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Eulerian polynomials of spherical type

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Abstract. The Eulerian polynomial of a finite Coxeter system (W, S) of rank n records, for each $k \in \{1, \ldots, n\}$, the number of elements $w \in W$ with an ascent set $\{s \in S \mid l(ws) > l(w)\}$ of size k, where $l(w)$ denotes the length of w with respect to S. The classical Eulerian polynomial occurs when the Coxeter group has type A_n, so W is the symmetric group on $n+1$ letters. Victor Reiner gave a formula for arbitrary Eulerian polynomials and showed how to compute them in the classical cases. In this note, we compute the Eulerian polynomial for any spherical type.

Let M be a Coxeter matrix of rank n. This means M is a symmetric $n \times n$ matrix with entries in $\mathbb{N}$ such that $M_{ii} = 1$ and $M_{ij} > 1$ if $i \neq j$. We also refer to M as a diagram, that is, an edge-labeled graph with nodes $\{1, \ldots, n\}$ and edge $\{i, j\}$ labeled M_{ij} whenever $M_{ij} > 2$. Our setting will involve Coxeter groups as introduced in [3]. Accordingly, we let (W, S) be a Coxeter system of type M. Then $\{1, \ldots, n\}$ and the set $S = \{s_1, \ldots, s_n\}$ of simple reflections are in bijective correspondence and we will often identify the two, so S can be viewed as the set of nodes of M. We also write $W(M)$ instead of W to record the dependence on M. If $W(M)$ is finite, then M is called spherical. The connected spherical diagrams M are A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$, E_n $(n = 6, 7, 8)$, F_4, G_2, H_n $(n = 3, 4)$, and $\mathrm{I}_2^{(m)}$ $(m \geq 3)$. The double occurrences in this list are $\mathrm{A}_2 = \mathrm{I}_2^{(3)}$, $\mathrm{B}_2 = \mathrm{I}_2^{(4)}$, and $\mathrm{G}_2 = \mathrm{I}_2^{(6)}$. The nodes of these diagrams are labeled as in [3]. In this note, we assume that M is spherical.

Two great assets of the study of Coxeter groups are the reflection representation ρ and the root system Φ. Both are related to the vector space $V = \oplus_i \mathbb{R}\alpha_i$ with formal basis α_i $(1 \leq i \leq n)$ supplied with the symmetric bilinear form $(\cdot, \cdot)$ determined by

$$(\alpha_i, \alpha_j) = -2\cos(2\pi/M_{ij})$$

for $1 \leq i, j \leq n$. The *reflection representation* of W is the group homomorphism ρ from W to the orthogonal group on V with respect to $(\cdot, \cdot)$ for which

$$\rho(s)\alpha_j = \alpha_j - (\alpha_j, \alpha_s)\alpha_s,$$

where j and s are nodes of M. This representation is faithful.

As M is spherical, $(\cdot,\cdot)$ is positive definite, so W may be viewed as a finite real orthogonal group in n dimensions. Now $\Phi = \bigcup_{s\in S} W\alpha_s$ is a *root system* in the sense of [6]; its members are called *roots*. The elements α_s for $s \in S$ are called the *simple roots*. In the case of a Weyl group, a root system in the sense of [3] can be obtained from Φ by adjusting the length of certain roots. The set of positive roots of Φ is defined to be $\Phi^+ = \Phi \cap (\oplus_s \mathbb{R}_{\geq 0}\alpha_s)$. It is well known that Φ is the disjoint union of Φ^+ and $-\Phi^+$.

For $j \in \{1,\ldots,n\}$, define p_j to be the number of elements $w \in W$ such that $\{s \in S \mid \rho(w)\alpha_s \in \Phi^+\}$ has size j. This number is related to the descent statistics discussed in [2, 10]. The *Eulerian polynomial* of type M is

$$P(M,t) = \sum_{j=0}^{n} p_j t^j.$$

If $M = \mathrm{A}_n$, then, $W \cong \Sigma_{n+1}$, the symmetric group on $n+1$ letters, and, as a W-set, Φ can be identified with the set of distinct ordered pairs (i,j) for $1 \leq i,j \leq n+1$ in such a way that the simple roots are the pairs $(i,i+1)$ for $1 \leq i \leq n$, and the positive roots are all (i,j) with $i<j$. In this case, p_i is the number of $\pi \in \Sigma_{n+1}$ such that $\pi(i) < \pi(i+1)$.

The coefficient of t^i in the polynomial $P(M,1+t)$ equals the number of i-dimensional faces of the polytope (permutahedron) associated to M. The corresponding toric variety has only even-dimensional Betti numbers; these are the coefficients of $P(M,t)$. The signature of the toric variety equals $P(M,-1)$; see [8, 1].

I am grateful to Prof. Hirzebruch for drawing my attention to this polynomial and his inspiring lecture at the Killing meeting in Münster, December 7, 2007, where he posed the problem of computing the Eulerian polynomial for $M = \mathrm{E}_8$. The results for Coxeter groups of classical types are known and appear in Theorem 4 below; the results for the exceptional spherical types are given in Table 1. We will derive all of these results from the following expression for the Eulerian polynomial in terms of standard parabolic subgroups of W. Here a *standard parabolic subgroup* of W is a subgroup W_J generated by a subset J of S. The formula is a special case of [9, Theorem 1], of which we give a proof that does not essentially differ from the original.

Proposition 1. *The Eulerian polynomial for spherical type M is determined by*

$$P(M,t) = \sum_{K\subseteq S} \frac{|W|(t-1)^{|K|}}{|W_K|},$$

where (W,S) is the Coxeter system of type M.

Proof. For $J \subseteq S$, define p_J to be the number of elements $w \in W$ such that $\{s \in S \mid \rho(w)\alpha_s \in \Phi^+\} = J$. For $w \in W$, let $l(w)$ be the minimum length q of an expression of w as a product $r_1 \cdots r_q$ of members r_i of S. The proof is based on two facts, which are well known in Coxeter group theory (cf. [3, 5, 7]). The first is the fact that $\rho(w)\alpha_s \in \Phi^+$ is equivalent to $l(ws) > l(w)$. The second is

the fact that, for given $J \subseteq S$, the set of elements $w \in W$ with $l(ws) > l(w)$ for each $s \in J$ is a complete set of distinguished coset representatives of W_J, the subgroup of W generated by all members of J. In particular, its size is $|W/W_J|$ and so, by inclusion/exclusion,

$$p_J = \sum_{K \supseteq J} (-1)^{|K|+|J|} |W/W_K| \text{ for each } J \subseteq S.$$

As a consequence,

$$\begin{aligned} P(M,t) &= \sum_{J \subseteq S} p_J t^{|J|} = \sum_{J \subseteq S} \sum_{K \supseteq J} (-1)^{|K|+|J|} |W/W_K| t^{|J|} \\ &= \sum_{K \subseteq S} (-1)^{|K|} |W/W_K| \sum_{J \subseteq K} (-t)^{|J|} \\ &= \sum_{K \subseteq S} (-1)^{|K|} |W/W_K| (1-t)^{|K|} \\ &= \sum_{K \subseteq S} |W/W_K| (t-1)^{|K|}. \end{aligned}$$

□

Here are some immediate observations on these polynomials.

Lemma 2. *The Eulerian polynomial $P(M,t)$ satisfies the following properties.*

(i) $P(M,1) = |W|$ *and* $p_0 = 1$.
(ii) $P(\mathrm{A}_1, t) = 1 + t$.
(iii) $P(\mathrm{I}_2^{(m)}, t) = 1 + (2m-2)t + t^2$.
(iv) If M is the disjoint and disconnected union of the diagrams M_1 and M_2 then $P(M,t) = P(M_1,t)P(M_2,t)$.
(v) $P(M,t) = t^n P(M, t^{-1})$.

Proof. (i) is clear from the definition. (ii) and (iii) follow directly from Proposition 1. (iv) follows from the decompositions $W(M) = W(M_1) \times W(M_2)$ and $\Phi = \Phi_1 \dot\cup \Phi_2$. (v) is equivalent to $p_j = p_{n-j}$ for each $j \in \{0, \ldots, n\}$, which follows from left multiplication by the longest element w_0. For, $\rho(w)\alpha_s \in \Phi^+$ if and only if $\rho(w_0 w)\alpha_s \in -\Phi^+$, so, for each $J \subseteq S$, left multiplication by w_0 gives a bijection between $\{w \in W \mid \{s \in S \mid \rho(w)\alpha_s \in \Phi^+\} = J\}$ and $\{w \in W \mid \{s \in S \mid \rho(w)\alpha_s \in \Phi^+\} = S \setminus J\}$, proving $p_J = p_{S \setminus J}$. □

If we apply Proposition 1 directly, we need to consider all 2^n subdiagrams of M. The following corollary of Proposition 1 reduces that number to the number of connected components on a given node of M. Fix $k \in S$ and let $\mathcal{K}(M,k)$ denote the collection of the empty set and the connected subsets of S containing k. For $I \in \mathcal{K}(M,k)$, write $N(I)$ for the set of elements of S equal to or connected with a member of I, with the understanding that $N(\varnothing) = \{k\}$. For $J \subseteq S$, we write $M \setminus J$ to denote the diagram induced by M on $S \setminus J$.

Corollary 3. *Let M be a spherical Coxeter diagram and (W,S) a Coxeter system. Then, for each $k \in S$,*

$$P(M,t) = \sum_{I \in \mathcal{K}(M,k)} \frac{|W|(t-1)^{|I|}}{|W_I| \cdot |W_{S \setminus N(I)}|} P(M \setminus N(I), t).$$

Proof. Using Proposition 1, we first sum over the connected components I of a subdiagram containing the node k and next over possible completions of I to a subset J of S. If I is a nonempty member of $\mathcal{K}(M,k)$, then, taking the sum over all subsets of S of which I is a connected component yields precisely the summand for I in the sum in the corollary. For $I = \varnothing$, the possible completions are the subsets of $S \setminus \{k\}$. This explains the choice $N(\varnothing) = \{k\}$ and the summand for that value of N. □

The following result gives efficient recursion formulas for the classical Weyl groups. Part (iii) is due to Stembridge; cf. [10, Section 4].

Theorem 4. *The polynomials $P(M,t)$ for M one of A_n $(n \geq 1)$, B_n $(n \geq 2)$, D_n $(n \geq 4)$ satisfy the following recursion, where $P(\mathrm{A}_{-1},t) = P(\mathrm{A}_0,t) = 1$.*

$$P(\mathrm{A}_n,t) = \sum_{i=0}^{n} \binom{n+1}{i+1} P(\mathrm{A}_{n-i-1},t)(t-1)^i.$$

$$P(\mathrm{B}_n,t) = \sum_{i=0}^{n} 2^{n-i} \binom{n}{i} P(\mathrm{A}_{n-i-1},t)(t-1)^i.$$

$$P(\mathrm{D}_n,t) = P(\mathrm{B}_n,t) - 2^{n-1} n t P(\mathrm{A}_{n-2},t)$$

Proof. We apply Corollary 3. In all cases, the summation index i equals the size of the connected component in $\mathcal{K}(M,n)$ whose induced subdiagram in M is a straight path starting at n directed towards 1; it has type A_i. For A_n, this gives

$$\begin{aligned} P(\mathrm{A}_n,t) &= \sum_{i=0}^{n} \frac{|W(\mathrm{A}_n)|(t-1)^i}{|W(\mathrm{A}_i)| \cdot |W(\mathrm{A}_{n-i-1})|} P(\mathrm{A}_{n-i-1},t) \\ &= \sum_{i=0}^{n} \binom{n+1}{i+1} P(\mathrm{A}_{n-i-1},t)(t-1)^i. \end{aligned}$$

For B_n, we find

$$\begin{aligned} P(\mathrm{B}_n,t) &= \sum_{i=0}^{n} \frac{|W(\mathrm{B}_n)|(t-1)^i}{|W(\mathrm{B}_i)| \cdot |W(\mathrm{A}_{n-i-1})|} P(\mathrm{A}_{n-i-1},t) \\ &= \sum_{i=0}^{n} 2^{n-i} \binom{n}{i} P(\mathrm{A}_{n-i-1},t)(t-1)^i. \end{aligned}$$

As for D_n, we set aside the members $\varnothing$ and $\{n\}$ of $\mathcal{K}(\mathrm{D}_n,n)$; the corresponding summands appear separately in the summation below. The summation index j $(3 \leq j \leq n)$ equals the size of the component in $\mathcal{K}(\mathrm{D}_n,n)$ distinct from the

straight path towards 1; it contains $n-2$ and $n-1$ and has type D_j. Of Lemma 2, we use (ii) for $P(\mathrm{A}_1,t)$ and (iv) for $P(\mathrm{A}_{n-3}\mathrm{A}_1,t)$.

$$\begin{aligned}
P(\mathrm{D}_n,t) &= \frac{|W(\mathrm{D}_n)|}{|W(\mathrm{A}_{n-1})|}P(\mathrm{A}_{n-1},t) + \frac{|W(\mathrm{D}_n)|(t-1)}{|W(\mathrm{A}_1)|\cdot|W(\mathrm{A}_{n-3}\mathrm{A}_1)|}P(\mathrm{A}_{n-3}\mathrm{A}_1,t)\\
&\quad + \sum_{i=2}^{n-1}\frac{|W(\mathrm{D}_n)|(t-1)^i}{|W(\mathrm{A}_i)|\cdot|W(\mathrm{A}_{n-i-2})|}P(\mathrm{A}_{n-i-2},t)\\
&\quad + \sum_{j=3}^{n}\frac{|W(\mathrm{D}_n)|(t-1)^j}{|W(\mathrm{D}_j)|\cdot|W(\mathrm{A}_{n-j-1})|}P(\mathrm{A}_{n-j-1},t)\\
&= 2^{n-1}P(\mathrm{A}_{n-1},t) + 2^{n-2}\binom{n}{2}(t^2-1)P(\mathrm{A}_{n-3},t)\\
&\quad + \sum_{i=2}^{n-1}2^{n-1}\binom{n}{i+1}(t-1)^iP(\mathrm{A}_{n-i-2},t)\\
&\quad + \sum_{j=3}^{n}2^{n-j}\binom{n}{j}(t-1)^jP(\mathrm{A}_{n-j-1},t)
\end{aligned}$$

We equate the two summations as $P(\mathrm{A}_{n-1},t)$ and $P(\mathrm{B}_n,t)$ up to some scalars and a few missing terms, as follows. By compensating for the $i=0$ and $i=1$ terms, we find that the first summation, over i, contributes

$$-2^{n-1}nP(\mathrm{A}_{n-2},t) - 2^{n-1}\binom{n}{2}(t-1)p(\mathrm{A}_{n-3},t) + 2^{n-1}P(\mathrm{A}_{n-1},t),$$

and, by compensating for $j=0,1,2$, we find that the last summation, over j, contributes

$$\begin{aligned}
&-2^nP(\mathrm{A}_{n-1},t) - 2^{n-1}n(t-1)P(\mathrm{A}_{n-2},t)\\
&\qquad\qquad - 2^{n-2}\binom{n}{2}(t-1)^2P(\mathrm{A}_{n-3},t) + P(\mathrm{B}_n,t).
\end{aligned}$$

Substituting these contributions for the summations in the above expression for $P(\mathrm{D}_n,t)$, we find

$$\begin{aligned}
P(\mathrm{D}_n,t) &= 2^{n-1}P(\mathrm{A}_{n-1},t) + 2^{n-2}\binom{n}{2}(t^2-1)P(\mathrm{A}_{n-3},t)\\
&\quad - 2^{n-1}nP(\mathrm{A}_{n-2},t) - 2^{n-1}\binom{n}{2}(t-1)p(\mathrm{A}_{n-3},t)\\
&\quad + 2^{n-1}P(\mathrm{A}_{n-1},t) - 2^nP(\mathrm{A}_{n-1},t) - 2^{n-1}n(t-1)P(\mathrm{A}_{n-2},t)\\
&\quad - 2^{n-2}\binom{n}{2}(t-1)^2P(\mathrm{A}_{n-3},t) + P(\mathrm{B}_n,t)\\
&= -2^{n-1}ntP(\mathrm{A}_{n-2},t) + P(\mathrm{B}_n,t).
\end{aligned}$$

□

TABLE 1. Eulerian polynomials of non-classical connected types

M	$P(M,t)$
H_3	$1+59t+59t^2+t^3$
F_4	$1+236t+678t^2+236t^3+t^4$
H_4	$1+1316t+4566t^2+1316t^3+t^4$
E_6	$1+1272t+12183t^2+24928t^3+12183t^4+1272t^5+t^6$
E_7	$1+17635t+309969t^2+1123915t^3+1123915t^4+309969t^5$ $+17635t^6+t^7$
E_8	$1+881752t+28336348t^2+169022824t^3+300247750t^4$ $+169022824t^5+28336348t^6+881752t^7+t^8$

As $\mathrm{G}_2=\mathrm{I}_2^{(6)}$ has been dealt with by Lemma 2(iii), it remains to consider the Coxeter diagram on non-classical types and rank at least 3.

Theorem 5. *For connected non-classical Coxeter diagrams M of rank $n\geq 3$, the polynomials $P(M,t)$ are as in Table 1.*

Proof. This follows from a systematic application of Corollary 3. For instance, for E_6, with $k=6$, we find nine components in $\mathcal{K}(\mathrm{E}_6,6)$, and compute

$$\begin{aligned}P(\mathrm{E}_6,t)&=\frac{51840}{1920}P(\mathrm{D}_5,t)+\frac{51840}{240}P(\mathrm{A}_4,t)(t-1)\\&\quad+\frac{51840}{72}P(\mathrm{A}_1\mathrm{A}_2,t)(t-1)^2+\frac{51840}{48}P(\mathrm{A}_1,t)(t-1)^3\\&\quad+\frac{51840}{120}(t-1)^4+\frac{51840}{720}(t-1)^5\\&\quad+\frac{51840}{240}P(\mathrm{A}_1,t)(t-1)^4+\frac{51840}{1920}(t-1)^5+\frac{51840}{51840}(t-1)^6\\&=1+1272t+12183t^2+24928t^3+12183t^4+1272t^5+t^6.\end{aligned}$$

The polynomial $P(\mathrm{H}_3,t)$ can also be determined directly from Lemma 2(i), (v). □

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