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Families index for Boutet de Monvel operators

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Abstract. We define the analytical and the topological indices for continuous families of operators in the C*-closure of the Boutet de Monvel algebra. Using techniques of C*-algebra, K-theory, and the Atiyah–Singer theorem for families of elliptic operators on a closed manifold, we prove that these two indices coincide.

Introduction

Boutet de Monvel's calculus [5] provides a pseudodifferential framework which encompasses the classical differential boundary value problems. In an extension of the concept of Lopatinski and Shapiro, it associates to each operator two symbols: a pseudodifferential principal symbol, which is a bundle homomorphism, and an operator-valued boundary symbol. Ellipticity requires the invertibility of both. In this case, the calculus allows the construction of a parametrix. If the underlying manifold is compact, elliptic elements define Fredholm operators, and the parametrices are Fredholm inverses. Boutet de Monvel showed how then the index can be computed in topological terms. The crucial observation is that elliptic operators can be mapped to compactly supported K-theory classes on the cotangent bundle over the interior of the manifold. The topological index map, applied to this class, then furnishes an integer which is equal to the index of the operator.

For the construction of the above map, Boutet de Monvel combined operator homotopies and classical (vector bundle) K-theory in a very refined way. It therefore came as a surprise that this map—which is neither obvious nor trivial—can also be obtained as a composition of various standard maps in K-theory for C*-algebras—which was not yet available when [5] was written. In fact, it turns out to be basically sufficient to have a precise understanding of the short exact sequence induced by the boundary symbol map, [16], see also [15].

In the spirit of the classical result of Atiyah and Singer [3] we introduce and consider in this article *families* of operators in Boutet de Monvel's calculus, an issue that has not been addressed in [5].

More specifically, we consider a compact manifold X with boundary and then a fiber bundle $Z \to Y$ with fiber X over a compact Hausdorff space Y. We are then studying fiberwise (elliptic) Boutet de Monvel operators, depending continuously on $y \in Y$. In order to be able to use the powerful tools of C*-algebra K-theory we define such an operator family A over Y as a continuous section of a bundle of C*-algebras over Y, a concept which is slightly more general than that of Atiyah and Singer, who equip the set of operators with a Fréchet-space topology. In fact, restricted to the case without boundary, our algebra of continuous families $\mathfrak{A}$ contains that of [3] as a dense subalgebra.

While the analytic index $\mathrm{ind}_a(A)$ of such an elliptic family A as an element of $K(Y)$ is easily defined following Atiyah [2] and Jänich [11], cp. Definition 3.2 below, it is less obvious how to obtain the topological description. Similar to Boutet de Monvel's approach, the essential step is the construction of a map which associates to an elliptic family an element of the compactly supported K-theory of the total space of the bundle of cotangent spaces over the interior of the underlying manifolds. We regard this map as a homomorphism defined on $K_1(\mathfrak{A}/\mathfrak{K})$, where $\mathfrak{K}$ denotes the ideal of continuous families which have values in compact operators. In its definition, we use a fact which builds upon an observation of Boutet de Monvel: There exists a natural subalgebra $\mathfrak{A}^\dagger$ of $\mathfrak{A}$ for which $K_*(\mathfrak{A}^\dagger/\mathfrak{K}) \cong K_*(\mathfrak{A}/\mathfrak{K})$ so that each elliptic family A in $\mathfrak{A}$ can be represented by a class $a \in K_1(\mathfrak{A}^\dagger/\mathfrak{K})$. Moreover, $\mathfrak{A}^\dagger/\mathfrak{K}$ is commutative which allows us to make the connection to classical (vector bundle) K-theory. Then $\mathrm{ind}_t(A)$ is defined by applying the classical construction of the topological index to a, compare Definition 3.3.

Our main result is then that these two indices are equal. To prove this, we reduce to the classical families index theorem of Atiyah and Singer [3]. We assign in a canonical way to A an index problem on a bundle of closed manifolds, namely the double of our original bundle of manifolds with boundary. We then show that this associated family has the same analytic as well as topological index as A. In this step we make once more use of the isomorphism $K_1(\mathfrak{A}/\mathfrak{K}) \cong K_1(\mathfrak{A}^\dagger/\mathfrak{K})$.

It is perhaps worth stressing that our index theorem does not use the Boutet de Monvel index theorem for boundary value problems, which can actually be obtained from ours by taking Y equal to one point. Taking the families index theorem for granted, Albin and Melrose derived a more refined formula for the Chern character of the index bundle in terms of symbolic data [1, Thm. 3.8].

The paper is structured as follows: Section 1 starts with a review of the Boutet de Monvel calculus for a single manifold. We introduce the C*-algebra $\mathcal{A}$ of Boutet de Monvel operators of order and class zero and the boundary symbol map γ. Section 2 gives the technical introduction of operator families in Boutet de Monvel's calculus over a compact Hausdorff space Y. We define them as the continuous sections into a bundle of operator algebras whose typical fiber is the C*-algebra $\mathcal{A}$. In order to keep the exposition simple, we first treat the case of scalar-valued operators. We introduce γ as the fiberwise symbol map and extend the results on the kernel and image of γ to the family situation.

While in the single operator case this was sufficient to compute the K-theory of $\mathcal{A}/\mathcal{K}$, the situation is more complicated in the families case. In fact, an important ingredient in [16] is the fact that whenever X is connected and $\partial X \neq \varnothing$ there exists a continuous section of S^*X°. This is no longer true in the families case. Instead, we prove in Theorem 2.11 the fact alluded to above: We define $\mathfrak{A}^\dagger$ as the C*-algebra generated by all sections whose pseudodifferential part is independent of the co-variable at the boundary and whose singular Green part vanishes. Then $\mathfrak{A}^\dagger/\mathfrak{K}$ is commutative. Moreover, we use a Mayer–Vietoris argument to show that the inclusion map induces an isomorphism

$$K_*(\mathfrak{A}^\dagger/\mathfrak{K}) \cong K_*(\mathfrak{A}/\mathfrak{K}). \tag{1}$$

In Section 3 we study the index problem. Again, we confine ourselves first to the case of trivial one-dimensional bundles. We introduce the analytic and topological index and, as our main result, prove that the analytic and the topological index are equal. To achieve this, we reduce with the help of a doubling procedure to the case of families of closed manifolds. This reduction is based on the fact that we can use the isomorphism in (1) to represent any element of $K_1(\mathfrak{A}/\mathfrak{K})$ as a K_1-class of $\mathfrak{A}^\dagger/\mathfrak{K}$. In Section 4 we finish by explaining the arguments needed for the general situation.

Two appendices give technical details about the structure group of our families and about the Künneth theorem we are using.

1. Boutet de Monvel calculus for a single manifold

In this section, we introduce notation and recall the case of single operators. Details can be found in the monographs of Rempel and Schulze [19] and Grubb [8] as well as in the short introduction [21].

Let X be a compact manifold of dimension n with boundary ∂X and interior X°. We equip X with a *collar* (i.e., a neighborhood U of the boundary and a diffeomorphism $\delta : U \to \partial X \times [0,1)$) which then induces the *boundary defining function* $x_n = \mathrm{pr}_{[0,1)} \circ \delta$. The variables of ∂X will be denoted x'. The collar is used to provide the double $2X$ of X with a (noncanonical) smooth structure. Recall that $2X$ is the union of two copies X^+ and X^- of X quotiented by identification of the two copies of ∂X.

An element in Boutet de Monvel's calculus is a matrix of operators

$$A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(X, E_1) \\ \oplus \\ C^\infty(\partial X, F_1) \end{matrix} \longrightarrow \begin{matrix} C^\infty(X, E_2) \\ \oplus \\ C^\infty(\partial X, F_2), \end{matrix} \tag{2}$$

acting between sections of vector bundles E_1, E_2 over X and F_1, F_2 over ∂X. In this article we shall focus on the case of endomorphisms, where $E_1 = E_2 = E$ and $F_1 = F_2 = F$. For convenience, we choose a Riemannian metric g on M and Hermitean metrics on E, F to later obtain fixed Hilbert spaces structures, although the results do not depend on these choices. The operator P_+ in the upper left corner is a truncated pseudodifferential operator, derived from a (classical) pseudodifferential operator P on $2X$. Given $u \in C^\infty(X,E)$, P_+u is defined as the composition r^+Pe^+u. Here e^+ extends u by zero to a function on $2X$, to which P is applied. The result then is restricted (via r^+) to X. In general it is not true that $P_+u \in C^\infty(X,E)$. In order to ensure this, P is required to satisfy the *transmission condition*: If $p \sim \sum p_j$ is the asymptotic expansion of the local symbol p of P into terms $p_j(x,\xi)$, which are positively homogeneous of degree j in ξ one requires that, for $x_n = 0$ and $\xi = (0, \pm 1)$ one has $D^\beta_x D^\alpha_\xi p_j(x', 0, 0, 1) = (-1)^{j-|\alpha|} D^\beta_x D^\alpha_\xi p_j(x', 0, 0, -1)$. As for the remaining entries, G is a singular Green operator, T a trace operator, K a potential operator, and S a pseudodifferential operator on the boundary.

Operators in Boutet de Monvel's calculus have an *order* and a *class* or *type*. There are invertible elements in the calculus which allow us to reduce both, order and class, to zero. These operators then form a $*$-subalgebra of the bounded operators on the Hilbert space $H := L^2(X,E) \oplus L^2(\partial X, F)$.

Definition 1.1. Let $\mathcal{A}^\circ(E,F)$ denote the algebra of the (polyhomogeneous) Boutet de Monvel operators of order and class zero on $H = L^2(X,E) \oplus L^2(\partial X, F)$, endowed with its natural Fréchet topology, and $\mathcal{A}(E,F)$ its C*-closure in the algebra of all bounded operators on H. We write $\mathcal{A}^\circ$ and $\mathcal{A}$ if $E = X \times \mathbb{C}$ is trivial one-dimensional and $F = 0$.

Let $A \in \mathcal{A}^\circ(E,F)$ be given as in (2). For each entry P, S, G, T, K we have a symbol. This is the usual one for P and S, while G, T, and K can be considered as operator-valued pseudodifferential operators on ∂X with classical symbols in the sense of Schulze [22].

These are defined as follows, see [21]: The principal pseudodifferential symbol $\sigma(A)$ of A is the restriction of the principal symbol of P to the cosphere bundle over X. In order to define the boundary principal symbol $\gamma(A)$ we first denote by p^0, g^0, t^0, k^0, and s^0 the principal symbols of P, G, T, K, and S, respectively. We let $E^0_{x',\xi'}$ be the pullback of $E|_{\{x_n=0\}}$ to the normal bundle of X, lifted to $(x',\xi') \in S^*\partial X$. For fixed $(x',\xi') \in S^*\partial X$, $\xi_n \mapsto p^0(x',0,\xi',\xi_n)$ is a function on the conormal line in (x',ξ'), acting on $E^0_{x',\xi'}$. It induces a truncated pseudodifferential operator

$$p^0(x',0,\xi',D_n)_+ = r^+ p^0(x',0,\xi',D_n) e^+ : L^2(\mathbb{R}_{\geq 0}, E^0_{x',\xi'}) \to L^2(\mathbb{R}_{\geq 0}, E^0_{x',\xi'}).$$

In local coordinates near the boundary we then define the boundary principal symbol $\gamma(A)(x',\xi') : L^2(\mathbb{R}_{\geq 0}, E^0_{x',\xi'}) \oplus F_{x',\xi'} \to L^2(\mathbb{R}_{\geq 0}, E^0_{x',\xi'}) \oplus F_{x',\xi'}$ by

$$(3) \qquad \gamma(A)(x',\xi') := \begin{pmatrix} p^0(x',0,\xi',D_n)_+ + g^0(x',\xi',D_n) & k^0(x',\xi',D_n) \\ t^0(x',\xi',D_n) & s^0(x',\xi') \end{pmatrix},$$

with D_n indicating that we let the symbol act as an operator with respect to the variable x_n only. Note that the operator $g^0(x',\xi',D_n)$ is compact and that $k^0(x',\xi',D_n)$, $t^0(x',\xi',D_n)$ and $s^0(x',\xi')$ even have finite rank. The operator $p^0(x',0,\xi',D_n)_+$ on the other hand is a Toeplitz type operator; it will not be compact unless $p^0 = 0$.

Denoting by $\mathcal{K} = \mathcal{K}(H)$ the ideal of compact operators on $\mathcal{L}(H)$, one has the following important estimate based on work by Gohberg [7], Seeley [23] and Grubb–Geymonat [9], see [19, 2.3.4.4, Thm. 1] for a proof:

$$\inf_{K\in\mathcal{K}} \|A + K\| = \max\{\|\sigma(A)\|_{\sup}, \|\gamma(A)\|_{\sup}\}, \tag{4}$$

where the sup-norms on the right hand side are over the cosphere bundles in X and ∂X, respectively. This estimate implies, in particular, that both symbols extend continuously to C*-algebra homomorphisms defined on $\mathcal{A}(E, F)$. For fixed (x', ξ') the range $\{\gamma(A)(x', \xi') \mid A \in \mathcal{A}\}$ forms an algebra of Wiener–Hopf type operators.

It also follows from this estimate that γ vanishes on $\mathcal{K}$. Since the entries of $\gamma(A)(x', \xi')$ induced by g^0, k^0, t^0 and s^0 are (pointwise) compact while that induced by p^0 is not (unless $p^0 = 0$), we conclude that a Boutet de Monvel operator A belongs to $\ker\gamma$ if and only if $\sigma(A)$ vanishes at the boundary. Based on this observation (see [15, Sec. 2] for details) one can show that σ induces an isomorphism

$$\ker\gamma/\mathcal{K} \cong C_0(S^*X^\circ). \tag{5}$$

The K-theory of the range of γ was described in [15, Sec. 3]. Let $\mathtt{b}$: $C(\partial X) \to \operatorname{Im}\gamma$ denote the C*-homomorphism that maps g to $\gamma(m(f))$, where $m(f)$ is the operator of multiplication by a function $f \in C(X)$ whose restriction to ∂X equals g. Then $\mathtt{b}$ induces a K-theory isomorphism.

2. K-Theory of the families C*-algebra

To simplify the exposition, we shall assume in this section that $E = X \times \mathbb{C}$ is the trivial one-dimensional line bundle and $F = 0$.

Let Diff(X) denote the group of diffeomorphisms of X, equipped with its usual Fréchet topology. Recall that $\delta : U \to \partial X \times [0, 1)$ is the collar fixed at the beginning of Section 1. Let G denote the subgroup of Diff(X) consisting of those ϕ such that $\delta \circ \phi \circ \delta^{-1} : \partial X \times [0, 1/2) \to \partial X \times [0, 1)$ is of the form $(x', x_n) \mapsto (\varphi(x'), x_n)$ for some diffeomorphism $\varphi : \partial X \to \partial X$. We are going to use two properties that each $\phi \in G$ satisfies: the boundary defining function is preserved ($x_n \circ \phi = x_n$ for $0 \le x_n \le 1/2$), and the canonical map $2\phi : 2X \to 2X$, defined by $2\phi \circ i_\pm = i_\pm \circ \phi$, where $i_\pm : X^\pm \to 2X$ are the two canonical embeddings of X in $2X$, is a diffeomorphism of $2X$.

Throughout this paper, $\pi : Z \to Y$ will denote a fiber bundle over the compact Hausdorff space Y with fiber X and structure group G. Note, however, that this choice of structure group is just for convenience and can always be (essentially uniquely) arranged for a general bundle with typical fiber X, see the Appendix A for details.

We denote $Z_y := \pi^{-1}(y)$. Each Z_y is a smooth manifold with boundary, noncanonically diffeomorphic to X. The restriction of π to $\partial Z = \bigcup_y \partial Z_y$ is a fiber bundle $\pi_\partial : \partial Z \to Y$ with fiber ∂X and structure group Diff(∂X).

Next we define a bundle of Hilbert spaces, and later a C*-algebra which will act on its space of sections. This is a bit delicate, as it depends on some further choices; therefore we give the details. We choose a continuous family of Riemannian metrics $(g_y)_{y\in Y}$ with corresponding measures μ_y on Z_y and define $H_y := L^2(Z_y, \mu_y)$. Recall that such a family (g_y) exists: we can patch them together using trivializations of the bundle and a partition of unity on Y, as the space of Riemannian metrics on X is convex.

The union $\mathfrak{H} = \bigcup_{y\in Y} H_y$ is a fiber bundle of topological vector spaces over Y, canonically associated to $\pi : Z \to Y$, with trivializations induced from the trivializations of π in the obvious way. The structure group is the group of invertible bounded operators on H, *equipped with the strong topology.*

Remark 2.1. That we obtain here the strong topology and not the norm topology comes from the fact that the changes of trivialization are implemented by pullback with the diffeomorphisms of G, and this is continuous in the strong, but not the norm topology. This makes our considerations about bundles of operators later quite cumbersome and requires to use the fact that we deal with pseudodifferential operators.

Moreover, the choice $(g_y)_{y\in Y}$ gives rise to a continuous family of inner products on $\mathfrak{H}$ inducing the given topology of the fibers H_y.

Let $\mathcal{A}_y$ be the Boutet de Monvel algebra of order and class zero on $L^2(Z_y)$. We want to define the bundle of Boutet de Monvel algebras $\aleph = \bigcup_{y\in Y} \mathcal{A}_y$ as locally trivial bundle with structure group the automorphism group of the C*-algebra $\mathcal{A}$ with the *norm topology*, associated to $Z \to Y$.

To achieve this, we need the diffeomorphism invariance of the Boutet de Monvel algebra in a precise form.

Definition 2.2. Given $\phi \in G$, let T_ϕ denote the bounded operator on $L^2(X)$ defined by $f \mapsto f \circ \phi^{-1}$.

Proposition 2.3. *We have a well defined continuous action (for the Fréchet topology on G and the norm topology on $\mathcal{A}$)*

$$G \times \mathcal{A} \ni (\phi, A) \mapsto T_\phi A T_\phi^{-1} \in \mathcal{A}.$$

Moreover, by restriction we get an action $G \times \mathcal{A}^\circ \to \mathcal{A}^\circ$.

Proof. This corresponds to [3, Prop. 1.3]. In fact, even if X is closed, Atiyah and Singer consider a slightly different situation in that they close $\mathcal{A}^\circ$ with respect to the operator norm of the action on all Sobolev spaces, while we only use the operator norm on L^2. Their argument still applies verbatim, since they treat the action on each Sobolev space separately.

Indeed, the proof of [3, Prop. 1.3] uses only a number of formal properties of the algebra of pseudodifferential operators which are also satisfied by the Boutet de Monvel algebra, and therefore applies in the same way to our general situation. To be more specific, let us list these properties:

(1) the Boutet de Monvel algebra $\mathcal{A}^\circ$ is diffeomorphism invariant, i.e. in particular $T_\phi A T_\phi^{-1} \in \mathcal{A}^\circ$ for $A \in \mathcal{A}^\circ$ and $\phi \in G$.

(2) Each T_ϕ is a bounded operator on $L^2(X)$ and the map $G \to \mathcal{L}(L^2(X))$ is strongly continuous. Moreover, for a sufficiently small open neighborhood of 1, the image has uniformly bounded norm. The proof of this fact as given in [3] works for compact manifolds with boundary exactly the same way as for closed manifolds.
(3) Let $\mathcal{V}_G$ denote the space of vector fields on X which, in the collar, pull back from vector fields on ∂X. The exponential map, defined with the help of Riemannian metrics which respect the collar structure, gives a local diffeomorphism (of Fréchet manifolds) between $\mathcal{V}_G$ and G.
(4) If $V \in \mathcal{V}_G$ and $A \in \mathcal{A}^\circ$ then the commutator $[A, V]$ belongs to $\mathcal{A}^\circ$ by the rules of the calculus, cp. [8, Thm. 2.7.6].

All these properties are either well known or easy to establish. □

Corollary 2.4. *We obtain the bundle* $\aleph = \bigcup_{y\in Y} \mathcal{A}_y$ *of topological algebras with bundle of subalgebras* $\aleph^\circ = \bigcup_{y\in Y} \mathcal{A}_y^\circ$, *modelled on* $(\mathcal{A}, \mathcal{A}^\circ)$ *with structure group the automorphism group of* $\mathcal{A}$ *with its norm topology and the automorphism group of* $\mathcal{A}^\circ$ *with its Fréchet topology. The local trivializations are induced by the local trivializations of* $\pi : Z \to Y$, *where a diffeomorphisms* $\alpha_y : Z_y \to X$ *induces a trivialization isomorphism* $\mathcal{A}_y \to \mathcal{A}$ *by conjugation with* T_{α_y}.

Moreover, the choice of metrics $(g_y)_{y\in Y}$ *induces a continuous family of norms on the fibers of* $\aleph$ *inducing the topology. With these norms the bundle becomes a bundle of* C^**-algebras.*

Proof. The statement about the bundle of topological algebras follows immediately from Proposition 2.3. Moreover, it is well known that each $\mathcal{A}_y$ is closed under taking adjoints in $\mathcal{L}(L^2(Z_y))$.

We now check that with this structure, we obtain a locally trivial bundle of C*-algebras. Fix a local trivialization with diffeomorphisms $\alpha_y : Z_y \to X$. If we pull back the inner products on H_y to $H = L^2(X)$ with the induced maps, then the corresponding Gram operator G_y, expressing this pullback inner product in terms of the original one on $L^2(X)$, is the multiplication with a smooth positive function m_y which depends continuously on y: the density of $\alpha_y^* \mu_y$ with respect to a chosen measure μ on X. Note that G_y belongs to $\mathcal{A}$ and its norm, which is just the supremum, depends continuously on y. Now compose the original trivialization of $\mathcal{A}_y$ with conjugation by $\sqrt{G_y}$ and the resulting trivialization will respect the C*-algebra structures, but inherit the norm continuity of transition maps. To summarize: with a canonical modification (given in terms of the inner products) we have obtained trivializations of our bundle $\aleph$ as a bundle of C*-algebras, as claimed. □

Definition 2.5. We denote by $\mathfrak{A}$ the set of continuous sections of the bundle $\aleph$ of C*-algebras. With the pointwise operations and the supremum norm, this becomes a C*-algebra. The underlying topological algebra is canonically associated to $\pi : Z \to Y$, the norm and the $*$-operation depend on the choice of the family of metrics $(g_y)_{y\in Y}$.

The principal symbol and the boundary principal symbol extend continuously to two families of C*-algebra homomorphisms

$$\sigma_y : \mathcal{A}_y \to C(S^*Z_y) \quad \text{and} \quad \gamma_y : \mathcal{A}_y \to C(S^*\partial Z_y, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))),$$

where S^* denotes cosphere bundle and $\mathcal{L}$ bounded operators. Here γ_y is well defined, since the structure group of the bundle $\pi : Z \to Y$ leaves the boundary defining function invariant, see [8, Thm. 2.4.11].

Let us denote by S^*Z the disjoint union of all S^*Z_y. This can canonically be viewed as the total space of a fiber bundle over Y with structure group G. One analogously defines $S^*\partial Z = \bigcup_y S^*\partial Z_y$ and $S^*Z^\circ = \bigcup S^*Z_y^\circ$.

Definition 2.6. Given $A \in \mathfrak{A}$, let σ_A be the function on S^*Z defined by piecing together all the σ_y's. Then $A \mapsto \sigma_A$ defines a C*-algebra homomorphism

$$\sigma : \mathfrak{A} \longrightarrow C(S^*Z).$$

One also gets, analogously,

$$\gamma : \mathfrak{A} \longrightarrow C(S^*\partial Z, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))).$$

Let $\mathfrak{K}$ denote the subalgebra of $\mathfrak{A}$ consisting of the sections $(A_y)_{y\in Y}$ such that A_y is compact for every $y \in Y$. It follows immediately from the corresponding statement for a single manifold that $\ker\sigma \cap \ker\gamma = \mathfrak{K}$. It is also straightforward to generalize the description of $\ker\gamma$ for a single manifold (5):

Theorem 2.7. *The principal symbol restricted to* $\ker\gamma$ *induces a C^*-algebra isomorphism*

$$\ker\gamma/\mathfrak{K} \simeq C_0(S^*Z^\circ). \tag{6}$$

Here $C_0(S^*Z^\circ)$ consists of the elements of $C(S^*Z)$ which, for every $y \in Y$, vanish on all points of S^*Z_y with base point belonging to ∂Z_y.

Regarding each $f \in C(Z)$ as a family of multiplication operators on $(H_y)_{y\in Y}$, furnishes an embedding of $C(Z)$ into $\mathfrak{A}$, which we denote $m : C(Z) \to \mathfrak{A}$. Mapping a $g \in C(\partial Z)$ to the boundary principal symbol of $m(f)$, where $f \in C(Z)$ is such that its restriction to ∂Z is g, defines the C*-algebra homomorphism $b : C(\partial Z) \to \operatorname{Im}\gamma$.

Theorem 2.8. *The homomorphisms $b_* : K_i(C(\partial Z)) \to K_i(\operatorname{Im}\gamma)$, $i = 0, 1$, induced by b are isomorphisms.*

Proof. Given an open set $U \subseteq Y$, let us denote by $\pi_U : Z_U = \pi^{-1}(U) \to U$ the restriction of π to U, by $\mathfrak{A}_U$ the algebra of sections in $\mathfrak{A}$ which vanish outside U and by γ_U the restriction of γ to $\mathfrak{A}_U$. Moreover we let

$$C_0(\partial Z_U) = \{f \in C(\partial Z) \mid \operatorname{supp} f \subseteq \pi_\partial^{-1}(U)\}$$

and write b_U for the restriction of b to $C_0(\partial Z_U)$. If the bundle π is trivial over U, then $\mathfrak{A}_U$ is isomorphic to $C_0(U, \mathcal{A})$ and, with respect to this isomorphism, b_U corresponds to the tensor product of the identity on $C_0(U)$ with the corresponding map for a single manifold, also denoted by b in [15, 16]. It is the content of [15, Cor. 8] that b induces a K-theory isomorphism onto the image of γ. It then follows from the Künneth formula for C*-algebras [20] that b_U induces isomorphisms $b_{U*} : K_i(C_0(\partial Z_U)) \longrightarrow K_i(\operatorname{Im}\gamma_U)$, $i = 0, 1$, see Proposition B.2 in Appendix B.

Now let $(\operatorname{Im}\gamma)_U$ denote the subset of $\operatorname{Im}\gamma$ consisting of those functions which vanish outside $\bigcup_{y\in U} S^*\partial Z_y$. It is obvious that $\operatorname{Im}\gamma_U \subseteq (\operatorname{Im}\gamma)_U$. Since both $\operatorname{Im}\gamma_U$ and $(\operatorname{Im}\gamma)_U$ are closed in $C(S^*\partial Z, \mathcal{L}(L^2(\mathbb{R}_{\geq 0})))$, to show that they are equal it suffices to show that the former is dense in the latter. This follows from the fact that multiplication by a complex continuous function with support contained in U maps $(\operatorname{Im}\gamma)_U$ to $\operatorname{Im}\gamma_U$. This simple observation implies that, for open sets U and V, we have a canonical C*-algebra isomorphism

$$\operatorname{Im}\gamma_{U\cap V} \cong \{(f, g) \in \operatorname{Im}\gamma_U \oplus \operatorname{Im}\gamma_V \mid f = g\}. \tag{7}$$

Now suppose that we have shown b_{U*} to be an isomorphism for some open U and that V is open and π trivial over V, and so in particular also over $U \cap V$. We then consider the two—thanks to (7)—commutative diagrams

$$\begin{array}{ccc} C_0(\partial Z_{U\cap V}) & \to & C_0(\partial Z_U) \\ \downarrow & & \downarrow \\ C_0(\partial Z_V) & \to & C_0(\partial Z_{U\cup V}) \end{array} \quad \text{and} \quad \begin{array}{ccc} \operatorname{Im}\gamma_{U\cap V} & \to & \operatorname{Im}\gamma_U \\ \downarrow & & \downarrow \\ \operatorname{Im}\gamma_V & \to & \operatorname{Im}\gamma_{U\cup V}. \end{array}$$

Because they are cartesian, we may extract from both diagrams cyclic exact Mayer–Vietoris sequences (see [4, 21.2.2] or [14, 7.2.1]), and we may use the K-theory maps induced by b_U, b_V, $b_{U\cap V}$ and $b_{U\cup V}$ to map the first cyclic sequence to the second. By assumption and the case of trivial bundles, the maps induced by b_U, b_V and $b_{U\cap V}$ are isomorphisms. It then follows from the five-lemma that also $b_{U\cup V}$ induces a K-theory isomorphism.

Since Y has a finite cover by open sets over which π is trivial, induction shows that b induces K-theory isomorphisms. □

Using Theorem 2.7, we obtain the following commutative diagram of C*-algebra homomorphisms, whose horizontal lines are exact:

$$\begin{array}{ccccccccc} 0 \longrightarrow & C_0(S^*Z^\circ) & \longrightarrow & \mathfrak{A}/\mathfrak{K} & \xrightarrow{\gamma} & \operatorname{Im}\gamma & \longrightarrow 0 \\ & \big\uparrow{\scriptstyle m^\circ} & & \big\uparrow{\scriptstyle m} & & \big\uparrow{\scriptstyle b} & \\ 0 \longrightarrow & C_0(Z^\circ) & \longrightarrow & C(Z) & \xrightarrow{r} & C(\partial Z) & \longrightarrow 0. \end{array}$$

We have denoted by r the map that pieces together all restrictions $r_y : C(Z_y) \to C(\partial Z_y)$, $y \in Y$, and by Z° the union $\bigcup_y Z_y^\circ$. Since the isomorphism (6) is induced by the principal symbol, and the principal symbol of an operator of multiplication by a function is the function itself, the map m° in the diagram above is actually the map of composition with the canonical projection $S^*Z^\circ \to Z^\circ$. We may apply the cone-mapping functor [16, Lemma 9] to the above diagram and get (using the same arguments that prove (11) in [16]) the following commutative diagram of cyclic exact sequences

$$
(8)\qquad
\begin{array}{ccc}
K_0(C_0(Z^\circ)) & \longrightarrow & K_0(C(Z)) \\
\downarrow m_*^\circ & & \downarrow m_* \\
K_0(C_0(S^*Z^\circ)) & \longrightarrow & K_0(\mathfrak{A}/\mathfrak{K}) \\
\downarrow & & \downarrow \\
K_1(Cm^\circ) & \xrightarrow{\cong} & K_1(Cm) \\
\downarrow & & \downarrow \\
K_1(C_0(Z^\circ)) & \longrightarrow & K_1(C(Z)) \\
\downarrow m_*^\circ & & \downarrow m_*^\circ \\
K_1(C_0(S^*Z^\circ)) & \longrightarrow & K_1(\mathfrak{A}/\mathfrak{K}) \\
\downarrow & & \downarrow \\
K_0(Cm^\circ) & \xrightarrow{\cong} & K_0(Cm) \\
\downarrow & & \downarrow \\
K_0(C_0(Z^\circ)) & \longrightarrow & K_0(C(Z)),
\end{array}
$$

where $\cong$ denotes isomorphism.

Up to this point, everything goes exactly as in the case of a single manifold, but here comes a difference: The homomorphism m_0 does not necessarily have a left inverse (in the case of a single manifold X, such a left inverse is defined by composition with a section of S^*X), and hence the cyclic exact sequences above do not have to split into short exact ones.

To proceed we now introduce the subalgebra $\mathfrak{A}^\dagger$ of $\mathfrak{A}$ and an associated subalgebra B of $C(S^*Z)$ with the properties outlined in the introduction: For each $y \in Y$, let B_y denote the subalgebra of $C(S^*Z_y)$ consisting of the functions which do not depend on the co-variable over the boundary, that is, an $f \in C(S^*Z_y)$ belongs to B_y if and only if the restriction of f to the points of S^*Z_y over ∂Z_y equals $g \circ p_y$, for some $g \in C(\partial Z_y)$, where $p_y : S^*Z_y \to Z_y$ is the canonical projection. We then define $\mathcal{A}_y^\dagger$ as the C*-subalgebra of $\mathcal{A}_y$ generated by $\{P_+ \mid P \text{ is a pseudodifferential operator with the transmission property and } \sigma_y(P_+) \in B_y\}$.

Definition 2.9. Let B denote the subalgebra of $C(S^*Z)$ consisting of the functions whose restriction to each S^*Z_y belongs to B_y. We let then $\mathfrak{A}^\dagger$ be the C*-subalgebra of $\mathfrak{A}$ consisting of the sections $(A_y)_{y\in Y}$ such that $A_y \in \mathcal{A}_y^\dagger$ for every $y \in Y$.

Proposition 2.10. *The C^*-algebra $\mathfrak{A}^\dagger/\mathfrak{K}$ is commutative, and the map*

$$\mathfrak{A}^\dagger/\mathfrak{K} \ni [A] \overset{\bar{\sigma}}{\longmapsto} \sigma(A) \in B$$

is a C^*-algebra isomorphism.

Proof. Let $P = (P_y)$ be a family of *pseudodifferential* operators with symbol independent of the co-variable over the boundary, i.e. a generator of $\mathfrak{A}^\dagger$. According to (3), $\gamma(P)$ can be considered as a function on ∂Z, acting for $z \in \partial Z$ on $L^2(\mathbb{R}_{\geq 0})$ by multiplication with $\gamma(P)(z)$. Moreover, for $z \in \partial Z$ we have

$\gamma(z) = \sigma(z)$ independent of the co-variable by assumption. It follows that the composed algebra homomorphism

$$\sigma : \mathfrak{A}^\dagger \xrightarrow{\sigma\oplus\gamma} C(S^*Z) \oplus C(S^*\partial Z, \mathcal{L}(L^2(\mathbb{R}_{\geq 0}))) \xrightarrow{\mathrm{pr}} C(S^*Z)$$

has the same kernel as $\sigma \oplus \gamma$, namely $\mathfrak{K}$ and so the map we consider is injective and in particular $\mathfrak{A}^\dagger/\mathfrak{K}$ is commutative. By the very definition of $\mathfrak{A}^\dagger$, $\sigma : \mathfrak{A}^\dagger \to B$ has dense image, as a morphism of C*-algebras it is therefore also surjective. □

This allows us to describe the K-theory of $\mathfrak{A}/\mathfrak{K}$:

Theorem 2.11. *The composition*

$$K_i(\mathfrak{A}/\mathfrak{K}) \xrightarrow{\iota_*^{-1}} K_i(\mathfrak{A}^\dagger/\mathfrak{K}) \xrightarrow{\bar{\sigma}_*} K_i(B)$$

is an isomorphism, $i = 0, 1$.

The proof makes use of the following proposition, which is easily established by a diagram chase, compare [10, Exer. 38 of Sec. 2.2]:

Proposition 2.12. *Let there be given a commutative diagram of abelian groups with exact rows,*

$$\begin{array}{ccccccccccc} \cdots & \to & A_i' & \xrightarrow{f_i'} & B_i' & \xrightarrow{g_i'} & C_i' & \xrightarrow{h_i'} & A_{i+1}' & \to & \cdots \\ & & \uparrow a_i & & \uparrow b_i & & \uparrow c_i & & \uparrow a_{i+1} & & \\ \cdots & \to & A_i & \xrightarrow{f_i} & B_i & \xrightarrow{g_i} & C_i & \xrightarrow{h_i} & A_{i+1} & \to & \cdots, \end{array}$$

where each c_i is an isomorphism. Then the sequence

$$\cdots \longrightarrow A_i \xrightarrow{(a_i,-f_i)} A_i' \oplus B_i \xrightarrow{\langle f_i', b_i\rangle} B_i' \xrightarrow{h_i c_i^{-1} g_i'} A_{i+1} \longrightarrow \cdots$$

is exact, where $\langle f_i', b_i\rangle$ is the map defined by $\langle f_i', b_i\rangle(\alpha,\beta) = f_i'(\alpha) + b_i(\beta)$.

We are now ready to prove Theorem 2.11. Applying Proposition 2.12 to the diagram (8), we get the exact sequence

$$(9)\qquad \begin{array}{ccccc} K_0(C_0(Z^\circ)) & \to & K_0(C(Z)) \oplus K_0(C_0(S^*Z^\circ)) & \to & K_0(\mathfrak{A}/\mathfrak{K}) \\ \uparrow & & & & \downarrow \\ K_1(\mathfrak{A}/\mathfrak{K}) & \leftarrow & K_1(C(Z)) \oplus K_1(C_0(S^*Z^\circ)) & \leftarrow & K_1(C_0(Z^\circ)). \end{array}$$

We next consider the following diagram of commutative C*-algebras

$$(10)\qquad \begin{array}{ccc} C_0(Z^\circ) & \xrightarrow{m^\circ} & C_0(S^*Z^\circ) \\ \downarrow & & \downarrow p_2 \\ C(Z) & \xrightarrow{p_1} & B. \end{array}$$

As $C_0(Z^\circ)$ is canonically isomorphic to

$$\{(f,g) \in C(Z) \oplus C_0(S^*Z^\circ) \mid \ p_1(f) = p_2(g)\},$$

we obtain a Mayer–Vietoris exact sequence associated to (10):

$$\begin{array}{ccccc} K_0(C_0(Z^\circ)) & \to & K_0(C(Z)) \oplus K_0(C_0(S^*Z^\circ)) & \to & K_0(B) \\ \uparrow & & & & \downarrow \\ K_1(B) & \leftarrow & K_1(C(Z)) \oplus K_1(C_0(S^*Z^\circ)) & \leftarrow & K_1(C_0(Z^\circ)). \end{array} \tag{11}$$

The map $\iota : B \cong \mathfrak{A}^\dagger/\mathfrak{K} \hookrightarrow \mathfrak{A}/\mathfrak{K}$ and the identity on the other K-theory groups furnish morphisms from the cyclic sequence (11) to the cyclic sequence (9). The five lemma then shows that the induced maps in K-theory are isomorphisms. Together with Proposition 2.10 we obtain the assertion. □

3. The Boutet de Monvel family index theorem

The index of a continuous function with values in Fredholm operators was defined by Jänich [11] and Atiyah [2]. Using the following Proposition 3.1, their definition can be extended to sections of our $\aleph$.

Proposition 3.1. *Let $\mathfrak{H}$ and $\mathfrak{A}$ be as above, $k \in \mathbb{N}$ and let $(A_y)_{y\in Y} \in M_k(\mathfrak{A})$ be such that, for each y, A_y is a Fredholm operator, where we interpret $M_k(\mathfrak{A})$ as the sections of the bundle with fiber $M_k(\mathcal{A}_y)$. Then there are continuous sections $s_1, \cdots, s_q$ of $\mathfrak{H}^k$ such that the maps*

$$\begin{array}{cccc} \tilde{A}_y : & H_y^k \oplus \mathbb{C}^q & \longrightarrow & H_y^k \oplus \mathbb{C}^q \\ & (v, \lambda) & \longmapsto & (A_y v + \sum_{j=1}^q \lambda_j s_j(y), 0) \end{array}$$

have image equal to $H_y^k \oplus 0$ for all $y \in Y$ and $(\ker \tilde{A}_y)_{y\in Y}$ is a (finite dimensional) vector bundle over Y.

Proof. Similar to [3, Prop. (2.2)] and to [2, Prop. A5]. □

Definition 3.2. Given $A = (A_y)_{y\in Y} \in \mathfrak{A}$ as in Proposition 3.1, we denote by $\ker \tilde{A}$ the bundle $(\ker \tilde{A}_y)_{y\in Y}$ and define

$$\operatorname{ind}_a(A) = [\ker \tilde{A}] - [Y \times \mathbb{C}^q] \in K(Y).$$

This is independent of the choices of q and of $s_1, \cdots, s_q$ and we call it *the analytical index of A*.

If $A = (A_y)_{y\in Y} \in M_k(\mathfrak{A})$ is a section such that each A_y is a Fredholm operator on H_y^k then the projection to $M_k(\mathfrak{A}/\mathfrak{K})$ is invertible and hence defines an element of $K_1(\mathfrak{A}/\mathfrak{K})$. Since $\operatorname{ind}_a(A)$ is invariant under stabilization, homotopies and perturbations by compact operator valued sections, we get a homomorphism

$$\operatorname{ind}_a : K_1(\mathfrak{A}/\mathfrak{K}) \longrightarrow K(Y). \tag{12}$$

Next we define the *topological* index, also as a homomorphism

$$\operatorname{ind}_t : K_1(\mathfrak{A}/\mathfrak{K}) \longrightarrow K(Y).$$

Let T^*Z denote the union of all T^*Z_y, and B^*Z the union of all B^*Z_y, equipped with their canonical topologies, where B^*Z_y denotes the bundle of closed unit balls of T^*Z_y. One may regard B^*Z as a compactification of T^*Z and identify the "points at infinity" with S^*Z.

Let $\sim$ denote the equivalence relation that identifies, for each $y \in Y$, all points of each ball of B^*Z_y which lies over a point of ∂Z_y. The C*-algebra B of Theorem 2.11 is isomorphic to the algebra of continuous functions on the quotient space $S^*Z/\sim$. Let $\beta : K_1(C(S^*Z/\sim)) \to K_0(C_0(T^*Z^\circ))$ denote the index map associated to the short exact sequence

$$0 \longrightarrow C_0(T^*Z^\circ) \longrightarrow C(B^*Z/\sim) \longrightarrow C(S^*Z/\sim) \longrightarrow 0,$$

where T^*Z° is the union over $y \in Y$ of all points of T^*Z_y which lie over interior points of Z_y and the map from $C(B^*Z/\sim)$ to $C(S^*Z/\sim)$ is induced by restriction.

Let $2Z$ denote the union $\bigcup_y 2Z_y$, where each $2Z_y$ is the double of Z_y, and $\pi_d : 2Z \to Y$ the canonical projection. This can be given the structure of a Diff$(2X)$-bundle, with trivializations obtained by "doubling" (as explained at the beginning of Section 2) the trivializations of the bundle $\pi : Z \to Y$. Each fiber $2Z_y$ is then equipped with the smooth structure induced by the trivializations of $\pi_d : 2Z \to Y$ and we can form the bundles T^*2Z and S^*2Z as the unions, respectively, of all cotangent bundles $T^*(2Z_y)$ and of all cosphere bundles $S^*(2Z_y)$, $y \in Y$. We denote by $\text{AS-ind}_t : K_0(C_0(T^*2Z)) \to K(Y)$ the composition of Atiyah and Singer's [3] topological families-index for the bundle of closed manifolds $2Z$ with the canonical isomorphism $K(T^*2Z) \simeq K_0(C_0(T^*2Z))$. Theorem 2.11 allows us to define the topological index:

Definition 3.3. The topological index ind_t is the following composition of maps

$$\text{ind}_t : K_1(\mathfrak{A}/\mathfrak{K}) \overset{\bar\sigma_* \circ \iota_*^{-1}}{\longrightarrow} K_1(C(S^*Z/\sim)) \overset{\beta}{\longrightarrow} K_0(C_0(T^*Z^\circ)) \overset{e_*}{\longrightarrow} K_0(C_0(T^*2Z)) \overset{\text{AS-ind}_t}{\longrightarrow} K(Y),$$

where $e : C_0(T^*Z^\circ) \to C_0(T^*2Z)$ denotes the map which extends by zero.

If $A = (A_y)_{y\in Y} \in \mathfrak{A}$ is a family of Fredholm operators we denote by $\text{ind}_t(A)$ the topological index evaluated at the element of $K_1(\mathfrak{A}/\mathfrak{K})$ that A defines.

Theorem 3.4. *Let $A = (A_y)_{y\in Y} \in \mathfrak{A}$ be a continuous family of Fredholm operators in the closure of the Boutet de Monvel algebra for each y. Then*

$$\text{ind}_a(A) = \text{ind}_t(A). \tag{13}$$

Proof. Our strategy is to derive the equality of the indices from the classical Atiyah–Singer index theorem for families [3, Thm. (3.1)]. To this end we define an operator family $\hat{A}$ acting on a vector bundle over the double of Z by a gluing technique involving the principal symbol family of A. We proceed in several steps. Step 1 consists of a few preliminary remarks on the choice of the representative of the K-theory class of A. In Step 2 we describe the construction of the bundle. We then define the operator family $\hat{A}$ over $2Z$ in

Step 3. Its topological index coincides with that of A as we shall see in Step 4. The equality of the analytic indices of A and $\hat{A}$ is the content of Step 5.

Step 1. We need to prove that ind_t and ind_a coincide on $K_1(\mathfrak{A}/\mathfrak{K})$. Using that $K_1(\mathfrak{A}/\mathfrak{K}) = K_1(\mathfrak{A}^\dagger/\mathfrak{K})$ by Theorem 2.11, an arbitrary element of $K_1(\mathfrak{A}/\mathfrak{K})$ is a class $[[A]]_1$ (the inner brackets denoting a class in the quotient by the compacts), for some operator family $A = (A_y)_{y\in Y} \in M_k(\mathfrak{A}^\dagger)$, $k \in \mathbb{N}$, such that, for each y, $A_y : H^k_y \to H^k_y$ is a Fredholm operator with symbol in B. It will be convenient to pick a representative with special properties. We denote by $C^\infty(S^*X/\sim)$ the subset of $C^\infty(S^*X)$ of functions which factor through $S^*X/\sim$, i.e. are independent of the co-variable at the boundary. The algebraic tensor product $C_0(U) \otimes C^\infty(S^*X/\sim)$ is dense in $C_0(U \times S^*X/\sim)$ for every open subset U of Y. Furthermore, the inclusion of the space of all elements in $C^\infty(S^*X/\sim)$ which are independent of the co-variable even in a neighborhood of ∂Z into $C^\infty(S^*X/\sim)$ is a homotopy equivalence. We can therefore assume that the symbol family $(\sigma_y(A_y))_{y\in Y}$ is given as a finite sum of elements supported in open subsets U of Y over which Z is trivial, and each of these is a pure tensor in $C_0(U) \otimes C^\infty(S^*X)$ which is independent of the co-variable near the boundary. Hence it suffices to prove equality for such an A.

Step 2. For each $y \in Y$, let Z^+_y and Z^-_y denote the two copies of Z_y which are glued together at ∂Z_y to form $2Z_y$. The map $i_y : \partial Z^+_y \to \partial Z^-_y$ identifies the two copies of ∂Z_y. We define E_y as the quotient of the disjoint union $Z^+_y \times \mathbb{C}^k \cup Z^-_y \times \mathbb{C}^k$ by the equivalence relation that identifies the pairs (x, v) and (x', w) if and only if they are equal or $x' = i_y(x)$, $x \in \partial Z^+_y$, and $w = \sigma_y(A_y)(x)v$ (remembering that at points of S^*Z_y over ∂Z_y, $\sigma_y(A_y)$ is independent of the covector variable). This set E_y naturally becomes a smooth vector bundle over Z_y. Let E denote the union of all E_y, which in the same way becomes a vector bundle over Y.

When defining families of smooth manifolds with smooth vector bundles, Atiyah and Singer make the technical assumption that the fiberwise vector bundles are isomorphic to a fixed vector bundle on the typical fiber. If Y is not connected, this is not necessarily satisfied. However, the isomorphism type of E_y depends only on the homotopy type of the map σ_y, in particular only on the component of the space of all continuous maps from ∂Z_y to $M_k(\mathbb{C})$ in which it lies. As Y is compact, it decomposes into finitely many open and closed subsets over each of which the isomorphism type of E_y is constant. As the K-theory of Y as well as $\mathfrak{A}/\mathfrak{K}$ split as direct sums under such disjoint union decompositions of Y, and as ind_a, ind_t respect this, we can restrict to one such subset of Y. Then we are canonically in the situation of [3, Def. 1.2], i.e. E is a smooth vector bundle over the family of smooth manifolds $2Z$.

Step 3. Let $\pi_s : S^*2Z \to 2Z$ denote the canonical projection and S^*Z^+ and S^*Z^-, respectively, the union of all $S^*Z^+_y$ and $S^*Z^-_y$, $y \in Y$. The bundle π_s^*E can be seen as the disjoint union of $S^*Z^+ \times \mathbb{C}^k$ and $S^*Z^- \times \mathbb{C}^k$ quotiented by the equivalence relation that identifies a boundary point (s, v) in $S^*Z^+ \times \mathbb{C}^k$

with $(s, \sigma_A(s) \cdot v)$ in $S^*Z^- \times \mathbb{C}^k$. Similarly, the bundle $S^*2Z \times \mathbb{C}^k$ can be seen as the disjoint union of $S^*Z^+ \times \mathbb{C}^k$ and $S^*Z^- \times \mathbb{C}^k$ quotiented by the equivalence relation that identifies a boundary point (s, v) in $S^*Z^+ \times \mathbb{C}^k$ with (s, v) in $S^*Z^- \times \mathbb{C}^k$. We then define $\hat{a} \in \mathrm{Hom}(\pi_s^*E,\, S^*2Z \times \mathbb{C}^k)$ by

$$\hat{a}(s,v) = \begin{cases} \sigma_A(s) \cdot v, & \text{if } (s,v) \in S^*Z^+ \times \mathbb{C}^k, \\ v, & \text{if } (s,v) \in S^*Z^- \times \mathbb{C}^k. \end{cases} \tag{14}$$

We want to show that $\hat{a}$ is the symbol of a continuous family of pseudodifferential operators. As any element of $\mathrm{Hom}(\pi_s^*E,\, S^*2Z\times\mathbb{C}^k)$, our $\hat{a}$ can be regarded as a family $(\hat{a}_y)_{y\in Y}$, $\hat{a}_y \in \mathrm{Hom}(\pi_s^*E_y,\, S^*2Z_y \times \mathbb{C}^k)$. It is easily checked that our definition of $\hat{a}$ indeed mends continuously at boundary points. But more is true. Since $\sigma_y(A_y)$ is smooth and independent of the co-variable near the boundary, each $\hat{a}_y$ is smooth. Moreover, since we assumed in Step 1 that a is a finite sum of local elementary tensors, we see that $\hat{a}$ is the symbol of an Atiyah–Singer family of pseudodifferential operators on $2Z$.[1]

Step 4. Let $\iota : K_0(C_0(T^*2Z)) \to K(B^*2Z, S^*2Z) \simeq K(T^*2Z)$ denote the canonical isomorphism (we refer to [4] and mainly [12] for topological K-theory definitions and notation). By Definition 3.3, it is enough to show that $\iota(e_*(\beta([\sigma_A]_1)))$ is equal to the element of $K(B^*2Z, S^*2Z)$ defined by the triple $(\pi_b^*E,\, B^*2Z \times \mathbb{C}^k,\, \hat{a})$, where $\pi_b : B^*2Z \to 2Z$ denotes the canonical projection.

The main step here is to understand $\beta([\sigma_A]_1)$. Now, σ_A can and will be considered as a function on $S^*Z/\sim$ with values in $Gl_k(\mathbb{C})$, representing an element in $K_1(C(S^*Z/\sim))$ and at the same time the corresponding element of the topological K-theory $K^1(S^*Z/\sim)$, [12, 3.2]. Recall from [12, 3.21] that for the pair of compact topological spaces $S^*Z/\sim\, \subset B^*Z/\sim$, the boundary map in topological K-theory assigns to σ_A the relative K-class $((B^*Z/\sim) \times \mathbb{C}^k, (B^*Z/\sim) \times \mathbb{C}^k, \sigma_A)$, corresponding under the excision isomorphism $K((B^*Z/\sim),\, (S^*Z/\sim)) \cong K(B^*Z, S^*Z)$ to $(B^*Z \times \mathbb{C}^k, B^*Z \times \mathbb{C}^k, \sigma_A)$, compare [12, 2.35]. Moreover, this corresponds to β under the isomorphism with C*-algebra K-theory. We next have to compute the map $e^{top} : K(B^*Z, S^*Z) \to K(B^*2Z, S^*2Z)$ in topological K-theory, representing $e_* : K_0(C_0(T^*Z)) \to K_0(C_0(T^*2Z))$. Recall, however, that $e^{top}(V, W, \tau)$ is given by any extension $\tilde{V}$ of V, $\tilde{W}$ of W to B^*2Z and an extension of τ to an isomorphism $\tilde{\tau}$ between $\tilde{V}$ and $\tilde{W}$ on all of $(B^*2Z \setminus B^*Z) \cup S^*Z$, $\tilde{\tau}$ finally restricted to S^*2Z. Finally, observe that $(\pi_b^*E, B^*2Z \times \mathbb{C}^k, \hat{a})$ provides exactly such an extension (as $\hat{a}$ extends as id over all of $B^*2Z \setminus B^*Z$) and therefore represents $\iota e_*(\beta([\sigma_A]))$, as we had to prove.

Step 5. In order to show that the analytic indices coincide, we will introduce yet another operator family. Since $\sigma(A)$ is independent of the co-variable near the boundary, there is an open set $U \subseteq 2Z$ containing $Z^- = \bigcup_y Z_y^-$ and a

[1]Recall that they use a slightly stricter definition of operator families: While we here require continuity of the family with respect to the $L^2(X)$-operator norm, they take into account the norms on the whole range of Sobolev spaces.

bundle isomorphism

$$\Phi : E|_U \longrightarrow U \times \mathbb{C}^k$$

such that the restriction of $\hat{a}$ to $\pi_s^{-1}(U)$ is equal to the pullback of Φ by π_s. Let $(\chi_y^+)_{y\in Y}$ and $(\chi_y^-)_{y\in Y}$ be continuous families of smooth functions on $2Z$ with $0 \le \chi_y^\pm \le 1$, $(\chi_y^+)^2 + (\chi_y^-)^2 = 1$. Moreover, let the support of each χ_y^+ be contained in the interior of Z_y^+ and $\chi_y^+ \equiv 1$ outside a neighborhood of ∂Z_y^+ in U. Then

$$\hat{B}_y = \chi_y^+ \hat{A}_y \chi_y^+ + \chi_y^- \Phi_y \chi_y^-$$

defines a family of pseudodifferential operators in the sense of Atiyah and Singer which has the same principal symbol—and hence the same analytic index—as $\hat{A}$.

For each $y \in Y$, we canonically identify the space $L^2(E_y)$ of L^2-sections of E_y with the direct sum $L^2(Z_y^+;\mathbb{C}^k) \oplus L^2(Z_y^-;\mathbb{C}^k)$ and denote by $e_y^\pm$ and $r_y^\pm$ the maps of extension by zero and restriction,

$$e_y^\pm : L^2(Z_y^\pm;\mathbb{C}^k) \to L^2(E_y) \quad \text{and} \quad r_y^\pm : L^2(2Z_y;\mathbb{C}^k) \to L^2(Z_y^\pm;\mathbb{C}^k).$$

Then $B_y = r_y^+ \hat{B}_y e_y^+$ defines a continuous family $B = (B_y)_{y\in Y}$ in $M_k(\mathfrak{A})$. As $\sigma(A) = \sigma(B)$ (and hence $\gamma(A) = \gamma(B)$), it suffices to prove that the analytic indices of B and $\hat{B}$ are equal.

Proposition (2.2) of [3], applied to the family $\hat{B}$ provides us with sections $s_y^j \in C^\infty(2Z_y;\mathbb{C}^k)$, $y \in Y$, $1 \le j \le q$, such that

$$\begin{array}{rcl} \hat{Q}_y : C^\infty(2Z_y;E_y) \oplus \mathbb{C}^q & \longrightarrow & C^\infty(2Z_y;\mathbb{C}^k) \\ (u;\lambda_1,\cdots,\lambda_q) & \longmapsto & \hat{B}_y(u) + \sum_{j=1}^q \lambda_j s_y^j \end{array}$$

is onto, $\ker \hat{Q} = (\ker \hat{Q}_y)_{y\in Y}$ is a vector bundle and the analytic index of $\hat{B}$ is equal to $[\ker \hat{Q}] - [Y \times \mathbb{C}^q]$. Now let $t_y^j = r_y^+ s_y^j \in C^\infty(Z_y;\mathbb{C}^k)$. The continuity with respect to y that we get from [3, Prop. (2.2)] is enough to ensure that $(t_y^j)_{y\in Y}$ is a continuous section of our bundle of Hilbert spaces $\bigcup_{y\in Y} L^2(Z_y;\mathbb{C}^k)$. We then define

$$\begin{array}{rcl} Q_y : L^2(Z_y;\mathbb{C}^k) \oplus \mathbb{C}^q & \longrightarrow & L^2(Z_y;\mathbb{C}^k) \\ (u;\lambda_1,\cdots,\lambda_q) & \longmapsto & B_y(u) + \sum_{j=1}^q \lambda_j t_y^j. \end{array}$$

Since B_y is elliptic, $\ker Q_y \subset C^\infty(Z_y;\mathbb{C}^k)$. Using that Φ_y is local, it is straightforward to check that

$$\hat{B}_y = e_y^+ r_y^+ \hat{B}_y e_y^+ r_y^+ + e_y^- r_y^- \hat{B}_y e_y^- r_y^- = e_y^+ B_y r_y^+ + e_y^- r_y^- \Phi_y e_y^- r_y^-$$

and, hence, $\ker Q_y$ and $\ker \hat{Q}_y$ are isomorphic for each y (because Φ is an isomorphism). Moreover, Q_y is also surjective: Given $v \in L^2(Z_y;\mathbb{C}^k)$, if $u \in L^2(2Z_y;E_y)$ is a preimage of $e_y^+ v$ under $\hat{Q}_y$, then $r_y^+ u$ is a preimage of v under Q_y. Hence the analytic index of B is given by $[\ker Q] - [Y \times \mathbb{C}^q]$. The bundles $\ker Q = (\ker Q_y)_{y\in Y}$ and $\ker \hat{Q}$ are isomorphic and then

$$\operatorname{ind}_a(B) = [\ker Q] - [Y \times \mathbb{C}^q] = [\ker \hat{Q}] - [Y \times \mathbb{C}^q] = \operatorname{ind}_a(\hat{B}),$$

as we wanted. □

4. NONTRIVIAL BUNDLES

In this section we discuss families of Boutet de Monvel operators acting between vector bundles. The case considered in the first two sections correspond to the case of trivial bundles over the manifolds and the zero bundle over the boundary.

In addition to the data assumed up to this point (a bundle of manifolds $\pi : Z \to Y$ with fiber X), we take smooth vector bundles E and F over X and ∂X, respectively. Let $\mathrm{Diff}(\partial X, F)$ denote the group of diffeomorphisms of F which map fibers to fibers linearly, and let G_E denote the group of diffeomorphisms of E which map fibers to fibers linearly and whose restrictions to the base belong to the group G defined on page 347. We equip $\mathrm{Diff}(\partial X, F)$ with its canonical topology [3, p. 123] and do the same construction for G_E. Note that there are homomorphisms "forget the action in the fiber" $h_\partial : \mathrm{Diff}(\partial X, F) \to \mathrm{Diff}(\partial X)$ and $h : G_E \to G$. Define the fiber product group

$$G_r := \{(\phi, \psi) \in \mathrm{Diff}(\partial X, F) \times G_E \mid h_\partial(\phi) = h(\psi)\}.$$

Let $(p : \tilde{E} \to Z;\ q : \tilde{F} \to \partial Z)$ be maps such that $(\pi \circ p : \tilde{E} \to Y;\ \pi_\partial \circ q : \tilde{F} \to Y)$ are bundles with, respectively, fibers E and F and structure group G_r. It follows that, for each pair of local trivializations (α, β) of $(\pi \circ p : \tilde{E} \to Y;\ F \to Y)$ there are local trivialization α_0 of $\pi : Z \to Y$ and β_0 of $\partial Z \to Y$ such that the diagram

$$\begin{array}{ccc} (\pi \circ p)^{-1}(U) & \xrightarrow{\alpha} & U \times E \\ \Big\downarrow p & & \Big\downarrow \\ \pi^{-1}(U) & \xrightarrow{\alpha_0} & U \times X \end{array} \tag{15}$$

commutes, where the right vertical arrow is the identity on U times the bundle projection on E. This defines a vector bundle structure for $p : \tilde{E} \to Z$. Moreover, for each $y \in Y$, the restriction of p to $\tilde{E}_y = (\pi \circ p)^{-1}(y)$ defines a smooth vector bundle $p_y : \tilde{E}_y \to Z_y$, isomorphic to $E \to X$. We obtain the corresponding result for the map q and get a vector bundle $q : \tilde{F} \to \partial Z$ and, for each $y \in Y$, a smooth vector bundle $q_y : \tilde{F}_y \to \partial Z_y$ isomorphic to $F \to \partial X$.

Choose now, in addition to the family of Riemannian metrics $(g_y)_{y \in Y}$ families of Hermitean metrics on E_y and F_y which depend continuously on $y \in Y$. Using them, we get families of Hilbert spaces $H_y := L^2(Z_y; E_y) \oplus L^2(\partial Z_y; F_y)$ which patch together to a bundle of Hilbert spaces. Let $\mathcal{A}(E, F)_y$ denote the C^*-subalgebra of the algebra of all bounded operators on H_y generated by the polyhomogeneous Boutet de Monvel operators of order and class zero.

Exactly as [3, Prop. 1.3] our Proposition 2.3 generalizes to the case of nontrivial bundles and their diffeomorphisms and is the basis for the generalization of Corollary 2.4 to the case of nontrivial bundles: the $\mathcal{A}(E, F)_y$ form in a canonical way a continuous bundle of C^*-algebras, which we continue to call $\aleph$ by abuse of notation.

Let $\mathfrak{A}$ denote the set of continuous sections of the bundle $\aleph$, forming again a C*-algebra with pointwise operations and supremum norm. The K-theory results of Section 2 can be extended to this more general setting using arguments similar to those used in [16]. In particular, the analytic and topological index given in Section 3 can also be defined as maps $K_1(\mathfrak{A}) \to K(Y)$. Theorem 3.4 then extends to this more general setting.

Remark 4.1. Variants of Theorem 3.4, the family index theorem for the Boutet de Monvel algebra for real K-theory or for equivariant K-theory should hold as well, and one should be able to derive them along the lines used in the present article.

Appendix A. Reduction of the structure group

Let, as in the main body of the text, X be a compact smooth manifold with boundary ∂X, and fix a collar diffeomorphism $\delta : U \to \partial X \times [0, 1)$ with collar coordinate x_n. Recall that G was defined as the subgroup of the diffeomorphism group $\mathrm{Diff}(X)$ of those diffeomorphisms which respect the product structure and collar coordinate for $x_n \in [0, 1/2)$. For convenience, in the text we were working with bundles of manifolds modelled on X and with structure group G, i.e. with a canonically defined collar of the boundary in each fiber of the bundle.

In this appendix, we prove that, for any bundle (over a paracompact space) with structure group $\mathrm{Diff}(X)$ we have a unique (up to isomorphism) reduction to the structure group G. In other words, the functor from bundles (over a given paracompact base) with structure group G to bundles with structure group $\mathrm{Diff}(X)$ which "forgets the collar" is an equivalence of categories. [This is similar to the (unique up to isomorphism) choice of a Riemannian metric on a given finite dimensional vector bundle: reduction of the structure group from $GL(n)$ to $O(n)$.]

It is well known that we get this unique reduction of structure group if the inclusion $G \to \mathrm{Diff}(X)$ is a homotopy equivalence, compare [6] for a rather refined version of this fact. We therefore show

Theorem A.1. *The inclusion $G \to \mathrm{Diff}(X)$ (and therefore the corresponding map $BG \to B\,\mathrm{Diff}(X)$ of classifying spaces) are homotopy equivalences.*

Proof. Observe first that G and $\mathrm{Diff}(X)$ as well as BG and $B\,\mathrm{Diff}(X)$ are paracompact Fréchet manifolds by [13, Sec. 41, 42, 44.21] (the reference is for $\mathrm{Diff}(X)$, but the proofs easily generalize to G). Therefore it suffices by [18, Thm. 15] to show that $G \to \mathrm{Diff}(X)$ is a weak homotopy equivalence and it follows automatically that it is a homotopy equivalence.

To show that the map is a weak homotopy equivalence, we have for a continuous map $f : K \to \mathrm{Diff}(X)$, where K is a compact CW-complex, to construct a homotopy f_s from $f_0 = f$ to an f_1 which takes values in G. Moreover, the homotopy should be constant on every CW-subcomplex K_0 of K where f already maps to G. Note that K_0 is a deformation retract of a neighborhood U, i.e. there is a homotopy $h : K \times [0,1] \to K$ from the identity to h_1 such that $h_1(U) = K_0$ and such that h_t is the identity on K_0. By precomposing with h_1 we can therefore assume that f maps the neighborhood U of K_0 to G.

Let us now construct the family f_t. Choose $\eta \in (0,1]$ such that $\tilde{f}(k) = \delta \circ f(k) \circ \delta^{-1}$ maps $\partial X \times [0,\eta)$ to $\partial X \times [0,1)$ for all $k \in K$ and write $\tilde{f}(k)(x',t) = (\varphi(x',t;k), \tau(x',t;k))$.

In two steps we shall now first deform τ to a function $\hat{\tau}$ which equals t for small t and then φ to a function which depends only on x' for small t.

Observe that, as $f(k)$ is a diffeomorphism of a manifold with boundary, $\frac{\partial \tau}{\partial t} > 0$ and therefore, by the compactness of K, if we choose η small enough, $C > \frac{\partial \tau}{\partial t} > c > 0$ for some $C > c > 0$ on all of $K \times \partial X \times [0,\eta)$.

Pick a smooth function $a : [0,\eta) \to [0,1]$ such that $a(t) \equiv 0$ for t close to zero, $a(t) \equiv 1$ for t close to η and such that

$$\hat{\tau}(x',t;k) = (1-a(t))t + a(t)\tau(x',t;k), \quad (x',t) \in \partial X \times [0,\eta),$$

satisfies $\partial\hat{\tau}(x',t;k)/\partial t \geq c/2$ for every $x' \in \partial X$ end every $k \in K$. To construct such an a, we use the uniform growth of τ: Choose, for some given $\varepsilon > 0$, the function a so that $(1-a)t$ is monotonely increasing on the interval $[0,4\varepsilon]$ with $(1-a)t = t$ on $[0,\varepsilon]$ and $(1-a)t = 2\varepsilon$ on $[3\varepsilon,4\varepsilon]$. Then a can be taken to be increasing with $a \equiv 0$ near 0 and $a(4\varepsilon) = 1/2$. Moreover, $\hat{\tau}$ is strictly increasing as τ is. Finally choose a on $[4\varepsilon,\eta]$ such that $(1-a)t$ monotonely decreases to 0 and equals zero on $[\eta-\varepsilon,\eta]$. Moreover, we arrange for the derivative $\partial_t((1-a)t)$ to be always $\geq -2\frac{2\varepsilon}{\eta-5\varepsilon}$. Here, a is necessarily increasing with $a \equiv 1$ near η. The derivative $\partial_t(a\tau)$ can therefore be estimated from below by $c/2$. For ε sufficiently small, we will have $2\frac{2\varepsilon}{\eta-5\varepsilon} < c$ and thus $\partial_t\hat{\tau}(x',t;k) > 0$ for all x', t, k. Note that $\hat{\tau}(x',t;k) = t$ for t close to zero, and $\hat{\tau}(x',t;k) = \tau(x',t;k)$ for t close to η, uniformly in k. We then let

$$\tau_s = s\hat{\tau} + (1-s)\tau, \quad 0 \leq s \leq 1.$$

Then $\frac{\partial \tau_s}{\partial t} \geq c/2$ on $K \times \partial X \times [0,\eta)$, if we assume without loss of generality that $c < 1$.

For the second step fix a smooth function $\rho : [0,1) \to [0,1)$ with $\rho(t) = 0$ for $t < \varepsilon$ and $\rho(t) = t$ for $t > 1-\varepsilon$. Next choose a smooth family of smooth functions ρ_s, $0 \leq s \leq 1$ such that ρ_0 is the identity and $\rho_1 = \rho$. By compactness, we have a uniform bound $|d\rho_s(t)/dt| \leq R$. For a given $\eta > 0$, define $\rho_s^\eta(t) : [0,\eta) \to [0,\eta)$, $t \mapsto \eta\rho_s(\eta^{-1}t)$. Then still $|d\rho_s^\eta/dt| \leq R$, even independently of η.

Let $\varphi_s^\eta(x',t) := \varphi(x',\rho_s^\eta(t))$ and $\tilde{f}_s^\eta(k)(x',t) = (\varphi_s^\eta(x',t), \tau_s(t))$. Then $\tilde{f}_s^\eta$ equals the given $\tilde{f}$ for t close to η. Therefore $f_s^\eta = \delta^{-1} \circ \tilde{f}_s \circ \delta$ extends for each s to a self-map of X. The Jacobians $\partial\tilde{f}_s^\eta$ and $\partial\tilde{f}$ are $n \times n$ matrices. For $t = 0$ we have $\frac{\partial\tau}{\partial x'} = \frac{\partial\tau_s}{\partial x'} = 0$, so that the first $n-1$ entries of the last row vanish in both cases, while $\frac{\partial\tau}{\partial t}$ and $\frac{\partial\tau_s}{\partial t}$ are strictly positive there. As $\partial\tilde{f}$ is invertible, so is $\frac{\partial\varphi(x',0)}{\partial x'}$, hence $\frac{\partial\varphi_s^\eta(x',t)}{\partial x'}|_{t=0} = \frac{\partial\varphi(x,\rho_s^\eta(t))}{\partial x'}|_{t=0}$, and hence $\partial\tilde{f}_s^\eta(k)(x',0)$. Noting that $\rho_s^\eta < \eta$, that $\frac{\partial\tau_s}{\partial x}(x',t)$ is independent of η, and that $\frac{\partial\rho_s^\eta}{\partial t}$ and hence $\frac{\partial \varphi_s^\eta}{\partial t}$ are bounded independently of η, we conclude from continuity that $\partial \tilde{f}_s^\eta(k)(x',t)$ is invertible on $\partial X \times [0,\eta)$, provided η is sufficiently small. So $f_s(k)$ is a submersion for all s,k.

We check that we actually constructed diffeomorphisms. We made our construction such that all the maps $f_s(k)$ are submersions which map the boundary to itself, therefore the image is an open subset of X. As X is compact, the image is also closed, and the map being a local diffeomorphism, is a covering map. Because it is homotopic to the diffeomorphism $f(k)$, it is a trivial covering map and therefore a diffeomorphism.

It is obvious that $f_0 = f$ and $f_1(k)$ lies in the variant of G where $1/2$ is replaced by $\eta - \epsilon$.

Next, we compose with a family of reparametrizations of the collar $[0,1)$ which stretches $[0,\eta-\epsilon)$ to $[0,1/2)$ such that in the end we really map to G. Note that our construction is carried out in such a way that for $k \in U$, where $f(k)$ was already in G, $f_s(k) \in G$ for all s, although, because of the last reparametrization step, not necessarily $f_s(k) = f(k)$.

Therefore, finally, we choose a function $\beta : K \to [0,1]$ which is 1 outside U and 0 on K_0 and replace the homotopy $f_s(k)$ with $f_{\beta(k)s}(k)$.

This yields the desired homotopy from $f_0 = f$ to an f_1 taking values in G. Moreover, the mapping is constant on K_0. □

Appendix B. The Künneth formula

By the "Künneth formula", we mean the following theorem of Schochet [20]:

Theorem B.1. *Let A and B be C^*-algebras with A in the smallest subcategory of the category of separable nuclear C^*-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed product by $\mathbb{Z}$ and by $\mathbb{R}$. Then there is a natural $\mathbb{Z}/2$-graded exact sequence*

$$0 \to K_*(A) \otimes K_*(B) \to K_*(A \otimes B) \to \operatorname{Tor}(K_*(A), K_*(B)) \to 0. \tag{16}$$

We use this Theorem to prove a statement made in the proof of Theorem 2.8:

Proposition B.2. $b_{U*} : K_i(C_0(\partial Z_U)) \to K_i(\operatorname{Im}\gamma_U)$ *is an isomorphism, $i = 0, 1$.*

Proof. Let $A = C_0(U)$ and $B = C(\partial X)$. Then $\operatorname{Im}\gamma_U$ is equal to $A \otimes C$, where C is the image of the boundary principal symbol map for the single manifold X. As explained in the introduction of [15], C can be regarded as a C*-subalgebra of $C(S^*\partial X) \otimes \mathcal{T}$, where $\mathcal{T}$ denotes the Toeplitz algebra. Since $\mathcal{T}$ belongs to the category defined in the statement of Theorem B.1 (see Examples 5.6.4 and 6.5.1 in [17]), we may apply Schochet's theorem for $A \otimes B$ and for $A \otimes C$.

Now let $\mathtt{b} : C(\partial X) \to C$ be the map analogous to the map b defined right before the statement of Theorem 2.8. In [15, Sec. 3], it is proven that $\mathtt{b}$ induces a K-theory isomorphism ($\mathtt{b}$ was denoted b in [15, 16]). Using that the exact sequence of Theorem B.1 is natural, we can map (16) to the corresponding sequence obtained by replacing B with C. Since the maps induced by $\mathtt{b}$ are isomorphisms, it follows from the five-lemma that the maps induced by $b_U = \mathrm{id}_A \otimes \mathtt{b}$ are also isomorphisms. □

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