The de Rham cohomology of
Drinfeld's half space

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(Communicated by Peter Schneider)

Abstract. Let $\mathcal{X} \subset \mathbb{P}_K^d$ be Drinfeld's half space over a p-adic field K. The de Rham cohomology of $\mathcal{X}$ was first computed by Schneider and Stuhler [20]. Afterwards there were given different proofs by Alon, de Shalit, Iovita and Spiess [1, 5, 15]. This paper presents yet another approach for the determination of these invariants by analyzing the de Rham complex of $\mathcal{X}$ from the viewpoint of results given in [17, 18]. Moreover, we treat as a generalization the dual BGG complex of a given algebraic representation in the sense of Faltings [7] respectively Schneider [19].

1. Introduction

Let p be a prime number and let K be a finite extension of the field of p-adic numbers $\mathbb{Q}_p$. We denote by

$$\mathcal{X} = \mathcal{X}_K^{(d+1)} = \mathbb{P}_K^d \setminus \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H)$$

(the complement of all K-rational hyperplanes in projective space) Drinfeld's half space [6] of dimension $d \geq 1$ over K. It is a rigid analytic variety over K which is equipped with an action of the p-adic Lie group $G = \mathrm{GL}_{d+1}(K)$. In [20] Schneider and Stuhler determined the cohomology of $\mathcal{X}$ for any "good" cohomology theory (e.g. the étale and the de Rham cohomology) as G-representations. Here they make only use of the "good" properties as homotopy invariance, existence of a product structure etc. It turns out that the de Rham cohomology is given by

$$(1) \qquad H_{\mathrm{dR}}^*(\mathcal{X}) = \bigoplus_{i=0}^{d} \mathrm{Hom}_K(v_{P_{(d+1-i,1,\ldots,1)}}^G, K)[-i].$$

Here $P_{(d+1-i,1,\ldots,1)}$ is the (lower) standard parabolic subgroup of G which corresponds to the decomposition $(d+1-i,1,\ldots,1)$ of $d+1$. Further for a parabolic subgroup $P \subset G$, the smooth generalized Steinberg representation v_P^G is

the unique irreducible quotient of the smooth unnormalized induced representation $i_P^G = \mathrm{ind}_P^G(K)$ with respect to the trivial P-representation [3, 4]. A few years later Alon, de Shalit, Iovita and Spiess [1, 5, 15] gave different proofs of this result by relating differential forms on $\mathcal{X}$ with harmonic cochains on the Bruhat–Tits building of G and considering logarithmic forms, respectively.

In this short notice we explain how we can determine the de Rham cohomology of $\mathcal{X}$ from its de Rham complex

$$(2) \qquad \Omega^\bullet(\mathcal{X}) : 0 \to \mathcal{O}(\mathcal{X}) \to \Omega^1(\mathcal{X}) \to \cdots \to \Omega^d(\mathcal{X}) \to 0$$

by applying some recent results given in [17, 18]. Here for $i = 0, \ldots, d$, the expression $\Omega^i(\mathcal{X}) = H^0(\mathcal{X}, \Omega^i)$ is the space of $\mathcal{X}$-valued sections of the usual homogeneous vector bundle Ω^i on projective space $\mathbb{P}_K^d$. Further the de Rham cohomology of $\mathcal{X}$ is the ordinary homology of the above complex since $\mathcal{X}$ is a Stein space. In contrast to the generalized Steinberg representations v_P^G the contributions $\Omega^i(\mathcal{X})$ in the de Rham complex are much bigger objects. Indeed they are reflexive K-Fréchet spaces with a continuous G-action [22]. Their strong duals $\Omega^i(\mathcal{X})'$, $i = 0, \ldots, d$, (i.e., the K-vector space of continuous linear forms equipped with the strong topology of bounded convergence) are locally analytic G-representations in the sense of Schneider and Teitelbaum [21]. More generally, the same holds true for arbitrary homogeneous vector bundles on $\mathbb{P}_K$. In [17] we constructed, for any such homogeneous vector bundle $\mathcal{E}$, a decreasing filtration by closed G-stable subspaces

$$(3) \qquad \mathcal{E}(\mathcal{X})^0 \supset \mathcal{E}(\mathcal{X})^1 \supset \cdots \supset \mathcal{E}(\mathcal{X})^{d-1} \supset \mathcal{E}(\mathcal{X})^d = H^0(\mathbb{P}^d, \mathcal{E})$$

on $\mathcal{E}(\mathcal{X})^0 = \mathcal{E}(\mathcal{X})$. As we will see in the next section the filtration behaves functorially in $\mathcal{E}$. Hence we get a filtered de Rham complex

$$(4) \qquad \left(0 \to \mathcal{O}(\mathcal{X})^j \to \Omega^1(\mathcal{X})^j \to \cdots \to \Omega^d(\mathcal{X})^j \to 0\right)_{j=0,\ldots,d}.$$

In this paper we analyze its induced spectral sequence

$$E_0^{p,q} = \mathrm{gr}^p(\Omega^{p+q}(\mathcal{X})) \Rightarrow H^{p+q}(\Omega^\bullet(\mathcal{X})),$$

cp. [10]. In the case of $d = 2$ this was also carried out by Schraen [23]. The main theorem of this paper is the following result.

Theorem 1.1. *The spectral sequence E_0 attached to the filtered de Rham complex* (4) *degenerates at E_1 and yields the cohomology formula* (1).

In the final section we replace the de Rham complex by the dual BGG complex attached to an algebraic representation in the sense of Faltings [7, 8] respectively Schneider [19]. More precisely, let $\lambda \in \mathbb{Z}^{d+1}$ be a dominant weight with corresponding irreducible algebraic representation $V(\lambda)$. Then we consider the complex

$$0 \to \mathcal{E}_\lambda(\mathcal{X}) \to \mathcal{E}_{w_1 \cdot \lambda}(\mathcal{X}) \to \cdots \to \mathcal{E}_{w_d \cdot \lambda}(\mathcal{X}) \to 0,$$

where the $\mathcal{E}_{w_i \cdot \lambda}$ are certain homogeneous vector bundles on $\mathbb{P}_K^d$ depending on the weight $w_i \cdot \lambda$ (for a precise description we refer to the final section). Schneider [19] proved that it is quasi-isomorphic to the complex $\Omega^\bullet(\mathcal{X}) \otimes V(\lambda)$.

It coincides with the de Rham complex (2) for $\lambda = 0$. In particular, the determination of the homology of $\mathcal{E}_{\bullet \cdot \lambda}(\mathcal{X})$ is not a surprising issue. Nevertheless, we get with the same proof:

Theorem 1.2. *Let $\lambda \in X^+$. Then the spectral sequence E_0 attached to the attached filtered complex degenerates at E_1 and one has*

$$H^*(\mathcal{E}_{\bullet \cdot \lambda}(\mathcal{X})) = \bigoplus_{i=0}^{d} \operatorname{Hom}_K(v_{P_{(d+1-i,1,\ldots,1)}}^G, V(\lambda))[-i].$$

2. THE PROOF OF THEOREM 1.1

We begin by recalling some terminology used in [17]. The following lines are an extract of [17, Section 1].

We consider the action of G on projection space $\mathbb{P}_K^d$ given by

$$g \cdot [q_0 : \cdots : q_d] := [q_0 : \cdots : q_d]g^{-1}.$$

We fix a homogeneous vector bundle $\mathcal{E}$ on $\mathbb{P}_K^d$ and let $\mathfrak{g} = \operatorname{Lie} G$ be the Lie algebra of G. Then $\mathcal{E}$ is naturally a $\mathfrak{g}$-module, i.e., there is a homomorphism of Lie algebras $\mathfrak{g} \to \operatorname{End}(\mathcal{E})$ which extends to the universal enveloping algebra $U(\mathfrak{g})$. Fix an integer $0 \leq j \leq d - 1$ and let

$$\mathbb{P}_K^j = V(X_{j+1}, \ldots, X_d) \subset \mathbb{P}_K^d$$

be the closed K-subvariety defined by the vanishing of the coordinate functions $X_{j+1}, \ldots, X_d$. Let $P_{j+1} = P_{(j+1,d-j)} \subset G$ be the (lower) standard-parabolic subgroup attached to the decomposition $(j+1, d-j)$ of $d+1$. It is clearly the stabilizer of $\mathbb{P}_K^j$ under the above action. Both the Zariski cohomology $H^*(\mathbb{P}_K^d \setminus \mathbb{P}_k^j, \mathcal{E})$ and the algebraic local cohomology $H_{\mathbb{P}_K^j}^*(\mathbb{P}_K^d, \mathcal{E})$ are thus equipped with an action of the semi-direct product $P_{(j+1,d-j)} \ltimes U(\mathfrak{g})$. Here the semi-direct product is as usual induced by the adjoint action of $P_{(j+1,d-j)}$ on $\mathfrak{g}$. Further the natural long exact sequence

$$(5) \qquad \cdots \to H^{i-1}(\mathbb{P}_K^d \setminus \mathbb{P}_K^j, \mathcal{E}) \to H_{\mathbb{P}_K^j}^i(\mathbb{P}_K^d, \mathcal{E})$$
$$\to H^i(\mathbb{P}_K^d, \mathcal{E}) \to H^i(\mathbb{P}_K^d \setminus \mathbb{P}_K^j, \mathcal{E}) \to \cdots$$

is equivariant with respect to this action. By general arguments in local cohomology theory [12], one deduces that

$$(6) \qquad H_{\mathbb{P}_K^j}^i(\mathbb{P}_K^d, \mathcal{E}) = \begin{cases} 0 & \text{for } i < d - j, \\ H^i(\mathbb{P}_K^d, \mathcal{E}) & \text{for } i > d - j. \end{cases}$$

In the case $i = d - j$, we have thus an exact sequence

$$0 \to H^{d-j-1}(\mathbb{P}_K^d, \mathcal{E}) \to H^{d-j-1}(\mathbb{P}_K^d \setminus \mathbb{P}_K^j, \mathcal{E})$$
$$\to H_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \mathcal{E}) \to H^{d-j}(\mathbb{P}_K^d, \mathcal{E}) \to 0.$$

We set

$$(7) \qquad \tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) := \ker\big(H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{E})\big)$$

$$\cong \operatorname{coker}\big(H^{d-j-1}(\mathbb{P}^d_K, \mathcal{E}) \to H^{d-j-1}(\mathbb{P}^d_K \setminus \mathbb{P}^j_K, \mathcal{E})\big)$$

which is consequently a $P_{(j+1,d-j)} \ltimes U(\mathfrak{g})$-module.

For an arbitrary parabolic subgroup $P \subset G$, let $\mathcal{O}^{\mathfrak{p}}$ be the full subcategory of the category $\mathcal{O}$ (in the sense of Bernstein, Gelfand, Gelfand [2]) consisting of $U(\mathfrak{g})$-modules of type $\mathfrak{p} = \operatorname{Lie} P$. We let $\mathcal{O}^{\mathfrak{p}}_{\text{alg}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ given by objects M such that all $\mathfrak{p}$-representations appearing in M are induced by finite-dimensional algebraic P-representations, cp. [18].

Lemma 2.1. *The $U(\mathfrak{g})$-module $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E})$ lies in the category $\mathcal{O}^{\mathfrak{p}_{(j+1,d-j)}}_{\text{alg}}$.*

Proof. This is an easy consequence of [17, Lem. 1.2.1] which states the existence of a finite-dimensional algebraic $P_{(j+1,d-j)}$-module which generates $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E})$ as $U(\mathfrak{g})$-module.   □

The next statement is the main result of [17]. For its formulation we need some more notation. Denote by $\operatorname{Rep}^{\ell a}_K(G)$ the category of locally analytic G-representations with coefficients in K. For a parabolic subgroup $P \subset G$, let

$$\operatorname{Ind}^G_P : \operatorname{Rep}^{\ell a}_K(P) \to \operatorname{Rep}^{\ell a}_K(G)$$

be the locally analytic induction functor [9]. Let $\operatorname{St}_{d-j} = v^{\operatorname{GL}_{d-j}}_B$ be the smooth Steinberg representation of $\operatorname{GL}_{d-j}(K)$, $j = 0, \ldots, d$. We consider St_{d-j} as a representation of $P_{(j+1,d-j)}$ via the trivial action of the unipotent radical of $P_{(j+1,d-j)}$ and the factor $\operatorname{GL}_{j+1}(K) \subset L_{(j+1,d-j)}$, respectively. We equip St_{d-j} with the finest locally convex topology so that it becomes a locally analytic P-representation [21]. Thus for any algebraic representation N of $P_{(j+1,d-j)}$, the tensor product $N \otimes \operatorname{St}_{d-j}$ is a locally analytic representation. In particular, this applies to the G-representation $H^{d-j}(\mathbb{P}^d_K, \mathcal{E})' \otimes v^G_{P_{(j+1,1,\ldots,1)}}$ which we also denote by $v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K, \mathcal{E})')$.

Theorem 2.2. *For $j = 0, \ldots, d-1$, there are extensions of locally analytic G-representations*

$$0 \to v^G_{P_{(j+1,1,\ldots,1)}}(H^{d-j}(\mathbb{P}^d_K, \mathcal{E})') \to (\mathcal{E}(\mathcal{X})^j / \mathcal{E}(X)^{j+1})' \to \operatorname{Ind}^G_{\underline{P_{j+1}}}(U'_j)^{\mathfrak{d}_j} \to 0.$$

Proof. This is [17, Thm. 1].   □

Here the $\underline{P_{j+1}}$-representation U'_j is a tensor product $N'_j \otimes \operatorname{St}_{d-j}$ of an algebraic $\underline{P_{j+1}}$-representation N'_j and St_{d-j}. The symbol $\mathfrak{d}_j$ indicates a system of differential equations depending on N_j. Here the representation N_j is characterized by the property that it generates the kernel of the natural homomorphism $H^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \to H^{d-j}(\mathbb{P}^d_K, \mathcal{E})$ as a module with respect to $U(\mathfrak{g})$.

This is exactly the starting point of the main construction in [18]. In fact, the locally analytic G-representation $\operatorname{Ind}^G_{\underline{P_{j+1}}}(U'_j)^{\mathfrak{d}_j}$ above can be characterized

as the image of the object $\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}) \times \mathrm{St}_{d-j}$ under a functor

$$\mathcal{F}^G_{P_{j+1}} : \mathcal{O}^{\mathfrak{p}_{j+1}}_{\mathrm{alg}} \times \mathrm{Rep}^\infty_K(L_{j+1}) \to \mathrm{Rep}^{\ell a}_K(G),$$

i.e.

$$\mathrm{Ind}^G_{P_{j+1}}(U'_j)^{\mathfrak{d}_j} = \mathcal{F}^G_{P_{j+1}}(\tilde{H}^{d-j}_{\mathbb{P}^j_K}(\mathbb{P}^d_K, \mathcal{E}), \mathrm{St}_{d-j}).$$

Here $\mathrm{Rep}^\infty_K(L_{j+1})$ is the category of smooth L_{j+1}-representations with coefficients over K.

Let us briefly recall the definition of this functor for an arbitrary parabolic subgroup $P \subset G$ with Levi decomposition $P = L \cdot U$. Let M be an object of $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Then there is a surjective map

$$\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to M$$

for some finite-dimensional algebraic P-representation $W \subset M$. Let V be a smooth L-representation. We consider V via the trivial action of U as a P-representation. As explained above the tensor product representation $W' \otimes_K V$ (where W' is the dual of W) is a locally analytic P-representation. Then

$$\mathcal{F}^G_P(M, V) = \mathrm{Ind}^G_P(W' \otimes_K V)^{\mathfrak{d}}$$

denotes the subset of functions $f \in \mathrm{Ind}^G_P(W' \otimes_K V)$ which are killed by the submodule $\mathfrak{d} = \ker(\phi)$. In [18] it is shown that this subset is a well-defined G-stable closed subspace of $\mathrm{Ind}^G_P(W' \otimes_K V)$ and has therefore a natural structure of a locally analytic G-representation. The resulting functor is contravariant in the first and covariant in the second variable. It is proved in [18, Prop. 4.10 a] that $\mathcal{F}^G_P$ is exact in both arguments.

Now we come to the functoriality aspect concerning the filtration (3) mentioned in the introduction.

Lemma 2.3. *Let $f : \mathcal{E} \to \mathcal{F}$ be a homomorphism of homogeneous vector bundles on $\mathbb{P}^d_K$. Then f is compatible with the filtrations, i.e., f induces G-equivariant homomorphisms $\mathcal{E}(\mathcal{X})^i \to \mathcal{F}(\mathcal{X})^i$, $i \geq 0$.*

Proof. The definition of the filtration involves only the geometry of $\mathcal{X}$ (being the complement of a hyperplane arrangement) and not the homogeneous vector bundle itself. In fact, the K-Fréchet space $\mathcal{E}(\mathcal{X}) = H^0(\mathcal{X}, \mathcal{E})$ appears in an exact sequence

$$0 \to H^0(\mathbb{P}^d_K, \mathcal{E}) \to H^0(\mathcal{X}, \mathcal{E}) \to H^1_\mathcal{Y}(\mathbb{P}^d_K, \mathcal{E}) \to H^1(\mathbb{P}^d_K, \mathcal{E}) \to 0.$$

We consider the K-Fréchet space $H^1_\mathcal{Y}(\mathbb{P}^d_K, \mathcal{E})$, where $\mathcal{Y} \subset \mathbb{P}^d_K$ is the "closed" complement of $\mathcal{X}$ in $\mathbb{P}^d_K$. The filtration is induced (by taking the preimage) by a similar one on $H^1_\mathcal{Y}(\mathbb{P}^d_K, \mathcal{E})$ which we briefly review. Here all geometric objects are considered as pseudo-adic spaces in the sense of [14].

Let $\{e_0, \ldots, e_d\}$ be the standard basis of $V = K^{d+1}$ and let Δ be the standard basis of simple roots with respect to the Borel subgroup of lower triangular matrices. For any $\alpha_i \in \Delta$, put $V_{\alpha_i} = V_i = \bigoplus_{j=0}^i K \cdot e_j$ and set $Y_{\alpha_i} = Y_i = \mathbb{P}(V_i)$. For any subset $I \subset \Delta$ with $\Delta \setminus I = \{\alpha_{i_1} < \cdots < \alpha_{i_r}\}$, let $Y_I = Y_{\alpha_{i_1}} = \mathbb{P}(V_{i_1})$. Furthermore, let P_I be the lower parabolic subgroup of G,

such that I coincides with the set of simple roots appearing in the Levi factor of P_I. Then gY_I is a closed subset of $\mathcal{Y}$ and we denote by $\Phi_{g,I} : gY_I \hookrightarrow \mathcal{Y}$ the corresponding embedding. Let $\mathbb{Z}$ be the constant sheaf on $\mathcal{Y}$ and set $\mathbb{Z}_{g,I} := (\Phi_{g,I})_*(\Phi_{g,I}^*(\mathbb{Z}))$. Then

$$\prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \subset \prod_{g \in G/P_I} \mathbb{Z}_{g,I}$$

denotes the subsheaf of locally constant sections with respect to the topological space G/P_I. In [17, §2.1] we proved that there is an acyclic resolution

$$0 \to \mathbb{Z} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I|=1}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I|=2}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I}$$

$$\to \cdots \to \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I|=d-1}} \prod_{g \in G/P_I}{}' \mathbb{Z}_{g,I} \to \prod_{g \in G/P_\varnothing}{}' \mathbb{Z}_{g,\varnothing} \to 0$$

of the constant sheaf $\mathbb{Z}$ on $\mathcal{Y}$. Let $i : \mathcal{Y} \hookrightarrow \mathbb{P}_K^d$ be the closed immersion. By applying the functor $\mathrm{Hom}(i_*(-), \mathcal{E})$ to this complex, we get a spectral sequence converging to $H^1_{\mathcal{Y}}(\mathbb{P}_K^d, \mathcal{E})$. Finally, the filtration on $H^1_{\mathcal{Y}}(\mathbb{P}_K^d, \mathcal{E})$ is just the one induced by this spectral sequence. It follows now easily from the construction that f is compatible with the filtrations on $\mathcal{E}(\mathcal{X})$ and $\mathcal{F}(\mathcal{X})$. $\square$

The de Rham complex (2) together with Lemma 2.3 induces complexes

$$0 \to \mathcal{O}(\mathcal{X})^j / \mathcal{O}(\mathcal{X})^{j+1} \to \Omega^1(\mathcal{X})^j / \Omega^1(\mathcal{X})^{j+1} \to \cdots \to \Omega^d(\mathcal{X})^j / \Omega^d(\mathcal{X})^{j+1} \to 0,$$

$j = 0, \ldots, d-1$, which form just the E_0-term of the spectral sequence attached to the filtered de Rham complex (4). Apart from the expressions $v_{P_{(j+1,1,\ldots,1)}}^G (H^{d-j}(\mathbb{P}_K^d, \Omega^i)')$, $i = 0, \ldots, d$, appearing in Theorem 2.2, this complex coincides by what we observed above with the dual of the complex

$$0 \to \mathcal{F}_{P_{j+1}}^G(\tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \Omega^d), \mathrm{St}_{d-j})$$

$$\to \cdots \to \mathcal{F}_{P_{j+1}}^G(\tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \Omega^1), \mathrm{St}_{d-j}) \to \mathcal{F}_{P_{j+1}}^G(\tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \mathcal{O}), \mathrm{St}_{d-j}) \to 0.$$

Proposition 2.4. *The above complex is acyclic.*

Proof. By the exactness of the functor $\mathcal{F}_P^G$ in the first entry it suffices to prove that the complex

$$0 \to \tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \mathcal{O}) \to \tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \Omega^1) \to \cdots \to \tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \Omega^d) \to 0$$

of $\mathfrak{g}$-modules is acyclic. Hence we have reduced the whole issue to a computation in coherent cohomology of projective space. Set

$$V = \mathbb{P}_K^d \setminus \mathbb{P}_K^j = \bigcup_{k=j+1}^d D(T_k),$$

where we denote as usual by $T_0, \ldots, T_d$ the homogeneous coordinate functions on $\mathbb{A}_K^{d+1}$. Then by identity (7) we have the description

$$\tilde{H}_{\mathbb{P}_K^j}^{d-j}(\mathbb{P}_K^d, \Omega^i) \cong \operatorname{coker}\big(H^{d-j-1}(\mathbb{P}_K^d, \Omega^i) \to H^{d-j-1}(V, \Omega^i)\big)$$

for all $i \geq 0$. On the other hand, we have the following well-known chain of identities:

$$(8) \qquad K = H^0(\mathbb{P}_K^d, \mathcal{O}) = H^1(\mathbb{P}_K^d, \Omega^1) = \cdots = H^d(\mathbb{P}_K^d, \Omega^d),$$

cp. [13]. All other cohomology groups vanish. Therefore it is enough to prove that the homology in degree $d - j - 1$ of the complex

$$0 \to H^{d-j-1}(V, \mathcal{O}) \to H^{d-j-1}(V, \Omega^1) \to \cdots \to H^{d-j-1}(V, \Omega^d) \to 0$$
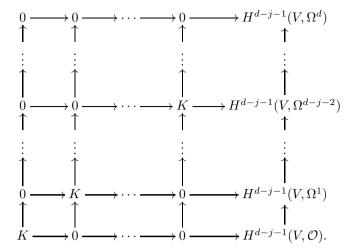
induces $H^{d-j-1}(\mathbb{P}_K^d, \Omega^{d-j-1}) = K$ and vanishes elsewhere. For this issue, we consider the double complex

$$
\begin{array}{ccccccc}
\displaystyle\bigoplus_{k=j+1}^d \Omega^d(D(T_k)) & \to & \displaystyle\bigoplus_{j+1 \leq k < l \leq d} \Omega^d(D(T_k) \cap D(T_l)) & \to & \cdots & \to & \Omega^d\Big(\displaystyle\bigcap_{k=j+1}^d D(T_k)\Big) \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
\displaystyle\bigoplus_{k=j+1}^d \Omega^i(D(T_k)) & \to & \displaystyle\bigoplus_{j+1 \leq k < l \leq d} \Omega^i(D(T_k) \cap D(T_l)) & \to & \cdots & \to & \Omega^i\Big(\displaystyle\bigcap_{k=j+1}^d D(T_k)\Big) \\
\uparrow & & \uparrow & & & & \uparrow \\
\vdots & & \vdots & & & & \vdots \\
\uparrow & & \uparrow & & & & \uparrow \\
\displaystyle\bigoplus_{k=j+1}^d \Omega^1(D(T_k)) & \to & \displaystyle\bigoplus_{j+1 \leq k < l \leq d} \Omega^1(D(T_k) \cap D(T_l)) & \to & \cdots & \to & \Omega^1\Big(\displaystyle\bigcap_{k=j+1}^d D(T_k)\Big) \\
\uparrow & & \uparrow & & & & \uparrow \\
\displaystyle\bigoplus_{k=j+1}^d \mathcal{O}(D(T_k)) & \to & \displaystyle\bigoplus_{j+1 \leq k < l \leq d} \mathcal{O}(D(T_k) \cap D(T_l)) & \to & \cdots & \to & \mathcal{O}\Big(\displaystyle\bigcap_{k=j+1}^d D(T_k)\Big).
\end{array}
$$

whose total complex gives rise to the de Rham cohomology of V, cp. [11]. Since $H_{\mathbb{P}_K^j}^k(\mathbb{P}_K^d, \Omega^i) = 0$ for all $k < d - j$ by identity (6), we see that

$$H^k(V, \Omega^i) = H^k(\mathbb{P}^d, \Omega^i)$$

for all such indices k. Evaluating the double complex along the horizontal lines, we get thus the following E_1-term:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & H^{d-j-1}(V,\Omega^d) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & K & \longrightarrow & H^{d-j-1}(V,\Omega^{d-j-2}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & & \vdots & & \vdots & & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & K & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & H^{d-j-1}(V,\Omega^1) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
K & \longrightarrow & 0 & \longrightarrow \cdots \longrightarrow & 0 & \longrightarrow & H^{d-j-1}(V,\mathcal{O}).
\end{array}
$$

But the de Rham cohomology of V is easily computed in another way. In fact, using the comparison isomorphism with Betti cohomology [11] and the long exact cohomology sequence for constant coefficients (5), we see that

$$
H^*_{\mathrm{dR}}(V) = \bigoplus_{i=0}^{d-j-1} K[-2i].
$$

The claim follows now easily. $\square$

For the proof of Theorem 1.1, we recall that

$$
E_0^{p,q} = \mathrm{gr}^p(\Omega^{p+q}(\mathcal{X})) \Rightarrow H^{p+q}(\Omega^\bullet(\mathcal{X}))
$$

is the induced spectral sequence of our filtered de Rham complex.

Corollary 2.5. *The E_1-term of the above spectral sequence has the shape*

$$
E_1^{p,q} = \begin{cases} \mathrm{Hom}_K(v^G_{P_{(d+1-p,1,\ldots,1)}}, K) & \text{for } q = 0, \\ 0 & \text{for } q \neq 0 \end{cases}
$$

for $p \geq 0$. Hence it degenerates at E_1 and we get formula (1).

This finishes the proof of Theorem 1.1.

3. A generalization: The dual BGG complex

In this final section we consider a generalization of what we have done before. We replace the de Rham complex (2) by the dual BGG complex attached to an algebraic representation in the sense of Faltings [7, 8] respectively Schneider [19]. For introducing this complex we have to introduce some more notation.

Let $\mathbf{G} = \mathrm{GL}_{d+1}$ considered as a linear algebraic group over K. Let $\mathbf{T} \subset \mathbf{G}$ be the diagonal torus and let $\mathbf{B} \subset \mathbf{G}$ be the Borel subgroup of lower triangular matrices. Denote by $\Phi \subset X^*(\mathbf{T})$ the corresponding set of roots of $\mathbf{G}$. Let

$\mathbf{B}^+ \subset \mathbf{G}$ the Borel subgroup of upper triangular matrices and let $\Delta^+ \subset \Phi$ be the set of simple roots with respect to $\mathbf{B}^+$. We consider the set

$$X^+ = \{\lambda \in X^*(\mathbf{T}) \mid (\lambda, \alpha^\vee) \geq 0 \text{ for all } \alpha \in \Delta^+\}$$

of dominant weights in $X^*(\mathbf{T})$. For $\lambda \in X^+$, we denote by $V(\lambda)$ the finite-dimensional irreducible algebraic $\mathbf{G}$-representation over K of highest weight λ, cp. [16]. We consider $V(\lambda)$ as a $\mathbf{G}$-representation in the sequel.

Let $\mathbf{P}_{(1,d)}$ be the stabilizer of the base point $[1 : 0 : \cdots : 0] \in \mathbb{P}_K^d(K)$ and let $\mathbf{L} = \mathbf{L}_{(1,d)} \subset \mathbf{P}_{(1,d)}$ be the Levi subgroup. Further let

$$X_L^+ = \{\lambda \in X^*(\mathbf{T}) \mid (\lambda, \alpha) \geq 0 \text{ for all } \alpha \in \Delta_L^+\}$$

be the set of $\mathbf{L}$-dominant weights where $\Delta_L^+ \subset \Delta$ consist of those simple roots which appear in $\mathbf{L}$. Every $\lambda \in X_L^+$ gives rise to a finite-dimensional irreducible algebraic $\mathbf{L}$-representation $V_L(\lambda)$. We consider it as a $\mathbf{P}$-module by letting act the unipotent radical trivially on it. Let

$$\pi : \mathbf{G} \to \mathbf{G}/\mathbf{P}_{(1,d)}$$

be the projection map and identify $\mathbf{G}/\mathbf{P}_{(1,d)}$ with $\mathbb{P}_K^d$. Let V be a finite-dimensional algebraic representation of $\mathbf{P}_{(1,d)}$. For a Zariski-open subset $U \subset \mathbb{P}_K^d$, put

$$\mathcal{E}_V(U) := \big\{\text{algebraic morphisms } f : \pi^{-1}(U) \to V \mid$$
$$f(gp) = p^{-1}f(g) \text{ for all } g \in \mathbf{G}(\overline{K}),\, p \in \mathbf{P}_{(1,d)}(\overline{K})\big\}.$$

Then $\mathcal{E}_V$ defines a homogeneous vector bundle on $\mathbb{P}_K^d$ and every homogeneous vector bundle is of this shape. We consider it at the same time as such an object over the rigid-analytic space $(\mathbb{P}_K^d)^{\mathrm{rig}}$. If $\lambda \in X_L^+$, then we set $\mathcal{E}_\lambda := \mathcal{E}_{V_L(\lambda)}$.

Let W be the Weyl group of $\mathbf{G}$ and consider the dot action $\cdot$ of W on $X^*(\mathbf{T})$ given by

$$w \cdot \chi = w(\chi + \rho) - \rho,$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$. Let $W_L \subset W$ be the Weyl group of L. Consider the set $^L W = W_L \backslash W$ of left cosets and the cycles

$$w_i := (1, 2, 3, \ldots, i+1) \in S_{d+1} \cong W,$$

$i = 0, \ldots, d$, which are just the representatives of shortest length in their cosets. If $\lambda \in X^+$ and $w \in {}^L W$, then $w \cdot \lambda \in X_L^+$. The dual BGG-complex of $\lambda \in X^+$ is given by the complex

$$0 \to \underline{V}(\lambda) \to \mathcal{E}_\lambda \to \mathcal{E}_{w_1 \cdot \lambda} \to \cdots \to \mathcal{E}_{w_d \cdot \lambda} \to 0.$$

Here $\underline{V}(\lambda)$ is the constant sheaf on $\mathbb{P}_K^d$ with values in $V(\lambda)$. By considering sections in $\mathcal{X}$, we get a complex

$$0 \to V(\lambda) \to \mathcal{E}_\lambda(\mathcal{X}) \to \mathcal{E}_{w_1 \cdot \lambda}(\mathcal{X}) \to \cdots \to \mathcal{E}_{w_d \cdot \lambda}(\mathcal{X}) \to 0.$$

It is proved in [19] that the complex $\mathcal{E}_{\bullet \cdot \lambda}(\mathcal{X})$ is quasi-isomorphic to the complex $\Omega^\bullet(\mathcal{X}) \otimes V(\lambda)$. The classical case is [7, 8]. For $\lambda = 0$, we get the usual de Rham complex.

Proof of Theorem 1.2. The proof is the same as above. Instead of the series of identities (8) we use this time the Borel–Weil–Bott theorem, cp. [16]. Indeed by considering the spectral sequence

$$(R^m \text{ind}_P^G)(R^n \text{ind}_B^P)(M) \Rightarrow R^n \text{ind}_B^G(M)$$

(cp. [16, Prop. 4.5 c]), we deduce that $H^i(\mathbb{P}_K^d, \mathcal{E}_{w\cdot\lambda}) = H^i(G/B, \mathcal{L}_{w\cdot\lambda})$ since $w \cdot \lambda \in X_L^+$ is L-dominant. Here $\mathcal{L}_{w\cdot\lambda}$ is the line bundle on G/B attached to the weight λ. Hence we get

$$H^i(\mathbb{P}_K^d, \mathcal{E}_{w_j\cdot\lambda}) = \begin{cases} H^0(\mathbb{P}_K^d, \mathcal{E}_\lambda) & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Moreover, the latter object has the description $H^0(\mathbb{P}_K^d, \mathcal{E}_\lambda) = V(\lambda)$. As for the interpretation of the de Rham cohomology of V we use the fact from [7, 8] that the complex $\mathcal{E}_{\bullet\cdot\lambda}(V)$ is quasi-isomorphic to $V(\lambda) \otimes \Omega^\bullet(V)$ instead. The claim follows. $\square$

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