A geometric description of the Atiyah–Hirzebruch spectral sequence for B-bordism

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Abstract. In this paper we give a geometric description of the general term and the differential of the Atiyah–Hirzebruch spectral sequence for B-bordism. This description is given in terms of bordism classes of maps from stratifolds. We illustrate that with a computational example. We also discuss the case of a general homology theory, where this description is given in terms of the Postnikov sections of the given theory.

1. Introduction

Generalized homology theories play an important role in mathematics. Some important examples of such theories are bordism theories, including stable homotopy. Despite their relatively simple description, computations are, in general, a hard task. One of the main tools for computations is the Atiyah–Hirzebruch spectral sequence (AHSS). As in other spectral sequences, the difficulty comes from computing the general terms $E_{p,q}^r$, which are certain subquotients, and the differentials $d_{p,q}^r$, which are defined using diagram chasing. The geometric nature of the theory is not completely lost in this description, but it is indirect.

In this paper we give an alternative description of the general term and the differential. Our description is geometric and makes use of stratifolds. Stratifolds, defined by Kreck (see [5] and also Section 5), are certain stratified spaces generalizing smooth manifolds. In addition to the top stratum, which is a smooth manifold, stratifolds have a singular part. One can form bordism theories using stratifolds instead of manifolds. If one imposes no restriction on the stratifolds aside of compactness, and possibly a B-structure on their top stratum, one obtains a trivial theory. The reason is the triviality of the coefficients, since every stratifold is the boundary of its cone, which is a stratifold with boundary. To avoid this, one can impose the condition that the

codimension one stratum is empty. Then the resulting theory is an ordinary homology theory.

Another condition one might look at is that all strata of codimension $0 < k < r$ are empty. For a fibration $B \to BO$, denote by $\Omega^{B(r)}$ the bordism theory of compact stratifolds with a B-structure on their top stratum (B-stratifolds), with all strata of codimension $0 < k < r + 2$ empty. For $r = \infty$ we simply write Ω^B. Given a CW complex X (whose k-th skeleton is denoted by X^k), for $r \geq 2$ denote

$$\hat{E}^r_{p,q} = \mathrm{Im}\big(\Omega^{B(q+r-2)}_{p+q}(X^p) \to \Omega^{B(q)}_{p+q}(X^{p+r-1})\big).$$

This is the r-th page of our spectral sequence. Next we describe the differential. Let $\hat{d}^r_{p,q} : \hat{E}^r_{p,q} \to \hat{E}^r_{p-r,q+r-1}$ be the homomorphisms induced by the map

$$[f : \mathcal{S} \to X^p] \mapsto [g \circ f|_{\partial W} : \partial W \to X^{p-1}],$$

where $\mathcal{S}$ is a compact B-stratifold of dimension $p + q$ representing an element in $\Omega^{B(q+r-2)}_{p+q}(X^p)$, W is the top stratum of S and $g : \partial W \to \mathrm{sing}(\mathcal{S})$ is the attaching map which is used for gluing W to $\mathrm{sing}(\mathcal{S})$, the singular part of $\mathcal{S}$. Note that $\dim(\mathrm{sing}(\mathcal{S})) \leq p - r$ so f is homotopic to a map f' with $f'(\mathrm{sing}(\mathcal{S})) \subseteq X^{p-r}$, in particular, the right side is an element in

$$\mathrm{Im}(\Omega^{B(q+2r-3)}_{p+q-1}(X^{p-r}) \to \Omega^{B(q+r-1)}_{p+q-1}(X^{p-1})).$$

We will show that $\hat{d}^r_{p,q}$ is well defined.

There is a natural transformation

$$\varphi : \hat{E}^2_{p,q} \to H_p(X, \Omega^B_q)$$

given as follows: An element α in $\Omega^{B(q)}_{p+q}(X^p)$ can be represented by a map $f : \mathcal{S} \to X^p$ which is smooth on the preimage of the top cells, where $\mathcal{S}$ is a B-stratifold of dimension $p + q$. Then we define $\varphi(\alpha) \in H_p(X, \Omega^B_q)$ to be the homology class represented by the cycle $\Sigma_\alpha([M_\alpha] \cdot c_\alpha)$, where each c_α is a p-cell and M_α is the preimage of a regular value in c_α with the induced B-structure. We will show that it is well defined.

Our main result is the following:

Theorem 1.1. *The pair* $(\hat{E}^r_{p,q}, \hat{d}^r_{p,q})$ *is a spectral sequence and there is a natural isomorphism of spectral sequences*

$$\phi : \hat{E}^r_{p,q} \to E^r_{p,q},$$

where the right-hand side is the standard AHSS as appears in [1]. *The natural transformation* $\varphi : \hat{E}^2_{p,q} \to H_p(X, \Omega^B_q)$ *is an isomorphism commuting with* ϕ *and the isomorphism* $E^2_{p,q} \to H_p(X, \Omega^B_q)$ *given in* [1].

The proof of Theorem 1.1 consists of several steps. The first is an interpretation of the AHSS in terms of the Postnikov tower, which for cohomology was done by Maunder. Given this, the next step is a geometric interpretation of the groups occurring from the Postnikov system. The last step is the relation between the differentials and the proof that the map ϕ commutes with the differentials.

In Section 7 we illustrate that by giving another computation of $\Omega_5^{\mathrm{fr}}(\mathbb{C}P^\infty)$, a result obtained by Liulevicius [8] using the Adams spectral sequence. Here is a simpler example of a computation in framed bordism. Suppose $f : S^{n-1} \to S^{n-k}$ is a smooth map, which represents a nontrivial element in framed bordism, where S^{n-1} has the trivial framing. The mapping cone, C_f, has the structure of a compact framed stratifold of dimension n with two strata, where the framing on the top stratum is the trivial one. For example, when $f : S^3 \to S^2$ is the Hopf map then $C_f \cong \mathbb{C}P^2$. The identity map $\mathrm{id} : C_f \to C_f$ represents an element in

$$\mathrm{Im}\big(\Omega_n^{\mathrm{fr}(r-2)}(C_f) \to \Omega_n^{\mathrm{fr}(0)}(C_f)\big) = \hat{E}_{n,0}^r$$

for $r \leq k$. Its differential is represented by the map $f : S^{n-1} \to S^{n-k}$, considered as an element in

$$\Omega_{n-1}^{\mathrm{fr}(r-1)}(C_f^{n-1}) = \Omega_{n-1}^{\mathrm{fr}(r-1)}(S^{n-k}).$$

When $r < k$ the differential is zero, since the mapping cylinder of f is a permitted null bordism. When $r = k$ this is a nontrivial element. To see this, assume that T is a null bordism. Then the singular part of T is of dimension at most $n-k-1$. By cellular approximation the singular part factors through the constant map, hence the map f is bordant to an element in the coefficients. But this is a contradiction, since this would imply that f is null bordant, since S^{n-1} with its framing is null bordant. By taking $k = 2$, we get that $\hat{d}_{n,0}^2$ is, in general, nontrivial for $n \geq 4$, and by taking $k = 3$, we get that $\hat{d}_{n,0}^3$ is, in general, nontrivial for $n \geq 5$. By analyzing all stable operations, one can see that this uniquely determines those differentials.

Steenrod's problem regarding realization of integral homology classes by maps from closed oriented manifolds [2] was answered negatively by Thom [13]. Replacing manifolds with stratifolds, one might ask the following:

> Given a class in integral homology of a CW complex, what is the minimal dimension of the singular part in a stratifold that represents it?

In other words, how "far" is it from being representable? As a corollary of our main theorem (Corollary 6.3) it follows that this is equivalent to the question of how many steps does the element survive in the AHSS, considered as an element in $E_{p,0}^2$.

Remark 1.2. An alternative, more direct, proof of this fact appears in the Appendix. For this we prove a smooth approximation theorem for stratifolds, which can be useful elsewhere.
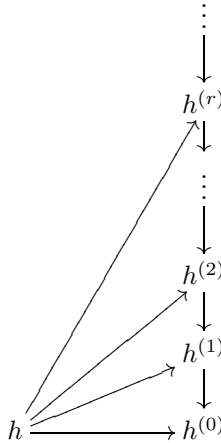
The organization of the paper is as follows: In Section 2 we discuss the basic properties of the Postnikov tower of a homology theory. In Section 3 we discuss things related to the Steenrod realization problem. In Section 4 we compare the AHSS with our spectral sequence. In Section 5 we review stratifolds, stratifold homology, and B-stratifolds. In Section 6 we describe the Postnikov tower of a B-bordism theory using B-stratifolds and prove our

main theorem. In Section 7 we give a computational example. In the Appendix we prove a smooth approximation theorem for maps between stratifolds.

2. The Postnikov tower of a homology theory

Remark 2.1. All spaces are assumed to be CW complexes, and for a CW complex X we denote by X^k its k-th skeleton.

Let h be a representable generalized homology theory. One can construct the Postnikov tower, which is a sequence of homology theories $h^{(r)}$ and natural transformations between them, which fit into the following diagram:

$$
\begin{array}{c}
\vdots \\
\downarrow \\
h^{(r)} \\
\uparrow \downarrow \\
\vdots \\
\downarrow \\
h^{(2)} \\
\downarrow \\
h^{(1)} \\
\downarrow \\
h \Longrightarrow h^{(0)}
\end{array}
$$

so that it has the following properties. The theories $h^{(r)}$ have the property that the map $h_n \to h_n^{(r)}$ is an isomorphism for $n \leq r$, and $h_n^{(r)}$ is trivial for $n > r$ (h_n stands for $h_n(pt)$, the n-th coefficient group). These properties determine $h^{(r)}$ completely. For a proof of existence and uniqueness see [11, Chap. II, Thm. 4.13 and Cor. 4.18].

Example 2.2. If h is a connective homology theory, i.e., $h_n = 0$ for $n < 0$, then $h^{(0)}$ is naturally isomorphic to homology with coefficients in h_0 (for CW complexes). If h is oriented bordism, Ω^{SO}, then $h^{(0)}$ is naturally isomorphic to integral homology. Later on we show that in the case of Ω^{SO}, or other B-bordism theories, the theories $h^{(r)}$ can be given a geometric description using B-stratifolds.

Remark 2.3. It follows from uniqueness that for $r' \geq r$ we have $(h^{(r')})^{(r)} = h^{(r)}$. This implies that in all our constructions and propositions one can replace h by $h^{(r')}$.

By induction, using excision, we get the following:

Lemma 2.4. *Let X be a CW complex. Then $h_n^{(r)}(X^k)$ is trivial if $k + r < n$.*

Again, using excision and induction, one proves the following:

Lemma 2.5. $h_{r+k}^{(r)}(X^k, X^{k-1}) \cong h_r \otimes C_k(X)$, where $C_k(X)$ is the k-th cellular chain group.

This gives a nice description of $h_{r+k}^{(r)}(X^k)$:

Lemma 2.6. *There is a natural isomorphism*
$$h_{r+k}^{(r)}(X^k) \cong \ker\bigl(h_r \otimes C_k(X) \to h_r \otimes C_{k-1}(X)\bigr).$$

Proof. Look at the exact sequence of the triple (X^k, X^{k-1}, X^{k-2}):
$$0 \to h_{r+k}^{(r)}(X^k, X^{k-2}) \to h_{r+k}^{(r)}(X^k, X^{k-1}) \to h_{r+k-1}^{(r)}(X^{k-1}, X^{k-2}).$$
We conclude that
$$h_{r+k}^{(r)}(X^k, X^{k-2}) \cong \ker\bigl(h_r \otimes C_k(X) \to h_r \otimes C_{k-1}(X)\bigr).$$
The lemma follows from the fact that the map $h_{r+k}^{(r)}(X^k) \to h_{r+k}^{(r)}(X^k, X^{k-2})$ is an isomorphism using Lemma 2.4. $\qquad\square$

Proposition 2.7. *Let X be a CW complex, then the map $h_n(X, X^k) \to h_n^{(r)}(X, X^k)$ is an isomorphism if $n \le k + r + 1$.*

Proof. We prove it for finite-dimensional CW complexes by induction on the dimension. This, together with the fact that for (additive) generalized homology theories we have [10]
$$h_*(X) = \mathrm{colim}(h_*(X^m))$$
will imply the statement for the case where X is infinite-dimensional.

The statement is trivial if $\dim(X) \le k$, since then both groups vanish. Assume that the statement is true for every CW complex Y such that $\dim(Y) = m-1 \ge k$, in particular for X^{m-1}. Using the long exact sequence for the triple (X^m, X^{m-1}, X^k), we get the following commutative diagram with exact rows:

$$
\begin{array}{ccc}
h_{n+1}(X^m, X^{m-1}) & \xrightarrow{\;(1)\;} & h_{n+1}^{(r)}(X^m, X^{m-1}) \\[1em]
\downarrow & & \downarrow \\[1em]
h_n(X^{m-1}, X^k) & \xrightarrow{\;(2)\;} & h_n^{(r)}(X^{m-1}, X^k) \\[1em]
\downarrow & & \downarrow \\[1em]
h_n(X^m, X^k) & \xrightarrow{\;(3)\;} & h_n^{(r)}(X^m, X^k) \\[1em]
\downarrow & & \downarrow \\[1em]
h_n(X^m, X^{m-1}) & \xrightarrow{\;(4)\;} & h_n^{(r)}(X^m, X^{m-1}) \\[1em]
\downarrow & & \downarrow \\[1em]
h_{n-1}(X^{m-1}, X^k) & \xrightarrow{\;(5)\;} & h_{n-1}^{(r)}(X^{m-1}, X^k).
\end{array}
$$

The maps (2) and (5) are isomorphisms by our assumption. We have the following commutative diagram, where all horizontal maps are isomorphisms by excision and Mayer–Vietoris:

$$
\begin{array}{ccccccc}
h_j(X^m, X^{m-1}) & \longrightarrow & \oplus h_j(D^m, S^{m-1}) & \longrightarrow & \oplus h_j(S^m, *) & \longrightarrow & \oplus h_{j-m}(S^0, *) \\
\downarrow{\scriptstyle(a)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle(b)} \\
h_j^{(r)}(X^m, X^{m-1}) & \longrightarrow & \oplus h_j^{(r)}(D^m, S^{m-1}) & \longrightarrow & \oplus h_j^{(r)}(S^m, *) & \longrightarrow & \oplus h_{j-m}^{(r)}(S^0, *).
\end{array}
$$

For $j = n$ the map (b) is an isomorphism, since $j - m \leq r$, so the same is true for (a) which is (4). For $j = n + 1$ either $\oplus h_{j-m}^{(r)}(S^0, *) = 0$ (if $n = k + r + 1$ and $m = k+1$), or (b) is an isomorphism. In any case (b) is surjective and the same is true for (a) which is equal to (1). Now we deduce by the five lemma that (3) is an isomorphism.   □

We use this isomorphism to define a natural transformation

$$\Phi : h_n^{(r)}(X) \to h_{n-1}(X^{n-r-1})$$

as the composition

$$h_n^{(r)}(X) \to h_n^{(r)}(X, X^{n-r-1}) \to h_n(X, X^{n-r-1}) \to h_{n-1}(X^{n-r-1}).$$

This is better seen using the following diagram:

$$
(1) \qquad
\begin{array}{ccccc}
h_n(X) & \longrightarrow & h_n(X, X^{n-r-1}) & \longrightarrow & h_{n-1}(X^{n-r-1}) \\
\downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow \\
h_n^{(r)}(X) & \longrightarrow & h_n^{(r)}(X, X^{n-r-1}) & \longrightarrow & h_{n-1}^{(r)}(X^{n-r-1}).
\end{array}
$$

The following might be a useful tool for computation:

Proposition 2.8. *The following sequence is exact:*

$$\cdots \longrightarrow h_n(X^{n-r-1}) \longrightarrow h_n(X) \longrightarrow h_n^{(r)}(X) \xrightarrow{\Phi} h_{n-1}(X^{n-r-1})$$

$$\longrightarrow h_{n-1}^{(r)}(X^{n-r-1}) \oplus h_{n-1}(X) \longrightarrow h_{n-1}^{(r)}(X) \longrightarrow h_{n-2}(X^{n-r-1}) \longrightarrow \cdots$$

(one can prolong the sequence to the left by the sequence of the pair).

Proof. Extend diagram (1) to the right. Note that the homomorphisms

$$h_k(X, X^{n-r-1}) \to h_k^{(r)}(X, X^{n-r-1})$$

are isomorphisms for all $k \leq n$ (Proposition 2.7) and that $h_n^{(r)}(X^{n-r-1})$ is trivial (Lemma 2.4). The exactness is obtained by diagram chase (see also [4, Chap. 2, Ex. 38]).   □

Corollary 2.9. $\mathrm{Im}\big(h_n(X) \to h_n^{(r)}(X)\big) \cong \mathrm{Im}\big(h_n(X) \to h_n(X, X^{n-r-1})\big).$

Proof. This follows from the fact that both kernels are equal to

$$\mathrm{Im}\big(h_n(X^{n-r-1}) \to h_n(X)\big),$$

using the exact sequence in Proposition 2.8 and the sequence for the pair.   □

3. STEENROD REALIZATION PROBLEM

Steenrod's question, which was mentioned in the introduction, can be re-phrased as asking whether the map $\Omega_n^{SO}(X) \to H_n(X)$ is surjective (here n is arbitrary and the homology is with integral coefficients). Note that this map is the localization map, using the identification $H_n \cong (\Omega^{SO})_n^{(0)}$. Using the exact sequence in Proposition 2.8, we see that this question is equivalent to the question whether the following map is injective:

$$\Omega_{n-1}^{SO}(X^{n-1}) \to H_{n-1}(X^{n-1}) \oplus \Omega_{n-1}^{SO}(X).$$

By exactness,

$$\ker\big(\Omega_{n-1}^{SO}(X^{n-1}) \to \Omega_{n-1}^{SO}(X)\big) = \mathrm{Im}\big(\Omega_n^{SO}(X, X^{n-1}) \to \Omega_{n-1}^{SO}(X^{n-1})\big).$$

By cellular approximation, the map $\Omega_n^{SO}(X^n, X^{n-1}) \to \Omega_n^{SO}(X, X^{n-1})$ is surjective, so

$$\ker\big(\Omega_{n-1}^{SO}(X^{n-1}) \to \Omega_{n-1}^{SO}(X)\big) = \mathrm{Im}\big(\Omega_n^{SO}(X^n, X^{n-1}) \to \Omega_{n-1}^{SO}(X^{n-1})\big).$$

The group $\Omega_n^{SO}(X^n, X^{n-1})$ is generated by the n-cells, so the right-hand side is generated by the attaching maps of the n-cells, hence all elements are spherical.

We deduce that Steenrod's problem (in dimension $n+1$) is equivalent to the following question.

Given a CW complex X, does the equality

$$\ker\big(\pi_n(X^n) \to \Omega_n^{SO}(X^n)\big) = \ker\big(\pi_n(X^n) \to H_n(X^n)\big)$$

hold?

Remark 3.1. The equivalence is in the sense that every element which belongs to the right side but not to the left side corresponds to a non-representable class in some X with the given n skeleton.

Let X be a CW complex and $[f : M \to X]$ an element in $\Omega_*^{SO}(X)$. The classifying map for the stable tangent bundle of M induces a map

$$\Omega_*^{SO}(X) \to H_*(X \times BSO).$$

This map is known to be a rational isomorphism (see, e.g., [7, Thm. 18.51]).

Taking X to be a point, this map is injective, i.e., one can detect the cobordism class of M by its image. It would be nice if this was true for every space X. Unfortunately, this is not the case:

Corollary 3.2. *The map* $\Omega_*^{SO}(X) \to H_*(X \times BSO)$ *need not be injective.*

Proof. Since we know that the answer to the Steenrod problem is negative, there is a space X and an integer n such that

$$\ker\big(\pi_n(X^n) \to \Omega_n^{SO}(X^n)\big) \subsetneq \ker\big(\pi_n(X^n) \to H_n(X^n)\big),$$

i.e., there is a strict inclusion. Let $[f : S^n \to X]$ be an element on the right side but not on the left side. Since the tangent bundle of a sphere is stably trivial, the map $S^n \to X \times BSO$ factors through $X \times *$, hence the image of $[S^n \to X]$ in $H_*(X \times BSO)$ is zero.   □

4. THE AHSS IN TERMS OF THE POSTNIKOV TOWER

For $r \geq 2$ let

$$\hat{E}_{p,q}^r = \mathrm{Im}\big(h_{p+q}^{(q+r-2)}(X^p) \to h_{p+q}^{(q)}(X^{p+r-1})\big),$$

and let $\hat{d}_{p,q}^r : \hat{E}_{p,q}^r \to \hat{E}_{p-r,q+r-1}^r$ be the homomorphism induced by the following diagram:

$$
\begin{array}{ccc}
h_{p+q}^{(q+r-2)}(X^p) & \longrightarrow & h_{p+q}^{(q)}(X^{p+r-1}) \\
\downarrow{\scriptstyle\Phi} & & \downarrow{\scriptstyle\Phi} \\
h_{p+q-1}^{(q+2r-3)}(X^{p-r}) \longrightarrow h_{p+q-1}^{(q+2r-3)}(X^{p-r+1}) & \longrightarrow & h_{p+q-1}^{(q+r-1)}(X^{p-1})
\end{array}
$$

where $\hat{E}_{p,q}^r$ is the image of the top row and $\hat{E}_{p-r,q+r-1}^r$ is the image of the composition of the bottom row. We would like to compare the pair $(\hat{E}_{p,q}^r, \hat{d}_{p,q}^r)$ with the AHSS. Maunder in [9] shows that the cohomological AHSS is naturally isomorphic to the spectral sequence given by the Postnikov filtration of the cohomology theory. In [9, §4.4] he gives a description of the general term. This can also be done in the case of homology, and one obtains the same groups as here. Maunder proves this by constructing an isomorphism between the exact couples. This way the isomorphism between the general terms is indirect, which makes it harder to identify the differentials in our case. Therefore, we give a construction of the natural isomorphism, and use it to show that the differentials agree.

Theorem 4.1. *The pair $(\hat{E}_{p,q}^r, \hat{d}_{p,q}^r)$ is a spectral sequence and there is a natural isomorphism of spectral sequences $\hat{E}_{p,q}^r \to E_{p,q}^r$, where on the right side we have the standard AHSS obtained by the exact couple.*

We start by proving the following lemmas:

Lemma 4.2. *Let X be a CW complex. Then the following maps are surjective:*
  (i) $h_n^{(r)}(X) \to h_n^{(r)}(X, X^{n-r-2})$,
  (ii) $h_n(X, X^{n-r-2}) \to h_n^{(r)}(X, X^{n-r-2})$.

Proof. (i) This follows from the long exact sequence for the pair and Lemma 2.4.

(ii) If we use the isomorphism $h_n(X, X^{n-r-2}) \to h_n^{(r+1)}(X, X^{n-r-2})$ given in Proposition 2.7, it is enough to show that the map $h_n^{(r+1)}(X, X^{n-r-2}) \to h_n^{(r)}(X, X^{n-r-2})$ is surjective. We look at the following diagram:

$$
\begin{array}{ccccc}
h_n^{(r+1)}(X, X^{n-r-2}) & \longrightarrow & h_n^{(r+1)}(X, X^{n-r-1}) & \longrightarrow & h_{n-1}^{(r+1)}(X^{n-r-1}, X^{n-r-2}) \\
\downarrow{\scriptstyle(1)} & & \downarrow{\scriptstyle(2)} & & \downarrow{\scriptstyle(3)} \\
h_n^{(r)}(X, X^{n-r-2}) & \xrightarrow{\ (4)\ } & h_n^{(r)}(X, X^{n-r-1}) & \longrightarrow & h_{n-1}^{(r)}(X^{n-r-1}, X^{n-r-2}).
\end{array}
$$

Here (2) and (3) are isomorphisms by Proposition 2.7, and (4) is injective since $h_n^{(r)}(X^{n-r-1}, X^{n-r-2})$ is trivial by Lemma 2.4. Now the lemma follows by a diagram chase. $\qquad\square$

Lemma 4.3. *Suppose we are given the following diagram:*

$$
\begin{array}{ccccc}
A' & & & & \\
\uparrow & \searrow{\scriptstyle f'} & & & \\
A & \xrightarrow{\ f\ } & B & \xrightarrow{\ g\ } & C
\end{array}
$$

If the bottom row is exact in B then $\operatorname{Im}(f')/\operatorname{Im}(f) \cong \operatorname{Im}(g \circ f')$.

Proof. We have

$$
\begin{aligned}
\operatorname{Im}(g \circ f') &\cong \operatorname{Im}(f')/\ker(g) \cap \operatorname{Im}(f') \\
&= \operatorname{Im}(f')/\operatorname{Im}(f) \cap \operatorname{Im}(f') = \operatorname{Im}(f')/\operatorname{Im}(f). \qquad\square
\end{aligned}
$$

Proof of Theorem 4.1. By [1, Chap. I, §7] we have the following natural isomorphism:

$$
E_{p,q}^r = \frac{\operatorname{Im}\big(h_{p+q}(X^p, X^{p-r}) \to h_{p+q}(X^p, X^{p-1})\big)}{\operatorname{Im}\big(h_{p+q+1}(X^{p+r-1}, X^p) \to h_{p+q}(X^p, X^{p-1})\big)}.
$$

We use Lemma 4.3 for the following diagram:

$$
\begin{array}{ccccc}
h_{p+q}(X^p, X^{p-r}) & & & & \\
\uparrow & \searrow & & {\scriptstyle f_1} & \\
h_{p+q+1}(X^{p+r-1}, X^p) & \longrightarrow & h_{p+q}(X^p, X^{p-1}) & \longrightarrow & h_{p+q}(X^{p+r-1}, X^{p-1}),
\end{array}
$$

where f_1 is the composition, to conclude that

$$
E_{p,q}^r \cong \operatorname{Im}(f_1).
$$

In order to simplify this expression, look at the following diagram:

$$\begin{array}{ccc}
h_{p+q}(X^p, X^{p-r}) \xrightarrow{\quad} h_{p+q}(X^{p+r-1}, X^{p-r}) \xrightarrow{\quad} h^{(q)}_{p+q}(X^{p+r-1}, X^{p-r}) \\
\downarrow \qquad \searrow^{f_1} \qquad \downarrow \qquad \qquad \downarrow^{(1)} \\
h_{p+q}(X^p, X^{p-1}) \xrightarrow{\quad} h_{p+q}(X^{p+r-1}, X^{p-1}) \xrightarrow{(2)} h^{(q)}_{p+q}(X^{p+r-1}, X^{p-1}),
\end{array}$$

where f_2 is the composition.

The homomorphism (1) is injective by the long exact sequence for the triple $(X^{p+r-1}, X^{p-1}, X^{p-r})$ and Lemma 2.4; (2) is an isomorphism by Proposition 2.7.

From (1) and (2) it follows that $E^r_{p,q} \cong \mathrm{Im}(f_1) \cong \mathrm{Im}(f_2)$. Look at the following diagram:

$$\begin{array}{ccc}
h^{(q+r-2)}_{p+q}(X^p) & \xrightarrow{\quad} & h^{(q)}_{p+q}(X^{p+r-1}) \\
\downarrow^{(2)} & & \downarrow^{(3)} \\
h_{p+q}(X^p, X^{p-r}) \xrightarrow{(1)} h^{(q+r-2)}_{p+q}(X^p, X^{p-r}) & \xrightarrow{\quad} & h^{(q)}_{p+q}(X^{p+r-1}, X^{p-r}).
\end{array}$$

Here, (1) and (2) are surjective by Lemma 4.2; (3) is an isomorphism by the long exact sequence for the pair and Lemma 2.4.

This implies that $\mathrm{Im}(f_2) \cong \hat{E}^r_{p,q}$. The fact that this isomorphism commutes with the differential follows from diagram chasing. $\qquad \square$

4.4. **The $\hat{E}^2$ page.** The case $r = 2$ has the following form:

$$\hat{E}^2_{p,q} = \mathrm{Im}\big(h^{(q)}_{p+q}(X^p) \to h^{(q)}_{p+q}(X^{p+1})\big).$$

Note that by the long exact sequence for the pair, we have

$$\mathrm{Im}\big(h^{(q)}_{p+q}(X^p) \to h^{(q)}_{p+q}(X^{p+1})\big)$$
$$\cong h^{(q)}_{p+q}(X^p)/\mathrm{Im}\big(h^{(q)}_{p+q+1}(X^{p+1}, X^p) \to h^{(q)}_{p+q}(X^p)\big),$$

and by Lemmas 2.5 and 2.6 this is naturally isomorphic to

$$\ker\big(h_q \otimes C_p(X) \to h_q \otimes C_{p-1}(X)\big)/\mathrm{Im}\big(h_q \otimes C_{p+1}(X) \to h_q \otimes C_p(X)\big),$$

which is by definition $H_p(X, h_q)$, as we know from the standard presentation of the AHSS. It is not hard to see this is compatible with the isomorphism $E^2_{p,q} \to H_p(X, \Omega^B_q)$ which appears in [1].

5. STRATIFOLDS, STRATIFOLD HOMOLOGY, AND GENERALIZATION TO B-STRATIFOLD THEORIES

Stratifolds are a generalization of manifolds. They were introduced by Kreck [5] and used in order to define a bordism theory, denoted by SH_*, which is naturally isomorphic to singular homology for CW complexes. Kreck also

defined a cohomology theory using stratifolds which is defined on the category of smooth oriented manifolds (without boundary but not necessarily compact). It is denoted by SH^* and is naturally isomorphic to singular cohomology.

5.1. Stratifolds and B-stratifolds.

Kreck defined stratifolds as spaces with a certain sheaf of functions, called the smooth functions, fulfilling certain properties, but for our purpose the following definition is enough (these stratifolds are also called p-stratifolds).

Stratifolds are constructed inductively in a similar way to the way we construct CW complexes. We start with a discrete set of points denoted by X^0 and define inductively the set of smooth functions, which in the case of X^0 are all real functions.

Suppose X^{n-1} together with a smooth set of functions is given. Let W be an n-dimensional smooth manifold, "the n stratum" with boundary and a collar c, and f a continuous map from the boundary of W to X^{n-1}. We require that f will be proper and smooth, which means that its composition with every smooth map from X^{n-1} is smooth. Define $X^n = X^{n-1} \cup_f W$. The smooth maps on X^n are defined to be those maps $g : X^n \to \mathbb{R}$ which are smooth when restricted to X^{n-1} and to W and such that for some $0 < \delta$ we have $gc(x, t) = gf(x)$ for all $x \in \partial W$ and $t < \delta$.

Among the examples of stratifolds are smooth manifolds, real algebraic varieties [3], and the one-point compactification of a smooth manifold (which is the interior of a manifold with boundary). The cone over a stratifold and the product of two stratifolds are again stratifolds.

We can also define stratifolds with boundary, which are analogous to manifolds with boundary. A main difference is that every stratifold is the boundary of its cone, which is a stratifold with boundary.

Given two stratifolds with boundary (T', S') and (T'', S'') and an isomorphism $f : S' \to S''$, there is a well-defined stratifold structure on the space $T' \cup_f T''$, that is called the gluing. On the other hand, given a smooth map $g : T \to \mathbb{R}$ such that there is a neighborhood of 0 which consists only of regular values then the preimages $g^{-1}((-\infty, 0]) = T'$ and $g^{-1}([0, \infty)) = T''$ are stratifolds with boundary, and T is isomorphic to the gluing $T' \cup_{\mathrm{id}} T''$.

To obtain singular homology we specialize our stratifolds in the following way: we use compact stratifolds, require that their top stratum will be oriented and the codimension one stratum will be empty.

One can generalize this in several ways. One way will be to require some B-structure on the top stratum, like a spin or a string structure. We will call such stratifolds B-stratifolds. Another way will be to restrict the allowed strata. In the next sections we will discuss some of these generalizations.

Remark 5.2. Regularity, a condition that is often required, is not needed here as was noted by Kreck in his preprint [6].

5.3. Stratifold homology.

Stratifold homology was defined by Kreck in [5]. We will describe here a variant of this theory called parametrized stratifold

homology, which is naturally isomorphic to it for CW complexes. In this paper we will refer to parametrized stratifold homology just as stratifold homology and use the same notation for it.

Definition 5.4. Let X be a topological space and $n \geq 0$, define $SH_n(X)$ to be $\{g : S \to X\}/ \sim$, i.e., bordism classes of maps $g : S \to X$, where S is a compact oriented stratifold of dimension n and g is a continuous map. We often denote the class $[g : S \to X]$ by $[S, g]$ or by $[S \to X]$. $SH_n(X)$ has a natural structure of an abelian group, where addition is given by disjoint union of maps and the inverse is given by reversing the orientation. If $f : X \to Y$ is a continuous map, then the induced map $f_* : SH_n(X) \to SH_n(Y)$ is given by composition.

Kreck has constructed a boundary operator and proved the following theorem (see [5, Chap. 5]).

Theorem 5.5 (Mayer–Vietoris)**.** *The following sequence is exact:*

$$\cdots \longrightarrow SH_n(U \cap V) \longrightarrow SH_n(U) \oplus SH_n(V) \longrightarrow SH_n(U \cup V)$$

$$\stackrel{\partial}{\longrightarrow} SH_{n-1}(U \cap V) \longrightarrow \cdots ,$$

where the first map is induced by inclusions and the second is the difference of the maps induced by inclusions.

SH_* with the boundary operator is a homology theory. Its main property is the following (see [5, Thm. 20.1]).

Theorem 5.6. *There is a natural isomorphism of homology theories*

$$\varphi : SH_* \to H_*.$$

The isomorphism φ is given by $\varphi_n([S, f]) = f_*([S])$, where $[S] \in H_n(S, \mathbb{Z})$ is the fundamental class of S.

Remark 5.7. One can replace stratifolds with B-stratifolds and all the construction will still work. One only needs to know how to give a B-structure to a boundary and to see that gluing along a boundary gives a B-structure. We denote these theories by SH_*^B and call them B-stratifold homology.

6. The Postnikov tower of a B-bordism theory

Define the following sequence of variants of B-stratifold homology:

$$\Omega_p^{B(k)}(X) = [f : S \to X]/ \sim,$$

where S is a compact B-stratifold which has empty strata in codimension strictly less than $k+2$ or, equivalently, its singular part is of dimension at most $p - k - 2$. The same condition for the codimension must hold for the bordism relation. Note that $\Omega_p^{B(0)}(X) = SH_p^B(X)$. In case $B = SO$ we omit the B from the notation. Clearly, there are the following natural transformations:

$$\Omega_p^B(X) \to \cdots \to \Omega_p^{B(2)}(X) \to \Omega_p^{B(1)}(X) \to \Omega_p^{B(0)}(X) = SH_p^B(X).$$

The map $\Omega_p^B \to \Omega_p^{B(r)}$ is an isomorphism for $p \le r$ since in this range the singular part of the cycles and bordisms in $\Omega_n^{B(r)}(pt)$ must be empty. $\Omega_p^{B(r)}$ is trivial for $p > r$ since then the cone of a cycle is an allowed null bordism. This proves the following proposition.

Proposition 6.1. $(\Omega^B)^{(r)} \cong \Omega^{B(r)}$. In particular, $SH^B \cong H(-, \Omega_0^B)$.

We use this geometric description of $(\Omega^B)^{(r)}$ in order to describe the general term in the AHSS for Ω^B: Recall our definition

$$\hat{E}_{p,q}^r = \mathrm{Im}\big(\Omega_{p+q}^{B(q+r-2)}(X^p) \to \Omega_{p+q}^{B(q)}(X^{p+r-1})\big)$$

and the differentials $\hat{d}_{p,q}^r : \hat{E}_{p,q}^r \to \hat{E}_{p-r,q+r-1}^r$ given by

$$[f : \mathcal{S} \to X^p] \mapsto [g \circ f|_{\partial W} : \partial W \to X^{p-1}],$$

as noted in the introduction. We prove our main theorem.

Proof of Theorem 1.1. The identification of the spectral sequences follows from Theorem 4.1 and Proposition 6.1. The differential $d_{p,q}^r$ is induced by the natural transformation

$$\Phi : \Omega_n^{B(r)}(X) \to \Omega_{n-1}^B(X^{n-r-1}),$$

which was discussed in Section 2. A direct translation of this natural transformation gives our differential $\hat{d}_{p,q}^r$ (This is easily seen by the fact that the inverse of the isomorphism

$$\Omega_n^B(X, X^{n-r-1}) \to \Omega_n^{B(r)}(X, X^{n-r-1})$$

is given by restriction to the top stratum). Hence the isomorphism commutes with the differentials. This implies that the pair $(\hat{E}_{p,q}^r, \hat{d}_{p,q}^r)$ is a spectral sequence. The identification of the second page follows from the identification in the general case appearing in Section 2. $\square$

We can compare this description of the differential to other known descriptions. One example is the d^2 differential in spin bordism in the rows $q = 0, 1$ (see [12, Lem. 2.3.2]):

$$d_{p,0}^2 : H_p(X, \mathbb{Z}) \to H_{p-2}(X, \mathbb{Z}/2)$$

is given by reduction mod 2 composed with the dual of sq^2;

$$d_{p,1}^2 : H_p(X, \mathbb{Z}/2) \to H_{p-2}(X, \mathbb{Z}/2)$$

is given by the dual of sq^2.

Let us describe $\hat{d}_{p,0}^2$. An element $\alpha \in H_p(X, \mathbb{Z})$ can be represented (uniquely, up to bordism) by a map from a compact p-dimensional spin stratifold $[S \to X]$. The differential is given by the restriction to the boundary of the top stratum, say $M^{p-1} \to X$. Since this map factors through the singular part of S, we can assume that the image is contained in X^{p-2}. Composition with the collapse map

$$X^{p-2} \to X^{p-2}/X^{p-3} \cong \bigvee S^{p-2}$$

gives a 1-cycle since a map $M^{p-1} \to S^{p-2}$ gives an element in $\Omega_1^{\text{Spin}} \cong \mathbb{Z}/2$. This gives a nice geometric description to the dual of sq^2, at least for classes which are in the image of the reduction map from integral homology.

6.2. **The filtration in homology.** When $q = 0$, we have that

$$\hat{E}_{p,0}^r = \text{Im}\big(\Omega_p^{B(r-2)}(X^p) \to SH_p^B(X^{p+r-1})\big).$$

When $r \geq 2$, this is equal to

$$\hat{E}_{p,0}^r = \text{Im}\big(\Omega_p^{B(r-2)}(X) \to SH_p^B(X)\big).$$

Assume that $\Omega_0^B \cong \mathbb{Z}$ so $SH_p^B(X) \cong H_p(X; \mathbb{Z})$, which is often the case, then we conclude:

Corollary 6.3. *For a CW complex X, the filtration in singular homology given by the AHSS for B-bordism*

$$E_{p,0}^\infty \subseteq \cdots \subseteq E_{p,0}^4 \subseteq E_{p,0}^3 \subseteq E_{p,0}^2 \cong H_p(X, \mathbb{Z}),$$

agrees with the filtration given by all classes in homology that are represented by maps from B-stratifolds with singular part of dimension at most $p - r - 2$.

In the Appendix we give a more straight-forward proof of this fact, see Corollary A.5.

7. AN EXAMPLE

To illustrate our description, we compute the group $\Omega_5^{\text{fr}}(\mathbb{C}P^\infty)$. This group was first computed by Liulevicius in [8], using the Adams spectral sequence. We start by computing

$$\hat{d}_{4,1}^2 : \hat{E}_{4,1}^2 \to \hat{E}_{2,2}^2.$$

Using Lemma 2.6, we have

$$\hat{E}_{4,1}^2 = \Omega_5^{\text{fr}(1)}(\mathbb{C}P^2) \cong h_1 \otimes C_4(\mathbb{C}P^\infty) \cong \mathbb{Z}/2,$$
$$\hat{E}_{2,2}^2 = \Omega_4^{\text{fr}(2)}(\mathbb{C}P^1) \cong h_2 \otimes C_2(\mathbb{C}P^\infty) \cong \mathbb{Z}/2.$$

We describe the generator of $\Omega_5^{\text{fr}(1)}(\mathbb{C}P^2)$. Let $\mathcal{S} = D^4 \times S^1 \cup_g \mathbb{C}P^1$, where D^4 has the standard framing, S^1 has the Lie group framing, and the gluing map $g : S^3 \times S^1 \to \mathbb{C}P^1$ is given by first projecting on S^3 and then the Hopf map. $\mathcal{S}$ is a framed stratifold of dimension 5 with a singular part of dimension 2. The map $f : \mathcal{S} \to \mathbb{C}P^2$ is given as follows: $f : D^4 \times S^1 \to D^4$ is the projection and $f : \mathbb{C}P^1 \to \mathbb{C}P^1$ is the identity. $(\mathcal{S}, f)$ is an allowed cycle in $\Omega_5^{\text{fr}(1)}(\mathbb{C}P^2)$ since the singularity is of codimension 3. The class $\hat{d}_{4,1}^2([\mathcal{S}, f])$ is given by $[S^3 \times S^1 \to \mathbb{C}P^1]$. This class is nontrivial since it is a fiber bundle where all fibers are non-null bordant, and hence $\hat{d}_{4,1}^2$ is an isomorphism.

Next, we compute the differentials coming out of $\hat{E}_{6,0}^r = \Omega_6^{\text{fr}(0)}(\mathbb{C}P^3) \cong \mathbb{Z}$, by constructing a generator for this group. Consider the fiber bundle

$$\mathbb{C}P^1 \to \mathbb{C}P^3 \xrightarrow{p} S^4.$$

Set $\mathcal{S} = \mathbb{C}P^3$ as a framed stratifold the following way: Choose a point $x \in S^4$ and set $\mathbb{C}P^1 \cong p^{-1}(x)$ to be the singular part of $\mathcal{S}$. Set $D = p^{-1}(S^4 \setminus \{x\})$

to be the top stratum of $\mathcal{S}$ together with the framing obtained from $S^4 \setminus \{x\}$ and the trivial framing of the fiber $\mathbb{C}P^1$. Note that by choosing a small disk centered at x, the preimage of its complement is a framed manifold with boundary ∂ and we can consider $\mathcal{S}$ as the gluing of this manifold to $\mathbb{C}P^1$ along the projection map $g : \partial \to \mathbb{C}P^1$. The identity map $f : \mathcal{S} \to \mathbb{C}P^3$ is a generator of $\Omega_6^{\mathrm{fr}(0)}(\mathbb{C}P^3)$. The fact that the singular part of $\mathcal{S}$ is of dimension 2 implies that this generator is mapped to zero by $\hat{d}_{6,0}^2$ and $\hat{d}_{6,0}^3$ (see Corollary 6.3). We are left with computing $\hat{d}_{6,0}^4$. This differential is induced by the following diagram:

$$
\begin{array}{ccc}
\Omega_6^{\mathrm{fr}(2)}(\mathbb{C}P^3) & \longrightarrow & \Omega_6^{\mathrm{fr}(0)}(\mathbb{C}P^4) \cong \mathbb{Z} \\
\downarrow & & \downarrow \\
\mathbb{Z}/24 \cong \Omega_5^{\mathrm{fr}(5)}(\mathbb{C}P^1) & \longrightarrow & \Omega_5^{\mathrm{fr}(3)}(\mathbb{C}P^2).
\end{array}
$$

Note that the only differential entering the $(2,3)$ position before the fourth page is coming from

$$
\hat{E}_{4,2}^2 = \Omega_6^{\mathrm{fr}(2)}(\mathbb{C}P^2) \cong \mathbb{Z}/2.
$$

Hence, the kernel of the bottom map in the diagram is either trivial or $12\mathbb{Z}/24$. We will show that if we lift a generator of $\Omega_6^{\mathrm{fr}(0)}(\mathbb{C}P^4)$ to $\Omega_6^{\mathrm{fr}(2)}(\mathbb{C}P^3)$ and then map it to $\Omega_5^{\mathrm{fr}(5)}(\mathbb{C}P^1)$, we get twice the generator. (Note that $\Omega_5^{\mathrm{fr}(5)}(\mathbb{C}P^1) \cong \Omega_5^{\mathrm{fr}}(\mathbb{C}P^1)$.) This will imply that $\hat{E}_{2,3}^5 \cong \mathbb{Z}/2$. Taking $[\mathcal{S} \to \mathbb{C}P^3]$ as a generator, the element in $\Omega_5^{\mathrm{fr}}(\mathbb{C}P^1)$ is given by the gluing map $[g : \partial \to \mathbb{C}P^1]$. Note that g is a (trivial) bundle whose fiber is S^3, so it is enough to analyze the framing on the fiber induced by the framing on ∂, which itself is induced by the framing on the preimage of the complement of a disk around x. It is well known that the bundle

$$
\mathbb{C}P^1 \to \mathbb{C}P^3 \xrightarrow{p} S^4
$$

is linear and it is obtained by the clutching function given by the generator in $\pi_3(SO(3))$, which is mapped to twice the generator in $\pi_3(SO)$. Thus the induced framing on the fiber is twice the generator in Ω_3^{fr}, and we are done.

Appendix A. Smooth approximation of maps between stratifolds

We give here a more direct proof of Corollary 6.3. The main tool is the approximation theorem.

Lemma A.1. *Every continuous map $f : M \to S$, where M is a manifold with boundary ∂M, S is a stratifold, and f is smooth when restricted to ∂M, is homotopic to a smooth map relative boundary.*

Proof. We prove it by induction on the dimension of S. This is clear if S is 0-dimensional, since then S is discrete, so f is locally constant. Assume that the assertion holds when S is of dimension smaller than n, and let $f : M \to S$ be a continuous map, where S is of dimension n.

By using the collar of ∂M, we can assume that f is smooth on a neighborhood of ∂M.

By construction, S is obtained from Σ, the singular part, and a manifold N of dimension n with a boundary ∂N and an (open) collar. Denote by U the singular part together with the collar, and by V the interior of N, that is $N \setminus \partial N$. This is an open cover of S, denote $f^{-1}(U) = U'$ and $f^{-1}(V) = V'$, then this is an open cover of M. We can choose a smooth function $g : M \to \mathbb{R}$ such that $g|_{U' \setminus V'} = 0$ and $g|_{V' \setminus U'} = 1$, and a regular value both of g and of $g|_{\partial M}$, say 0.5. Denote by P its preimage. Then P is a manifold with a boundary, denoted by ∂P. By a standard approximation argument one can show that f is homotopic relative boundary to a map $\tilde{f} : M \to S$ with the following properties:

 (i) $\tilde{f}$ is smooth in a neighborhood of $\partial M \cup g^{-1}([0.5, 1])$;
 (ii) $g^{-1}([0.5, 1])$ is mapped into V;
(iii) $g^{-1}([0, 0.5])$ is mapped into U.

We are left with smoothing $g^{-1}([0, 0.5))$. We can find a manifold M' of dimension n with boundary $\partial M'$, embedded in $g^{-1}([0, 0.5))$ such that $\tilde{f}$ is smooth outside of M' and in a neighborhood of $\partial M'$. The manifold M' is mapped to U, which is smoothly homotopy equivalent to Σ. This implies that it is homotopic to a new map, which is smooth outside of M' and in a neighborhood of its boundary, and that the image of M' is contained in Σ. Now we use the inductive step to smooth this map using the fact that Σ is of dimension $< n$. $\qquad\square$

Proposition A.2. *Every continuous map $f : S \to S'$, where S and S' are stratifolds, is homotopic to a smooth map.*

Proof. This is proved by induction on the dimension of S using Lemma A.1. $\quad\square$

Proposition A.3. *Any locally finite, finite-dimensional countable CW complex is homotopy equivalent to a stratifold with the same set of cells.*

Proof. This is proved by induction on the dimension using the fact that the attaching maps can be made smooth using Lemma A.1. $\qquad\square$

Lemma A.4. *Let X^d be a CW complex of dimension d and M^m a closed oriented manifold of dimension $m > d$ together with a map $f : M^m \to X^d$. The element (M^m, f) represents the zero element in $SH_m(X^d)$, and there exists a null bordism, which has singular part of dimension $\leq d$.*

Proof. The first assertion is clear since $SH_m(X^d) = 0$. First assume that X^d is a compact stratifold and f is smooth. In this case the null bordism can be taken to be the mapping cylinder $\text{Cyl}(f)$ whose boundary is M^m and whose singular part is X^d. The general case follows from Proposition A.3 since the image of f is contained in some finite subcomplex that is homotopy equivalent to a stratifold. $\qquad\square$

Corollary A.5. *The filtrations given by the images*

$$\Omega_p(X) \to \cdots \to \Omega_p^{(2)}(X) \to \Omega_p^{(1)}(X) \to \Omega_p^{(0)}(X) \cong H_p(X, \mathbb{Z})$$

and

$$\Omega_p(X) \to \cdots \to \Omega_p(X, X^{p-3}) \to \Omega_p(X, X^{p-2}) \to H_p(X, X^{p-2}) \cong H_p(X, \mathbb{Z})$$

are equal.

Proof. Given $[S, f] \in \Omega_p^{(k)}(X)$ we can assume by cellular approximation that S^{p-k-2}, the $p-k-2$ skeleton of S, is mapped to X^{p-k-2}, the $p-k-2$ skeleton of X. Denote by $(N^p, \partial N^p)$ the manifold we used to get the top stratum of S. Then there is an induced map

$$(N^p, \partial N^p) \xrightarrow{f} (X, X^{p-k-2}).$$

By the definition of the bordism relation, we have

$$[(S, \varnothing), f] = [(N^p, \partial N^p), f] \in \Omega_p(X, X^{p-k-2}).$$

Let $[(N^p, \partial N^p), f] \in \Omega_p(X, X^{p-k-2})$ be any element. By Lemma A.4, $(\partial N^p, f|_{\partial N^p})$ represents the zero element in $SH_m(X^{p-k-2})$, and there exists a null bordism, $(S, \partial N^p)$, which has singular part of dimension $\leq p - k - 2$. Gluing $(N^p, \partial N^p)$ and $(S, \partial N^p)$, we get a stratifold $N^p \cup_{\partial N^p} S$ of dimension p with a singular part of dimension $\leq p - k - 2$ mapped to X, hence an element in $\Omega_p^{(k)}(X)$, such that its image in $SH_p(X, X^{p-k-2})$ equals the image of $[(N^p, \partial N^p), f]$. $\qquad\square$

Lemma A.4 and its proof have another nice corollary:

Corollary A.6. *Let X be a CW complex having cells only in even dimensions. Every homology class can be represented by a map from a stratifold having only even dimensional strata. This can be restated in the following way: let SH_*^{even} be the bordism theory of stratifolds with strata only in even codimensions. Then the map $SH_*^{\mathrm{even}}(X) \to SH_*(X)$ is surjective.*

Proof. In this case the odd dimensional homology groups are trivial, so we may only look at even dimensional ones. Given $\alpha \in SH_{2k}(X)$, we represent it using a stratifold $[S, f]$, and like we did before, we can replace the singular part of S with the $2k - 2$ skeleton of X without changing the homology class. $\qquad\square$

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