_I}^{-1}R^+ = R^+ \setminus ({w_I}^{-1}R^- \cap R^+) = R^+ \setminus R_I^+ = \{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$$

with $i_1 < \ldots < i_l$, we obtain

$$U(\mathbb{F}_q)_I = X_{\alpha_{i_1}} \ldots X_{\alpha_{i_l}}.$$

Hence, it suffices to show that X_α normalizes X_β for $\alpha \in R^+$ and $\beta \in R^+ \setminus R_I^+$. As T is split over $\mathbb{F}_q$, the root system of the algebraic group coincides with that of the BN-pair, which is defined by means of the Weyl group $W_0 = N(\mathbb{F}_q)/T(\mathbb{F}_q) \cong N/T$. Now, if $\{U_\alpha\}_{\alpha \in R}$ are the root subgroups of the reductive group G relative to T, we have $U \cap U^{w_0 s_\alpha} = U_\alpha$ for each simple $\alpha \in \Delta$ by [Spr09, Proposition 8.2.1], which implies $B \cap B^{w_0 s_\alpha} = TU_\alpha$. Hence, X_α, the set of p-elements in $B(\mathbb{F}_q) \cap B^{w_0 s_\alpha}(\mathbb{F}_q) = T(\mathbb{F}_q)U_\alpha(\mathbb{F}_q)$, is equal to $U_\alpha(\mathbb{F}_q)$ for simple α, which implies the corresponding equality for general $\alpha \in \Phi$, because ${}^wU_\alpha = U_{w\alpha}$ for each $w \in W_0$. This implies the commutator relation

$$[X_\alpha, X_\beta] \subseteq \langle X_\gamma : \gamma \in (\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap R \rangle \subseteq U(\mathbb{F}_q)_I,$$

by [Spr09, Proposition 8.2.3], which shows that X_α normalizes X_β, because β and hence each γ as above contains at least one simple root not corresponding to an element of I.

□

By [CE04, Prop. 6.3 (i)] there exist elements $n_\alpha \in X_\alpha X_{-\alpha} X_\alpha \cap N(\mathbb{F}_q)$ for all $\alpha \in \Delta$ such that the image of $n_\alpha = n_{s_\alpha}$ in W_0 is s_α. We will fix such a system of representatives and denote by $S_0^{(1)}$ the set of all n_α.

Chapter 2

Galois Representations

2.1 Representations of W_L

We will now briefly review some generalities on absolute Galois groups over local fields and the corresponding Weil groups. For more details, see [Tat79] and [Del73, chap. 2]. Denote by $G_L = \mathrm{Gal}(\overline{L}/L)$ the absolute Galois group of L, where L still denotes a finite extension of $\mathbb{Q}_p$. We have a well defined surjective group homomorphism from G_L to $G_{\overline{\mathbb{F}}_q} = \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ given by $\sigma \mapsto [x + \mathfrak{m}_{\overline{L}} \mapsto \sigma(x) + \mathfrak{m}_{\overline{L}}]$. Its kernel I_L is called the inertia subgroup. Hence, we have the exact sequence

$$1 \to I_L \to G_L \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \to 1.$$

The group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is isomorphic to $\hat{\mathbb{Z}}$ topologically generated by the (arithmetic) Frobenius automorphism $x \mapsto x^q$. Denote by φ a lift of this Frobenius in G_L. We will also call φ a Frobenius. The choice of φ gives a unique continuous homomorphism $\hat{\mathbb{Z}} \to G_L$ mapping 1 to φ, which allows us to speak of $\hat{\mathbb{Z}}$-powers of φ. Each element of G_L is uniquely a product of a $\hat{\mathbb{Z}}$-power of φ and and an element of I_L. In other words we obtain a splitting of the above exact sequence, i.e. $G_L \cong I_L \rtimes \hat{\mathbb{Z}}$ as abstract groups.

Definition 2.1.1. The Weil group W_L of L is the subgroup of G_L which is generated by φ and I_L. We endow W_L with the unique topology such that I_L is open in W_L and carries the profinite topology induced from G_L.

I_L has a unique pro-p-Sylow subgroup P_L. This is a normal subgroup in W_L: Any conjugate of P_L is contained in I_L, because I_L is normal in W_L. But this conjugate is also pro-p, hence contained in P_L.

The quotient I_L/P_L is (non-canonically) isomorphic to $\prod_{l \neq p} \mathbb{Z}_l \cong \varprojlim_{(p,n)=1} \mathbb{Z}/n\mathbb{Z}$. Note that the definition of W_L does not depend on the choice of φ and that $W_L \subseteq G_L$ is a dense subset in the profinite topology of G_L. We also remark that the topology on W_L is strictly finer than the subspace topology induced from G_L, because I_L is open in W_L but not open in G_L, as it is not of finite index in the compact group G_L.

Another way to phrase this definition of the Weil group is to define W_L as the preimage of $\mathbb{Z}$ under the projection $G_L \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ which induces the exact sequence

$$1 \to I_L \to W_L \to \mathbb{Z} \to 0.$$

Again, the choice of φ defines a splitting such that $W_L \cong I_L \rtimes \mathbb{Z}$. The topology described above is the product topology induced by any such splitting, where $\mathbb{Z}$ is endowed with the discrete topology.

In the following, we will discuss the representation theory of W_L. We will begin by comparing the representation theory of W_L in characteristic p to that of G_L. Unless otherwise stated, a representation of W_L resp. G_L will always be a continuous representation of W_L resp. G_L on a finite dimensional $\overline{\mathbb{F}}_q$-vector space V where $\mathrm{GL}(V)$ carries the discrete topology.

Lemma 2.1.2. *Each open subgroup of W_L which is of finite index is also open in the induced topology of G_L.*

Proof. We first treat the case that our given subgroup is normal. So let $U \subseteq W_L$ be such an open normal subgroup of finite index. As U is open and normal in W_L, $I_0 := I_L \cap U$ is open in I_L and normal in W_L. On the other hand, being of finite index, U must contain some nontrivial power of φ, say φ^m for some $m > 0$. Hence, U contains the subgroup $\varphi^{m\mathbb{Z}} I_0$ of W_L.

Because of $\varphi^{m\mathbb{Z}} I_0 = \varphi^{m\hat{\mathbb{Z}}} I_0 \cap W_L$ it suffices to show that $\varphi^{m\hat{\mathbb{Z}}} I_0$ is open in G_L. On the one hand, it is closed as it is the product of the compact subgroups I_0 and $\varphi^{m\hat{\mathbb{Z}}}$. On the other hand its index in G_L is $m[I_L : I_0]$, hence finite. Both together imply that it is an open subgroup of G_L.

Now we turn to the general case. Let $\{\sigma_1, \dots, \sigma_r\}$ be a system of representatives for $U \backslash W_L$. Then all the subgroups $\sigma_i U \sigma_i^{-1}$ have finite index in G. Hence their intersection $U' := \bigcap_i \sigma_i U \sigma_i^{-1} \subseteq U$ is of finite index in W_L by Poincaré's Theorem. On the other hand it is normal in W_L by construction and the finite intersection of the open subgroups $\sigma_i U \sigma_i^{-1}$, hence open in W_L. By the first case U' is open in the induced topology and thus the same is true for U. □

Proposition 2.1.3. *The category of finite-dimensional G_L-representations and the category of finite-dimensional W_L-representations are equivalent.*

Proof. First of all, any representation of G_L restricts to a representation of W_L. This restriction is also a continuous representation, because the topology of W_L is finer than the subspace topology. Hence, we obtain a functor from G_L-representations to W_L-representations.

On the other hand, we claim that each W_L-representation uniquely extends to a representation of G_L. Then this will obviously define a quasi-inverse functor. As W_L is dense in G_L, it is clear that an extension will be unique, if it exists.

So, let $\rho : W_L \to \mathrm{GL}(V)$ be a continuous representation for some finite dimensional $\overline{\mathbb{F}}_q$-vector space V. $\mathrm{GL}(V)$ is isomorphic to $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$, where n is the dimension of V. This implies, that every element, in particular $\rho(\varphi)$, in $\mathrm{GL}(V)$ is of finite order. Additionally, I_L is compact and hence its image under ρ is compact and discrete, thus finite. We see that the image of ρ is a finite subgroup of $\mathrm{GL}(V)$.

By Lemma 2.1.2, $ker(\rho)$ is also open in the subspace topology of W_L obtained from G_L, which means that ρ is also continuous in the subspace topology.

This implies $W_L \cap \overline{ker(\rho)} = ker(\rho)$ and we get an injective group homomorphism $W_L/ker(\rho) \to G_L/\overline{ker(\rho)}$. Because $\overline{ker(\rho)}$ contains some power of φ, the quotient is generated by φ and I_L, and the map is surjective. Now, we may extend ρ as the composite

$$G_L \to G_L/\overline{ker(\rho)} \cong W_L/ker(\rho) \to G_L(V).$$

□

Note that the argumentation of the previous proposition relies only on the fact that the image is finite. In particular, this equivalence of categories also holds if we replace the coefficients of our representations by $\overline{\mathbb{F}}_l$ for any other prime l.

The classification of irreducible W_L-representations resp. G_L-representations over $\overline{\mathbb{F}}_q$ is known, using Serre's fundamental characters [Ser71]. See for example [Vig97, 1.14, 1.15] for W_L-representations or [Ber10, Lemma 2.1.4] for G_L-representations. We will give a different method to phrase this classification which suits our needs better. To do this, we begin by remarking that any irreducible representation of W_L is trivial on P_L: On the one hand, every continuous representation of a pro-p group has a non trivial fixed vector in characteristic p. On the other hand, the vectors fixed by P_L are a subrepresentation because P_L is normal in W_L. The irreducibility implies that P_L acts trivially.

The group W_L/P_L is topologically generated by a topological generator α of I_L/P_L and φ. The only relation between those generators is $\varphi\alpha\varphi^{-1} = \alpha^q$ (cf. [Iwa55, Thm. 2 (i)]). For the remainder of this section, we fix such an element α in addition to our choice of φ.

In the following, we will be interested in classifying the irreducible representations of W_L. However, parts of this can be done more generally for any reductive group G^* over $\overline{\mathbb{F}}_q$. We use the superscript * here to indicate that G^* is supposed to be understood as the dual group of another reductive group G. By doing the following more generally we also avoid the confusion which arises from the fact that GL_n is dual to itself and hence it is difficult to see that representations should be understood as homomorphisms into the dual group of GL_n.

Any continuous $\rho : W_L/P_L \to \mathrm{G}^*(\overline{\mathbb{F}}_q)$ is given by a tuple $(s, u) \in \mathrm{G}^*(\overline{\mathbb{F}}_q)^2$, where s is the image of φ and u is the image of α. As a consequence, we have $sus^{-1} = u^q$. Note that this relation is invariant under conjugating s and u by the same element of $G^*(\overline{\mathbb{F}}_q)$. Also note that s and u both have finite order because we can embed G^* into some $\mathrm{GL}_m(\overline{\mathbb{F}}_q)$ where every element has finite order.

Conversely, if we are given a tuple $(s, u) \in (G^*)^2$ with the relation $sus^{-1} = u^q$ this defines a continuous homomorphism from W_L/P_L to G^*. Let m be the order of u. Then we obtain a continuous group homomorphism

$$I_L/P_L \to (I_L/P_L)/(I_L/P_L)^m \cong \mathbb{Z}/m\mathbb{Z} \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$$

sending α to u bearing in mind that $I_L/P_L \cong \prod_{l\neq p} \mathbb{Z}_l$. We can extend this to $W_L/P_L \cong \varphi^{\mathbb{Z}} \ltimes I_L/P_L$ by determining the image of φ as s. This is indeed a homomorphism because of $sus^{-1} = u^q$ and it is continuous because its kernel contains a subgroup which is of finite index in I_L/P_L.

Lemma 2.1.4. *Let $(s, u) \in G^*$ such that $sus^{-1} = u^q$. Then u is semisimple.*

Proof. Using Jordan decomposition, we can write $u = u_1u_2$ with commuting u_1 and u_2 such that u_1 is semisimple and u_2 is unipotent. Since any unipotent element is of p-power order, we have $u_2^{q^r} = 1$ for sufficiently large r. Then,

$$s^r us^{-r} = u^{q^r} = u_1^{q^r} u_2^{q^r} = u_1^{q^r}$$

shows that u is conjugate to a semisimple element, hence semisimple itself. □

Now suppose that G^* is equipped with a Frobenius map F^* and has a split maximal torus T^*. Denote by W_0^* its Weyl group. Further let $\rho : W_L/P_L \to G^*$ be a continuous homomorphism. Then, Lemma 2.1.4 implies that there exists a $g \in G^*$ such that the image of I_L/P_L under ${}^g\rho$ is contained in T^*.

Recall, that we have actions of W_0^* and F^* on T^*. These induce actions of W_0^* and F^* on $\mathrm{Hom}^{cont}(I_L/P_L, T^*)$. Explicitly, we have

$$w\psi = [\sigma \mapsto w\psi(\sigma)]$$

for $w \in W_0^*$ and $\psi \in \mathrm{Hom}^{cont}(I_L/P_L, T^*)$ and

$$F\psi = [\sigma \mapsto F\psi(\sigma)]$$

for all $\psi \in \mathrm{Hom}^{cont}(I_L/P_L, T^*)$.

Lemma 2.1.5. *Suppose we are given $g_1, g_2 \in G^*$ such that ${}^{g_1}\rho(I_L/P_L) \subseteq T^*$ and ${}^{g_2}\rho(I_L/P_L) \subseteq T^*$. Then there exists a $w \in W_0^*$ such that ${}^{wg_1}\rho_{|I_L/P_L} = {}^{g_2}\rho_{|I_L/P_L}$.*

Proof. The proof can be done analogously to that of [Car93][Prop. 3.7.1], which states that the semisimple conjugacy classes in G^* are in bijection with the Weyl group orbits of T^*.

It suffices to show that $t_1 := {}^{g_1}\rho(\alpha)$ and $t_2 := {}^{g_2}\rho(\alpha)$ are conjugate by some element of W_0^* for a topological generator α of I_L/P_L. Let $g := g_2g_1^{-1}$ so that we have ${}^g t_1 = t_2$. By the Bruhat decomposition there exists a unique $w \in W_0^*$ such that $g \in U^*_{w^{-1}}wB^*$ and for a fixed lift n of w in the normalizer of T^* there are unique $u \in U^*$, $t \in T^*$ and $u' \in U^*_{w-1}$ such that

$$g = u'ntu.$$

This implies

$$u'ntut_1 = t_2u'ntu,$$

or equivalently

$$u'n(tt_1)(t_1^{-1}ut_1) = (t_2u't_2^{-1})n(n^{-1}t_2nt)u$$

and by uniqueness we obtain

$$tt_1 = n^{-1}t_2nt$$

and hence $t_2 = nt_1n^{-1}$ which yields the claim. □

As a consequence of the lemma, ρ defines a W_0^*-orbit of homomorphisms from I_L/P_L to T^* which we will denote by $W_0^* res_{I_L/P_L}(\rho)$.

As T^* is split, F^* is the q-power map and we obtain

$$sus^{-1} = u^q = F^*(u)$$

so that $W_0 res_{I_L/P_L}(\rho)$ is F^*-stable. We will denote the set of F^*-stable W_0-orbits of continuous homomorphisms from I_L/P_L to T^* by $(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*}$. We have constructed a well-defined map

$$\mathrm{Hom}^{cont}(W_L/P_L, G^*) \to (\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*}.$$

On the other hand, we have a map

$$\mathrm{Hom}^{cont}(W_L/P_L, G^*) \to \mathrm{Hom}^{cont}(W_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times))$$

mapping $\rho \in \mathrm{Hom}^{cont}(W_L/P_L, G^*)$ to $\sigma \mapsto \mathrm{ev}_{\rho(\sigma)}$, where $\mathrm{ev}_{\rho(\sigma)}$ denotes the evaluation at $\rho(\sigma)$.

Analogously, we can define a map

$$\mathrm{Hom}^{cont}(I_L/P_L, T^*) \to \mathrm{Hom}^{cont}(I_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times)).$$

This map is constant on W_0^*-orbits because $\overline{\mathbb{F}}_q^\times$ is commutative. Together, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}^{cont}(W_L/P_L, G^*) & \to & \mathrm{Hom}^{cont}(W_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times)) \\ \downarrow & & \downarrow \\ (\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0)^F & \to & \mathrm{Hom}^{cont}(I_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times)) \end{array}$$

in which the right vertical map is given by restriction. This diagram allows us to define the fiber product

$$(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^F \times_{\mathrm{Hom}^{cont}(I_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times))} \mathrm{Hom}^{cont}(W_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times)).$$

In the following, we aim to describe the natural map from $\mathrm{Hom}^{cont}(W_L/P_L, G^*)$ into this fiber product in the case of $G = \mathrm{GL}_n$ (and thus $G^* = \mathrm{GL}_n$).

Proposition 2.1.6. *Let $(s, u) \in \mathrm{GL}_n(\overline{\mathbb{F}}_q)^2$ such that $sus^{-1} = u^q$.*

(i) Assume that $\overline{\mathbb{F}}_q^n$ has no nontrivial subspaces invariant under s and u. Then, there exists some $g \in \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ such that

$$gug^{-1} = \begin{pmatrix} y & & & \\ & y^q & & \\ & & \ddots & \\ & & & y^{q^{n-1}} \end{pmatrix},$$

and

$$gsg^{-1} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ (-1)^n z & & & & 0 \end{pmatrix},$$

where $y \in \mathbb{F}_{q^n} \setminus \cup_{m<n} \mathbb{F}_{q^m}$ *and* $z \in \overline{\mathbb{F}}_q^\times$.

(ii) Let (s', u') *be another tuple as above defining elements* y' *and* z' *as in (i). Then* (s, u) *and* (s', u') *are conjugate by some element of* $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ *if and only if* $z = z'$ *and the* y *and* y' *are* $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$*-conjugate.*

(iii) If we are given a tuple (s, u) *as in part (i),* $\overline{\mathbb{F}}_q^n$ *has no nontrivial subspaces stable under* s *and* u.

Proof. (i): From Lemma 2.1.4, we know that u is semisimple. First of all, we will show that the multiplicities of all its eigenvalues are one. So let y be an eigenvalue of multiplicity m and $j \geq 1$ be maximal such that the elements $y, y^q, \ldots, y^{q^{j-1}}$ are pairwise distinct. By the relation $sus^{-1} = u^q$, the eigenvalues of u are stable under taking q-th powers and all q-power-powers of y occur with the same multiplicity. After conjugation, we may assume, that

$$u = diag(y, \ldots, y, y^q, \ldots, y^q, \ldots, y^{q^{j-1}}, \ldots, y^{q^{j-1}}, x_1, \ldots, x_l),$$

where all the powers of y appear m times and the x_i are not of the form y^{q^i} for any i. Now write

$$s = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with $A \in M_{jm \times jm}(\overline{\mathbb{F}}_q)$, $B \in M_{jm \times l}(\overline{\mathbb{F}}_q)$, $C \in M_{l \times jm}(\overline{\mathbb{F}}_q)$ and $D \in M_{l \times l}(\overline{\mathbb{F}}_q)$. Using $su = u^q s$, we see in particular that

$$C \cdot diag(y, \ldots, y, y^q, \ldots, y^q, \ldots, y^{q^{j-1}}) = diag(x_1^q, \ldots, x_l^q) \cdot C.$$

Writing down the matrix multiplication and using that $x_i \notin \{y, \ldots, y^{q^{j-1}}\}$ for all $1 \leq i \leq j$, we see that $C = 0$. This gives us a nontrivial subspace stable by u and s if $l \geq 1$, so $l = 0$.

Now, if M is the diagonal $m \times m$ matrix having the unique eigenvalue y, we have $u = diag(M, M^q, \ldots, M^{q^{j-1}})$. Let us write $s = A = (A_{r,s})_{r,s}$ with $m \times m$ matrices $A_{r,s}$. Then, the relation $sus^{-1} = u^q$ is equivalent to

$$A_{r,s} \cdot M^{q^{s-1}} = M^{q^r} \cdot A_{r,s} \text{ for all } 1 \leq r, s \leq j,$$

or equivalently $A_{r,s} = M^{q^r - q^{s-1}} A_{r,s}$, because M is central in $\mathrm{GL}_m(\overline{\mathbb{F}}_q)$. The minimality of j implies that $M^{q^r - q^{s-1}} \neq 1$ unless $s = r + 1$ und hence, we have $A_{r,s} = 0$ unless $s = r + 1$.

Write A_r for $A_{r-1,r}$ and A_1 for $A_{j,1}$. Now, let $v_1 \in \overline{\mathbb{F}}_q^m$ be an eigenvector for $A_j A_{j-1} \ldots A_2 A_1$ and for $2 \leq r \leq j$, let

$$v_r = A_{r-1} A_{r-2} \ldots A_2 A_1 v_1.$$

Denote by e_i the standard basis of $\overline{\mathbb{F}}_q^n$ resp. $\overline{\mathbb{F}}_q^m$ and write

$$v_r = \sum_{i=1}^{m} v_{r,i} e_i \in \overline{\mathbb{F}}_q^m.$$

We define vectors $w_r \in \overline{\mathbb{F}}_q^m$ by

$$w_r := \sum_{i=1}^{m} v_{r,i} e_{(r-1)m+i}$$

for $1 \leq r \leq j$. This definition implies $v_1 = w_1$, viewing $\overline{\mathbb{F}}_q^m$ as a subspace of $\overline{\mathbb{F}}_q^n$ in the natural way. We have $sw_r = w_{r+1}$ for $2 \leq r < j$ and $sw_j = A_j A_{j-1} \dots A_2 A_1 v_1$, viewing the element on the right hand side as a vector in $\overline{\mathbb{F}}_q^n$. But the latter lies in $\overline{\mathbb{F}}_q^\times w_1$, because of the choice of v_1 as an eigenvector for $A_j A_{j-1} \dots A_2 A_1$. This means that the subspace generated by all w_r is stable under s and it obviously is stable under u. This is only possible if $j = n$. In particular, all the A_r are nonzero scalars. The choice of j ensures that $y^{q^t} \neq 1$ for $t < n$, i.e. y is not contained in any proper subfield of $\mathbb{F}_{q^n}$.

So the only thing which remains to be shown is that we can achieve by conjugation that all the A_i for $i \neq 1$ can be chosen to be 1. This can be done by conjugating with

$$diag(1, A_2, A_2 A_3, \dots, A_2 \dots A_n).$$

This does not affect the form of u. Then, s obtains the form of the assumption with $z = (-1)^{n+1} A_1 \dots A_n$.

(ii): Assume the tuples are conjugate. Then $z = \det(s) = \det(s') = z'$. As u and u' are conjugate, their eigenvalues are the same, so $\{y, y^q, \dots y^{q^{n-1}}\} = \{y', y'^q, \dots, y'^{q^{n-1}}\}$.

On the other hand, assume that $z = z'$ and $y = y'^{q^r}$ for some r. After conjugation, we may assume that

$$u = \begin{pmatrix} y & & & \\ & y^q & & \\ & & \ddots & \\ & & & y^{q^{n-1}} \end{pmatrix}, u' = \begin{pmatrix} y^{q^r} & & & \\ & y^{q^{r+1}} & & \\ & & \ddots & \\ & & & y^{q^{r+n-1}} \end{pmatrix}$$

and

$$s = s' = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ (-1)^n z & & & & 0 \end{pmatrix}.$$

By conjugating with a Weyl group element, we can assume that $u = u'$ at the cost of interchanging one of the ones on the secondary diagonal with the entry $(-1)^n z$ in the matrix s'. By conjugating with a diagonal matrix as in the proof of (i) we can bring s' to its old form without changing u'.

(iii): Let (s,u) be as in (i). We may assume $g=1$. Let $v \in \overline{\mathbb{F}}_q^n$ be nonzero and write $v = \sum_{i=1}^n a_i e_i$. If v is a multiple of some e_i, we are done because then we get all the e_i up to scalar from applying u repeatedly to v.

Now, if v is not a multiple of e_n, $v - y^{-q^{n-1}} uv$ is nonzero, so we may assume $a_n = 0$. Similarly, v is either a scalar multiple of e_{n-1} or we may assume $a_{n-1} = a_n = 0$. Continuing this process, we are reduced to the case where v is either a scalar multiple of some e_i with $2 \leq i \leq n$ or $a_2 = \ldots = a_n = 0$, which means that v is a scalar multiple of e_1. In either case this proves our claim. □

We remark that, the conjugacy class of (s,u) depends on our choices for φ and α. For both cases, this will be so even if $n=1$. The difference between two Frobenius lifts is an element of I_L/P_L. So, it will be sufficient to give a continuous homomorphism $W_L/P_L \to \overline{\mathbb{F}}_q$ which is not trivial on I_L/P_L. For this choose a prime l_0 such that l_0 divides $q-1$ and consider the composite

$$W_L/P_L \cong \mathbb{Z} \ltimes \prod_{l \neq p} \mathbb{Z}_l \twoheadrightarrow \mathbb{Z}_{l_0} \twoheadrightarrow \mathbb{Z}/l_0\mathbb{Z} \hookrightarrow \mathbb{Z}/(q-1)\mathbb{Z} \cong \mathbb{F}_q^\times,$$

where the first isomorphism is given by our generators and the injection can be arbitrarily chosen.

This also gives an example for a homomorphism for which u depends on the choice of a topological generator. To see this, we only need to observe that for any $a \in \mathbb{Z}_{l_0}^\times$, the element $(1,\ldots,1,a,1\ldots)$ is a topological generator of $\prod_{l\neq p} \mathbb{Z}_l$ and that the projection map $\mathbb{Z}_l \to \mathbb{Z}/l\mathbb{Z}$ is not constant on $\mathbb{Z}_l^\times$.

Proposition 2.1.6 classifies irreducible W_L-representations up to isomorphism depending on our choices. Now we can describe the image in the fiber product.

To do this we will slightly reinterpret the involved sets of homomorphisms. For $G = \mathrm{GL}_n$, we have canonical isomorphisms

$$X(G^*) = \mathrm{Hom}(G^*, \mathbb{G}_m) = \mathrm{Hom}(G^*/(G^*)', \mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$$

where the third isomorphism is given by the determinant and this induces a canonical isomorphisms

$$\mathrm{Hom}^{cont}(W_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times)) = \mathrm{Hom}^{cont}(W_L/P_L, \overline{\mathbb{F}}_q^\times)$$

and

$$\mathrm{Hom}^{cont}(I_L/P_L, \mathrm{Hom}(X(G^*), \overline{\mathbb{F}}_q^\times)) = \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times).$$

Thus, we can reinterpret our fiber product as

$$(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*} \times_{\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times)} \mathrm{Hom}^{cont}(W_L/P_L, \overline{\mathbb{F}}_q^\times).$$

Now we can reformulate Proposition 2.1.6 to give a classification of irreducible n-dimensional W_L-representations independent of our choices:

Theorem 2.1.7. *The canonical map from irreducible W_L/P_L-representations into the fiber product*

$$(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*} \times_{\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times)} \mathrm{Hom}^{cont}(W_L/P_L, \overline{\mathbb{F}}_q^\times)$$

is injective. Its image consists of those pairs, such that the corresponding F^*-stable W_0^*-orbit consists of $n!$ elements and F^* acts by an n-cycle on each representative.

Proof. Firstly, we show the injectivity. If $\rho : W_L/P_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ is an irreducible representation, we need to show that we can recover the conjugacy class of (s, u) as in Proposition 2.1.6 from the image of ρ in the fiber product. Let (ψ_1, ψ_2) be the image of ρ. Then, we can recover y as in Proposition 2.1.6 up to $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$-conjugacy as an eigenvalue of $\psi_1(\alpha)$ and z as $\det(\psi_2(\varphi))$. This defines (s, u) up to conjugacy by part (ii) of Proposition 2.1.6.

It is an immediate consequence of Proposition 2.1.6 that for each irreducible ρ the corresponding orbit is of length $n!$ and that the Frobenius acts as an n-cycle on each representative of that orbit.

Now, let $(W_0^*\psi_1, \psi_2)$ be a tuple in the claimed image with a chosen representative ψ_1 and let $\psi_1(\alpha) = diag(y_1, \ldots, y_n)$. The fact that $W_0^*\psi_1$ is of length $n!$ means that the y_i are pairwise distinct. After renumbering the y_i, i.e. choosing a different representative of the orbit, the other condition translates as $y_i^q = y_{i+1}$ for $1 \leq i \leq n$ where $y_{n+1} := y_1$. Because all the y_i are distinct, we get that all the y_i lie in $F_{q^n} \setminus \cup_{m<n}\mathbb{F}_{q^m}$. Hence, we obtain an element $u = diag(y, y^q, \ldots, y^{q^{n-1}}) \in G^*(\overline{\mathbb{F}}_q)$ where $y = y_1$.

Now, if $z = \det(\psi_2(\varphi))$, we can define s as in Proposition 2.1.6 and obtain $sus^{-1} = u^q$. Now this tuples gives as a preimage.

□

We remark that the map fails to be injective if we drop the assumption about irreducibility. For example we may consider the semisimple non-irreducible representations of dimension 2 give by the parameters

$$s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, u = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

with $a \in \mathbb{F}_q^\times$. These are obviously all non-isomorphic, but have the same image in the fiber product.

Having classified the irreducible W_L-representations, we will now discuss their fields of definition.

Lemma 2.1.8. *Let ρ be an irreducible W_L-representation of dimension n and let $z = \det(\rho(\alpha))$. Then ρ can be defined over $\mathbb{F}_q(z)$.*

Proof. Let $\chi = tr(\rho)$ be the corresponding character. We may replace W_L by $W_L/ker(\rho)$ and assume that we are considering the representation of a finite group this way. It is known that the Schur index of an irreducible representation of a finite group over a finite field is 1 (see e.g. [Bra45], section 2), i.e. it can be defined over the field $\mathbb{F}_p[\{\chi(g) : g \in W_L\}]$.

Modulo $\ker(\rho)$, each element of W_L is of the form $\varphi^l\alpha^m$ for some $l, m \in \mathbb{N}_0$ and we get

$$\chi(\varphi^l\alpha^m) = \begin{cases} 0 & \text{if } n \nmid m \\ tr_{\mathbb{F}_{q^n}/\mathbb{F}_q}(y)^l z^{\frac{m}{n}} & \text{if } n \mid m \end{cases}$$

by Proposition 2.1.6 (i) for some $y \in \mathbb{F}_{q^n}$. The right hand side is contained in $\mathbb{F}_q(z)$. $\square$

We will conclude this section by comparing the classification given here to the one given by Berger for $L = \mathbb{Q}_p$ in [Ber10]. This is done as follows: For $n \geq 1$, we chose elements $\pi_n \in \mathbb{Q}_p$ such that $\pi_n^{p^n-1} = -p$. This allows us to define $\omega_n : I_{\mathbb{Q}_p} \to \overline{\mathbb{F}}_p^\times$ by mapping $g \in I_{\mathbb{Q}_p}$ to the residue class of $\frac{g(\pi_n)}{\pi_n}$. This definition does not depend on the choice of π_n. We call ω_n the fundamental character of level n. The image of ω_n is equal to $\mathbb{F}_{p^n}$. Note that ω_1 is the mod-p cyclotomic character. Hence, it extends to a character of $\mathbb{G}_{\mathbb{Q}_p}$ by setting

$$\omega_1 : G_{\mathbb{Q}_p} \to (\mathbb{Z}/p\mathbb{Z})^\times = \mathbb{F}_p^\times, g \mapsto \kappa(g), \text{ where } g(\zeta_p) = \zeta_p^{\kappa(g)}$$

for a p-th root of unity ζ_p.

For a fixed integer $n \geq 1$, an integer $1 \leq h \leq p^n - 2$ is called primitive if the characters $\omega_n^h, \omega_n^{ph}, \dots, \omega_n^{p^{n-1}h}$ are pairwise distinct. As the image of ω_n is $\mathbb{F}_{p^n}$ this is equivalent to saying that for a generator x of $\mathbb{F}_{p^n}$, all the elements $x^h, x^{hp}, \dots, x^{hp^{n-1}}$ are pairwise distinct, i.e. x^h is not contained in any proper subfield of $\mathbb{F}_{p^n}$.

For $\lambda \in \overline{\mathbb{F}}_p$, let μ_λ be the unramified character which sends φ to λ^{-1}. Then, we have the following classification of irreducible $G_{\mathbb{Q}_p}$-representations:

Proposition 2.1.9. *(i) For each $n \geq 1$ and each primitive $1 \leq h \leq p^n - 2$ there exists a representation $\mathrm{ind}(w_n^h)$ which is determined uniquely up to isomorphism by the conditions*

- $\det(\mathrm{ind}(\omega_n^h)) = \omega_1^h$
- $\mathrm{ind}(\omega_n^h)|_{I_{\mathbb{Q}_p}} = \bigoplus_{i=0}^{n-1} \omega_n^{p^i h}$.

(ii) Every absolutely irreducible n-dimensional $G_{\mathbb{Q}_p}$-representation is isomorphic to $\mathrm{ind}(\omega_n^h) \otimes \mu_\lambda$ for some primitive $1 \leq h \leq p^n - 2$ and some $\lambda \in \overline{\mathbb{F}}_p$.

Proof. This is Corollary 2.1.5 in [Ber10] and the preceding discussion. $\square$

Now, fix some $n \geq 1$ and let $1 \leq h \leq p^n - 2$ be primitive. We may assume $\omega_1(\varphi) = 1$ by possibly changing φ by some element of $I_{\mathbb{Q}_p}$, because the image of ω_1 is $\mathbb{F}_p^\times$. Then the representation defined by $y = \omega_n^h(v)$ and $z = \omega_1(\varphi) = 1$ as in Proposition 2.1.6 (i) is equal to $\det(\mathrm{ind}(\omega_n^h))$ by definition. The same conjugation argument as in the proof of Proposition 2.1.6 (i) then shows, that the representation defined by $y = \omega_n^h(v)$ and $z = (-1)^n \lambda^{-n}$ is isomorphic to $\mathrm{ind}(\omega_n^h) \otimes \mu_\lambda$.

2.2 Representations of I_L

Let G and G^* be algebraic groups with Frobenius maps F and F^* and maximal tori T and T^* in duality to each other given by $\delta : X(T) \to Y(T^*)$ and $\epsilon : Y(T) \to X(T^*)$. We may identify the finite Weyl groups of (G, T) and (G^*, T^*) leaving out the isomorphism in our notation. Recall that $\delta(w(\chi)) = w(\delta(\chi))$ for $\chi \in X(T)$ with this identification.

We have to start with a lemma connecting inertia groups and the unit groups of the corresponding fields: Let $Z := \varprojlim_{(p,m)=1} \mu_m(\overline{\mathbb{F}}_q)$, where $\mu_m(\overline{\mathbb{F}}_q)$ denotes the group of m-th roots of unity of $\overline{\mathbb{F}}_q$ and the transition maps are

$$\mu_{m_1}(\overline{\mathbb{F}}_q) \to \mu_{m_2}(\overline{\mathbb{F}}_q), x \mapsto x^{\frac{m_1}{m_2}},$$

when m_2 divides m_1. Further, we can define $V := Z \otimes \mathbb{Q}$.

Lemma 2.2.1. *There is a surjective homomorphism $s : I_L \to Z$ with kernel P_L given by the congruence*

$$\frac{\sigma(x)}{x} \equiv (s(\sigma)^r)_q$$

for all $x \in \overline{L}$ modulo $\mathfrak{m}_{\overline{L}}$, where $v(x) = \frac{r}{q}$. Hence, it induces an isomorphism $I_L/P_L \to Z$ which we will also denote by s.

Proof. See [Del73, 2.2.2 b)] for the existence of s and the assumption that P_L is its kernel. Let $(\zeta_r)_r \in Z$ be any element. Let L^{nr} be the maximal unramified extension of L and

$$\tilde{L} = L^{nr}(\pi_L^{1/m} : (m,p) = 1).$$

Then there exists $\sigma : \tilde{L} \to \tilde{L}$ trivial on L^{nr} such that $\sigma(\pi_L^{1/m}) = z_m \pi_L^{1/m}$, because z lies in the projective limit Z, and we can extend σ to an automorphism of $\overline{L}$, which we shall also call σ. This lies in I_L because it fixes L^{nr} and we have $\frac{\sigma(\pi_L^{1/m})}{\pi_L^{1/m}} = z_m$. This implies that $s(\sigma) = z$. □

Lemma 2.2.2. *(i) There exists a canonical isomorphism $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} Z \cong \overline{\mathbb{F}}_q^\times$. It is given by $\frac{n}{m} \otimes x \mapsto (x^n)_m$.*

(ii) There exists a canonical short exact sequence

$$0 \to Z \to V \to \overline{\mathbb{F}}_q^\times \to 0.$$

(iii) There exists a canonical short exact sequence

$$0 \to Y(T) \otimes_{\mathbb{Z}} Z \to Y(T) \otimes_{\mathbb{Z}} V \to T \to 0.$$

(iv) There exists a canonical isomorphism

$$(Y(T) \otimes_{\mathbb{Z}} Z)/(F-1) \cong T^F.$$

Proof. (i): The given map is easily seen to be well-defined. On the other hand, we can find a system of primitive m-th roots of unity ζ_m, such that $\zeta = (\zeta_m)_{(m,p)=1}$ lies in Z. This way, we obtain a map from $\overline{\mathbb{F}}_q^\times$ to $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} Z$ by sending $a = \zeta_m^n$ to $\zeta \otimes \frac{n}{m}$. Elementary calculations show that this map is well-defined and it is obviously an inverse to the map of the assertion. In particular it does not depend on the choice of ζ.

(ii): Since $Z \cong \prod_{l \neq p} \mathbb{Z}_l$ is torsion free and hence flat as a $\mathbb{Z}$-module, we obtain this from the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

and (i).

(iii): This follows from tensoring (ii) with $Y(T)$, the fact that $Y(T)$ is free and hence flat over $\mathbb{Z}$ and using the canonical isomorphism $Y(T) \otimes \overline{\mathbb{F}}_q \cong T$.

(iv): We have the commutative diagram

$$\begin{array}{ccccccccc} & & & & & & T^F & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & Y(T) \otimes_{\mathbb{Z}} Z & \longrightarrow & Y(T) \otimes_{\mathbb{Z}} V & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y(T) \otimes_{\mathbb{Z}} Z & \longrightarrow & Y(T) \otimes_{\mathbb{Z}} V & \longrightarrow & T & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & (Y(T) \otimes_{\mathbb{Z}} Z)/(F-1) & & & & & & \end{array}$$

where the vertical maps are the ones induced by $F-1$ on T resp. $Y(T)$ and the identity on V resp. Z. Using the snake lemma, it suffices to show that the middle vertical arrow is an isomorphism.

Some finite power of F acts as a p^f-th power map ($f \geq 1$) on T and hence on $Y(T)$, so F, viewed as an endomorphism of $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ does not have 1 as an eigenvalue. Hence $F-1$ is injective and thus bijective on $Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$. Thus, $F-1$ is bijective on $Y(T) \otimes_{\mathbb{Z}} V \cong Y(T) \otimes_{\mathbb{Z}} \mathbb{Q} \otimes_{\mathbb{Z}} Z$. $\square$

Lemma 2.2.3. *(i) There exists a (non-canonical) isomorphism*

$$\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \cong \overline{\mathbb{F}}_q^\times$$

depending on the choice of a topological generator of I_L/P_L.

(ii) There is a canonical isomorphism

$$\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes I_L/P_L \cong \overline{\mathbb{F}}_q^\times$$

given by $f \otimes \sigma \mapsto f(\sigma)$.

Proof. (i): Let α be a topological generator of I_L/P_L. Then we take evaluation at α as our map above. Clearly, the map is injective. Now let $x \in \overline{\mathbb{F}}_q$ be of order m. There exists a surjective continuous homomorphism $I_L/P_L \to \mathbb{Z}/m\mathbb{Z}$, because $(m,p) = 1$. Composing this homomorphism from $\mathbb{Z}/m\mathbb{Z}$ to $\overline{\mathbb{F}}_q^\times$, which maps the image of v in $\mathbb{Z}/m\mathbb{Z}$ to x yields a preimage of x.

(ii): Let α be as in the proof of (i) and denote by γ the homomorphism of the assertion and by β the homomorphism $x \mapsto (\alpha \mapsto x) \otimes \alpha$ from $\overline{\mathbb{F}}_q^\times$ to $\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes I_L/P_L$.

Clearly $\gamma \circ \beta = id_{\overline{\mathbb{F}}_q^\times}$. On the other hand, let $\sigma \in I_L/P_L$ and $x \in \overline{\mathbb{F}}_q^\times$ of order m. Then there exists an $n \in \mathbb{Z}$ such that $\sigma \equiv \alpha^n \mod (I_L/P_L)^m$. We have

$$\beta(\gamma((\alpha \mapsto x) \otimes \sigma)) = \beta(x^n) = (\alpha \mapsto x^n) \otimes \alpha = (\alpha \mapsto x) \otimes \alpha^n = (\alpha \mapsto x) \otimes \sigma$$

and hence $\beta \circ \gamma = id_{\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times)}$. $\square$

For the upcoming proposition, we need to define Frobenius- and Weyl group actions on the group $\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)$. We do this by using the respective actions on $Y(T)$. Let $\psi \in \mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)$, $\gamma \in Y(T)$ and $z \in Z$. Then we set

$$(F\psi)(\gamma \otimes z) = \psi(F(\gamma) \otimes z)$$

and

$$(w\psi)(\gamma \otimes z) = \psi(w^{-1}\gamma \otimes z)$$

for $w \in W_0$. Here we choose the sign in the W_0-action so that we can obtain a left action again.

Proposition 2.2.4. *There is a canonical isomorphism*

$$\mathrm{Hom}^{cont}(I_L/P_L, T^*) \cong \mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times).$$

The image of $\varphi \in \mathrm{Hom}^{cont}(I_L/P_L, T^*)$ *is given by*

$$\gamma \otimes s(\rho) \mapsto \epsilon(\gamma)\varphi(\rho),$$

It is equivariant for the Weyl group and Frobenius actions on both sides.

Proof. We endow T^*, $\overline{\mathbb{F}}_q^\times$ and $Y(T) \otimes \overline{\mathbb{F}}_q^\times$ with the discrete topology. Using the previous lemma, we have

$$\begin{aligned}
\mathrm{Hom}^{cont}(I_L/P_L, T^*) &= \mathrm{Hom}^{cont}(I_L/P_L, Y(T^*) \otimes \overline{\mathbb{F}}_q^\times) \\
&= \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes Y(T^*) \\
&= \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes X(T) \\
&= \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes \mathrm{Hom}(Y(T), \mathbb{Z}) \\
&= \mathrm{Hom}(Y(T), \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times)) \\
&= \mathrm{Hom}^{cont}(Y(T) \otimes I_L/P_L, \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes I_L/P_L) \\
&= \mathrm{Hom}^{cont}(Y(T) \otimes I_L/P_L, \overline{\mathbb{F}}_q^\times) \\
&= \mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times).
\end{aligned}$$

In the third line from the bottom, $Y(T) \otimes I_L/P_L$ is equipped with the product topology induced by the choice of a basis of $Y(T)$. The topology does not depend on the choice of this basis. $\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes I_L/P_L$ is endowed with the discrete topology. It is then seen by a direct computation, that the map $f \mapsto f \otimes id_{I_L/P_L}$ is an isomorphism from $\mathrm{Hom}(Y(T), \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times))$ to $\mathrm{Hom}^{cont}(Y(T) \otimes I_L/P_L, \mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times) \otimes I_L/P_L)$.

For the explicit description it suffices to treat that case that $\gamma = \gamma_i$ is in an element of a chosen basis $\{\gamma_1, \ldots, \gamma_n\} \subseteq Y(T)$ with related basis $\{\chi_1^* = \epsilon(\gamma_1), \ldots, \chi_n^* = \epsilon(\gamma_n)\} \subseteq X(T^*)$. Additionally, let $\{\gamma_1^*, \ldots, \gamma_n^*\} \subseteq Y(T^*)$ be the basis dual to $\{\chi_1^*, \ldots, \chi_n^*\}$ with respect to the pairing $\langle \cdot, \cdot \rangle$. Following the above isomorphisms, one sees that if $\varphi(\rho) = \prod \gamma_i^*(t_i)$ with $t_i \in \overline{\mathbb{F}}_q$, the image ψ of φ is given by

$$\psi(\gamma_i \otimes s(\rho)) = t_i = \chi_i^*(\varphi(\rho)).$$

We see the Frobenius equivariance by remarking that

$$F^*(\varphi)(\rho) = F^*(\varphi(\rho)) = \prod F^*(\gamma_j^*)(t_j)$$

and hence

$$\begin{aligned} F(\psi)(\gamma_i \otimes s(\rho)) &= \psi(F(\gamma_i) \otimes s(\rho)) = F^*(\chi_i^*)(\varphi(\rho)) \\ &= \prod_{j=1}^n t_j^{\langle F^*\chi_i^*, \gamma_j^* \rangle} = \prod_{j=1}^n t_j^{\langle \chi_i^*, F^*\gamma_j^* \rangle} = \chi_i^*(\prod_{j=1}^n F^*\gamma_j^*(t_i)) \\ &= \chi_i^*(F^*(\varphi)(\rho)). \end{aligned}$$

Now turning to the W_0-equivariance, we have for $w \in W_0$:

$$\begin{aligned} (w\psi)(\gamma_i \otimes s(\rho)) &= \psi(w^{-1}\gamma_i \otimes s(\rho)) = w^{-1}\chi_i^*(\varphi(\rho)) \\ &= \prod_{j=1}^n t_j^{\langle w^{-1}\chi_i^*, \gamma_j^* \rangle} = \prod_{j=1}^n t_j^{\langle \chi_i^*, w\gamma_j^* \rangle} = \chi_i^*(\prod_{j=1}^n w\gamma_j^*(t_i)) \\ &= \chi_i^*(w\varphi(\rho)). \end{aligned}$$

□

Corollary 2.2.5. *We have a canonical isomorphism of sets of Frobenius stable Weyl group orbits*

$$(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*} \cong (\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)/W_0)^F.$$

Proof. This is immediate by the previous proposition. □

Fix any $g \in G$. Then ${}^gT \subseteq G$ is a maximal torus. We obtain isomorphisms

$$X(T) \to X({}^gT), \chi \mapsto {}^g\chi := [{}^gt \mapsto \chi(t)]$$

and

$$Y(T) \to Y({}^gT), \gamma \mapsto {}^g\gamma := [t \mapsto {}^g\gamma(t)]$$

The latter induces an isomorphism

$$\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times) \to \mathrm{Hom}^{cont}(Y({}^gT) \otimes Z, \overline{\mathbb{F}}_q^\times)$$

given by $\rho \mapsto g\rho := [{}^g\gamma \otimes z \mapsto \rho(\gamma \otimes z)]$. This is compatible with our notations introduced before in the following way: If $w \in W_0$ and $n \in N(T)$ lifts w, we get $w\rho = n\rho$. In particular, we get $t\rho = \rho$ for all $t \in T$.

Now suppose we are given an F^*-stable W_0^*-orbit in $\mathrm{Hom}^{cont}(I_L/P_L, T^*)$, i.e. an element of

$$(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*} = (\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)/W_0)^F.$$

Recall that by Corollary 2.1.7 this can be obtained from an irreducible W_L-representation in the case of $G = \mathrm{GL}_n$. Let ρ be a representative in the right hand side. Then, we can find a $w \in W_0$ such that

$$F(\rho) = w^{-1}\rho.$$

By Lang's Theorem we can find a $g \in G$ such that $g^{-1}F(g)$ is a lift of w. Then, we get

$$\begin{aligned} F(g\rho) &= F(g)F(\rho) = F(g)w^{-1}\rho = F(g)F(g^{-1})g\rho \\ &= g\rho \in (\mathrm{Hom}^{cont}(Y({}^gT) \otimes Z, \overline{\mathbb{F}}_q^\times))^F. \end{aligned}$$

We will write T_w instead of ${}^gT$ which will be justified by the upcoming lemma. From Lemma 2.2.2 (iv), we get

$$\begin{aligned} \mathrm{Hom}^{cont}(Y(T_w) \otimes Z, \overline{\mathbb{F}}_q^\times)^F &= \mathrm{Hom}^{cont}((Y(T_w) \otimes Z)/(F-1), \overline{\mathbb{F}}_q^\times) \\ &= \mathrm{Hom}(T_w(\mathbb{F}_q), \overline{\mathbb{F}}_q^\times). \end{aligned}$$

Thus, we get a pair (T_w, θ) with a torus T_w and a character $\theta : T_w(\mathbb{F}_q) \to \overline{\mathbb{F}}_q$. We will say that we obtain T_w from twisting T with w. Indeed, up to G^F-conjugacy, T_w only depends only on w and not on the choice of g above. More precisely, we have:

Lemma 2.2.6. *The tuple (T_w, θ) defined above is well-defined up to G^F-conjugacy.*

Proof. This can be proven analogously to [Car93, Prop. 3.3.3 (i)]. For sake of completeness we give a proof here. Hence, let ρ, ρ' be different choices for representatives of an F-stable W_0-orbit in $\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)$. Then, we can find some $n \in N(T)$ such that $\rho' = n\rho$. As before, let $w \in W_0$ such that $F(\rho) = w^{-1}\rho$. Then we have $F(\rho') = F(n)w^{-1}n^{-1}\rho'$.

We choose $g, g' \in G$ such that $g^{-1}F(g)$ resp. $g'^{-1}F(g')$ lifts w resp. $nwF(n)^{-1}T$. This implies

$$n^{-1}g'^{-1}F(g')F(n)F(g^{-1})g \in T$$

or equivalently

$$gn^{-1}g'^{-1}F(g')F(n)F(g^{-1}) \in {}^gT.$$

By Lang's Theorem applied to the group ${}^gT$, there exists a $t \in T$ such that

$$gn^{-1}g'^{-1}F(g')F(n)F(g^{-1}) = (gtg^{-1})^{-1}F(gtg^{-1}) = gt^{-1}g^{-1}F(g)F(t)F(g^{-1}),$$

which simplifies to

$$gtn^{-1}g'^{-1} = F(gtn^{-1}g'^{-1}),$$

i.e. $gtn^{-1}g'^{-1} \in G^F$. Hence, $g\rho = gtn^{-1}g'^{-1}g'\rho'$ and $g'\rho'$ are conjugate over G^F and so are the corresponding tori and characters. □

2.3 The GL_n-Case

We will now continue making the construction of the previous subsection explicit for irreducible Weil group representation of dimension n and show how to apply the previous results to our situation. Suppose we are given an irreducible representation $\rho_0 : W_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$. As before, we write $\chi_i \in X(T)$ for the character sending a diagonal matrix to its i-th entry and $\gamma_i \in Y(T)$ for the cocharacter which sends $y \in \overline{\mathbb{F}}_q$ to the diagonal matrix with (i,i)-entry y and all other diagonal entries 1.

Choose a generator α of I_L/P_L. By Proposition 2.1.6, the restriction of ρ_0 to I_l/P_L is given by $\alpha \mapsto diag(y, y^q, \ldots, y^{q^{n-1}})$ with some $y \in \mathbb{F}_q^n$ which is not contained in any proper subfield. By the explicit description of Lemma 2.2.4, the restriction of ρ_0 to I_L/P_L corresponds to

$$\phi_y := [\gamma_i \otimes s(\alpha) \mapsto y^{q^{i-1}}] \in \mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times).$$

We have

$$F(\phi_y) = \phi_y^q = \phi_{y^q} = w^{-1}\phi_y.$$

with $w = (1, 2, \ldots, n)$. This is a Coxeter element in W_0. Note that by conjugating ρ_0 with Weyl group elements, i.e. replacing ρ by another isomorphic representation, we could have obtained any Coxeter element, i.e. n-cycle. Moreover, we obtain the same w element for any irreducible representation of W_L (up to conjugacy).

Let $g \in G$ such that $g^{-1}F(g)$ is a lift of w. If ω is the $\mathbb{F}_{q^n}^\times$-component of $s(\alpha)$, ω is a generator of $\mathbb{F}_{q^n}^\times$. Hence, there exists a unique primitive $m \in \{1, \ldots, q^n - 2\}$ such that $y = \omega^m$. We can uniquely write

$$m = a_1 + a_2 q + \ldots + a_n q^{n-1}$$

with $0 \le a_1, \ldots, a_n \le q - 1$.

Lemma 2.3.1. *With w as above, we habe $T_w^F \cong \mathbb{F}_{q^n}^\times$. If $\mu = a_1\chi_1 + \ldots + a_n\chi_n$, θ is the restriction of ${}^g\mu$ to T_w^F.*

Proof. We keep the notations introduced before the lemma. $T_w = {}^gT$ and hence ${}^g diag(t_1, \ldots, t_n) \in T_w^F$ if and only if

$${}^g diag(t_1, \ldots, t_n) = {}^{F(g)} diag(t_1^q, \ldots, t_n^q),$$

which is equivalent to

$$diag(t_1, \ldots, t_n) = {}^w diag(t_1^q, \ldots, t_n^q) = diag(t_n^q, t_1^q, \ldots, t_{n-1}^q).$$

In particular, $t_1^{q^n} = t_1 \in \mathbb{F}_{q^n}$ and the points fixed by the Frobenius are the elements $diag(t, t^q, \ldots, t^{q^{n-1}})$ with $t \in \mathbb{F}_{q^n}^\times$. This gives an obvious isomorphism $T_w^F \cong \mathbb{F}_{q^n}^\times$.

To determine the character θ of T_w^F, we have to make the constructions before explicit. First of all, we will do this for the isomorphism

$$T_w^F \cong (Y(T_w) \otimes_{\mathbb{Z}} Z)/(F - 1)$$

given by Lemma 2.2.2 (iv). This is given by the snake lemma in the following way:

Let ${}^g diag(t, t^q, \ldots, t^{q^{n-1}}) \in T_w^F$. This element is contained in T_w, hence it is the image of some element of $Y(T_w) \otimes_{\mathbb{Z}} V = Y(T_w) \otimes \mathbb{Q} \otimes Z$ under the natural map. We may choose

$$\sum_{i=1}^{n} {}^g\gamma_i \otimes \frac{1}{q^n - 1} \otimes \bar{t}^{q^{i-1}} = (\sum_{i=1}^{n} q^{i-1} \cdot {}^g\gamma_i) \otimes \frac{1}{q^n - 1} \otimes \bar{t}$$

as a preimage, where $\bar{t}$ is any element of Z with $(\bar{t})_{q^n-1} = t$. We need to apply $F-1$ to this element. Because of

$$F({}^g\gamma_i) = {}^{gg^{-1}F(g)}F(\gamma_i) = q \cdot {}^{gw}\gamma_i = q \cdot {}^g\gamma_{i+1}$$

(where the indices are always supposed to be read modulo n), we obtain

$$\begin{aligned}(F-1)(\sum_{i=1}^{n} q^{i-1} \cdot {}^g\gamma_i) \otimes \frac{1}{q^n-1} \otimes \bar{t} &= (\sum_{i=1}^{n} q^{i-1}(F({}^g\gamma_i) - {}^g\gamma_i)) \otimes \frac{1}{q^n-1} \otimes \bar{t} \\ &= (\sum_{i=1}^{n} q^{i} \cdot {}^g\gamma_{i+1} - q^{i-1} \cdot {}^g\gamma_i) \otimes \frac{1}{q^n-1} \otimes \bar{t} \\ &= (q^n - 1) \cdot {}^g\gamma_1 \otimes \frac{1}{q^n-1} \otimes \bar{t} = {}^g\gamma_1 \otimes 1 \otimes \bar{t},\end{aligned}$$

which is contained in $Y(T_w) \otimes Z$. Its residue class modulo $F-1$ is the image of ${}^g diag(t, t^q, \ldots, t^{q^{n-1}}) \in T_w^F$.

Now, ρ_0 defines a continuous homomorphism from I_L/P_L to $\overline{\mathbb{F}}_q^\times$, which is given by y as above. It corresponds to the map $\gamma_i \otimes s(\alpha) \mapsto y^{q^{i-1}}$ by Lemma 2.2.4. Hence, by the above calculation, θ is given by

$${}^g diag(\omega, \ldots, \omega^{q^{n-1}}) \mapsto y = \omega^m = {}^g\mu({}^g(diag(\omega, \ldots, \omega^{q^{n-1}}))).$$

□

Corollary 2.3.2. *θ is in general position, i.e. no nontrivial element of $N(T_w)^F/T_w^F$ fixes θ.*

Proof. A straight-forward calculation in the symmetric group shows that an element n of $N(T_w) = {}^gN(T)$ is fixed by F if and only if ${}^{g^{-1}}nT$ lies in the subgroup generated by w. Hence, $N(T_w)^F/T_w^F$ is a cyclic group of order n generated by the element ${}^gw$ which maps ${}^g\chi_i$ to ${}^g\chi_{i+1}$ and no power of this element fixes θ because of the primitivity of m. □

2.4 Projective Representations

A projective W_L-representation of dimension n is a continuous homomorphism $\rho : W_L \to \mathrm{PGL}_n(\overline{\mathbb{F}}_q)$, where $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ is endowed with the discrete topology. We call a projective representation irreducible if its image is not contained in a nontrivial parabolic subgroup. Note that a representation $\rho_0 : W_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ is irreducible if and only if its image is not contained in a nontrivial parabolic subgroup. We have the following relation between irreducible W_L-representations and irreducible projective W_L-representations.

Lemma 2.4.1. *The map*

$$\{irreducible\ \rho_0 : W_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)\} \to \{irreducible\ \rho : W_L \to \mathrm{PGL}_n(\overline{\mathbb{F}}_q)\}$$

induced by projection modulo the center is well-defined and surjective.

Proof. We first check that the map is well-defined. Assume that ρ_0 maps to ρ and the image of ρ is contained in a nontrivial parabolic subgroup. By conjugating ρ, we may assume that the image of ρ_0 is contained in a parabolic subgroup containing the standard Borel of $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$. The preimage of this is a parabolic subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ and hence ρ is not irreducible.

For the surjectivity, we remark that ρ_0 is given by a tuple $(s, u) \in \mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ such that $sus^{-1} = u^q$. Thus, we can find $(s_0, u_0) \in \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ such that $s_0 u_0 s_0^{-1} = u_0^q z$ for some $z \in Z(\mathrm{GL}_n(\overline{\mathbb{F}}_q))$. We can find $\tilde{z} \in Z(\mathrm{GL}_n(\overline{\mathbb{F}}_q))$ such that $\tilde{z}^{q-1} = z$, i.e.

$$s_0(u_0\tilde{z})s_0^{-1} = (u_0\tilde{z})^q.$$

Now assume that there are $(s_0, u_0\tilde{z})$-stable subspaces of $\overline{\mathbb{F}}_q^n$. Then, s_0 and $u_0\tilde{z}$ are contained in a nontrivial parabolic subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$ and hence their images s and u are contained in a nontrivial parabolic subgroup of $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ in contradiction to our assumption. □

Denote by T' the standard split maximal torus of $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$, i.e. $T' = T \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and by T'^* the standard split maximal torus of $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$, i.e. T'^* is the projection of T^* in $\mathrm{PGL}_n(\overline{\mathbb{F}}_q)$. The choice of these tori induces a duality between the algebraic groups PGL_n and SL_n. We may identify the finite Weyl groups of SL_n, PGL_n and GL_n, but will continue to write W_0^* when we speak of the Weyl group of PGL_n or of GL_n considered as the dual group.

Proposition 2.4.2. *The diagram*

$$\begin{array}{ccc}
\{irreducible\ \rho_0 : W_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)\} & \to & \{irreducible\ \rho : W_L \to \mathrm{PGL}_n(\overline{\mathbb{F}}_q)\} \\
\downarrow & & \downarrow \\
(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*} & \to & (\mathrm{Hom}^{cont}(I_L/P_L, T'^*)/W_0^*)^{F^*} \\
\downarrow & & \downarrow \\
(\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)/W_0)^F & \to & (\mathrm{Hom}^{cont}(Y(T') \otimes Z^\times, \overline{\mathbb{F}}_q^\times)/W_0)^F
\end{array}$$

is commutative.

Proof. The commutativity of the upper square is obvious: We can conjugate the respective restrictions to I_L/P_L into the respective tori by the "same" element of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$. The commutativity of the second diagram follows from the explicit description in Proposition 2.2.4.

□

Corollary 2.4.3. *Let $\rho_0 : W_L \to \mathrm{PGL}_n(\overline{\mathbb{F}}_q)$ be irreducible inducing $\rho : W_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ such that ρ_0 defines the tuple (T_w, θ). Then ρ defines the tuple (T'_w, θ') where θ' is the restriction of θ to $T'_w = T_w \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)$. The tuple (T'_w, θ') is in general position.*

Proof. Let ϕ be a representative of an F-stable W_0-orbit in $\mathrm{Hom}^{cont}(Y(T')\otimes Z^\times, \overline{\mathbb{F}}_q^\times)$ and $\phi_0 \in \mathrm{Hom}^{cont}(Y(T)\otimes Z, \overline{\mathbb{F}}_q^\times)$ a lift. Then there exists some $w \in W_0$ such that $F\phi_0 = w^{-1}\phi_0$ and hence $F\phi = w^{-1}\phi$. Then, we can choose $g \in \mathrm{SL}_n$ such that $g^{-1}F(g)$ lifts w (considered as an element of the finite Weyl group of GL_n or SL_n).

Then

$$T'_w = {}^gT' = {}^g(T \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q)) = T_w \cap \mathrm{SL}_n(\overline{\mathbb{F}}_q).$$

Finally, we see the assertion on the character by going through the construction of the identification $\mathrm{Hom}^{cont}(Y(T_w)\otimes Z, \overline{\mathbb{F}}_q^\times)^F = \mathrm{Hom}(T_w(\mathbb{F}_q), \overline{\mathbb{F}}_q^\times)$ given by the snake lemma.

To see that θ' is in general position, we need to show that no nontrivial element of the group $N({}^gT')^F/{}^gT'^F$ generated by the n-cycle $w = {}^g(1,\ldots,n)$ acts nontrivially on θ'. Let $\mu \in X({}^gT)$ be a lift of θ and $\mu' \in X({}^gT')$ be its restriction. Assume $w^i \neq 1$ fixes θ'. Then $w^i\mu' - \mu' \in (q-1)X({}^gT')$. We may even assume $w^i\mu' - \mu' = 0$ by changing μ by some element of $(q-1)X(T)$ which does not change θ. Hence,

$$w^i\mu - \mu \in Q \cap X(T)^{W_0} = 0,$$

which contradicts the fact that θ is in general position by Corollary 2.3.2. $\square$

Chapter 3

G-Modules and Deligne-Lusztig Characters

In the last chapter we have seen how to construct a $G(\mathbb{F}_q)$-conjugacy-class (T_w, θ) from a continuous homomorphism $\rho : W_L/P_L \to G^*$ where $T_w \subseteq \mathrm{GL}_n(\overline{\mathbb{F}}_q)$ is a maximal torus and $\theta : T_w(\mathbb{F}_q) \to \mathbb{F}_q^\times$ is a character of its $\mathbb{F}_q$-rational points. To such a datum, Deligne and Lusztig have associated a virtual character of $\mathrm{GL}_n(\mathbb{F}_q)$ over an algebraically closed field in characteristic zero. We will use this in the following to construct a virtual representation of $G(\mathbb{F}_q)$ in characteristic p, which turns out to be an irreducible representation if ρ is an irreducible representation. To do this we have to start by repeating some generalities on algebraic representations of algebraic groups and their Grothendieck groups.

3.1 G-Modules

As before, let G be a group with Frobenius map F and fix a Borel subgroup B. By a G-module, we mean an algebraic representation of G over $\overline{\mathbb{F}}_q$. Any $\lambda \in X(T)$ can be considered as a character of B^- via inflation using the projection map $B^- \to T$. Thus, for $\lambda \in X(T)_+$, we have the Weyl module

$$\begin{aligned} W(\lambda) &= \mathrm{ind}_{B^-}^G(\lambda) \\ &= \{f \in \mathrm{Hom}(G, \mathbb{G}_a) : f(bg) = \lambda(b)f(g) \text{ for all } g \in G, b \in B^-\}. \end{aligned}$$

It is a finite dimensional $\overline{\mathbb{F}}_q$-vector space. It becomes a G-module by right translation. For a dominant $\lambda \in X(T)_+$ let $F(\lambda) = \mathrm{soc}_G W(\lambda)$. This is a simple G-module. Recall the partial order on $X(T)$ given by

$$x_1 \leq x_2 \text{ if and only if } x_2 - x_1 \in \sum_{\alpha \in \Phi^+} \mathbb{N}_0 \alpha.$$

We have the following classification of simple G-modules:

Theorem 3.1.1. *(i) Any simple G-module is isomorphic to $F(\lambda)$ for some $\lambda \in X(T)_+$. If $F(\lambda) \cong F(\mu)$ with $\lambda, \mu \in X(T)_+$, we have $\lambda = \mu$.*

(ii) Let $\lambda, \mu \in X(T)_+$. If $F(\mu)$ is a composition factor of $W(\lambda)$, $\mu \leq \lambda$. $F(\lambda)$ occurs as a composition factor of $W(\lambda)$ with multiplicity one.

Proof. (i) is [Jan87, II, Cor. 2.7]. (ii) follows from [Jan87, II, Prop. 2.2 b)] and [Jan87, II, Prop. 2.4 b)]. □

We call λ the highest weight of $F(\lambda)$. Under an extra condition on G, this induces a classification of the irreducible representations of the finite group G^F. To formulate this we need to introduce a subset of dominant weights. For this, let $r \in \mathbb{N}$ and set

$$X_r(T) = \{\lambda \in X(T) : 0 \leq \langle \lambda, \check{\alpha} \rangle < p^r \text{ for all } \alpha \in \Delta\}.$$

Note that this is in general not a finite set: For example, if $G = \mathrm{GL}_n$, we have $X(T)^{W_0} = \{\lambda \in X(T) : \langle \lambda, \check{\alpha} \rangle = 0 \text{ for all } \alpha \in \Delta\} \cong \mathbb{Z}$, and $X_r(T)$ is closed under addition of these elements.

Theorem 3.1.2. *Let $r \in \mathbb{N}$ such that $q = p^r$ and suppose that $G' = [G, G]$ is simply connected. For $\lambda \in X_r(T)$, the G^F-module $F(\lambda)$ is simple. Any simple G^F-module is of this form. $F(\lambda) \cong F(\mu)$ as G^F-modules if and only if $\lambda - \mu \in (q-1)X(T)^{W_0}$.*

Proof. See [Her09, Thm. 3.10] □

Explicitly, this means for $G = \mathrm{GL}_n$:

Corollary 3.1.3. *The irreducible $\mathrm{GL}_n(\mathbb{F}_q)$-modules are of the form $F(a_1, \ldots, a_n)$ with $0 \leq a_i - a_{i+1} \leq q - 1$. We have $F(a_1, \ldots, a_n) \cong F(b_1, \ldots, b_n)$ if and only if $(a_1, \ldots, a_n) - (b_1, \ldots, b_n) \in (q-1, \ldots, q-1)\mathbb{Z}$.*

3.2 p-Alcoves

For the representation theory of the groups G^F, it is convenient to consider alcoves relative to p. We will introduce this language now. Everything below can be found in greater detail in [Jan87][II.6] or [Her09][3.2].

For $\beta \in \Phi$ and $n \in \mathbb{Z}$, we can define the affine reflection $s_{\beta,n}$ on $X(T)$ (or $X(T) \otimes \mathbb{R}$) by

$$s_{\beta,n}(\lambda) = \lambda - (\langle \lambda, \check{\beta} \rangle - n)\beta = s_\beta(\lambda) + n\beta$$

for all $\lambda \in X(T)$. Denote by W_p the subgroup of $W_0 \ltimes X(T)$ generated by $\{s_{\beta,pn} : \beta \in \Phi, n \in \mathbb{Z}\}$, i.e. $W_p = W_0 \ltimes pQ$. We call W_p the p-affine Weyl group. Note that W_p is isomorphic to the affine Weyl group $W_0 \ltimes Q$ of the dual root datum. However it will be convenient for the following to include the p in the definition of the group rather than the considered action. The groups $W_0 \ltimes X(T)$ and W_p act on $X(T)$ and $X(T) \otimes \mathbb{R}$ naturally by affine maps.

As for the finite Weyl group we have the “dot“-action of W_p on the sets $X(T)$ and $X(T) \otimes \mathbb{R}$:

$$w \cdot \lambda := w(\lambda + \rho) - \rho,$$

where $\rho \in X(T) \otimes \mathbb{R}$ is chosen such that $\langle \rho, \check{\alpha} \rangle = 1$ for $\alpha \in \Phi$ simple, e.g. $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Using the dot action W_p acts by affine reflections. In other words, $s_{\beta,np}$ acts as the reflection with respect to the affine hyperplane

$$H_{\beta,np} := \{\lambda \in X(T) \otimes \mathbb{R} : \langle \lambda + \rho, \check{\beta} \rangle = np\}.$$

As usual, we can define alcoves as the connected components of the complement of all these affine hyperplanes in $X(T) \otimes \mathbb{R}$. To clarify that they are alcoves with respect to the W_p-action, we will call them p-alcoves. There is a distinct p-alcove, the so-called "lowest p-alcove" given by

$$C_0 := \{\lambda \in X(T) \otimes \mathbb{R} : 0 < \langle \lambda + \rho, \check{\alpha} \rangle < p \text{ for all } \alpha \in \Phi^+\}.$$

This name is justified, because C_0 is the unique minimal dominant p-alcove with respect to the order relation $\uparrow$ that will be introduced below. As in the usual affine setting, one checks that W_p operates on the set of p-alcoves and that $\overline{C_0}$ (and hence the closure of any other p-alcove) is a fundamental domain for the W_p-action on the set of p-alcoves.

We define the restricted region to be the set

$$A_{res} := \{\lambda \in X(T) \otimes \mathbb{R} : 0 < \langle \lambda + \rho, \check{\alpha} \rangle < p \text{ for all } \alpha \in \Delta\}$$

and we call a p-alcove restricted if it is contained in A_{res}.

We will now need an order relation on $X(T)$ induced by the W_p-action: Let $\lambda, \mu \in X(T)$. We say that $\lambda \uparrow \mu$ if $\lambda = \mu$ or if there are $\{s_i = s_{\beta_i, pn_i}\}_{1 \leq i \leq r}$ such that

$$\lambda \leq s_1 \cdot \lambda \leq s_2 s_1 \cdot \lambda \leq \ldots \leq s_r \ldots s_1 \cdot \lambda = \mu.$$

Clearly, if $\lambda \uparrow \mu$, we have $\lambda \leq \mu$ and $\lambda \in W_p \cdot \mu$. However the converse is not true in general.

If additionally, $C_0 \cap X(T)$ is nonempty, and C_1, C_2 are two p-alcoves, choose some $\lambda \in C_1 \cap X(T)$. Then there is a unique $\mu \in W_p \lambda \cap C_2$. We will define $C_1 \uparrow C_2$ if and only if $\lambda \uparrow \mu$. This is easily checked to be independent of the choice of λ. For small p it can happen that $C_0 \cap X(T)$ is empty. The definition can also be extended to this case ([Jan87][II.6.5]) but we shall not need this generalization.

The first important application for the order relation $\uparrow$ is the "strong linkage principle":

Proposition 3.2.1. *Let $\lambda, \mu \in X(T)_+$. If $F(\lambda)$ is a constituent of $W(\mu)$, we have $\lambda \uparrow \mu$.*

Proof. Se [Jan87][II.6.13]. □

We are now going to treat the GL_n-case in low dimensions as an example. This is all contained in [Her09][3.2]. Let us first treat the GL_2-case. Here we have

$$A_{res} = C_0 = \{(a, b) \in \mathbb{R}^2 : -1 < a - b < p - 1\}.$$

The following proposition explicitly determines the simple $\mathrm{GL}_2(\mathbb{F}_p)$-modules.

Proposition 3.2.2. *(i) Let $(a,b) \in X_1(T)$, i.e. $0 \leq a-b \leq p-1$. Then the G-module $W(a,b) = F(a,b)$ is simple and isomorphic to* $\mathrm{Sym}^{a-b}\overline{\mathbb{F}}_p^2 \otimes \det^b$.

(ii) The irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-representations are precisely the $\mathrm{Sym}^{a-b}\overline{\mathbb{F}}_p^2 \otimes \det^b$ with $(a,b) \in X_1(T)$, i.e. $0 \leq a-b \leq p-1$.

Proof. (i): By the strong linkage principle (Proposition 3.2.1), the Weyl module $F(a,b)$ for $(a,b) \in X_1(T)$ has no other constituents except for $F(a,b)$, because (a,b) lies in $\overline{C_0}$. Its multiplicity is one by [Jan87][II.6.16] and hence $W(a,b) = F(a,b)$ is simple.

If P is a homogeneous polynomial of degree $a-b$,

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \mapsto (g_1g_4 - g_2g_3)^b P(g_1, g_2)$$

lies in $W(a,b)$ and these elements form a nonzero subrepresentation which is isomorphic to $\mathrm{Sym}^{a-b}\overline{\mathbb{F}}_p^2 \otimes \det^b$. Since $W(a,b)$ is irreducible, the claim follows.

(ii): This follows from (i) and Theorem 3.1.1 and part (i). □

Let us now turn to the case of GL_3: The two restricted p-alcoves are

$$C_0 = \{(a,b,c) \in X(T) \otimes \mathbb{R} : -1 < a-b; -1 < b-c; a-c < p-1\}$$

and

$$C_1 := \{(a,b,c) \in \mathbb{R}^3 : p-1 < a-c; a-b < p-1; b-c < p-1\}.$$

Proposition 3.2.3. *Let $(a,b,c) \in X_1(T)$.*

(i) If $(a,b,c) \in C_1$, there is an exact sequence

$$0 \to F(a,b,c) \to W(a,b,c) \to F(c+p-2, b, a-p+2) \to 0.$$

(ii) If $(a,b,c) \notin C_1$, i.e. it is contained in C_0 or in the boundary of C_1, we have $W(a,b,c) = F(a,b,c)$.

Proof. See [Her09][3.18]. □

If $G = \mathrm{GL}_4$, the restricted p-alcoves are the following:

$$C_0 = \{(a,b,c,d) - \rho \in \mathbb{Z}^4 \otimes \mathbb{R} : 0 < a-b; 0 < b-c; 0 < c-d; a-d < p\},$$

$$C_1 = \{(a,b,c,d) - \rho \in \mathbb{Z}^4 \otimes \mathbb{R} : 0 < b-c; p < a-d; a-c < p; b-d < p\},$$

$$C_2 = \{(a,b,c,d) - \rho \in \mathbb{Z}^4 \otimes \mathbb{R} : 0 < c-d; p < a-c; a-b < p; b-d < p\},$$

$$C_3 = \{(a,b,c,d) - \rho \in \mathbb{Z}^4 \otimes \mathbb{R} : 0 < a-b; p < b-d; c-d < p; a-c < p\},$$

$$C_4 = \{(a,b,c,d) - \rho \in \mathbb{Z}^4 \otimes \mathbb{R} : p < a-c; p < b-d; b-c < p; a-d < 2p\},$$

$$C_5 = \{(a,b,c,d) - \rho \in \mathbb{Z}^4 \otimes \mathbb{R} : 2p < a-d; a-b < p; b-c < p; c-d < p\}.$$

We define elements in W_p by

$$w_{0,1} := (s_{\alpha_1+\alpha_2+\alpha_3}, p(\alpha_1+\alpha_2+\alpha_3)),$$

$$w_{1,2} := w_{3,4} := (s_{\alpha_1+\alpha_2}, p(\alpha_1+\alpha_2)),$$
$$w_{1,3} := w_{2,4} := (s_{\alpha_2+\alpha_3}, p(\alpha_2+\alpha_3)),$$
$$w_{4,5} := (s_{\alpha_1+\alpha_2+\alpha_3}, 2p(\alpha_1+\alpha_2+\alpha_3)),$$

Straightforward calculations show that

$$w_{i,j} \cdot C_i = C_j$$

whenever $w_{i,j}$ is defined. From this, one easily gets that

$$C_0 \uparrow C_1 \uparrow \begin{matrix} C_2 \\ C_3 \end{matrix} \uparrow C_4 \uparrow C_5.$$

3.3 Irreducible Representations of $\mathrm{GL}_n(\mathbb{F}_q)$

We know a classification of the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ from 3.1.2 (ii). In this section, we are going to give another classification in terms of BN-pairs and illuminate the connection between those classifications. The reason for us to do this, is that the second classification is suited better for establishing a connection between representations of $\mathrm{GL}_n(\mathbb{F}_q)$ and modules over finite Hecke algebras later.

This classification is due to Curtis for a finite group with a restricted split BN-pair in characteristic p. Let S_0 be the system of simple reflections. For $s = s_\alpha \in S_0$, we have the subgroups $U_s = U_\alpha = U \cap {}^{sw_0}U$ and $B_s = B \cap {}^{sw_0}B$. We set $T_s = T_\alpha = T \cap B_s$. Recall that there exist elements $n_{s_\alpha} = n_\alpha \in X_\alpha X_{-\alpha} X_\alpha \cap N(\mathbb{F}_q)$ for all $\alpha \in \Delta$ such that the image of n_α in W_0 is s_α.

Theorem 3.3.1. *The irreducible representations of* $\mathrm{GL}_n(\mathbb{F}_q)$ *are in bijective correspondence to tuples* (λ, I) *where* λ *is a character of* $T(\mathbb{F}_q)$ *and*

$$I \subseteq \{s \in S_0 : \lambda(T_s(\mathbb{F}_q)) = \{1\}\}.$$

We denote this representation by $F_{(\lambda,I)}$. *It is characterized by the following property:*

There exists an element $m \in F_{(\lambda,I)}$ *such that* $bm = \lambda(b)m$ *for all* $b \in B(\mathbb{F}_q)$ *(viewing* λ *as a character of* $B(\mathbb{F}_q)$ *via inflation) and*

$$\sum_{u \in U_s(\mathbb{F}_q)} u n_s m = \begin{cases} 0, & \text{if } s \notin I \\ -m, & \text{if } s \in I. \end{cases}$$

Proof. All of this is contained in [Cur70]. Theorem 4.3 shows that any irreducible representation is given by a weight m as above and that this weight determines $F_{(\lambda,I)}$ up to isomorphism. Theorem 5.7 states that the weights can be classified by tuples (λ, I) as above. □

Since we have two classifications of the irreducible representations of $\mathrm{GL}_n(\mathbb{F}_q)$ any $F(a_1, \ldots, a_n)$ as in Corollary 3.1.3 is isomorphic to precisely one $F_{(\lambda,I)}$. We now make this bijection explicit:

Proposition 3.3.2. *Let $\mu = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ with $0 \leq a_i - a_{i+1} \leq q-1$ which we view as a character of the algebraic torus T. Assume that $F(\mu) \cong F_{(\lambda,I)}$ for some $\lambda : T(\mathbb{F}_q) \to \overline{\mathbb{F}}_q$ and $I \subseteq \{s \in S_0 : \lambda(T_s) = \{1\}\}$. Then, we have*

$$\lambda = \mu_{|T(\mathbb{F}_q)} \text{ and } I = \{s_i \in S : a_i - a_{i+1} = q-1\}.$$

Proof. By [Jan87, II. Prop. 2.4], $F(\mu)^U = F(\mu)_\mu$ is one dimensional. So if we choose $0 \neq m \in F(\mu)^U$, B acts on m by μ and hence $B(\mathbb{F}_q)$ acts by λ on m.

As there are no roots in the case $n = 1$ the statement about the set I is trivially true then. Let us now consider the case $n = 2$. If $(a_1, a_2) = (a + (q-1), a)$, let $M = F(a + (p-1), a) = \mathrm{Sym}^{p-1}\overline{\mathbb{F}}_q^2 \otimes \det^a$. Denote by $M^{(i)}$ the representation with underlying vector space M and the action of $\mathrm{GL}_n(\mathbb{F}_q)$ being given by $g \cdot m := F_p^i(g)m$ where F_p is the p-power Frobenius map. Then Steinberg's Tensor Product Theorem [Jan87, II Cor. 3.17] states that

$$F(a + (q-1), a) \cong M \otimes_{\overline{\mathbb{F}}_q} M^{(1)} \otimes_{\overline{\mathbb{F}}_q} \ldots \otimes_{\overline{\mathbb{F}}_q} M^{(r-1)},$$

hence has dimension $q = p^r$. If $a_1 - a_2 < q - 1$ we get an isomorphism

$$F(a_1, a_2) \cong M_0 \otimes_{\overline{\mathbb{F}}_q} M_1^{(1)} \otimes_{\overline{\mathbb{F}}_q} \ldots \otimes_{\overline{\mathbb{F}}_q} M_{r-1}^{(r-1)}$$

where all M_i are of dimension less or equal to p and at least one of them has dimension strictly smaller than p by Steinberg's Tensor Product Theorem. Hence, the dimension of $F(a_1, a_2)$ is strictly smaller than q.

Now, only two cases can occur. Either $I = \{s\} = S_0$ or $I = \emptyset$. [CE04, Thm. 6.12 (ii)] states that we are in the first case if and only if $F(a_1, a_2)$ is of dimension q.

Let us now deduce the general case from this. Let $s = (i, i+1) \in S_0$. [Jan87, II 2.11] states that $F(a_1, \ldots, a_n)^{U_s}$ is the simple module with highest weight $(a_1, \ldots, a_n)$ for the Levi subgroup

$$L_s = diag(\mathbb{G}_m, \ldots \mathbb{G}_m, \mathrm{GL}_2, \mathbb{G}_m, \ldots, \mathbb{G}_m),$$

where GL_2 is in the i-th and the $(i+1)$-st row and column. But $F(a_i, a_{i+1})$ with L_s-action given by

$$(t_1, \ldots, t_{i-1}, g, t_{i+2}, \ldots, t_n) \cdot m := \prod_{j \neq i, i+1} t_j^{a_j} gm$$

is a simple module of highest weight $(a_1, \ldots, a_n)$ and thus isomorphic to $F(a_1, \ldots, a_n)^{U_s}$. By the natural inclusion $\mathrm{GL}_2 \subseteq L_s$, it becomes the simple module of highest weight (a_i, a_{i+1}). Now, the case of $n = 2$ yields the assumption for s, as the condition defining if $s \in I$ can be read inside L_s. $\square$

3.4 Grothendieck Groups

Let $\mathcal{C}$ be any abelian category such that the isomorphism classes of objects in $\mathcal{C}$ are a set. Denote by $\hat{\mathcal{C}}$ the set of isomorphism classes of objects in $\mathcal{C}$. Then the Grothendieck group of $\mathcal{C}$ is defined by

$$\mathrm{G}_0(\mathcal{C}) = \mathbb{Z}[\hat{\mathcal{C}}]/\langle B - A - C : 0 \to A \to B \to C \to 0 \text{ is exact}\rangle.$$

We will now apply this construction when $\mathcal{C}$ is either the category of finite dimensional algebraic G-modules or the category of finite dimensional K-representations of a finite group H for an algebraically closed field K. We will denote these groups $\mathrm{G}_0(G)$ resp. $\mathrm{G}_0(K[H])$. In both cases the isomorphism classes of simple objects of the corresponding categories form a basis of the Grothendieck groups. We will call elements of these groups virtual representations. Additionally, the tensor product of representations defines a multiplication which endows the respective Grothendieck groups with ring structures in both cases.

In the second case, we can consider the character of any representation of H over K by mapping $h \in H$ to its trace. This extends canonically to the Grothendieck group. The characters afforded by the isomorphism classes of irreducible representations are linearly independent (cf. [CR90][Thm. 17.3]). Hence, the map from $\mathrm{G}_0(K[H])$ to the set of class functions on H mapping a virtual representation to its character is injective.

$\{F(\lambda)\}_{\lambda \in X(T)_+}$ forms a basis of the Grothendieck group of G-modules by Theorem 3.1.1. If G' is simply connected, let $X_r(T)^0$ be a system of representatives for $X_r(T)$ modulo the equivalence relation

$$\lambda \cong \mu \Leftrightarrow \lambda - \mu \in (q-1)X_0(T).$$

By Theorem 3.1.2, the $\{F(\lambda)\}_{\lambda \in X_r(T)^0}$ are a basis for $\mathrm{G}_0(\overline{\mathbb{F}}_q[G^F])$.

For any G-module M and $\lambda \in X(T)$ we can consider the T-submodule

$$M_\lambda = \{m \in M : tm = \lambda(t)m \text{ for all } t \in T\}.$$

This allows us to define the formal character

$$\mathrm{ch}(M) := \sum_{\lambda \in X(T)} \dim(M_\lambda)e^\lambda \in \mathbb{Z}[X(T)]^{W_0}.$$

Here, we use e^λ as a symbol for the character λ considered as an element of $\mathbb{Z}[X(T)]$. We do this because we write the group law in $X(T)$ additively which corresponds to the multiplication in $\mathbb{Z}[X(T)]$. This results in formulas like $e^{\lambda_1}e^{\lambda_2} = e^{\lambda_1+\lambda_2}$.

Proposition 3.4.1. *The formal character induces a ring isomorphism*

$$\mathrm{ch} : \mathrm{G}_0(G) \to \mathbb{Z}[X(T)]^{W_0}.$$

Proof. See [Jan87, II, 5.7]. □

For $\lambda \in X(T)$, we can define a Weyl module in the Grothendieck group of G-modules by
$$W(\lambda) = \sum_i (-1)^i (R^i \mathrm{ind}_{B^-}^G)(\lambda)),$$
where $R^i\mathcal{F}$ denotes the ith right derived functor of $\mathcal{F}$ for any additive functor $\mathcal{F}$. Note that we do not require λ to be dominant. This is consistent with the previous definition for dominant λ as
$$(R^i \mathrm{ind}_{B^-}^G)(\lambda) = 0$$
for $i > 0$ in this case. There should be no confusion whether we mean the genuine representation or the element of the Grothendieck group by $W(\lambda)$. The formal character of a G-module can be computed using Weyl's Character-Formula:

Proposition 3.4.2. *For $\lambda \in X(T)$ we have*
$$\mathrm{ch}(W(\lambda)) = \frac{\sum_{w \in W_0} (-1)^{\mathrm{l}(w)} \cdot e^{w(\lambda+\rho)}}{\sum_{w \in W_0} (-1)^{\mathrm{l}(w)} \cdot e^{w(\rho)}}.$$

Proof. See [Jan87, II, Prop. 5.10]. □

Recall the dot-action of W_0 on $X(T)$. Then this formula implies that
$$W(w \cdot \lambda) = (-1)^{\mathrm{l}(w)} W(\lambda).$$

As the restriction is an exact functor from the category of finite G-modules to the category of finite dimensional G^F-representations, we obtain a homomorphism
$$\mathrm{res}_{G^F}^G : \mathrm{G}_0(G) \to \mathrm{G}_0(\overline{\mathbb{F}}_q[G^F]).$$

This allows us to view $F(\lambda)$ or $W(\lambda)$ as an element of $\mathrm{G}_0(\overline{\mathbb{F}}_q[G^F])$. To avoid misunderstandings, we may write $F_G(\lambda)$ resp. $F_{G^F}(\lambda)$ and similarly for Weyl modules. Note that $F_{G^F}(\lambda)$ is not simple in general.

Now let us assume $G = \mathrm{GL}_n$, such that $G' = \mathrm{SL}_n$. Again, the restriction to $G' \subseteq G$ resp. ${G'}^F \subseteq G^F$ induces homomorphisms $\mathrm{res}_{G'}^G$ and $\mathrm{res}_{G'^F}^{G^F}$. Then $T' = T \cap G'$ is a split maximal torus in G'. We have a (non-canonically) split exact sequence
$$0 \to X(T)^{W_0} \to X(T) \to X(T') \to 0$$
where the right map is given by restriction of characters. We will denote this map by $\lambda \mapsto \bar{\lambda}$.

Lemma 3.4.3. *(i)* $\mathrm{res}_{G'^F}^{G'} \circ \mathrm{res}_{G'}^G = \mathrm{res}_{G'^F}^{G^F} \circ \mathrm{res}_{G^F}^G = \mathrm{res}_{G'^F}^G$.

(ii) $\mathrm{res}_{G'}^G(F_G(\lambda)) = F_{G'}(\bar{\lambda})$ *for $\lambda \in X(T)_+$. In particular,* $\mathrm{res}_{G'}^G$ *is surjective.*

(iii) $\mathrm{res}_{G'}^G(W_G(\lambda)) = W_{G'}(\bar{\lambda})$ *for $\lambda \in X(T)_+$.*

(iv) $\mathrm{res}_{G'^F}^{G^F}(F_{G^F}(\lambda)) = F_{G'^F}(\bar{\lambda})$ *for $\lambda \in X(T)_+$. In particular,* $\mathrm{res}_{G'^F}^{G^F}$ *is surjective.*

(v) $\mathrm{res}_{G'^F}^{G^F}(W_{G^F}(\lambda)) = W_{G'^F}(\bar{\lambda})$ *for $\lambda \in X(T)_+$.*

Proof. (i) is obvious.

(ii): We have $G = G'Z(G)$. By [Jan87][Prop. 2.8], $Z(G)$ acts by a character. Hence, $F(\lambda)$ is a simple module for G' by restriction. Clearly, $\bar{\lambda}$ is the highest weight for the restriction of $F(\lambda)$ to T' which shows (ii).

(iii): We can identify the finite Weyl groups of G and G'. $\lambda \mapsto \bar{\lambda}$ is W_0-equivariant and hence induces a map

$$\operatorname{res}^T_{T'} : \mathbb{Z}[X(T)]^{W_0} \to \mathbb{Z}[X(T')]^{W_0}.$$

Obviously, this commutes with the formal character map, i.e.

$$\operatorname{res}^T_{T'} \circ \operatorname{ch}_G = \operatorname{ch}_{G'} \circ \operatorname{res}^G_{G'}.$$

But it is clear from Weyl's character formula 3.4.2 that $\operatorname{res}^T_{T'}$ maps $\operatorname{ch}(W_G(\lambda))$ to $\operatorname{ch}(W_{G^F}(\bar{\lambda}))$.

(iv):

$$\begin{aligned} F_{G'^F}(\bar{\lambda}) &= \operatorname{res}^{G'}_{G'^F}(F_{G'}(\bar{\lambda})) \overset{(ii)}{=} \operatorname{res}^{G'}_{G'^F} \circ \operatorname{res}^G_{G'}(F_G(\lambda)) \\ &\overset{(i)}{=} \operatorname{res}^{G^F}_{G'^F} \circ \operatorname{res}^G_{G^F}(F_G(\lambda)) = \operatorname{res}^{G^F}_{G'^F}(F_{G^F}(\lambda)). \end{aligned}$$

The second claim follows from the fact that every irreducible ${G'}^F$-representation is of the form $F_{G'^F}(\mu)$ for some $\mu \in X(T')_+$.

(v) follows similarly from (i) and (iii). □

Proposition 3.4.4. *(i) The Weyl modules $\{W_G(\lambda)\}_{\lambda \in X(T)_+}$ are a basis of* $\mathrm{G}_0(G)$.

(ii) The Weyl modules $\{W_{G^F}(\lambda)\}_{\lambda \in X_r(T)^0}$ are a basis of $\mathrm{G}_0(\overline{\mathbb{F}}_q[G^F])$. More precisely, there exists a total ordering extending $\leq$ such that the base change matrix between $\{W_{G^F}(\lambda)\}_{\lambda \in X_r(T)^0}$ and $\{F_{G^F}(\lambda)\}_{\lambda \in X_r(T)^0}$ is upper triangular with ones on the diagonal.

Proof. (i): Consider the endomorphism of the abelian group $\mathrm{G}_0(G)$ given by $F(\lambda) \mapsto W(\lambda)$ for all $\lambda \in X(T)_+$. We need to show that it is an isomorphism. But by Theorem 3.1.1 (ii), the matrix representing this endomorphism is upper triangular with ones on the diagonal in any total order extending $\leq$ on $X(T)_+$.

(ii): We first show this for G' instead of G. Let $V(\lambda) = W(-w_0\lambda)^*$. By [Jan87, II, 2.13], $V(\lambda) = W(\lambda)$ in $\mathrm{G}_0(\overline{\mathbb{F}}_q[{G'}^F])$. Hence, they have the same composition factors. By [Won72, Thm. 3E] any composition factor $F_{G'^F}(\mu)$ of $V_{G'^F}(\lambda)$ (and thus of $W_{G'^F}(\lambda)$) for $\lambda \in X_r(T)$ satisfies $\mu \leq \lambda$ and $\mu \in X_r(T)$. Moreover, by the same source, $F_{G'}(\lambda)$ occurs with multiplicity one. Now the claim follows for G' as in (i).

In the general case, choose a splitting $X(T) \cong X(T') \oplus X(T)^{W_0}$. We endow $X(T)^{W_0}$ with the order relation given by equality and $X(T)$ with the lexicographical order coming from this and the natural order on $X(T')$, i.e.

$$(\lambda_0, \overline{\lambda}) \preceq (\mu_0, \overline{\mu}) \Leftrightarrow \begin{cases} \overline{\lambda} < \overline{\mu} \\ \text{or} \\ \overline{\lambda} = \overline{\mu} \text{ and } \lambda_0 = \mu_0. \end{cases}$$

Now, let

$$W_{G^F}(\lambda) = \sum_{\mu \in X_r(T)^0} a_\mu F_{G^F}(\mu)$$

and

$$W_{G'^F}(\overline{\lambda}) = \sum_{\mu' \in X_r(T')} b_{\mu'} F_{G'^F}(\mu').$$

Note that all a_μ and $b_{\mu'}$ are non-negative integers as they are multiplicities of composition factors of the genuine modules $W_{G^F}(\lambda)$ resp. $W_{G'^F}(\overline{\lambda})$. By replacing some μ by other representatives of their equivalence classes modulo $(q-1)X(T)^{W_0}$, we may assume that $\mu \preceq \lambda$ for all μ with $a_\mu \neq 0$. Let $\mu_0 \in X_r(T)^0$ such that $a_{\mu_0} \neq 0$. Applying Lemma 3.4.3 (iv) and (v), we see that $b_{\overline{\mu_0}} = \sum_{\overline{\mu} = \overline{\mu_0}} a_\mu$.

If $\bar{\mu}_0 < \bar{\lambda}$, we have $\mu_0 \prec \lambda$ by construction. If $\bar{\lambda} = \bar{\mu}_0$, the simply connected case yields $b_{\overline{\mu_0}} = 1$. But as $F_G(\lambda)$ occurs with multiplicity one in $W_G(\lambda)$, we have $a_\lambda \geq 1$ which enforces $a_\lambda = 1$ and hence $\mu_0 = \lambda$.

This shows that the matrix representing $F(\lambda) \mapsto W(\lambda)$ for $\lambda \in X_r(T)^0$ is upper triangular with ones on the diagonal for any total order extending $\preceq$ and we are done as before. $\square$

3.5 Deligne-Lusztig Characters and Jantzen's Formula

As before, let G be a reductive group with Frobenius map F and suppose we are given a G^F-conjugacy class (T_w, θ) with a maximal (possibly non-split) torus $T_w \subseteq G$ obtained from twisting T with $w \in W_0$ and a character $\theta : T_w(\mathbb{F}_q) \to \overline{\mathbb{F}}_q^\times$.

Once and for all, we fix some prime $l \neq p$ and an isomorphism of abstract fields $\overline{L} = \overline{\mathbb{Q}}_p \cong \overline{\mathbb{Q}}_l$. Then, the Teichmüller map $\overline{\mathbb{F}}_q^\times \to \overline{\mathbb{Q}}_p^\times$ defines an embedding $\overline{\mathbb{F}}_q^\times \to \overline{\mathbb{Q}}_l^\times$ by which we will view θ as a character with values in $\overline{\mathbb{Q}}_l^\times$. To such a tuple (T_w, θ), Deligne and Lusztig assign a virtual representation

$$R^\theta(w) \in \mathrm{G}_0(\overline{\mathbb{Q}}_l[G^F]) \cong \mathrm{G}_0(\overline{\mathbb{Q}}_p[G^F])$$

(cf. [DL76, Def. 1.9]). Using the theory developed by Deligne and Lusztig, we get an immediate consequence of what we have calculated before:

The resulting virtual $\overline{\mathbb{Q}}_p$-representation is in fact independent of the chosen isomorphism: If we denote by $[\cdot] : \overline{\mathbb{F}}_p^\times \to \overline{\mathbb{Q}}_p^\times$ the Teichmüller map and by $\phi : \overline{\mathbb{Q}}_p \to \overline{\mathbb{Q}}_l$ the chosen isomorphism, $R^\theta(w)$, considered as a character with values in $\overline{\mathbb{Q}}_l$ is given by

$$R^\theta(w)(g) = \frac{1}{\#T^F} \sum_{t \in T^F} \phi[\theta(t^{-1})] \mathcal{L}(g, t)$$

with integers $\mathcal{L}(g,t)$, the Lefschetz numbers of (g,t) acting on the affine algebraic variety $\tilde{X} = \Lambda^{-1}(U)$. Hence, if we view $R^\theta(w)$ as a character with values in $\overline{\mathbb{Q}}_p$, it will be given by

$$R^\theta(w)(g) = \frac{1}{\#T^F} \sum_{t \in T^F} [\theta(t^{-1})] \mathcal{L}(g, t),$$

which is independent of the chosen isomorphism.

Corollary 3.5.1. *The virtual character associated with an irreducible representation of W_L resp. irreducible projective W_L-representation is a cuspidal $\mathrm{GL}_n(\mathbb{F}_q)$ resp. $\mathrm{SL}_n(\mathbb{F}_q)$-representation after possibly multiplying with -1.*

Proof. In both cases (T_w, θ) resp. (T'_w, θ') is in general position. As the $\mathbb{F}_q$-rank of T_w resp. T'_w is 1 resp. 0 those tori cannot be contained in nontrivial parabolic subgroups defined over $\mathbb{F}_q$. Now Theorem 8.13 in [DL76] yields the claim. □

From now on we will always view the $R^\theta(w)$ as virtual representations over $\overline{\mathbb{Q}}_p$ because this enables us to reduce them to $\overline{\mathbb{F}}_q$: Let V be such a $\overline{\mathbb{Q}}_p$-representation of G^F and choose a lattice $M \subseteq V$. Then we can can define its reduction mod p as $M/\overline{\mathbb{Z}}_p M$. However, this depends on the choice of M, but its image in $\mathrm{G}_0(\overline{\mathbb{F}}_q[G^F])$ does not by the Brauer-Nesbitt Theorem: This is for example stated in [CR90][16.16] in a slightly differing setting: There $\overline{\mathbb{Q}}_p$ is replaced by the quotient field K of a discrete valuation ring R which has a residue field of characteristic p. By taking K large enough we can achieve that M is defined over $\mathcal{O}_K$, say $M = M_0 \otimes_{\mathcal{O}_K} \overline{\mathbb{Z}}_p$ for some lattice M_0 inside a finite dimensional K-representation V_0 with $V_0 \otimes_K \overline{\mathbb{Q}}_p = V$, and that the residue field of K contains $\overline{\mathbb{F}}_q$. After possibly enlarging K (e.g. if K is a splitting field for G^F), the isomorphism-class of V_0 does not depend on any choices. But then the characters of $M/\overline{\mathbb{Z}}_p M$ and $M_0/\pi_K M_0$ are obviously the same and the latter is independent of any choices by [CR90][Prop. 16.16].

Jantzen has examined how those virtual representations reduce to virtual representations over $\overline{\mathbb{F}}_q$. We begin with giving them a different parametrization. Firstly, let $g \in G$ such that $g^{-1}F(g)$ is a lift of w in G. Recall that $T_w = {}^gT$. By [DM91, Prop. 13.7 (i)] the restriction $X(T_w) \to \mathrm{Hom}(T_w^F, \overline{\mathbb{F}}_q^\times)$ is surjective. Hence, we find a $\mu \in X(T)$ such that

$$\theta = [T_w^F \to \overline{\mathbb{F}}_q^\times, t \mapsto \mu(g^{-1}tg)].$$

We can multiply $R^\theta(w)$ with ± 1 such that its value at 1 becomes positive and denote the resulting character by $R_w(\mu)$.

From now on assume for simplicity that G is split. The group $X(T) \rtimes W_0$ acts on the set $W_0 \times X(T)$ by

$$^{(\chi,\sigma)}(w, \mu) = (\sigma w \sigma^{-1}, \sigma\mu + (q - \sigma w \sigma^{-1})\chi).$$

If two elements of (w, μ) and (w', μ') in $W_0 \times X(T)$ are conjugate under this action, they define the same Deligne-Lusztig character, i.e. we have $R_w(\mu) = R_{w'}(\mu')$ (cf. [Jan81][3.1]).

Jantzen has given a formula for the reductions of the virtual representations $R_w(\mu)$ mod p if G is a connected semisimple simply connected algebraic group defined and split over $\mathbb{F}_p$. He has also deduced the more general case where only G' needs to be simply connected. This has been published by Herzig. From now on, assume that either $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$.

So, let T be the standard split torus in G. Since $G' = \mathrm{SL}_n$ is simply connected, we can find elements $\omega_\alpha \in X(T)$ for any simple root α such that $\langle \omega_\alpha, \check{\beta} \rangle = \delta_{\alpha,\beta}$ for any simple root β. For $\sigma \in W_0$, we set

$$\rho_\sigma = \sum_{\substack{\alpha \in B \\ \sigma^{-1}\alpha \in R^-}} \omega_\alpha \text{ and } \epsilon_\sigma = \sigma^{-1}\rho_\sigma.$$

We set $\rho := \rho_{w_0}$ where w_0 is the longest element of W_0. This goes hand in hand with our notation above because $\langle \rho, \check{\beta} \rangle = 1$ for each simple root β. Consider the matrix

$$\begin{aligned}(m_{\sigma,\tau})_{\sigma,\tau \in W_0} :&= (\det(\tau)\mathrm{ch}W(-\epsilon_{w_0\sigma} + \epsilon_\tau - \rho))_{\sigma,\tau \in W_0} \\ &= (\mathrm{ch}W(\tau(-\epsilon_{w_0\sigma} + \epsilon_\tau) - \rho))_{\sigma,\tau \in W_0}\end{aligned}$$

with entries in $\mathbb{Z}[X(T)]$. It is upper triangular and unipotent in some ordering of W_0 and hence invertible (See [Her09][Appendix 3.3]). We can consider the inverse matrix with entries $\gamma_{\sigma,\tau}$. With this matrix we can express the reduction of $R_w(\mu)$ by the restrictions of Weyl modules to the finite groups $\mathrm{GL}_n(\mathbb{F}_q)$ resp. $\mathrm{SL}_n(\mathbb{F}_q)$.

Proposition 3.5.2. *Let $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$. In $\mathrm{G}_0(\overline{\mathbb{F}}_q[G(\mathbb{F}_q)])$, we have*

$$\overline{R_w(\mu)} = \sum_{\sigma,\tau \in W_0} \mathrm{ch}^{-1}(\gamma_{\sigma,\tau}) W(\sigma(\mu - w\epsilon_{w_0\tau}) - \rho + q\rho_\sigma).$$

Proof. See [Her09, Thm. 5.2]. □

This formula becomes very complicated for growing n. However it is quite manageable and useful for explicit computations in low dimensions. We will now give its explicit form for $G = \mathrm{GL}_2$ and $G = \mathrm{GL}_3$ when $w = s = (1,2)$ resp. $w = (1,2,3)$ in which case we shall apply it later.

We start with the GL_2-case. For the unique root α, we may choose $\omega_\alpha = \chi_1$ and we get $\rho_1 = 0$, $\rho = \rho_s = \chi_1$, $\epsilon_1 = 0$ and $\epsilon_s = s\chi_1 = \chi_2$. So, using Weyl's character formula, we can calculate

$$m_{1,1} = \mathrm{ch}W(-\epsilon_s + \epsilon_1 - \rho) = \mathrm{ch}W(-\chi_1 - \chi_2) = e^{-\chi_1 - \chi_2},$$

$$m_{1,s} = -\mathrm{ch}W(-\epsilon_s + \epsilon_s - \rho) = 0,$$
$$m_{s,1} = \mathrm{ch}W(-\epsilon_1 + \epsilon_1 - \rho) = 0,$$

$$\begin{aligned}m_{s,s} &= \det(s)\mathrm{ch}W(-\epsilon_1 + \epsilon_s - \rho) = \mathrm{ch}W(s \cdot (\epsilon_s - \rho)) \\ &= \mathrm{ch}W(s\epsilon_s - \rho) = \mathrm{ch}W(0) = e^0.\end{aligned}$$

Inserting this into the statement of the proposition, we get

$$\begin{aligned}\overline{R_s(w)} &= W(\mu - s\epsilon_s - \rho + \chi_1 + \chi_2) + W(s\mu + (q-1)\rho) \\ &= W(\mu - \rho + s\rho) + W(s\mu + (q-1)\rho).\end{aligned}$$

Let us now give the explicit version of Jantzen's formula for GL_3. We identify $X(T)$ with $\mathbb{Z}^3$ by the choice of the basis $\{\chi_1, \chi_2, \chi_3\}$. Let $\alpha = (1,-1,0)$ and $\beta = (0,1,-1)$ be the simple roots and $s_1 = s_\alpha$, $s_2 = s_\beta$ the corresponding simple reflections. We may take $\omega_\alpha = (1,0,0)$ and $\omega_\beta = (1,1,0)$ which implies $\rho_1 = 0$, $\rho_{s_1} = \omega_\alpha = (1,0,0)$, $\rho_{s_2} = \omega_\beta = (1,1,0)$, $\rho_{s_1 s_2} = \omega_\alpha = (1,0,0)$, $\rho_{s_2 s_1} = \omega_\beta = (1,1,0)$ and $\rho_{w_0} = \omega_\alpha + \omega_\beta = (2,1,0)$ and hence $\epsilon_1 = 0$, $\epsilon_{s_1} = (0,1,0)$, $\epsilon_{s_2} = (1,0,1)$, $\epsilon_{s_1 s_2} = (0,0,1)$, $\epsilon_{s_2 s_1} = (0,1,1)$ and $\epsilon_{w_0} = (0,1,2)$. The matrix $(m_{\sigma,\tau})_{\sigma,\tau}$ is diagonal for GL_3 (cf. [Her09][5.1]) and hence we only need to calculate the diagonal entries:

$$\begin{aligned}
m_{1,1} &= \mathrm{ch}W(-\epsilon_{w_0} - \rho) = \mathrm{ch}W(-2,-2,-2),\\
m_{s_1,s_1} &= \mathrm{ch}W(s_1(-\epsilon_{s_1 s_2} + \epsilon_{s_1}) - \rho) = \mathrm{ch}W(-1,-1,-1),\\
m_{s_2,s_2} &= \mathrm{ch}W(s_2(-\epsilon_{s_2 s_1} + \epsilon_{s_2}) - \rho) = \mathrm{ch}W(-1,-1,-1),\\
m_{s_1 s_2,s_1 s_2} &= \mathrm{ch}W(s_1 s_2(-\epsilon_{s_1} + \epsilon_{s_1 s_2}) - \rho) = \mathrm{ch}W(-1,-1,-1),\\
m_{s_2 s_1,s_2 s_1} &= \mathrm{ch}W(s_2 s_1(-\epsilon_{s_2} + \epsilon_{s_2 s_1}) - \rho) = \mathrm{ch}W(-1,-1,-1),\\
m_{w_0,w_0} &= \mathrm{ch}W(w_0 \epsilon_{w_0} - \rho) = \mathrm{ch}W(0,0,0).
\end{aligned}$$

Inserting our calculations before into the formula from Proposition 3.5.2 for $\mu = (a,b,c)$, we obtain

$$\begin{aligned}
\overline{R_w(a,b,c)} =& W(a-2,b+1,c+1) + W(b+q-1,a-1,c+1)\\
&+ W(a+q-2,c+q-1,b+1) + W(c+q-2,a,b+1)\\
&+ W(b+q-2,c+q,a) + W(c+2(q-1),b+q-1,a).
\end{aligned}$$

This illustrates that Jantzen's formula becomes more complicated even when moving from $n=2$ to $n=3$. This effect is only due to the growth of the Weyl group. Obviously alone the order of W_0 becomes very big quickly. For $n>3$ there arises another difficulty: The matrix $(m_\sigma,\tau)_{\sigma,\tau}$ is no longer diagonal, so that there are way more nonzero summands.

Despite this, the Jordan-Hölder constituents of $\overline{R_w(\mu)}$ can still be described in generic situations with results by Herzig. For the remainder of this section assume $q=p$. We begin by remarking that we can identify the p-alcoves for varying p with each other by the bijective map $X(T)\otimes\mathbb{R} \to X(T)\otimes\mathbb{R}, \mu-\rho \mapsto p^{-1}\mu - \rho$. If

$$C = \{\mu \in X(T)\otimes\mathbb{R} : n_\alpha < \langle \mu+\rho, \check{\alpha}\rangle < (n_\alpha+1)p \text{ for all } \alpha\in\Phi^+\}$$

is any p-alcove with integers $n_\alpha \in \mathbb{Z}$ we will say that $\mu \in X(T)\otimes\mathbb{R}$ lies δ-deep in C if

$$n_\alpha + \delta < \langle \mu+\rho, \check{\alpha}\rangle < (n_\alpha+1)p - \delta \text{ for all } \alpha\in\Phi^+.$$

A statement formulated for varying p is said to be true for μ sufficiently deep in C if there is a $\delta>0$, independent of p, such that the statement is true for all δ-deep $\mu\in C$. The following proposition describes the Jordan-Hölder constituents of $\overline{R_w}(\mu)$ if μ lies sufficiently deep in an p-alcove.

Proposition 3.5.3. *Suppose that C is a p-alcove and that $\mu\in X(T)$ lies sufficiently deep in C. Then the Jordan-Hölder constituents of $\overline{R_w(\mu)}$ are the $F(\lambda)$ with λ restricted such that there exist $\sigma\in W_0$, $\nu\in X(T)$ with $\sigma\cdot(\mu+(w-p)\nu-\rho)$ dominant and*

$$\sigma\cdot(\mu+(w-p)\nu-\rho) \uparrow w_0\cdot(\lambda - p\rho).$$

Proof. See [Her09][5.7]. $\square$

Chapter 4

Hecke Algebras

4.1 The finite Hecke Algebra

Let G be a connected reductive group defined and split over $\mathbb{F}_q$ with corresponding Frobenius map F. We keep the notations B, U and T as introduced before. We have the representation $\overline{\mathbb{F}}_q[U^F\backslash G^F]$, where $g \in G^F$ acts by multiplication with g^{-1} from the right. Note that $\overline{\mathbb{F}}_q[U^F\backslash G^F]$ is isomorphic to the induced representation $\mathrm{Ind}_{U^F}^{G^F}\overline{\mathbb{F}}_q$.

Recall that (G^F, B^F, N^F, S_0) is a strongly split BN-pair of characteristic p by Lemma 1.5.3. In the context of finite Hecke algebras, we will write $W_0^{(1)}$ instead of $N(\mathbb{F}_q)$. We are doing this because the superscript $?^{(1)}$ will always denote an extension of a group by the torus $T(\mathbb{F}_q)$. The length function extends from W_0 to $W_0^{(1)}$ by inflation. Recall that for each simple root α, we have defined the subgroups $X_\alpha = U_\alpha^F$, $G_\alpha = \phi_\alpha(\mathrm{SL}_2)^F$ and $T_\alpha = T^F \cap G_\alpha$ of G^F. If $s = s_\alpha$ is the corresponding simple reflection, we will also replace the subscript α by s, i.e. we will write T_s for T_α and so on.

Definition 4.1.1. The $\overline{\mathbb{F}}_q$-algebra

$$\mathcal{H}_0^{(1)} = \mathrm{End}_{G^F}(\overline{\mathbb{F}}_q[U^F\backslash G^F])$$

is called the finite Hecke algebra of G.

Recall that

$$n_\alpha = u_\alpha(1)u_\alpha(-1)u_\alpha(1) = \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in X_\alpha X_{-\alpha} X_\alpha \cap N(T)^F$$

for all $\alpha \in \Delta$ and the image of $n_\alpha = n_{s_\alpha}$ in W_0 is s_α. Denote by $S_0^{(1)}$ the set of all n_α for $\alpha \in \Delta$.

Theorem 4.1.2. *There exist elements $(\tau_n)_{n \in W_0^{(1)}}$ in $\mathcal{H}_0^{(1)}$ such that $\mathcal{H}_0^{(1)}$ has the following presentation:*

- $\mathcal{H}_0^{(1)} = \bigoplus_{n \in W_0^{(1)}} \overline{\mathbb{F}}_q \tau_n$ *as $\overline{\mathbb{F}}_q$-vector spaces.*

- $\tau_n^2 = -\#T_\alpha^{-1}(\sum_{t\in T_\alpha} \tau_{nt})$ *for all* $n = n_\alpha \in S_0^{(1)}$.
- $\tau_{n_1}\tau_{n_2} = \tau_{n_1 n_2}$ *if* $l(n_1 n_2) = l(n_1) + l(n_2)$.

Proof. See [CE04, Prop. 6.8 (i)] and [CE04, Thm. 6.10 (ii)]. □

In the following, by an $\mathcal{H}_0^{(1)}$-module, we will always mean a right module and denote by $\mathrm{G}_0(\mathcal{H}_0^{(1)})$ the Grothendieck group of finitely generated $\mathcal{H}_0^{(1)}$ right-modules. This convention is due to the fact that the functor $\mathrm{Hom}_{\overline{\mathbb{F}}_q[G^F]}(\overline{\mathbb{F}}_q[U^F\backslash G^F], \cdot)$ makes a representation M of G^F to an $\mathcal{H}_0^{(1)}$-right module by

$$\mathrm{Hom}_{\overline{\mathbb{F}}_q[G^F]}(\overline{\mathbb{F}}_q[U^F\backslash G^F], M) \times \mathcal{H}_0^{(1)} \to \mathrm{Hom}_{\overline{\mathbb{F}}_q[G^F]}(\overline{\mathbb{F}}_q[U^F\backslash G^F], M), (f, \tau) \mapsto f(\tau(\cdot)).$$

By Frobenius reciprocity we have canonical isomorphisms

$$\mathrm{Hom}_{\overline{\mathbb{F}}_q[G^F]}(\overline{\mathbb{F}}_q[U^F\backslash G^F], M) \cong \mathrm{Hom}_{\overline{\mathbb{F}}_q[U^F]}(\overline{\mathbb{F}}_q, M) \cong M^{U^F}.$$

A classification for the simple $\mathcal{H}_0^{(1)}$-modules is known, which we are going to describe in the following. Let $\lambda : T^F \to \overline{\mathbb{F}}_q^\times$ be a character. We can consider the set

$$S_0^\lambda = \{s \in S_0 : \lambda(T_s) = 1\}$$

and some subset $I \subseteq S_0^\lambda$. Then, we obtain a character $\psi_{(\lambda,I)} : \mathcal{H}_0^{(1)} \to \overline{\mathbb{F}}_q$ defined by

$$\psi_{(\lambda,I)}(\tau_n) = \begin{cases} (-1)^l \lambda^{-1}(t), & \text{if } n = n_{s_1} \dots n_{s_l} t \text{ with } s_1, \dots, s_l \in I, t \in T \\ 0, & \text{if } n \notin \langle (n_i)_{i\in I}, T\rangle. \end{cases}$$

Theorem 4.1.3. *Every simple module of* $\mathcal{H}_0^{(1)}$ *is of the form* $\psi_{(\lambda,I)}$ *for some character* $\lambda : T(\mathbb{F}_q) \to \overline{\mathbb{F}}_q^\times$ *and a subset* $I \subseteq S_0^\lambda$. *They are pairwise non-isomorphic.*

Proof. See [CE04, Thm. 6.10 (iii)]. □

Theorem 4.1.4. *(i) The functor* $\mathrm{Hom}_{\overline{\mathbb{F}}_q[G^F]}(\overline{\mathbb{F}}_q[U^F\backslash G^F], \cdot)$ *from the category of finite dimensional representations of* G^F *to the category of finitely generated* $\mathcal{H}_0^{(1)}$*-right modules induces a bijection between the isomorphism classes of simple objects of both categories.*

(ii) Under the bijection of (i) the simple module $F_{(\lambda,I)}$ *is mapped to* $\psi_{(\lambda,I)}$.

Proof. See [CE04, Thm. 6.12]. □

Corollary 4.1.5. *The assignment* $F_{(\lambda,I)} \mapsto \psi_{(\lambda,I)}$ *induces an isomorphism between the Grothendieck groups* $\mathrm{G}_0(\overline{\mathbb{F}}_q[G(\mathbb{F}_q)])$ *and* $\mathrm{G}_0(\mathcal{H}_0^{(1)})$.

4.2 The pro-p Iwahori Hecke Algebra

In this section, we describe the structure of pro-p Iwahori Hecke algebras. For more details on anything stated here, see [Vig05]. Let G be a split connected reductive group split over L. For convenience, further assume that the root datum of G is irreducible. As before, let T be a maximal torus split over L, N the normalizer of T and $B = TU$ a Borel subgroup defined over L containing T with unipotent radical U.

The groups G, T and U can be defined over $\mathcal{O}_L$. We fix such a model. Hence, we may speak of $\mathcal{O}_L$- and $\mathbb{F}_q$-rational points.

The Iwahori subgroup $I \subseteq G(\mathcal{O}_L)$ is the preimage of $B(\mathbb{F}_q)$ with respect to the reduction map and its unique pro-p Sylow subgroup $I^{(1)}$, the preimage of $U(\mathbb{F}_q)$, is called the pro-p Iwahori subgroup.

$N(L)/T(L)$ identifies with the finite Weyl group W_0 of G. The group $N(L)/T(\mathcal{O}_L)$ is isomorphic to the semidirect product $W_0 \ltimes Y(T)$ with the natural action of W_0 on $Y(T)$. We can consider $Y(T)$ as a subgroup of W by mapping a cocharacter γ to $\gamma(\pi_L)$, where $\pi_L \in L$ is a prime element.

As before, we will use the notation we^y for the element $(w, y) \in W_0 \ltimes Y(T)$. This has the advantage that we can write the multiplication of cocharacters multiplicatively in W and additively in $Y(T)$, i.e. $e^x e^y = e^{x+y}$.

We will need to consider the group $W^{(1)} = N(L)/T(1 + \pi_L\mathcal{O}_L)$. It is related to W by the exact sequence

$$1 \to T(\mathbb{F}_q) \to W^{(1)} \to W \to 1.$$

Note that this sequence does not split in general. However, it splits for $G = \mathrm{GL}_n$.

As before, we have the elements $n_{s_\alpha} = n_\alpha = \phi_\alpha\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $h_\alpha(t) = \phi_\alpha\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \check{\alpha}(t)$ for $\alpha \in \Phi$. We will write n_s for n_α if $s = s_\alpha \in S_0$ and $n_s = h_{\alpha_0}(\pi_L)n_{\alpha_0}$ if $s = s_0$ and $h_s = h_\alpha$ for $s = s_\alpha \in S_0$ and $h_{s_0} = h_{\alpha_0}$. We set $T_s(\mathbb{F}_q) := h_s(\mathbb{F}_q^\times)$ if s is either a reflection, i.e. of the form s_α for some $\alpha \in \Phi$ or $s = s_0$. This is compatible with the notations of the previous section.

Definition 4.2.1. The integral pro-p Hecke algebra is the ring

$$\mathcal{H}^{(1)}_{\mathbb{Z}} := \mathrm{End}_{\mathbb{Z}[G(L)]}\mathbb{Z}[I^{(1)}\backslash G(L)],$$

where $g \in G$ acts on $I^{(1)}\backslash G(L)$ by multiplying with g^{-1} from the right. Its scalar extension $\mathcal{H}^{(1)} = \mathcal{H}^{(1)}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q$ is the pro-p Iwahori Hecke algebra.

Both of these algebras have a nice description in terms of generators and relations. We have the decomposition $G(L) = \coprod_{w \in W^{(1)}} I^{(1)}wI^{(1)}$ (cf. [Vig05, Thm.6]), which implies that $\mathcal{H}^{(1)}_{\mathbb{Z}}$ identifies with the free $\mathbb{Z}$-module generated by the double cosets $I^{(1)}wI^{(1)}$ for $w \in W^{(1)}$. Denote by $\tau_w \in \mathcal{H}^{(1)}_{\mathbb{Z}}$ the element corresponding to $I^{(1)}wI^{(1)}$ and also its image in $\mathcal{H}^{(1)}$.

Theorem 4.2.2. *$\mathcal{H}^{(1)}_{\mathbb{Z}}$ has the following presentation:*

- $\mathcal{H}_{\mathbb{Z}}^{(1)} = \bigoplus_{w \in W^{(1)}} \mathbb{Z}\tau_w$ *as* $\mathbb{Z}$*-modules.*
- $\tau_\sigma^2 = q\tau_{\sigma^2} + \tau_\sigma \sum_{t \in T_s(\mathbb{F}_q)} \tau_{tn_s^{-1}\sigma}$ *for all* $\sigma \in S^{(1)}$ *lifting* $s \in S$*.*
- $\tau_{w_1}\tau_{w_2} = \tau_{w_1w_2}$ *if* $l(w_1w_2) = l(w_1) + l(w_2)$*.*

Proof. See [Vig05, Thm. 1] □

Corollary 4.2.3. $\mathcal{H}^{(1)}$ *has the following presentation:*

- $\mathcal{H}^{(1)} = \bigoplus_{w \in W^{(1)}} \overline{\mathbb{F}}_q\tau_w$ *as* $\overline{\mathbb{F}}_q$*-vector spaces.*
- $\tau_\sigma^2 = \tau_\sigma \sum_{t \in T_s(\mathbb{F}_q)} \tau_{tn_s^{-1}\sigma}$ *for all* $\sigma \in S^{(1)}$ *lifting* $s \in S$*.*
- $\tau_{w_1}\tau_{w_2} = \tau_{w_1w_2}$ *if* $l(w_1w_2) = l(w_1) + l(w_2)$*.*

We will refer to the relation in the the second line of the presentation as the quadratic relations and those in the third line as the braid relations.

In particular, we see that $\mathcal{H}_0^{(1)}$ identifies with the subalgebra of $\mathcal{H}^{(1)}$ generated by all $\{\tau_{n_s}\}_{s \in S_0}$ and all $\{\tau_t\}_{t \in T(\mathbb{F}_q)}$, which is isomorphic to $\mathcal{H}_0^{(1)}$.

4.3 Idempotents and Inclusions

The non pro-p affine Iwahori Hecke algebras of split reductive groups over p-adic fields do not change when L is replaced by a finite extension: They are given by a presentation only depending on the root datum. However, this is no longer true for pro-p Iwahori Hecke algebras because the finite torus is involved in the presentation of $\mathcal{H}^{(1)}$.

Let L' be a finite extension with inertia degree f and let

$$W^{(f)} = N(L')/T(1 + \pi_{L'}\mathcal{O}_{L'}) = W \ltimes T(\mathbb{F}_{q^f}).$$

Then, $W^{(f)}$ and hence the pro-p Iwahori Hecke algebra only depends on the inertia degree of L'/L. Thus we may speak of pro-p Iwahori Hecke algebras $\mathcal{H}^{(i)}$ for each integer $i \geq 1$. If we want to specify in which algebra an expression is supposed to be read, we write $\tau_w^{(i)}$ instead of τ_w.

One could naively try to use the injective homomorphism $\tau_w^{(1)} \mapsto \tau_w^{(i)}$ of vector spaces induced by the inclusion $W^{(1)} \subseteq W^{(i)}$ to obtain an injection of algebras. However, this is not compatible with the quadratic relation, because the sum in this relation depends on the residue field.

Now, let m be the rank of T and denote by $\hat{T}(\mathbb{F}_{q^i})$ the set of characters from $T(\mathbb{F}_{q^i})$ to $\overline{\mathbb{F}}_q^\times$. For each $\lambda \in \hat{T}(\mathbb{F}_{q^i})$, we can define

$$\epsilon_\lambda^{(i)} := (-1)^m \sum_{t \in T(\mathbb{F}_{q^i})} \lambda^{-1}(t)\tau_t \in \overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})] \subseteq \mathcal{H}^{(i)}.$$

An immediate calculation shows that $\{\epsilon_\lambda^{(i)}\}_{\lambda \in \hat{T}(\mathbb{F}_{q^i})}$ is a system of pairwise orthogonal idempotents with $1 = \sum_{\lambda \in \hat{T}(\mathbb{F}_{q^i})} \epsilon_\lambda$. $W^{(i)}$ operates on $\hat{T}(\mathbb{F}_{q^i})$ from the left by $(w\lambda)(t) := \lambda(w^{-1}tw)$. Note that $\lambda(w^{-1}tw)$ makes sense because $T(\mathbb{F}_{q^i})$ is normal in $W^{(i)}$.

Lemma 4.3.1. *For $\lambda \in \hat{T}(\mathbb{F}_{q^i})$ and $w \in W^{(i)}$, we have $\epsilon_\lambda^{(i)} \tau_w^{(i)} = \tau_w^{(i)} \epsilon_{w^{-1}\lambda}^{(i)}$.*

Proof. It suffices to treat the case $i = 1$. Since the length of each $t \in T(\mathbb{F}_q)$ is zero, we get by the braid relations:

$$\begin{aligned}
\epsilon_\lambda \tau_w &= (-1)^m \Big(\sum_{t \in T(\mathbb{F}_q)} \lambda^{-1}(t) \tau_t \Big) \tau_w = (-1)^m \sum_{t \in T(\mathbb{F}_q)} \lambda^{-1}(t) \tau_{tw} \\
&= (-1)^m \sum_{t \in T(\mathbb{F}_q)} \lambda^{-1}(t) \tau_{ww^{-1}tw} = (-1)^m \sum_{t \in T(\mathbb{F}_q)} \lambda^{-1}(t) \tau_w \tau_{w^{-1}tw} \\
&= \tau_w (-1)^m \sum_{t \in T(\mathbb{F}_q)} \lambda^{-1}(t) \tau_{w^{-1}tw} = \tau_w (-1)^m \sum_{t \in T(\mathbb{F}_q)} \lambda^{-1}(wtw^{-1}) \tau_t \\
&= \tau_w \epsilon_{w^{-1}\lambda}
\end{aligned}$$

□

Denote by $\tilde{S}$ the set of all elements of $s_\alpha \in W_0$, i.e. the set of all conjugates of S_0 in W_0. As W_0 is a factor group of $W^{(1)}$, $W^{(1)}$ acts on $\tilde{S}$ by conjugation. Recall that we have defined a torus T_s for each $s \in \tilde{S}$. For $s \in \tilde{S}$, and two integers i dividing j, we set

$$\begin{aligned}
\epsilon_s^{(i,j)} :&= \sum_{\lambda \in \hat{T}(\mathbb{F}_{q^j}), \lambda(T_s(\mathbb{F}_{q^j}))=1} \epsilon_\lambda^{(j)} + \sum_{\lambda \in \hat{T}(\mathbb{F}_{q^j}), \lambda(T_s(\mathbb{F}_{q^i}))\neq 1} \epsilon_\lambda^{(j)} \\
&= 1 - \sum_{\lambda \in \hat{T}(\mathbb{F}_{q^j}), \lambda(T_s(\mathbb{F}_{q^j}))\neq 1, \lambda(T_s(\mathbb{F}_{q^i}))=1} \epsilon_\lambda^{(j)}.
\end{aligned}$$

Because all summands are pairwise orthogonal idempotents, the $\epsilon_s^{(i,j)}$ are idempotents. These will help us to define an injective algebra homomorphism from $\mathcal{H}^{(i)}$ to $\mathcal{H}^{(j)}$. The plan is to use the naive map but multiplying by $\epsilon_s^{(i,j)}$. "Leaving out" the idempotents in the sum in the second line fixes the quadratic relation but the map will still remain injective because we have not left out too much.

Lemma 4.3.2. *(i) For $s \in \tilde{S}$ and $w \in W^{(j)}$ we have: $\epsilon_s^{(i,j)} \tau_w^{(j)} = \tau_w^{(j)} \epsilon_{w^{-1}sw}^{(i,j)}$.*

(ii) $\epsilon_s^{(i,j)} \tau_{n_s}^{(j)} = \tau_{n_s}^{(j)} \epsilon_s^{(i,j)}$ for all $s \in S_0$ and $\epsilon_{s_{\alpha_0}}^{(i,j)} \tau_{n_{s_0}}^{(j)} = \tau_{n_{s_0}}^{(j)} \epsilon_{s_{\alpha_0}}^{(i,j)}$.

Proof. (i): Using Lemma 4.3.1 we obtain

$$\begin{aligned}
\epsilon_s^{(i,j)}\tau_w^{(j)} &= \left(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_s(\mathbb{F}_{q^j}))=1}\epsilon_\lambda^{(j)} + \sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_s(\mathbb{F}_{q^i}))\neq 1}\epsilon_\lambda^{(j)}\right)\tau_w^{(j)} \\
&= \tau_w^{(j)}\left(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_s(\mathbb{F}_{q^j}))=1}\epsilon_{w^{-1}\lambda}^{(j)} + \sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_s(\mathbb{F}_{q^i}))\neq 1}\epsilon_{w^{-1}\lambda}^{(j)}\right) \\
&= \tau_w^{(j)}\left(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(w^{-1}T_s(\mathbb{F}_{q^j})w)=1}\epsilon_\lambda^{(j)} + \sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(w^{-1}T_s(\mathbb{F}_{q^i})w)\neq 1}\epsilon_\lambda^{(j)}\right) \\
&= \tau_w^{(j)}\left(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_{w^{-1}sw}(\mathbb{F}_{q^j}))=1}\epsilon_\lambda^{(j)} + \sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_{w^{-1}sw}(\mathbb{F}_{q^i})w)\neq 1}\epsilon_\lambda^{(j)}\right) \\
&= \tau_w^{(j)}\epsilon_{w^{-1}sw}^{(j)}
\end{aligned}$$

(ii): This follows from (i) taking $w = n_s$ for for $s \in S$. □

Recall that $W_{aff}^{(i)}$ is the preimage of W_{aff} in $W^{(1)}$ and that $\Omega^{(i)}$ is the preimage of Ω. Let $\mathcal{H}_{aff}^{(i)}$ be the subspace of basis $(\tau_w)_{w\in W_{aff}^{(i)}}$. Using induction on the word length and the presentation of $\mathcal{H}^{(i)}$ one sees that $\mathcal{H}_{aff}^{(i)}$ is a subalgebra of $\mathcal{H}^{(i)}$. We have an isomorphism

$$\mathcal{H}^{(i)} \cong \overline{\mathbb{F}}_q[\Omega^{(i)}]\hat{\otimes}_{\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]}\mathcal{H}_{aff}^{(i)}$$

where $\hat{\otimes}$ denotes the usual tensor product as a module over $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$ and the multiplication is twisted such that

$$\tau_u \otimes \tau_w \cdot \tau_{u'} \otimes \tau_{w'} = \tau_{uu'} \otimes \tau_{u'^{-1}wu'}\tau_{w'}$$

(see Corollary 3 in [Vig05]).

We will now use this decomposition to construct our map from $\mathcal{H}^{(i)}$ to $\mathcal{H}^{(j)}$ by constructing it first on the affine part. So, define $\varphi_{i,j} : \mathcal{H}_{aff}^{(i)} \to \mathcal{H}_{aff}^{(j)}$ as the $\overline{\mathbb{F}}_q$-linear extension of

$$\varphi_{i,j}(\tau_w^{(i)}) = \left(\prod_{(\alpha,k)\in\Phi_{aff}^+\cap w\Phi_{aff}^-}\epsilon_{s_\alpha}^{(i,j)}\right)\tau_w^{(j)},$$

where $W_{aff}^{(1)}$ acts on Φ_{aff} by inflation. We can make this definition more explicit: If $w = tn_{s_1}\dots n_{s_r}$ is reduced we know the positive affine roots mapped to negative affine roots by w^{-1} (see [Bor09][Lemma 2.2.13]) and we obtain

$$\varphi_{i,j}(\tau_w^{(i)}) = \epsilon_{s_1}^{(i,j)}\dots\epsilon_{s_1\dots s_r\dots s_1}^{(i,j)}\tau_w^{(j)}.$$

For example, we have $\varphi_{i,j}(\tau_{n_s}^{(i)}) = \epsilon_s^{(i,j)}\tau_{n_s}^{(j)}$ for any simple reflection s. This is obviously $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$-linear. Our first goal is to show that this map is multiplicative.

Lemma 4.3.3. *(i) Let $T' \subseteq T(\mathbb{F}_{q^j})$ be any subgroup, λ a character of T' and $\epsilon_\lambda = (\#T')^{-1}\sum_{t\in T'}\lambda^{-1}(t)\tau_t^{(j)}$. Further let $\mu \in \hat{T}(\mathbb{F}_{q^j})$. Then we have*

$$\epsilon_\lambda \epsilon_\mu^{(j)} = \begin{cases} \epsilon_\mu^{(j)}, & \text{if } \mu_{|T'} = \lambda, \\ 0, & \text{if } \mu_{|T'} \neq \lambda. \end{cases}$$

(ii) For $s \in \tilde{S}$, we set $\epsilon_{1,s}^{(i)} = \epsilon_1$ as in part (i) for $T' = T_s(\mathbb{F}_{q^{(i)}})$; then

$$\epsilon_{1,s}^{(i)}\epsilon_s^{(i,j)} = \epsilon_{1,s}^{(j)}\epsilon_s^{(i,j)}.$$

Proof. (i): Let $t_1,\dots,t_r$ be a system of representatives for $T(\mathbb{F}_{q^j})/T'$. We get

$$\begin{aligned}
\epsilon_\lambda\epsilon_\mu^{(j)} &= \epsilon_\lambda(-1)^m \sum_{t\in T(\mathbb{F}_{q^j})} \mu^{-1}(t)\tau_t^{(j)} \\
&= \epsilon_\lambda(-1)^m \sum_{t\in T'}\sum_{l=1}^{r} \mu^{-1}(tt_l)\tau_{tt_l}^{(j)} \\
&= \epsilon_\lambda(-1)^m \sum_{t\in T'}\mu^{-1}(t)\tau_t^{(j)}\sum_{l=1}^{r}\mu^{-1}(t_l)\tau_{t_l}^{(j)} \\
&= \epsilon_\lambda\epsilon_{\mu_{|T'}}\sum_{l=1}^{r}\mu^{-1}(t_l)\tau_{t_l}^{(j)}.
\end{aligned}$$

Now the product $\epsilon_\lambda\epsilon_{\mu_{|T'}}$ is either 0 or ϵ_λ. In the first case this yields the result immediately and in the second case we have to read the above calculation backwards without the factor ϵ_λ at the beginning of each line.

(ii): Using part (i) and the definition of $\epsilon_s^{(i,j)}$ we get

$$\epsilon_{1,s}^{(i)}\epsilon_s^{(i,j)} = \sum_{\lambda\in\hat{T}(\mathbb{F}_{q^j}),\lambda(T_s(\mathbb{F}_{q^i}))=1} \epsilon_\lambda^{(j)} = \epsilon_{1,s}^{(j)}\epsilon_s^{(i,j)}.$$

□

Proposition 4.3.4. *$\varphi_{i,j} : \mathcal{H}_{aff}^{(i)} \to \mathcal{H}_{aff}^{(j)}$ is a homomorphism of $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$-algebras.*

Proof. By $\overline{\mathbb{F}}_q[T(\mathbb{F}_q)]$-linearity it suffices to show that

$$\varphi_{i,j}(\tau\tau_w^{(i)}) = \varphi_{i,j}(\tau)\varphi_{i,j}(\tau_w^{(i)})$$

for $\tau \in \mathcal{H}_{aff}^{(i)}$ and $w \in W_{aff}^{(i)}$. By induction on the length of w it suffices to treat the case $w = n_s \in S^{(1)}$, because this case implies for $l(wn_s) = l(w) - 1$

$$\begin{aligned}
\varphi_{i,j}(\tau\tau_w^{(i)}) &= \varphi_{i,j}(\tau\tau_{wn_s^{-1}}^{(i)}\tau_{n_s}^{(i)}) = \varphi_{i,j}(\tau\tau_{wn_s^{-1}}^{(i)})\varphi(\tau_{n_s}^{(i)}) \\
&= \varphi_{i,j}(\tau)\varphi_{i,j}(\tau_{wn_s^{-1}}^{(i)})\varphi_{i,j}(\tau_{n_s}^{(i)}) = \varphi_{i,j}(\tau)\varphi_{i,j}(\tau_{wn_s^{-1}}^{(i)}\tau_{n_s}^{(i)}) \\
&= \varphi_{i,j}(\tau)\varphi_{i,j}(\tau_w^{(i)}).
\end{aligned}$$

Hence, we are reduced to showing the "$w = n_s$"-case and we may assume that $\tau = \tau_v^{(i)}$ for some $v \in W_{aff}^{(1)}$ by linearity.

Let us first assume that $l(vn_s) = l(v) + 1$ and assume that $v = tn_{s_1} \dots n_{s_r}$ is reduced such that $vn_s = tn_{s_1} \dots n_{s_r} n_s$ is reduced. We obtain using Lemma 4.3.2 (i):

$$\begin{aligned}\varphi_{i,j}(\tau_v^{(i)} \tau_{n_s}^{(i)}) &= \varphi_{i,j}(\tau_{vn_s}^{(i)}) = \epsilon_{s_1}^{(i,j)} \dots \epsilon_{s_1 \dots s_r \dots s_1}^{(i,j)} \epsilon_{s_1 \dots s_r s s_r \dots s_1}^{(i,j)} \tau_{vn_s}^{(j)} \\ &= \epsilon_{s_1}^{(i,j)} \dots \epsilon_{s_1 \dots s_r \dots s_1}^{(i,j)} \tau_v^{(j)} \epsilon_s^{(i,j)} \tau_{n_s}^{(j)} = \varphi_{i,j}(\tau_v^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)}).\end{aligned}$$

On the other hand, if $l(vn_s) = l(v) - 1$, we have

$$\varphi_{i,j}(\tau_v^{(i)} \tau_{n_s}^{(i)}) = \varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)} \tau_{n_s}^{(i)2}) = -\varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)} \epsilon_{1,s}^{(i)} \tau_{n_s}^{(i)}) = -\varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)} \tau_{n_s}^{(i)}) \epsilon_{1,s}^{(i)}.$$

By the first case this becomes

$$\varphi_{i,j}(\tau_v^{(i)} \tau_{n_s}^{(i)}) = -\varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)}) \epsilon_{1,s}^{(i)} = -\varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)}) \epsilon_{1,s}^{(i)} \epsilon_s^{(i,j)} \tau_{n_s}^{(j)}$$

With Lemma 4.3.3 (ii) we can conclude that

$$\varphi_{i,j}(\tau_v^{(i)} \tau_{n_s}^{(i)}) = -\varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)}) \epsilon_{1,s}^{(j)} \epsilon_s^{(i,j)} \tau_{n_s}^{(j)} = \varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)}) \epsilon_s^{(i,j)2} \tau_{n_s}^{(j)2} = \varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)})^2$$

and using the first case again we obtain

$$\begin{aligned}\varphi_{i,j}(\tau_v^{(i)} \tau_{n_s}^{(i)}) &= \varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)}) = \varphi_{i,j}(\tau_{vn_s^{-1}}^{(i)} \tau_{n_s}^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)}) \\ &= \varphi_{i,j}(\tau_v^{(i)}) \varphi_{i,j}(\tau_{n_s}^{(i)}).\end{aligned}$$

This finished the proof. □

Our next goal is to see that $\varphi_{i,j}$ is injective. For this we have to examine how much information is lost by multiplying with $\epsilon_s^{(i,j)}$ and compute their product.

Lemma 4.3.5. *(i) Let $I \subseteq \tilde{S}$ be a nonempty subset. For $J \subseteq I$, denote by $M_{I,J}$ the set of all $\lambda \in \hat{T}(\mathbb{F}_{q^j})$ such that $\lambda(T_s(\mathbb{F}_{q^j})) = 1$ for $s \in J$ and $\lambda(T_s(\mathbb{F}_{q^i})) \neq 1$ for $s \in I \setminus J$. Then*

$$\prod_{s \in I} \epsilon_s^{(i,j)} = \sum_{J \subseteq I} \sum_{\lambda \in M_{I,J}} \epsilon_\lambda^{(j)}.$$

(ii) $\epsilon_\lambda^{(i)} \prod_{s \in I} \epsilon_s^{(i,j)} \neq 0$ for any $\lambda \in \hat{T}(\mathbb{F}_{q^i})$.

Proof. (i): We proceed by induction on the cardinality of I. If I consists of a single reflection, the claim is precisely the definition of $\epsilon_s^{(i,j)}$. For the general case fix some $\bar{s} \in I$ and let $I_0 := I \setminus \{\bar{s}\}$. By the induction hypothesis, we get

$$\begin{aligned}\prod_{s \in I} \epsilon_s^{(i,j)} &= (\prod_{s \in I_0} \epsilon_s^{(i,j)}) \epsilon_{\bar{s}}^{(i,j)} = (\sum_{J \subseteq I_0} \sum_{\lambda \in M_{I_0,J}} \epsilon_\lambda^{(j)})(\sum_{J \subseteq \{\bar{s}\}} \sum_{\lambda \in M_{\{\bar{s}\},J}} \epsilon_\lambda^{(j)}) \\ &= \sum_{J \subseteq I_0} (\sum_{\lambda \in M_{I,J \cup \{\bar{s}\}}} \epsilon_\lambda^{(j)} + \sum_{\lambda \in M_{I,J}} \epsilon_\lambda^{(j)}) = \sum_{J \subseteq I} \sum_{\lambda \in M_{I,J}} \epsilon_\lambda^{(j)}.\end{aligned}$$

(ii): Let $J_0 = \{s \in I : \lambda(T_s(\mathbb{F}_{q^i})) = 1\}$. It suffices to show that there exists a $\mu \in \hat{T}(\mathbb{F}_{q^j})$ such that $\mu_{|T(\mathbb{F}_{q^i})} = \lambda$ and $\mu(T_s(\mathbb{F}_{q^j})) = 1$ if $s \in J_0$, because then $\epsilon_\lambda^{(i)} \prod_{s\in I} \epsilon_s^{(i,j)}$ is a sum of orthogonal idempotents which contains $\epsilon_\mu^{(j)}$ precisely once as a summand by Lemma 4.3.3 (i) and part (i) due to the fact that $\mu \in M_{I,J_0}$ and all the $M_{I,J}$ are pairwise disjoint.

Now, after the choice of a basis of $X(T)$, λ is given by m integers well-defined modulo $q^i - 1$. If we choose these integers all in $\{0, \ldots, q^m - 2\}$ and restrict the corresponding algebraic character to $T(\mathbb{F}_{q^j})$ we obtain a lift with the needed properties. □

Proposition 4.3.6. *The $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$-algebra homomorphism $\varphi_{i,j} : \mathcal{H}_{aff}^{(i)} \to \mathcal{H}_{aff}^{(j)}$ is injective.*

Proof. We choose a system of representatives $\dot{W}_{aff} = \{\dot{w}\}_{w\in W_{aff}}$ for $W_{aff}^{(i)}/T(\mathbb{F}_{q^i})$. Then $\mathcal{H}_{aff}^{(i)}$ is a free $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$-module with basis $\{\tau_{\dot{w}}^{(i)}\}_{w\in W_{aff}}$. Now let

$$\begin{aligned} 0 &= \varphi_{i,j}\Big(\sum_{\dot{w}\in\dot{W}_{aff}} \Big(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^i})} c_{w,\lambda}\epsilon_\lambda^{(i)}\Big)\tau_{\dot{w}}^{(j)}\Big) \\ &= \sum_{\dot{w}\in\dot{W}_{aff}} \Big(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^i})} c_{w,\lambda}\epsilon_\lambda^{(i)}\Big)\Big(\prod_{(\alpha,k)\in\Phi_{aff}^+\cap w\Phi_{aff}^-} \epsilon_{s_\alpha}^{(i,j)}\Big)\tau_{\dot{w}}^{(j)}. \end{aligned}$$

Then

$$\Big(\sum_{\lambda\in\hat{T}(\mathbb{F}_{q^i})} c_{w,\lambda}\epsilon_\lambda^{(i)}\Big)\Big(\prod_{(\alpha,k)\in\Phi_{aff}^+\cap\Phi_{aff}^-} \epsilon_{s_\alpha}^{(i,j)}\Big) = 0$$

for each $w \in W_{aff}$ and hence all $c_{w,\lambda} = 0$ by Lemma 4.3.5 (ii). □

Now we can extend $\varphi_{i,j}$ to $\mathcal{H}^{(i)}$ by

$$\begin{aligned} \mathcal{H}^{(i)} = \overline{\mathbb{F}}_q[\Omega^{(i)}] \otimes_{\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]} \mathcal{H}_{aff}^{(i)} &\xrightarrow{id\otimes\varphi_{i,j}} \overline{\mathbb{F}}_q[\Omega^{(i)}] \otimes_{\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]} \mathcal{H}_{aff}^{(j)} \\ &= \overline{\mathbb{F}}_q[\Omega^{(i)}] \otimes_{\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]} \overline{\mathbb{F}}_q[T(\mathbb{F}_{q^j})] \otimes_{\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^j})]} \mathcal{H}_{aff}^{(j)} \\ &= \overline{\mathbb{F}}_q[\Omega^{(j)}] \otimes_{\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^j})]} \mathcal{H}_{aff}^{(j)} = \mathcal{H}^{(j)}. \end{aligned}$$

We will also denote this homomorphism by $\varphi_{i,j}$.

Theorem 4.3.7. *$\varphi_{i,j} : \mathcal{H}^{(i)} \to \mathcal{H}^{(j)}$ an injective homomorphism of $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$-algebras.*

Proof. The injectivity follows from the injectivity of $\varphi_{i,j} : \mathcal{H}_{aff}^{(i)} \to \mathcal{H}_{aff}^{(i)}$ and the fact that $\overline{\mathbb{F}}_q[\Omega^{(i)}]$ is free and hence flat over $\overline{\mathbb{F}}_q[T(\mathbb{F}_{q^i})]$. For $u \in \Omega^{(i)}$ and $w \in W_{aff}^{(i)}$ we have

$$\tau_u^{(j)}\varphi_{i,j}(\tau_{u^{-1}wu}^{(i)}) = \varphi_{i,j}(\tau_w^{(i)})\tau_u^{(j)}$$

by Lemma 4.3.2 (ii) and the definition of $\varphi_{i,j}$. This shows that $\varphi_{i,j}$ is indeed multiplicative. □

Remark 4.3.8. An analogous construction yields injective algebra homomorphisms for the case of finite Hecke algebras. It is even simpler in this case because the intermediate step using the affine Hecke algebra is not needed because of the absence of the group $\Omega^{(i)}$ in this setting.

4.4 Supersingular $\mathcal{H}^{(1)}$-Modules

For any commutative ring R with unit, let $\mathcal{H}_R^{(1)}$ be the R-algebra $\mathcal{H}_{\mathbb{Z}}^{(1)} \otimes_{\mathbb{Z}} R$. We take $R = \mathbb{Z}[q^{-1/2}]$. As R is torsion free and hence flat over $\mathbb{Z}$, $\mathcal{H}_{\mathbb{Z}}^{(1)} \subseteq \mathcal{H}_R^{(1)}$ is a subalgebra. According to [Vig05, Prop. 4], there exists a unique ring morphism $\theta = \theta^- : R[Y(T)^{(1)}] \to \mathcal{H}_R^{(1)}$ such that $\theta(y) = \tau_y$ whenever $y \in Y(T)^{(1)}$ is antidominant, by which we mean that its image in $Y(T)$ is antidominant. If $w = e^y v \in W^{(1)}$ with $v \in W_0^{(1)}$ and $y \in Y(T)^{(1)}$, we set

$$E_w = q^{\frac{l(w)-l(e^y)-l(v)}{2}}\theta(y)\tau_v.$$

It can be shown (cf. [Vig05, Prop. 7]) that the elements E_w lie in $\mathcal{H}_{\mathbb{Z}}^{(1)}$. We will write shortly E_y for E_{e^y} when $y \in Y(T)^{(1)}$.

Theorem 4.4.1. *(i) $(E_w)_{w\in W^{(1)}}$ is a $\mathbb{Z}$-basis of $\mathcal{H}_{\mathbb{Z}}^{(1)}$. It is called the Bernstein-basis.*

(ii) $(E_y)_{y\in Y(T)^{(1)}}$ is a $\mathbb{Z}$-basis of $\mathcal{A}_{\mathbb{Z}}^{(1)} := \mathrm{im}(\theta) \cap \mathcal{H}_{\mathbb{Z}}^{(1)}$. $\mathcal{A}_{\mathbb{Z}}^{(1)}$ is presented by the relations

$$E_{y_1}E_{y_2} = q^{\frac{l(e^{y_1})+l(e^{y_2})-l(e^{y_1+y_2})}{2}}E_{y_1+y_2} \text{ for } y_1, y_2 \in Y(T)^{(1)}.$$

Proof. See [Vig05, Thm. 2, Lemma 3]. □

The Bernstein basis allows us to define an operation of W_0 on $\mathcal{A}_{\mathbb{Z}}^{(1)}$ by

$$wE_y := E_{w(y)}.$$

Using this we can describe the center of $\mathcal{H}_{\mathbb{Z}}^{(1)}$.

Theorem 4.4.2. *(i) The center $\mathcal{Z}_{\mathbb{Z}}^{(1)}$ of $\mathcal{A}_{\mathbb{Z}}^{(1)}$ is equal to $(\mathcal{A}_{\mathbb{Z}}^{(1)})^{W_0}$. It has the $\mathbb{Z}$-basis $z_{\{y\}} := \sum_{x\in W_0 y} E_x$, where y runs through the W_0-orbits in $Y(T)^{(1)}$. $\mathcal{A}_{\mathbb{Z}}^{(1)}$ and $\mathcal{H}_{\mathbb{Z}}^{(1)}$ are finitely generated $\mathcal{Z}_{\mathbb{Z}}^{(1)}$-modules.*

(ii) The center of $\mathcal{H}^{(1)}$ is $\mathcal{Z}^{(1)} = \mathcal{Z}_{\mathbb{Z}}^{(1)} \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q$.

Proof. See [Vig14, Thm. 1.2, 1.3]. □

By an $\mathcal{H}^{(1)}$ (or $\mathcal{H}_{aff}^{(1)}$)-module we will always denote a right module. This convention is due to the fact that the functor of $I^{(1)}$-invariants naturally yields $\mathcal{H}^{(1)}$-right modules. Now, let M be any $\mathcal{H}^{(1)}$-module with a central character. This will be the case in particular, if M is simple and finite dimensional (the last assumption being a consequence of the first). As $\mathcal{Z}^{(1)}$ is the center of $\mathcal{H}^{(1)}$, $\mathcal{Z}^{(1)}$ acts by a character $\omega = \omega_M : \mathcal{Z}^{(1)} \to \overline{\mathbb{F}}_q$ on M which we will call the central character of M.

Definition 4.4.3. (i) A character $\omega : \mathcal{Z}^{(1)} \to \overline{\mathbb{F}}_q$ is called supersingular if $\omega(z_{\{y\}}) = 0$ for each $y \in Y(T)^{(1)}$ with $l(y) > 0$.

(ii) A nonzero $\mathcal{H}^{(1)}$-module M with a central character ω_M is called supersingular if ω_M is supersingular.

The notion of supersingularity can also be extended to modules without a central character (see [Oll12][5.10]). However, this is technically more complicated and we shall not need the definition in this generality because we will only be interested in simple supersingular modules. There is a nice description of simple supersingular modules involving characters of $\mathcal{H}^{(1)}_{aff}$. Let us first give a classification of characters of the affine Hecke algebra. For this, we consider the set of all tuples (λ, I) such that $\lambda : T(\mathbb{F}_q) \to \overline{\mathbb{F}}_q^\times$ is a character and $I \subseteq S$.

Proposition 4.4.4. *The $\overline{\mathbb{F}}_q$-characters of $\mathcal{H}^{(1)}_{aff}$ are all defined over $\mathbb{F}_q$. They are parametrized by pairs (λ, I) such that $I \subseteq S^\lambda := \{s \in S : \lambda(T_s(\mathbb{F}_q)) = 1\}$. The character $\chi = \chi_{(\lambda,I)}$ associated with (λ, I) is given by:*

- $\chi(\tau_t) = \lambda^{-1}(t)$ *for all* $t \in T(\mathbb{F}_q)$.
- $\chi(\tau_{n_s}) = 0$ *if* $s \in S \setminus I$.
- $\chi(\tau_{n_s}) = -1$ *if* $s \in I$.

Proof. See [Vig05, Prop. 2]. □

Note that this classification induces an obvious map from characters of $\mathcal{H}^{(1)}_0$ to characters of $\mathcal{H}^{(1)}_{aff}$.

There are two distinguished characters of $\mathcal{H}^{(1)}_{aff}$, namely $\chi_{triv} = \chi_{(1,\emptyset)}$ and $\chi_{sign} = \chi_{(1,S)}$, where 1 denotes the trivial character of $T(\mathbb{F}_q)$. Now suppose, λ_0 is a character of $T(\mathbb{F}_q)$ trivial on T_s for each $s \in S$. Then, we can define the twist of $\chi_{(\lambda,I)}$ by λ_0 as $\chi_{(\lambda_0\lambda,I)}$. Note that we still have $I \subseteq S^{\lambda_0\lambda}$ because of the condition imposed on λ_0. We will say that χ is a twist of χ', if there exists some λ_0 as above such that χ' is the twist of χ by λ_0. This yields a classification of supersingular simple modules:

Theorem 4.4.5. *Suppose that the root system of G is irreducible. A simple $\mathcal{H}^{(1)}$-module is supersingular if and only if it contains a character of $\mathcal{H}^{(1)}_{aff}$ that is not a twist of χ_{triv} or χ_{sign}.*

Proof. This is [Oll12] Thm. 5.14. □

We will now give an operation of Ω on the tuples (λ, I). For $u \in \Omega$, we set ${}^uI := uIu^{-1}$ which is at least contained in S, as Ω normalizes S. Additionally, we let Ω act on the characters of $T(\mathbb{F}_q)$ by the projection to W_0, i.e. if $u = we^y$, ${}^u\lambda(t) = \lambda(w^{-1}tw)$. So, we can define ${}^u(\lambda, I) = ({}^u\lambda, {}^u I)$.

Lemma 4.4.6. *(i) The Ω-action restricts to an action on those tuples (λ, I) such that $I \subseteq S^\lambda := \{s \in S : \lambda(T_s(\mathbb{F}_q)) = 1\}$.*

(ii) $\Omega \cap Y(T)$ acts trivially on the tuples (λ, I) such that $I \subseteq S^\lambda$.

Proof. (i): Let $u = we^y \in \Omega$, $s \in S$ and $\alpha \in \Phi$ such that $h_s = \check{\alpha}$. We need to show that ${}^u\lambda(T_{{}^us}(\mathbb{F}_q)) = 1$. We have

$$T_{{}^us} = T_{ws_\alpha w^{-1}e^{y'}} = T_{ws_\alpha w^{-1}} = T_{s_{w(\alpha)}} = \mathrm{im}((w\alpha)\check{}) = {}^w\mathrm{im}(\check{\alpha}) = {}^wT_s,$$

where $y' = 0$ if ${}^us \in S_0$ and $y' = \alpha_0$ if ${}^us = s_0$. This implies

$$T_{{}^us}(\mathbb{F}_q) = {}^wT_s(\mathbb{F}_q),$$

and hence

$${}^u\lambda(T_{{}^us}(\mathbb{F}_q)) = \lambda({}^{w^{-1}w}T_s(\mathbb{F}_q)) = \lambda(T_s(\mathbb{F}_q)) = 1.$$

(ii): By definition, $\Omega \cap Y(T)$ acts trivially on all λ. Hence, it suffices to show that $\Omega \cap Y(T)$ is central in W. For this it suffices that any $y \in \Omega \cap Y(T)$ commutes with any $s \in S_0$, as $Y(T)$ is abelian. We can compute the length of y by

$$0 = l(e^y) = |\sum_{\alpha \in R^+} (\alpha, y)|$$

(cf. [Bor09, Satz 2.3.4]). This yields $(\alpha, y) = 0$ for all $\alpha \in \Delta$ and hence $s_\alpha(y) = 0$, which implies $e^y s_\alpha = s_\alpha e^y$. □

Part (ii) of the previous lemma shows that the image of Ω under the projection $W \twoheadrightarrow W_0$ acts on the tuples (λ, I) such that $I \subseteq S^\lambda$. We will denote this image by Ω_0. This is a finite abelian group. Denote its order by n. Note that this will not cause any confusion when $G = \mathrm{GL}_n$, as Ω_0 is generated by an n-cycle in this case.

Lemma 4.4.7. *(i) $(Y(T)^{(1)})^{W_0}$ is contained in the center of $W^{(1)}$.*

(ii) $(Y(T)^{(1)})^{W_0} \subseteq \Omega^{(1)}$.

(iii) $\Omega^{(1)} \cap Y(T)^{(1)} = \langle T(\mathbb{F}_q), (Y(T)^{(1)})^{W_0}\rangle$ and $(Y(T)^{(1)})^{W_0} \cap T(\mathbb{F}_q) = T(\mathbb{F}_q)^{W_0}$.

Proof. (i): This follows immediately from the facts that $Y(T)^{(1)}$ is commutative as a quotient of $T(L)$, and that $W^{(1)}$ is generated by $W_0^{(1)}$ and $Y(T)^{(1)}$.

(ii): Let $y \in (Y(T)^{(1)})^{W_0}$ with image $y_0 \in Y(T)^{W_0}$. Then $l(y_0) = 0$, which implies $y_0 \in \Omega$ and hence $y \in \Omega^{(1)}$.

(iii): The inclusion from the right to the left is clear by (ii). Any element $\Omega^{(1)} \cap Y(T)^{(1)}$ has its image in $\Omega \cap Y(T) = Y(T)^{W_0}$ under the natural projection by definition. Hence it is congruent to some element of $(Y(T)^{(1)})^{W_0}$ modulo $T(\mathbb{F}_q)$. We have the second equality as

$$(Y(T)^{(1)})^{W_0} \cap T(\mathbb{F}_q) = (Y(T)^{(1)} \cap T(\mathbb{F}_q))^{W_0} = T(\mathbb{F}_q)^{W_0}.$$

□

Proposition 4.4.8. *There exists a canonical bijection between*

- *the Ω_0-orbits of cardinality n of triples (λ, I, ω) such that $I \subseteq S^\lambda$ and $\omega : (Y(T)^{(1)})^{W_0} \to \overline{\mathbb{F}}_q^\times$ is a character which coincides with λ^{-1} on $T(\mathbb{F}_q)^{W_0}$ and*

- *the isomorphism classes of simple $\mathcal{H}^{(1)}$-modules of dimension n containing a character of $\mathcal{H}^{(1)}_{aff}$.*

Such a module M associated with (λ, I, ω) satisfies

$$M \cong \bigoplus_{(\mu,J)\in\Omega_0(\lambda,I)} \chi_{(\mu,J)}$$

as an $\mathcal{H}^{(1)}_{aff}$-module. If M is supersingular, and given by a triple (λ, I, ω), its central character $\chi_{\lambda,\omega}$ is determined by

$$\chi_{\lambda,\omega}(z_{\{y\}}) = \begin{cases} \omega(u)\sum_{t'\in W_0 t}\lambda^{-1}(t'), & \text{if } y = tu \text{ with } t \in T(\mathbb{F}_q), \\ & u \in (Y(T)^{(1)})^{W_0},\ i.e.\ l(y) = 0 \\ 0, & \text{if } l(y) > 0. \end{cases}$$

Proof. This is [Vig05, Prop. 3]. As this proposition is of central importance for us, we will give a detailed proof here. We first construct a map from the first to the second set. So let (λ, I, ω) be a representative of such an orbit and $\chi = \chi_{(\lambda,I)}$. Let $\tilde{\mathcal{H}}^{(1)}$ be the submodule of $\mathcal{H}^{(1)}$ generated by the elements τ_{uw} with $u \in (Y(T)^{(1)})^{W_0}$ (viewed as an element of $W^{(1)}$) and $w \in W^{(1)}_{aff}$. This is indeed a subalgebra of $\mathcal{H}^{(1)}$, as $(Y(T)^{(1)})^{W_0} \subseteq \Omega^{(1)}$ normalizes $W^{(1)}_{aff}$. Since $(Y(T)^{(1)})^{W_0}$ is contained in the center of $W^{(1)}$, there exists a unique extension of χ and ω to a character of $\tilde{\mathcal{H}}^{(1)}$, which we will denote by χ_ω.

Now we set

$$M(\chi_\omega) := \chi_\omega \otimes_{\tilde{\mathcal{H}}^{(1)}} \mathcal{H}^{(1)}.$$

As $v := 1 \otimes 1$ is an eigenvector for the character χ of $\mathcal{H}^{(1)}_{aff}$, this module contains a character of $\mathcal{H}^{(1)}_{aff}$. If $u_1 \in \Omega^{(1)}$ is a lift of $u \in \Omega$, $v\tau_{u_1}$ is an eigenvector for $\chi_{u^{-1}(\lambda,I)}$. As these elements are a basis of $M(\chi_\omega)$ as an $\overline{\mathbb{F}}_q$-vector space, and the number of those characters is precisely n by assumption, we have $\dim M(\chi_\omega) = n$.

So, it remains to show that $M(\chi_\omega)$ is simple. Let $M' \subseteq M(\chi_\omega)$ be a nonzero submodule and $v = \sum_{i=1}^r v_i a_i \in M'$, where $r \geq 1$, all v_i are characters of $\mathcal{H}^{(1)}_{aff}$ associated with elements of the Ω_0-orbit of (λ, I) and all $a_i \neq 0$. If $r = 1$, we are done, because this implies that M' contains all the Ω_0-conjugates of χ. This shows in particular, that $M(\chi_\omega)$ does not depend on our initial choice of a representative (λ, I). Thus, we may assume $r \geq 2$. As the v_i are distinct, we can find an $f \in \mathcal{H}^{(1)}_{aff}$ such that $v_r(f) \neq v_{r-1}(f)$ and assume $v_r(f) \neq 0$ without loss of generality. This implies

$$0 \neq v - vf\frac{1}{v_r(f)} \in M' \cap \sum_{i=1}^{r-1} v_i\overline{\mathbb{F}}_q.$$

Induction leads to the case $r = 1$.

Conversely, let M be a module as in the statement of the proposition and $\chi = \chi_{(\lambda,I)}$ a character contained in M. As above, M contains all the $\chi_{u(\lambda,I)}$ for $u \in \Omega_0$ and the vector space generated by those characters is a submodule. Hence, its dimension, which coincides with the length of the Ω_0-orbit of (λ, I), must be n

and as there is only one orbit, this does not depend on the choice of the character χ. As M is simple, it contains a central character and by restriction, we obtain $\omega : (Y(T)^{(1)})^{W_0} \to \overline{\mathbb{F}}_q^\times$.

Now, we check that these constructions are inverse to each other. Obviously, the first one followed by the second one yields the identity. So let us start with a simple n-dimensional module M containing $\chi = \chi_{(\lambda,I)}$ on which $(Y(T)^{(1)})^{W_0}$ acts by ω. As M and $M(\chi_\omega)$ are simple, it suffices to show, that there exists a nontrivial homomorphism between them. For this we observe that

$$\begin{aligned} 0 \neq \mathrm{Hom}_{\tilde{\mathcal{H}}^{(1)}}(\chi_\omega, M) &= \mathrm{Hom}_{\tilde{\mathcal{H}}^{(1)}}(\chi_\omega, \mathrm{Hom}_{\mathcal{H}^{(1)}}(\mathcal{H}^{(1)}, M)) \\ &= \mathrm{Hom}_{\mathcal{H}^{(1)}}(\chi_\omega \otimes_{\tilde{\mathcal{H}}^{(1)}} \mathcal{H}^{(1)}, M) = \mathrm{Hom}_{\mathcal{H}^{(1)}}(M(\chi_\omega), M). \end{aligned}$$

The description of the central characters is obvious. □

Note that if $n > 1$ in the previous proposition, the supersingularity of M is automatically given by Theorem 4.4.5 because sign- and trivial characters have Ω_0-orbits of length 1.

4.5 Supersingular Modules for GL_n

We will now specialize to the case, where $G = \mathrm{GL}_n$ with $n \geq 2$. As there is a canonical splitting of $W \to W^{(1)}$, we may view $Y(T)$ and $(Y(T))^{W_0}$ as subgroups of $Y(T)^{(1)}$.

Fix $\lambda : T(\mathbb{F}_q) \to \overline{\mathbb{F}}_q^\times$ and $\omega : (Y(T))^{W_0} \to \overline{\mathbb{F}}_q^\times$. Note that, by the choice of π_L, fixing ω is the same as fixing $z = \omega(diag(\pi_L, \ldots, \pi_l)) \in \overline{\mathbb{F}}_q^\times$. We have a homomorphism of $\overline{\mathbb{F}}_q$-vector spaces $\chi_{\lambda,\omega} : \mathcal{Z}^{(1)} \to \overline{\mathbb{F}}_q$ given by

$$\chi_{\lambda,\omega}(z_{\{y\}}) = \begin{cases} \omega(u) \sum_{t' \in W_0 t} \lambda^{-1}(t'), & \text{if } y = tu \text{ with } t \in T(\mathbb{F}_q), \\ & u \in (Y(T))^{W_0}, \text{ i.e. } l(y) = 0 \\ 0, & \text{if } l(y) > 0. \end{cases}$$

Note that $\chi_{\lambda,\omega} = \chi_{{}^w\lambda,\omega}$ for all $w \in W_0$. We know that $\chi_{\lambda,\omega}$ is a character if there is an $I \subset S^\lambda$ such that $\#\Omega_0(\lambda, I) = n$ by Proposition 4.4.8. This will not be true in general. However, we can always achieve this by replacing λ by another representative of its W_0-orbit, which leads to the same $\chi_{\lambda,\omega}$:

First consider the case that ${}^w\lambda \neq \lambda$ for all $w \in W_0 \setminus \{1\}$. Then the elements ${}^u\lambda$ for $u \in \Omega_0$ are all distinct and we may take $I = \emptyset$. Now assume, that we can find $1 \neq w \in W_0$ such that ${}^w\lambda = \lambda$. This means that we can find $1 \leq i < j \leq n$ such that (i, j) acts trivially on λ. By conjugation, we can assume $i = 1$ and $j = 2$. Then, $(1, 2) \in S^\lambda$ and we can take $I = \{(1, 2)\}$ which gives the desired result as $\#\Omega_0 I = n$. As the map $\chi_{\lambda,\omega}$ does not depend on the representative $\lambda \in W_0\lambda$, this shows that they are all supersingular characters of $\mathcal{Z}^{(1)}$.

Now we want to answer the question when two such supersingular characters coincide. First we need a technical lemma:

Lemma 4.5.1. *Let R be any integral domain and $a_1, \ldots, a_n \in R$. For any $1 \leq i \leq n$, the multiplicity of a_i as a root of*

$$\sum_{i=0}^{n}(-1)^i X^i \sum_{j_1<\ldots<j_{n-i}} \prod_{k=1}^{n-i} a_{j_k}$$

is precisely $\#\{j \in \{1 \ldots, n\} : a_i = a_j\}$.

Proof. We have to show

$$\sum_{i=0}^{n}(-1)^i X^i \sum_{j_1<\ldots<j_{n-i}} \prod_{k=1}^{n-i} a_{j_k} = \prod_{i=1}^{n}(a_i - X).$$

We proceed by induction on n. For $n = 1$, the claim is trivial and by induction, it suffices to remark that

$$\begin{aligned}
&(\sum_{i=0}^{n-1}(-1)^i X^i \sum_{j_1<\ldots<j_{n-1-i}} \prod_{k=1}^{n-1-i} a_{j_k})(a_n - X) \\
&= \sum_{i=0}^{n-1}(-1)^i X^i \sum_{j_1<\ldots<j_{n-i}=n} \prod_{k=1}^{n-i} a_{j_k}) \\
&- \sum_{i=0}^{n-1}(-1)^i X^{i+1} \sum_{j_1<\ldots<j_{n-1-i}} \prod_{k=1}^{n-1-i} a_{j_k} \\
&= \sum_{i=0}^{n-1}(-1)^i X^i \sum_{j_1<\ldots<j_{n-i}=n} \prod_{k=1}^{n-i} a_{j_k}) \\
&+ \sum_{i=1}^{n}(-1)^i X^i \sum_{j_1<\ldots<j_{n-i}\neq n} \prod_{k=1}^{n-i} a_{j_k} \\
&= \sum_{i=0}^{n}(-1)^i X^i \sum_{j_1<\ldots<j_{n-i}} \prod_{j=1}^{n-i} a_j
\end{aligned}$$

□

Proposition 4.5.2. *Every supersingular character of $\mathcal{Z}^{(1)}$ is of the the form $\chi_{\lambda,\omega}$ for some $\lambda : T(\mathbb{F}_q) \to \overline{\mathbb{F}}_q^\times$ and $\omega : (Y(T))^{W_0} \to \overline{\mathbb{F}}_q^\times$. We have $\chi_{\lambda_1,\omega_1} = \chi_{\lambda_2,\omega_2}$ if and only if $\omega_1 = \omega_2$ and $W_0\lambda_1 = W_0\lambda_2$.*

Proof. As $\mathcal{H}^{(1)}$ is finitely generated as a $\mathcal{Z}^{(1)}$-module every supersingular character χ of $\mathcal{Z}^{(1)}$ is the central character of some supersingular simple finite dimensional module, a quotient of $\chi \otimes_{\mathcal{Z}^{(1)}} \mathcal{H}^{(1)}$. But those are of the form $\chi_{\lambda,\omega}$ by Proposition 4.4.8.

Additionally, we already know the "if"-statement. As $(Y(T))^{W_0}$ is a subgroup of ${\mathcal{Z}^{(1)}}^\times$, we can recover ω from χ. So it suffices to show that $W_0\lambda$ can be recovered from χ. Let α be a generator of $\mathbb{F}_q^\times$. Let t_i be the diagonal matrix with (i, i)-entry α

and all other diagonal entries equal to 1. If we let $\alpha^{m_i} = \lambda^{-1}(t_i)$, $W_0\lambda$ is determined by the numbers m_i modulo $q-1$ with their multiplicities. Denoting by $M_{(i)}$ the diagonal matrix with the first i entries equal to α and the other ones equal to 1, we get these from χ by considering the polynomial

$$f(X) = \sum (-1)^i X^i \chi(z_{\{M_{(n-i)}\}}) = \sum (-1)^i X^i \sum_{1 \leq j_1 < \ldots < j_{n-i} \leq n} \prod_{k=1}^{n-i} \alpha^{m_{j_k}}$$

and using the previous lemma. □

With the help of $\chi_{\lambda,\omega}$, we can define the algebra

$$\mathcal{H}^{(1)}_{\lambda,\omega} := \mathcal{H}^{(1)} \otimes_{\mathcal{Z}^{(1)}} \chi_{\lambda,\omega}.$$

This is a finite dimensional $\overline{\mathbb{F}}_q$-algebra by Theorem 4.4.2. There is an obvious dimension-preserving bijection between (simple) modules with central character $\chi_{\lambda,\omega}$ over $\mathcal{H}^{(1)}$ and (simple) modules over $\mathcal{H}^{(1)}_{\lambda,\omega}$. Thus, studying n-dimensional simple supersingular $\mathcal{H}^{(1)}$-modules is equivalent to studying n-dimensional simple $\mathcal{H}^{(1)}_{\lambda,\omega}$-modules for all (λ, ω).

We will now examine how simple supersingular modules behave under our inclusion construction of section 4.3. Let $\varphi = \varphi_{(1,i)} : \mathcal{H}^{(1)} \to \mathcal{H}^{(i)}$ be that inclusion for some $i \geq 1$. For a simple supersingular $\mathcal{H}^{(i)}$-module M of dimension n we can ask the question if M is also simple and supersingular considered as a module over $\mathcal{H}^{(1)}$. Let $\chi_{(\mu,I)}$ be a character of $\mathcal{H}^{(i)}_{aff}$ contained in M and let λ be the restriction of μ to $T(\mathbb{F}_q)$. Clearly, $I \subseteq S^\lambda$. However it is not necessary that the Ω_0-orbit of (λ, I) consists of n elements. For example, if $i > 1$ and $\mu(diag(t_1, \ldots, t_n)) = t_1^{q-1}$, and $I = \emptyset$, we have $\#\Omega_0(\mu, I) = n$ and $\#\Omega_0(\lambda, I) = 1$ if $i > 1$. As it turns out this condition already decides our question:

Proposition 4.5.3. *With the notations above, the following are equivalent:*

(i) M is simple over $\mathcal{H}^{(1)}$

(ii) M is simple and supersingular over $\mathcal{H}^{(1)}$

(iii) $\#\Omega_0(\lambda, I) = n$

Proof. Assume (i) and let $v \in M$ be an eigenvector for $\psi_{(\mu,I)}$ with some $\mu \in \hat{T}(\mathbb{F}_{q^i})$ and $I \subseteq S^\mu$. We will show that v is an eigenvector for $\chi_{(\lambda,I)}$ which proves (ii). As $\phi(\tau_t^{(i)}) = \tau_t^{(i)}$ for $t \in T(\mathbb{F}_q)$, we have $v\tau_t^{(i)} = v\lambda^{-1}(t)$. If $s \notin I$, we have

$$v\varphi(\tau_{n_s}^{(1)}) = v\epsilon_s^{(1,i)}\tau_{n_s}^{(i)} = v\tau_{n_s}^{(i)}\epsilon_s^{(1,i)} = 0.$$

On the other hand, if $s \in I$ we need to show that $v\epsilon_s^{(1,i)} = v$. Recall that

$$\epsilon_s^{(1,i)} := \sum_{\sigma \in \hat{T}(\mathbb{F}_{q^i}), \sigma(T_s(\mathbb{F}_{q^i}))=1} \epsilon_\sigma^{(i)} + \sum_{\sigma \in \hat{T}(\mathbb{F}_{q^i}), \sigma(T_s(\mathbb{F}_q)) \neq 1} \epsilon_\sigma^{(i)}$$

For any $\sigma \in \hat{T}(\mathbb{F}_{q^i})$, we have

$$\chi_{(\mu,I)}(\epsilon_\sigma^{(i)}) = (-1)^n \chi_{(\mu,J)}(\sum_{t \in T(\mathbb{F}_{q^i})} \sigma^{-1}(t)\tau_t) = (-1)^n \sum_{t \in T(\mathbb{F}_{q^i})} \mu^{-1}(t)\sigma^{-1}(t) = \delta_{\mu^{-1},\sigma}.$$

As $s \in I$, $\mu(T_s(\mathbb{F}_{q^i})) = 1$ and hence $\chi_{(\mu,I)}(\epsilon_s^{(1,i)}) = 1$, which yields $v\epsilon_s^{(1,i)} = v$.

Let us now assume (ii). As before, M contains the character $\psi_{(\lambda,I)}$ as an $\mathcal{H}_{aff}^{(1)}$-submodule. By the description of Proposition 4.4.8, we have

$$M \cong \bigoplus_{(\sigma,J)\in\Omega_0(\lambda,I)} \chi_{(\sigma,J)}$$

and we get $\#\Omega_0(\lambda, I) = \dim(M) = n$.

Finally, (i) follows from (iii) as in the proof of Proposition 4.4.8 where it is shown that $M(\chi_\omega)$ is simple.

□

4.6 Computations on Große-Klönne's Functor

For this section, assume that $G = \mathrm{GL}_n$ and $L = \mathbb{Q}_p$. In [GK13], Große-Klönne has constructed two contravariant functors – each corresponding to one of the extremal simple roots in the Dynkin diagram of GL_n and a generator of the group Ω – from the category of $\mathcal{H}^{(1)}$-modules of finite length to étale (φ, Γ)-modules over Fontaine's ring $\mathcal{O}_\mathcal{E}$ which induces a bijection between simple supersingular $\mathcal{H}^{(1)}$-modules of dimension n and irreducible n-dimensional Galois representations. In this section we will make this bijection explicit so that we can compare our results with it later.

The proofs of Theorem 8.5 and 8.7 in [GK13] give an explicit description how to compute the tuple (λ, I) from a given Galois representation. We start with a Galois representation $\rho = \mathrm{ind}(\omega_n^{h_0}) \otimes \mu_\beta$ as in Proposition 2.1.9 with $0 \le h_0 \le p^n - 1$ primitive and $\beta \in \overline{\mathbb{F}}_q^\times$. The corresponding simple supersingular $\mathcal{H}^{(1)}$-modules are defined by triples (λ, I, ω) as in Proposition 4.4.8. Denote those tuples afforded by the choice of the root α_1 resp. α_{n-1} by $(\lambda_+(\rho), I_+(\rho), \omega_+(\rho))$ resp. $(\lambda_-(\rho), I_-(\rho), \omega_-(\rho))$. In the following, we shall restrict to carrying out the calculations for the bijection given by the choice of α_1. The other one is calculated completely analogously.

We can write the p-adic expansion of h_0

$$h_0 = a_1 + a_2 p + \ldots + a_n p^{n-1}.$$

As $\mathrm{ind}(\omega_n^{h_0})$ does not change when we multiply h_0 by p by Proposition 2.1.9 (i), we may permute the coefficients a_i cyclically and thus assume that a_n is minimal among the a_i. We get

$$h_0 = a_n \frac{p^n - 1}{p - 1} + \sum_{j=1}^{n-1} (a_j - a_n) p^{j-1},$$

where all coefficients are between 0 and $p-1$ because of the minimality of a_n. Let $h := \sum_{j=1}^{n-1}(a_j - a_n)p^{j-1}$. Because of $\omega_n^{\frac{p^n-1}{p-1}} = \omega_1$ we get

$$\operatorname{ind}(\omega_n^{h_0}) \otimes \mu_\beta = \operatorname{ind}(\omega_n^{h+a_n\frac{p^n-1}{p-1}}) \otimes \mu_\beta = \operatorname{ind}(\omega_n^h) \otimes \omega_1^{a_n}\mu_\beta.$$

The associated supersingular $\mathcal{H}^{(1)}$-module is described as follows: Let

$$h(p-1) = i_0 + i_1 p \ldots + i_{n-1}p^{n-1}$$

with $0 \leq i_j \leq p-1$ and $k_j = p-1-i_{n-j}$ for $1 \leq j \leq n$. Then λ_+ is the unique character of $T(\mathbb{F}_q)$ such that $\lambda_+(h_{s_{i-1}}(x)) = x^{k_i}$ and $\lambda_+(diag(1,\ldots,1,x)) = x$ for $x \in \overline{\mathbb{F}}_q^\times$. Here the s_i are given as in section 1.4. Further set $I_+ := \{s_i \in S : k_{i+1} = p-1\}$ and let $b = (-1)^{n-1}\lambda(diag(-1,\ldots,-1))\prod_{i=1}^{n-1} k_i!\beta^n$. Let ω_+ be character of $Y(T)^{(1)}$ which maps $diag(\pi_L,\ldots,\pi_L)$ to b and extends the restriction of λ_+^{-1} to $T(\mathbb{F}_q)^{W_0}$. Then the associated $\mathcal{H}^{(1)}$-module is given by the tuple $(\lambda_+(\rho), I_+(\rho), \omega_+(\rho) = (\lambda_+, I_+, \omega_+)$.

Now we will give explicit formulas for λ and I involving the a_i. We have

$$h(p-1) = \sum_{j=1}^{n-1}(a_j - a_n)p^j - \sum_{j=1}^{n-1}(a_j - a_n)p^{j-1} = \sum_{j=0}^{n-1}(a_j - a_{j+1})p^j$$

where we set $a_0 := a_n$. Let us define numbers $\delta_0, \ldots, \delta_{n-1}$ inductively by $\delta_0 = 0$ and

$$\delta_{j+1} := \begin{cases} 1, & \text{if } a_j - a_{j+1} - \delta_j < 0 \\ 0, & \text{if } a_j - a_{j+1} - \delta_j \geq 0 \end{cases}$$

and set $\delta_n = 0$. By construction,

$$h(p-1) = \sum_{j=0}^{n-1}(p\delta_{j+1} + a_j - a_{j+1} - \delta_j)p^j$$

is the p-adic expansion of $h(p-1)$ and hence we get $i_j = p\delta_{j+1} + a_j - a_{j+1} - \delta_j$ and

$$k_j = p - 1 - i_{n-j} = p(1 - \delta_{n-j-1}) - (1 - \delta_{n-j})a_{n-j} + a_{n-j-1}.$$

This way we can compute the character

$$\begin{aligned} \lambda_+(diag(t_1,\ldots,t_n)) &= \lambda(h_{s_1}(t_1)h_{s_2}(t_1t_2)\ldots h_{s_{n-1}}(t_1\ldots t_{n-1})diag(1,\ldots,1,t_1\ldots t_n)) \\ &= t_1^{k_2}(t_1t_2)^{k_3}\ldots(t_1\ldots t_{n-1})^{k_n}(t_1\ldots t_n)^{a_n} \\ &= t_1^{k_2+\ldots+k_n+a_n}t_2^{k_3+\ldots+k_n+a_n}\ldots t_{n-1}^{k_n+a_n}t_n^{a_n} \end{aligned}$$

We can further calculate the exponents: For $l \geq 2$ we have

$$\begin{aligned} k_l + \ldots + k_n &= (n-l+1)(p-1) - \sum_{j=0}^{n-1} i_j \\ &= (n-l+1)(p-1) - (p\delta_{n-l+1} + \sum_{j=1}^{n-l} p\delta_l - \sum_{j=1}^{n-1} \delta_j - \delta_0 + \sum_{j=0}^{n-l} a_j - \sum_{j=1}^{n-l+1} a_j) \\ &= (n-l+1)(p-1) - (p\delta_{n-l+1} + (p-1)\sum_{j=1}^{n-l} \delta_j + a_n - a_{n-l+1}) \end{aligned}$$

and hence

$$\begin{aligned} k_l + \ldots + k_n + a_n &= (n-l+1)(p-1) - (p-1)\sum_{j=1}^{n-l} -p\delta_{n-l+1} + a_{n-l+1} \\ &= (p-1)(n-l+1 - \sum_{j=1}^{n-l+1} \delta_l) - \delta_{n-l+1} + a_{n-l+1}. \end{aligned}$$

So, the character λ is given by

$$\lambda_+(diag(t_1, \ldots, t_n)) = t_1^{a_{n-1}-\delta_{n-1}} t_2^{a_{n-2}-\delta_{n-2}} \ldots t_{n-1}^{a_1-\delta_1} t_n^{a_n-\delta_n}$$

and $s_i \in I_+$ if and only if $k_{i+1} = p-1$.

The tuple (λ_+, I_+) obtained in this way is Ω_0-conjugate to the tuple (μ, J) given by

$$\mu(diag(t_1, \ldots, t_n) = t_1^{a_n-\delta_n} t_2^{a_{n-1}-\delta_{n-1}} \ldots t_{n-1}^{a_2-\delta_2} t_n^{a_1-\delta_1}$$

and $s_j \in J$ if and only if $k_j = p-1$.

We can summarize the results of our calculations:

Lemma 4.6.1. *With our notations above, we have*

$$\lambda_+(\rho)(diag(t_1, \ldots, t_n) = t_1^{a_n-\delta_n} t_2^{a_{n-1}-\delta_{n-1}} \ldots t_{n-1}^{a_2-\delta_2} t_n^{a_1-\delta_1},$$

$$I_+(\rho) = \{s_i \in S_0 : p\delta_{n-i+1} - a_{n-i} - a_{n-i+1} - \delta_{n-i} = 0\},$$

and $\omega_+(\rho)$ is the character of $Y(T)^{(1)}$ given by $\lambda_+^{-1}(\rho)_{|T(\mathbb{F}_q)^{W_0}}$ on $T(\mathbb{F}_q)^{W_0}$ such that

$$\omega_+(\rho)(diag(\pi_L, \ldots, \pi_L)) = (-1)^{n-1}\lambda(diag(-1, \ldots, -1)) \prod_{i=1}^{n-1} k_i! \beta^n.$$

Additionally, we have

$$\lambda_-(\rho)(diag(t_1, \ldots, t_n) = t_1^{-a_1+\delta_1} t_2^{-a_2+\delta_2} \ldots t_{n-1}^{-a_{n-1}+\delta_{n-1}} t_n^{-a_n+\delta_n},$$

$$I_-(\rho) = \{s_i \in S_0 : p\delta_i - a_{i+1} - a_i - \delta_{i+1} = 0\},$$

and $\omega_-(\rho)$ is the character of $Y(T)^{(1)}$ given by $\lambda_-^{-1}(\rho)_{|T(\mathbb{F}_q)^{W_0}}$ on $T(\mathbb{F}_q)^{W_0}$ such that

$$\omega_-(\rho)(diag(\pi_L, \ldots, \pi_L)) = (-1)^{n-1}\lambda(diag(-1, \ldots, -1)) \prod_{i=1}^{n-1} k_i! \beta^n.$$

Proof. We have seen the "+"-case and the other case is treated completely analogously. □

We emphasize the fact that, even in the GL_2-case, one obtains two different functors. This is due to the fact that these functors are defined by the choice of the same root but different choices for the generator of Ω.

4.7 Maps between Grothendieck Groups

For this section, assume that $G = \mathrm{GL}_n$ and $n \geq 2$. Let $\mathcal{H}_\pi^{(1)}$ be the subalgebra of $\mathcal{H}^{(1)}$ generated by $\mathcal{H}_{aff}^{(1)}$ and $\tau_{\pi_L \cdot id}^{\pm 1}$. This is the subalgebra called $\tilde{\mathcal{H}}^{(1)}$ in the proof of Proposition 4.4.8 because the subgroup generated by $\pi_L \cdot id$ identifies with $Y(T)^{W_0}$. It is the unique subgroup of index n in Ω. Then, we have inclusions of Hecke algebras

$$\mathcal{H}_0^{(1)} \subseteq \mathcal{H}_{aff}^{(1)} \subseteq \mathcal{H}_\pi^{(1)} \subseteq \mathcal{H}^{(1)}.$$

For each of these algebras denote by $\mathrm{G}_0(?)$ the Grothendieck group of the category of finite dimensional right modules over these algebras. As restriction of modules to a subalgebra is an exact functor, we obtain homomorphisms of Grothendieck groups

$$\mathrm{G}_0(\mathcal{H}^{(1)}) \to \mathrm{G}_0(\mathcal{H}_\pi^{(1)}) \to \mathrm{G}_0(\mathcal{H}_{aff}^{(1)}) \to \mathrm{G}_0(\mathcal{H}_0^{(1)}).$$

We will denote each of these maps by res, e.g. we have

$$\mathrm{res}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} : \mathrm{G}_0(\mathcal{H}^{(1)}) \to \mathrm{G}_0(\mathcal{H}_\pi^{(1)}).$$

We will now construct maps in the other direction. Recall that the characters $\psi_{(\lambda,I)}$ are a basis for $\mathcal{H}_0^{(1)}$ by Theorem 4.1.3. Here λ is a character of $T(\mathbb{F}_q)$ and $I \subseteq S_0^\lambda$. Now, we may also view I as a subset of S^λ and thus define the character $\chi_{(\lambda,I)}$ of $\mathcal{H}_{aff}^{(1)}$ by this pair. Hence, we obtain a homomorphism from $\mathrm{G}_0(\mathcal{H}_0^{(1)})$ to $\mathrm{G}_0(\mathcal{H}_{aff}^{(1)})$ as the linear extension of $\psi_{(\lambda,I)} \mapsto \chi_{(\lambda,I)}$ which we will denote by $\mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}$.

For the map from $\mathrm{G}_0(\mathcal{H}_{aff}^{(1)})$ to $\mathrm{G}_0(\mathcal{H}_\pi^{(1)})$ fix some $z \in \overline{\mathbb{F}}_q^\times$. $\tau_{\pi_L \cdot id}^{\mathbb{Z}}$ identifies with the subgroup $Y(T)^{W_0} = Y(T) \cap \Omega$ of Ω. Hence $\mathcal{H}_\pi^{(1)}$ is isomorphic to the algebra $\overline{\mathbb{F}}_q[Y(T)^{W_0}] \otimes_{\overline{\mathbb{F}}_q} \mathcal{H}_{aff}^{(1)}$ and each $\mathcal{H}_{aff}^{(1)}$-module has a unique extension to a module over $\mathcal{H}_\pi^{(1)}$ such that τ_{π_L} acts by multiplication with z. This defines an exact functor and hence a homomorphism

$$\mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}} : \mathrm{G}_0(\mathcal{H}_{aff}^{(1)}) \to \mathrm{G}_0(\mathcal{H}_\pi^{(1)}).$$

Finally, $\mathcal{H}^{(1)} = \overline{\mathbb{F}}_q[\Omega] \otimes_{\overline{\mathbb{F}}_q} \mathcal{H}_{aff}^{(1)}$ is a free module of rank n over $\mathcal{H}_\pi^{(1)} = \overline{\mathbb{F}}_q[\Omega \cap Y(T)] \otimes_{\overline{\mathbb{F}}_q} \mathcal{H}_{aff}^{(1)}$. So the functor $\otimes_{\mathcal{H}_\pi^{(1)}} \mathcal{H}^{(1)}$ is exact and respects finite dimensional modules. Hence we obtain a group homomorphism $\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} : \mathrm{G}_0(\mathcal{H}_\pi^{(1)}) \to \mathrm{G}_0(\mathcal{H}^{(1)})$.

Now assume that we are given a character $\psi_{(\lambda,I)}$ of $\mathcal{H}_0^{(1)}$ such that the Ω_0-orbit of (λ, I) has cardinality n when we view I as a subset of S^λ. The proof of Proposition 4.4.8 shows that the image of $\psi_{(\lambda,I)}$ in $\mathrm{G}_0(\mathcal{H}^{(1)})$ is the simple supersingular module associated with the triple (λ, I, ω), where $\omega : Y(T)^{W_0} \to \overline{\mathbb{F}}_q^\times$ is given by $\omega(diag(\pi_L, \dots, \pi_L)) = z$. However this does not work when the Ω_0-orbit of (λ, I) is not of maximal length.

There are tuples (λ, I) with $I \subseteq S^\lambda$ such that $\#\Omega_0(\lambda, I) = n$ but $\#\Omega_0(\lambda, I \cap S_0) < n$. For example we can take $(\lambda = 1, I = \{s_0\})$. But it is true that each simple supersingular module of dimension n such that τ_{π_L} acts by multiplication with z is contained in the image of $\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} \circ \mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}} \circ \mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}$. To see this it suffices to show that each Ω_0-orbit of cardinality n contains some (λ, I) such that $s_0 \notin I$. Assume the contrary. Because Ω_0 acts transitively on S, each representative of the orbit is of the form (λ, S) for some λ. This means that λ is invariant under conjugation with each simple reflection and hence with W_0. So we have $\Omega_0(\lambda, S) = 1$ in contradiction to our assumption.

Chapter 5

$\mathcal{H}^{(1)}$-Modules associated with Weil Group Representations

5.1 The Strategy

Let $G = \mathrm{GL}_n$ and keep the notations that were introduced in the previous chapters. So far, we have done the following: The irreducible Galois representations of dimension n are given by a subset of the fibre product

$$(\mathrm{Hom}^{cont}(I_L/P_L, T^*)/W_0^*)^{F^*} \times_{\mathrm{Hom}^{cont}(I_L/P_L, \overline{\mathbb{F}}_q^\times)} \mathrm{Hom}^{cont}(W_L/P_L, \overline{\mathbb{F}}_q^\times)$$

by Theorem 2.1.7. Then we have reinterpreted the first factor as

$$(\mathrm{Hom}^{cont}(Y(T) \otimes Z, \overline{\mathbb{F}}_q^\times)/W_0)^F$$

in Proposition 2.2.4 and its corollary. By Lemma 2.2.6, this defines a $\mathrm{GL}_n(\mathbb{F}_q)$-conjugacy class of a tuple (T_w, θ) where T_w is a torus of $\mathbb{F}_q$-rank 1 and θ is a character of $T_w(\mathbb{F}_q)$. We have explicitly determined this tuple in section 2.3.

By section 3.5, we obtain a virtual representation of $\mathrm{GL}_n(\mathbb{F}_q)$ over $\overline{L}$ which we can reduce modulo p to $\mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_n(\mathbb{F}_q)]) \cong \mathrm{G}_0(\mathcal{H}_0^{(1)})$. From there we can proceed as described in section 4.7 to obtain an element of $\mathrm{G}_0(\mathcal{H}^{(1)})$. For this we need an element $z \in \overline{\mathbb{F}}_q$ in order to declare the action of τ_{π_L}. We obtain this from the other factor of our fibre product $\mathrm{Hom}^{cont}(W_L/P_L, \overline{\mathbb{F}}_q^\times) = \mathrm{Hom}^{cont}(W_L^{ab}, \overline{\mathbb{F}}_q^\times)$ in the beginning: If φ is the Frobenius corresponding to π_L and the reciprocity isomorphism $L^\times \cong W_L^{ab}$ of local class field theory, we define z as the image of φ in $\overline{\mathbb{F}}_q^\times$.

Now we have constructed a map from irreducible n-dimensional Galois representations to the Grothendieck group of finite dimensional modules over $\mathcal{H}^{(1)}$. As it turns out, this composite does not behave in a nice way as will be illustrated for the GL_2-case in the next section. However, we will see that a little tweak, a "shift"-map on $\mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_n(\mathbb{F}_q)])$ will give us a way nicer behavior and establish a bijection between irreducible n-dimensional Galois representations and simple supersingular modules of dimension n in the case $n = 2$.

5.2 The GL_2-Case

Now we will examine the construction given in the previous section for GL_2 by making everything explicit. Let ρ be a continuous irreducible representation of W_L of dimension 2 and $m = a + bq$ as in section 2.3. By Proposition 2.1.6, m is not a multiple of $q+1$, i.e. $a \neq b$. We may assume that $a > b$ without changing the isomorphism class of ρ.

By Lemma 2.3.1 and the preceding discussion, the tuple (T_w, θ) is given in the following way: $w = s$ is the non trivial element of $W_0 = S_2$ and if $g \in \mathrm{GL}_2(\overline{\mathbb{F}}_q)$ such that $g^{-1}F(g)$ is a lift of s and θ is given as the restriction of ${}^g(a\chi_1 + b\chi_2)$ to $T_w(\mathbb{F}_q) = {}^gT(\mathbb{F}_q)$. We can now use Jantzen‘s formula (Proposition 3.5.2) and the explicit version of the discussion afterwards to calculate the reduction mod p of the corresponding Deligne-Lusztig character. We will now identify $X(T)$ with $\mathbb{Z}^2$, so that we will write $W(m,n)$ instead of $W(m\chi_1 + n\chi_2)$. Hence, we have

$$\overline{R_s(\mu)} = W(a-1, b+1) + W(b+q-1, a).$$

If $a-b \geq 2$, both weights occurring in the formula above are dominant. If $a = b+1$, the formula

$$-W(a-1,a) = \det(s)W(a-1,a) = W(s \cdot (a-1,a)) = W(a-1,a)$$

implies that the first summand is 0.

Given that we are aiming for a bijection between irreducible 2-dimensional Galois representations and supersingular 2-dimensional $\mathcal{H}^{(1)}$-modules, the first problem becomes apparent. For simplicity let us stick to the case $q = p$. We have

$$W(a-1, b+1) = F(a-1, b+1)$$

(unless $a = b+1$) and

$$W(b+p-1, a) = F(b+p-1, a)$$

and the inequalities $a - 1 - (b+1) = a - b - 2 < p - 1$, because $a - 2 \leq p - 1$, and $b + p - 1 - a < p - 1$, because $a > b$. By Proposition 3.3.2 this means that the corresponding characters for the finite Hecke algebra are always of the form $\psi_{(\lambda,\emptyset)}$. Hence, by construction, surjectivity cannot be achieved. Additionally, the two summands correspond to different characters of the finite torus and not all characters of the finite torus arise in this way. Namely, those trivial on $T_s(\mathbb{F}_p)$ are missing.

Now let $q = p^r$ be a general power of p again. We will introduce a “shift“-map on $\mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_2(\mathbb{F}_q)])$ which fixes all of these problems. Let

$$(a,b) \in X_r(T) = \{(a,b) \in X(T) : 0 \leq a - b < q\}.$$

Then $(a, b-1) \in X_1(T)$ unless $a - b = q - 1$. Thus, we can define the map $\chi : \mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_n(\mathbb{F}_q)]) \to \mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_n(\mathbb{F}_q)])$ by

$$\chi(F(a,b)) := \begin{cases} F(a, b-1), & \text{if } a - b \neq q - 1 \\ 0, & \text{if } a - b = q - 1. \end{cases}$$

When we want to apply this definition to reductions of Deligne-Lusztig characters obtained as above, we are always in the first case: If $F(c,d)$ is a summand of $W(a-1,b+1)$ (with $a > b+1$) resp. $W(b+q-1,a)$, then $(c,d) \leq (a-1,b+1)$ resp. $(c,d) \leq (b+q-1,a)$ by the strong linkage principle, so that $c-d \leq a-1-(b-1) \leq q-2$ resp. $c-d \leq b+q-1-a \leq q-2$. So the second part of the definition is rather for the sake of completeness and not relevant in our applications.

So now, we can include the shift map χ in our construction by applying it to $\overline{R_s(\mu)}$ and then using the constructions from section 4.7. Let us treat the simpler case $q = p$ first. Assume that $a = b+1$. Then $\overline{R_s(\mu)} = F(a-2+p,a)$ and hence

$$\chi(\overline{R_s(\mu)}) = F(a-1+(p-1),a-1).$$

By Proposition 3.3.2 the corresponding $\mathcal{H}_0^{(1)}$-module is $\psi_{(\lambda,\{s\})}$ with

$$\lambda(diag(t_1,t_2)) = t_1^{a-1}t_2^{a-1}.$$

So, we get that

$$\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}}\mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi(R_s(\mu))) = M_{(\lambda,\{s\},z)},$$

where $M_{(\lambda,I,z)}$ is the 2-dimensional supersingular $\mathcal{H}^{(1)}$-module given by the tuple (λ, I) on which τ_{π_L} acts by z.

On the other hand assume that $a > b+1$. Then we have

$$\chi(\overline{R_s(\mu)}) = F(a-1,b) + F(b+p-1,a-1).$$

Using Proposition 3.3.2 again, we see that the corresponding $\mathcal{H}_0^{(1)}$-modules are $\psi_{(\lambda,\emptyset)}$ and $\psi_{({}^s\lambda,\emptyset)}$ where $\lambda(diag(t_1,t_2)) = t_1^{a-1}t_2^{b}$, which implies that

$$\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}}\mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi(\overline{R_s(\mu)})) = 2M_{(\lambda,\emptyset,z)}.$$

In each of the cases we can speak of the supersingular simple 2-dimensional $\mathcal{H}^{(1)}$-module associated with an irreducible 2-dimensional Galois-representation ρ and denote it by $M(\rho)$.

Theorem 5.2.1. *Assume that $L/\mathbb{Q}_p$ is totally ramified, i.e. $q = p$. The assignment $\rho \mapsto M(\rho)$ is a bijection between irreducible 2-dimensional W_L-representations and irreducible 2-dimensional supersingular simple $\mathcal{H}^{(1)}$-modules. If $L = \mathbb{Q}_p$, we have $M(\rho) = M_{(\lambda_+(\rho),I_+(\rho),\omega_+(\rho))}$.*

Proof. All of this follows from the above explicit descriptions. It suffices to show that the Galois representations such that the determinant of the Frobenius is z correspond to the supersingular modules on which τ_{π_L} operates by z. The above calculations show that we can reconstruct a and b from the character λ associated with a supersingular simple module of dimension 2 and thus $\rho \mapsto M(\rho)$ is injective. In the "$a = b+1$"-case, a ranges from 1 to $p-1$, so the characters λ obtained from this are all the characters with $S^\lambda = S$. In the "$a > b+1$"-case, λ ranges over all characters with $S^\lambda = \emptyset$.

That $\rho \mapsto M(\rho)$ coincides with Große-Klönne's functor follows from the explicit descriptions given in this section and in 4.6, because $a > b$ implies $\delta_1 = 1$ and $\delta_0 = 0$. Of course, this also implies bijectivity with Große-Klönne's results. $\square$

Remark 5.2.2. If q is a general power of p, $\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} \mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}} \mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi(\overline{R_s(\mu)}))$ is no longer a multiple of a simple supersingular module. This is due to the fact that $W(a-1,b)$ and $W(b+p-1,a-1)$ are not simple in this case. But by Proposition 3.4.4, there exists an automorphism ϕ of $\mathrm{G}_0(\overline{\mathbb{F}}_q[\mathrm{GL}_2(\overline{\mathbb{F}}_q)])$ mapping $W(\lambda)$ to $F(\lambda)$ for $\lambda \in X_r(T)^0$ such that $\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} \mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}} \mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi(\phi(\overline{R_s(\mu)})))$ is the multiple of supersingular simple module of dimension 2 with the same argument as above. ϕ is represented by a upper triangular unipotent matrix with respect to the basis given by the irreducible representations (in a suitable ordering). As a consequence we obtain that there exists an automorphism Φ of $\mathrm{G}_0(\mathcal{H}^{(1)})$ given by an upper triangular unipotent matrix in the basis given by the simple $\mathcal{H}^{(1)}$-modules such that $\Phi(\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} \mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}} \mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi(\overline{R_s(\mu)})))$ is simple, supersingular and of dimension 2.

The case of $G = \mathrm{SL}_2$ can be treated similarly. For this, assume that $q = p$. Let $\rho : W_L \to \mathrm{PSL}_2(\mathbb{F}_p)$ be a projective Weil group representation. By Lemma 2.4.1, there exists some $\rho_0 : W_L \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ which reduces to ρ modulo the center of $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$. As in section 2.3 we associate with it the integer $m = a + bp$ such that $0 \le b < a \le p-2$. Let T' be the standard split torus of SL_n. Further denote by (T_s, θ) resp. (T'_s, θ') the associated tuples consisting of a maximal torus and a character of $\mathbb{F}_p$-rational points of that torus for GL_n resp. SL_n. Further let μ resp. μ' be algebraic characters of T resp. T' which give rise to θ resp. θ' after conjugation to T_s resp. T'_s. From Proposition 3.5.2 and Corollary 3.4.3 we can deduce that

$$\overline{R_w(\mu')} = \overline{R_w(\mu)}_{|\mathrm{SL}_n(\mathbb{F}_p)}.$$

If we identify $X(T')$ with $\mathbb{Z}$ by mapping the unique simple root α to 2, we thus obtain

$$\overline{R_s(\mu')} = W(a-b-2) + W(p-1-a+b).$$

As in the GL_2-case, the first summand is 0 if $a = b+1$.

Again, we can define a shift map $\chi : \mathrm{G}_0(\overline{\mathbb{F}}_p[\mathrm{SL}_n(\mathbb{F}_p)]) \to \mathrm{G}_0(\overline{\mathbb{F}}_p[\mathrm{SL}_n(\mathbb{F}_p)])$ by

$$\chi(F(a)) := \begin{cases} F(a+1), & \text{if } a \neq p-1 \\ 0, & \text{if } a = p-1. \end{cases}$$

This shift is compatible with the shift map for GL_2 such that the obvious diagram involving shift maps and restrictions commutes. As for GL_n, one obtains a homomorphism $\mathrm{inc}_{\mathcal{H}_0^{(1)\prime}}^{\mathcal{H}^{(1)\prime}}$ from $\mathrm{G}_0(\mathcal{H}_0^{(1)\prime})$ to $\mathrm{G}_0(\mathcal{H}_{aff}^{(1)\prime}) = \mathrm{G}_0(\mathcal{H}^{(1)\prime})$ where we denote the corresponding Hecke algebras for SL_2 by ?′. Defining $\lambda : T'(\mathbb{F}_p) \to \mathbb{F}_p^\times$ by $\lambda(diag(t, t^{-1}) = t^{b-a+1}$, we obtain

$$\mathrm{inc}_{\mathcal{H}_0^{(1)'}}^{\mathcal{H}^{(1)'}}(\chi(\overline{R_s(\mu')}) = \begin{cases} \chi_{(1,\{s\})}, & \text{if } a = b+1 \\ \chi_{(\lambda,\emptyset)} + \chi_{({}^s\lambda,\emptyset)}, & \text{if } a \neq b+1. \end{cases}$$

The modules occurring here as summands are all supersingular and the only supersingular simple module not occurring here is $\chi_{(1,\{s_0\})}$ by Theorem 4.4.5. The "missing" supersingular character is due to our construction exactly as we have seen in the GL_n-cases.

Also, we immediately see that the map $\rho \mapsto \mathrm{inc}_{\mathcal{H}_0^{(1)'}}^{\mathcal{H}^{(1)'}}(\chi(\overline{R_s(\mu')})$ from irreducible projective W_L-representations of dimension 2 to $\mathrm{G}_0(\mathcal{H}^{(1)'})$ is injective. The characters which are the summands in the second case are an L-packet as introduced by Koziol in [Koz13] Def. 6.4.

5.3 The GL_3-Case

Now we will imitate the phenomena we have seen using the shift map for GL_2 for the GL_3-situation. For the whole of this section, assume $q = p$. So, let ρ be an irreducible continuous representation of W_L of dimension 3 and let $m = a + bp + cp^2$ be as in section 2.3.

Recall from 3.5 that

$$\begin{aligned} \overline{R_w(a,b,c)} =& W(a-2, b+1, c+1) + W(b+q-1, a-1, c+1) \\ &+ W(a+q-2, c+q-1, b+1) + W(c+q-2, a, b+1) \\ &+ W(b+q-2, c+q, a) + W(c+2(q-1), b+q-1, a). \end{aligned}$$

with $w = (1,2,3) = s_1 s_2$, is the reduction of the associated Deligne-Lusztig character mod p. As cyclic permutation does not change the isomorphism class of ρ, we may assume that either

(I) $a \geq b \geq c$ or

(II) $a \leq b \leq c$.

The primitivity of m implies that $a = b = c$ cannot occur. Thus, we may assume that either $a > b \geq c$ in case (I) or $a < b \leq c$ in case (II).

We will begin by evaluating Jantzen's formula in the first case. So assume for now that we are in case (I), i.e. $a > b \geq c$. We can use Proposition 3.2.3 and the identity $\det(\sigma)W(\lambda) = W(\sigma \cdot \lambda)$ to express the Weyl modules as sums of simple modules. This is quite tedious but simple work so we will only do it for the first summand in the formula. Recall that the two restricted p-alcoves are

$$C_0 = \{(a,b,c) \in X(T) \otimes \mathbb{R} : -1 < a-b; -1 < b-c; a-c < p-1\}$$

and

$$C_1 := \{(a,b,c) \in \mathbb{R}^3 : p-1 < a-c; a-b < p-1; b-c < p-1\}.$$

So we are considering the Weyl-module $W(a-2, b+1, c+1)$. We can have the following cases:

(1) $a = b+1, c = b$:

$$\begin{aligned} W(a-2,b+1,c+1) &= W(c-1,c+1,c+1) \\ &= \det(s_1)W(s_1 \cdot (c-1,c+1,c+1)) \\ &= -W(c,c,c+1) = -\det(s_2)W(s_2 \cdot (c,c,c+1)) \\ &= W(c,c,c+1), \end{aligned}$$

so $W(a-2, b+1, c+1) = 0$.

(2) $a = b+1, c < b$:

$$\begin{aligned} W(a-2,b+1,c+1) &= W(a-2,a,c+1) = \det(s_1)W(a-1,a-1,c+1) \\ &= -F(a-1,b,c+1) \end{aligned}$$

because $(a-1, a-1, c+1) \in C_0$.

(3) $a = b+2$:

$$\begin{aligned} W(a-2,b+1,c+2) &= W(b,b+1,c+1) = \det(s_1)W(s_1 \cdot (b,b+1,c+1)) \\ &= -W(b,b+1,c+1), \end{aligned}$$

so $W(a-2, b+1, c+1) = 0$.

(4) $a \geq b+3$ (the generic case)

$$W(a-2,b+1,c+1) = F(a-2,b+1,c+1).$$

Note that in all of the first three cases, the weight (a, b, c) lies "close" to the boundary of C_0. That is why we can consider case (4) as the generic one. Now we can do completely analogous calculations for all six summands. Instead of giving these calculations in detail we will list all cases which can occur by different numerical relations between the parameters. We have the following possibilities:

(A) $a > b+2, b > c+1, a-c < p-1$ (the generic case):

$$\begin{aligned} \overline{R_w(a,b,c)} =& F(a-2,b+1,c+1) + F(c+p-2,a,b+1) \\ &+ F(b+p-2,c+p,a) + F(c+2(p-1),b+p-1,a) \\ &+ F(a-1,b,c+1) + F(b+p-1,a-1,c+1) \\ &+ F(c+p-1,a-1,b+1) + F(a+p-2,c+p-1,b+1) \\ &+ F(b+p-1,c+p-1,a). \end{aligned}$$

(B) $a > b+2, b > c+1, a-c = p-1$:

$$\begin{aligned} \overline{R_w(a,b,c)} =& F(a-2,b+1,c+1) + F(b+p-2,c+p,a) \\ &+ F(c+2(p-1),b+p-1,a) + F(a-1,b,c+1) \\ &+ F(b+p-1,a-1,c+1) + F(c+p-1,a-1,b+1) \\ &+ F(a+p-2,c+p-1,b+1) + F(b+p-1,c+p-1,a). \end{aligned}$$

(C) $a > b+2, b = c+1, a-c < p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(a-2,b+1,c+1) + F(c+p-2,a,b+1)\\ &+ F(c+2(p-1),b+p-1,a) + F(a-1,b,c+1)\\ &+ F(b+p-1,a-1,c+1) + F(c+p-1,a-1,b+1)\\ &+ F(a+p-2,c+p-1,b+1) + F(b+p-1,c+p-1,a).\end{aligned}$$

(D) $a > b+2, b = c+1, a-c = p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(a-2,b+1,c+1) + F(c+2(p-1),b+p-1,a)\\ &+ F(a-1,b,c+1) + F(b+p-1,a-1,c+1)\\ &+ F(c+p-1,a-1,b+1) + F(a+p-2,c+p-1,b+1)\\ &+ F(b+p-1,c+p-1,a).\end{aligned}$$

(E) $a > b+2, b = c, a-c < p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(a-2,b+1,c+1) + F(c+p-2,a,b+1)\\ &+ F(c+2(p-1),b+p-1,a) + F(b+p-1,a-1,c+1)\\ &+ F(a+p-2,c+p-1,b+1).\end{aligned}$$

(F) $a > b+2, b = c, a-c = p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(a-2,b+1,c+1) + F(c+2(p-1),b+p-1,a)\\ &+ F(b+p-1,a-1,c+1) + F(a+p-2,c+p-1,b+1).\end{aligned}$$

(G) $a = b+2, b > c+1, a-c < p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c+p-2,a,b+1) + F(b+p-2,c+p,a)\\ &+ F(c+2(p-1),b+p-1,a) + F(a-1,b,c+1)\\ &+ F(b+p-1,a-1,c+1) + F(c+p-1,a-1,b+1)\\ &+ F(a+p-2,c+p-1,b+1) + F(b+p-1,c+p-1,a).\end{aligned}$$

(H) $a = b+2, b > c+1, a-c = p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(b+p-2,c+p,a) + F(c+2(p-1),b+p-1,a)\\ &+ F(a-1,b,c+1) + F(b+p-1,a-1,c+1)\\ &+ F(c+p-1,a-1,b+1) + F(a+p-2,c+p-1,b+1)\\ &+ F(b+p-1,c+p-1,a).\end{aligned}$$

(I) $a = b+2, b = c+1, a-c < p-1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c+p-2,a,b+1) + F(c+2(p-1),b+p-1,a)\\ &+ F(a-1,b,c+1) + F(b+p-1,a-1,c+1)\\ &+ F(c+p-1,a-1,b+1) + F(a+p-2,c+p-1,b+1)\\ &+ F(b+p-1,c+p-1,a).\end{aligned}$$

(J) $a = b + 2, b = c + 1, a - c = p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c + 2(p-1), b + p - 1, a) + F(a-1, b, c+1)\\ &+ F(b+p-1, a-1, c+1) + F(c+p-1, a-1, b+1)\\ &+ F(a+p-2, c+p-1, b+1) + F(b+p-1, c+p-1, a).\end{aligned}$$

(K) $a = b + 2, b = c, a - c < p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c+p-2, a, b+1) + F(c + 2(p-1), b+p-1, a)\\ &+ F(b+p-1, a-1, c+1) + F(a+p-2, c+p-1, b+1).\end{aligned}$$

(L) $a = b + 2, b = c, a - c = p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c + 2(p-1), b+p-1, a) + F(b+p-1, a-1, c+1)\\ &+ F(a+p-2, c+p-1, b+1).\end{aligned}$$

(M) $a = b + 1, b > c + 1, a - c < p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c+p-2, a, b+1) + F(b+p-2, c+p, a)\\ &+ F(c + 2(p-1), b+p-1, a) + F(b+p-1, a-1, c+1)\\ &+ F(a+p-2, c+p-1, b+1).\end{aligned}$$

(N) $a = b + 1, b > c + 1, a - c = p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(b+p-2, c+p, a) + F(c + 2(p-1), b+p-1, a)\\ &+ F(b+p-1, a-1, c+1) + F(a+p-2, c+p-1, b+1).\end{aligned}$$

(O) $a = b + 1, b = c + 1, a - c < p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c+p-2, a, b+1) + F(c + 2(p-1), b+p-1, a)\\ &+ F(b+p-1, a-1, c+1) + F(a+p-2, c+p-1, b+1).\end{aligned}$$

(P) $a = b + 1, b = c + 1, a - c = p - 1$:

$$\begin{aligned}\overline{R_w(a,b,c)} =& F(c + 2(p-1), b+p-1, a) + F(b+p-1, a-1, c+1)\\ &+ F(a+p-2, c+p-1, b+1).\end{aligned}$$

(Q) $a = b + 1, b = c, a - c < p - 1$:

$$\overline{R_w(a,b,c)} = F(c+p-2, a, b+1) + F(c + 2(p-1), b+p-1, a).$$

(R) $a = b + 1, b = c, a - c = p - 1$:

$$\overline{R_w(a,b,c)} = F(c + 2(p-1), b+p-1, a).$$

Although this is all quite technical, the upshot is the following: The generic case is case (A). In all other cases, the weight (a,b,c) lies close to the boundary of C_0. The closer we move towards the boundary of the p-alcove C_0, the more summands become 0. Also there is one summand which "survives" in any case, namely $F(c+2(p-1),b+p-1,a)$. This summand is connected to the character $\mu = (a,b,c)$ in a simple way: We have $w_0\mu = (c,b,a)$ and hence $(c+2(p-1),b+p-1,a)$ and $w_0\mu$ induce the same character on $T(\mathbb{F}_p)$. More precisely, $(c+2(p-1),b+p-1,a)$ is the unique restricted weight, inducing the same character as $w_0\mu$ on $T(\mathbb{F}_p)$.

Now, let us turn to the case (II) where $a < b \leq c$. We have the problem that the weights occurring in the Weyl modules are not restricted anymore. We can solve this by taking duals. Lemma 2.1 and Lemma 10.1 of [Her06] explicitly describe the characters of the cuspidal representations $R_w(\mu)$. This explicit description makes it clear that the character of the dual of $R_w(\mu)$ is the character of $R_w(-\mu)$. We shall denote the dual by the superscript *. On the other hand, $W(\lambda)^* = W(-w_0\lambda)$ by [Jan87, II 2.13] and thus, the same is true for the simple modules. So we can derive the decomposition into simple modules.

For example, we have in the case (A') where $b > a+2, c > b+1, c-a < p-1$:

$$\begin{aligned}
\overline{R_w(a,b,c)} &= \overline{R_w(-a,-b,-c)}^* \\
&= (F(-a-2,-b+1,-c+1) + F(-c+p-2,-a,-b+1) \\
&+ F(-b+p-2,-c+p,-a) + F(-c+2(p-1),-b+p-1,-a) \\
&+ F(-a-1,-b,-c+1) + F(-b+p-1,-a-1,-c+1) \\
&+ F(-c+p-1,-a-1,-b+1) + F(-a+p-2,-c+p-1,-b+1) \\
&+ F(-b+p-1,-c+p-1,-a))^* \\
&= F(c-1,b-1,a+2) + F(b-1,a,c-p+2) \\
&+ F(a,c-p,b-p+2) + F(a,b-(p-1),c-2(p-1)) \\
&+ F(c-1,b,a+1) + F(c-1,a+1,b-(p-1)) \\
&+ F(b-1,a+1,c-(p-1) + F(b-1,c-(p-1),a-p+2) \\
&+ F(a,c-(p-1),b-(p-1).
\end{aligned}$$

We can proceed in a completely analogous way in all other cases (which we shall omit here) and obtain cases (B')-(R') in the obvious way, i.e. we interchange the variables in each of the defining inequalities.

As it turns out we need to define two different shift maps depending on whether we are in case (I) or (II). So let us define

$$\chi_+(F(a,b,c)) := \begin{cases} F(a,b,c-1), & \text{if } (a,b,c-1) \in X_1(T), \\ 0, & \text{if } (a,b,c-1) \notin X_1(T). \end{cases}$$

and

$$\chi_-(F(a,b,c)) := \begin{cases} F(a,b-1,c-1), & \text{if } (a,b-1,c-1) \in X_1(T), \\ 0, & \text{if } (a,b-1,c-1) \notin X_1(T). \end{cases}$$

As in the GL_2-case one sees that the second part of the definition does not occur applying the shift maps to $R_w(\mu)$ by going through the cases. Now we can apply the respective shift maps and consider the corresponding elements of $\mathrm{G}_0(\mathcal{H}_0^{(1)})$. The cases are discussed similarly. We will discuss examples so that all occurring phenomena can be seen in these examples.

For example in case (A) this yields the following: For $(i,j,k) \in \mathbb{Z}^3$ let $\psi_{(i,j,k),I}$ be the character $\psi_{(\lambda_{(i,j,k)},I)}$ with $\lambda_{(i,j,k)}(diag(t_1,t_2,t_3)) = t_1^i t_2^j t_3^k$. Then we get

$$\begin{aligned}\chi_+(\overline{R_w(a,b,c)}) &= \psi_{(c,b,a-1),\emptyset} + \psi_{w(c,b,a-1),\emptyset} + \psi_{w^2(c,b,a-1),\emptyset}\\ &+ \psi_{(a-1,b,c),\emptyset} + \psi_{w(a-1,b,c),\emptyset} + \psi_{w^2(a-1,b,c),\emptyset}\\ &+ \psi_{(a-2,b+1,c),\emptyset} + \psi_{(c-1,a,b),\emptyset} + \psi_{(b-1,c+1,a-1),\emptyset}.\end{aligned}$$

So, the summands in the first and the second line are the restrictions of the supersingular modules defined by the tuples $(\lambda_{(c,b,a-1)},\emptyset)$ resp. $(\lambda_{(a-1,b,c)},\emptyset)$ (and an action of τ_{π_L} which is not seen anymore after restricting to $\mathcal{H}_0^{(1)}$). Also these are the restrictions of the two simple supersingular 3-dimensional modules $M_{(\lambda_+(\rho),I_+(\rho),\omega_+(\rho))}$ and $M_{(\lambda_-(\rho)^{-1},I_-(\rho),\omega_-(\rho))}$ obtained by Große-Klönne's functors. This follows immediately from the explicit description in Lemma 4.6.1. Note that it is however not true that the supersingular modules given by Große-Klönne's functors are contained in $\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}}\mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi_+(\overline{R_w(\mu)}))$. This is due to the fact that τ_{π_L} does not operate by the determinant of the Frobenius in Große-Klönne's construction which is true for our construction by definition. For the other three summands we cannot give a similar interpretation. We have

$$\begin{aligned}\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}}\mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi_+(\overline{R_w(\mu)})) &= 3M_{\lambda_{(c,b,a-1)},\emptyset,z} + 3M_{\lambda_{(a-1,b,c)},\emptyset,z} + \dots\\ &= 3M_{\lambda_+(\rho),I_+(\rho),z)} + 3M_{\lambda_-(\rho)^{-1},I_-(\rho),z)} + \dots\end{aligned}$$

where the dots symbolize the image of the three summands in the bottom line. In the case (A'), the situation behaves completely analogously. Namely, we get

$$\begin{aligned}\chi_-(\overline{R_w(a,b,c)}) &= \psi_{(c-1,b-1,a),\emptyset} + \psi_{w(c-1,b-1,a),\emptyset} + \psi_{w^2(c-1,b-1,a),\emptyset}\\ &+ \psi_{(a,b-1,c-1),\emptyset} + \psi_{w(a,b-1,c-1),\emptyset} + \psi_{w^2(a,b-1,c-1),\emptyset}\\ &+ \psi_{(c-1,b-2,a+1),\emptyset} + \psi_{b-1,a-1,c),\emptyset} + \psi_{(a,c-2,b),\emptyset}.\end{aligned}$$

Again, this contains the restrictions of the supersingular simple modules

$$M_{(\lambda_+(\rho),I_+(\rho),\omega_+(\rho))}$$

and

$$M_{(\lambda_-(\rho)^{-1},I_-(\rho),\omega_-(\rho))};$$

we have

$$\begin{aligned}\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}}\mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}}(\chi_-(\overline{R_w(\mu)})) &= 3M_{\lambda_{(c-1,b-1,a)},\emptyset,z} + 3M_{\lambda_{(a,b-1,c-1)},\emptyset,z} + \dots\\ &= 3M_{\lambda_+(\rho),I_+(\rho),z)} + 3M_{\lambda_-(\rho)^{-1},I_-(\rho),z)} + \dots .\end{aligned}$$

So, in the generic cases we obtain two supersingular simple three-dimensional $\mathcal{H}^{(1)}$-modules corresponding to each irreducible three-dimensional Galois representation. Note that we always have $I = \emptyset$ in the generic cases. As for GL_2, the cases with nonempty I occur when μ is close to the boundary of its p-alcove. An example for this is the case (E) where $b = c$: We get

$$\chi_+(\overline{R_w(a,b,c)}) = \psi_{(c,b,a-1),\{s_1\}} + \psi_{(a-1,c,b),\{s_2\}} + \psi_{(b,a-1,c),\emptyset} + \psi_{(a-2,b+1,c),\emptyset} + \psi_{(c-1,a,b),\emptyset}.$$

Similar to the generic case, $\chi_+(\overline{R_w(a,b,c)})$ contains all the restrictions of $M_{\lambda_{(c,b,a-1)},\{s_1\},z}$: We have

$$M_{\lambda_{(c,b,a-1)},\{s_1\},z} \cong \chi_{\lambda_{(c,b,a-1)},\{s_1\}} \oplus \chi_{\lambda_{(a-1,c,b)},\{s_2\}} \oplus \chi_{\lambda_{(b,a-1,c,)},\{s_0\}}$$

as $\mathcal{H}^{(1)}_{aff}$-modules. As the affine reflection s_0 cannot be "seen" by $\mathcal{H}^{(1)}_0$, we obtain the odd looking summand $\psi_{(b,a-1,c),\emptyset}$. Again, the reductions of $M_{(\lambda_+(\rho),I_+(\rho),\omega_+(\rho))}$ and $M_{(\lambda_-(\rho)^{-1},I_-(\rho),\omega_-(\rho))}$ are all contained in $\chi_+(\overline{R_w(a,b,c)})$. Note that this time $M_{(\lambda_+(\rho),I_+(\rho),\omega_+(\rho))}$ and $M_{(\lambda_-(\rho)^{-1},I_-(\rho),\omega_-(\rho))}$ coincide: They are given by $(\lambda_+(\rho) = \lambda_{(c,b,a-1)}, I_+(\rho) = \{s_1\})$ and $(\lambda_-(\rho)^{-1}, I_-(\rho)) = (\lambda_{(a-1,b,c)}, \{s_2\})$. But now $b = c$ so both those tuples are Ω_0 conjugate.

The only cases where the restrictions of the modules assigned to ρ do not all occur in $\chi_+(\overline{R_w(a,b,c)})$ resp. $\chi_-(\overline{R_w(a,b,c)})$ are the cases closest to the boundary, namely (Q), (R); (Q') and (R'). For example in case (R), we have

$$\chi_+(\overline{R_w(a,b,c)}) = F(c+2(p-1), b+p-1, a)) = M_{\lambda_{(c,b,a-1)},\{s_1,s_2\}}$$

This is the restriction of one of the three summands of

$$M_{(\lambda_+(\rho),I_+(\rho),\omega_+(\rho))} = M_{(\lambda_-(\rho)^{-1},I_-(\rho),\omega_-(\rho))}.$$

All other cases are treated similarly and no phenomena we have not discussed yet occur. We summarize the results:

Theorem 5.3.1. *Denote by*

$$\chi_\pm(\overline{R_w(a,b,c)}) := \begin{cases} \chi_+(\overline{R_w(a,b,c)}) & \text{in case (I)} \\ \chi_-(\overline{R_w(a,b,c)}) & \text{in case (II).} \end{cases}$$

Then $M_{(\lambda_+(\rho),I_+(\rho),z)}$ *and* $M_{(\lambda_-(\rho)^{-1},I_-(\rho),z)}$ *are contained in the virtual module* $\mathrm{ind}_{\mathcal{H}^{(1)}_\pi}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}^{(1)}_{aff}}^{\mathcal{H}^{(1)}_\pi}\mathrm{inc}_{\mathcal{H}^{(1)}_0}^{\mathcal{H}^{(1)}_{aff}}(\chi_\pm(\overline{R_w(a,b,c)}))$. *Their multiplicities are:*

- 3 *in the cases (A), (B), (C), (D), (G), (H), (I), (J) and the corresponding "prime"-cases,*
- 2 *in the cases (E), (F), (K), (L), (M), (N), (O), (P) and the corresponding "prime"-cases,*
- 1 *in the cases (Q), (R), (Q') and (R').*

In particular, each supersingular irreducible $\mathcal{H}^{(1)}$*-module is contained in one*

$$\mathrm{ind}_{\mathcal{H}^{(1)}_\pi}^{\mathcal{H}^{(1)}}\mathrm{inc}[z]_{\mathcal{H}^{(1)}_{aff}}^{\mathcal{H}^{(1)}_\pi}\mathrm{inc}_{\mathcal{H}^{(1)}_0}^{\mathcal{H}^{(1)}_{aff}}(\chi_\pm(\overline{R_w(a,b,c)})).$$

5.4 The generic Case for GL_4

Let $G = \mathrm{GL}_n$, $q = p$ and ρ be an irreducible n-dimensional Galois representation with corresponding $w = (1, 2, \ldots, n)$ and $\mu = (a_1, \ldots, a_n)$. For the remainder of this section, assume that μ lies sufficiently deep in a p-alcove.

Lemma 5.4.1. *Assume that μ lies sufficiently deep in C_0. Then $F(w_0\mu + (p-1)\rho)$ occurs as a summand of $\overline{R_w(\mu)}$.*

Proof. Here we use Proposition 3.5.3. Taking $\nu = 0$ and and $\sigma = 1$, we have to show that

$$\mu - \rho \uparrow w_0 \cdot (w_0\mu + (p-1)\rho - p\rho).$$

But we even have equality here. □

The cases where μ cannot be chosen inside C_0 are more complicated. Without changing the isomorphism-class of ρ, we may assume that a_n is already minimal among the a_i. Further, there exists a $\tau \in W_0$ such $\tau\mu$ lies in C_0. Assume again that $\tau\mu$ lies sufficiently deep in C_0. We have

$$\overline{R_w(\mu)} = \overline{R_{\tau v \tau^{-1}}(\tau\mu)}.$$

We will try to answer the question if $\overline{R_w(\mu)}$ contains $F(\lambda)$ for some $\lambda \in X_1(T)$ which restricts to the same character of $T(\mathbb{F}_p)$ as $w_0\mu$. Up to $X(T)^{W_0}$, we can only have $\lambda = w_0\mu + (p-1)\rho_{w_0\tau^{-1}}$. To use Proposition 3.5.3, we need to find $\nu \in X(T)$ and $\sigma \in W_0$ as in that proposition. For this set $\nu := \tau w^{-1} w_0 \rho_{w_0\tau^{-1}w_0}$ and choose $\sigma \in W_0$ such that $\sigma(\tau\mu - p\nu)$ is dominant. As $\tau\mu$ lies sufficiently deep in C_0, $\sigma \cdot (\tau\mu - \rho + (\tau w \tau^{-1})\nu - p\nu)$ is also dominant, because the difference is independent of p. By Proposition 3.5.3, we have to show that

$$\sigma \cdot (\tau\mu - \rho + (\tau w \tau^{-1})\nu - p\nu) \uparrow w_0 \cdot (\lambda - p\rho).$$

We can at least show the following:

Lemma 5.4.2. *There exists $\tilde{w} \in W_p$ such that*

$$\tilde{w} \cdot \sigma \cdot (\tau\mu - \rho + (\tau w \tau^{-1})\nu - p\nu) = w_0 \cdot (\lambda - p\rho).$$

Proof. We define $\tilde{w}$ as an element of $W_0 \ltimes X(T)$:

$$\begin{aligned}\tilde{w} &= w_0 e^{-p\rho + p\rho_{w_0\tau^{-1}}} w_0 \tau^{-1} e^{p\nu} \sigma^{-1} \\ &= e^{p w_0(\rho_{w_0\tau^{-1}} - \rho)} \tau^{-1} e^{p\nu} \sigma^{-1} \\ &= e^{p w_0 (w-1) \rho_{w_0\tau^{-1}w_0}} \tau^{-1} \sigma^{-1}.\end{aligned}$$

To show that $\tilde{w}$ lies in W_p, we have to show $w_0(w-1)\rho_{w_0\tau_{-1}w_0} \in Q$ and that means that it suffices to show $(w-1)\rho_{w_0\tau_{-1}w_0} \in Q$. This is actually a general fact about root data: For each $x \in X(T)$ and $w \in W_0$, $(w-1)x \in Q$, see e.g. Lemma 1.1.11 in [Bor09]. Hence, $\tilde{w} \in W_p$.

Now we show that $\tilde{w}$ satisfies the equation from the claim. We have

$$\begin{aligned}\tilde{w}\cdot\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) &= w_0e^{-p\rho+p\rho_{w_0\tau^{-1}}}w_0\tau^{-1}\cdot(\tau\mu-\rho+\tau w\tau^{-1}\nu)\\ &= w_0\cdot(w_0\mu+w_0w\tau^{-1}\nu-\rho+p\rho_{w_0\tau^{-1}}-p\rho)\\ &= w_0\cdot(w_0\mu+(p-1)\rho_{w_0\tau^{-1}}-p\rho)\\ &= w_0\cdot(\lambda-p\rho).\end{aligned}$$

In the second line from the bottom, we have used that

$$w_0w\tau^{-1}\nu-\rho=\rho_{w_0\tau^{-1}w_0}-\rho=-\rho_{w_0\tau^{-1}}.$$

□

Proposition 5.4.3. *If $G=\mathrm{GL}_4$ and μ lies sufficiently deep in a p-alcove, $\overline{R_w(\mu)}$ contains $F(\lambda)$ with $\lambda=w_0\mu+(p-1)\rho_{w_0\tau^{-1}}$.*

Proof. By the discussion before, we have to show that

$$\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu)\uparrow w_0\cdot(\lambda-p\rho).$$

By the previous lemma, it is enough to show that the p-alcove of the left hand-side lies below the p-alcove of the right hand side with respect to $\uparrow$. As μ lies sufficiently deep in its p-alcove, we may replace the right hand side by $\sigma(\tau\mu-p\nu)$ without changing its p-alcove. Because a_4 is minimal among the a_i, we get $\tau(4)=4$.

The case where $\tau=1$ is Lemma 5.4.1. If $\tau=s_1$, we get

$$\lambda=w_0\mu+(p-1)(\omega_{\alpha_1}+\omega_{\alpha_2})$$

and

$$\begin{aligned}w_0\cdot(\lambda-p\rho) &= \mu-(p-1)w_0\omega_{\alpha_3}-\rho\\ &= (a_1,a_2-(p-1),a_3-(p-1),a_4-(p-1))-\rho\in C_2.\end{aligned}$$

On the other hand, we get $\nu=\omega_{\alpha_3}$, $\sigma=w$ and this leads to

$$\sigma(\tau\mu-p\nu)=(a_4,a_2-p,a_1-p,a_3-p)\in C_0\uparrow C_2.$$

Analogous calculations yield the claim in the other cases: If $\tau=s_2$,

$$\sigma(\tau\mu-p\nu)\in C_2\uparrow C_4\ni w_0\cdot(\lambda-p\rho).$$

If $\tau=s_1s_2s_1$,

$$\sigma(\tau\mu-p\nu)\in C_2\uparrow C_5\ni w_0\cdot(\lambda-p\rho).$$

If $\tau=s_1s_2$,

$$\sigma(\tau\mu-p\nu)\in C_0\uparrow C_3\ni w_0\cdot(\lambda-p\rho).$$

If $\tau=s_2s_1$,

$$\sigma(\tau\mu-p\nu)\in C_0\uparrow C_1\ni w_0\cdot(\lambda-p\rho).$$

□

It is now tentative to define shift maps in the following way: We define integers for $j \geq 0$

$$\delta_{j+1} := \begin{cases} 1, & \text{if } \tau\alpha_j \in \Phi^+ \\ 0, & \text{if } \tau\alpha_j \in \Phi^- \end{cases}$$

and $\delta := \delta_\tau := (\delta_n, \ldots, \delta_1)$. Note that these δ_i are exactly those from 4.6. We define

$$\chi_\tau(F(\mu)) := \begin{cases} F(\mu - \delta_\tau), & \text{if } \mu - \delta_\tau \in X_1(T) \\ 0, & \text{if } \mu - \delta_\tau \notin X_1(T) \end{cases}$$

The following is an immediate consequence from the calculations of 4.6 and Proposition 5.4.3:

Corollary 5.4.4. *Assume $G = \mathrm{GL}_4$, and that μ lies sufficiently deep in a p-alcove such that $\tau\mu$ is dominant. Then,*

$$\mathrm{ind}_{\mathcal{H}_\pi^{(1)}}^{\mathcal{H}^{(1)}} \mathrm{inc}[z]_{\mathcal{H}_{aff}^{(1)}}^{\mathcal{H}_\pi^{(1)}} \mathrm{inc}_{\mathcal{H}_0^{(1)}}^{\mathcal{H}_{aff}^{(1)}} (\chi_\tau(\overline{R_w(\mu)}))$$

contains $M_{(\lambda_+(\rho), I_+(\rho), z)}$.

It is however not true that $\chi_\tau(\overline{R_w(\mu)})$ contains the restriction of $M_{(\lambda_+(\rho), I_+(\rho), z)}$. This is shown by the following counterexample, Assume that $\tau = s_2$. The restriction of the associated module contains $F(\lambda)$ with $\lambda = ww_0\mu + (p-1)\rho_{ww_0\tau^{-1}} + \delta - w\delta$ which we will show not be contained in $\overline{R_w(\mu)} = \overline{R_{\tau v \tau^{-1}}(\tau\mu)}$. Assume the contrary:

We have $\delta = (0, 1, 0, 1)$. and $\rho_{ww_0\tau^{-1}} = \omega_{\alpha_2}$ which implies

$$w_0 \cdot (\lambda - p\rho) = (a_2 + 1, a_3 - p, a_4 - p + 2, a_1 - 2p + 1) - \rho \in C_2.$$

By assumption, there exists $\sigma \in W_0$ and $\nu \in X(T)$ such that

$$\sigma \cdot (\tau\mu - \rho + (\tau w \tau^{-1})\nu - p\nu) \uparrow w_0 \cdot (\lambda - p\rho),$$

so the left hand side must be contained in C_0, C_1 or C_2.

In the C_2-case we have to choose σ such that $\sigma\tau\mu = w_0ww_0\mu$, i.e. $\sigma = w_0ww_0\tau^{-1}$, because otherwise, say for example $s_1\sigma = w_0ww_0\tau^{-1}$ we would have for $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$:

$$s_1\sigma \cdot (\tau\mu - \rho + (\tau w \tau^{-1})\nu - p\nu) = w_0 \cdot (\lambda - p\rho),$$

or explicitly

$$(a_3 - p\nu_2 + \nu_3, a_2 - p\nu_3 + \nu_1, a_4 - p\nu_4 + \nu - 2 + 2, a_1 - p\nu_1 + \nu_4) = (a_2 + 1, a_3 - p, a_4 - p + 2, a_1 - 2p + 1).$$

For $a_i << p$ this implies $\nu = (2, 0, 1, 1)$ which implies again $a_3 = a_2 + 1$ which we can avoid by choosing a μ further apart from the boundary of its p-alcove. This way one sees that $\sigma = w_0ww_0\tau^{-1}$ in the C_2-case and similarly one shows that $\sigma = s_{\alpha_1+\alpha_2}w_0ww_0\tau^{-1}$ in the C_1-case and $\sigma = s_{\alpha_1+\alpha_2+\alpha_3}s_{\alpha_1+\alpha_2}w_0ww_0\tau^{-1}$ in the C_0-case. Let us go through these cases:

Case 1: $\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) \in C_2$. In order for $\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu)$ to be dominant, we have to choose $\nu = (2,1,0,1)$ and we obtain

$$\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) = (a_2+2, a_3-p, a_4-(p-1), a_1-2p+1)-\rho \neq w_0\cdot(\lambda-p\rho).$$

Case 2: $\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) \in C_1$. In order for $\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu)$ to be dominant, we have to choose $\nu = (2,1,1,0)$ and we obtain

$$\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) = (a_2+2, a_3-p+1, a_4-(p-1), a_1-2p)-\rho \neq w_0\cdot(\lambda-p\rho).$$

Case 3: $\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) \in C_0$. In order for $\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu)$ to be dominant, we have to choose $\nu = 0$ and we obtain

$$\sigma\cdot(\tau\mu-\rho+(\tau w\tau^{-1})\nu-p\nu) = (a_2+p, a_3, a_4, a_1-p)-\rho \neq w_0\cdot(\lambda-p\rho).$$

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