Mathematik

On Finitely Summable K-Homology

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

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– 2012 –

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Tag der mündlichen Prüfung:	29. Juni 2012
Tag der Promotion:	29. Juni 2012

Abstract

We define finitely summable K-homology K^*_{fin} for the category of topological $*$-algebras in terms of homotopy classes of finitely summable Fredholm modules and study various properties of this theory: we show that K^*_{fin} is invariant under stabilization with the algebras of Schatten class operators, but that it is not additive with respect to countable direct sums of topological $*$-algebras. We calculate the finitely summable K-homology of AF-algebras and discuss the theory for manifolds. Moreover, we consider classes of algebras for which K^*_{fin} degenerates. In particular, we prove that there cannot exist any finitely summable Fredholm modules of interest over the convolution algebra $\ell^1(\Gamma)$ of any discrete group Γ.

Zusammenfassung

Wir definieren für die Kategorie der topologischen $*$-Algebren endlich summierbare K-Homologie-Gruppen K^*_{fin}, deren Elemente durch Homotopieklassen endlich summierbarer Fredholm-Moduln gegeben sind, und studieren einige ihrer Eigenschaften: Wir zeigen, dass K^*_{fin} invariant ist unter Stabilisierung mit Schatten-Klassen, im Allgemeinen jedoch nicht additiv für abzählbar unendliche direkte Summen von topologischen $*$-Algebren ist. Des Weiteren berechnen wir die endlich summierbare K-Homologie von AF-Algebren und behandeln den Fall von Mannigfaltigkeiten. Darüber hinaus untersuchen wir einige Klassen von Algebren, für die K^*_{fin} degeneriert. Insbesondere beweisen wir, dass abgesehen von trivialen Beispielen keine endlich summierbaren Fredholm-Moduln über der Faltungsalgebra $\ell^1(\Gamma)$ einer diskreten Gruppe Γ existieren können.

Acknowledgements

First, I would like to thank my advisor Joachim Cuntz. Apart from giving me the opportunity to work on this subject, his guidance and encouragement throughout the years were of great help.

I thank Michael Puschnigg for kindly inviting me to Marseille and the many interesting discussions we had.

Working with my colleagues was always a great pleasure. They contributed a lot to making the good times more enjoyable and the bad times less distressing. Especially, I have to thank Martin Engbers.

I am deeply indebted to those who first taught me the joy of mathematics, in particular my father, Gert Broelemann, Inge Hachtel, Heiner Platzbecker, and Gisela Schuller.

Most importantly though, I would like to thank my parents and Birgit for all their support.

Contents

Introduction

The index theorem of Atiyah and Singer states that the Fredholm index of an elliptic differential operator on a closed smooth manifold agrees with its topological index, which is computed in terms of the K-theory class associated to the symbol of the operator. Applying the Chern character, a transformation of the K-theory of a manifold to its de Rham cohomology, yields an explicit formula for the index in terms of characteristic classes.

K-homology is a generalized homology theory dual to K-theory. Its cycles are given by Fredholm modules, which can be thought of as abstract elliptic operators: each elliptic operator determines a Fredholm module, and the index of the operator can be expressed by the duality pairing between K-theory and the K-homology class of the associated Fredholm module.

Like K-theory, K-homology naturally extends to non-commutative spaces. In his groundbreaking work on non-commutative differential geometry [Con85], Connes defines a non-commutative Chern character, which assigns to a Fredholm module a class in periodic cyclic cohomology, the non-commutative analogue of de Rham homology. Using this character, he obtains an explicit trace formula for the duality pairing, which leads to non-commutative generalizations of the index formula by Atiyah and Singer.

However, there is one drawback to this construction: to define the Chern character, the Fredholm module must satisfy the strong regularity condition of being finitely summable. It is therefore of great interest to determine which K-homology classes of a non-commutative space have a finitely summable Fredholm module as a representative. Moreover, it is desirable to have a homology theory at hand on which Connes' Chern character is well-defined, as this is not the case for standard K-homology.

One obvious approach to these questions is to restrict the class of Fredholm modules one considers in the definition of K-homology to only those which are finitely summable. We call the theory which is obtained this way 'finitely summable K-homology'. In this thesis we strive to get a better understanding of this theory about which very little is known so far.

Basic concepts. Let D be an elliptic first-order differential operator between vector bundles V and W over a smooth, closed manifold M of dimension n.

Elliptic theory states that D is a Fredholm operator. It is the aim of index theory to compute its Fredholm index.

Atiyah proposed to condense the properties of D which are crucial for the theory into the notion of an abstract elliptic operator. These abstract elliptic operators became the building blocks of Kasparov's analytic K-homology [Kas75], where they are called Fredholm modules. Little later, this development culminated in Kasparov's celebrated bivariant K-theory, encompassing K-theory and K-homology.

We can easily describe how to obtain a Fredholm module from D: if φ is the representation of $C(M)$ by multiplication on the L^2-sections[†] $L^2(M, V \oplus W)$ of $V \oplus W$, then the Fredholm module associated to D is given as the triple

$$\mathscr{F}_D := \left(\varphi,\, L^2(M, V \oplus W),\, F := \begin{bmatrix} 0 & D_0^* \\ D_0 & 0 \end{bmatrix}\right) \qquad \text{where} \;\; D_0 := \frac{\overline{D}}{\sqrt{I + \overline{D}^* \overline{D}}}$$

and $\overline{D}$ denotes the closure of D.

By elliptic theory, $(I + \overline{D}^* \overline{D})^{1/2}$ has a compact inverse, which easily implies that $F^2 - I$ is a compact operator as well. Moreover, since D is of first order, the commutators between D and operators of multiplication by smooth functions are given by multiplication with the symbol of D, which is a bounded operator from $L^2(M, V)$ to $L^2(M, W)$. Using this and the compactness of $(I + \overline{D}^* \overline{D})^{-1/2}$, one can show that the commutators of F with φ are not only bounded but even compact.

These two properties,

$$F^2 - I \in \mathcal{K}(L^2(M, V \oplus W)) \quad \text{and} \quad [F, \varphi(C(M))] \subseteq \mathcal{K}(L^2(M, V \oplus W)),$$

are the defining properties for $\mathscr{F}_D$ to be a Fredholm module over $C(M)$. Moreover, the direct sum decompositions of $L^2(M, V \oplus W) = L^2(M, V) \oplus L^2(M, W)$ and φ, with respect to which F is an odd operator, add additional grading information to $\mathscr{F}_D$ making it a so-called even Fredholm module.

The even K-homology group $K^0(C(M))$ of $C(M)$ is obtained by considering all even Fredholm modules over $C(M)$, forming equivalence classes of homotopic or unitarily equivalent modules, and then taking the Grothendieck group with addition given by the direct sum of Fredholm modules. The odd K-homology group $K^1(C(M))$ is defined similarly by considering ungraded Fredholm modules over $C(M)$.

One can easily check that the Fredholm index of D is the same as the graded index of F.[‡] As the Fredholm index is homotopy invariant and additive for direct sums of Fredholm operators, the index of D can be computed using any representative of $[\mathscr{F}_D] \in K^0(C(M))$.

[†] With respect to arbitrary metrics on M, V, and W.

[‡] I.e. the Fredholm index of D_0.

The duality pairing between K-homology and K-theory, commonly referred to as the index pairing,

$$\mathrm{Ind} : K^0(C(M)) \times K_0(C(M)) \longrightarrow \mathbb{Z}$$

generalizes the operation of computing the Fredholm index of F. In particular, the index of D agrees with $\mathrm{Ind}([\mathscr{F}_D], [1_{C(M)}])$.

Connes' formula for the index pairing is given as follows. Choose any representative $\mathscr{F}' = (\varphi', \mathcal{H}', F')$ of $[\mathscr{F}_D]$ satisfying $F'^* = F'$ and $F'^2 = I$ (such a representative can always be found). If p is a projection in $C^\infty(M)$ and γ is the grading operator[†] of $L^2(M, V \oplus W)$, then the index pairing between $[\mathscr{F}_D]$ and $[p]$ can be computed as

$$\mathrm{Ind}([\mathscr{F}_D], [p]) = \frac{(-1)^k}{2} \operatorname{Tr}\Big(\gamma F' \underbrace{[F', \varphi(p)][F', \varphi(p)] \ldots [F', \varphi(p)]}_{(2k+1)-\text{times}}\Big) \qquad (*)$$

where $2k+1 > n$ and Tr denotes the usual trace of Hilbert space operators (this formula easily generalizes to matrix algebras over $C^\infty(M)$). Note that if one interprets the commutators as taking derivatives and Tr as a kind of non-commutative integral, this formula can be seen as an integral over a non-commutative differential form.

For this formula to be well-defined, it is necessary that the argument of the trace belongs to the subalgebra $\mathcal{L}^1(L^2(M, V \oplus W)) \subseteq \mathcal{K}(L^2(M, V \oplus W))$ of trace-class operators, i.e. those compact operators whose sequence of characteristic values is ℓ^1-summable.

In the case of compact manifolds, this poses no problem. A refinement of Rellich's Lemma leads to an asymptotic lower bound on the growth of the eigenvalues of $(I + \overline{D}^* \overline{D})^{1/2}$. Its inverse lies in $\mathcal{L}^{n+1}(L^2(M, V))$, the class of compact operators whose sequence of characteristic values is ℓ^{n+1}-summable. This implies that $\mathscr{F}_D$ is $(n+1)$-summable over $C^\infty(M)$:

$$F^2 - I \in \mathcal{L}^{n+1}(L^2(M, V \oplus W)) \quad \text{and} \quad [F, \varphi(C^\infty(M))] \subseteq \mathcal{L}^{n+1}(L^2(M, V \oplus W)).$$

If a Fredholm module is p-summable for some $p < \infty$, we say that the module is finitely summable. We define the even finitely summable K-homology group $K^0_{fin}(C^\infty(M))$ of $C^\infty(M)$ to be the Grothendieck group of the equivalence classes of all finitely summable even Fredholm modules over $C^\infty(M)$ with respect to the same equivalence relations as for K-homology and with addition given again by the direct sum. $K^1_{fin}(C^\infty(M))$ is defined analogously.

Note that the commutator relation $[F, \varphi(f)] \in \mathcal{L}^{n+1}(L^2(M, V \oplus W))$ does not extend to $C(M)$ since $\mathcal{L}^{n+1}(L^2(M, V \oplus W))$ is not a closed subspace of $\mathcal{B}(L^2(M, V \oplus W))$ as $\mathcal{K}(L^2(M, V \oplus W))$ is. Thus, D does not define a finitely summable module over $C(M)$.

[†] $\gamma := I_{L^2(V)} \oplus -I_{L^2(W)}$.

Now, if we choose $\mathscr{F}'$ to be $(n+1)$-summable as well (which is always possible), formula $(*)$ makes sense since $(\mathcal{L}^{n+1})^{2k+1} \subseteq \mathcal{L}^1$.

In the language of cyclic homology, the $(2k+1)$-linear functional

$$\frac{(-1)^k}{2} \operatorname{Tr}\left(\gamma F' \underbrace{[F', \varphi(\cdot)][F', \varphi(\cdot)] \ldots [F', \varphi(\cdot)]}_{(2k+1)-\text{times}}\right)$$

is a cyclic cocycle on $C^\infty(M)$ whose class in $HP^0(C^\infty(M))$, the periodic cyclic cohomology of $C^\infty(M)$, is called the Chern-Connes character $ch^0(\mathscr{F}')$ of $\mathscr{F}'$ (up to a normalization constant depending on k). Together with the Chern character in K-theory, the index pairing transforms into the duality pairing between $HP^0(C^\infty(M))$ and $HP_0(C^\infty(M))$:

$$\operatorname{Ind}([\mathscr{F}_D], [p]) \ = \ \langle ch^0(\mathscr{F}'), \, ch_0([p]) \rangle.$$

In contrast to K-theory, we cannot expect that ch^0 extends to a well-defined map from $K^0(C^\infty(M))$ to $HP^0(C^\infty(M))$ since Connes' formula is only valid for finitely summable Fredholm modules. However, we can extend it to a well-defined map on finitely summable K-homology

$$ch^0 : K^0_{fin}(C^\infty(M)) \longrightarrow HP^0(C^\infty(M)).$$

A similar character ch^1 exists for $K^1_{fin}(C^\infty(M))$, and we obtain for both, even and odd K-homology, the following commutative diagram, in which the lower map is the duality pairing between periodic cyclic homology and cohomology:

$$\begin{array}{ccccc} K^*_{fin}(C^\infty(M)) & \times & K_*(C^\infty(M)) & \xrightarrow{\operatorname{Ind}} & \mathbb{Z} \\ \downarrow{\scriptstyle ch^*} & & \downarrow{\scriptstyle ch_*} & & \cap\!\mid \\ HP^*(C^\infty(M)) & \times & HP_*(C^\infty(M)) & \longrightarrow & \mathbb{C} \end{array}$$

It does not matter if we write $K_*(C^\infty(M))$ or $K_*(C(M))$ in this diagram, since K-theory is invariant under passing to 'smooth' subalgebras like $C^\infty(M)$. Yet, K^*_{fin}, like cyclic homology, does not make much sense as a theory for C^*-algebras: in general there are very few Fredholm modules which are finitely summable over a whole C^*-algebra.

The following questions are immediate:

— What classes of 'smooth' algebras are adequate for studying K^*_{fin}?

— If $\mathcal{A}$ is a dense subalgebra of a C^*-algebra A, how do $K^*_{fin}(\mathcal{A})$ and $K^*(A)$ relate? In many cases there is a natural map from $K^*_{fin}(\mathcal{A})$ to $K^*(A)$ by extending Fredholm modules over $\mathcal{A}$ to modules over A. Is this map surjective, i.e. can every K-homology class of A be realized as a finitely summable module over $\mathcal{A}$? Are there cases where this map is not injective (making K^*_{fin} a finer invariant than K^*)?

— How can K^*_{fin} be computed? In particular, is K^*_{fin} well-behaved with respect to constructions like short exact sequences, direct sums, stabilization or inductive limits?

In the course of this thesis, we will discuss these questions from various points of view.

Previous work. Literature treating finitely summable K-homology is very sparse. Most notable are Douglas' and Voiculescu's work on smooth sphere extensions [DV81], where they give a lower bound for the summability of non-trivial Fredholm modules over odd spheres, and more recently, Puschnigg's proof that there are (basically) no finitely summable Fredholm modules over the group algebras of lattices in higher rank Lie groups [Pus11]. We review these results in Sections 5.2 and 7.3. Apart from that, we are only aware of the paper by Salinas [Sal83], where a similar theory in the language of extensions is introduced but not extensively studied, and the approach by Wang [Wan95], which seems to have been unsuccessful as the announced papers containing his most important results never appeared.

Summary. We give a brief summary of the contents of this thesis.

Chapter 1 contains some prerequisites which we shall need further on. In particular, we introduce the Schatten classes of finitely summable operators $\mathcal{L}^p(\mathcal{H})$ and discuss some properties of pre-C^*-algebras. While we define K^*_{fin} for arbitrary topological $*$-algebras, pre-C^*-algebras come up naturally when dealing with K^*_{fin} since every finitely summable module has a pre-C^*-algebra as maximal domain (see Proposition 1.4.14). Moreover, the algebras of smooth functions on manifolds fall into this class. This makes them good candidates for non-commutative algebras of 'smooth' functions.

Apart from the contents of Chapter 1, we only assume some basic knowledge of functional analysis and C^*-algebra theory in most parts of this thesis. For the reader's convenience, we supplement some results we use in the appendix.

After a short account of standard K-homology and a few of its features, we introduce finitely summable K-homology in Chapter 2. We spend some time discussing various choices of equivalence relations and give a normalization result for finitely summable modules.

Chapter 3 collects various elementary results on K^*_{fin}. We prove that K^*_{fin} is well-behaved with respect to direct sums and unitization, and that K^*_{fin} is stable under taking the projective tensor product with the algebras $\mathcal{L}^p(\mathcal{H})$.

In Chapter 4 we compute K^*_{fin} for the class of AF-algebras. More precisely, we show the following: if $A = \overline{\bigcup_n A_n}$ is an AF-algebra, then $K^*_{fin}(\bigcup_n A_n)$ agrees with $K^*(A)$. While this does not sound very surprising, the proof requires some work.

The surprising difficulty in understanding K^*_{fin} for AF-algebras stems from the fact that K^*_{fin} is not very well-behaved with respect to direct limits. In particular, we show in Chapter 5 that K^*_{fin} is not σ-additive in general, even for algebraic direct sums. As a related result we prove that even if K^*_{fin} is σ-additive for a specific algebraic direct sum, this by no means implies that we can pass to smooth completions of this direct sum (like sequences of rapid decay).

We discuss the finitely summable K-homology of manifolds in Chapter 6. In the first section we fill in some details we left out in this introduction and give an account of the well-known fact that every K-homology class of a closed manifold is represented by a finitely summable module. In the second section we discuss the difficulties that arise if one tries to prove that finitely summable K-homology and standard K-homology of a closed manifold really are isomorphic.

In the last chapter we consider classes of algebras for which K^*_{fin} degenerates. These include algebras which contain ideals with bounded approximate units, C^*-algebras, amenable Banach algebras and the convolution algebra $\ell^1(\Gamma)$ for any countable discrete group Γ.

Conclusion. While we can give answers to the questions we have raised in special cases, we cannot say very much about K^*_{fin} in general. In particular, the questions if K^*_{fin} is diffeotopy invariant, how it behaves under suspensions and if it has good exactness properties are still open. The main difficulty in approaching these problems is the lack of a finitely summable analogue of Kasparov's Technical Theorem, as we discuss in Section 2.2. This theorem lies at the very heart of Kasparov's K-homology, being the central tool for constructing the product in K-homology and proving homotopy invariance or half-exactness of K^*. As long as no adequate replacement is found, general answers probably cannot be given.

Nevertheless, we have covered the topic from a broad range of perspectives and thus hope to give the reader a better impression of what can be expected of K^*_{fin} and what cannot.

Conventions and Notation

0	is not a natural number
$\mathbb{N}_0$	the set of natural numbers and 0
$\lceil x \rceil$	the smallest $n \in \mathbb{Z}$ such that $x \leq n$
δ_{ij}	1 if $i = j$, otherwise 0
$\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$	algebras
$A, B, C, \ldots$	completions of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$
$\mathscr{F}, \mathscr{D}, \mathscr{G}, \ldots$	Fredholm modules
$\mathcal{A}^+$	the algebra $\mathcal{A}$ with a unit adjoined
$\tilde{\mathcal{A}}$	$\mathcal{A}$ if $\mathcal{A}$ is unital, $\mathcal{A}^+$ if $\mathcal{A}$ is not unital
$P^\perp$	$I - P$
F-Ind	Fredholm index
Ind	index map (Definition 2.2.4)
$\mathrm{GL}(\mathcal{A})$	invertible elements of $\mathcal{A}$
$\mathcal{B}(\mathcal{H})$	bounded operators on $\mathcal{H}$
$\mathcal{D}_{\mathcal{X}}$, $\mathcal{D}_{\mathcal{X}}^p$	equations (1.12), (1.13)
$\mathcal{K}(\mathcal{H})$	compact operators on $\mathcal{H}$
$\mathcal{L}^p(\mathcal{H})$	p-summable operators on $\mathcal{H}$ (Definition 1.1.4)
$\mathcal{R}(\mathcal{H})$	finite-rank operators on $\mathcal{H}$
$\mathbb{F}^*(\mathcal{A})$	Fredholm modules over $\mathcal{A}$ (Definition 2.1.2)
$\mathbb{D}^*(\mathcal{A})$	degenerate modules (Definition 2.1.4)
$\mathbb{F}_p^*(\mathcal{A})$	p-summable modules (Definition 2.3.3)

$\widetilde{\mathbb{F}}_p^*(\mathcal{A})$	p-summable, normalized modules (Theorem 2.4.1)
$K^*(\mathcal{A})$	K-Homology of $\mathcal{A}$ (Definition 2.1.7)
$K_p^*(\mathcal{A})$	p-summable K-Homology of $\mathcal{A}$ (2.3.7)
$K_{fin}^*(\mathcal{A})$	finitely summable K-Homology of $\mathcal{A}$ (2.3.7)
path	a continuous map $\gamma : [0,1] \longrightarrow X$ into some topological space X
smooth path	an infinitely differentiable path in a topological vector space such that all its derivatives vanish at the boundary points 0 and 1[†]
system of matrix units	Definition A.2.1
system of matrix units for A	Definition A.2.3

[†]This definition has the advantage that we can concatenate two smooth paths and obtain again a smooth path. However, this is a mere technicality: every infinitely differentiable path (with bounded derivatives) can be reparameterized such that it becomes smooth in the sense of our definition.

Chapter 1

Preliminaries

1.1 Schatten class operators

In this section we give a short introduction to the theory of Schatten class (or p-summable) operators. The standard reference on this topic is the monograph by Simon [Sim05], to which we will also refer for most proofs. The first three chapters contain all we need.

Consider a compact operator T on a (possibly finite-dimensional) Hilbert space $\mathcal{H}$. Its absolute value $|T|$ is thus a positive compact operator. The spectral theorem for compact operators states that $|T|$ is diagonal, 0 is the only accumulation point of $\mathrm{sp}(T)$, and that all eigenspaces of $|T|$ are finite-dimensional with the possible exception of its kernel. In other words: there is a decreasing sequence of non-zero positive numbers $\{\mu_k(T)\}$ (namely the sequence of non-zero eigenvalues of $|T|$ counted by multiplicity) and a sequence of orthonormal vectors $\{\xi_k\}$ such that

$$|T|\xi = \sum_k \mu_k(T)(\xi \mid \xi_k)\xi_k \qquad \text{for all } \xi \in \mathcal{H}.$$

Note that this sum can also be finite.

Since we can write T in its polar decomposition as $T = V|T|$ with a partial isometry V, it follows that T must be of the form

$$T\xi = \sum_k \mu_k(T)(\xi \mid \xi_k)\eta_k \qquad \text{for all } \xi \in \mathcal{H}$$

where $\{\eta_k\} := \{V\xi_k\}$ is another sequence of orthonormal vectors. Any operator of this form is obviously compact and the numbers $\{\mu_k(T)\}$ must be the eigenvalues of its absolute value (except for 0). Thus, we obtain the following theorem:

1.1.1 Theorem. *Any compact operator T on a Hilbert space $\mathcal{H}$ is of the form*

$$T\xi = \sum_k \mu_k(T)(\xi \mid \xi_k)\eta_k \qquad \text{for all } \xi \in \mathcal{H} \tag{1.1}$$

with a decreasing sequence of non-zero positive numbers $\{\mu_k(T)\}$ and sequences of orthonormal vectors $\{\xi_k\}$, $\{\eta_k\}$.

The sequence $\{\mu_k(T)\}$ is independent of the choice of $\{\xi_k\}, \{\eta_k\}$ and is called the sequence of characteristic values of T.

The following min-max-formula holds for the characteristic values of a compact operator:

1.1.2 Proposition. *If T is a compact operator on a Hilbert space $\mathcal{H}$, then the characteristic values of T are given by*

$$\mu_k(T) = \min_{\substack{\mathcal{F}\subseteq\mathcal{H} \text{ subsp.} \\ \dim \mathcal{F}=k-1}} \max_{\substack{\xi\in\mathcal{F}^\perp \\ \|\xi\|=1}} \|T\xi\|. \tag{1.2}$$

Proof. Let T be of the form (1.1). Then, by choosing $\mathcal{F} := \operatorname{span}\{\xi_1, \dots, \xi_{k-1}\}$, we see that the left-hand side of (1.2) must be larger than or equal its right-hand side (with the minimum replaced by the corresponding infimum).

On the other hand, let any $(k-1)$-dimensional subspace $\mathcal{F}$ of $\mathcal{H}$ be given, and let $\mathcal{E}_k := \operatorname{span}\{\xi_1, \dots, \xi_k\}$. The intersection $\mathcal{F}^\perp \cap \mathcal{E}_k$ cannot be zero-dimensional: otherwise, the quotient map $\pi : \mathcal{H} \longrightarrow \mathcal{H}/\mathcal{F}^\perp$ would be injective on $\mathcal{E}_k$, but $\mathcal{H}/\mathcal{F}^\perp \cong \mathcal{F}$ is of smaller dimension than $\mathcal{E}_k$.

Since $\mathcal{F}^\perp \cap \mathcal{E}_k \neq \{0\}$, we can find a unit vector ξ' in the intersection, and as T is bounded from below on $\mathcal{E}_k$ by $\mu_k(T)$, we obtain

$$\max_{\substack{\xi\in\mathcal{F}^\perp \\ \|\xi\|=1}} \|T\xi\| \geq \|T\xi'\| \geq \mu_k(T).$$

The minimum is attained for $\mathcal{F} := \operatorname{span}\{\xi_1, \dots, \xi_{k-1}\}$. □

1.1.3 Corollary. *If T and S are compact operators and V is an arbitrary bounded operator, then*

$$\mu_k(VT) \leq \|V\|\mu_k(T) \quad \text{and} \quad \mu_k(TV) \leq \|V\|\mu_k(T) \qquad \text{for all } k \tag{1.3}$$

as well as

$$\mu_{k+l+1}(T+S) \leq \mu_{k+1}(T) + \mu_{l+1}(S) \qquad \text{for all } k, l \geq 0. \tag{1.4}$$

Proof. The first inequalities are obvious. For proving (1.4), fix k and l. Then there is a k-dimensional subspace $\mathcal{F}_T$ and an l-dimensional subspace $\mathcal{F}_S$ of $\mathcal{H}$ such that

$$\mu_{k+1}(T) = \max_{\substack{\xi\in\mathcal{F}_T^\perp \\ \|\xi\|=1}} \|T\xi\| \qquad \text{and} \qquad \mu_{l+1}(S) = \max_{\substack{\xi\in\mathcal{F}_S^\perp \\ \|\xi\|=1}} \|S\xi\|.$$

Since $\mathcal{F}_T + \mathcal{F}_S$ is at most $(k+l)$-dimensional, we conclude that

$$\begin{aligned}\mu_{k+l+1}(T+S) &\leq \max_{\substack{\xi \in (\mathcal{F}_T+\mathcal{F}_S)^\perp \\ \|\xi\|=1}} \|(T+S)\xi\| \\ &\leq \max_{\substack{\xi \in (\mathcal{F}_T+\mathcal{F}_S)^\perp \\ \|\xi\|=1}} \|T\xi\| + \max_{\substack{\xi \in (\mathcal{F}_T+\mathcal{F}_S)^\perp \\ \|\xi\|=1}} \|S\xi\| \\ &\leq \mu_{k+1}(T) + \mu_{l+1}(S).\end{aligned}$$

□

For $p \in [1, \infty)$, a p-summable operator is simply a compact operator whose sequence of characteristic values is p-summable:

1.1.4 Definition. Let $\mathcal{H}$ be a Hilbert space. For $1 \leq p < \infty$ and $T \in \mathcal{K}(\mathcal{H})$ we define

$$\|T\|_p := \|\{\mu_k(T)\}\|_p = \Big(\sum_k \mu_k(T)^p\Big)^{\frac{1}{p}}.$$

The pth Schatten class is given by

$$\mathcal{L}^p(\mathcal{H}) := \{T \in \mathcal{K}(\mathcal{H}) \mid \|T\|_p < \infty\}.$$

Elements of $\mathcal{L}^p(\mathcal{H})$ are called p-summable operators. If T is p-summable for some p, we call T finitely summable.

1.1.5 Remarks.

1. This definition as well as the statements in the rest of this section extend in an obvious way to compact operators between different Hilbert spaces (one way to see this is to regard operators between Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as elements of $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$).
2. If we want to point out that an operator T acting on a Hilbert space $\mathcal{H}$ is p-summable, we will often just write $T \in \mathcal{L}^p$ instead of $T \in \mathcal{L}^p(\mathcal{H})$.
3. The Schatten classes are named after Robert Schatten. Also see his classic monograph [Sch70].

With the little we have proven about the characteristic values of compact operators, we can already deduce some important properties of the Schatten classes:

Let $T, S \in \mathcal{L}^p(\mathcal{H})$. From (1.4) we obtain

$$\begin{aligned}\mu_{2m+1}(T+S) &\leq \mu_{m+1}(T) + \mu_{m+1}(S) \\ \mu_{2m}(T+S) &\leq \mu_m(T) + \mu_{m+1}(S),\end{aligned}$$

which leads to the estimate

$$\|\{\mu_k(T+S)\}\|_p^p \leq 2^{p+1}\|\{\mu_k(T)\}\|_p^p + 2^{p+1}\|\{\mu_k(S)\}\|_p^p. \tag{1.5}$$

Thus, $\mathcal{L}^p(\mathcal{H})$ is a vector space. Inequality (1.3) implies that $\mathcal{L}^p(\mathcal{H})$ is an ideal in $\mathcal{B}(\mathcal{H})$ (of course, not a closed one).

From estimate (1.5) we obtain $\|T+S\|_p \leq 2^{1+1/p}\,(\|T\|_p + \|S\|_p)$. With a bit of extra work, one can get rid of the factor $2^{1+1/p}$:

1.1.6 Proposition. *$\|\cdot\|_p$ is a norm on $\mathcal{L}^p(\mathcal{H})$ turning it into a Banach space. $\mathcal{L}^p(\mathcal{H})$ is an ideal in $\mathcal{B}(\mathcal{H})$ and*

$$\|UTV\|_p \leq \|U\| \cdot \|T\|_p \cdot \|V\| \qquad \textit{for all } T \in \mathcal{L}^p(\mathcal{H}),\ U, V \in \mathcal{B}(\mathcal{H}).$$

Proof. The triangle inequality for $\|\cdot\|_p$ follows easily from the fact that $\|\cdot\|_p$ can be computed as

$$\|T\|_p = \sup_{\substack{(\xi_k),(\eta_k)\\ \text{o.n. bases}}} \|\{(\eta_k \mid T\xi_k)\}\|_p$$

where the supremum is taken over all possible pairs of orthonormal bases of $\mathcal{H}$.

The completeness of $\mathcal{L}^p(\mathcal{H})$ can also be easily deduced from this representation of $\|\cdot\|_p$ [Sim05, Proposition 2.6 and Theorem 2.7]. □

1.1.7 Proposition. *If $p < q$, then*

$$\|T\|_q \leq \|T\|_p \qquad \textit{for all } T \in \mathcal{K}(\mathcal{H}),$$

hence $\mathcal{L}^p(\mathcal{H}) \subseteq \mathcal{L}^q(\mathcal{H})$.

Proof. Let $T \in \mathcal{L}^p(\mathcal{H})$ with $\|T\|_p = 1$ be given. In particular, we have $|\mu_k(T)| \leq 1$ for each k, so

$$1 = \|T\|_p^p = \sum_k |\mu_k(T)|^p \geq \sum_k |\mu_k(T)|^q = \|T\|_q^q.$$

□

If T is a positive bounded operator and $\{\xi_k\}$ is an orthonormal basis, the quantities $(\xi_k \mid T\xi_k)$ are positive for all k. This implies that the trace

$$\operatorname{Tr}(T) = \sum_{k=1}^{\infty} (\xi_k \mid T\xi_k)$$

of T is a positive, possibly infinite number, and it is easily seen that $\operatorname{Tr}(T)$ is independent of the choice of the orthonormal basis. In particular, if T is compact, choosing an orthonormal basis of eigenvectors reveals that $\operatorname{Tr}(T) = \|T\|_1$. Since the characteristic values of an arbitrary compact T are the eigenvalues of $|T|$, we conclude:

1.1.8 Proposition. *If $T \in \mathcal{K}(\mathcal{H})$, then*

$$\|T\|_p = (\operatorname{Tr}|T|^p)^{\frac{1}{p}}. \tag{1.6}$$

1.1.9 Remark. The right-hand side of (1.6) makes sense for any bounded operator T. It is finite if and only if $T \in \mathcal{L}^p(\mathcal{H})$.

It turns out that Tr can be extended by the same formula to all operators in $\mathcal{L}^1(\mathcal{H})$, not only positive ones [Sim05, Theorem 3.1]. Therefore, $\mathcal{L}^1(\mathcal{H})$ is often called the set of trace-class operators.

The trace of TS is the same as the trace of ST, like in the finite-dimensional setting:

1.1.10 Proposition. *If $T, S \in \mathcal{B}(\mathcal{H})$ have the property that TS and ST lie in $\mathcal{L}^1(\mathcal{H})$, then*

$$\operatorname{Tr} TS = \operatorname{Tr} ST.$$

Proof. [Sim05, Corollary 3.8]. □

If TS lies in $\mathcal{L}^1$, this does not automatically imply that ST lies in $\mathcal{L}^1$. Consider, for example, two isometries T, S with orthogonal range. Then $T^*S = 0 \in \mathcal{L}^1$, whereas ST^* is a partial isometry of infinite rank.

Like for commutative L^p-spaces, there is a Hölder inequality for the p-norms:

1.1.11 Proposition (Non-Commutative Hölder Inequality)**.** *If the numbers $1 \le p, q, r < \infty$ satisfy $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $T \in \mathcal{L}^p(\mathcal{H})$, $S \in \mathcal{L}^q(\mathcal{H})$, then*

$$\|TS\|_r \le \|T\|_p \cdot \|S\|_q.$$

Proof. [Sim05, Theorem 2.8]. □

In particular, if $1 = \frac{1}{p} + \frac{1}{q}$, Hölder's inequality implies that TS lies in $\mathcal{L}^1$, thus $\operatorname{Tr}(TS)$ defines a bilinear pairing between $\mathcal{L}^p$ and $\mathcal{L}^q$. If T and S are diagonal with respect to the same orthonormal basis $\{\xi_k\}$, which means that T and S are of the form

$$T\xi = \sum_{k=1}^{\infty} t_k(\xi \mid \xi_k)\xi_k \quad \text{and} \quad S\xi = \sum_{k=1}^{\infty} s_k(\xi \mid \xi_k)\xi_k\,,$$

then we have

$$\operatorname{Tr}(TS) = \sum_{k=1}^{\infty} t_k s_k\,.$$

Thus, $\operatorname{Tr}(TS)$ generalizes the duality pairing between $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$. In fact, one can prove:

1.1.12 Proposition. *Let $1 < p, q < \infty$ such that $1 = \frac{1}{p} + \frac{1}{q}$. Then $\mathcal{L}^q(\mathcal{H})$ can be identified with the dual space of $\mathcal{L}^p(\mathcal{H})$ under the duality pairing*

$$\mathcal{L}^p(\mathcal{H}) \times \mathcal{L}^q(\mathcal{H}) \longrightarrow \mathbb{C}$$
$$(T, S) \longmapsto \mathrm{Tr}(TS).$$

In particular, $\mathcal{L}^p(\mathcal{H})$ is reflexive.

The same pairing yields the dualities

$$\mathcal{K}(\mathcal{H})^* = \mathcal{L}^1(\mathcal{H}) \quad \textit{and} \quad \mathcal{L}^1(\mathcal{H})^* = \mathcal{B}(\mathcal{H}).$$

Proof. [Sim05, Theorem 3.2]. □

Finally, we shall need the following non-commutative analogue of Fatou's Lemma:

1.1.13 Proposition (Non-Commutative Fatou Lemma)**.** *Let $\{A_\lambda\}$ be a net in $\mathcal{L}^p(\mathcal{H})$ which converges weakly to $A \in \mathcal{B}(\mathcal{H})$. If $\sup_\lambda \|A_\lambda\|_p < \infty$, then $A \in \mathcal{L}^p(\mathcal{H})$ and*

$$\|A\|_p \leq \sup_\lambda \|A_\lambda\|_p.$$

Proof. [Sim05, Theorem 2.7]. □

1.2 Representations of $*$-algebras

In this section we will establish a few elementary results about the representation theory of topological $*$-algebras.

1.2.1 Definition. A $*$-algebra $\mathcal{A}$ is a complex algebra with an anti-linear map $* : \mathcal{A} \longrightarrow \mathcal{A}$ such that

$$* \circ * = \mathrm{id}_\mathcal{A} \quad \text{and} \quad (xy)^* = y^* x^* \qquad \text{for all } x, y \in \mathcal{A}.$$

If $\mathcal{A}$ carries a Hausdorff vector space topology making algebra multiplication and the involution $*$ continuous maps, we call $\mathcal{A}$ endowed with this topology a topological $*$-algebra.

We call an ideal of $\mathcal{A}$ any subspace of $\mathcal{A}$ which is a two-sided ideal in the sense of ring theory and which is closed under the involution.[†] An ideal of a topological $*$-algebra is closed if it is closed as a linear subspace.

1.2.2 Definition. A $*$-representation (short: representation) of a $*$-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is an algebra homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ satisfying

$$\varphi(x^*) = \varphi(x)^* \qquad \text{for all } x \in \mathcal{A}.$$

If $\mathcal{A}$ is a topological $*$-algebra and φ is continuous, then we call φ a continuous $*$-representation of $\mathcal{A}$.

[†]Such ideals are often called $*$-ideals, but since all ideals we consider are $*$-ideals, we drop the $*$.

All results in this section are well-known in the C^*-algebra case and we claim no originality in proving them in the slightly more general setting of topological $*$-algebras, in which they are certainly known as well. We begin by establishing the notion of the support of a $*$-representation and prove that if $\mathcal{A}$ has an ideal $\mathcal{J}$, then each representation of $\mathcal{A}$ has the support of its restriction to $\mathcal{J}$ as an invariant subspace.

1.2.3 Proposition. *Let $\mathcal{A}$ be a $*$-algebra, $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ a $*$-representation, and let $\mathcal{J}$ be an Ideal of $\mathcal{A}$.*

If P is the orthogonal projection of $\mathcal{H}$ onto $\overline{\varphi(\mathcal{J})\mathcal{H}}$,[†] *then*

$$[\varphi(x), P] = 0 \quad \text{and} \quad \varphi(j)P^{\perp} = 0 \qquad \text{for all } x \in \mathcal{A},\ j \in \mathcal{J}.$$

Thus, φ can be decomposed as the direct sum of representations on $P\mathcal{H}$ and $P^{\perp}\mathcal{H}$, and φ restricts to the zero representation of $\mathcal{J}$ on $P^{\perp}\mathcal{H}$.

Proof. For any $\varphi(j)\xi \in \varphi(\mathcal{J})\mathcal{H}$ and $x \in \mathcal{A}$ we have

$$(P\varphi(x))\varphi(j)\xi = \varphi(x)\varphi(j)\xi$$

since $\mathcal{J}$ is an ideal in $\mathcal{A}$. Thus, by continuity, $P\varphi(x)$ and $\varphi(x)$ agree on $P\mathcal{H}$, which means that

$$P\varphi(x)P = \varphi(x)P \qquad \text{for all } x \in \mathcal{A}.$$

Substituting x^* for x and using that φ is a $*$-representation we get

$$P\varphi(x)P = P\varphi(x).$$

Therefore, $P\varphi(x) = \varphi(x)P$ for all $x \in \mathcal{A}$. Moreover, for any $j \in \mathcal{J}$ we have $P\varphi(j) = \varphi(j)$ and in particular

$$\varphi(j)P^{\perp} = P^{\perp}\varphi(j) = 0.$$

□

1.2.4 Corollary. *If $\mathcal{A}$ is a $*$-algebra, $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ a $*$-representation, and P the projection onto $\overline{\varphi(\mathcal{A})\mathcal{H}}$, then*

$$P\varphi(x) = \varphi(x)P = \varphi(x) \qquad \text{for all } x \in \mathcal{A}.$$

P is called the support projection of φ.

[†] The (topological) closure of $\varphi(\mathcal{J})\mathcal{H}$ is indeed a vector subspace of $\mathcal{H}$: if $\varphi(x)\xi$, $\varphi(y)\eta \in \varphi(\mathcal{J})\mathcal{H}$ and if $\{u_\lambda\}$ is an approximate unit for the C^*-algebra $\overline{\varphi(\mathcal{J})}$, then

$$\varphi(x)\xi + \varphi(y)\eta = \lim_\lambda u_\lambda\varphi(x)\xi + \lim_\lambda u_\lambda\varphi(y)\eta = \lim_\lambda u_\lambda(\varphi(x)\xi + \varphi(y)\eta) \in \overline{\overline{\varphi(\mathcal{J})}\mathcal{H}} = \overline{\varphi(\mathcal{J})\mathcal{H}}.$$

By continuity, this extends to all of $\overline{\varphi(\mathcal{J})\mathcal{H}}$.

We call a net $\{u_\lambda\}$ in a topological $*$-algebra $\mathcal{A}$ a left-approximate unit if $\lim_{\lambda\to\infty} u_\lambda x = x$ for all $x \in \mathcal{A}$. We call $\{u_\lambda\}$ bounded if it is bounded as a subset of $\mathcal{A}$ in the sense of Definition A.1.2.

1.2.5 Proposition. *Let $\mathcal{A}$ be a topological $*$-algebra with a bounded left-approximate unit $\{u_\lambda\}$. If φ is a continuous $*$-representation of $\mathcal{A}$ with support projection P, then*

$$\operatorname*{s-lim}_{\lambda\to\infty} \varphi(u_\lambda) = P$$

in the strong operator topology on $\mathcal{B}(\mathcal{H})$.

Proof. For any $x \in \mathcal{A}$, $\xi \in \mathcal{H}$ we have

$$\lim_{\lambda\to\infty} \varphi(u_\lambda)\varphi(x)\xi = \lim_{\lambda\to\infty} \varphi(u_\lambda x)\xi = \varphi(x)\xi. \tag{1.7}$$

Since $\{u_\lambda\}$ is bounded and φ is continuous, the operators $\varphi(u_\lambda)$ are uniformly bounded in norm. Combined with (1.7), this implies that $\varphi(u_\lambda)\xi$ converges to ξ for any $\xi \in P\mathcal{H}$. Since moreover $\varphi(u_\lambda)\eta = 0$ for all $\eta \in P^\perp\mathcal{H}$, the proposition follows. □

1.2.6 Corollary. *If φ is a $*$-representation of a $*$-algebra $\mathcal{A}$ with support projection P, then P lies in the strong closure of $\varphi(\mathcal{A})$.*

Proof. The strong closure of $\varphi(\mathcal{A})$ agrees with the strong closure of $\overline{\varphi(\mathcal{A})}$ which, being a C^*-algebra, has a bounded approximate unit. The identity representation of $\overline{\varphi(\mathcal{A})}$ has the same support as φ. □

1.2.7 Proposition. *Let $\mathcal{A}$ be a topological $*$-algebra. If $\mathcal{J}$ is an ideal of $\mathcal{A}$ with a bounded left-approximate unit $\{u_\lambda\}$, then each continuous $*$-representation of $\mathcal{J}$ extends uniquely to a continuous $*$-representation of $\mathcal{A}$ with the same support.*

Proof. Let φ be a continuous $*$-representation of $\mathcal{J}$ on $\mathcal{H}$ and P its support projection. For any $x \in \mathcal{A}$ define an operator $\varphi(x)$ on the linear span of $\varphi(\mathcal{J})\mathcal{H}$ by setting

$$\varphi(x)\Big(\sum_{k=1}^{n} \varphi(j_k)\xi_k\Big) := \sum_{k=1}^{n} \varphi(xj_k)\xi_k \in P\mathcal{H} \tag{1.8}$$

for $j_k \in \mathcal{J}$, $\xi_k \in \mathcal{H}$ $(1 \le k \le n)$. To show that this is well-defined, assume that $\sum_{k=1}^{n'} \varphi(j'_k)\xi'_k = \sum_{k=1}^{n} \varphi(j_k)\xi_k$. Since $\{u_\lambda\}$ is a left-approximate unit for $\mathcal{J}$, we know that $xu_\lambda j_k$ converges to xj_k and that $xu_\lambda j'_k$ converges to xj'_k. Hence, by the continuity of φ,

$$\begin{aligned}\sum_{k=1}^{n}\varphi(xj_k)\xi_k &= \sum_{k=1}^{n}\lim_{\lambda\to\infty}\varphi(xu_\lambda j_k)\xi_k\\ &= \lim_{\lambda\to\infty}\varphi(xu_\lambda)\sum_{k=1}^{n}\varphi(j_k)\xi_k\\ &= \lim_{\lambda\to\infty}\varphi(xu_\lambda)\sum_{k=1}^{n'}\varphi(j'_k)\xi'_k\\ &= \sum_{k=1}^{n'}\lim_{\lambda\to\infty}\varphi(xu_\lambda j'_k)\xi'_k = \sum_{k=1}^{n'}\varphi(xj'_k)\xi'_k.\end{aligned} \tag{1.9}$$

By definition, $\varphi(x)$ is linear on the span of $\varphi(\mathcal{J})\mathcal{H}$, and (1.9) also shows that $\varphi(xu_\lambda)$ converges strongly to $\varphi(x)$ on this space. It is moreover clear that formula (1.8) extends the action of $\mathcal{J}$ on the span of $\varphi(\mathcal{J})\mathcal{H}$.

Since the net $\{u_\lambda\}$ is bounded in $\mathcal{J}$ and the multiplication in $\mathcal{A}$ is continuous, xu_λ is bounded in $\mathcal{J}$. Hence, by the continuity of φ, there is a $C < \infty$ such that

$$\|\varphi(xu_\lambda)\| \leq C \qquad \text{for all } \lambda.$$

Therefore, we also have $\|\varphi(x)\| \leq C$ on the span of $\varphi(\mathcal{J})\mathcal{H}$, so $\varphi(x)$ can be continuously extended to a bounded operator on $P\mathcal{H}$ and then to an operator on $\mathcal{H}$ by extending it with the zero operator on the complement of $P\mathcal{H}$.

Next, since

$$\begin{aligned}\varphi(\mu_1 x+\mu_2 y)\varphi(j)\xi :=&\varphi((\mu_1 x+\mu_2 y)j)\xi\\ =&\mu_1\varphi(xj)\xi+\mu_2\varphi(yj)\xi =: \mu_1\varphi(x)\varphi(j)\xi+\mu_2\varphi(y)\varphi(j)\psi\end{aligned}$$

and

$$\varphi(xy)\varphi(j)\xi := \varphi(xyj)\xi =: \varphi(x)\varphi(yj)\xi =: \varphi(x)\varphi(y)\varphi(j)\xi$$

for $\mu_1,\mu_2 \in \mathbb{C}$, $x, y \in \mathcal{A}$, we see that φ is linear and multiplicative. Since φ is multiplicative on $\mathcal{A}$ and a $*$-homomorphism on $\mathcal{J}$, we conclude that

$$\varphi(x^*)\varphi(j)\xi = \varphi(x^*j)\xi = \varphi(j^*x)^*\xi = (\varphi(j^*)\varphi(x))^*\xi = \varphi(x)^*\varphi(j)\xi$$

for all $x \in \mathcal{A}$, $j \in \mathcal{J}$, and $\xi \in \mathcal{H}$. Thus, φ is indeed a $*$-representation of $\mathcal{A}$.

Finally, we prove the continuity of φ. Let $\varepsilon > 0$ be given. Since φ is continuous on $\mathcal{J}$, $\varphi^{-1}(B_\varepsilon(0)) \cap \mathcal{J}$ is a neighbourhood of 0 in $\mathcal{J}$. Multiplication in $\mathcal{A}$ is continuous, so there are neighbourhoods $U \subseteq \mathcal{A}, V \subseteq \mathcal{J}$ of 0 such that

$$U \cdot V \subseteq \varphi^{-1}(B_\varepsilon(0)) \cap \mathcal{J}.$$

Since $\{u_\lambda\}$ is bounded in $\mathcal{J}$, there is an $r < \infty$ such that $\{u_\lambda\} \subseteq rV$. Hence,

$$\frac{1}{r}U \cdot \{u_\lambda\} \subseteq \varphi^{-1}(B_\varepsilon(0)) \cap \mathcal{J},$$

which implies that

$$\|\varphi(xu_\lambda)\| < \varepsilon \qquad \text{for all } \lambda \text{ and } x \in \frac{1}{r}U.$$

As $\varphi(x)$ is the strong limit of $\varphi(xu_\lambda)$, we conclude that

$$\|\varphi(x)\| \leq \varepsilon \qquad \text{for all } x \in \frac{1}{r}U.$$

Thus, φ is continuous.

There can only be one extension of φ to $\mathcal{A}$ with the same support since equation (1.8) already determines such φ for all $x \in \mathcal{A}$. □

1.3 Holomorphic functional calculus

As we have already indicated in the introduction, the category of C^*-algebras is not an adequate setting for finitely summable K-homology: neither the Schatten classes $\mathcal{L}^p(\mathcal{H})$ nor algebras of smooth functions like $C^\infty(M)$ are C^*-algebras. Therefore, functional calculus with arbitrary continuous functions will not be at our disposal. However, we will make use of a functional calculus with holomorphic functions which is available for a larger class of algebras, including arbitrary Banach algebras:

1.3.1 Definition (Holomorphic functional calculus)**.** If x is an element of a unital Banach algebra A and $f : U \longrightarrow \mathbb{C}$ is holomorphic on some neighbourhood U of $\mathrm{sp}(x) \subseteq \mathbb{C}$, then define

$$f(x) := \frac{1}{2\pi i} \int_\Gamma f(z)(z-x)^{-1}\,\mathrm{d}z \tag{1.10}$$

where Γ is a Cauchy contour in $U \setminus \mathrm{sp}(x)$ such that $\mathrm{sp}(x)$ lies in the domain enclosed by Γ.

It is clear that the integral in (1.10) converges as inversion is continuous in $\mathrm{GL}(A)$. By standard arguments from function theory, one sees that this definition is indeed independent of the choice of Γ.

If A is not unital, we can embed A into its unitization A^+ and obtain an element $f(x) \in A^+$. With π being the quotient map $A^+ \longrightarrow \mathbb{C}$, we have

$$\pi(f(x)) = \frac{1}{2\pi i} \int_\Gamma f(z)(z-0)^{-1}\,\mathrm{d}z = f(0)$$

by the continuity of π and Cauchy's integral formula. This implies:

1.3.2 Proposition. *If f is holomorphic on a neighbourhood of* $\mathrm{sp}(x)$ *with* $f(0) = 0$*, then* $f(x) \in A$.

We list some important properties of holomorphic functional calculus:

1.3.3 Proposition. *Let $x \in A$, $\lambda \in \mathbb{C}$ and let f, g be functions which are holomorphic on a neighbourhood of* $\operatorname{sp}(x)$*. Then:*

(i) $(\lambda f + g)(x) = \lambda f(x) + g(x)$

(ii) $(fg)(x) = f(x)g(x)$

(iii) $\operatorname{sp} f(x) = f(\operatorname{sp}(x))$

(iv) If f can be written as a power series $f(z) = \sum_{k=0}^{\infty} \mu_k z^k$ with radius of convergence greater than $\|x\|$, then $f(x) = \sum_{k=0}^{\infty} \mu_k x^k$.

If A is a C^*-algebra and x is normal, then (for holomorphic functions) the holomorphic functional calculus for x agrees with the continuous functional calculus for x.

A nice exposition of holomorphic functional calculus containing a proof of this proposition can be found in [GGK90, Chapter 1].

1.4 Pre-C^*-algebras

Pre-C^*-algebras are C^*-normed algebras that are closed under holomorphic functional calculus. Since the typical K-theory constructions for projections and unitaries only depend on holomorphic functional calculus (see Section 1.5), pre-C^*-algebras posses a well-behaved K-theory. In fact, the K-theory of a pre-C^*-algebra agrees with the K-theory of its C^*-algebra completion (Theorem 1.5.9).

The closure under holomorphic functional calculus can also be seen as a minimal requirement for an algebra to be a (non-commutative) algebra of smooth functions. In particular, if M is a smooth manifold, then $C_0^\infty(M)$ is a pre-C^*-algebra.

While we define finitely summable K-homology for arbitrary topological $*$-algebras, every p-summable Fredholm module $\mathscr{F} = (\varphi, \mathcal{H}, F)$ extends to a p-summable module over a pre-C^*-algebra. Moreover, from a dual point of view, F lies in the pre-C^*-algebra of all operators that commute with φ up to p-summable operators. This allows us to perform certain deformations of F while preserving the commutation relation with φ.

1.4.1 Definition. Let $\mathcal{A}$ be a normed algebra and A its norm completion. Since A is a Banach algebra, we can define by holomorphic functional calculus an element $f(x) \in A$ for each $x \in \mathcal{A}$ and each function f which is holomorphic on a neighbourhood of $\operatorname{sp}_A(x)$ (with $f(0) = 0$ if $\mathcal{A}$ is non-unital). If $f(x) \in \mathcal{A}$ for each such x and f, we say that $\mathcal{A}$ is closed under holomorphic functional calculus.

1.4.2 Definition. A *pre-C^*-algebra* is a $*$-algebra $\mathcal{A}$ endowed with a C^*-norm such that $M_n(\mathcal{A})$ with its induced C^*-norm is closed under holomorphic functional calculus for all $n \in \mathbb{N}$.

1.4.3 Remarks. 1. It follows in particular from the definition that for any $x \in \mathcal{A}$ we have $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_A(x)$.

2. In the literature, the term 'pre-C^*-algebra' sometimes refers to the class of all C^*-normed $*$-algebras.

3. Our definition is the same as in [Bla98], where such algebras are called 'local C^*-algebras', but it seems that this term has come out of fashion.

The following two propositions from [Sch92] show that if $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_A(x)$ for all $x \in \mathcal{A}$, then this often already implies that $\mathcal{A}$ is a pre-C^*-algebra.

1.4.4 Proposition. *Let $\mathcal{A}$ be a C^*-normed $*$-algebra and A its completion. If $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_A(x)$ for all $x \in \mathcal{A}$, then*

$$\mathrm{sp}_{M_n(\mathcal{A})}(x) = \mathrm{sp}_{M_n(A)}(x) \qquad \text{for all } x \in M_n(\mathcal{A}),\ n \in \mathbb{N}.$$

Proof. The non-unital case can be easily reduced to the unital case, so assume that $\mathcal{A}$ is unital. Moreover, we prove the proposition only for $n = 2$. For arbitrary n, the claim follows by embedding $M_n(\mathcal{A})$ into $M_{2^k}(\mathcal{A})$ where $n \leq 2^k$.

First, let

$$x = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \in M_2(\mathcal{A}) \cap M_2(A)^{-1} \quad \text{with} \quad \|1_{M_2(\mathcal{A})} - x\| < 1$$

be given. Then we also have $\|1_{\mathcal{A}} - x_{11}\| < 1$, which implies that x_{11} is invertible in A. By assumption, its inverse lies in $\mathcal{A}$. This means that we can factorize x over $M_2(\mathcal{A})$ as

$$\begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x_{21}x_{11}^{-1} & 1 \end{bmatrix} \begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} - x_{21}x_{11}^{-1}x_{12} \end{bmatrix} \begin{bmatrix} 1 & x_{11}^{-1}x_{12} \\ 0 & 1 \end{bmatrix}.$$

The left and right factors are obviously invertible in $M_2(\mathcal{A})$. Because x is invertible in $M_2(A)$, the factor in the middle must also be invertible in $M_2(A)$ and, being diagonal, in $M_2(\mathcal{A})$. Thus, x is invertible in $M_2(\mathcal{A})$.

Now, let $x \in M_2(\mathcal{A}) \cap M_2(A)^{-1}$ be arbitrary. Since $\mathcal{A}$ is dense in A, there is a $y \in M_2(\mathcal{A})$ with $\|x^{-1} - y\| < 1/\|x\|$. Then $\|1_{M_2(\mathcal{A})} - xy\| = \|x(x^{-1} - y)\| < 1$. By the argument above, there exists an inverse $z \in M_2(\mathcal{A})$ to xy. Thus, $yz \in M_2(\mathcal{A})$ is the inverse of x. □

1.4.5 Definition. A Fréchet algebra is a topological algebra whose underlying topological vector space is a Fréchet space (we only require that multiplication is continuous but not that the topology is generated by a sequence of submultiplicative semi-norms).

1.4.6 Proposition. *Let $\mathcal{A}$ be a C^*-normed $*$-algebra with completion A such that $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_A(x)$ for all $x \in \mathcal{A}$.*

If $\mathcal{A}$ can be endowed with a Fréchet algebra topology stronger than its norm topology, then $\mathcal{A}$ is a pre-C^*-algebra.

Proof. Let $\mathcal{A}$ be given with its Fréchet algebra topology, and assume without loss of generality that $\mathcal{A}$ is unital. The inclusion ι of $\mathcal{A}$ into A is continuous by assumption. Thus, $\mathrm{GL}(\mathcal{A}) = \iota^{-1}(\mathrm{GL}(A))$ is open in $\mathcal{A}$, which makes it possible to find a complete metric on $\mathrm{GL}(\mathcal{A})$ which induces the subspace topology of $\mathrm{GL}(\mathcal{A}) \subseteq \mathcal{A}$.† It can be shown that a group with a complete metric that makes multiplication continuous has continuous inversion [Wae71].

Therefore, the integrands in (1.10) are continuous and the completeness of $\mathcal{A}$ implies that $\mathcal{A}$ is closed under holomorphic functional calculus.

The Fréchet topology on $\mathcal{A}$ induces a Fréchet algebra topology on $M_n(\mathcal{A})$ stronger than the norm topology. Using Proposition 1.4.4 we thus see that $M_n(\mathcal{A})$ must be closed under holomorphic functional calculus by the same argument. □

1.4.7 Corollary. *If M is a smooth manifold, then $C_0^\infty(M)$ is a pre-C^*-subalgebra of $C_0(M)$.*

Proof. If $U \subseteq \mathbb{R}^n$ is open, then $C_0^\infty(U)$ can be made a Fréchet algebra using the semi-norms

$$|f|_k := \sum_{|\alpha|=k} \|\partial^\alpha f\|_\infty \qquad \text{for all } f \in C_0^\infty(U),\ k \in \mathbb{N}_0.$$

If M is an arbitrary smooth manifold, cover M with coordinate patches and define on each patch semi-norms by the same formula. □

1.4.8 Proposition. *If φ is a $*$-homomorphism from a pre-C^*-algebra $\mathcal{A}$ into a C^*-normed algebra $\mathcal{B}$, then φ is automatically continuous. In particular, any such $*$-homomorphism extends to a $*$-homomorphism between the completions A, B of $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Using the invariance of the spectrum, the proof is the same as for C^*-algebras:

$$\|x\|^2 = \|x^*x\| = \rho_A(x^*x) = \rho_{\mathcal{A}}(x^*x) \geq \rho_B(\varphi(x)^*\varphi(x)) = \|\varphi(x)\|^2$$

for all $x \in \mathcal{A}$ with $\rho(x)$ denoting the spectral radius of x. □

1.4.9 Proposition. *If $\mathcal{A}$ is a pre-C^*-algebra and $\mathcal{J}$ a closed ideal of $\mathcal{A}$, then $\mathcal{J}$ is a pre-C^*-algebra as well.*

Proof. Let J be the norm-closure of $\mathcal{J}$ in A. Consider $j \in \mathcal{J}$ and a function f holomorphic on a neighbourhood of $\mathrm{sp}_J(j)$ with $f(0) = 0$. Then we have $f(j) \in J$ and, since $\mathrm{sp}_J(j) \cup \{0\} = \mathrm{sp}_A(j) \cup \{0\}$, also $f(j) \in \mathcal{A}$. Moreover, $\mathcal{J} = J \cap \mathcal{A}$ since $\mathcal{J}$ is closed in $\mathcal{A}$. Thus, $f(j) \in \mathcal{J}$. If $\mathcal{J}$ is unital and f holomorphic on $\mathrm{sp}_J(j)$ with $f(0) \neq 0$, then $f(j) = (f - f(0))(j) + f(0) \cdot 1_{\mathcal{J}} \in \mathcal{J}$.

The same arguments hold for matrix algebras over $\mathcal{J}$. □

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†Consider the embedding $\varphi : \mathrm{GL}(\mathcal{A}) \longrightarrow \mathcal{A} \times \mathbb{R},\ x \longmapsto (x, d(x, \partial\,\mathrm{GL}(\mathcal{A}))^{-1})$ where d denotes a metric on $\mathcal{A}$ which induces the Fréchet space topology. The pullback of the product metric on $\mathcal{A} \times \mathbb{R}$ via φ has this property.

Schmitt proves in [Sch91] that quotients of pre-C^*-algebras are again pre-C^*-algebras. We reproduce the proof in the following proposition and its corollary with a little more verbosity.

The proof is based on the fact that closedness under holomorphic functional calculus is equivalent to being closed under functional calculus with power series, a property which is easily seen to pass to quotients.

1.4.10 Definition. Let $\mathcal{A}$ be a unital normed algebra. If $f(a) \in \mathcal{A}$ for any $a \in \mathcal{A}$ and any f holomorphic on a disk around the origin of radius greater than $\|a\|$, we say that $\mathcal{A}$ is closed under functional calculus with power series.

1.4.11 Proposition. *If $\mathcal{A}$ is closed under functional calculus with power series, then $\mathcal{A}$ is closed under holomorphic functional calculus.*

Proof. First, note that the following is true in general: if $h_n : [0,1] \longrightarrow \mathbb{C}$ ($n \in \mathbb{N}$) are uniformly bounded continuous functions and $\sum_{n=1}^{\infty} \xi_n$ is an absolutely converging series in some Banach space, then

$$\int_0^1 \sum_{n=1}^{\infty} h_n(t)\xi_n \, \mathrm{d}t = \sum_{n=1}^{\infty} \left(\int_0^1 h_n(t) \, \mathrm{d}t \right) \cdot \xi_n. \tag{1.11}$$

Denote by A the completion of $\mathcal{A}$. The function $f(x) = 1/(1-x)$ is holomorphic on the open disk of radius 1 around 0. So if $\|1_{\mathcal{A}} - x\| < 1$, then $x^{-1} = f(1_{\mathcal{A}} - x)$ lies in $\mathcal{A}$. As in the proof of Proposition 1.4.4, it follows that $\mathrm{sp}_{\mathcal{A}}(x) = \mathrm{sp}_A(x)$ for all $x \in \mathcal{A}$.

Let now f be an arbitrary function that is holomorphic on an open neighbourhood U of $\mathrm{sp}_{\mathcal{A}}(x)$, and let Γ be a piecewise linear Cauchy contour in U enclosing $\mathrm{sp}_{\mathcal{A}}(x)$ (such a contour can always be found). We have to prove that the integral

$$\int_\Gamma f(z)(z-x)^{-1} \, \mathrm{d}z$$

converges in $\mathcal{A}$. To this end, parameterize Γ by linear paths $\gamma_j : [0,1] \longrightarrow \mathbb{C}$ such that

$$|\gamma_j(1) - \gamma_j(0)| < \frac{1}{\|(\gamma_j(0) - x)^{-1}\|} \qquad \text{for all } j.$$

Then we can write $(\gamma_j(t) - x)^{-1}$ as

$$(\gamma_j(t) - x)^{-1} = (\gamma_j(0) - x)^{-1} \sum_{l=0}^{\infty} \Big[(\gamma_j(0) - \gamma_j(t))(\gamma_j(0) - x)^{-1} \Big]^l.$$

Abbreviating $\gamma_j(1) - \gamma_j(0)$ by δ and using (1.11), we thus obtain

$$\begin{aligned}\int_{\Gamma_j} f(z)(z-x)^{-1}\,\mathrm{d}z &= \int_0^1 f(\gamma(t)) \sum_{l=0}^{\infty} (-\delta t)^l (\gamma_j(0)-x)^{-l-1} \cdot \delta\,\mathrm{d}t \\ &= \int_0^1 \sum_{l=0}^{\infty} \Big(f(\gamma(t))(-t)^l\Big) \Big(\delta(\gamma_j(0)-x)^{-1}\Big)^{l+1}\,\mathrm{d}t \\ &= \sum_{l=0}^{\infty} \Big(\int_0^1 f(\gamma(t))(-t)^l\,\mathrm{d}t\Big) \Big(\delta(\gamma_j(0)-x)^{-1}\Big)^{l+1} \\ &= g\left(\delta(\gamma_j(0)-x)^{-1}\right)\end{aligned}$$

where

$$g(z) := \sum_{l=0}^{\infty} \Big(\int_0^1 f(\gamma(t))(-t)^l\,\mathrm{d}t\Big) z^{l+1}.$$

g is a power series with radius of convergence at least 1. By assumption, the integral must therefore lie in $\mathcal{A}$. □

1.4.12 Corollary. *If $\mathcal{A}$ is a pre-C^*-algebra and $\mathcal{J}$ a closed ideal of $\mathcal{A}$, then $\mathcal{A}/\mathcal{J}$ is also a pre-C^*-algebra.*

Proof. First assume that $\mathcal{A}$ is unital. By the last proposition, it suffices to show that $\mathcal{A}/\mathcal{J}$ is closed under functional calculus with power series. For this let $[x] \in \mathcal{A}/\mathcal{J}$ and let f be holomorphic on a disk of radius r greater than $\|[x]\|$. There is a representative $x' \in \mathcal{A}$ of $[x]$ such that $\|x'\| < r$. Hence, we can apply f to x', and since $\mathcal{A}$ is a pre-C^*-algebra, $f(x')$ lies in $\mathcal{A}$. Thus, $f([x]) = [f(x')] \in \mathcal{A}/\mathcal{J}$.

If $\mathcal{A}$ is a non-unital pre-C^*-algebra, then $\mathcal{A}^+$ is a pre-C^*-algebra as well. Thus, $\mathcal{A}/\mathcal{J}$ is a closed ideal of the pre-C^*-algebra $\mathcal{A}^+/\mathcal{J}$ and therefore a pre-C^*-algebra.

The same arguments hold for matrix algebras. □

1.4.13 Proposition. *If $\mathcal{A}_i$ $(i \in \mathcal{I})$ is a system of pre-C^*-subalgebras of a C^*-algebra B, then $\bigcap_{i\in\mathcal{I}} \mathcal{A}_i$ is a pre-C^*-algebra.*

If $\mathcal{A}_i$ $(i \in \mathcal{I})$ is an increasing union of pre-C^*-subalgebras of B, then $\bigcup_{i\in\mathcal{I}} \mathcal{A}_i$ is a pre-C^*-algebra.

Proof. Obvious. □

If $\mathcal{X}$ is a set of operators on a Hilbert space $\mathcal{H}$, then

$$\mathcal{D}_{\mathcal{X}} := \Big\{ T \in \mathcal{B}(\mathcal{H}) \;\Big|\; [X,T] \in \mathcal{K}(\mathcal{H}) \text{ for all } X \in \mathcal{X} \cup \mathcal{X}^* \Big\} \tag{1.12}$$

is easily seen to be a C^*-algebra. As a final result for this section, we show that the set of operators which commute with $\mathcal{X} \cup \mathcal{X}^*$ up to p-summable operators forms a pre-C^*-subalgebra† of $\mathcal{D}_{\mathcal{X}}$.

†As we will see later on, $\mathcal{D}^p_{\mathcal{X}}$ is not necessarily dense in $\mathcal{D}_{\mathcal{X}}$.

1.4.14 Proposition. *If $X \in \mathcal{B}(\mathcal{H})$, then*

$$\mathcal{D}^p_X := \Big\{ T \in \mathcal{B}(\mathcal{H}) \;\Big|\; [X,T] \in \mathcal{L}^p(\mathcal{H}) \text{ and } [X^*,T] \in \mathcal{L}^p(\mathcal{H}) \Big\}$$

is a pre-C^*-algebra.

Proof. $\mathcal{D}^p_X$ is a $*$-algebra since $\mathcal{L}^p(\mathcal{H})$ is an ideal and

$$[X,TS] = T[X,S] + [X,T]S \quad \text{and} \quad [X,T^*] = -[X^*,T]^*$$

for all $T,S \in \mathcal{B}(\mathcal{H})$. If $T \in \mathcal{D}^p_X$ is invertible as an operator on $\mathcal{H}$, then

$$[X,T^{-1}] = -T^{-1}[X,T]T^{-1} \in \mathcal{L}^p(\mathcal{H}).$$

This means that $\operatorname{sp}_{\mathcal{D}^p_X}(T) = \operatorname{sp}_{\mathcal{B}(\mathcal{H})}(T)$ for all $T \in \mathcal{D}^p_X$.

Moreover, the norm

$$\|T\|_* := \|T\| + \|[X,T]\|_p + \|[X^*,T]\|_p$$

is submultiplicative on $\mathcal{D}^p_X$ because

$$\begin{aligned}
&\|TS\| + \|[X,TS]\|_p + \|[X^*,TS]\|_p \\
&\qquad = \|TS\| + \|T[X,S] + [X,T]S\|_p + \|T[X^*,S] + [X^*,T]S\|_p \\
&\qquad \le \|T\|\|S\| + \|T\|\|[X,S]\|_p + \|[X,T]\|_p\|S\| \\
&\qquad\qquad\qquad\qquad + \|T\|\|[X^*S]\|_p + \|[X^*,T]\|_p\|S\| \\
&\qquad \le \Big(\|T\| + \|[X,T]\|_p + \|[X^*,T]\|_p\Big)\Big(\|S\| + \|[X,S]\|_p + \|[X^*,S]\|_p\Big)
\end{aligned}$$

for all $S,T \in \mathcal{D}^p_X$.

If T_n is a Cauchy sequence in $\mathcal{D}^p_X$ with respect to $\|\cdot\|_*$, then T_n converges in operator norm to a bounded operator T. Thus, $[X,T_n]$ converges in $\mathcal{B}(\mathcal{H})$ to $[X,T]$. Since $\mathcal{L}^p(\mathcal{H})$ is complete, $[X,T_n]$ also converges in $\mathcal{L}^p(\mathcal{H})$ to an operator K. However, since the topology induced by $\|\cdot\|_p$ is stronger than the topology induced by $\|\cdot\|$, both limits must agree. So $[X,T] \in \mathcal{L}^p(\mathcal{H})$ and, by the same argument, $[X^*,T] \in \mathcal{L}^p(\mathcal{H})$. This shows that $\mathcal{D}^p_X$ endowed with $\|\cdot\|_*$ is a Banach algebra. Thus, $\mathcal{D}^p_X$ is a pre-C^*-algebra by Proposition 1.4.6. □

1.4.15 Corollary. *If $\mathcal{X}$ is a set of operators on a Hilbert space $\mathcal{H}$, then*

$$\mathcal{D}^p_{\mathcal{X}} := \Big\{ T \in \mathcal{B}(\mathcal{H}) \;\Big|\; [X,T] \in \mathcal{L}^p(\mathcal{H}) \text{ for all } X \in \mathcal{X} \cup \mathcal{X}^* \Big\} \tag{1.13}$$

is a pre-C^*-algebra.

Proof. $\mathcal{D}^p_{\mathcal{X}} = \bigcap_{X \in \mathcal{X}} \mathcal{D}^p_X$ and Proposition 1.4.13. □

1.5 Projections and unitaries in pre-C^*-algebras

We collect some standard results about homotopies of projections and unitaries in pre-C^*-algebras. The following proofs are taken from [Bla98]. Our main point in repeating these results is to give the reader the opportunity to convince himself that all arguments work as well in the setting of pre-C^*-algebras as they do for C^*-algebras (for which he might be familiar with them).

In the whole section $\mathcal{A}$ will denote a pre-C^*-algebra.

1.5.1 Proposition. *Every invertible $w \in \mathcal{A}$ admits a polar decomposition $w = u|w|$.*

Proof. w^*w is invertible, so $\sqrt{\cdot}$ is holomorphic on $\operatorname{sp}(w^*w)$. Thus, $|w|$ and therefore $u = w|w|^{-1}$ lie in $\mathcal{A}$. □

1.5.2 Proposition. *Every invertible $w \in \mathcal{A}$ is homotopic to a unitary via a path of invertibles. Two unitaries which are homotopic via a path of invertibles are homotopic via a path of unitaries.*

Proof. If w is invertible and $w = u|w|$ its polar decomposition, then w and u are connected via the path $w_t = u((1-t)|w| + t)$.

If w_t is a path of invertibles connecting unitaries u and v, then by the continuity of functional calculus, $|w_t|$ is also a continuous path. Thus, $u_t = w_t|w_t|^{-1}$ is a continuous path of unitaries connecting u and v. □

1.5.3 Proposition. *If $u \in \mathcal{A}$ is unitary and homotopic to $1_{\mathcal{A}}$ via a path of unitaries, then there are positive $a_1, \dots, a_n \in \mathcal{A}$ such that*

$$u = e^{ia_1} \cdots e^{ia_n}.$$

Proof. If $\|1 - u\| < 2$, then $-1 \notin \operatorname{sp} u$. Thus, the branch of the complex logarithm taking $S^1 \setminus \{-1\}$ to $(-i\pi, i\pi)$ is holomorphic on $\operatorname{sp}(u)$, so $\log(u) \in \mathcal{A}$. Let $a_1 := 2\pi \cdot 1_{\mathcal{A}} + \frac{1}{i}\log(u)$ to obtain $u = e^{ia_1}$.

If u and v are connected via a path of unitaries, by the compactness of $[0,1]$ we can find unitaries $u = u_0, u_1, \dots, u_n = v$ with $\|u_{i+1}u_i^* - 1\| = \|u_{i+1} - u_i\| < 2$, and the general case follows from what we have already shown. □

1.5.4 Corollary. *Any two unitaries $u, v \in \mathcal{A}$ that are homotopic via a path of invertibles can be joined by paths $e^{ita_0}u_0, \dots, e^{ita_{n-1}}u_{n-1}$ where $a_0, \dots, a_{n-1}$ are positive and $u = u_0, u_1, \dots, u_n = v$ unitaries with $e^{ia_j}u_j = u_{j+1}$ for $j = 0, \dots, n-1$.*

1.5.5 Proposition. *Every idempotent in $\mathcal{A}$ is homotopic to a projection via a path of idempotents. Two projections in $\mathcal{A}$ which are homotopic via a path of idempotents are homotopic via a path of projections.*

Proof. Let e be an idempotent in $\mathcal{A}$. Since $\mathcal{A}$ is a pre-C^*-algebra, the element $z = 1+(e-e^*)(e^*-e)$ must be invertible in $\tilde{\mathcal{A}}$. Then $p = ee^*z^{-1}$ is a projection in $\mathcal{A}$, which is homotopic to e via the path $(1 - tp + te)e(1 + tp - te)$.

If e_t is a path of idempotents between projections p_0 and p_1, set again $z_t = 1 + (e_t - e_t^*)(e_t^* - e_t)$. Then $p_t = e_t e_t^* z_t^{-1}$ is a path of projections from p_0 to p_1. □

1.5.6 Proposition. *If a, b are self-adjoint and w is invertible in $\mathcal{A}$ with $waw^{-1} = b$, then there is a unitary u in $\mathcal{A}$ such that $uau^* = b$. In fact, u can be chosen to be the unitary part of the polar decomposition of w.*

Proof. We have $wa = bw$ and by taking adjoints $aw^* = w^*b$. Thus, we get $w^*wa = w^*bw = aw^*w$. This implies that $|w|$ commutes with a. If $w = u|w|$ is the polar decomposition of w, we conclude that

$$uau^* = w|w|^{-1}au^* = wa|w|^{-1}u^* = bw|w|^{-1}u^* = b.$$

□

1.5.7 Proposition. *If p, q are projections in $\mathcal{A}$ such that $\|p - q\| < 1$, then p and q are homotopic via $u_t p u_t^*$ where u_t is a path of unitaries in $\tilde{\mathcal{A}}$ with $u_0 = 1$.*

Proof. Let $v = (2q - 1)(2p - 1) + 1$. Then $\|1 - \frac{v}{2}\| = \|(2q - 1)(q - p)\| < 1$, so v is invertible. If $w_t = t\frac{v}{2} + 1 - t$, then w_t is also invertible for $0 \leq t \leq 1$ as $\|1 - w_t\| = \|t(1 - \frac{v}{2})\| < 1$. Because $vp = 2qp = qv$, we see that $w_t p w_t^{-1}$ is a path from p to q. If u_t is the unitary part of the polar decomposition of w_t, then $u_t p u_t^*$ is the path we are looking for by the last proposition. □

1.5.8 Corollary. *Any two projections p, q homotopic in $\mathcal{A}$ via a path of idempotents can be connected via paths $e^{ita_0}p_0e^{-ita_0}, \ldots, e^{ita_{n-1}}p_{n-1}e^{-ita_{n-1}}$ where $a_0, \ldots, a_{n-1}$ are positive and $p = p_0, p_1, \ldots, p_n = q$ projections with $e^{ia_j}p_je^{-ia_j} = p_{j+1}$ for $j = 0, \ldots n - 1$.*

As these propositions indicate, pre-C^*-algebras play nicely with K-theory constructions. In fact, for K-theory calculations of C^*-algebras, we can always restrict to a dense pre-C^*-subalgebra:

1.5.9 Theorem. *If $\mathcal{A}$ is dense in A, then the inclusion $\iota : \mathcal{A} \longrightarrow A$ induces isomorphisms*

$$\iota_* : K_*(\mathcal{A}) \longrightarrow K_*(A).$$

Proof. (Sketch) Assume that A is unital. As $M_n(\mathcal{A})$ is dense in $M_n(A)$, we can find to each projection p of $M_n(A)$ a self-adjoint $p' \in M_n(\mathcal{A})$ near p such that 0 and 1 do not lie in the same connected component of $\mathrm{sp}(p')$. Using holomorphic functional calculus on p', we find a projection p'' near p' which is homotopic to p in $M_n(A)$ by Proposition 1.5.7. Thus, ι_0 is surjective.

If $p, q \in M_n(\mathcal{A})$ are homotopic in $M_n(A)$, dissect a connecting path into small pieces at points p_i such that $\|p_i - p_{i+1}\| < 1/2$. Find projections $p'_i \in M_n(\mathcal{A})$ with $\|p'_i - p_i\| < 1/4$. Then p'_i and p'_{i+1} are homotopic in $M_n(\mathcal{A})$, again by Proposition 1.5.7. This shows that p and q can be connected in $M_n(\mathcal{A})$ by a path of projections, proving that ι_0 is injective.

Similar arguments work for ι_1. □

Chapter 2

Finitely Summable K-Homology

We introduce finitely summable K-homology. In the first section we review the basic definitions of K-homology and also settle some notation we will use later on. In Section 2.2 we discuss two prominent features of K-homology: the Kasparov product and the index pairing. We will argue why we cannot expect to have a product for finitely summable K-homology. Section 2.3 contains the definition of finitely summable K-homology. In Section 2.4 we discuss how some standard normalization procedures for Fredholm modules carry over to our theory.

2.1 K-Homology

The purpose of this and the following section is to give a brief overview of K-homology, providing the most important definitions and stating some central properties. We have by no means the space to give an adequate introduction to the subject, as it can be found in [HR00] or, more condensed and in the full bivariant setting, in [Bla98]. However, we do hope that we can give the reader a sufficient foundation for understanding what follows. This section also serves the purpose of establishing some notation.

Topological K-theory, the classification of vector bundles up to stabilization, is a generalized cohomology theory on the category of finite CW-complexes. Spanier-Whitehead duality dictates the existence of a dual homology theory. This theory is called K-homology.

There are two different but mostly equivalent approaches to a concrete realization of K-homology, which both, like K-theory, generalize naturally to non-commutative spaces, i.e. C^*-algebras. Since the transition from spaces to algebras reverses arrows, K-homology then becomes a cohomology theory for C^*-algebras, which we denote by K^*.

The starting point for the first approach to K-homology is the classification of essentially normal operators on a Hilbert space by unitary equivalence up to compact operators. Brown, Douglas and Fillmore observed that such an operator defines an extension of the C^*-algebra of continuous functions on the essential spectrum of the operator by the algebra of compact operators and that the classification problem for operators with the same essential spectrum X can be reduced to classifying these extensions of $C(X)$ up to an appropriate equivalence relation [BDF77].

Under this equivalence relation, the class of extensions of $C(X)$ by the algebra of compact operators becomes an abelian group denoted by $\mathrm{Ext}(C(X))$. By defining higher order Ext-groups via suspensions of $C(X)$, Ext becomes a homology theory, which can be shown to agree with K-homology. This is the 'Ext-picture' of K-homology. We will not work with it except for the proof of Theorem 4.1.1, where we will say a few more words about it.

The second approach, which is the one we will work with, comes from index theory. Index theory is concerned with calculating the Fredholm indices of elliptic differential operators over smooth manifolds. Based on ideas of Atiyah to condense the properties of elliptic operators which are relevant for index theory into a more abstract notion, Kasparov defined homology groups in terms of equivalence classes of so-called Fredholm modules [Kas75]. These groups turn out to be another realization of K-homology.

2.1.1 Definition. A Hilbert space $\mathcal{H}$ is called $\mathbb{Z}_2$-graded (or just graded) if it is equipped with a decomposition into a positive and a negative subspace

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-.$$

An operator on $\mathcal{H}$ is called even if it maps $\mathcal{H}_+$ into $\mathcal{H}_+$ and $\mathcal{H}_-$ into $\mathcal{H}_-$. If an operator on $\mathcal{H}$ maps $\mathcal{H}_+$ into $\mathcal{H}_-$ and $\mathcal{H}_-$ into $\mathcal{H}_+$, it is called odd.

If $\mathcal{H}$ is a graded Hilbert space, then $\mathcal{H}^{op}$ is the same Hilbert space $\mathcal{H}$ endowed with the opposite grading: the positive subspace of $\mathcal{H}$ is the negative subspace of $\mathcal{H}^{op}$ and the negative subspace of $\mathcal{H}$ is the positive subspace of $\mathcal{H}^{op}$. In particular, the identity operator on $\mathcal{H}$ becomes an odd operator, when viewed as a map between $\mathcal{H}$ and $\mathcal{H}^{op}$.

2.1.2 Definition. An odd Fredholm module over a C^*-algebra A is a triple $(\varphi, \mathcal{H}, F)$ where $\varphi : A \longrightarrow \mathcal{B}(\mathcal{H})$ is a $*$-representation of A on a (possibly finite-dimensional) separable Hilbert space $\mathcal{H}$ and F is a bounded operator on $\mathcal{H}$ satisfying the relations

$$\varphi(x)(F^2 - I) \in \mathcal{K}(\mathcal{H}) \quad \text{and} \quad [F, \varphi(x)] \in \mathcal{K}(\mathcal{H}) \qquad \text{for all } x \in A. \tag{2.1}$$

An odd Fredholm module together with a $\mathbb{Z}_2$-grading of $\mathcal{H}$ is called an even Fredholm module if φ is even and F is odd with respect to this grading. In other words, an even Fredholm module is an odd Fredholm module of the form

$$\left(\varphi_+ \oplus \varphi_-,\, \mathcal{H}_+ \oplus \mathcal{H}_-,\, \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\right).$$

We denote the class of odd Fredholm modules over A by $\mathbb{F}^1(A)$ and the class of even Fredholm modules by $\mathbb{F}^0(A)$.

2.1.3 Remarks. 1. Defining Fredholm modules as triples $(\varphi, \mathcal{H}, F)$ is a bit redundant, since $\mathcal{H}$ is already given as the representation space of φ. We have opted for this more verbose definition to always have a name for this space at hand.

2. There are quite a few variations of how Fredholm modules are defined. For example, F is often required to be self-adjoint or self-adjoint up to compact operators. We have chosen here the most general definition and show further below that all these definitions lead to the same homology groups.[†]

As we will demonstrate in Chapter 6, elliptic differential operators over a closed manifold M give rise to Fredholm modules over $C(M)$ (one obtains F by making the differential operator bounded). This justifies to think of Fredholm modules as abstract elliptic operators over (possibly) non-commutative spaces.

If a Fredholm module is constructed from an elliptic operator, the relations in (2.1) are not only satisfied up to compact operators but even up to p-summable operators, where p depends on the dimension of the manifold. This observation is the starting point of finitely summable K-homology.

Now, to define Kasparov's K-homology groups $K^*(A)$, we first need to introduce the notions of degeneracy and direct sum of Fredholm modules:

2.1.4 Definition. A Fredholm module $(\varphi, \mathcal{H}, F) \in \mathbb{F}^*(A)$ is called degenerate if

$$F^* = F, \quad F^2 - I = 0 \quad \text{and} \quad [F, \varphi(x)] = 0 \quad \text{for all } x \in A.$$

The classes of degenerate odd and even Fredholm modules over A will be denoted by $\mathbb{D}^*(A) \subseteq \mathbb{F}^*(A)$.

2.1.5 Definition. For $(\varphi, \mathcal{H}, F), (\varphi', \mathcal{H}', F') \in \mathbb{F}^*(A)$ define

$$(\varphi, \mathcal{H}, F) \oplus (\varphi', \mathcal{H}', F') := (\varphi \oplus \varphi', \mathcal{H} \oplus \mathcal{H}', F \oplus F').$$

In the even case, the direct sum is to be understood as a direct sum of graded Hilbert spaces (the even (resp. odd) subspace of $\mathcal{H} \oplus \mathcal{H}'$ is the direct sum of the even (resp. odd) subspaces of $\mathcal{H}$ and $\mathcal{H}'$).

It is obvious that $(\varphi, \mathcal{H}, F) \oplus (\varphi', \mathcal{H}', F') \in \mathbb{F}^*(A)$. If both summands are degenerate, their direct sum is degenerate as well.

[†]See Theorem 2.4.1 and Remark 2.4.8.

Next, we define some equivalence relations on the sets[†] of odd and even Fredholm modules:

2.1.6 Definition. 1. Two odd modules $(\varphi, \mathcal{H}, F), (\varphi', \mathcal{H}', F') \in \mathbb{F}^1(A)$ are called unitarily equivalent if there is a unitary $U : \mathcal{H} \longrightarrow \mathcal{H}'$ such that

$$F' = UFU^* \quad \text{and} \quad \varphi'(x) = U\varphi(x)U^* \quad \text{for all } x \in A.$$

Two even modules are called unitarily equivalent if, moreover, U respects the grading, sending the even (resp. odd) subspace of $\mathcal{H}$ to the even (resp. odd) subspace of $\mathcal{H}'$. We write $(\varphi, \mathcal{H}, F) \sim_u (\varphi', \mathcal{H}', F')$.

2. $(\varphi, \mathcal{H}, F), (\varphi', \mathcal{H}', F') \in \mathbb{F}^*(A)$ are said to differ by degenerate modules if there are $(\psi, \mathcal{N}, G), (\psi', \mathcal{N}', G') \in \mathbb{D}^*(A)$ such that

$$(\varphi, \mathcal{H}, F) \oplus (\psi, \mathcal{N}, G) = (\varphi', \mathcal{H}', F') \oplus (\psi', \mathcal{N}', G').$$

We denote this by $(\varphi, \mathcal{H}, F) \sim_d (\varphi', \mathcal{H}', F')$.

3. $(\varphi, \mathcal{H}, F), (\varphi, \mathcal{H}, F') \in \mathbb{F}^*(A)$ are called operator homotopic if there is a continuous path $F_t \in \mathcal{B}(\mathcal{H})$ with $F_0 = F$ and $F_1 = F'$ such that $(\varphi, \mathcal{H}, F_t) \in \mathbb{F}^*(A)$ for all $t \in [0, 1]$. We write $(\varphi, \mathcal{H}, F) \sim_{oh} (\varphi, \mathcal{H}, F')$.

4. If $\sim_1, \sim_2, \ldots, \sim_k$ are equivalence relations, we define $\langle \sim_1, \sim_2, \ldots, \sim_k \rangle$ to be the transitive closure of the $\sim_j$. So $x \langle \sim_1, \sim_2, \ldots, \sim_k \rangle y$ if and only if there exists a finite chain

$$x \sim_{i_0} x_1 \sim_{i_1} \ldots \sim_{i_{n-1}} x_n \sim_{i_n} y$$

with $i_j \in \{1, 2, \ldots, k\}$.

It is immediately apparent that these relations are indeed equivalence relations.

If a Fredholm module $\mathscr{F}$ is defined by an elliptic operator D, then the index data[‡] of D is preserved by $\mathscr{F}$. As the Fredholm index is invariant under unitary equivalence and homotopies, additive with respect to direct sums, and zero for degenerate modules, the following definition is plausible (compare Definition 2.2.4).

[†]The vigilant reader will have noticed that $\mathbb{F}^*(A)$ and $\mathbb{D}^*(A)$ are no sets at all but honest classes since there are class-many separable Hilbert spaces. The same is true for the equivalence classes of Fredholm modules we are going to define. Thus, to define $K^*(A)$ as the set of these equivalence classes is completely meaningless!

If you should feel uncomfortable now, there is an easy remedy: choose for each at most countable cardinality a fixed Hilbert space and consider only Fredholm modules which are defined on these spaces. If you moreover choose fixed unitaries which identify direct sums of these spaces with other Hilbert spaces of this set, you are out of trouble. We happily leave the details to readers with a romantic attitude towards formalism.

[‡]The (graded) Fredholm index of D, but also the index of D after twisting it with some vector bundle.

2.1.7 Definition. The K-homology groups $K^0(A)$ and $K^1(A)$ of a C^*-algebra A are given by

$$K^*(A) := \mathbb{F}^*(A) \;/\; \langle \sim_u, \sim_d, \sim_{oh} \rangle.$$

It is obvious that the direct sum operation on Fredholm modules is compatible with all equivalence relations introduced so far. Therefore, we can define the sum of elements $\mathbf{x}, \mathbf{y} \in K^*(\mathcal{A})$ to be the class of the direct sum of two of their representatives. The following proposition shows that $K^0(A)$ and $K^1(A)$ really form a group when equipped with this addition.

2.1.8 Proposition. *The sets $K^*(A)$ form abelian groups with addition given by the direct sum of representatives.*

Proof. First note that, by dividing out $\sim_d$ and $\sim_u$, all degenerate odd (resp. even) modules are equivalent. The class $\mathbf{0} \in K^*(A)$ they represent is a neutral element under addition.

If $\mathbf{x} = [(\varphi, \mathcal{H}, F)] \in K^*(A)$, then the inverse of $\mathbf{x}$ is represented by $(\varphi, \mathcal{H}^{op}, -F)$. This is because

$$\left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H}^{op},\, \begin{bmatrix} \cos(\frac{\pi}{2}t)F & \sin(\frac{\pi}{2}t)I \\ \sin(\frac{\pi}{2}t)I & -\cos(\frac{\pi}{2}t)F \end{bmatrix}\right)$$

is an operator homotopy connecting the direct sum of the modules to a degenerate module.

Finally, $K^*(A)$ is abelian since a direct sum of modules $\mathscr{F} \oplus \mathscr{F}'$ is unitarily equivalent to $\mathscr{F}' \oplus \mathscr{F}$. □

2.1.9 Remark. In the introduction we have defined $K^*(A)$ as the Grothendieck group of $\mathbb{F}^*(A) \;/\; \langle \sim_u, \sim_{oh} \rangle$. Both definitions agree: if $\mathscr{D}$ is a degenerate Fredholm module over A, then $\bigoplus_{\mathbb{N}} \mathscr{D}$ is a Fredholm module as well. Since $\mathscr{D} \oplus \bigoplus_{\mathbb{N}} \mathscr{D} \sim_u \bigoplus_{\mathbb{N}} \mathscr{D}$, cancellation implies that the class of $\mathscr{D}$ must be zero.

2.1.10 Definition. If $\alpha : A \longrightarrow B$ is a $*$-homomorphism and $\mathscr{F} = (\varphi, \mathcal{H}, F)$ a Fredholm module over B, then define the pullback of $\mathscr{F}$ via α to be the module over A given by

$$\alpha^* \mathscr{F} := (\varphi \circ \alpha, \mathcal{H}, F)$$

This pullback operation descends to K-homology, making K^0 and K^1 contravariant functors from the category of C^*-algebras to the category of abelian groups.

2.2 Product and index pairing in K-homology

One of the remarkable features of Kasparov's K-homology is the existence of a bilinear product

$$\times : K^i(A) \times K^j(B) \longrightarrow K^{i+j}(A \otimes B) \tag{2.2}$$

(the sum $i + j$ is to be understood modulo 2 and $A \otimes B$ denotes the minimal C^*-algebraic tensor product of A and B). More precisely, the following theorem is true:

2.2.1 Theorem (Kasparov)**.** *For any separable C^*-algebras A and B there are bilinear maps of the form (2.2) such that the following statements are true:*

(i) If A is a separable C^*-algebra and $\mathbf{1}$ is the generator of $K^0(\mathbb{C})$ of index 1,[†] then under the isomorphisms

$$\mathbb{C} \otimes A \cong A \cong A \otimes \mathbb{C}$$

we have for any $\mathbf{x} \in K^*(A)$

$$\mathbf{1} \times \mathbf{x} = \mathbf{x} = \mathbf{x} \times \mathbf{1}.$$

(ii) The product is natural in the following sense: if $\alpha : A \longrightarrow A'$ and $\beta : B \longrightarrow B'$ are $*$-homomorphisms between separable C^*-algebras, $\mathbf{x} \in K^*(A')$ and $\mathbf{y} \in K^*(B')$, then

$$(\alpha \otimes \beta)^*(\mathbf{x} \times \mathbf{y}) = \alpha^*\mathbf{x} \times \beta^*\mathbf{y}.$$

(iii) The product is associative: if $\mathbf{x} \in K^*(A)$, $\mathbf{y} \in K^*(B)$ and $\mathbf{z} \in K^*(C)$, then under the isomorphism $(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ we obtain

$$(\mathbf{x} \times \mathbf{y}) \times \mathbf{z} = \mathbf{x} \times (\mathbf{y} \times \mathbf{z}).$$

Proof. [HR00, Chapter 9]. □

The K-homology groups $K^*(A)$ are just special cases of Kasparov's bivariant K-theory groups. In this setting, we have

$$K^*(A) = KK^*(A, \mathbb{C}),$$

and the product becomes an instance (the external product) of the even more powerful bivariant product in KK-theory

$$KK^i(A_1, B_1 \otimes C) \times KK^j(C \otimes A_2, B_2) \longrightarrow KK^{i+j}(A_1 \otimes A_2, B_1 \otimes B_2).$$

Much of K-homology revolves around the product, and more than one deep property of K-homology can be derived from its properties. As an example, we indicate how to prove the homotopy invariance of $K^*(A)$ using the product:

[†] The index paring with $1 \in \mathbb{C}$ (Definition 2.2.4) is an isomorphism between $K^0(\mathbb{C})$ and $\mathbb{Z}$.

2.2.2 Theorem. *If $\alpha_t : A \longrightarrow B$ is a homotopy of $*$-homomorphisms between separable C^*-algebras, then*

$$\alpha_0^* = \alpha_1^* : K^*(B) \longrightarrow K^*(A).$$

Proof. (Sketch) Assume we have already proven that the evaluation maps

$$ev_0, ev_1 : C([0,1]) \longrightarrow \mathbb{C}$$

of $C([0,1])$ at 0 and 1 satisfy

$$ev_0^* = ev_1^* : K^0(\mathbb{C}) \longrightarrow K^0(C([0,1])).$$

The homotopy α_t is nothing but a $*$-homomorphism

$$\alpha : A \longrightarrow C([0,1]) \otimes B,$$

and

$$\alpha_i = (\mathrm{ev}_i \otimes \mathrm{id}_B) \circ \alpha \qquad \text{for } i = 0, 1.$$

Thus, for $\mathbf{x} \in K^*(B)$ we have

$$\begin{aligned}\alpha_0^*\mathbf{x} &= \alpha^* \circ (\mathrm{ev}_0 \otimes \mathrm{id}_B)^*(\mathbf{1} \times \mathbf{x}) \\ &= \alpha^* \circ (\mathrm{ev}_0^*\,\mathbf{1} \times \mathbf{x}) = \alpha^* \circ (\mathrm{ev}_1^*\,\mathbf{1} \times \mathbf{x}) \\ &= \alpha^* \circ (\mathrm{ev}_1 \otimes \mathrm{id}_B)^*(\mathbf{1} \times \mathbf{x}) = \alpha_1^*\mathbf{x},\end{aligned}$$

using properties (i) and (ii) of Theorem 2.2.1.

For a full proof see [HR00, Theorem 9.3.3]. □

Bott periodicity of K^* can be proven by a similar use of the product (in fact, one even uses the product to prove Bott periodicity for $\mathbb{C}$). See [HR00, Theorem 9.5.2] and also [Ati68].

If $\mathscr{F}_1 = (\varphi_1, \mathcal{H}_1, F_1)$ and $\mathscr{F}_2 = (\varphi_2, \mathcal{H}_2, F_2)$ are even Fredholm modules over separable C^*-algebras A_1 and A_2, then a representative of the product $[\mathscr{F}_1] \times [\mathscr{F}_2]$ can be constructed as

$$\mathscr{F}_1 \times \mathscr{F}_2 := \Big(\varphi_1 \hat{\otimes} \varphi_2, \mathcal{H}_1 \hat{\otimes} \mathcal{H}_2, N_1(F_1 \hat{\otimes} I) + N_2(I \hat{\otimes} F_2)\Big).$$

Here, N_1, N_2 are certain positive operators and $\hat{\otimes}$ denotes the graded tensor product.† In particular, since F_1 and F_2 are odd, $F_1 \hat{\otimes} I$ and $I \hat{\otimes} F_2$ anticommute.

†If A, B are graded algebras and $a_1, a_2 \in A$, $b_1, b_2 \in B$ homogeneous (i.e. either odd or even) elements, then multiplication in $A \hat{\otimes} B$ is defined as

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) := (-1)^{\partial b_1 \partial a_2} a_1 a_2 \hat{\otimes} b_1 b_2$$

where ∂x is 0 or 1 depending on whether x is even or odd.

Assuming that F_1 and F_2 are symmetries, we have

$$(F_1\hat{\otimes}I + I\hat{\otimes}F_2)^2 = F_1^2\hat{\otimes}I + I\hat{\otimes}F_2^2 = 2 \cdot I\hat{\otimes}I.$$

Thus, if we set $N_1 = N_2 = I/\sqrt{2}$, we obtain a symmetry again. However, the commutators with $\varphi_1\hat{\otimes}\varphi_2$ will only lie in $\mathcal{K}(\mathcal{H}_1)\hat{\otimes}\varphi_2(A) + \varphi_1(A)\hat{\otimes}\mathcal{K}(\mathcal{H}_2)$ and therefore will not be compact in general.

One can easily check that the following conditions on N_1, N_2 suffice to make $\mathscr{F}_1 \times \mathscr{F}_2$ a Fredholm module:

(i) N_1, N_2 are even and $N_1^2 + N_2^2 = I$.

(ii) $[N_1, F_1\hat{\otimes}I], [N_1, I\hat{\otimes}F_2] \in \mathcal{K}(\mathcal{H}_1\hat{\otimes}\mathcal{H}_2)$

(iii) $[N_1, \varphi_1(A_1)\hat{\otimes}\varphi_2(A_2)] \subseteq \mathcal{K}(\mathcal{H}_1\hat{\otimes}\mathcal{H}_2)$

(iv) $N_1 \cdot \mathcal{K}(\mathcal{H}_1)\hat{\otimes}I \subseteq \mathcal{K}(\mathcal{H}_1\hat{\otimes}\mathcal{H}_2)$ and $N_2 \cdot I\hat{\otimes}\mathcal{K}(\mathcal{H}_2) \subseteq \mathcal{K}(\mathcal{H}_1\hat{\otimes}\mathcal{H}_2)$.

Such operators N_1, N_2 can be found for separable A_1, A_2 by a theorem of Kasparov, which is known as Kasparov's Technical Theorem (or Kasparov's Technical Lemma).

Kasparov's Technical Theorem is an existence theorem for non-commutative partitions of unity. A basic version of it can be stated as follows.

2.2.3 Theorem (Kasparov's Technical Theorem)**.** *Let a separable Hilbert space $\mathcal{H}$, separable C^*-algebras $B_1, B_2 \subseteq \mathcal{B}(\mathcal{H})$ and a separable subset Δ of $\mathcal{B}(\mathcal{H})$ be given such that $B_1 \cdot B_2 \subseteq \mathcal{K}(\mathcal{H})$ and $[\Delta, B_1] \subseteq B_1$. Then there is an $X \in \mathcal{B}(\mathcal{H})$ with $0 \leq X \leq I$ and*

$$XB_1 \subseteq \mathcal{K}(\mathcal{H}), \qquad (I - X)B_2 \subseteq \mathcal{K}(\mathcal{H}), \qquad [\Delta, X] \subseteq \mathcal{K}(\mathcal{H}).$$

Proof. [HR00, Theorem 3.8.1]. □

To obtain N_1 and N_2 from this theorem, choose

$$\Delta := \{F_1\hat{\otimes}I, I\hat{\otimes}F_2, \gamma\} \cup \varphi_1(A_1)\hat{\otimes}\varphi_2(A_2)$$

where γ denotes the grading operator of $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ (the symmetry which is the identity on the positive subspace of $\mathcal{H}_1\hat{\otimes}\mathcal{H}_2$ and minus the identity on its negative subspace). Let B_1 be the smallest C^*-subalgebra of $\mathcal{K}(\mathcal{H}_1)\hat{\otimes}\mathcal{B}(\mathcal{H}_2)$ which contains $\mathcal{K}(\mathcal{H}_1)\hat{\otimes}I$ and satisfies $[\Delta, B_1] \subseteq B_1$. Let $B_2 := I\hat{\otimes}\mathcal{K}(\mathcal{H}_2)$. Then we can set

$$N_1 := \left(\frac{X + \gamma X\gamma}{2}\right)^{\frac{1}{2}} \quad \text{and} \quad N_2 := (I - N_1^2)^{\frac{1}{2}}$$

[HR00, Proposition 9.2.3 and Proposition 9.2.5].

Theorem 2.2.3 can also be used to prove excision in K-homology [HR00, Theorem 5.4.5].

The proof of Kasparov's Technical Theorem crucially depends on C^*-algebra techniques. In particular, one needs to construct an approximate unit of $\mathcal{K}(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ which is quasi-central[†] for Δ.

If one wanted to construct a product of finitely summable Fredholm modules (Definition 2.3.3) which is again finitely summable, $\mathcal{K}(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ would have to be replaced by $\mathcal{L}^p(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ everywhere above. In particular, an approximate unit of $\mathcal{L}^p(\mathcal{H}_1 \hat{\otimes} \mathcal{H}_2)$ would have to be found which is quasi-central for Δ. However, this is not possible in general, even for a single operator. In fact, Voiculescu's k_p precisely measures the obstruction for the existence of such approximate units [Voi79].

All in all, there does not seem to be much hope to define a product for finitely summable K-homology. This is the main reason why working with finitely summable K-homology is much harder than working with standard K-homology.

* * *

A second important feature of K-homology is its duality pairing with K-theory. Being defined in terms of Fredholm indices, this pairing is also called the 'index pairing'. If a K-homology class is constructed from a differential operator, the index data of the operator can be calculated via the index pairing of its K-homology class with K-theory. Therefore, it is of particular interest to obtain formulas for calculating this index pairing (see the introduction to this thesis for further discussion).

There are two index pairings, an odd one between $K^1(A)$ and $K_1(A)$ and an even one between $K^0(A)$ and $K_0(A)$. We will only need the even one:

2.2.4 Definition and Proposition. Let A be a unital C^*-Algebra and

$$\mathscr{F} = \left(\varphi_+ \oplus \varphi_-,\, \mathcal{H}_+ \oplus \mathcal{H}_-,\, \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} \right)$$

an even Fredholm module over A. If p is a projection in $M_n(A)$, define $\mathrm{Ind}_{\mathscr{F}}(p)$ to be the Fredholm index

$$\mathrm{Ind}_{\mathscr{F}}(p) := \text{F-Ind}\Big[(\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi_-)(p) \cdot (I_n \otimes U) \cdot (\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi_+)(p) \Big]$$

where $(\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi_-)(p) \cdot (I_n \otimes U) \cdot (\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi_+)(p)$ is regarded as a map between the spaces $(\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi_+)(p)(\mathbb{C}^n \otimes \mathcal{H}_+)$ and $(\mathrm{id}_{M_n(\mathbb{C})} \otimes \varphi_-)(p)(\mathbb{C}^n \otimes \mathcal{H}_-)$.

This definition descends to a well-defined bilinear map

$$K^0(A) \times K_0(A) \longrightarrow \mathbb{Z},$$

called the index pairing between $K^0(A)$ and $K_0(A)$.

[†]If I is an ideal in A and $x \in A$, then an approximate unit u_λ of I is quasi-central for x if $\lim_\lambda [u_\lambda, x] = 0$.

Proof. First, let p be a projection in A. We have

$$\begin{bmatrix} \varphi_+(p)V\varphi_-(p)U\varphi_+(p) - \varphi_+(p) & 0 \\ 0 & \varphi_-(p)U\varphi_+(p)V\varphi_-(p) - \varphi_-(p) \end{bmatrix}$$
$$= (\varphi_+ \oplus \varphi_-)(p) \cdot \left(\begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} \cdot (\varphi_+ \oplus \varphi_-)(p) \cdot \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} \cdot (\varphi_+ \oplus \varphi_-)(p) - I \right) \in \mathcal{K}.$$

This means that $\varphi_-(p)U\varphi_+(p)$ is up to compact operators an invertible map between $\varphi_+(p)\mathcal{H}_+$ and $\varphi_-(p)\mathcal{H}_-$ and therefore Fredholm.

If p is a projection in $M_n(A)$, the same argument applies since

$$\mathscr{F}^{(n)} := \left(\mathrm{id}_{M_n(\mathbb{C})} \otimes (\varphi_+ \oplus \varphi_-),\, \mathbb{C}^n \otimes (\mathcal{H}_+ \oplus \mathcal{H}_-),\, I_n \otimes \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix} \right)$$

is a Fredholm module over $M_n(A)$. Moreover, it is clear that the index is invariant under the embedding of $p \in M_n(A)$ into $M_{n+1}(A)$. In particular, for the rest of the proof it suffices to consider only projections in A since we can always replace A by $M_n(A)$ and $\mathscr{F}$ by $\mathscr{F}^{(n)}$.

Next, observe that if $q = upu^*$ for some unitary $u \in A$, then

$$\begin{aligned} \text{F-Ind } \varphi_-(upu^*)U\varphi_+(upu^*) &= \text{F-Ind } \varphi_-(p)\varphi_-(u^*)U\varphi_+(u)\varphi_+(p) \\ &= \text{F-Ind } \varphi_-(p)U\varphi_+(u^*)\varphi_+(u)\varphi_+(p) \\ &= \text{F-Ind } \varphi_-(p)U\varphi_+(p) \end{aligned}$$

since the Fredholm index is invariant under compact perturbations (of course, the Fredholm indices have to be taken with respect to the right domains and codomains).

Furthermore, if q is orthogonal to p, then

$$\varphi_-(p+q)U\varphi_+(p+q)$$

differs by a compact operator from

$$\varphi_-(p)U\varphi_+(p) + \varphi_-(q)U\varphi_+(q).$$

Thus, $\mathrm{Ind}_{\mathcal{F}}$ is a homomorphism from the semigroup of unitary equivalence classes of projections in $M_\infty(A)$ to $\mathbb{Z}$. Since $\mathbb{Z}$ is a group, this homomorphism descends to a homomorphism from the Grothendieck group of $M_\infty(A)$, namely $K_0(A)$, to $\mathbb{Z}$.

It is clear that if $\mathscr{F}'$ is another even Fredholm module over A, we have $\mathrm{Ind}_{\mathscr{F}\oplus\mathscr{F}'} = \mathrm{Ind}_{\mathscr{F}} + \mathrm{Ind}_{\mathscr{F}'}$ since everything decomposes as direct sums. By homotopy invariance of the Fredholm index, we see that $\mathrm{Ind}_{\mathscr{F}} = \mathrm{Ind}_{\mathscr{F}'}$ if $\mathscr{F}$ and $\mathscr{F}'$ are operator-homotopic Fredholm modules.

Finally, unitarily equivalent modules obviously have the same index map and if $\mathscr{F}$ is degenerate, then

$$\varphi_+(p)V\varphi_-(p)U\varphi_+(p) = \varphi_+(p) \quad \text{and} \quad \varphi_-(p)U\varphi_+(p)V\varphi_-(p) = \varphi_-(p).$$

Thus, $\varphi_-(p)U\varphi_+(p)$ is invertible as a map from $\varphi_+(p)\mathcal{H}_+$ to $\varphi_-(p)\mathcal{H}_-$ and therefore has index 0.

These observations show that $\mathrm{Ind}_{\mathscr{F}}$ is independent of the class of $\mathscr{F}$ in $K^0(A)$, which concludes the proof. □

One can prove that the index map is actually a special case of the bivariant Kasparov product: if one identifies $K_0(A)$ with $KK^0(\mathbb{C}, A)$ and $K^0(A)$ with $KK^0(A, \mathbb{C})$ as well as $KK^0(\mathbb{C}, \mathbb{C})$ with $\mathbb{Z}$, the index pairing becomes the product

$$KK^0(\mathbb{C}, A) \times KK^0(A, \mathbb{C}) \longrightarrow KK^0(\mathbb{C}, \mathbb{C}).$$

2.2.5 Remark. The index map is natural in the following sense: let α be a $*$-homomorphism between unital C^*-algebras A and B, $\mathbf{x} \in K^0(B)$ and $\mathbf{p} \in K_0(A)$. Then it follows immediately from the definition of the index map that

$$\mathrm{Ind}_{\mathbf{x}}(\alpha_* \mathbf{p}) = \mathrm{Ind}_{\alpha^* \mathbf{x}}(\mathbf{p}).$$

Using this, we can extend the definition of the index map to non-unital C^*-algebras A: if we denote by ι the inclusion of A in A^+, define

$$\mathrm{Ind}_{\iota^* \mathbf{x}}(\mathbf{p}) := \mathrm{Ind}_{\mathbf{x}}(\iota_* \mathbf{p}) \qquad \text{for all } \mathbf{p} \in K_0(A).$$

This is indeed well-defined: we have the short exact sequence[†]

$$0 \longleftarrow K^0(A) \xleftarrow{\iota^*} K^0(A^+) \xleftarrow{\pi^*} K^0(\mathbb{C}) \longleftarrow 0,$$

therefore ι^* is surjective. If $\iota^* \mathbf{x} = \iota^* \mathbf{y}$, then there is a $\mathbf{z} \in K^0(\mathbb{C})$ with $\pi^* \mathbf{z} = \mathbf{x} - \mathbf{y}$. Thus, $\mathrm{Ind}_{\mathbf{x}-\mathbf{y}}(\iota_* \mathbf{p}) = \mathrm{Ind}_{\mathbf{z}}(\pi_* \iota_* \mathbf{p}) = 0$.

2.3 Definition of finitely summable K-homology

For the rest of this chapter $\mathcal{A}$ denotes any topological $*$-algebra.

Since we want to define finitely summable K-homology for arbitrary topological $*$-algebras, we first have to extend the notion of a Fredholm module to such algebras. The definition carries over verbatim from Definition 2.1.2 with the only exception that we require the representation to be continuous since this is not automatic for arbitrary topological $*$-algebras:

2.3.1 Definition. An odd Fredholm module over a topological $*$-algebra $\mathcal{A}$ is a triple $(\varphi, \mathcal{H}, F)$ where $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a continuous $*$-representation of $\mathcal{A}$ on a (possibly finite-dimensional) separable Hilbert space $\mathcal{H}$ and F is a bounded operator on $\mathcal{H}$ satisfying the relations

$$\varphi(x)(F^2 - I) \in \mathcal{K}(\mathcal{H}) \quad \text{and} \quad [F, \varphi(x)] \in \mathcal{K}(\mathcal{H}) \qquad \text{for all } x \in \mathcal{A}.$$

[†]This is proven in Proposition 3.2.1 for finitely summable K-homology. The same proof works for K-homology.

An odd Fredholm module together with a $\mathbb{Z}_2$-grading of $\mathcal{H}$ is called an even Fredholm module if φ is even and F is odd with respect to this grading. In other words, an even Fredholm module is an odd Fredholm module of the form

$$\left(\varphi_+ \oplus \varphi_-,\, \mathcal{H}_+ \oplus \mathcal{H}_-,\, \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\right).$$

We denote the class of odd Fredholm modules over $\mathcal{A}$ by $\mathbb{F}^1(\mathcal{A})$, the class of even Fredholm modules by $\mathbb{F}^0(\mathcal{A})$.

2.3.2 Remark. Everything in this chapter and some of what we say in later chapters is also true in a purely algebraic setting. We restrict to topological algebras and continuous representations at this point mainly to not have several variants of finitely summable K-homology floating around.

2.3.3 Definition. We call a Fredholm module $(\varphi, \mathcal{H}, F) \in \mathbb{F}^*(\mathcal{A})$ *p*-summable ($1 \leq p < \infty$) if

$$\varphi(x)(F^2 - I) \in \mathcal{L}^p \quad \text{and} \quad [F, \varphi(x)] \in \mathcal{L}^p \quad \text{for all } x \in \mathcal{A}.$$

We denote the classes of odd and even p-summable Fredholm modules over $\mathcal{A}$ by $\mathbb{F}^*_p(\mathcal{A})$. A Fredholm module is called finitely summable if it is p-summable for some p.

From now on we shall always assume that $1 \leq p < \infty$. Whenever a statement or definition contains the unquantified variable p, it is to be understood that it holds for all such p.

Degenerate modules and direct sums of Fredholm modules over topological $*$-algebras are defined exactly as for C^*-algebras in Definitions 2.1.4 and 2.1.5. Note that direct sums of p-summable modules are again p-summable.

We define two more equivalence relations for p-summable Fredholm modules:

2.3.4 Definition.

1. If $(\varphi, \mathcal{H}, F), (\varphi, \mathcal{H}, F') \in \mathbb{F}^*_p(\mathcal{A})$ are operator homotopic via a continuous path $F_t \in \mathcal{B}(\mathcal{H})$ such that $(\varphi, \mathcal{H}, F_t) \in \mathbb{F}^*_p(\mathcal{A})$ for all $t \in [0, 1]$, we write $(\varphi, \mathcal{H}, F) \sim_{oh,p} (\varphi, \mathcal{H}, F')$.

2. For $(\varphi, \mathcal{H}, F), (\varphi', \mathcal{H}', F') \in \mathbb{F}^*_p(\mathcal{A})$ write $(\varphi, \mathcal{H}, F) \sim_{d+oh,p+u} (\varphi', \mathcal{H}', F')$ if there are degenerate modules $(\psi, \mathcal{N}, G), (\psi', \mathcal{N}', G') \in \mathbb{D}^*(\mathcal{A})$ such that $(\varphi, \mathcal{H}, F) \oplus (\psi, \mathcal{N}, G)$ is operator homotopic via p-summable modules to a module which is unitarily equivalent to $(\varphi', \mathcal{H}', F') \oplus (\psi', \mathcal{N}', G')$.

2.3.5 Proposition. $\sim_{d+oh,p+u}$ *is an equivalence relation.*

Proof. Only the transitivity of $\sim_{d+oh,p+u}$ might not be completely obvious.

For $i = 1, 2, 3$ let $(\varphi_i, \mathcal{H}_i, F_i) \in \mathbb{F}_p^*(\mathcal{A})$ be given with

$$(\varphi_1, \mathcal{H}_1, F_1) \sim_{d+oh,p+u} (\varphi_2, \mathcal{H}_2, F_2) \sim_{d+oh,p+u} (\varphi_3, \mathcal{H}_3, F_3).$$

This means that there are

$$(\psi_1, \mathcal{N}_1, G_1), (\psi_{2,1}, \mathcal{N}_{2,1}, G_{2,1}), (\psi_{2,2}, \mathcal{N}_{2,2}, G_{2,2}), (\psi_3, \mathcal{N}_3, G_3) \in \mathbb{D}^*(\mathcal{A}),$$

p-summable operator homotopies

$$(\varphi_1 \oplus \psi_1, \mathcal{H}_1 \oplus \mathcal{N}_1, F_t^{(1)}) \quad \text{and} \quad (\varphi_2 \oplus \psi_{2,2}, \mathcal{H}_2 \oplus \mathcal{N}_{2,2}, F_t^{(2)}),$$

as well as unitaries

$$U_1 : \mathcal{H}_1 \oplus \mathcal{N}_1 \longrightarrow \mathcal{H}_2 \oplus \mathcal{N}_{2,1} \quad \text{and} \quad U_2 : \mathcal{H}_2 \oplus \mathcal{N}_{2,2} \longrightarrow \mathcal{H}_3 \oplus \mathcal{N}_3$$

such that:

(i) $F_0^{(1)} = F_1 \oplus G_1$

(ii) $\Big(U_1(\varphi_1 \oplus \psi_1)U_1^*, \mathcal{H}_2 \oplus \mathcal{N}_{2,1}, U_1 F_1^{(1)} U_1^*\Big) = \Big(\varphi_2 \oplus \psi_{2,1}, \mathcal{H}_2 \oplus \mathcal{N}_{2,1}, F_2 \oplus G_{2,1}\Big)$

(iii) $F_0^{(2)} = F_2 \oplus G_{2,2}$

(iv) $\Big(U_2(\varphi_2 \oplus \psi_{2,2})U_2^*, \mathcal{H}_3 \oplus \mathcal{N}_3, U_2 F_1^{(2)} U_2^*\Big) = \Big(\varphi_3 \oplus \psi_3, \mathcal{H}_3 \oplus \mathcal{N}_3, F_3 \oplus G_3\Big)$

Let S be the unitary

$$S : \mathcal{H}_2 \oplus \mathcal{N}_{2,1} \oplus \mathcal{N}_{2,2} \longrightarrow \mathcal{H}_2 \oplus \mathcal{N}_{2,2} \oplus \mathcal{N}_{2,1}$$

which flips the last two summands, and define

$$\tilde{U}_1 := S(U_1 \oplus I) : \mathcal{H}_1 \oplus \mathcal{N}_1 \oplus \mathcal{N}_{2,2} \longrightarrow \mathcal{H}_2 \oplus \mathcal{N}_{2,2} \oplus \mathcal{N}_{2,1}.$$

We then have

$$\tilde{U}_1(\varphi_1 \oplus \psi_1 \oplus \psi_{2,2})\tilde{U}_1^* = \varphi_2 \oplus \psi_{2,2} \oplus \psi_{2,1} \tag{2.3}$$

and

$$\tilde{U}_1(F_1^{(1)} \oplus G_{2,2})\tilde{U}_1^* = F_0^{(2)} \oplus G_{2,1}$$

by properties (ii) and (iii). Thus, we can concatenate the two operator homotopies

$$\Big(\varphi_1 \oplus \psi_1 \oplus \psi_{2,2}, \mathcal{H}_1 \oplus \mathcal{N}_1 \oplus \mathcal{N}_{2,2}, F_t^{(1)} \oplus G_{2,2}\Big)$$

and

$$\Big(\varphi_1 \oplus \psi_1 \oplus \psi_{2,2}, \mathcal{H}_1 \oplus \mathcal{N}_1 \oplus \mathcal{N}_{2,2}, \tilde{U}_1^*(F_t^{(2)} \oplus G_{2,1})\tilde{U}_1\Big)$$

to an operator homotopy

$$\mathscr{F}_t := \Big(\varphi_1 \oplus \psi_1 \oplus \psi_{2,2}, \mathcal{H}_1 \oplus \mathcal{N}_1 \oplus \mathcal{N}_{2,2}, \tilde{F}_t\Big).$$

Property (i) implies that $\mathscr{F}_t$ starts at

$$\Big(\varphi_1 \oplus \psi_1 \oplus \psi_{2,2}, \mathcal{H}_1 \oplus \mathcal{N}_1 \oplus \mathcal{N}_{2,2}, F_1 \oplus G_1 \oplus G_{2,2}\Big).$$

By Property (iv) and (2.3), we have

$$(U_2 \oplus I)\tilde{U}_1(\varphi_1 \oplus \psi_1 \oplus \psi_{2,2})\tilde{U}_1^*(U_2 \oplus I)^* = \varphi_3 \oplus \psi_3 \oplus \psi_{2,1}$$

and

$$(U_2 \oplus I)\tilde{U}_1 \left(\tilde{U}_1^*(F_1^{(2)} \oplus G_{2,1})\tilde{U}_1\right) \tilde{U}_1^*(U_2 \oplus I)^* = F_3 \oplus G_3 \oplus G_{2,1}.$$

Therefore, the endpoint of $\mathscr{F}_t$ is unitarily equivalent to

$$\Big(\varphi_3 \oplus \psi_3 \oplus \psi_{2,1}, \mathcal{H}_3 \oplus \mathcal{N}_3 \oplus \mathcal{N}_{2,1}, F_3 \oplus G_3 \oplus G_{2,1}\Big).$$

□

2.3.6 Corollary.

$$\langle \sim_u, \sim_d, \sim_{oh,p} \rangle \;=\; \sim_{d+oh,p+u} \;\; .$$

2.3.7 Definition. The finitely summable K-homology groups of $\mathcal{A}$ are defined as

$$K_p^*(\mathcal{A}) := \mathbb{F}_p^*(\mathcal{A}) \,/\, \langle \sim_u, \sim_d, \sim_{oh,p} \rangle$$
$$K_{fin}^*(\mathcal{A}) := \varinjlim_p K_p^*(\mathcal{A}).$$

The inductive limit is taken with respect to the maps $K_p^*(\mathcal{A}) \longrightarrow K_q^*(\mathcal{A})$ which are induced by the inclusion $\mathbb{F}_p^*(\mathcal{A}) \subseteq \mathbb{F}_q^*(\mathcal{A})$ for $p < q$.

In particular, two finitely summable modules $\mathscr{F}, \mathscr{F}'$ represent the same class in $K_{fin}^*(\mathcal{A})$ if and only if there is some p such that $\mathscr{F} \sim_{d+oh,p+u} \mathscr{F}'$.

The direct sum of two p-summable modules is again p-summable and therefore defines an addition on $K_p^*(\mathcal{A})$ as for K-homology. We obtain:

2.3.8 Proposition. *The sets $K_p^*(\mathcal{A})$ and $K_{fin}^*(\mathcal{A})$ form abelian groups with addition given by the direct sum of representatives.*

Proof. For $K_p^*(\mathcal{A})$ the proof is exactly the same as for standard K-homology (Proposition 2.1.8). $K_{fin}^*(\mathcal{A})$ is an abelian group because it is an inductive limit of abelian groups. □

If $\alpha : \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous $*$-homomorphism between topological $*$-algebras, and $\mathscr{F} \in \mathbb{F}_p^*(\mathcal{B})$, then $\alpha^*\mathscr{F}$ is a p-summable module over $\mathcal{A}$. Thus, K_p^* and K_{fin}^* become contravariant functors from the category of topological $*$-algebras and continuous $*$-homomorphisms to the category of abelian groups.

2.3.9 Proposition. *For any $\mathscr{F} \in \mathbb{F}_p^*(\mathcal{A})$ we have $[\mathscr{F}] = \mathbf{0} \in K_p^*(\mathcal{A})$ if and only if there is a degenerate $\mathscr{D} \in \mathbb{D}^*(\mathcal{A})$ such that $\mathscr{F} \oplus \mathscr{D}$ is operator homotopic to a degenerate module via a path of p-summable modules.*

We have, $[\mathscr{F}] = \mathbf{0} \in K_{fin}^*(\mathcal{A})$ if and only if there is a $\mathscr{D} \in \mathbb{D}^*(\mathcal{A})$ such that $\mathscr{F} \oplus \mathscr{D}$ is operator homotopic to a degenerate module via a path of q-summable modules for some $1 \leq q < \infty$.

Proof. This follows directly from Corollary 2.3.6. □

2.3.10 Proposition. *If $\mathscr{F} = (\varphi, \mathcal{H}, F), \mathscr{F}' = (\varphi, \mathcal{H}, F') \in \mathbb{F}_p^*(\mathcal{A})$ and*

$$\varphi(x)(F - F') \in \mathcal{L}^p \qquad \text{for all } x \in \mathcal{A},$$

then

$$[\mathscr{F}] = [\mathscr{F}'] \in K_p^*(\mathcal{A}).$$

We say that $\mathscr{F}'$ is a p-summable perturbation of $\mathscr{F}$ with respect to φ.

Proof. The linear path between F and F' defines a p-summable operator homotopy. □

Let $\mathcal{A}$ be a topological $*$-algebra which is at the same time a pre-C^*-algebra with C^*-completion A. If $\mathscr{F} = (\varphi, \mathcal{H}, F)$ is any Fredholm module over $\mathcal{A}$, then φ is automatically continuous in the C^*-topology (Proposition 1.4.8) and therefore extends to a $*$-representation of A.† Moreover, if $\{x_n\}$ is a sequence in $\mathcal{A}$ converging to $x \in A$, then

$$[F, \varphi(x_n)] \xrightarrow{n\to\infty} [F, \varphi(x)] \quad \text{and} \quad \varphi(x_n)(F^2 - I) \xrightarrow{n\to\infty} \varphi(x)(F^2 - I)$$

as bounded operators on $\mathcal{H}$. Since $\mathcal{K}(\mathcal{H})$ is closed in $\mathcal{B}(\mathcal{H})$, we see that $\mathscr{F}$ extends to a Fredholm module over A. Thus, we obtain natural maps

$$\Phi_p : K_p^*(\mathcal{A}) \longrightarrow K^*(A) \quad \text{and} \quad \Phi : K_{fin}^*(\mathcal{A}) \longrightarrow K^*(A).$$

2.3.11 Definition. We call the maps Φ_p and Φ the comparison maps between $K_p^*(\mathcal{A})$ (resp. $K_{fin}^*(\mathcal{A})$) and $K^*(A)$. We say that every class in $K^*(A)$ is representable by a finitely summable module over $\mathcal{A}$ if Φ is surjective.

†Of course, if the topology on $\mathcal{A}$ is induced by the C^*-norm, then this is trivially true, whether $\mathcal{A}$ is a pre-C^*-algebra or not.

2.4 Normalization of finitely summable modules

In this section we prove that the standard normalization procedures which are known for K-homology are also applicable in the finitely summable setting. We will see that, compared to ordinary K-homology, a bit of extra care has to be taken when working with $K_p^*(\mathcal{A})$.

Denote by $\widetilde{\mathbb{F}}_p^1(\mathcal{A})$ those modules $(\varphi, \mathcal{H}, F) \in \mathbb{F}_p^1(\mathcal{A})$ for which $F = F^*$ and $F^2 = I$, and by $\widetilde{\mathbb{F}}_p^0(\mathcal{A})$ those modules in $\mathbb{F}_p^0(\mathcal{A})$ which are of the form

$$\left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, F = \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\right)$$

with a unitary U.

For $(\varphi, \mathcal{H}, F), (\varphi, \mathcal{H}, F') \in \widetilde{\mathbb{F}}_p^*(\mathcal{A})$ we write $(\varphi, \mathcal{H}, F) \sim_{soh,p} (\varphi, \mathcal{H}, F')$ if there is an operator homotopy $\mathscr{F}_t = (\varphi, \mathcal{H}, F_t)$ with $\mathscr{F}_t \in \widetilde{\mathbb{F}}_p^*(\mathcal{A})$ for all t such that the paths $F_t : [0,1] \longrightarrow \mathcal{B}(\mathcal{H})$ and $[F_t, \varphi(x)] : [0,1] \longrightarrow \mathcal{L}^p(\mathcal{H})$ are smooth for each $x \in \mathcal{A}$. We call this a smooth operator homotopy.

Define

$$\begin{aligned} \widetilde{K_p^*}(\mathcal{A}) &:= \widetilde{\mathbb{F}}_p^*(\mathcal{A}) \,/\, \langle \sim_u, \sim_d, \sim_{soh,p} \rangle \\ \widetilde{K_{fin}^*}(\mathcal{A}) &:= \varinjlim_p \widetilde{K_p^*}(\mathcal{A}) \end{aligned}$$

where in the even case we require the degenerate modules in the definition of $\sim_d$ to lie in $\widetilde{\mathbb{F}}_p^0(\mathcal{A})$. $\widetilde{K_p^*}(\mathcal{A})$ and $\widetilde{K_{fin}^*}(\mathcal{A})$ are groups by the same argument as for Proposition 2.3.8.

2.4.1 Theorem. *The natural maps*

$$\begin{aligned} \widetilde{K_p^*}(\mathcal{A}) &\longrightarrow K_p^*(\mathcal{A}) \\ \widetilde{K_{fin}^*}(\mathcal{A}) &\longrightarrow K_{fin}^*(\mathcal{A}) \end{aligned}$$

induced by the inclusions $\widetilde{\mathbb{F}}_p^*(\mathcal{A}) \subseteq \mathbb{F}_p^*(\mathcal{A})$ *are isomorphisms.*

In other words: we can define $K_p^*(\mathcal{A})$ and $K_{fin}^*(\mathcal{A})$ by only allowing Fredholm modules $(\varphi, \mathcal{H}, F)$ with symmetries F and restricting to smooth operator homotopies.

To proof the surjectivity of this map, we need to make the operator F of any given module $(\varphi, \mathcal{H}, F)$ a symmetry. The standard procedure, assuming that F is already self-adjoint and contractive, would be to replace the module by

$$\left(\varphi \oplus 0,\, \mathcal{H} \oplus \mathcal{H}^{op},\, \begin{bmatrix} F & \sqrt{1-F^2} \\ \sqrt{1-F^2} & -F \end{bmatrix}\right) \tag{2.4}$$

as described in [HR00, Chapter 8]. However, the following example shows that we cannot expect this module to be p-summable again.

2.4.2 Example. On $\ell^2(\mathbb{Z})$ let X be the self-adjoint unitary given by

$$X(e_n) := e_{-n},$$

where e_n denotes the characteristic function of $\{n\}$, and define F as

$$F(e_n) := \begin{cases} (1 - \frac{1}{n})e_n & n > 0 \\ e_n & \text{otherwise} \end{cases}.$$

We have $F^* = F$, $\|F\| = 1$, and

$$(I - F^2)e_n = \begin{cases} (\frac{2}{n} - \frac{1}{n^2})e_n & n > 0 \\ 0 & \text{otherwise} \end{cases},$$

so $I - F^2 \in \mathcal{L}^2$. Moreover,

$$[F, X](e_n) = [F - I, X](e_n) = \begin{cases} 0 & n = 0 \\ \frac{1}{n}e_{-n} & n \neq 0 \end{cases},$$

hence $[F, X] \in \mathcal{L}^2$. Thus, if we let $\mathcal{A}$ be the $*$-algebra generated by X, then $(\mathrm{id}_{\mathcal{A}}, \ell^2(\mathbb{Z}), F)$ is a 2-summable Fredholm module over $\mathcal{A}$.

On the other hand, we have $X(I - F^2)^{\frac{1}{2}} \notin \mathcal{L}^2$, so the module

$$\left(\mathrm{id}_{\mathcal{A}} \oplus 0,\ \ell^2(\mathbb{Z}) \oplus \ell^2(\mathbb{Z}),\ \begin{bmatrix} F & \sqrt{I - F^2} \\ \sqrt{I - F^2} & -F \end{bmatrix}\right)$$

is not 2-summable.

While (2.4) is not a p-summable Fredholm module in general, we can use a variation of this construction to stay within $\mathbb{F}_p^*(\mathcal{A})$:

2.4.3 Lemma. *If $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}_p^*(\mathcal{A})$, then*

$$\mathscr{F}' := (\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}^{op}, F') \text{ where } F' := \begin{bmatrix} \frac{3}{2}F - \frac{1}{2}F^3 & (I - F^2)(I - \frac{1}{4}F^2) \\ I - F^2 & -(\frac{3}{2}F - \frac{1}{2}F^3) \end{bmatrix}$$

is a p-summable odd (resp. even) Fredholm module over $\mathcal{A}$ such that ${F'}^2 = I$ and

$$[\mathscr{F}] = [\mathscr{F}'] \in K_p^*(\mathcal{A}).$$

Proof. Direct computation shows that $\mathscr{F}'$ is a p-summable Fredholm module and that ${F'}^2 = I$. Moreover, $\mathscr{F}'$ is a p-summable perturbation of

$$(\varphi, \mathcal{H}, F) \oplus (0, \mathcal{H}^{op}, 0)$$

with respect to $\varphi \oplus 0$, and the second summand is zero in $K_p^*(\mathcal{A})$. □

How to find such an F'? With the *Ansatz*

$$F' = \begin{bmatrix} A & C \\ B & -A \end{bmatrix}$$

where A, B, C are polynomials in F, we get that $F'^2 = I$ if and only if $BC = I - A^2$. Now A should be a p-summable perturbation of F with respect to $\varphi(\mathcal{A})$, so we assume $A = F + D(I - F^2)$. Moreover, we would like B and C to be p-summable with respect to φ. This will be the case if they both contain $I - F^2$ as a factor. This leads to the condition that

$$(I - F^2)^2 \quad \text{divides} \quad I - [F + D(I - F^2)]^2\,,$$

which is fulfilled for $D = \frac{1}{2}F$.

Proof of Theorem 2.4.1. First, note that the maps $\widetilde{K}_p^*(\mathcal{A}) \longrightarrow K_p^*(\mathcal{A})$ commute with the connecting maps $K_p^*(\mathcal{A}) \longrightarrow K_q^*(\mathcal{A})$ $(p < q)$. Thus, if the theorem is true for every $K_p^*(\mathcal{A})$, it must also hold for their inductive limit $K_{fin}^*(\mathcal{A})$.

To prove the theorem for $K_p^*(\mathcal{A})$, we consider the odd and even case separately:

$\boldsymbol{* = 1}$. Let $\mathscr{F}$ be an odd p-summable Fredholm module. By Lemma 2.4.3, there is an $\mathscr{F}' = (\varphi', \mathcal{H}', F') \in \mathbb{F}_p^*(\mathcal{A})$ such that $[\mathscr{F}'] = [\mathscr{F}] \in K_p^1(\mathcal{A})$ and $F'^2 = I$.

Corollary 1.4.15 states that $\mathcal{D}^p_{\varphi'(\mathcal{A})} = \{X \in \mathcal{B}(\mathcal{H}') \,|\, [\varphi'(\mathcal{A}), X] \subseteq \mathcal{L}^p(\mathcal{H}')\}$ is a pre-C^*-algebra. $P' := \frac{1}{2}(F' + I)$ is an idempotent in $\mathcal{D}^p_{\varphi'(\mathcal{A})}$, so by Proposition 1.5.5 there is a continuous path $P'_t \in \mathcal{D}^p_{\varphi'(\mathcal{A})}$ of idempotents from P' to a projection. From it, we obtain the operator homotopy $(\varphi', \mathcal{H}', 2P'_t - 1)$ between $\mathscr{F}'$ and a module in $\widetilde{\mathbb{F}}_p^1(\mathcal{A})$. This proves surjectivity.

Now, let $\mathscr{F} \in \widetilde{\mathbb{F}}_p^1(\mathcal{A})$ be a module with $[\mathscr{F}] = \mathbf{0} \in K_p^1(\mathcal{A})$. By Proposition 2.3.9 there is a $\mathscr{D} \in \mathbb{D}^1(\mathcal{A})$ such that $\mathscr{F} \oplus \mathscr{D}$ is operator homotopic to another degenerate $\mathscr{D}' \in \mathbb{D}^1(\mathcal{A})$. Since $[\mathscr{D}] = [\mathscr{D}'] = \mathbf{0} \in \widetilde{K}_p^1(\mathcal{A})$, it suffices to show that $\mathscr{F} \oplus \mathscr{D}$ and $\mathscr{D}'$ represent the same class in $\widetilde{K}_p^1(\mathcal{A})$ to prove injectivity.

Let $(\varphi, \mathcal{H}, F) := \mathscr{F} \oplus \mathscr{D}$, $(\varphi, \mathcal{H}, G) := \mathscr{D}'$ and let F_t be the path implementing the operator homotopy. Differing by a degenerate module, $(\varphi, \mathcal{H}, F)$ and $(\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, F \oplus -F)$ represent the same class in $\widetilde{K}_p^1(\mathcal{A})$. Again by Lemma 2.4.3,

$$P'_t := \frac{1}{2}\left(\begin{bmatrix} \frac{3}{2}F_t - \frac{1}{2}F_t^3 & (I - F_t^2)(I - \frac{1}{4}F_t^2) \\ I - F_t^2 & -(\frac{3}{2}F_t - \frac{1}{2}F_t^3) \end{bmatrix} + I_2\right)$$

is a homotopy of idempotents in $\mathcal{D}_{\varphi\oplus 0(\mathcal{A})}$ connecting $P' := \frac{1}{2}(F \oplus (-F) + I_2)$ to $Q' := \frac{1}{2}(G \oplus (-G) + I_2)$. By Corollary 1.5.8, we can find in $\mathcal{D}_{\varphi\oplus 0(\mathcal{A})}$ positive $A_0, \ldots, A_{n-1}$ and projections $P' = P'_0, P'_1, \ldots, P'_n = Q'$ such that

$e^{iA_j}P'_j e^{-iA_j} = P'_{j+1}$ $(i = 0, \dots, n-1)$. Defining $F'_j := 2P'_j - 1$, we obtain operator homotopies

$$(\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, e^{itA_j}F'_j e^{-itA_j}) \qquad \text{for } j = 0, \dots, n-1 \tag{2.5}$$

connecting $(\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, F'_j)$ to $(\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, F'_{j+1})$.

Note that $e^{\pm itA_j} : [0,1] \longrightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is smooth, having nth derivative $(\pm iA_j)^n e^{\pm itA_j}$. Hence, $e^{itA_j}F'_j e^{-itA_j} : [0,1] \longrightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ is smooth as well (the derivatives at 0 and 1 do not vanish, so one has to reparameterize the paths to obtain smooth paths in the sense of our definition on page 8).

Finally, we show that the map $[\varphi(x), e^{\pm itA_j}] : [0,1] \longrightarrow \mathcal{L}^p(\mathcal{H} \oplus \mathcal{H})$ is smooth for every $x \in \mathcal{A}$ with nth derivative $[\varphi(x), (\pm iA_j)^n e^{\pm itA_j}]$ (here again the path has to be reparameterized). This then implies the smoothness of $[\varphi(x), e^{itA_j}F'_j e^{-itA_j}] : [0,1] \longrightarrow \mathcal{L}^p(\mathcal{H} \oplus \mathcal{H})$.

We have

$$\begin{aligned}
&\Big\| \Big[\varphi(x), \frac{1}{h}\Big((\pm iA_j)^n e^{\pm i(t+h)A_j} - (\pm iA_j)^n e^{\pm itA_j} \Big) - (\pm iA_j)^{n+1} e^{\pm itA_j} \Big] \Big\|_p \\
&\quad = \Big\| \Big[\varphi(x), (\pm iA_j)^n e^{\pm itA_j} \frac{1}{h}(e^{\pm ihA_j} - I - \pm hiA_j) \Big] \Big\|_p \\
&\quad \le \Big\| \Big[\varphi(x), (\pm iA_j)^n e^{\pm itA_j} \Big] \Big\|_p \cdot \Big\| \frac{1}{h}(e^{\pm ihA_j} - I) - \pm iA_j \Big\| \\
&\qquad\qquad + \Big\| (\pm iA_j)^n e^{\pm itA_j} \Big\| \cdot \Big\| \Big[\varphi(x), \frac{1}{h}(e^{\pm ihA_j} - I - \pm hiA_j) \Big] \Big\|_p .
\end{aligned}$$

The first summand tends to zero for $h \to 0$ since $e^{\pm itA_j} : [0,1] \longrightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ has the derivative $\pm iA_j e^{\pm itA_j}$. For the p-norm in the second summand we get

$$\begin{aligned}
\Big\| \Big[\varphi(x), \frac{1}{h}(e^{\pm ihA_j} - I - \pm hiA_j) \Big] \Big\|_p &= \Big\| \Big[\varphi(x), \sum_{k=2}^{\infty} \frac{(\pm iA_j)^k h^{k-1}}{k!} \Big] \Big\|_p \\
&\le \sum_{k=2}^{\infty} k \|A_j\|^{k-1} \|[\varphi(x), A_j]\|_p \frac{|h|^{k-1}}{k!} \\
&= \|[\varphi(x), A_j]\|_p (e^{|h|\|A_j\|} - 1) \xrightarrow{h \to 0} 0.
\end{aligned}$$

Thus, the homotopies in (2.5) are smooth (after reparameterization) and

$$[\mathscr{F} \oplus \mathscr{D}] = [(\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, F'_0)] = [(\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, F'_n)] = [\mathscr{D}'] = \mathbf{0} \in \widetilde{K}^1_p(\mathcal{A}).$$

$* = \mathbf{0}$. Let $\mathscr{F} = (\varphi_+ \oplus \varphi_-, \mathcal{H}_+ \oplus \mathcal{H}_-, F)$ be an even p-summable Fredholm module. By Lemma 2.4.3, we can again assume that $F^2 = I$. Furthermore, consider the degenerate module

$$\mathscr{D} := \left(\psi \oplus \psi, \mathcal{N} \oplus \mathcal{N}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right) \tag{2.6}$$

where $\mathcal{N} = \bigoplus_{\mathbb{N}} \mathcal{H}_+ \oplus \bigoplus_{\mathbb{N}} \mathcal{H}_-$ and $\psi = \bigoplus_{\mathbb{N}} \varphi_+ \oplus \bigoplus_{\mathbb{N}} \varphi_-$. $\mathscr{F} \oplus \mathscr{D}$ is unitarily equivalent to a module of the form $\mathscr{F}' = (\psi \oplus \psi, \mathcal{N} \oplus \mathcal{N}, F')$ with $F'^2 = I$, so

$$F' = \begin{bmatrix} 0 & U'^{-1} \\ U' & 0 \end{bmatrix} \quad \text{where } U' \in \mathcal{D}^p_{\psi(\mathcal{A})} \text{ is invertible.}$$

By Proposition 1.5.2, there is a continuous path U'_t of invertibles in $D^p_{\psi(\mathcal{A})}$ from U' to a unitary U'_1, which leads to the operator homotopy

$$\left(\psi \oplus \psi, \mathcal{N} \oplus \mathcal{N}, \begin{bmatrix} 0 & U_t'^{-1} \\ U'_t & 0 \end{bmatrix}\right)$$

from $\mathscr{F}'$ to a module in $\widetilde{\mathbb{F}}^0_p(\mathcal{A})$. This proves surjectivity.

Now, let $\mathscr{F} \in \widetilde{\mathbb{F}}^0_p(\mathcal{A})$ be a module with $[\mathscr{F}] = \mathbf{0} \in K^0_p(\mathcal{A})$. By Proposition 2.3.9 there is a $\mathscr{D} \in \mathbb{D}^0(\mathcal{A})$ such that $\mathscr{F} \oplus \mathscr{D}$ is operator homotopic to another degenerate $\mathscr{D}' \in \mathbb{D}^0(\mathcal{A})$. Adding a degenerate module of the form (2.6), we can assume that $\mathscr{D}, \mathscr{D}' \in \widetilde{\mathbb{F}}^0_p(\mathcal{A})$. Using Lemma 2.4.3 again, we finally arrive at an operator homotopy

$$\left(\varphi \oplus \varphi, \mathcal{H} \oplus \mathcal{H}, F = \begin{bmatrix} 0 & U_t^{-1} \\ U_t & 0 \end{bmatrix}\right)$$

between modules in $\widetilde{\mathbb{F}}^0_p(\mathcal{A})$ equivalent to $\mathscr{F} \oplus \mathscr{D}$ and $\mathscr{D}'$. The unitaries U_0 and U_1 are thus connected by a path of invertibles in $\mathcal{D}^p_{\varphi(\mathcal{A})}$.

By Corollary 1.5.8 there are positive $A_0, \dots, A_{n-1} \in \mathcal{D}^p_{\varphi(\mathcal{A})}$ and unitary $U_0 = V_0, V_1, \dots, V_n = U_1 \in \mathcal{D}^p_{\varphi(\mathcal{A})}$ such that $e^{iA_j} V_j = V_{j+1}$ $(i = 0, \dots, n-1)$. Defining

$$F_j := \begin{bmatrix} 0 & V_j^* \\ V_j & 0 \end{bmatrix}$$

we get operator homotopies

$$\left(\varphi \oplus \varphi, \mathcal{H} \oplus \mathcal{H}, \begin{bmatrix} 0 & V_j^* e^{-itA_j} \\ e^{itA_j} V_j & 0 \end{bmatrix}\right) \qquad \text{for } j = 0, \dots, n-1$$

connecting $(\varphi \oplus \varphi, \mathcal{H} \oplus \mathcal{H}, F_j)$ to $(\varphi \oplus \varphi, \mathcal{H} \oplus \mathcal{H}, F_{j+1})$. The arguments from the odd case show that these homotopies are smooth (after reparameterization), so $\mathscr{F} \oplus \mathscr{D}$ and $\mathscr{D}'$ are equivalent in $\widetilde{K}^0_p(\mathcal{A})$. □

2.4.4 Corollary. *If $\mathscr{F} = (\varphi, \mathcal{H}, F)$, $\mathscr{F}' = (\varphi, \mathcal{H}, F') \in \widetilde{\mathbb{F}}^1_p(\mathcal{A})$ are operator homotopic via a path of symmetries, then there is a unitary $U \in \mathcal{D}^p_{\varphi(\mathcal{A})}$ such that $F' = UFU^*$.*

2.4.5 Remark. In our proof we have made use the fact that p-summable Fredholm modules are related to projections and unitaries in the algebras $\mathcal{D}^p_{\varphi(\mathcal{A})}$. This connection between K-homology and the K-theory of such dual algebras was first observed by Paschke in [Pas81].

2.4.6 Remark. Theorem 2.4.1 implies that the Chern-Connes character we discussed in the introduction is indeed well-defined for finitely summable K-homology. See [Con85, Theorem I.5.2].

2.4.7 Proposition. *If $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}_p^*(\mathcal{A})$ and P is an (even) projection onto a closed subspace of $\mathcal{H}$ containing the support of φ, then*

$$[\mathscr{F}] = [(P\varphi, P\mathcal{H}, PFP)] \in K_p^*(\mathcal{A}).$$

Proof. Write $\mathscr{F}$ as

$$(\varphi, \mathcal{H}, F) = \left(P\varphi \oplus 0,\, \mathcal{H},\, \begin{bmatrix} PFP & PFP^\perp \\ P^\perp FP & P^\perp FP^\perp \end{bmatrix}\right).$$

Since $\varphi(\mathcal{A})PFP^\perp \subseteq \mathcal{L}^p$, we see that $\mathscr{F}$ is with respect to φ a p-summable perturbation of $(P\varphi, P\mathcal{H}, PFP) \oplus (0, P^\perp\mathcal{H}, P^\perp FP^\perp)$. The second summand is zero in $K_p^*(\mathcal{A})$. □

2.4.8 Remark. Theorem 2.4.1, Corollary 2.4.4, and Proposition 2.4.7 are obviously true for standard K-homology as well if one replaces $\mathcal{L}^p$ everywhere by the algebra of compact operators and the dual algebras $\mathcal{D}^p_{\varphi(\mathcal{A})}$ by $\mathcal{D}_{\varphi(A)}$.

Chapter 3

Basic Properties

This chapter covers some basic properties of finitely summable K-homology. We start by considering Fredholm modules $(\varphi, \mathcal{H}, F)$ where F commutes exactly with φ, and we compute K_p^* of algebras over which every p-summable Fredholm module is a p-summable perturbation of such a module (we will use this result in Chapter 7). Section 3.2 covers the behaviour of K_{fin}^* with respect to unitizations and direct sums. In Section 3.3 we prove that, given a p-summable Fredholm module $(\varphi, \mathcal{H}, F)$ over a topological $*$-algebra $\mathcal{A}$, the p-norms of the commutators $[F, \varphi(x)]$ are bounded if the topology of $\mathcal{A}$ satisfies the condition of being barreled. Stability of K_{fin}^* under taking tensor products with algebras of Schatten class operators is proven in Section 3.4. The last section contains two technical lemmas which we will use in the next chapter.

In this chapter $\mathcal{A}$ will always denote a topological $*$-algebra.

3.1 Nearly degenerate Fredholm modules

3.1.1 Definition. Let $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}_p^*(\mathcal{A})$ be a p-summable Fredholm module. We call $\mathscr{F}$ nearly degenerate if

$$[F, \varphi(x)] = 0 \qquad \text{for all } x \in \mathcal{A}.$$

3.1.2 Lemma. *If $\mathscr{F} = (\varphi, \mathcal{H}, F)$ is nearly degenerate, then there is a nearly degenerate $\mathscr{F}' = (\varphi', \mathcal{H}', F')$ such that $[\mathscr{F}] = [\mathscr{F}'] \in \mathbb{F}_p^*(\mathcal{A})$ and $F'^* = F'$.*

Proof. We can prove this using a slight variation of the proof of Theorem 2.4.1.

$\boldsymbol{* = 1}$. Consider the module from Lemma 2.4.3 given by

$$\mathscr{F}' := (\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, F')$$

where

$$F' := \begin{bmatrix} \frac{3}{2}F - \frac{1}{2}F^3 & (I - F^2)(I - \frac{1}{4}F^2) \\ I - F^2 & -(\frac{3}{2}F - \frac{1}{2}F^3) \end{bmatrix}. \tag{3.1}$$

Note that the idempotent $P' := \frac{1}{2}(F' + I)$ lies not only in the pre-C^*-algebra $\mathcal{D}^p_{\varphi\oplus 0(\mathcal{A})}$ but also in $M_2(C^*(F,I))$. The intersection of both algebras is a pre-C^*-algebra as well. Therefore, we can find a homotopic projection $\tilde{P} \in \mathcal{D}^p_{\varphi\oplus 0(\mathcal{A})} \cap M_2(C^*(F,I))$, from which we obtain a module $\tilde{\mathscr{F}} := (\varphi \oplus 0, \mathcal{H} \oplus \mathcal{H}, \tilde{F})$ with self-adjoint

$$\tilde{F} := 2(\tilde{P} - 1) \in \mathcal{D}^p_{\varphi\oplus 0(\mathcal{A})} \cap M_2(C^*(F,I))$$

representing the same class in $K^1_p(\mathcal{A})$ as $\mathscr{F}$.

Now, $[\varphi(\mathcal{A}), F] = \{0\}$ implies that $[\varphi(\mathcal{A}), C^*(F,I)] = \{0\}$. So if we write $\tilde{F}$ as

$$\tilde{F} = \begin{bmatrix} \tilde{F}_{11} & \tilde{F}_{21}^* \\ \tilde{F}_{21} & \tilde{F}_{22} \end{bmatrix},$$

then $[\varphi(\mathcal{A}), \tilde{F}_{11}] = \{0\}$ and $\tilde{F}_{11}^* = \tilde{F}_{11}$. Thus, by Proposition 2.4.7, $(\varphi, \mathcal{H}, \tilde{F}_{11})$ is a nearly degenerate, self-adjoint module which is equivalent to $\mathscr{F}$.

$* = \mathbf{0}$. After adding a degenerate module as in the proof of Theorem 2.4.1, we can assume that

$$\mathscr{F} = \left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & V \\ U & 0 \end{bmatrix}\right)$$

and $[\varphi(\mathcal{A}), C^*(U,V,I)] = \{0\}$. Applying Lemma 2.4.3 gives us an equivalent module

$$\mathscr{F}' = \Big((\varphi \oplus \varphi) \oplus (0 \oplus 0), (\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H})^{op}, F'\Big)$$

where F' is again of the form (3.1). In particular, F' is an odd operator with respect to the grading on $(\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H})^{op}$, we have $F'^2 = I$ and $F' \in M_4(C^*(U,V,I))$. Taking the part of F' which maps the positive subspace (the first and fourth summand) to the negative subspace (the second and third summand), we obtain an invertible operator U' with

$$U' \in \mathcal{D}^p_{\varphi\oplus 0(\mathcal{A})} \cap M_2(C^*(U,V,I)).$$

As above, we find a homotopy of U' to a unitary in $\mathcal{D}^p_{\varphi\oplus 0(\mathcal{A})} \cap M_2(C^*(U,V,I))$ which defines an operator homotopy between $\mathscr{F}'$ and a module

$$\tilde{\mathscr{F}} = \Big((\varphi \oplus \varphi) \oplus (0 \oplus 0), (\mathcal{H} \oplus \mathcal{H}) \oplus (\mathcal{H} \oplus \mathcal{H})^{op}, \tilde{F}\Big).$$

$\tilde{F}$ satisfies $\tilde{F}^2 = I$, $\tilde{F}^* = \tilde{F}$, and $\tilde{F} \in M_4(C^*(U,V,I))$. Again denoting the compression of $\tilde{F}$ to the first two copies of $\mathcal{H}$ by $\tilde{F}_{11}$, $(\varphi \oplus \varphi, \mathcal{H} \oplus \mathcal{H}, \tilde{F}_{11})$ is an even nearly degenerate module with self-adjoint $\tilde{F}_{11}$ that is equivalent to $\mathscr{F}$. □

3.1.3 Proposition. *If $\mathscr{F} \in \mathbb{F}_p^1(\mathcal{A})$ is nearly degenerate, then*

$$[\mathscr{F}] = \mathbf{0} \in K_p^1(\mathcal{A}).$$

If $\mathscr{F} \in \mathbb{F}_p^0(\mathcal{A})$ is nearly degenerate, then there is a module

$$\mathscr{F}' = (\varphi'_+ \oplus \varphi'_-, \mathcal{H}' \oplus \mathcal{H}', 0) \in \mathbb{F}_p^0(\mathcal{A}) \tag{3.2}$$

such that $[\mathscr{F}] = [\mathscr{F}'] \in K_p^0(\mathcal{A})$. ($\varphi'_+$ and φ'_- can be chosen to be non-degenerate.)

Proof. Let $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}_p^1(\mathcal{A})$ be nearly degenerate. We can assume $F^* = F$ by Lemma 3.1.2. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$f(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}.$$

By Borel functional calculus, $f(F)$ is a symmetry with $[\varphi(\mathcal{A}), f(F)] = \{0\}$. Moreover, defining $g : \mathbb{R} \longrightarrow \mathbb{R}$ as

$$g(x) := \begin{cases} \frac{1}{1+x} & x \geq 0 \\ \frac{-1}{1-x} & x < 0 \end{cases},$$

we have $f(x) - x = (1 - x^2)g(x)$. Thus,

$$\varphi(x)(f(F) - F) = \varphi(x)(I - F^2)g(F) \in \mathcal{L}^p \qquad \text{for all } x \in \mathcal{A}.$$

This means that $\mathscr{F}$ is with respect to φ a p-summable perturbation of the degenerate module $(\varphi, \mathcal{H}, f(F))$ and thus represents the zero class in $K_p^1(\mathcal{A})$.

Next, let $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}_p^0(\mathcal{A})$ be an even, nearly degenerate module. The argument from the odd case does not carry over since the function f is not odd and therefore $f(F)$ might not be odd, either. Instead, we modify $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ to

$$f(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \qquad \text{and} \qquad g(x) := \begin{cases} \frac{1}{1+x} & x > 0 \\ 0 & x = 0 \\ \frac{-1}{1-x} & x < 0 \end{cases}.$$

Again we can write $f(x) - x = (1 - x^2)g(x)$, so $[\mathscr{F}] = [(\varphi, \mathcal{H}, f(F))] \in K_p^0(\mathcal{A})$.

Let $P := \chi_{\{0\}}(F)$ where $\chi_{\{0\}}$ denotes the characteristic function of the Borel set $\{0\}$. P is an even projection commuting with φ and $f(F)$, hence

$$(\varphi, \mathcal{H}, f(F)) = (P\varphi, P\mathcal{H}, 0) \oplus (P^\perp\varphi, P^\perp\mathcal{H}, P^\perp f(F))$$

and thus $[(\varphi, \mathcal{H}, f(F))] = [(P\varphi, P\mathcal{H}, 0)] \in K_p^0(\mathcal{A})$ since the second summand is degenerate.

Finally, by applying Proposition 2.4.7 we can restrict the module to the support of $P\varphi$ to make the representation non-degenerate. $\square$

3.1.4 Definition. Let $\mathbb{FR}(\mathcal{A})$ be the monoid of unitary equivalence classes of continuous, non-degenerate, finite-dimensional $*$-representations of $\mathcal{A}$ with the direct sum as monoid operation.

Denote by $\mathrm{FR}(\mathcal{A})$ the Grothendieck group of $\mathbb{FR}(\mathcal{A})$.

Remember the definition of $\widetilde{\mathbb{F}}_p^*(\mathcal{A})$ (Theorem 2.4.1): $\widetilde{\mathbb{F}}_p^1(\mathcal{A})$ consists of those p-summable modules $(\varphi, \mathcal{H}, F)$ over $\mathcal{A}$ which satisfy $F^* = F$ and $F^2 = I$. The elements of $\widetilde{\mathbb{F}}_p^0(\mathcal{A})$ are p-summable even modules over $\mathcal{A}$ of the form

$$\left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\right)$$

with unitary U.

From the preceding proposition we can conclude:

3.1.5 Theorem. *Let $\mathcal{A}$ be unital. If every module in $\widetilde{\mathbb{F}}_p^*(\mathcal{A})$ is a p-summable perturbation (with respect to the representation) of a nearly degenerate module, then $K_p^1(\mathcal{A}) = 0$ and there is a natural isomorphism*

$$K_p^0(\mathcal{A}) \cong \mathrm{FR}(\mathcal{A}).$$

Proof. Keeping Theorem 2.4.1 in mind, the odd case is already contained in Proposition 3.1.3. For the even case, note that the map

$$\varphi \mapsto (\varphi \oplus 0, \mathcal{H} \oplus 0, 0)$$

sending a continuous, finite-dimensional $*$-representation $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ to an even p-summable Fredholm module over $\mathcal{A}$ descends to a homomorphism from $\mathbb{FR}(\mathcal{A})$ to $K_p^0(\mathcal{A})$. Since $K_p^0(\mathcal{A})$ is a group, this yields a well-defined homomorphism

$$\Phi : \mathrm{FR}(\mathcal{A}) \longrightarrow K_p^0(\mathcal{A}).$$

As $-[(\varphi, \mathcal{H}, F)] = [(\varphi, \mathcal{H}^{op}, -F)]$, we see that $\Phi\left([\varphi] - [\psi]\right)$ is represented by the module

$$(\varphi \oplus \psi, \mathcal{H} \oplus \mathcal{H}', 0).$$

Thus, the surjectivity of Φ follows directly from Proposition 3.1.3 (note that if the representations φ'_+ and φ'_- in (3.2) are non-degenerate, then they have to be finite-dimensional since $\mathcal{A}$ is unital).

For proving the injectivity of Φ, assume $\Phi\left([\varphi] - [\psi]\right) = \mathbf{0}$, and let

$$\mathscr{F} := \left(\varphi \oplus \psi, \mathcal{H} \oplus \mathcal{H}', 0\right).$$

Let $\mathcal{J} := \ker(\varphi \oplus \psi)$ and $\mathcal{B} := \mathcal{A}/\mathcal{J}$ endowed with the quotient topology. Denote by π the quotient map. Since $\mathcal{B}$ is algebraically isomorphic to the finite-dimensional C^*-algebra $(\varphi \oplus \psi)(\mathcal{A})$ and there is only one Hausdorff vector space topology on finite-dimensional vector spaces, the topology on $\mathcal{B}$ is given by a C^*-norm making it a C^*-algebra isomorphic to $(\varphi \oplus \psi)(\mathcal{A})$.

φ and ψ factor through π, so there are φ', ψ' such that $\varphi = \varphi' \circ \pi$ and $\psi = \psi' \circ \pi$.

If we define $\mathscr{F}'$ to be the module $\mathscr{F}' := (\varphi' \oplus \psi', \mathcal{H} \oplus \mathcal{H}', 0)$ over $\mathcal{B}$, we have

$$\mathbf{0} = \Phi([\varphi] - [\psi]) = [\mathscr{F}] = \pi^*[\mathscr{F}'].$$

To proof the theorem it therefore suffices to show: (i) $[\mathscr{F}'] = \mathbf{0} \in K_p^0(\mathcal{B})$ implies $[\varphi] = [\psi]$. (ii) π^* is injective.

(i) As a finite-dimensional C^*-algebra, $\mathcal{B}$ is of the form $\mathcal{B} = \bigoplus_{i=1}^d M_{n_i}(\mathbb{C})$. Let e_i be arbitrary minimal projections in $M_{n_i}(\mathbb{C})$. Under the assumption $[\mathscr{F}'] = \mathbf{0}$, we know that $\mathrm{Ind}_{\mathscr{F}'}(e_i) = 0$ for all i. This means by definition of the index map that

$$\operatorname{rk} \varphi'(e_i) = \operatorname{rk} \psi'(e_i) \qquad \text{for } i = 1, \dots, d.$$

As φ' and ψ' are non-degenerate by definition of $\mathrm{FR}(\mathcal{A})$, this implies that φ' and ψ' are unitarily equivalent and hence also φ and ψ.

(ii) By Theorem 2.4.1, every class of $K_p^0(\mathcal{B})$ is represented by a module of the form

$$\mathscr{G}' := (\rho' \oplus \rho', \mathcal{H} \oplus \mathcal{H}, G') \in \widetilde{\mathbb{F}}_p^0(\mathcal{B})$$

with a symmetry G'. Suppose that $\pi^*[\mathscr{G}'] = \mathbf{0}$. Then by Proposition 2.3.9[†] and Theorem 2.4.1, there is a degenerate

$$\mathscr{D} = (\mu, \mathcal{N}, N) \in \widetilde{\mathbb{F}}_p^0(\mathcal{A})$$

such that $\pi^*\mathscr{G}' \oplus \mathscr{D}$ is operator homotopic to a degenerate module via a path G_t of symmetries. Let P be the orthogonal projection onto $\overline{\mu(\mathcal{J})\mathcal{N}}$. By Proposition 1.2.3, we can define a $*$-representation μ' of $\mathcal{B}$ on $P^\perp\mathcal{N}$ by

$$\mu'(x') := P^\perp \mu(x) \qquad \text{where } x' = \pi(x).$$

Since P lies in the weak closure of $\mu(\mathcal{A})$, it commutes exactly with N, and we obtain a degenerate module

$$\mathscr{D}' := (\mu', P^\perp\mathcal{N}, P^\perp N P^\perp)$$

over $\mathcal{B}$.

[†]Strictly speaking, we do not use Proposition 2.3.9 but the analogous statement for $\widetilde{K}_p^*(\mathcal{A})$, which has, of course, exactly the same proof.

Let $P' := 0 \oplus 0 \oplus P \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{N})$. We want to show that

$$\mathscr{G}'_t := \Big(\rho' \oplus \rho' \oplus \mu',\, \mathcal{H} \oplus \mathcal{H} \oplus P^\perp \mathcal{N},\, P'^\perp G_t P'^\perp\Big)$$

is an operator homotopy between $\mathscr{G}' \oplus \mathscr{D}'$ and a degenerate module over $\mathcal{B}$. Since $\mathscr{G}'_1$ is degenerate by the same argument as above, we only need to show that $\mathscr{G}'_t$ really is a p-summable module for every $t \in [0,1]$.

Fix one t, and to shorten notation let

$$\rho := (\rho' \oplus \rho') \circ \pi \oplus \mu,$$

so P' is the projection onto $\overline{\rho(\mathcal{J})(\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{N})}$.

By assumption, $(\rho, \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{N}, G_t)$ is (with respect to ρ) a p-summable perturbation of some nearly degenerate module $(\rho, \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{N}, \tilde{G}_t)$. Since P' lies in the weak closure of $\rho(\mathcal{J})$, P' commutes exactly with $\tilde{G}_t$. P' also commutes exactly with ρ. Thus, for any $x \in \mathcal{A}$ we obtain

$$\rho(x)[P', G_t] = [P', \rho(x) G_t] = [P', \rho(x)(G_t - \tilde{G}_t)] \in \mathcal{L}^p,$$

and therefore

$$\rho(x)(P'^\perp G_t P'^\perp G_t P'^\perp - P'^\perp) \in \mathcal{L}^p.$$

This proves that $\mathscr{G}'_t$ is indeed a p-summable module over $\mathcal{B}$ for any t.

□

3.1.6 Corollary.

$$K^0_p(\mathbb{C}) = \mathbb{Z} \quad \textit{and} \quad K^1_p(\mathbb{C}) = 0$$

as well as

$$K^0_{fin}(\mathbb{C}) = \mathbb{Z} \quad \textit{and} \quad K^1_{fin}(\mathbb{C}) = 0.$$

Proof. Given any p-summable module $\mathscr{F} = (\varphi, \mathcal{H}, F)$ over $\mathbb{C}$, the module

$$\Big(\varphi, \mathcal{H}, \varphi(1) F \varphi(1) + \varphi(1)^\perp F \varphi(1)^\perp\Big)$$

is a nearly degenerate, p-summable perturbation of $\mathscr{F}$. Therefore, Theorem 3.1.5 applies.

The statement for $K^*_{fin}(\mathbb{C})$ follows immediately since the isomorphisms $\Phi : \mathrm{FR}(A) \longrightarrow K^0_p(\mathbb{C})$ commute with the connecting maps $K^0_p(\mathbb{C}) \longrightarrow K^0_q(\mathbb{C})$ $(p < q)$. □

3.2 Unitization and direct sums

By adjoining a unit to $\mathcal{A}$, we obtain the split exact sequence

$$0 \longrightarrow \mathcal{A} \overset{\iota}{\longrightarrow} \mathcal{A}^+ \overset{\pi}{\longrightarrow} \mathbb{C} \longrightarrow 0.$$

As for standard K-Homology, this sequence induces an exact sequence in finitely summable K-homology:

3.2.1 Proposition. *The sequence*

$$0 \longleftarrow K_p^*(\mathcal{A}) \overset{\iota^*}{\longleftarrow} K_p^*(\mathcal{A}^+) \overset{\pi^*}{\longleftarrow} K_p^*(\mathbb{C}) \longleftarrow 0$$

is split exact, so

$$K_p^1(\mathcal{A}^+) = K_p^1(\mathcal{A}) \quad \textit{and} \quad K_p^0(\mathcal{A}^+) = K_p^0(\mathcal{A}) \oplus \mathbb{Z}.$$

The same is true for K_{fin}^*.

Proof. Define a split $e : K_p^*(\mathcal{A}) \longrightarrow K_p^*(\mathcal{A}^+)$ as follows: if $\mathbf{x} \in K_p^*(\mathcal{A})$ is represented by $(\varphi, \mathcal{H}, F)$ with $F^* = F$, $F^2 = I$, then let

$$e(\mathbf{x}) := [(\varphi^+, \mathcal{H}, F)] \in K_p^*(\mathcal{A}^+)$$

where φ^+ denotes the extension of φ to $\mathcal{A}^+$ by setting $\varphi^+(1) := I$. The well-definedness of e is guaranteed by Theorem 2.4.1.

By construction we have $\iota^* \circ e = \mathrm{id}_{K_p^*(\mathcal{A})}$, so ι^* is surjective.

Next, let $\mathbf{y} \in K_p^*(\mathcal{A}^+)$ be given. By Theorem 2.4.1, there is a module $(\varphi, \mathcal{H}, F)$ with $F^* = F$, $F^2 = I$ representing $\mathbf{y}$. By definition, $e \circ \iota^*(\mathbf{y})$ is represented by $(\hat{\varphi}, \mathcal{H}, F)$ where φ and $\hat{\varphi}$ agree on $\mathcal{A}$, but $\hat{\varphi}(1) = I$.

Since $[\varphi(1), F] \in \mathcal{L}^p$ and $[\varphi(1), \hat{\varphi}(\mathcal{A}^+)] = \{0\}$, we see that $(\hat{\varphi}, \mathcal{H}, F)$ is a p-summable perturbation of

$$\begin{aligned}\Big(\varphi(1)\hat{\varphi}, \varphi(1)\mathcal{H}, \varphi(1)F\varphi(1)\Big) &\oplus \Big(\varphi(1)^\perp\hat{\varphi}, \varphi(1)^\perp\mathcal{H}, \varphi(1)^\perp F\varphi(1)^\perp\Big) \\ &= \Big(\varphi, \varphi(1)\mathcal{H}, \varphi(1)F\varphi(1)\Big) \oplus \Big(\varphi(1)^\perp\hat{\varphi}, \varphi(1)^\perp\mathcal{H}, \varphi(1)^\perp F\varphi(1)^\perp\Big).\end{aligned}$$

The first summand is in the class of $\mathbf{y}$ by Proposition 2.4.7. So if $\iota^*\mathbf{y} = \mathbf{0}$, then

$$\mathbf{0} = \mathbf{y} + \Big[\Big(\varphi(1)^\perp\hat{\varphi}, \varphi(1)^\perp\mathcal{H}, \varphi(1)^\perp F\varphi(1)^\perp\Big)\Big].$$

Now, $\varphi(1)^\perp\hat{\varphi}(\mathcal{A}) = \{0\}$ implies that $\varphi(1)^\perp\hat{\varphi}$ factorizes over π. Thus, the second summand lies in the image of π^*, which proves exactness in the middle.

Finally, π^* is injective by the functoriality of K_p^* and the split-exactness of the algebra extension.

The statement for K_{fin}^* follows immediately. $\square$

Finitely summable K-homology is also well-behaved with respect to direct sums:

3.2.2 Proposition. *Given the direct sum exact sequence*

$$0 \longrightarrow \mathcal{A}_1 \underset{\iota_1}{\overset{\pi_1}{\rightleftarrows}} \mathcal{A}_1 \oplus \mathcal{A}_2 \underset{\pi_2}{\overset{\iota_2}{\rightleftarrows}} \mathcal{A}_2 \longrightarrow 0,$$

the sequence

$$0 \longleftarrow K_p^*(\mathcal{A}_1) \underset{\iota_1^*}{\overset{\pi_1^*}{\leftrightarrows}} K_p^*(\mathcal{A}_1 \oplus \mathcal{A}_2) \underset{\pi_2^*}{\overset{\iota_2^*}{\leftrightarrows}} K_p^*(\mathcal{A}_2) \longleftarrow 0$$

is exact as well. In particular,

$$K_p^*(\mathcal{A}_1 \oplus \mathcal{A}_2) = K_p^*(\mathcal{A}_1) \oplus K_p^*(\mathcal{A}_2).$$

The same is true for K_{fin}^*.

Proof. By the functoriality of K_p^*, we only need to show that $\operatorname{Im} \pi_2^* = \ker \iota_1^*$.

Let $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}_p^*(\mathcal{A}_1 \oplus \mathcal{A}_2)$ be given and let P be the support projection of $\varphi \circ \iota_1$. $\mathscr{F}$ is with respect to φ a p-summable perturbation of

$$\mathscr{F}_1 \oplus \mathscr{F}_2 := (P\varphi, P\mathcal{H}, PFP) \oplus (P^\perp\varphi, P^\perp\mathcal{H}, P^\perp F P^\perp).$$

We have $\iota_1^*[\mathscr{F}_2] = \mathbf{0}$ and thus $\iota_1^*[\mathscr{F}] = \mathbf{0}$ can only hold if $\iota_1^*[\mathscr{F}_1] = \mathbf{0}$. But since $[\mathscr{F}_1] \in \operatorname{Im} \pi_1^*$, this is only possible if $[\mathscr{F}_1] = \mathbf{0}$, so $[\mathscr{F}] = [\mathscr{F}_2] \in \operatorname{Im} \pi_2^*$.

The statement for K_{fin}^* follows immediately. □

Later on, we shall see that this proposition does not carry over to countable direct sums.

3.3 Barreled algebras

We call a topological algebra barreled if it is barreled as a topological vector space (Definition A.1.3). In particular, all Fréchet algebras are barreled. As an immediate consequence of the closed graph theorem, which also is valid for linear maps with barreled domains, we obtain the following boundedness property for finitely summable modules over these algebras:

3.3.1 Proposition. *If $\mathcal{A}$ is a barreled topological $*$-algebra and $(\varphi, \mathcal{H}, F)$ a p-summable Fredholm module over $\mathcal{A}$, then the maps*

$$\begin{aligned} D : \mathcal{A} &\longrightarrow \mathcal{L}^p(\mathcal{H}) \\ x &\longmapsto [F, \varphi(x)] \end{aligned} \qquad \text{and} \qquad \begin{aligned} S : \mathcal{A} &\longrightarrow \mathcal{L}^p(\mathcal{H}) \\ x &\longmapsto \varphi(x)(F^2 - I) \end{aligned}$$

are continuous.

In particular, for every bounded subset $\mathcal{X} \subseteq \mathcal{A}$ there is a constant C such that

$$\|[F, \varphi(x)]\|_p < C \quad \textit{and} \quad \|\varphi(x)(F^2 - I)\|_p < C \qquad \textit{for all } x \in \mathcal{X}.$$

Proof. Let $\{x_n\} \subseteq \mathcal{A}$ be a net converging to $x \in \mathcal{A}$ such that $D(x_n)$ converges to T in $\mathcal{L}^p(\mathcal{H})$. Denote by ι the embedding of $\mathcal{L}^p(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$. Since ι is continuous, $\iota \circ D(x_n)$ converges to T in $\mathcal{B}(\mathcal{H})$. But $\iota \circ D$ is continuous as well, so $T = D(x)$. The same argument holds for S and thus the graphs of D and S are closed in $\mathcal{A} \times \mathcal{L}^p(\mathcal{H})$. Since $\mathcal{A}$ is barreled and $\mathcal{L}^p(\mathcal{H})$ is a Banach space, the closed graph theorem (Theorem A.1.7) implies the continuity of D and S. □

3.4 Stability

In this section we show that K^*_{fin} is stable under taking tensor products with $\mathcal{L}^p$ for arbitrary $1 \leq p < \infty$.

To simplify notation, we will assume throughout this section that $\mathcal{L}^p$ denotes the fixed algebra $\mathcal{L}^p(\ell^2(\mathbb{N}))$.

We denote by $\mathfrak{e}_i$ the ith basis vector of the canonical orthonormal basis of $\ell^2(\mathbb{N})$ and by e_{ij} the corresponding matrix units for $\mathcal{L}^p$ satisfying the relations

$$e_{ij}\mathfrak{e}_k = \delta_{jk}\mathfrak{e}_i \qquad \text{for all } i, j, k \in \mathbb{N}.$$

Moreover, we abbreviate e_{ii} by e_i.

3.4.1 Lemma. *If $S, T \in \mathcal{L}^p$, then the operator $S \otimes T \in \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$ is p-summable and*

$$\|S \otimes T\|_p = \|S\|_p \cdot \|T\|_p.$$

Proof. Since S and T are compact, there are families of orthonormal vectors $\{\xi_n\}, \{\eta_n\}, \{\xi'_n\}, \{\eta'_n\}$ such that

$$S(\xi) = \sum_n \mu_n(S)(\xi \mid \xi_n)\eta_n \quad \text{and} \quad T(\xi) = \sum_n \mu_n(T)(\xi \mid \xi'_n)\eta'_n$$

for all $\xi \in \ell^2(\mathbb{N})$. $\{\xi_m \otimes \xi'_n\}$ and $\{\eta_m \otimes \eta'_n\}$ are families of orthonormal vectors in $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ and

$$(S \otimes T)(\xi) = \sum_m \sum_n \mu_m(S)\mu_n(T)(\xi \mid \xi_m \otimes \xi'_n)(\eta_m \otimes \eta'_n)$$

for all $\xi \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$.

This means that $S \otimes T$ has (up to order) the characteristic values $\mu_m(S)\mu_n(T)$ and thus

$$\begin{aligned}\|S \otimes T\|_p^p &= \sum_m \sum_n (\mu_m(S)\mu_n(T))^p \\ &= \Big(\sum_m \mu_m(S)^p\Big)\Big(\sum_n \mu_n(T)^p\Big) = \|S\|_p^p \cdot \|T\|_p^p.\end{aligned}$$

□

3.4.2 Lemma. *Let $\mathcal{A}$ be a $*$-algebra and $\tilde{\varphi} : \mathcal{L}^p \odot \mathcal{A} \longrightarrow \tilde{\mathcal{H}}$ a non-degenerate $*$-representation of the algebraic tensor product $\mathcal{L}^p \odot \mathcal{A}$. Then there is a $*$-representation $\varphi : \mathcal{A} \longrightarrow \mathcal{H}$ of $\mathcal{A}$ such that up to unitary equivalence*

$$\tilde{\varphi} = \mathrm{id}_{\mathcal{L}^p} \otimes \varphi.$$

Proof. Let P be the support projection of the representation of $\mathcal{A}$ on $\tilde{\mathcal{H}}$ given by $x \mapsto \tilde{\varphi}(e_1 \otimes x)$, and let φ be the restriction of this representation to $\mathcal{H} := P\tilde{\mathcal{H}}$.

We define isometries $S_i : \mathcal{H} \longrightarrow \tilde{\mathcal{H}}$ by the rule

$$S_i(\tilde{\varphi}(e_1 \otimes x)\xi) := \tilde{\varphi}(e_{i1} \otimes x)\xi \tag{3.3}$$

and linear and continuous extension.

Note that the assignment in (3.3) is indeed well-defined: for any $x, y \in \mathcal{A}$, $\xi, \eta \in \tilde{\mathcal{H}}$ we have

$$\begin{aligned}\Big(\tilde{\varphi}(e_{i1} \otimes x)\xi \,\Big|\, \tilde{\varphi}(e_{j1} \otimes y)\eta\Big) &= \delta_{ij}\Big(\tilde{\varphi}(e_1 \otimes y^*x)\xi \,\Big|\, \eta\Big) \\ &= \delta_{ij}\Big(\tilde{\varphi}(e_1 \otimes x)\xi \,\Big|\, \tilde{\varphi}(e_1 \otimes y)\eta\Big).\end{aligned} \tag{3.4}$$

So if $\sum_{k=1}^m \tilde{\varphi}(e_1 \otimes x_k)\xi_k = \sum_{l=1}^n \tilde{\varphi}(e_1 \otimes y_l)\eta_l$, then

$$\begin{aligned}&\Big\|\sum_{k=1}^m \tilde{\varphi}(e_{i1} \otimes x_k)\xi_k - \sum_{l=1}^n \tilde{\varphi}(e_{i1} \otimes y_l)\eta_l\Big\|^2 \\ &= \Big(\sum_{k=1}^m \tilde{\varphi}(e_{i1} \otimes x_k)\xi_k - \sum_{l=1}^n \tilde{\varphi}(e_{i1} \otimes y_l)\eta_l \,\Big|\, \sum_{k=1}^m \tilde{\varphi}(e_{i1} \otimes x_k)\xi_k - \sum_{l=1}^n \tilde{\varphi}(e_{i1} \otimes y_l)\eta_l\Big) \\ &= \Big(\sum_{k=1}^m \tilde{\varphi}(e_1 \otimes x_k)\xi_k - \sum_{l=1}^n \tilde{\varphi}(e_1 \otimes y_l)\eta_l \,\Big|\, \sum_{k=1}^m \tilde{\varphi}(e_1 \otimes x_k)\xi_k - \sum_{l=1}^n \tilde{\varphi}(e_1 \otimes y_l)\eta_l\Big) \\ &= 0.\end{aligned}$$

Equation (3.4) also shows that the S_i are indeed isometric and have orthogonal range.

If η is orthogonal to the image of S_j, then for any $i \in \mathbb{N}$ and $x \in \mathcal{A}$ we have

$$\begin{aligned}\|\tilde{\varphi}(e_{ij} \otimes x)\eta\|^2 &= \Big(\tilde{\varphi}(e_{ij} \otimes x)\eta \,\Big|\, \tilde{\varphi}(e_{ij} \otimes x)\eta\Big) \\ &= \Big(\eta \,\Big|\, \tilde{\varphi}(e_{j1} \otimes x^*)\tilde{\varphi}(e_{1j} \otimes x)\eta\Big) \\ &= \Big(\eta \,\Big|\, S_j(\tilde{\varphi}(e_{1j} \otimes x^*x)\eta)\Big) = 0.\end{aligned}$$

Thus, the closed span of the ranges of the S_j must be $\tilde{\mathcal{H}}$ since $\tilde{\varphi}$ is non-degenerate. Therefore, we can define a unitary

$$\begin{aligned}U : \ell^2(\mathbb{N}) \otimes \mathcal{H} &\longrightarrow \tilde{\mathcal{H}} \\ \mathfrak{e}_i \otimes \xi &\longmapsto S_i(\xi).\end{aligned}$$

For any $\tilde{\varphi}(e_{k1} \otimes y)\xi \in S_k\mathcal{H}$ we then have

$$\begin{aligned}&U(e_{ij} \otimes \varphi(x))U^*(\tilde{\varphi}(e_{k1} \otimes y)\xi) \\ &\qquad = U(e_{ij} \otimes \varphi(x))(\mathfrak{e}_k \otimes \tilde{\varphi}(e_1 \otimes y)\xi) \\ &\qquad = \delta_{jk} U(\mathfrak{e}_i \otimes \tilde{\varphi}(e_1 \otimes xy)\xi) \\ &\qquad = \delta_{jk}\tilde{\varphi}(e_{i1} \otimes xy)\xi \\ &\qquad = \tilde{\varphi}(e_{ij} \otimes x)\tilde{\varphi}(e_{k1} \otimes y)\xi,\end{aligned}$$

i.e.

$$U\,(\mathrm{id}_{\mathcal{L}^p} \otimes \varphi)(y)\,U^* = \tilde{\varphi}(y) \qquad \text{for all } y \in \mathcal{L}^p \odot \mathcal{A}.$$

□

Let $\mathcal{A}$ be a locally convex $*$-algebra whose topology is generated by a family of semi-norms ν_i $(i \in I)$. The projective tensor product $\mathcal{L}^q \hat{\otimes} \mathcal{A}$ is the completion of the algebraic tensor product $\mathcal{L}^q \odot \mathcal{A}$ endowed with the locally convex topology generated by the family of semi-norms $\|\cdot\|_q \otimes \nu_i$ where

$$(\|\cdot\|_q \otimes \nu_i)(y) := \inf \Big\{ \sum_{j=1}^{n} \|T_j\|_q \nu_i(x_j) \,\Big|\, y = \sum_{j=1}^{n} T_j \otimes x_j \Big\}$$

for all $y \in \mathcal{L}^q \odot \mathcal{A}$.

We can continuously embed $\mathcal{A}$ into $\mathcal{L}^q \hat{\otimes} \mathcal{A}$ via the map

$$\begin{aligned}\iota : \mathcal{A} &\longrightarrow \mathcal{L}^q \hat{\otimes} \mathcal{A} \\ x &\longmapsto e_1 \otimes x\end{aligned}$$

(the concrete choice of the minimal projection in $\mathcal{L}^q$ does not matter, of course). The embedding ι induces isomorphisms in finitely summable K-homology as the following theorem shows.

3.4.3 Theorem. *If $\mathcal{A}$ is a barreled, locally convex $*$-algebra, then for any $1 \leq q < \infty$ the embedding ι of $\mathcal{A}$ into $\mathcal{L}^q \hat{\otimes} \mathcal{A}$ induces isomorphisms*

$$K^*_{fin}(\mathcal{A}) \overset{\iota^*}{\longleftarrow} K^*_{fin}(\mathcal{L}^q \hat{\otimes} \mathcal{A}).$$

Proof. Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ be a continuous $*$-representation of $\mathcal{A}$. φ induces a $*$-representation $\mathrm{id}_{\mathcal{L}^q} \otimes \varphi$ of $\mathcal{L}^q \odot \mathcal{A}$ on $\ell^2(\mathbb{N}) \otimes \mathcal{H}$ sending $T \otimes x$ to $T \otimes \varphi(x)$. Note that by Lemma 3.4.1 there is a continuous semi-norm ν on $\mathcal{A}$ such that

$$\|T \otimes \varphi(x)\| = \|T\| \cdot \|\varphi(x)\| \leq \|T\|_q \nu(x)$$

for all $T \in \mathcal{L}^q$ and $x \in \mathcal{A}$. Thus, if $y = \sum_{i=1}^n T_i \otimes x_i$, then

$$\|(\mathrm{id}_{\mathcal{L}^q} \otimes \varphi)(y)\| \leq \sum_{i=1}^n \|T_i \otimes \varphi(x_i)\| \leq \sum_{i=1}^n \|T_i\|_q \nu(x_i),$$

and, taking the infimum over all such representations of y,

$$\|(\mathrm{id}_{\mathcal{L}^q} \otimes \varphi)(y)\| \leq (\|\cdot\|_q \otimes \nu)(y).$$

This shows that $\mathrm{id}_{\mathcal{L}^q} \otimes \varphi$ is continuous for the projective topology on $\mathcal{L}^q \odot \mathcal{A}$ and thus extends to a continuous $*$-representation of $\mathcal{L}^q \hat{\otimes} \mathcal{A}$.

Now, if $\mathscr{F} = (\varphi, \mathcal{H}, F)$ is a p-summable module over $\mathcal{A}$ with $p \geq q$, then by Proposition 3.3.1 there is a continuous semi-norm ν on $\mathcal{A}$ such that

$$\|[F, \varphi(x)]\|_p \leq \nu(x) \quad \text{and} \quad \|\varphi(x)(F^2 - I)\|_p \leq \nu(x) \quad \text{for all } x \in \mathcal{A}.$$

Setting $\tilde{\varphi} := \mathrm{id}_{\mathcal{L}^q} \otimes \varphi$, $\tilde{F} := I \otimes F$, we obtain

$$\|[\tilde{F}, \tilde{\varphi}(T \otimes x)]\|_p = \|T \otimes [F, \varphi(x)]\|_p = \|T\|_p \|[F, \varphi(x)]\|_p \leq \|T\|_q \nu(x)$$

and

$$\|\tilde{\varphi}(T \otimes x)(\tilde{F}^2 - I)\|_p = \|T \otimes (\varphi(x)(F^2 - I))\|_p \leq \|T\|_q \nu(x).$$

As above, we conclude that

$$\|[\tilde{F}, \tilde{\varphi}(y)]\|_p \leq (\|\cdot\|_q \otimes \nu)(y) \quad \text{and} \quad \|\tilde{\varphi}(y)(\tilde{F}^2 - I)\|_p \leq (\|\cdot\|_q \otimes \nu)(y)$$

for all $y \in \mathcal{L}^q \hat{\otimes} \mathcal{A}$. Thus, $\tilde{\mathscr{F}} := (\tilde{\varphi}, \ell^2(\mathbb{N}) \otimes \mathcal{H}, \tilde{F})$ is a p-summable module over $\mathcal{L}^q \hat{\otimes} \mathcal{A}$.

This operation of extending $\mathscr{F}$ to $\tilde{\mathscr{F}}$ is compatible with direct sums, unitary equivalence, and operator homotopies and hence induces a group homomorphism $\rho : K^*_{fin}(\mathcal{A}) \longrightarrow K^*_{fin}(\mathcal{L}^q \hat{\otimes} \mathcal{A})$. Moreover, it is easy to check that ι^* is a left-inverse to ρ, so ρ is injective.

To show that ρ is surjective, let $\tilde{\mathscr{F}} = (\tilde{\varphi}, \tilde{\mathcal{H}}, \tilde{F})$ be any p-summable module over $\mathcal{L}^q \hat{\otimes} \mathcal{A}$ with self-adjoint $\tilde{F}$. We can assume that $p \in \mathbb{N}$ is even and greater than q. Assuming moreover that $\tilde{\varphi}$ is non-degenerate (Proposition 2.4.7) and using Lemma 3.4.2, we can write $\tilde{\mathscr{F}}$ as

$$\tilde{\mathscr{F}} = (\mathrm{id}_{\mathcal{L}^q} \otimes \varphi, \ell^2(\mathbb{N}) \otimes \mathcal{H}, \tilde{F}).$$

Let

$$F' := \sum_{i=1}^{\infty} (e_{i1} \otimes I)\tilde{F}(e_{1i} \otimes I).$$

We want to show that

$$\mathscr{F}' := (\mathrm{id}_{\mathcal{L}^q} \otimes \varphi, \ell^2(\mathbb{N}) \otimes \mathcal{H}, F')$$

is a p^2-summable module over $\mathcal{L}^q \hat{\otimes} \mathcal{A}$ which is a p^2-summable perturbation of $\tilde{\mathscr{F}}$ with respect to $\mathrm{id}_{\mathcal{L}^q} \otimes \varphi$. If we then identify $\mathcal{H}$ with $\mathfrak{e}_1 \otimes \mathcal{H}$ and define F to be the restriction of $(e_1 \otimes I)\tilde{F}(e_1 \otimes I)$ to $\mathcal{H}$, then

$$F'(\mathfrak{e}_i \otimes \xi) = (e_{i1} \otimes I)\tilde{F}(e_{1i} \otimes I)(\mathfrak{e}_i \otimes \xi) = (e_{i1} \otimes I)(\mathfrak{e}_1 \otimes F\xi) = \mathfrak{e}_i \otimes F\xi.$$

Thus $F' = I \otimes F$, and $[\mathscr{F}']$ lies in the image of ρ, proving the theorem.

To prove our claim, note that $\mathcal{L}^q \hat{\otimes} \mathcal{A}$ is barreled by Proposition A.1.6 so that we can apply Proposition 3.3.1 again. Hence, there is a continuous semi-norm ν on $\mathcal{A}$ such that

$$\|T \otimes \varphi(x)\| \leq \|T\|_q \nu(x) \qquad \text{for all } T \in \mathcal{L}^q,\ x \in \mathcal{A} \tag{3.5}$$

and

$$\|[\tilde{F}, T \otimes \varphi(x)]\|_p \leq \|T\|_q \nu(x) \qquad \text{for all } T \in \mathcal{L}^q,\ x \in \mathcal{A}. \tag{3.6}$$

Since the involution is continuous, there is another continuous semi-norm $\nu' \geq \nu$ on $\mathcal{A}$ such that

$$\nu(x^*) \leq \nu'(x) \qquad \text{for all } x \in \mathcal{A}. \tag{3.7}$$

Now we have

$$\begin{aligned} \|[F', T \otimes \varphi(x)]\|_p &= \|T \otimes [F, \varphi(x)]\|_p \\ &= \|T\|_p \cdot \|[(e_1 \otimes I)\tilde{F}(e_1 \otimes I), e_1 \otimes \varphi(x)]\|_p \\ &\leq \|T\|_q \cdot \|e_1 \otimes I\| \cdot \|[\tilde{F}, e_1 \otimes \varphi(x)]\|_p \cdot \|e_1 \otimes I\| \\ &\leq \|T\|_q \cdot \|e_1\|_q \nu(x) = \|T\|_q \nu(x) \end{aligned}$$

and thus

$$\|[F', (\mathrm{id}_{\mathcal{L}^q} \otimes \varphi)(y)]\|_p \leq (\|\cdot\|_q \otimes \nu)(y) \qquad \text{for all } y \in \mathcal{L}^q \hat{\otimes} \mathcal{A}. \tag{3.8}$$

To show that $\tilde{F}$ and F' are p^2-summable perturbations with respect to $\mathrm{id}_{\mathcal{L}^q} \otimes \varphi$, first suppose that $T := \sum_{j=1}^{\infty} a_j e_j \in \mathcal{L}^q$ with $a_j \in \mathbb{R}$ and that $x \in \mathcal{A}$ is self-adjoint.

Abbreviating $T \otimes \varphi(x)$ by X and $\tilde{F} - F'$ by Y, we have

$$\begin{aligned} \|(T \otimes \varphi(x))(\tilde{F} - F')\|_{p^2}^{p^2} &= \||XY|^{p^2}\|_1 \\ &= \||XY|^p\|_p^p \\ &= \|(YX^2Y)^{\frac{1}{2}p}\|_p^p, \end{aligned}$$

and after permuting the factors:

$$\begin{aligned} &\leq C(\|X\|^{p-1}\|Y\|^{p-1}\|[X,Y]\|_p + \|X^pY^p\|_p)^p \\ &\leq C'(\|T\|_q^p \nu(x)^p + \|X^pY\|_p)^p \end{aligned}$$

where in the last line we used (3.5), (3.6) and (3.8). The constants C and C' only depend on p and the norm of $\tilde{F} - F'$.

We estimate the second summand by

$$\begin{aligned} &\|X^pY\|_p \\ &= \Big\| \sum_{i=1}^{\infty} |a_i|^p (e_i \otimes \varphi(x)^{p-1}) \Big((e_i \otimes \varphi(x))\tilde{F} - (e_{i1} \otimes \varphi(x))\tilde{F}(e_{1i} \otimes I) \Big) \Big\|_p \\ &\leq \sum_{i=1}^{\infty} |a_i|^p \nu(x)^{p-1} \Big(\|[e_{i1} \otimes \varphi(x), \tilde{F}]\|_p + \|[e_i \otimes \varphi(x), \tilde{F}]\|_p \Big) \\ &\leq \sum_{i=1}^{\infty} 2|a_i|^p \nu(x)^p \leq 2\|T\|_q^p \nu(x)^p, \end{aligned}$$

so all in all we get

$$\|(T \otimes \varphi(x))(\tilde{F} - F')\|_{p^2} \leq C''\|T\|_q \nu(x).$$

If $S \in \mathcal{L}^q$ is arbitrary, then we can find a $T \in \mathcal{L}^q$ as above and unitaries U, V such that $S = UTV$. Then

$$\begin{aligned} &\|(S \otimes \varphi(x))(\tilde{F} - F')\|_{p^2} \\ &\qquad \leq \|U \otimes I\| \|(TV \otimes \varphi(x))(\tilde{F} - F')\|_{p^2} \\ &\qquad \leq \|[TV \otimes \varphi(x), \tilde{F} - F']\|_{p^2} + \|(\tilde{F} - F')(TV \otimes \varphi(x))\|_{p^2} \\ &\qquad \leq \|[TV \otimes \varphi(x), \tilde{F} - F']\|_p + \|(V^*T \otimes \varphi(x))(\tilde{F} - F')\|_{p^2} \\ &\qquad \leq 2\|TV\|_q \nu(x) + \|V^* \otimes I\| C'' \|T\|_q \nu(x) \\ &\qquad = (2 + C'')\|S\|_q \nu(x). \end{aligned}$$

Finally, if x is not self-adjoint, then from $\nu\left(\frac{x^*+x}{2}\right) \leq \frac{\nu'(x)+\nu(x)}{2} \leq \nu'(x)$ and in the same way $\nu\left(\frac{x^*-x}{2}\right) \leq \nu'(x)$, we conclude that for any $S \in \mathcal{L}^q, x \in \mathcal{A}$

$$\|(S \otimes \varphi(x))(\tilde{F} - F')\|_{p^2} \leq C'''\|S\|_q \nu'(x)$$

and thus

$$\left\|(\mathrm{id}_{\mathcal{L}^q} \otimes \varphi)(y)(\tilde{F} - F')\right\|_{p^2} \leq C'''(\|\cdot\|_q \otimes \nu')(y) < \infty \qquad \text{for all } y \in \mathcal{L}^q \hat{\otimes}_\pi \mathcal{A}.$$

From this we also obtain that $(\mathrm{id}_{\mathcal{L}^q} \otimes \varphi)(y)(F'^2 - I) \in \mathcal{L}^{p^2}$ for all $y \in \mathcal{L}^q \hat{\otimes} \mathcal{A}$, so $\mathscr{F}'$ is indeed a p^2-summable Fredholm module, which concludes the proof. $\square$

3.5 Miscellaneous results

We end this chapter with two technical lemmas that we will use in the next chapter.

3.5.1 Lemma. *If $\varphi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a $*$-representation and $P \in \mathcal{B}(\mathcal{H})$ a projection such that*

$$P\varphi(x^*x)P - P\varphi(x^*)P\varphi(x)P \in \mathcal{L}^p \qquad \textit{for all } x \in \mathcal{A},$$

then

$$P\varphi(x)P^\perp \in \mathcal{L}^{2p} \quad \textit{and} \quad P^\perp\varphi(x)P \in \mathcal{L}^{2p} \qquad \textit{for all } x \in \mathcal{A}.$$

The statement remains true if $\mathcal{L}^p$ and $\mathcal{L}^{2p}$ are replaced by $\mathcal{K}(\mathcal{H})$ or $\mathcal{R}(\mathcal{H})$.

Proof. Write φ with respect to the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$ as

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}.$$

Since φ is a $*$-representation, we have $\varphi_{12}(x^*) = \varphi_{21}(x)^*$ and

$$\begin{bmatrix} \varphi_{11}(x^*x) & \varphi_{12}(x^*x) \\ \varphi_{21}(x^*x) & \varphi_{22}(x^*x) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(x^*)\varphi_{11}(x) + \varphi_{12}(x^*)\varphi_{21}(x) & * \\ * & * \end{bmatrix}$$

for any $x \in \mathcal{A}$. Therefore, the assumption implies that

$$\varphi_{21}(x)^*\varphi_{21}(x) = \varphi_{12}(x^*)\varphi_{21}(x) \in \mathcal{L}^p,$$

thus $\varphi_{21}(x) \in \mathcal{L}^{2p}$ and $\varphi_{12}(x) = (\varphi_{21}(x^*))^* \in \mathcal{L}^{2p}$. $\square$

In general, there is no smaller $q < 2p$ such that $P\varphi(x)P^\perp \in \mathcal{L}^q$: consider, for example, the $*$-algebra generated by

$$X := \begin{bmatrix} \sqrt{I - T^2} & T \\ T & -\sqrt{I - T^2} \end{bmatrix}$$

with $0 < T < I$ and $T \in \mathcal{L}^{2p}$, but $T \notin \mathcal{L}^q$ for any q smaller than $2p$. Then the identity representation of this algebra and $P := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ satisfy the conditions of Lemma 3.5.1, but $PXP^\perp \notin \mathcal{L}^q$ for $q < 2p$.

3.5.2 Lemma. *Let $\mathscr{F} = (\varphi, \mathcal{H}, F)$ and $\mathscr{F}' = (\varphi', \mathcal{H}', F')$ be p-summable odd Fredholm modules over $\mathcal{A}$ such that F and F' are symmetries. If for $P := \frac{1}{2}(F + I)$, $P' := \frac{1}{2}(F' + I)$ there is a unitary*

$$U : P\mathcal{H} \longrightarrow P'\mathcal{H}'$$

such that

$$UP\varphi(x)PU^* - P'\varphi'(x)P' \in \mathcal{L}^p \qquad \text{for all } x \in \mathcal{A},$$

then $\mathscr{F}$ and $\mathscr{F}'$ represent the same class in $K^1_p(\mathcal{A})$.

The analogous statement for $K^1(\mathcal{A})$ is true as well.

Proof. First, note that

$$\tilde{\mathscr{F}} := \Big(\varphi \oplus 0, (P\mathcal{H} \oplus P^\perp\mathcal{H}) \oplus P'^\perp\mathcal{H}', (I \oplus -I) \oplus -I\Big)$$

is the direct sum of $\mathscr{F}$ with a degenerate module. Similarly, if $\tilde{U}$ is the unitary

$$\tilde{U} := U \oplus I : P\mathcal{H} \oplus P'^\perp\mathcal{H}' \longrightarrow P'\mathcal{H}' \oplus P^{\perp\prime}\mathcal{H}',$$

then

$$\tilde{\mathscr{F}}' := \Big(\tilde{U}^*\varphi'(\cdot)\tilde{U} \oplus 0, (P\mathcal{H} \oplus P'^\perp\mathcal{H}') \oplus P^\perp\mathcal{H}, (I \oplus -I) \oplus -I\Big)$$

is the direct sum of a module which is unitarily equivalent to $\mathscr{F}'$ and a degenerate module. If S is the unitary

$$S : P\mathcal{H} \oplus P^\perp\mathcal{H} \oplus P'^\perp\mathcal{H} \longrightarrow P\mathcal{H} \oplus P'^\perp\mathcal{H}' \oplus P^\perp\mathcal{H}$$

which flips the last two summands, then $\tilde{\mathscr{F}}'$ is unitarily equivalent to the module

$$\tilde{\mathscr{F}}'' := \Big(S^*(\tilde{U}^*\varphi'(\cdot)\tilde{U} \oplus 0)S, P\mathcal{H} \oplus P^\perp\mathcal{H} \oplus P'^\perp\mathcal{H}', I \oplus -I \oplus -I\Big).$$

$\tilde{\mathscr{F}}$ and $\tilde{\mathscr{F}}''$ satisfy the conditions of the proposition with $U = I$, so we see that is suffices to prove the proposition under the assumption that $\mathcal{H} = \mathcal{H}'$, $F = F'$ and $U = I$.

Decompose φ and φ' with respect to $\mathcal{H} = P\mathcal{H} \oplus P^\perp\mathcal{H}$ as

$$\varphi = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \quad \text{and} \quad \varphi' = \begin{bmatrix} \varphi'_{11} & \varphi'_{12} \\ \varphi'_{21} & \varphi'_{22} \end{bmatrix},$$

and consider the operator homotopy

$$\mathscr{F}_t := \left(\begin{bmatrix} \varphi_{11} & \varphi_{12} & & \\ \varphi_{21} & \varphi_{22} & & \\ & & \varphi'_{11} & \varphi'_{12} \\ & & \varphi'_{21} & \varphi'_{22} \end{bmatrix}, \mathcal{H} \oplus \mathcal{H}, \begin{bmatrix} \cos(\pi t)I & & \sin(\pi t)I & \\ & -I & & \\ \sin(\pi t)I & & -\cos(\pi t)I & \\ & & & I \end{bmatrix} \right).$$

$\mathscr{F}_0$ is a representative of $[\mathscr{F}] - [\mathscr{F}'] \in K^1_p(\mathcal{A})$, whereas for $t = 1$ we obtain a degenerate module. Explicit calculation shows that $\mathscr{F}_t$ is in fact a p-summable module for any t (bearing in mind that by assumption $\varphi_{11}(x) - \varphi'_{11}(x) \in \mathcal{L}^p$ and that $\varphi_{12}(x), \varphi_{21}(x), \varphi'_{12}(x), \varphi'_{21}(x) \in \mathcal{L}^p$ since $\mathscr{F}$ and $\mathscr{F}'$ are p-summable). $\square$

Chapter 4

AF-Algebras

AF-Algebras are the inductive limits of finite-dimensional C^*-Algebras. In other words, every AF-Algebra A can be written as $A = \overline{\bigcup_n A_n}$ with an increasing sequence A_n ($n \in \mathbb{N}$) of finite-dimensional C^*-subalgebras. Since every A_n, being a C^*-algebra, is a pre-C^*-algebra, the same is true for their union. So every AF-Algebra $\overline{\bigcup_n A_n}$ comes equipped with $\bigcup_n A_n$ as a dense pre-C^*-subalgebra.

In this chapter we will analyse the relation between the finitely summable K-homology of $\bigcup_n A_n$ and the ordinary K-homology of A.

The algebra $\bigcup_n A_n$ is not uniquely determined by A, of course (think of $A = \mathcal{K}(\mathcal{H})$ and $M_\infty(\mathbb{C})$ with respect to two different orthonormal bases as dense subalgebras). However, they are unique up to unitary equivalence:

4.0.3 Proposition. *If an AF-algebra A can be written as*

$$A = \overline{\bigcup_n A_n} = \overline{\bigcup_n B_n}$$

with increasing sequences of finite-dimensional C^*-subalgebras A_n and B_n, then for every $\varepsilon > 0$ there is a unitary $u \in \tilde{A}$ with $\|u - 1_{\tilde{A}}\| < \varepsilon$ such that

$$u \cdot \bigcup_n A_n \cdot u^* = \bigcup_n B_n.$$

Proof. [Dav96, Theorem III.3.5]. □

Such a u can be constructed recursively with the help of the following lemma:

4.0.4 Lemma. *For every $\varepsilon > 0$ and $n \in \mathbb{N}$ there is a $\delta > 0$ such that the following is true:*
Let A, B be C^*-subalgebras of a common unital C^*-algebra D with $\dim A \leq n$ such that there is a system of matrix units $\{e_{ij}^{(k)}\}$ for A with $\operatorname{dist}(e_{ij}^{(k)}, B) < \delta$

for all i, j, k. Then there is a unitary $u \in C^*(A, B, 1_D)$ with $\|u - 1_D\| < \varepsilon$ such that

$$u\ A\ u^* \subseteq B.$$

Moreover, u can be chosen to commute with $A \cap B$.

Proof. [Dav96, Lemma III.3.2 and Corollary III.3.3]. □

The proof of Lemma 4.0.4 works by first finding matrix units in B which lie near the given matrix units. Then a unitary is constructed which translates the old matrix units to the new ones in B.

We do not reproduce the proof of Lemma 4.0.4 as we will prove a variation of the statement in Section 4.1, anyway. The part of finding the unitary is contained in Lemma 4.1.4, while the construction of adequate matrix units is contained in the proof of Theorem 4.1.5.

Throughout this chapter $A = \overline{\bigcup_n A_n}$ will denote an AF-algebra with an increasing sequence A_n $(n \in \mathbb{N})$ of finite-dimensional C^*-subalgebras.

4.1 The odd case

In this section we compute $K_p^1(\bigcup_n A_n)$.

4.1.1 Theorem. *Every element of $K^1(A)$ can be represented by a 1-summable Fredholm module over $\bigcup_n A_n$.*

Proof. Let $\mathbf{x} \in K^1(A)$ be given. By Theorem 2.4.1 and Remark 2.4.8, $\mathbf{x}$ has a representative $(\varphi, \mathcal{H}, F)$ with $F^* = F$ and $F^2 = I$. Let $P := 1/2(F + I)$. Adding a degenerate module if necessary, we may assume that $P\varphi(x)P \notin \mathcal{K}(\mathcal{H})$ for any $x \in A \setminus \{0\}$.

Such a module defines an extension of A by $\mathcal{K}(P\mathcal{H})$ as follows. Consider $E \subset \mathcal{B}(P\mathcal{H})$ given by

$$E := P\varphi(A)P + \mathcal{K}(P\mathcal{H}).$$

E is a $*$-algebra since φ commutes with P up to compact operators. Denote by q the quotient map from $\mathcal{B}(P\mathcal{H})$ to the Calkin-Algebra $\mathcal{B}(P\mathcal{H})/\mathcal{K}(P\mathcal{H})$ and by σ the c.p.c.[†] map from A to E sending x to $P\varphi(x)P$. Again because φ and P commute up to compact operators, $q \circ \sigma$ is a $*$-homomorphism onto $q(E)$. It is injective since $P\varphi(x)P \notin \mathcal{K}(\mathcal{H})$ for $x \in A \setminus \{0\}$. Therefore, we obtain the C^*-algebra extension

$$0 \longrightarrow \mathcal{K}(P\mathcal{H}) \xrightarrow{\subseteq} E \underset{\pi}{\overset{\sigma}{\rightleftarrows}} A \longrightarrow 0$$

with $\pi = (q \circ \sigma)^{-1} \circ q$ (E is complete since $q(E)$ is closed in $\mathcal{B}(P\mathcal{H})/\mathcal{K}(P\mathcal{H})$).

[†]Completely positive contractive (Definition A.3.1).

Extensions of AF-algebras are again AF-algebras [Dav96, Theorem III.6.3]. Therefore, since A and $\mathcal{K}(P\mathcal{H})$ are AF, E is also AF and there is an increasing sequence of finite-dimensional C^*-subalgebras E_n of E such that $E = \overline{\bigcup_n E_n}$. By a standard result on ideals in inductive limits of C^*-algebras, we can write $\mathcal{K}(P\mathcal{H}) = \overline{\bigcup_n \mathcal{K}_n}$ where $\mathcal{K}_n := \mathcal{K}(P\mathcal{H}) \cap E_n$ [Dav96, Lemma III.4.1].

Note that the $\mathcal{K}_n$ are finite-dimensional C^*-algebras of compact operators. However, this is only possible if they are, in fact, algebras of finite-rank operators (every compact projection has finite rank). Therefore, we have the extension

$$0 \longrightarrow \bigcup_n \mathcal{K}_n \xrightarrow{\subseteq} \bigcup_n E_n \xrightarrow{\pi} \bigcup_n \pi(E_n) \longrightarrow 0,$$

where $\bigcup_n \mathcal{K}_n$ is an algebra of finite-rank operators.

A priori, there is no reason why $\bigcup_n \pi(E_n)$ should be equal to $\bigcup_n A_n$, but this can be fixed easily: since $\bigcup_n E_n$ is dense in E, we have $\overline{\bigcup_n \pi(E_n)} = A$. According to Proposition 4.0.3, there is a unitary $u \in \tilde{A}$ homotopic to the identity such that

$$u \cdot \bigcup_n \pi(E_n) \cdot u^* = \bigcup_n A_n.$$

Lift u to a unitary $U \in \tilde{E}$. Replacing E_n by UE_nU^* and $\mathcal{K}_n$ by $U\mathcal{K}_nU^*$, we arrive at the extension

$$0 \longrightarrow \bigcup_n \mathcal{K}_n \xrightarrow{\subseteq} \bigcup_n E_n \xrightarrow{\pi} \bigcup_n A_n \longrightarrow 0.$$

Now we are done if we can find a c.p.c. split $\sigma' : \bigcup_n A_n \longrightarrow \bigcup_n E_n$ of this extension: let φ' be a Stinespring dilation (Theorem A.3.3) of σ' on $\mathcal{H}' := P\mathcal{H} \oplus \mathcal{N}$ and P' the projection of $\mathcal{H}'$ onto $P\mathcal{H}$. Since σ' splits the extension,

$$P'\varphi'(xy)P' - P'\varphi'(x)P'\varphi'(y)P' = \sigma'(xy) - \sigma'(x)\sigma'(y) \in \mathcal{R}(P\mathcal{H})$$

holds for all $x, y \in \bigcup_n A_n$. Thus, $(\varphi', \mathcal{H}', 2P' - I)$ is a 1-summable Fredholm module over $\bigcup_n A_n$ by Lemma 3.5.1. Lemma 3.5.2 shows that this module extends to a representative of the same K-homology class of A as $(\varphi, \mathcal{H}, F)$.

To construct σ', observe that, since each E_n is finite-dimensional and $\mathcal{K}_n$ is an ideal in E_n, E_n decomposes as $E_n = \mathcal{K}_n \oplus Q_n$ for some finite-dimensional Q_n isomorphic to $\pi(E_n)$ via π. Regarding the union $\bigcup_n E_n$ as an inductive limit, we obtain an inductive system of the form

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathcal{K}_{n-1} & \xrightarrow{k_{n-1}} & \mathcal{K}_n & \xrightarrow{k_n} & \mathcal{K}_{n+1} & \longrightarrow & \cdots \\
 & \nearrow & \oplus & \nearrow^{\tilde{q}_{n-1}} & \oplus & \nearrow^{\tilde{q}_n} & \oplus & \nearrow & \\
\cdots & \longrightarrow & Q_{n-1} & \xrightarrow{q_{n-1}} & Q_n & \xrightarrow{q_n} & Q_{n+1} & \longrightarrow & \cdots
\end{array}$$

with connecting maps $e_n := (k_n, q_n + \tilde{q}_n)$.

The projections π_n of $\mathcal{K}_n \oplus Q_n$ onto Q_n define maps to the sub-diagram

$$\ldots \longrightarrow Q_{n-1} \xrightarrow{q_{n-1}} Q_n \xrightarrow{q_n} Q_{n+1} \longrightarrow \ldots$$

which are compatible with the connecting maps. The inductive limit of this diagram is isomorphic to $\bigcup_n A_n$ via π. Therefore, it suffices to find c.p.c. maps $\sigma'_n : Q_n \longrightarrow \mathcal{K}_n \oplus Q_n$ which split π_n and which are compatible with the connecting maps.

Since Q_n and Q_{n+1} are finite-dimensional, we can find a c.p.c. left-inverse $s_n : Q_{n+1} \longrightarrow Q_n$ for the injection q_n:

Let $\{e_{ij}^{(k)}\}$ be a system of matrix units for Q_n, choose minimal projections $p^{(k)} \in Q_{n+1}$ with $p^{(k)} \leq q_n(e_1^{(k)})$, and let $p := \sum_{k,i} q_n(e_{i1}^{(k)}) p^{(k)} q_n(e_{1i}^{(k)})$. Then the compression $\gamma_{n+1} : Q_{n+1} \longrightarrow Q_{n+1}$ sending $x \in Q_{n+1}$ to pxp is c.p.c., and $\gamma_{n+1} \circ q_n$ is a $*$-isomorphism onto the image of γ_{n+1}. Thus, we can set $s_n := (\gamma_{n+1} \circ q_n)^{-1} \circ \gamma_{n+1}$.

Now, let ι_{Q_n} denote the embedding of Q_n into E_n and define recursively

$$\sigma'_1 := \iota_{Q_1} \quad \text{and} \quad \sigma'_{n+1} := (k_n, \tilde{q}_n) \circ \sigma'_n \circ s_n + \iota_{Q_{n+1}}.$$

Each σ'_n is obviously c.p.c. and splits π_n, so we only need to check compatibility with the connecting maps:

$$\begin{aligned}
\sigma'_{n+1} \circ q_n - e_n \circ \sigma'_n &= \Big((k_n, \tilde{q}_n) \circ \sigma'_n \circ s_n + \iota_{Q_{n+1}} \Big) \circ q_n - e_n \circ \sigma'_n \\
&= (k_n, \tilde{q}_n) \circ \sigma'_n + \iota_{Q_{n+1}} \circ q_n - (k_n, q_n + \tilde{q}_n) \circ \sigma'_n \\
&= \iota_{Q_{n+1}} \circ q_n - (0, q_n) \circ \Big((k_{n-1}, \tilde{q}_{n-1}) \circ \sigma'_{n-1} \circ s_{n-1} + \iota_{Q_n} \Big) \\
&= \iota_{Q_{n+1}} \circ q_n - (0, q_n) \circ \iota_{Q_n} \\
&= 0.
\end{aligned}$$

□

To answer the question whether every p-summable module over $\bigcup_n A_n$ which is zero in $K^1(A)$ is already zero in $K_p^1(\bigcup_n A_n)$, we need a bit more preparation. We start with two simple lemmas about operators which lie near projections or near partial isometries:

4.1.2 Lemma. *If $P \in \mathcal{B}(\mathcal{H})$ is a projection and $X \in \mathcal{B}(\mathcal{H})$ is self-adjoint, then*

$$\|\chi_{[1/2,\infty)}(X) - X\| \leq \|X - P\|,$$

where $\chi_{[1/2,\infty)}$ denotes the characteristic function of the interval $[1/2, \infty)$.

If $X^2 - X \in \mathcal{I}$ where $\mathcal{I} = \mathcal{L}^p(\mathcal{H})$ or $\mathcal{I} = \mathcal{K}(\mathcal{H})$, then

$$\chi_{[1/2,\infty)}(X) - X \in \mathcal{I}.$$

Proof. Abbreviate $\|X - P\|$ by d. Since $\mathrm{sp}(P) = \{0, 1\}$, we have

$$\mathrm{sp}(X) \subseteq [-d, d] \cup [1 - d, 1 + d].$$

Thus, $|\chi_{[1/2,\infty)} - \mathrm{id}_{\mathbb{R}}|$ is bounded on $\mathrm{sp}(X)$ by d. Since Borel functional calculus is norm decreasing, the first claim follows.

To prove the second claim, define functions $p, q : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$p(x) := \begin{cases} \frac{1}{x-1} & x < \frac{1}{2} \\ \frac{1}{x} & x \geq \frac{1}{2} \end{cases} \quad \text{and} \quad q(x) := x^2 - x.$$

Then

$$X - \chi_{[1/2,\infty)}(X) = p(X)q(X) = p(X)(X^2 - X) \in \mathcal{I}.$$

□

4.1.3 Lemma. *Let $T \in \mathcal{B}(\mathcal{H})$ and a projection $P \in \mathcal{B}(\mathcal{H})$ be given such that $TP = T$. Then the partial isometry part V of the polar decomposition $T = V|T|$ satisfies*

$$\|V - T\| \leq \|P - T^*T\|^{\frac{1}{2}}.$$

If $P - T^*T \in \mathcal{I}$ where $\mathcal{I} = \mathcal{L}^p(\mathcal{H})$ or $\mathcal{I} = \mathcal{K}(\mathcal{H})$, then

$$V - T \in \mathcal{I}.$$

Proof. Abbreviate $\|P - T^*T\|$ by d. We have $T^*T = PT^*TP$, so T^*T restricts to an operator on $P\mathcal{H}$ whose spectrum is contained in $[\max(0, 1 - d), 1 + d]$. Checking that $|1 - \sqrt{\cdot}|$ is bounded on $[\max(0, 1 - d), 1 + d]$ by $\sqrt{d}$, we obtain

$$\|V - T\| \leq \|V\| \cdot \|P - (T^*T)^{\frac{1}{2}}\| \leq \sqrt{d}.$$

To prove the second claim, note that $V - T$ factors as

$$\begin{aligned} V - T = (V - T)P = V\left(I - (T^*T)^{\frac{1}{2}}\right)P &= V\left(I + (T^*T)^{\frac{1}{2}}\right)^{-1}(I - T^*T)P \\ &= V\left(I + (T^*T)^{\frac{1}{2}}\right)^{-1}(P - T^*T). \end{aligned}$$

□

Next, we prove a lemma stating that two systems of matrix units which lie close enough to each other can be transformed into each other by a unitary near the identity. This is a slight variation of a part of the proof of Lemma 4.0.4.

4.1.4 Lemma. *For any $N \in \mathbb{N}$ and $\varepsilon > 0$ there is a $\delta > 0$ such that the following statement is true:*
If $\{E_{ij}^{(k)}\}$ and $\{F_{ij}^{(k)}\}$ are two systems of matrix units of operators on a Hilbert

space $\mathcal{H}$ with the same indices and with in total N projections $E_i^{(k)}$ (resp. $F_i^{(k)}$) such that

$$\|E_{ij}^{(k)} - F_{ij}^{(k)}\| < \delta \qquad \text{for all } i,j,k,$$

then there is a unitary $U \in \mathcal{B}(\mathcal{H})$ with

$$UE_{ij}^{(k)}U^* = F_{ij}^{(k)} \quad \text{for all } i,j,k \qquad \text{and} \qquad \|U - I\| < \varepsilon.$$

If moreover $\mathcal{I} = \mathcal{L}^p(\mathcal{H})$ or $\mathcal{I} = \mathcal{K}(\mathcal{H})$ and

$$E_{ij}^{(k)} - F_{ij}^{(k)} \in \mathcal{I} \qquad \text{for all } i,j,k,$$

then U can be chosen to satisfy

$$U - I \in \mathcal{I}.$$

For $\varepsilon \leq 1$, an appropriate δ is $\frac{\varepsilon^2}{25N^2}$.

Proof. Define

$$V := \sum_k \sum_i F_{i1}^{(k)} E_{1i}^{(k)} + \Big(I - \sum_k \sum_i F_i^{(k)}\Big)\Big(I - \sum_k \sum_i E_i^{(k)}\Big).$$

Then for $\delta \leq \frac{1}{2N}$ we have

$$\begin{aligned} \|V - I\| &\leq \sum_k \sum_i \|(F_{i1}^{(k)} - E_{i1}^{(k)})\| + \sum_k \sum_i \|F_i^{(k)} - E_i^{(k)}\| \\ &< 2N\delta \leq 1, \end{aligned}$$

so V is invertible and has the polar decomposition $V = U|V|$ with a unitary U. Moreover, Lemma 4.1.3 implies that

$$\begin{aligned} \|U - I\| &\leq \|U - V\| + \|V - I\| \leq \|V^*V - I\|^{\frac{1}{2}} + \|V - I\| \\ &< (\|V^* - I\|\|V\| + \|V - I\|)^{\frac{1}{2}} + 2N\delta \\ &< (6N\delta)^{\frac{1}{2}} + 2N\delta < 5N\delta^{\frac{1}{2}}. \end{aligned}$$

Thus, $\|U - I\| < \varepsilon$ if $\delta \leq \varepsilon^2/(25N^2)$.

Check that

$$VE_{ij}^{(k)} = F_{i1}^{(k)} E_{1j}^{(k)} = F_{ij}^{(k)} V,$$

hence

$$V^*VE_{ij}^{(k)} = V^*F_{ij}^{(k)}V = E_{ij}^{(k)}V^*V.$$

The second identity implies that $|V|^{-1}$ commutes with all $E_{ij}^{(k)}$ and therefore

$$UE_{ij}^{(k)}U^* = V|V|^{-1}E_{ij}^{(k)}U^* = VE_{ij}^{(k)}|V|^{-1}U^* = F_{ij}^{(k)}V|V|^{-1}U^* = F_{ij}^{(k)}$$

for all i,j,k.

If $F_{ij}^{(k)} - E_{ij}^{(k)} \in \mathcal{I}$ for all i,j,k, then $I - V \in \mathcal{I}$. In particular we have $I - V^*V \in \mathcal{I}$ and thus Lemma 4.1.3 implies that $U - V \in \mathcal{I}$. □

Now we can finish the computation of $K^1_p(\bigcup_n A_n)$:

4.1.5 Theorem. *Let $\mathscr{F}$ be a p-summable Fredholm module over $\bigcup_n A_n$. If $[\mathscr{F}] = \mathbf{0} \in K^1(A)$, then $[\mathscr{F}] = \mathbf{0} \in K^1_p(\bigcup_n A_n)$.*

Proof. Let $\mathscr{F} = (\varphi, \mathcal{H}, F) \in \mathbb{F}^1_p(\bigcup_n A_n)$ with $[\mathscr{F}] = \mathbf{0} \in K^1(A)$ be given, and extend it to a Fredholm module over A. By Theorem 2.4.1 we can assume that $F^* = F$ and $F^2 = I$. Let $P := 1/2(F + I)$.

$[\mathscr{F}] = \mathbf{0} \in K^1(A)$ implies by Theorem 2.4.1 and Remark 2.4.8 that (after adding a degenerate module) we can assume that $\mathscr{F}$ is operator homotopic to degenerate module via a path of symmetries. Thus, by Corollary 2.4.4 there must be a unitary $U \in \mathcal{D}_{\varphi(A)}$ such that $(\varphi, \mathcal{H}, UFU^*)$ is a degenerate module. This means that

$$[F, U^*\varphi(x)U] = 0 \quad \text{and} \quad U^*\varphi(x)U - \varphi(x) \in \mathcal{K}(\mathcal{H}) \qquad \text{for all } x \in A.$$

Let σ and σ_0 be the compressions of φ and $U^*\varphi(\cdot)U$ to $P\mathcal{H}$. Since F commutes with $U^*\varphi(\cdot)U$, we know that σ_0 is a $*$-homomorphism. Moreover, we have

$$\sigma_0(x) - \sigma(x) \in \mathcal{K}(P\mathcal{H}) \qquad \text{for all } x \in A$$

(in the Ext-picture of K-homology this means that σ_0 is a $*$-homomorphism split for the trivial extension associated to $\mathscr{F}$).

Now, by twisting σ_0 we want to find a $*$-homomorphism $\sigma' : A \longrightarrow \mathcal{B}(P\mathcal{H})$ with $\sigma'(x) - \sigma(x) \in \mathcal{L}^p$ for all $x \in \bigcup_n A_n$. Then the degenerate module $(\sigma', P\mathcal{H}, I)$ will be in the same class of $K^1_p(\bigcup_n A_n)$ as $\mathscr{F}$ (Lemma 3.5.2), proving the theorem.

To define σ', we will construct recursively $*$-homomorphisms σ_n from A to $\mathcal{B}(P\mathcal{H})$ with the following properties:

(i) $\sigma_n(x) - \sigma(x) \in \mathcal{K}(P\mathcal{H})$ for all $x \in A$

(ii) $\sigma_n(x) - \sigma(x) \in \mathcal{L}^p(P\mathcal{H})$ for all $x \in A_n$

(iii) $\sigma_n(x) = \sigma_{n-1}(x)$ for all $x \in A_{n-1}$ (if $n > 1$)

Then we can just define σ' to be $\sigma_n(x)$ for $x \in A_n$.

Assume σ_{n-1} has already been constructed, and let $\{f^{(k)}_{ij}\}$ be matrix units for A_n with in total d projections.

Our first aim is to show that we can find for each $0 < \delta < 1$ matrix units $\{F^{(k)}_{ij}\}$ with the same indices as $\{f^{(k)}_{ij}\}$ satisfying

$$\|F^{(k)}_{ij} - \sigma_{n-1}(f^{(k)}_{ij})\| < \delta \quad \text{and} \quad F^{(k)}_{ij} - \sigma(f^{(k)}_{ij}) \in \mathcal{L}^p \qquad \text{for all } i, j, k. \tag{4.1}$$

We will then use Lemma 4.1.4 to find a unitary which translates the $\sigma_{n-1}(f^{(k)}_{ij})$ to the $F^{(k)}_{ij}$. Along the lines of the proof of Lemma 4.0.4, we construct the $F^{(k)}_{ij}$ in two steps:

Step 1. First, we find the projections $F_i^{(k)}$. To this end, let $F_{i_s}^{(k_s)}$ $(s = 1, \ldots, d)$ be an enumeration of these projections. The $F_{i_s}^{(k_s)}$ we construct will have the additional property that

$$\|F_{i_s}^{(k_s)} - \sigma_{n-1}(f_{i_s}^{(k_s)})\| < \delta_s \quad \text{where} \quad \delta_s := (16d)^{-(3+d-s)}\delta^2. \tag{4.2}$$

Since $\sigma_{n-1}(f_{i_s}^{(k_s)}) - \sigma(f_{i_s}^{(k_s)}) \in \mathcal{K}(P\mathcal{H})$ and $\mathcal{L}^p(P\mathcal{H})$ is dense in $\mathcal{K}(P\mathcal{H})$, we can find self-adjoint $\tilde{F}_{i_s}^{(k_s)}$ such that

$$\|\tilde{F}_{i_s}^{(k_s)} - \sigma_{n-1}(f_{i_s}^{(k_s)})\| < \frac{1}{16}\delta_s \quad \text{and} \quad \tilde{F}_{i_s}^{(k_s)} - \sigma(f_{i_s}^{(k_s)}) \in \mathcal{L}^p.$$

The latter implies since $\sigma(f_{i_s}^{(k_s)})^2 - \sigma(f_{i_s}^{(k_s)}) \in \mathcal{L}^p$ that we also have

$$\left(\tilde{F}_{i_s}^{(k_s)}\right)^2 - \tilde{F}_{i_s}^{(k_s)} \in \mathcal{L}^p.$$

Defining

$$F_{i_1}^{(k_1)} := \chi_{[1/2,\infty)}(\tilde{F}_{i_1}^{(k_1)}),$$

Lemma 4.1.2 implies $\|F_{i_1}^{(k_1)} - \tilde{F}_{i_1}^{(k_1)}\| < \frac{1}{16}\delta_1$ and $F_{i_1}^{(k_1)} - \tilde{F}_{i_1}^{(k_1)} \in \mathcal{L}^p$. Thus, $F_{i_1}^{(k_1)}$ satisfies (4.1) and (4.2).

For the other $F_{i_s}^{(k_s)}$ we proceed in the same way, but we have to ensure that we end up with pairwise orthogonal projections. To this end, assume that we have already constructed $F_{i_1}^{(k_1)}, \ldots, F_{i_{s-1}}^{(k_{s-1})}$. Let

$$P_s := \sum_{r=1}^{s-1} F_{i_r}^{(k_r)} \quad \text{and} \quad p_s := \sum_{r=1}^{s-1} f_{i_r}^{(k_r)}.$$

Then define

$$F_{i_s}^{(k_s)} := \chi_{[1/2,\infty)}(P_s^\perp \tilde{F}_{i_s}^{(k_s)} P_s^\perp),$$

which is obviously a projection orthogonal to the earlier defined $F_{i_r}^{(k_r)}$. Since

$$P_s - \sigma(p_s),\ \tilde{F}_{i_s}^{(k_s)} - \sigma(f_{i_s}^{(k_s)}),\ \sigma(p_s)\sigma(f_{i_s}^{(k_s)}),\ \sigma(f_{i_s}^{(k_s)})\sigma(p_s) \in \mathcal{L}^p,$$

we also have

$$P_s^\perp \tilde{F}_{i_s}^{(k_s)} P_s^\perp - \tilde{F}_{i_s}^{(k_s)} \in \mathcal{L}^p,$$

hence $P^\perp \tilde{F}_{i_s}^{(k_s)} P^\perp - \sigma(f_{i_s}^{(k_s)}) \in \mathcal{L}^p$. Again, this implies

$$\left(P_s^\perp \tilde{F}_{i_s}^{(k_s)} P_s^\perp\right)^2 - P_s^\perp \tilde{F}_{i_s}^{(k_s)} P_s^\perp \in \mathcal{L}^p. \tag{4.3}$$

Moreover,

$$\begin{aligned}
&\|P_s^\perp \tilde{F}_{i_s}^{(k_s)} P_s^\perp - \tilde{F}_{i_s}^{(k_s)}\| \\
&\qquad \le 2\|P_s - \sigma_{n-1}(p_s)\| \cdot \|\tilde{F}_{i_s}^{(k_s)}\| + \|\sigma_{n-1}(p_s)^\perp \tilde{F}_{i_s}^{(k_s)} \sigma_{n-1}(p_s)^\perp - \tilde{F}_{i_s}^{(k_s)}\| \\
&\qquad < 4(s-1)\delta_{s-1} + 2\|\tilde{F}_{i_s}^{(k_s)} - \sigma_{n-1}(f_{i_s}^{(k_s)})\| \\
&\qquad < \frac{1}{4}\delta_s + \frac{1}{8}\delta_s = \frac{3}{8}\delta_s,
\end{aligned}$$

and we obtain

$$\|P_s^\perp \tilde{F}_{i_s}^{(k_s)} P_s^\perp - \sigma_{n-1}(f_{i_s}^{(k_s)})\| < \frac{1}{2}\delta_s. \tag{4.4}$$

Equations (4.3) and (4.4) imply together with Lemma 4.1.2 that $F_{i_s}^{(k_s)}$ satisfies (4.1) and (4.2).

Having constructed the projections $F_i^{(k)}$, let us note that (4.2) in particular implies that each $F_i^{(k)}$ satisfies

$$\|F_i^{(k)} - \sigma_{n-1}(f_i^{(k)})\| < \delta_d = (16d)^{-3}\delta^2 \le 2^{-12}\delta^2.$$

Step 2. To construct the partial isometries $F_{ij}^{(k)}$, find as above operators $\tilde{F}_{i1}^{(k)}$ such that

$$\|\tilde{F}_{i1}^{(k)} - \sigma_{n-1}(f_{i1}^{(k)})\| < \frac{1}{3}2^{-10}\delta^2 \quad \text{and} \quad \tilde{F}_{i1}^{(k)} - \sigma(f_{i1}^{(k)}) \in \mathcal{L}^p. \tag{4.5}$$

$F_{i1}^{(k)}$ has to be a partial isometry between $F_1^{(k)}$ and $F_i^{(k)}$, so we first cut $\tilde{F}_{i1}^{(k)}$ down to

$$\hat{F}_{i1}^{(k)} := F_i^{(k)} \tilde{F}_{i1}^{(k)} F_1^{(k)}.$$

Since the $\tilde{F}_{i1}^{(k)}$ are p-summable perturbations of the $\sigma(f_{i1}^{(k)})$, the $F_i^{(k)}$ are p-summable perturbations of the $\sigma(f_i^{(k)})$ and since

$$\sigma(f_i^{(k)})\sigma(f_{i1}^{(k)})\sigma(f_1^{(k)}) - \sigma(f_{i1}^{(k)}), \quad \sigma(f_{i1}^{(k)})^*\sigma(f_{i1}^{(k)}) - \sigma(f_1^{(k)}) \in \mathcal{L}^p,$$

we obtain

$$\hat{F}_{i1}^{(k)} - \sigma(f_{i1}^{(k)}) \in \mathcal{L}^p \quad \text{and} \quad \hat{F}_{i1}^{(k)*}\hat{F}_{i1}^{(k)} - F_1^{(k)} \in \mathcal{L}^p. \tag{4.6}$$

Moreover,

$$\begin{aligned}
\|\hat{F}_{i1}^{(k)} - \tilde{F}_{i1}^{(k)}\| &< 2 \cdot 2^{-12}\delta^2 \|\tilde{F}_{i1}^{(k)}\| + \|\sigma_{n-1}(f_i^{(k)})\tilde{F}_{i1}^{(k)}\sigma_{n-1}(f_1^{(k)}) - \tilde{F}_{i1}^{(k)}\| \\
&< 2^{-10}\delta^2 + 2\|\tilde{F}_{i1}^{(k)} - \sigma_{n-1}(f_{i1}^{(k)})\| \\
&< 2^{-10}\delta^2 + \frac{2}{3} \cdot 2^{-10}\delta^2,
\end{aligned}$$

hence

$$\|\hat{F}_{i1}^{(k)} - \sigma_{n-1}(f_{i1}^{(k)})\| < 2^{-9}\delta^2$$

and

$$\begin{aligned}\|\hat{F}_{i1}^{(k)*}\hat{F}_{i1}^{(k)} - F_1^{(k)}\| &< 2^{-9}\delta^2\|\hat{F}_{i1}^{(k)}\| + 2^{-9}\delta^2 + \|\sigma_{n-1}(f_1^{(k)}) - F_1^{(k)}\| \\ &< 2^{-8}\delta^2 + 2^{-9}\delta^2 + 2^{-12}\delta^2 < 2^{-7}\delta^2.\end{aligned} \tag{4.7}$$

Thus, if we define $F_{i1}^{(k)}$ to be the partial isometry part of the polar decomposition of $\hat{F}_{i1}^{(k)}$, (4.6) and (4.7) together with Lemma 4.1.3 imply that

$$\|F_{i1}^{(k)} - \hat{F}_{i1}^{(k)}\| < 2^{-3}\delta \quad \text{and} \quad F_{i1}^{(k)} - \hat{F}_{i1}^{(k)} \in \mathcal{L}^p,$$

thus

$$\|F_{i1}^{(k)} - \sigma_{n-1}(f_{i1}^{(k)})\| < 2^{-2}\delta \quad \text{and} \quad F_{i1}^{(k)} - \sigma(f_{i1}^{(k)}) \in \mathcal{L}^p.$$

Moreover, the estimate (4.7) implies that $\hat{F}_{i1}^{(k)*}\hat{F}_{i1}^{(k)}$ is invertible on $F_1^{(k)}(P\mathcal{H})$, so the source projection of $F_{i1}^{(k)}$ is $F_1^{(k)}$. In the same way one can see that $F_i^{(k)}$ is the range projection of $F_{i1}^{(k)}$. This shows that

$$F_{ij}^{(k)} := F_{i1}^{(k)}F_{j1}^{(k)*}$$

are matrix units satisfying (4.1).

End of Proof. Now we are ready to apply Lemma 4.1.4. Let δ be the number which is given by Lemma 4.1.4 for $N = d$ and $\varepsilon = 1/(50d^2)$. Use the above construction to find matrix units $F_{ij}^{(k)}$ satisfying (4.1) for this δ. In particular,

$$F_{ij}^{(k)} - \sigma_{n-1}(f_{ij}^{(k)}) = \left(F_{ij}^{(k)} - \sigma(f_{ij}^{(k)})\right) + \left(\sigma(f_{ij}^{(k)}) - \sigma_{n-1}(f_{ij}^{(k)})\right) \in \mathcal{K}(P\mathcal{H})$$

for all i, j, k. Thus, the lemma provides us with a unitary U_1 such that $U_1 - I \in \mathcal{K}(P\mathcal{H})$ and

$$U_1\sigma_{n-1}(f_{ij}^{(k)})U_1^* = F_{ij}^{(k)} \qquad \text{for all } i, j, k.$$

Since A_n is the linear span of the matrix units $\{f_{ij}^{(k)}\}$, we obtain

$$U_1\sigma_{n-1}(x)U_1^* - \sigma(x) \in \mathcal{L}^p \qquad \text{for all } x \in A_n.$$

Moreover, $U_1 - I \in \mathcal{K}(P\mathcal{H})$ implies that

$$U_1\sigma_{n-1}(x)U_1^* - \sigma(x) \in \mathcal{K}(P\mathcal{H}) \qquad \text{for all } x \in A.$$

Thus, for $n = 1$ we can just define $\sigma_1(\cdot) := U_1\sigma_0(\cdot)U_1^*$. However, for $n > 1$ we have to ensure that σ_n extends σ_{n-1}.

If $\{e_{ij}^{(k)}\}$ are matrix units for A_{n-1}, we have

$$\|U_1\sigma_{n-1}(e_{ij}^{(k)})U_1^* - \sigma_{n-1}(e_{ij}^{(k)})\| \leq 2\|U_1 - I\| < \frac{1}{25d^2}.$$

Moreover,

$$\begin{aligned} &U_1\sigma_{n-1}(e_{ij}^{(k)})U_1^* - \sigma_{n-1}(e_{ij}^{(k)}) \\ &\quad = \Big(U_1\sigma_{n-1}(e_{ij}^{(k)})U_1^* - \sigma(e_{ij}^{(k)})\Big) + \Big(\sigma(e_{ij}^{(k)}) - \sigma_{n-1}(e_{ij}^{(k)})\Big) \in \mathcal{L}^p. \end{aligned}$$

Since A_{n-1} is a subalgebra of A_n, $\{e_{ij}^{(k)}\}$ cannot contain more than d projections. Thus, we can apply Lemma 4.1.4 again for $\varepsilon = 1$ to obtain a unitary U_2 such that

$$U_2U_1\sigma_{n-1}(e_{ij}^{(k)})U_1^*U_2^* = \sigma_{n-1}(e_{ij}^{(k)}) \qquad \text{for all } i, j, k,$$

and $U_2 - I \in \mathcal{L}^p$. If we define

$$\sigma_n(x) := U_2U_1\sigma_{n-1}(x)U_1^*U_2^* \qquad \text{for all } x \in A,$$

then σ_n must agree with σ_{n-1} on A_{n-1}. And since $U_2 - I \in \mathcal{L}^p$, we have for $x \in A_n$

$$\sigma_n(x) - \sigma(x) = \Big(\sigma_n(x) - U_1\sigma_{n-1}(x)U_1^*\Big) + \Big(U_1\sigma_{n-1}(x)U_1^* - \sigma(x)\Big) \in \mathcal{L}^p,$$

and

$$\sigma_n(x) - \sigma(x) \in \mathcal{K}(P\mathcal{H}) \qquad \text{for all } x \in A.$$

□

4.2 The even case

To understand the even case we will make use of the fact that the even K-homology of an AF-algebra A is determined by its index map. By this we mean that the index map

$$\mathrm{Ind} : K^0(A) \longrightarrow \mathrm{Hom}(K_0(A), \mathbb{Z})$$

is an isomorphism. One way to see this is as follows.

Being a σ-additive cohomology theory on the category of separable nuclear C^*-algebras, K^* has an exact Milnor $\varprojlim^1$-sequence for inductive limits

$$0 \longrightarrow \varprojlim{}^1 K^1(A_n) \longrightarrow K^0(A) \xrightarrow{\gamma} \varprojlim K^0(A_n) \longrightarrow 0.$$

γ is induced by the maps $K^0(A) \xrightarrow{\iota_n^*} K^0(A_n)$ where ι_n denotes the embedding of A_n into A (see [Bla98, Theorem 21.3.2 and Example 21.1.2 (c)] or [Sch84, Theorem 7.1]). Since each A_n is finite-dimensional, we have $K^1(A_n) = 0$ and thus $\varprojlim^1 K^1(A_n) = 0$. It is easy to verify that the index maps

$$\mathrm{Ind} : K^0(A_n) \longrightarrow \mathrm{Hom}(K_0(A_n), \mathbb{Z})$$

are isomorphisms. By their naturality, they induce an isomorphism $\varprojlim \mathrm{Ind}$ between the projective limits $\varprojlim K^0(A_n)$ and

$$\varprojlim \mathrm{Hom}(K_0(A_n), \mathbb{Z}) \cong \mathrm{Hom}(\varinjlim K_0(A_n), \mathbb{Z}) \cong \mathrm{Hom}(K_0(A), \mathbb{Z}).$$

Checking the definitions, one sees that the isomorphism $(\varprojlim \mathrm{Ind}) \circ \gamma$ is the index map for $K^0(A)$.

4.2.1 Theorem. *Every element of $K^0(A)$ can be represented by a 1-summable even Fredholm module over $\bigcup_n A_n$.*

Proof. To simplify notation we shall assume that A is unital and that its unit is shared by each A_n. By Proposition 3.2.1, this means no loss of generality.

Let $\mathscr{F}$ be any even Fredholm module over $A = \overline{\bigcup_n A_n}$ and $\mathrm{Ind}_{\mathscr{F}}$ the associated index map. By the preceding discussion and since $K_0(A) = K_0(\bigcup_n A_n)$ (Theorem 1.5.9), we are done if we can construct a 1-summable even Fredholm module over $\bigcup_n A_n$ which has the same index map on each $A_m \subseteq \bigcup_n A_n$.

To this end, let $\mathcal{H}$ be a separable infinite-dimensional Hilbert space. We will recursively construct $*$-homomorphisms

$$\varphi_n^{\pm} : A_n \longrightarrow \mathcal{B}(\mathcal{H})$$

such that $\varphi_n^{\pm}$ extends $\varphi_{n-1}^{\pm}$. Then

$$\mathscr{F}' := \left(\varinjlim \varphi_n^+ \oplus \varinjlim \varphi_n^-, \, \mathcal{B}(\mathcal{H}) \oplus \mathcal{B}(\mathcal{H}), \, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right) \tag{4.8}$$

will be the Fredholm module we are after.

To make our construction work, we will ensure that the $\varphi_n^{\pm}$ enjoy the following property for each n:

> There is a system of matrix units $\{e_{ij}^{(k)}\}$ for A_n and pairwise orthogonoal projections $P^{(k)}, P^{\pm(k)} \in \mathcal{B}(\mathcal{H})$ such that for all i, k:

$$\begin{aligned}
&\text{(i)} \quad \varphi_n^{\pm}(e_1^{(k)}) = P^{(k)} + P^{\pm(k)} \\
&\text{(ii)} \quad \varphi_n^{+}(e_{i1}^{(k)})P^{(k)} = \varphi_n^{-}(e_{i1}^{(k)})P^{(k)} \\
&\text{(iii)} \quad \mathrm{rk}\, P^{(k)} = \infty, \ \mathrm{rk}\, P^{\pm(k)} < \infty \\
&\text{(iv)} \quad \begin{cases} \mathrm{rk}\, P^{+(k)} = \mathrm{Ind}_{\mathscr{F}}(e_1^{(k)}), \ P^{-(k)} = 0 & \text{if } \mathrm{Ind}_{\mathscr{F}}(e_1^{(k)}) \geq 0 \\ \mathrm{rk}\, P^{-(k)} = |\mathrm{Ind}_{\mathscr{F}}(e_1^{(k)})|, \ P^{+(k)} = 0 & \text{if } \mathrm{Ind}_{\mathscr{F}}(e_1^{(k)}) < 0 \end{cases}
\end{aligned} \tag{4.9}$$

Since each $e_{ij}^{(k)}$ can be written as $e_{i1}^{(k)}e_{j1}^{(k)*}$, properties (i)–(iii) imply that φ_n^+ and φ_n^- differ only by finite-rank operators. Thus, (4.8) well be a 1-summable Fredholm module over $\bigcup_n A_n$. As the $e_1^{(k)}$ generate the K-theory of A_n, property (iv) guarantees that $\mathrm{Ind}_{\mathscr{F}}$ and $\mathrm{Ind}_{\mathscr{F}'}$ agree on $K_0(A_n)$ by definition of the index map:

$$\begin{aligned}&\mathrm{Ind}_{\mathscr{F}'}(e_1^{(k)})\\&\quad= \text{F-Ind}\left[\varphi_n^-(e_1^{(k)})I\varphi_n^+(e_1^{(k)}) : (P^{(k)}+P^{+(k)})\mathcal{H} \longrightarrow (P^{(k)}+P^{-(k)})\mathcal{H}\right]\\&\quad= \mathrm{rk}\,P^{+(k)} - \mathrm{rk}\,P^{-(k)}.\end{aligned}$$

It is clear that we can find such $\varphi_1^\pm$ for A_1, but it might be a little less obvious that $\varphi_{n+1}^\pm$ with property (4.9) can be found which extend $\varphi_n^\pm$. The construction goes as follows.

Let $\varphi_n^\pm$ be given, satisfying property (4.9) with matrix units $\{e_{ij}^{(k)}\}$. Take a system $\{f_{rs}^{(l)}\}$ of compatible matrix units for A_{n+1} (Proposition A.2.4), and assume that the matrix units are numbered in such a way that for each l there is a k with $f_1^{(l)} \le e_1^{(k)}$.

Decompose $P^{(k)}+P^{+(k)}$ and $P^{(k)}+P^{-(k)}$ with projections $Q_r^{(l)}, Q_r^{\pm(l)}$ as

$$P^{(k)}+P^{\pm(k)} = \sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} Q_r^{(l)} + Q_r^{\pm(l)}$$

such that for each l, r:

(a) $Q_r^{(l)} \le P^{(k)}$, $\mathrm{rk}\,Q_r^{(l)} = \infty$

(b) $\begin{cases}\mathrm{rk}\,Q_r^{+(l)} = \mathrm{Ind}_{\mathscr{F}}(f_1^{(l)}),\ Q_r^{-(l)} = 0 & \text{if } \mathrm{Ind}_{\mathscr{F}}(f_1^{(l)}) \ge 0\\ \mathrm{rk}\,Q_r^{-(l)} = |\mathrm{Ind}_{\mathscr{F}}(f_1^{(l)})|,\ Q_r^{+(l)} = 0 & \text{if } \mathrm{Ind}_{\mathscr{F}}(f_1^{(l)}) < 0\end{cases}$

This is only possible since the $P^{(k)}$ are infinite-dimensional by property (iii) and

$$\begin{aligned}\sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} \mathrm{rk}\,Q_r^{+(l)} - \sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} \mathrm{rk}\,Q_r^{-(l)} &= \sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} \mathrm{Ind}_{\mathscr{F}}(f_r^{(l)})\\ &= \mathrm{Ind}_{\mathscr{F}}\Big(\sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} f_r^{(l)}\Big)\\ &= \mathrm{Ind}_{\mathscr{F}}(e_1^{(k)}) = \mathrm{rk}\,P^{+(k)} - \mathrm{rk}\,P^{-(k)}\end{aligned}$$

by property (iv), thus

$$\sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} \mathrm{rk}\,Q_r^{+(l)} - \mathrm{rk}\,P^{+(k)} = \sum_{\substack{(l,r)\\ f_r^{(l)}\le e_1^{(k)}}} \mathrm{rk}\,Q_r^{-(l)} - \mathrm{rk}\,P^{-(k)}.$$

Now define

$$\varphi_{n+1}^{\pm}(f_1^{(l)}) := Q_1^{(l)} + Q_1^{\pm(l)},$$

satisfying property (i). Properties (iii) and (iv) are satisfied by properties (a) and (b).

For each $f_r^{(l)}$ ($r \neq 1$) so that there is some k with $f_r^{(l)} \leq e_1^{(k)}$ choose arbitrary partial isometries

$$\varphi_{n+1}^{\pm}(f_{r1}^{(l)}) : \left(Q_1^{(l)} + Q_1^{\pm(l)}\right)\mathcal{H} \longrightarrow \left(Q_r^{(l)} + Q_r^{\pm(l)}\right)\mathcal{H}$$

such that φ_{n+1}^{+} and φ_{n+1}^{-} restrict to the same map from $Q_1^{(l)}\mathcal{H}$ onto $Q_r^{(l)}\mathcal{H}$. We can choose these partial isometries because all $Q_r^{+(l)}$ (resp. $Q_r^{-(l)}$) have the same rank for fixed l by (b). Note that all $\varphi_{n+1}^{\pm}(f_{r1}^{(l)})$ satisfy property (ii).

If $f_r^{(l)}$ is arbitrary with $f_r^{(l)} \leq e_i^{(k)}$ for some i, k such that $i \neq 1$, define

$$\varphi_{n+1}^{\pm}(f_{r1}^{(l)}) := \varphi_n^{\pm}(e_{i1}^{(k)}) \cdot \varphi_{n+1}^{\pm}(e_{1i}^{(k)} f_{r1}^{(l)}).$$

By construction, if $e_{1i}^{(k)} f_{r1}^{(l)} = f_{s1}^{(l)}$, then $\varphi_{n+1}^{+}(e_{1i}^{(k)} f_{r1}^{(l)})$ and $\varphi_{n+1}^{-}(e_{1i}^{(k)} f_{r1}^{(l)})$ restrict to the same map from $Q_1^{(l)}$ to $Q_s^{(l)}$. $Q_s^{(l)}$ is contained in $P^{(k)}$ by property (a), and $\varphi_n^{+}(e_{i1}^{(k)})$ and $\varphi_n^{-}(e_{i1}^{(k)})$ agree on $P^{(k)}$ by property (ii). Thus, all $\varphi_{n+1}^{\pm}(f_{r1}^{(l)})$ satisfy property (ii).

Extend $\varphi_{n+1}^{\pm}$ to $*$-homomorphisms from A_{n+1} to $\mathcal{B}(\mathcal{H})$. As we have seen, $\varphi_{n+1}^{\pm}$ satisfy all properties of (4.9). So to finish the proof we only have to check that $\varphi_{n+1}^{\pm}$ extend $\varphi_n^{\pm}$.

Fix i, k and assume that $e_{i1}^{(k)} = \sum_{m=1}^{d} f_{s_m r_m}^{(l_m)}$. In particular,

$$e_1^{(k)} = \sum_{m=1}^{d} f_{r_m}^{(l_m)} \quad \text{and} \quad e_{1i}^{(k)} f_{s_m 1}^{(l_m)} = f_{r_m 1}^{(l_m)} \quad \text{for all } 1 \leq m \leq d.$$

Hence,

$$\begin{aligned}
\varphi_{n+1}^{\pm}(e_{i1}^{(k)}) &= \sum_{m=1}^{d} \varphi_{n+1}^{\pm}(f_{s_m r_m}^{(l_m)}) \\
&= \sum_{m=1}^{d} \varphi_{n+1}^{\pm}(f_{s_m 1}^{(l_m)}) \cdot \varphi_{n+1}^{\pm}(f_{r_m 1}^{(l_m)})^* \\
&= \sum_{m=1}^{d} \varphi_n^{\pm}(e_{i1}^{(k)}) \cdot \varphi_{n+1}^{\pm}(f_{r_m 1}^{(l_m)}) \cdot \varphi_{n+1}^{\pm}(f_{r_m 1}^{(l_m)})^* \\
&= \varphi_n^{\pm}(e_{i1}^{(k)}) \sum_{m=1}^{d} \varphi_{n+1}^{\pm}(f_{r_m}^{(l_m)}) \\
&= \varphi_n^{\pm}(e_{i1}^{(k)}) \sum_{\substack{(l,r) \\ f_r^{(l)} \leq e_1^{(k)}}} Q_r^{(l)} + Q_r^{\pm(l)} \\
&= \varphi_n^{\pm}(e_{i1}^{(k)}).
\end{aligned}$$

□

4.2.2 Theorem. *Let $\mathscr{F}$ be an even p-summable Fredholm module over $\bigcup_n A_n$. If $[\mathscr{F}] = \mathbf{0} \in K^0(A)$, then $[\mathscr{F}] = \mathbf{0} \in K^0_p(\bigcup_n A_n)$*

Proof. Because of Theorem 2.4.1, we can assume $\mathscr{F}$ to be of the form

$$\mathscr{F} = \left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\right)$$

with unitary U. Since $[\mathscr{F}] = \mathbf{0} \in K^0(A)$, we know in particular that $\mathrm{Ind}_{\mathscr{F}} = 0$. We can use this to construct unitaries U_n which commute exactly with $\varphi(A_n)$ and which are p-summable perturbations of U as follows.

Let $\{e^{(k)}_{ij}\}$ be a system of matrix units for A_n. Let $\tilde{U}^{(k)}_n := \varphi(e^{(k)}_1)U\varphi(e^{(k)}_1)$. $\tilde{U}^{(k)}_n$ is, up to p-summable operators, unitary as an operator on $\varphi(e^{(k)}_1)\mathcal{H}$. By definition of the index map, $\mathrm{Ind}_{\mathscr{F}}(e^{(k)}_1) = 0$ means that its Fredholm index is 0. Thus, we can add a finite-rank operator on $\varphi(e^{(k)}_1)\mathcal{H}$ to make $\tilde{U}^{(k)}_n$ invertible. After that, we can use Lemma 4.1.3 to find a unitary $\hat{U}^{(k)}_n$ on $\varphi(e^{(k)}_1)\mathcal{H}$ such that $\tilde{U}^{(k)}_n - \hat{U}^{(k)}_n \in \mathcal{L}^p$. Since U is unitary, $\mathrm{Ind}_{\mathscr{F}} = 0$ also implies that the compression of U to $(I - \sum_{k,i}\varphi(e^{(k)}_i))\mathcal{H}$ must have index 0 as well. Hence, we can find a unitary $\hat{U}_n$ on this space which agrees with $(I - \sum_{k,i}\varphi(e^{(k)}_i))U(I - \sum_{k,i}\varphi(e^{(k)}_i))$ up to p-summable operators.

Now just define

$$U_n := \sum_{k,i} \varphi(e^{(k)}_{i1})\hat{U}^{(k)}_n\varphi(e^{(k)}_{1i}) + \hat{U}_n.$$

One easily checks that U_n is a unitary which commutes exactly with $\varphi(A_n)$ such that $U - U_n \in \mathcal{L}^p$.

Let $U_0 := U$ and note that $[\varphi(\bigcup_i A_i), U_{n+1}U^*_n] \subseteq \mathcal{L}^p$ for each $n \in \mathbb{N}_0$. Since $U_{n+1}U^*_n - I \in \mathcal{L}^p$, there is a branch of the complex logarithm which is holomorphic on a neighbourhood of $\mathrm{sp}(U_{n+1}U^*_n)$. Let

$$X_n := \log(U_{n+1}U^*_n).$$

Using Corollary 1.4.15, we conclude

$$\Big[\varphi(\bigcup_i A_i), X_n\Big] \subseteq \mathcal{L}^p, \quad \|X_n\| \le 2\pi, \quad \text{and} \quad e^{X_n}U_n = U_{n+1}.$$

Now define

$$\mathscr{F}' := \left(\bigoplus_{n=0}^{\infty}(\varphi \oplus \varphi),\, \bigoplus_{n=0}^{\infty}(\mathcal{H} \oplus \mathcal{H}),\, \bigoplus_{n=0}^{\infty}\begin{bmatrix} 0 & U^*_n \\ U_n & 0 \end{bmatrix}\right).$$

By construction, each U_n commutes exactly with $\varphi(A_i)$ for $n \ge i$. Therefore, $\mathscr{F}'$ is a p-summable even Fredholm module over $\bigcup_n A_n$.

Further, since the norms of the X_n are bounded, the path

$$F'_t := \bigoplus_{n=0}^{\infty} \begin{bmatrix} 0 & U_n^* e^{-tX_n} \\ e^{tX_n} U_n & 0 \end{bmatrix}$$

is continuous. Moreover, $\varphi(A_i)$ commutes exactly with every X_n for $n \geq i$. Thus, F'_t defines a p-summable operator homotopy between $\mathscr{F}'$ and

$$\mathscr{F}'' := \left(\bigoplus_{n=0}^{\infty} (\varphi \oplus \varphi), \bigoplus_{n=0}^{\infty} (\mathcal{H} \oplus \mathcal{H}), \bigoplus_{n=0}^{\infty} \begin{bmatrix} 0 & U_{n+1}^* \\ U_{n+1} & 0 \end{bmatrix} \right).$$

But $\mathscr{F} \oplus \mathscr{F}''$ is unitarily equivalent to $\mathscr{F}'$, so

$$\mathscr{F} \oplus \mathscr{F}' \sim_{oh,p} \mathscr{F} \oplus \mathscr{F}'' \sim_u \mathscr{F}'.$$

Since $K_p^0(\bigcup_n A_n)$ is a group, we conclude that $[\mathscr{F}] = \mathbf{0} \in K_p^0(\bigcup_n A_n)$. □

4.2.3 Corollary. *The comparison maps*

$$K_p^*(\bigcup_n A_n) \longrightarrow K^*(A) \quad \textit{and} \quad K_{fin}^*(\bigcup_n A_n) \longrightarrow K^*(A)$$

are isomorphisms.

4.2.4 Remarks.

1. We have actually proven that every class of $K^*(A)$ can even be represented by a module $(\varphi, \mathcal{H}, F)$ over $\bigcup_n A_n$ satisfying

$$F^2 - I \in \mathcal{R}(\mathcal{H}) \quad \text{and} \quad [F, \varphi(x)] \in \mathcal{R}(\mathcal{H}) \quad \text{for all } x \in \bigcup_n A_n.$$

2. In the proof of Theorem 4.2.2 we have only used that $[\mathscr{F}] \in K^0(A)$ implies $\mathrm{Ind}_{\mathscr{F}} = 0$. Thus, replacing $\mathcal{L}^p(\mathcal{H})$ everywhere by $\mathcal{K}(\mathcal{H})$ yields a proof that the even K-homology of an AF-algebra is determined by its index map without referring to the Milnor $\varprojlim^1$-sequence.

Chapter 5

Direct Sums

In this chapter we study the finitely summable K-homology of infinite direct sums of topological $*$-algebras.

The first section concerns direct sums of countably many copies of $\mathbb{C}$. As we have seen in Chapter 4, every K-homology class of the direct sum $\bigoplus_{\mathbb{N}} \mathbb{C}$ can be represented by a finitely summable Fredholm module over the algebraic direct sum $\bigoplus_{\mathbb{N}}^{alg} \mathbb{C}$. We show that there is no larger subalgebra of $\bigoplus_{\mathbb{N}} \mathbb{C}$ which still has this property.

In the second section we give an example of a family of pre-C^*-algebras $\mathcal{A}_i$ such that every K-homology class of the completions A_i of $\mathcal{A}_i$ can be represented by a finitely summable module over $\mathcal{A}_i$, but not every K-homology class of $\bigoplus_{\mathbb{N}} A_i$ can be represented by a finitely summable module over $\bigoplus_{i\in\mathbb{N}}^{alg} \mathcal{A}_i$. In particular, K^*_{fin} is not σ-additive in general, even for algebraic direct sums.

5.1 Direct sums of $\mathbb{C}$

It is a well-known fact that K-Homology is σ-additive [Bla98, Theorem 19.7.1]. This means that for countably many separable C^*-algebras A_i ($i \in \mathbb{N}$) the K-homology of their direct sum is given by

$$K^*\Big(\bigoplus_{i\in\mathbb{N}} A_i\Big) = \prod_{i\in\mathbb{N}} K^*(A_i).$$

The isomorphism is implemented by the map

$$\begin{aligned} \Psi : K^*\Big(\bigoplus_{i\in\mathbb{N}} A_i\Big) &\longrightarrow \prod_{i\in\mathbb{N}} K^*(A_i) \\ \mathbf{x} &\longmapsto (\iota_i^*\mathbf{x})_{i\in\mathbb{N}} \end{aligned}$$

where ι_i is the embedding of A_i into the direct sum. In particular,

$$K^0\Big(\bigoplus_{\mathbb{N}} \mathbb{C}\Big) = \prod_{\mathbb{N}} \mathbb{Z} \quad \text{and} \quad K^1\Big(\bigoplus_{\mathbb{N}} \mathbb{C}\Big) = 0.$$

Denote by $\bigoplus_{\mathbb{N}}^{alg}\mathbb{C}$ the algebraic direct sum of countably many copies of $\mathbb{C}$. Since $\bigoplus_{\mathbb{N}}\mathbb{C}$ is an AF-algebra, we know by the results of Chapter 4 that the comparison maps

$$K^*_{fin}\Big(\bigoplus_{\mathbb{N}}^{alg}\mathbb{C}\Big) \longrightarrow K^*\Big(\bigoplus_{\mathbb{N}}\mathbb{C}\Big)$$

are isomorphisms. In particular, K^*_{fin} is σ-additive for this direct sum.

One might wish to extend this result to slightly larger smooth subalgebras of $\bigoplus_{\mathbb{N}}\mathbb{C}$, for example the algebra of rapidly decreasing sequences. However, it turns out that $\bigoplus_{\mathbb{N}}^{alg}\mathbb{C}$ is in fact the largest subalgebra of $\bigoplus_{\mathbb{N}}\mathbb{C}$ for which we can maintain σ-additivity:

5.1.1 Theorem. *Let $\mathcal{A}$ be a pre-C^*-subalgebra of $\bigoplus_{\mathbb{N}}\mathbb{C}$ which is strictly larger than $\bigoplus_{\mathbb{N}}^{alg}\mathbb{C}$. Then the comparison map*

$$K^0_{fin}(\mathcal{A}) \longrightarrow K^0\Big(\bigoplus_{\mathbb{N}}\mathbb{C}\Big)$$

is not surjective.

Proof. Since $\mathcal{A}$ is strictly larger than $\bigoplus_{\mathbb{N}}^{alg}\mathbb{C}$, there is an $x = (x_i)_{i\in\mathbb{N}} \in \mathcal{A}$ such that $x_i \neq 0$ for infinitely many i. Let $\{\lambda_n\}$ be the set $\{|x_i|\} \setminus \{0\}$ ordered as a strictly decreasing sequence, and let

$$k_i := \begin{cases} 0 & x_i = 0 \\ \left\lceil \left(\frac{n}{\lambda_n - \lambda_{n+1}}\right)^n \right\rceil & |x_i| = \lambda_n\,. \end{cases}$$

We are going to show that there cannot be a finitely summable even Fredholm module over $\mathcal{A}$ which represents the class $\mathbf{x} \in K^0\left(\bigoplus_{\mathbb{N}}\mathbb{C}\right)$ given by

$$\mathrm{Ind}_{\iota_i^*\mathbf{x}}(1) = k_i \qquad \text{for all } i \in \mathbb{N}$$

where $\iota_i : \mathbb{C} \longrightarrow \bigoplus_{\mathbb{N}}\mathbb{C}$ denotes the embedding of $\mathbb{C}$ into $\bigoplus_{\mathbb{N}}\mathbb{C}$ as its ith summand.

Assume that such a p-summable module exists. By Theorem 2.4.1, we can assume $\mathscr{F}$ to be of the form

$$\mathscr{F} = \left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\right)$$

with unitary U. Denote the element of $\bigoplus_{\mathbb{N}}\mathbb{C}$ which is zero everywhere except for the ith summand, where it is 1, by e_i. Thus, $x = \sum_{i=1}^{\infty} x_i e_i$. Finally, define p_n $(n \in \mathbb{N})$ to be the projections

$$p_n := \sum_{|x_i| \geq \lambda_n} e_i.$$

By assumption we know for

$$T_n := \varphi(p_n)U^*\Big(\sum_{i=1}^{\infty} x_i\varphi(e_i)\Big)U\varphi(p_n) - \sum_{|x_i|\geq\lambda_n} x_i\varphi(e_i)$$

that

$$\begin{aligned}\|T_n\|_p &= \|\varphi(p_n)\,(U^*\varphi(x)U - \varphi(x))\,\varphi(p_n)\|_p \\ &\leq \|(U^*\varphi(x)U - \varphi(x))\|_p < \infty\,.\end{aligned}$$

Therefore, to prove our claim it suffices to show that $\|T_n\|_p$ must get arbitrarily large when n tends to infinity.

To this end, first note that by the choice of $\mathbf{x}$,

$$\begin{aligned}\mathrm{Ind}_{\mathbf{x}}(p_n) = \sum_{|x_i|\geq\lambda_n} \mathrm{Ind}_{\mathbf{x}}(e_i) &= \sum_{|x_i|\geq\lambda_n} k_i \\ &\geq \left\lceil \Big(\frac{n}{\lambda_n - \lambda_{n+1}}\Big)^n \right\rceil.\end{aligned}$$

Remember that $\mathrm{Ind}_{\mathbf{x}}(p_n)$ is given by

$$\mathrm{Ind}_{\mathbf{x}}(p_n) = \text{F-Ind}\Big[\,\varphi(p_n)U\varphi(p_n) : \varphi(p_n)\mathcal{H} \longrightarrow \varphi(p_n)\mathcal{H}\,\Big],$$

so $\varphi(p_n)U\varphi(p_n)$ must have an at least $\lceil(n/(\lambda_n - \lambda_{n+1}))^n\rceil$-dimensional kernel when regarded as an operator on $\varphi(p_n)\mathcal{H}$. If we denote the projection onto this kernel by K_n, then

$$\begin{aligned}&\Big\|\varphi(p_n)U^*\Big(\sum_{i=1}^{\infty} x_i\varphi(e_i)\Big)U\varphi(p_n)K_n\Big\| \\ &\qquad\qquad = \Big\|\varphi(p_n)U^*\Big(\sum_{|x_i|<\lambda_n} x_i\varphi(e_i)\Big)U\varphi(p_n)K_n\Big\| \\ &\qquad\qquad \leq \Big\|\sum_{|x_i|<\lambda_n} x_i\varphi(e_i)\Big\| = \lambda_{n+1}.\end{aligned}$$

On the other hand, $\sum_{|x_i|\geq\lambda_n} x_i\varphi(e_i)$ is bounded from below on $K_n\mathcal{H}$ by λ_n. Therefore, for any unit vector $\xi \in K_n\mathcal{H}$, we find that

$$\begin{aligned}\|T_n\xi\| &\geq \Big|\,\Big\|\sum_{|x_i|\geq\lambda_n} x_i\varphi(e_i)\xi\Big\| - \Big\|\varphi(p_n)U^*\Big(\sum_{i=1}^{\infty} x_i\varphi(e_i)\Big)U\varphi(p_n)K_n\xi\Big\|\,\Big| \\ &\geq \lambda_n - \lambda_{n+1}.\end{aligned}$$

Hence, T_n is bounded from below on $K_n\mathcal{H}$ by $\lambda_n - \lambda_{n+1}$. We conclude that for $n \geq p$ we have

$$\|T_n\|_p \geq \|T_nK_n\|_p \geq (\lambda_n - \lambda_{n+1}) \cdot (\dim K_n\mathcal{H})^{\frac{1}{p}} \geq n.$$

□

5.1.2 Corollary. *If $\mathcal{A}$ is a pre-C^*-subalgebra of $\bigoplus_{\mathbb{N}} \mathbb{C}$ containing $\bigoplus_{\mathbb{N}}^{alg} \mathbb{C}$ which is at the same time a Fréchet $*$-algebra, then the comparison map*

$$K^0_{fin}(\mathcal{A}) \longrightarrow K^0\Big(\bigoplus_{\mathbb{N}} \mathbb{C}\Big)$$

is not surjective.

Proof. Let $\mathcal{A}$ be a Fréchet $*$-algebra containing $\bigoplus_{\mathbb{N}}^{alg} \mathbb{C}$ and $|\cdot|_n$ ($n \in \mathbb{N}$) semi-norms generating the topology of $\mathcal{A}$. Let $\mu_i := 1 + \max_{1 \le n \le i} |e_i|_n$.

It is easy to see that the sum $\sum_{i=1}^{\infty} 2^{-i} \mu_i^{-1} e_i$ converges to some element x in $\mathcal{A}$. By the continuity of the multiplication in $\mathcal{A}$, we have

$$e_j x = 2^{-j} \mu_j^{-1} e_j \neq 0 \qquad \text{for all } j \in \mathbb{N}.$$

Thus, x cannot lie in $\bigoplus_{\mathbb{N}}^{alg} \mathbb{C}$, and $\mathcal{A}$ must be strictly larger than $\bigoplus_{\mathbb{N}}^{alg} \mathbb{C}$. □

5.2 Algebraic direct sums

In the last section we have seen that K^*_{fin} is not very well-behaved with respect to completed direct sums. In view of the results of Chapter 4, one might still hope that finitely summable K-homology is σ-additive for algebraic direct sums.

In the case of K_p^*, it is at least clear that the map

$$\begin{aligned} \Psi : K_p^*\Big(\bigoplus_{i\in\mathbb{N}}^{alg} \mathcal{A}_i\Big) &\longrightarrow \prod_{i\in\mathbb{N}} K_p^*(\mathcal{A}_i) \\ \mathbf{x} &\longmapsto (\iota_i^* \mathbf{x})_{i\in\mathbb{N}}, \end{aligned}$$

is surjective: if $\mathscr{F}_i = (\varphi_i, \mathcal{H}_i, F_i)$ are p-summable Fredholm modules over $\mathcal{A}_i$ with $\|F_i\| \le 1$ ($i \in \mathbb{N}$), then

$$\mathscr{F} := \Big(\bigoplus_{i\in\mathbb{N}} \varphi_i, \bigoplus_{i\in\mathbb{N}} \mathcal{H}_i, \bigoplus_{i\in\mathbb{N}} F_i\Big) \tag{5.1}$$

is a p-summable module over $\bigoplus_{i\in\mathbb{N}}^{alg} \mathcal{A}_i$ which satisfies $\Psi([\mathscr{F}]) = ([\mathscr{F}_i])$.[†]

[†]The problem with proving the injectivity of Ψ is the following: if $(\varphi_i, \mathcal{H}_i, F_t^{(i)})$ are operator homotopies of p-summable modules over $\mathcal{A}_i$, then one has to show that $(\bigoplus \varphi_i, \bigoplus \mathcal{H}_i, \bigoplus F_0^{(i)})$ and $(\bigoplus \varphi_i, \bigoplus \mathcal{H}_i, \bigoplus F_1^{(i)})$ represent the same class of $K_p^*(\bigoplus^{alg} \mathcal{A}_i)$. However, the path $\bigoplus F_t^{(i)}$ is not necessarily continuous in the norm topology of $\mathcal{B}(\bigoplus \mathcal{H}_i)$.

For the ordinary K-homology of direct sums of separable C^*-algebras, this is no obstruction: if we assume the $F_t^{(i)}$ to be uniformly bounded in norm, then $\bigoplus F_t^{(i)}$ is at least strong-$*$-continuous. One can show using the Kasparov product that Fredholm modules which are connected by a strong-$*$-continuous operator homotopy represent the same K-homology class [Bla98, Definition 17.2.2 and Theorem 18.5.3].

However, this argument does not work for K^*_{fin} since the modules $\mathscr{F}_i$ might only be p-summable with p getting arbitrarily large for increasing i. Then the module (5.1) would not be p-summable for any p.

In fact, we can give a simple counterexample to the σ-additivity of K^*_{fin} using the following classical observation by Douglas and Voiculescu:

5.2.1 Theorem. *For $n > 1$ there is no $(2n-2)$-summable Fredholm module over $C^\infty(S^{2n-1})$ which represents a non-zero class in $K^1(C(S^{2n-1}))$.*

Proof. [DV81]. □

More precisely, the authors prove that there are no non-trivial $(n-1)$-smooth extension of the $(2n-1)$-sphere by $\mathcal{K}(\mathcal{H})$. An extension of the form

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow E \longrightarrow C(S^{2n-1}) \longrightarrow 0$$

with $E \subseteq \mathcal{B}(\mathcal{H})$ is called p-smooth if the complex coordinate functions z_i $(i = 1, \dots, n)$ of $S^{2n-1} \subseteq \mathbb{C}^n$ admit lifts $T_i \in E$ such that

$$[T_i, T_j] \in \mathcal{L}^p \quad \text{and} \quad [T_i^*, T_j] \in \mathcal{L}^p \qquad \text{for all } 1 \leq i, j \leq n.$$

An extension is called trivial if it admits a split which is a $*$-homomorphism.

Remembering how to construct extensions from Fredholm modules as we have discussed in the proof of Theorem 4.1.1, it is easy to see that every $(2n-2)$-summable module over $C^\infty(S^{2n-1})$ defines an $(n-1)$-smooth extension of $C(S^{2n-1})$.† As Lemma 3.5.2 implies that modules which induce trivial extensions are zero in K-homology, Theorem 5.2.1 follows.

Also note that by Bott periodicity of K-homology we have

$$K^1(C(S^{2n-1})) \cong K^1(C(S^1)) \cong \mathbb{Z}.$$

In particular, $K^1(C(S^{2n-1}))$ does not vanish for every $n \in \mathbb{N}$. Moreover, every class of $K^1(C(S^{2n-1}))$ can be represented by a finitely summable module over $C^\infty(S^{2n-1})$ as we demonstrate in Section 6.1.

With the help of these observations, we can construct a counterexample to the σ-additivity of K^*_{fin} as follows.

5.2.2 Proposition. *Not every class of $K^1(\bigoplus_{n\in\mathbb{N}}^\infty C(S^{2n-1}))$ can be represented by a finitely summable Fredholm module over the algebraic direct sum $\bigoplus_{n\in\mathbb{N}}^{alg} C^\infty(S^{2n-1})$.*

Proof. By the σ-additivity of K^*, there is a class $\mathbf{x} \in K^1(\bigoplus_{n\in\mathbb{N}} C(S^{2n-1}))$ such that $\iota_n^*\mathbf{x}$ is not zero in $K^1(C(S^{2n-1}))$ for every $n \in \mathbb{N}$. If there was a p-summable module $\mathscr{F}$ over $\bigoplus_{n\in\mathbb{N}}^{alg} C^\infty(S^{2n-1})$ which represents $\mathbf{x}$, then $\iota_n^*\mathscr{F}$ would be a p-summable module over $C^\infty(S^{2n-1})$ representing $\iota_n^*\mathbf{x}$ for each $n \in \mathbb{N}$. This is impossible for $2n-2 \geq p$. □

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†If $(\varphi, \mathcal{H}, F) \in \mathbb{F}^1_{2n-2}(C^\infty(S^{2n-1}))$, $F^* = F$, $F^2 = I$, and $P := 1/2(F + I)$, then $[P\varphi(z_i)P, P\varphi(z_j)P] = -P\varphi(z_i)P^\perp\varphi(z_j)P + P\varphi(z_j)P^\perp\varphi(z_i)P \in \mathcal{L}^{2n-2} \cdot \mathcal{L}^{2n-2} \subseteq \mathcal{L}^{n-1}$ since z_i and z_j commute.

Chapter 6

Manifolds

It is common folklore that for a closed manifold M every class of $K^*(C(M))$ can be represented by a finitely summable Fredholm module over $C^\infty(M)$. The reason is that every class of $K^*(C(A))$ is represented by an elliptic pseudo-differential operator over M, and such operators define finitely summable Fredholm modules over $C^\infty(M)$. We explain this in more detail in the first section.

In the second section we discuss the injectivity of the comparison map between $K^*_{fin}(C^\infty(M))$ and $K^*(C(M))$. Unfortunately, we can neither prove injectivity nor give a counterexample, but we hope to shed light on some of the difficulties that arise.

6.1 Representability by finitely summable modules

Let M be a smooth, closed (compact, without boundary) manifold of dimension n. Since $C^\infty(M)$ is a pre-C^*-subalgebra of $C(M)$ (Corollary 1.4.12), we can compare $K^*_{fin}(C^\infty(M))$ to $K^*(C(M))$. We want to prove:

6.1.1 Theorem. *Every K-homology class of $C(M)$ can be represented by a finitely summable Fredholm module over $C^\infty(M)$.*

The proof of this theorem requires some knowledge of pseudo-differential operators and topological K-theory. We cannot develop all required technical machinery here, but we will try to make it clear how the finite summability comes into play.

If $U \subseteq \mathbb{R}^n$ is open, a differential operator of order $m \in \mathbb{N}$ between $C_0^\infty(U)^k$ and $C_0^\infty(U)^l$ is an operator of the form

$$\begin{aligned} P : (C_0^\infty(U))^k &\longrightarrow (C_0^\infty(U))^l \\ f &\longmapsto \sum_{|\alpha| \le m} a_\alpha D^\alpha f. \end{aligned} \tag{6.1}$$

Here, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ denotes a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, and $D^\alpha = (-i\partial_1)^{\alpha_1} \ldots (-i\partial_n)^{\alpha_n}$. The a_α are smooth $M_{l,k}(\mathbb{C})$ valued functions on U. We call the matrix valued polynomial

$$\sigma_P^{tot}(x,\xi) := \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha$$

the total symbol of P.

A differential operator between a k-dimensional vector bundle V and an l-dimensional vector bundle W over M is an operator

$$P : C^\infty(M,V) \longrightarrow C^\infty(M,W)$$

which is local (i.e. $(Pf)(x)$ only depends on f on an arbitrary small neighbourhood of x) and which in local coordinates is of the form (6.1).

The total symbol of a differential operator does not transform nicely under coordinate changes. However, the principal symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

transforms like a covector. For differential operators on manifolds the principal symbol can therefore be defined invariantly as an element

$$\sigma_P \in \mathrm{Hom}(\pi^* V, \pi^* W)$$

where $\pi : T^*M \longrightarrow M$ denotes the base space projection of the cotangent bundle T^*M of M onto M.

Returning to the case $U \subseteq \mathbb{R}^n$ and by applying the Fourier transform to (6.1), we obtain

$$Pf(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma_P^{tot}(x,\xi) \hat{f}(\xi)\, \mathrm{d}\xi. \tag{6.2}$$

A pseudo-differential operator on U is an operator of the same form (6.2), but we allow a larger class of symbols, not only polynomials in ξ. There are quite a few symbol classes to choose from, which lead to slightly different theories, but they all share important properties similar to differential operators. One possible choice is the class of symbols σ for which each matrix entry σ_{ij} belongs to the Hörmander classes $S^m_{1,0}(U \times \mathbb{R}^n)$ of smooth functions on $U \times \mathbb{R}^n$ satisfying

$$|\partial_\xi^\alpha \partial_x^\beta \sigma_{ij}(x,\xi)| \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|} \qquad \text{for all } x \in K,\ \xi \in \mathbb{R}^n$$

where $K \subseteq U$ is compact, α, β are arbitrary multi-indices and $C_{\alpha,\beta,K}$ are constants depending only on α, β and K. The order m can be any real number.

Like differential operators, pseudo-differential operators on manifolds can be defined as operators which are of the form (6.2) when restricted to a coordinate chart. The condition of being local is replaced by the notion of pseudo-locality. Again, a principal symbol can be defined invariantly, but only up to lower order terms, as an element of

$$S^m_{1,0}(\mathrm{Hom}(\pi^*V, \pi^*W)) \,/\, S^{m-1}_{1,0}(\mathrm{Hom}(\pi^*V, \pi^*W)).$$

Further details on pseudo-differential operators can be found in the monograph [Shu01], among many others.

* * *

The proof of Theorem 6.1.1 can be broken down into two parts: first, every elliptic pseudo-differential operator on M defines a finitely summable Fredholm module over $C^\infty(M)$. Second, every K-homology class of M can be represented by such an operator.

For the first part, we need to introduce the Sobolev spaces

$$H^s(M, V) \subseteq L^2(M, V)$$

for vector bundles V over M (to define $L^2(M, V)$ we can equip M and V with arbitrary metrics).

We begin with the special case of L^2-functions on the n-torus $\mathbb{T}^n$ with values in $\mathbb{R}^k$. By taking the L^2-scalar product, these functions define $\mathbb{R}^k$-valued distributions on $\mathbb{T}^n$. The sth Sobolev space $H^s(\mathbb{T}^n, \mathbb{R}^k)$ is then the subspace of those functions in $L^2(\mathbb{T}^n, \mathbb{R}^k)$ whose sth distributional derivatives in all directions are again represented by L^2-functions. With the scalar product

$$(f \mid g)_s := \sum_{|\alpha| \leq s} \int_{\mathbb{T}^n} (\partial^\alpha f \mid \partial^\alpha g),$$

where the sum is taken over all multi-indices α of length less or equal than s, $H^s(\mathbb{T}^n, \mathbb{R}^k)$ becomes a Hilbert space.

Let now V be a k-dimensional vector bundle over M. To define $H^s(M, V)$, let $\{U_i\}$ be a finite covering of M by coordinate patches trivializing V. Embedding the U_i into $\mathbb{T}^n$, we obtain trivializations $\varphi_i : V|_{U_i} \longrightarrow \mathbb{T}^n \times \mathbb{R}^k$. Let $\{u_i\} \in C^\infty(M)$ be a partition of unity subordinate to $\{U_i\}$. If $f \in C^\infty(M, V)$ is a smooth section of V, then we can push forward $u_i f$ via φ_i to a function in $C^\infty(\mathbb{T}^n, \mathbb{R}^k)$ by extending it with 0 outside the range of φ_i. Define

$$(f \mid g)_s := \sum_i (\varphi_{i*}(u_i f) \mid \varphi_{i*}(u_i g))_s \qquad \text{for } f, g \in C^\infty(M, V),$$

and let $H^s(M, V)$ be the completion of $C^\infty(M, V)$ with respect to this scalar product.

We have made various choices in the definition of $H^s(M, V)$. Different choices of trivializations, embeddings into the torus and of partitions of unity lead to different scalar products on $H^s(M, V)$, but all these scalar products yield equivalent norms. In particular, $H^0(M, V)$ can be identified with $L^2(M, V)$.

Rellich's Lemma states that the embedding of $H^1(M,V)$ into $L^2(M,V)$ is compact. The crucial point for proving Theorem 6.1.1 is that this embedding is even finitely summable, as it was already observed by Weyl [Wey12].

6.1.2 Theorem (Finitely summable Rellich Lemma)**.** *If M is a closed manifold of dimension n and V is a vector bundle over M, then the inclusion*

$$\iota : H^1(M,V) \longrightarrow L^2(M,V)$$

is p-summable for every $p > n$.

Proof. If V is of dimension k, denote by $\tilde{\iota}$ the embedding of $H^1(\mathbb{T}^n, \mathbb{R}^k)$ into $L^2(\mathbb{T}^n, \mathbb{R}^k)$. Let $\{\varphi_i\}$ be trivializations and $\{u_i\}$ a partition of unity as above. Then we can write

$$\iota(f) = \sum_i \varphi_i^* \circ \tilde{\iota} \circ \varphi_{i*}(u_i f) \qquad \text{for all } f \in H^1(M,V)$$

with continuous maps

$$\varphi_{i*}(u_i \cdot\ ) : H^1(M,V) \longrightarrow H^1(\mathbb{T}^n, \mathbb{R}^k) \ \text{ and } \ \varphi_i^* : L^2(\mathbb{T}^n, \mathbb{R}^k) \longrightarrow L^2(M,V).$$

Thus, it suffices to prove that $\tilde{\iota}$ is p-summable for each $p > n$. Moreover, we can obviously assume that $k = 1$.

Remember that an orthonormal basis of $L^2(\mathbb{T}^n)$ (with respect to the Haar measure) is given by the functions

$$e_{\mathbf{k}}(z_1, \ldots, z_n) := z_1^{k_1} \ldots z_n^{k_n} \qquad \text{for } z_1, \ldots, z_n \in \mathbb{T}^1 \subseteq \mathbb{C}$$

with $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Moreover,

$$\partial_j e_{\mathbf{k}} = i k_j e_{\mathbf{k}}.$$

Thus, the H^1-scalar product between these basis vectors is given by

$$(e_{\mathbf{k}} \mid e_{\mathbf{r}})_1 = (e_{\mathbf{k}} \mid e_{\mathbf{r}}) + \sum_{j=1}^n (ik_j e_{\mathbf{k}} \mid ir_j e_{\mathbf{r}}) = \delta_{\mathbf{kr}} \Big(1 + \sum_{j=1}^n k_j^2\Big) e_{\mathbf{k}}.$$

This means that the functions

$$f_{\mathbf{k}} := \Big(1 + \sum_{j=1}^n k_j^2\Big)^{-1/2} e_{\mathbf{k}}$$

form an orthonormal basis of $H^1(\mathbb{T}^n)$. As $\tilde{\iota}$ sends $f_{\mathbf{k}}$ to $(1 + \sum_{j=1}^n k_j^2)^{-1/2} e_{\mathbf{k}}$, it has (up to order) the characteristic values

$$\mu_{\mathbf{k}} = \Big(1 + \sum_{j=1}^n k_j^2\Big)^{-1/2} \le \Big(n \sqrt[n]{(1/n + k_1^2) \ldots (1/n + k_n^2)}\Big)^{-1/2}.$$

Thus,

$$\sum_{\mathbf{k} \in \mathbb{Z}^n} |\mu_{\mathbf{k}}|^p \le n^{-\frac{p}{2}} \Big(\sum_{k_1 \in \mathbb{Z}} (1/n + k_1^2)^{-\frac{p}{2n}}\Big) \ldots \Big(\sum_{k_n \in \mathbb{Z}} (1/n + k_n^2)^{-\frac{p}{2n}}\Big) < \infty.$$

□

One reason for the importance of the Sobolev spaces is that they are natural domains for differential operators. In fact, if $P : C^\infty(M,V) \longrightarrow C^\infty(M,W)$ is a differential operator of order m between vector bundles V and W over M, then P extends to a bounded operator

$$P : H^{s+m}(M,V) \longrightarrow H^s(M,W),$$

as it is immediate from the definitions. This is also easily seen to be true for pseudo-differential operators.

Let $P : C^\infty(M,V) \longrightarrow C^\infty(M,W)$ be a pseudo-differential operator of order m. P is called elliptic if and only if it has a pseudo-differential operator pseudo-inverse $Q : C^\infty(M,W) \longrightarrow C^\infty(M,V)$ of order $-m$ such that $QP - I$ and $PQ - I$ are smoothing operators, i.e. pseudo-differential operators of order d for any $d \in \mathbb{R}$.

In particular, if P is of order 0, so is Q and both extend to bounded operators between $L^2(M,V)$ and $L^2(M,W)$. Let φ_V, φ_W be the representations of $C^\infty(M)$ on $L^2(M,V)$ and $L^2(M,W)$ by multiplication. We want to prove that

$$\mathscr{F} := \left(\varphi_V \oplus \varphi_W,\ L^2(M,V) \oplus L^2(M,W),\ F := \begin{bmatrix} & Q \\ P & \end{bmatrix} \right)$$

is an $(n+1)$-summable even Fredholm module over $C^\infty(M)$.

First, note that as a smoothing operator, $QP - I$ is in particular a pseudo-differential operator on V of order -1. Thus, it maps $L^2(M,V)$ continuously into $H^1(M,V)$. As an operator on $L^2(M,V)$ it therefore factorizes as

$$QP - I : L^2(M,V) \longrightarrow H^1(M,V) \xrightarrow{\iota} L^2(M,V),$$

making it a $(n+1)$-summable operator by Rellich's Lemma. The same argument holds for $PQ - I$, so $F^2 - I \in \mathcal{L}^{n+1}$.

Next, note that F and $\varphi_V(f) \oplus \varphi_W(f)$ are pseudo-differential operators of order 0 on $L^2(M,V) \oplus L^2(M,W) \cong L^2(M, V \oplus W)$. Since the commutator of two pseudo-differential operators of order m is an operator of order $m-1$, $[\varphi_V(f) \oplus \varphi_W(f), F]$ is a pseudo-differential operator of order -1. By the same argument as above, the commutator must therefore be $(n+1)$-summable.

Thus, we have assigned to each elliptic pseudo-differential operator of order 0 over M a class in $K^0_{n+1}(C^\infty(M))$.

If $P : C^\infty(M,V) \longrightarrow C^\infty(M,V)$, is a (formally) self-adjoint elliptic operator of order 0, then one easily sees that its class in $K^0_{n+1}(C^\infty(M))$ must be zero. However, we can assign to P a class in $K^1_{n+1}(C^\infty(M))$ as follows.

We have already seen that P extends to a self-adjoint bounded operator on $L^2(M,V)$ that, up to $(n+1)$-summable operators, is invertible and commutes with φ. In particular, 0 is an isolated point in $\mathrm{sp}(P) \subseteq \mathbb{R}$.

Thus, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by

$$f(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

is holomorphic on a neighbourhood of $\mathrm{sp}(P)$, and Corollary 1.4.15 implies that $F := f(P)$ commutes with φ up to $(n+1)$-summable operators as well. Hence,

$$(\varphi, L^2(M, V), F)$$

is a $(n+1)$-summable module over $C^\infty(M)$.

These constructions can also be extended to operators of higher order: if P is a classical elliptic pseudo-differential operator of order greater than 0, then we can consider

$$\frac{P}{\sqrt{I + P^*P}},$$

which is by [See67] an elliptic operator of order 0, and construct an $(n+1)$-summable module from it as above.

To prove that every class of $K^*(C^\infty(M))$ arises in this way, first note that the compactly supported K-theory of a locally compact space X can be described in terms of homotopy classes of complexes

$$0 \longrightarrow V_0 \xrightarrow{\alpha_0} V_1 \xrightarrow{\alpha_1} \dots \xrightarrow{\alpha_{n-1}} V_n \longrightarrow 0,$$

where the V_i are complex vector bundles over X and the α_i are vector bundle maps satisfying $\alpha_{i+1} \circ \alpha_i = 0$ such that the sequence is exact outside a compact subset of X [Seg68]. In [AS68] the authors show that in the case of $K_0(C_0(T^*M))$ one can restrict to complexes of the form

$$0 \longrightarrow \pi^*V \xrightarrow{\alpha} \pi^*W \longrightarrow 0$$

with complex vector bundles V, W over M and α defined outside the zero section of π^*V where it is invertible and satisfies

$$\alpha(\lambda v) = \alpha(v) \qquad \text{for all } \lambda > 0. \tag{6.3}$$

Moreover, by standard differential topology arguments, one can always assume that V, W are smooth vector bundles and α is a smooth bundle map. In particular, α belongs to the symbol class $S^0_{1,0}(\mathrm{Hom}(\pi^*V, \pi^*W))$ (after smoothing it on a small neighbourhood of the zero-section of π^*V).

Using a partition of unity and working in local coordinates, we can construct an order 0 pseudo-differential operator $P_\alpha : C^\infty(M, V) \longrightarrow C^\infty(M, W)$ with principal symbol α (up to operators of order -1). The invertibility of α implies that P_α is elliptic and thus defines a class $[P_\alpha] \in K^0(C(M))$. Since operators of order -1 are compact, $[P_\alpha]$ does not depend on how exactly we have constructed P_α.

This construction descends to a map

$$\mathcal{P} : K_0(C_0(T^*M)) \longrightarrow K^0(C(M)).$$

Computing this map on spheres and using Mayer-Vietoris sequences on both sides, one can show that $\mathcal{P}$ is an isomorphism, proving Theorem 6.1.1 in the even case [BD82a],[BD82b].

Similarly, $K_1(C_0(T^*M))$ can be described by maps

$$\alpha : \pi^*V \longrightarrow \pi^*V$$

where V is a complex vector bundle over M and α is a self-adjoint automorphism satisfying (6.3). As above, we can construct from α a self-adjoint elliptic operator on V of order 0, which defines a class in $K^1(C(M))$. Again, this construction induces an isomorphism between $K_1(C_0(T^*M))$ and $K^1(C(M))$.

6.2 Injectivity

Unfortunately, we cannot say if the comparison map between $K^*_{fin}(C^\infty(M))$ and $K^*(C(M))$ is injective in general or not. In fact, we cannot even compute K^0_{fin} for $C^\infty(S^1)$, the smooth suspension $C_0^\infty((0,1))$, or the smooth cone $C_0^\infty([0,1))$. Note that appropriate exact sequences, homotopy invariance, and Bott periodicity are not available for K^*_{fin} since the proofs of all these properties depend in one way or another on the Kasparov product or at least Kasparov's Technical Theorem (see Section 2.2). In the following we discuss some further problems that arise.

First, reconsider the proof that $\mathcal{P} : K_*(C_0(T^*M)) \longrightarrow K^*(C(M))$ is an isomorphism. It is true that $\mathcal{P}$ factors as

$$K_*(C_0(T^*M)) \xrightarrow{\mathcal{P}_{fin}} K^*_{fin}(C^\infty(M)) \xrightarrow{\Phi} K^*(C(M)),$$

where Φ denotes the comparison map between K^*_{fin} and K^*. Since $\mathcal{P}$ is an isomorphism, $\mathcal{P}_{fin}$ must be injective in particular. However, we cannot prove its surjectivity like above as neither we have a Mayer-Vietoris sequence for K^*_{fin}, nor can we compute K^*_{fin} of arbitrary spheres (the latter would involve Bott periodicity).

From a more abstract point of view, the isomorphism between the K-theory of $C_0(T^*M)$ and the K-homology of $C(M)$ can be seen as a case of Poincaré duality in KK-theory: it is possible to construct KK-elements

$$\Delta \in KK(C(M) \otimes C_0(T^*M), \mathbb{C}) \quad \text{and} \quad \hat{\Delta} \in KK(\mathbb{C}, C(M) \otimes C_0(T^*M))$$

whose Kasparov products with each other satisfy

$$\hat{\Delta} \times_{C_0(T^*M)} \Delta = \mathbf{1} \in KK(C(M), C(M))$$
$$\hat{\Delta} \times_{C(M)} \Delta = \mathbf{1} \in KK(C_0(T^*M), C_0(T^*M))$$

[Con94, Section 6.1.4.β].

Using only the formal properties of the product, one can conclude that the maps

$$\begin{aligned} K_*(C_0(T^*M)) &\longrightarrow K^*(C(M)) \\ \mathbf{x} &\longmapsto \mathbf{x} \times_{C_0(T^*M)} \Delta \end{aligned} \quad \text{and} \quad \begin{aligned} K^*(C(M)) &\longrightarrow K_*(C_0(T^*M)) \\ \mathbf{x} &\longmapsto \hat{\Delta} \times_{C(M)} \mathbf{x} \end{aligned}$$

must be inverse to each other [Eme01, Theorem 2.4]. Thus, the proof that $K_*(C_0(T^*M))$ and $K^*(C(M))$ are isomorphic is reduced to the computation of the Kasparov products of Δ with $\hat{\Delta}$. However, without a product which is compatible with K^*_{fin}, this argument is obviously of not much use to us (as we have seen, we can define a potential isomorphism between $K_*(C_0(T^*M))$ and $K^*_{fin}(C^\infty(M))$, but it is the formal properties of a product for K^*_{fin} we are lacking to complete the proof).

* * *

One possible approach to prove the injectivity of the comparison map might be as follows. Assume that $\mathscr{F} = (\varphi, \mathcal{H}, F)$ and $\mathscr{F}' = (\varphi, \mathcal{H}, F')$ are p-summable Fredholm modules over $C^\infty(M)$ which are operator homotopic as ordinary Fredholm modules via a path of symmetries F_t. By definition, F_t lies in $\mathcal{D}_{\varphi(C(M))}$ for every t. If we could approximate elements of $\mathcal{D}_{\varphi(C(M))}$ in operator norm by elements of $\mathcal{D}^q_\varphi(C^\infty(M))$ for some q, then we could also connect F_0 and F_1 by a path $\tilde{F}_t$ of symmetries in $\mathcal{D}^q_{\varphi(C^\infty(M))}$:

Choose $0 = t_0 < t_1 < \ldots < t_n = 1$ such that $\|F_t - F_{t+1}\| < 1/2$. Approximate F_t by an $\tilde{F}_t \in D^q_{\varphi(C^\infty(M))}$ such that $\|\tilde{F}_t - F_t\| \leq 1/4$. Using holomorphic functional calculus, we can assume that $\tilde{F}_t$ is a symmetry. We then have $\|\tilde{F}_t - \tilde{F}_{t+1}\| < 1$, which by Proposition 1.5.7 implies that $\tilde{F}_t$ and $\tilde{F}_{t+1}$ can be joined by a path of symmetries in $\mathcal{D}^q_{\varphi(C^\infty(M))}$.

However, it turns out that it is impossible to approximate operators in $\mathcal{D}_{\varphi(C^\infty(M))}$ by elements of $\mathcal{D}^q_{\varphi(C^\infty(M))}$ in general. To prove this, we need a classical result of Voiculescu which implies that there is up to finitely summable perturbations only one representation of $C^\infty(M)$ which does not contain any compact operator:

6.2.1 Theorem. *If $X_1, \ldots, X_n$ ($n \geq 2$) are commuting self-adjoint operators on a separable Hilbert space $\mathcal{H}$, then there is an approximate unit $\{A_n\}$ of $\mathcal{L}^n(\mathcal{H})$ which is quasi-central for $X_1, \ldots, X_n$.*

In other words: there is an increasing sequence of positive, contractive finite-rank operators $\{A_i\}$ strongly converging to I such that

$$\big\|[A_i, X_k]\big\|_n \xrightarrow{i \to \infty} 0 \qquad \textit{for all } \ 1 \leq k \leq n.$$

Proof. [Voi79, Theorem 4.2]. □

6.2.2 Theorem. *Let A be a unital commutative C^*-algebra. If $x_1, \ldots, x_n \in A$ $(n \geq 2)$ are self-adjoint and φ, ψ unital, faithful $*$-representations of A on separable Hilbert spaces $\mathcal{H}, \mathcal{H}'$ such that no non-zero compact operator lies in their images, then there is a unitary*

$$U : \mathcal{H} \longrightarrow \mathcal{H}'$$

satisfying

$$U\varphi(x_i)U^* - \psi(x_i) \in \mathcal{L}^n \qquad \textit{for } i = 1, \ldots, n.$$

Proof. This follows from the p-summable version of Voiculescu's non-commutative Weyl-von Neumann Theorem [Voi79, Corollary 2.5] and Theorem 6.2.1. □

6.2.3 Theorem. *Let A be an infinite-dimensional, commutative C^*-algebra generated by finitely many self-adjoint elements $x_1, \ldots, x_n$. If $\rho : A \longrightarrow \mathcal{B}(\mathcal{H})$ is a faithful representation of A on a separable Hilbert space $\mathcal{H}$ such that $\operatorname{Im} \rho$ contains no compact operator except 0, then*

$$\bigcup_p \mathcal{D}^p_{\{\rho(x_1),\ldots,\rho(x_n)\}} \quad \textit{is not dense in} \quad \mathcal{D}_{\{\rho(x_1),\ldots,\rho(x_n)\}}.$$

In other words: not every operator commuting with $\rho(x_1), \ldots, \rho(x_n)$ up to compact operators can be approximated by operators which commute with $\rho(x_1), \ldots, \rho(x_n)$ up to finitely summable operators.

Proof. Abbreviate $\mathcal{D}_{\{\rho(x_1),\ldots,\rho(x_n)\}}$ by $\mathcal{D}$ and $\mathcal{D}^p_{\{\rho(x_1),\ldots,\rho(x_n)\}}$ by $\mathcal{D}^p$.

Let P denote the support projection of ρ. If $T \in \mathcal{D}, S \in \mathcal{D}^p$, then we have $\|PTP - PSP\| \leq \|T - S\|$ and $PTP \in \mathcal{D}$, $PSP \in \mathcal{D}^p$ since P commutes with ρ. Thus, if the theorem is true for the compression of ρ to its support, then it must also be true for ρ. We can therefore assume that ρ is non-degenerate.

If A is not already unital, extend ρ to a unital $*$-representation ρ^+ of A^+ and add 1_{A^+} to the set $\{x_1, \ldots, x_n\}$. $\operatorname{Im} \rho^+$ also cannot contain a non-zero compact operator: otherwise, there is a self-adjoint $x \in A$ such that $I + \rho(x)$ is compact. Then there must be an isolated point $\lambda \in \operatorname{sp}(I + \rho(x))$ which is an eigenvalue of a finite-dimensional eigenspace of $I + \rho(x)$. The corresponding spectral projection P_λ lies in $\operatorname{Im} \rho^+$. Since $\operatorname{Im} \rho$ is an ideal in $\operatorname{Im} \rho^+$ and $\operatorname{Im} \rho$ contains no non-zero compact operator, we conclude that $(\operatorname{Im} \rho)P_\lambda = \{0\}$. But this implies that ρ is degenerate.

All in all, we can assume that A and ρ are unital.

Since A is infinite-dimensional, $\operatorname{spec}(A)$ is infinite. As $x_1, \ldots, x_n$ generate A, there is an x_k with infinite spectrum. Thus, as a function on $\operatorname{spec}(A)$, x_k has an infinite range. Since $\operatorname{spec}(A)$ is compact, there is in particular a converging sequence $\{\lambda_i\} \subseteq \operatorname{spec}(A)$ such that $\{x_k(\lambda_i)\}$ has no repetitions and moreover $\lim_{i\to\infty} \lambda_i$ does not belong to $\{\lambda_i\}$.

Let

$$\alpha : \coprod_{\mathbb{N}} \{pt\} \longrightarrow \operatorname{spec}(A)$$

be the map which sends the ith point to λ_i. Note that α induces a surjective $*$-homomorphism

$$\alpha^* : A \longrightarrow \Big(\bigoplus_{\mathbb{N}} \mathbb{C}\Big)^+.$$

Moreover, let

$$y := \alpha^*(x_k - \lim_{i\to\infty} x_k(\lambda_i) \cdot 1_A) \in \bigoplus_{\mathbb{N}} \mathbb{C}.$$

Since $x_k(\lambda_i)$ has no repetitions, y does not lie in the algebraic direct sum of the $\mathbb{C}$s.

Finally, let

$$\mathscr{F} = \left(\varphi \oplus \varphi,\, \mathcal{H}' \oplus \mathcal{H}',\, \begin{bmatrix} 0 & V^* \\ V & 0 \end{bmatrix}\right)$$

with unitary V be a representative of the K-homology class $\mathbf{x}$ of $\bigoplus_{\mathbb{N}} \mathbb{C}$ which is constructed in the proof of Theorem 5.1.1 using the element $y \in \bigoplus_{\mathbb{N}} \mathbb{C} \backslash \bigoplus_{\mathbb{N}}^{alg} \mathbb{C}$. Extend φ to a unital $*$-representation of $(\bigoplus_{\mathbb{N}} \mathbb{C})^+$ on $\mathcal{H}'$, which we also denote by φ.

Note that ρ and $\varphi \circ \alpha^* \oplus \rho$ satisfy the conditions of Theorem 6.2.2. Hence, there is a unitary $U : \mathcal{H} \longrightarrow \mathcal{H}' \oplus \mathcal{H}$ such that

$$U\rho(x_i)U^* - \varphi \circ \alpha^*(x_i) \oplus \rho(x_i) \in \mathcal{L}^{\max(n,2)} \qquad \text{for } i = 1, \ldots, n.$$

In particular,

$$[\rho(x_i), U^*(V \oplus 0)U] = U^*[U\rho(x_i)U^*, V \oplus 0]U \in \mathcal{K}(\mathcal{H}) \qquad \text{for } i = 1, \ldots, n,$$

thus $U^*(V \oplus I)U \in \mathcal{D}$.

Now assume that $\bigcup_p \mathcal{D}^p$ is dense in $\mathcal{D}$. Then there is a $p \geq \max(n, 2)$ and a $W \in \mathcal{D}^p$ such that

$$\|U^*(V \oplus 0)U - W\| < 1.$$

This implies that

$$\|V \oplus 0 - UWU^*\| < 1 \quad \text{and} \quad UWU^* \in \mathcal{D}^p_{\{\varphi\circ\alpha^*(x_i)\oplus\rho(x_i)\}}.$$

Denoting by P the projection onto $\mathcal{H}'$, we arrive at

$$\|V - PUWU^*P\| < 1 \quad \text{and} \quad PUWU^*P \in \mathcal{D}^p_{\{\varphi\circ\alpha^*(x_i)\}} \subseteq \mathcal{D}_{\varphi(\bigoplus_{\mathbb{N}} \mathbb{C})}.$$

This implies that

$$\mathscr{F}' := \left(\varphi \oplus \varphi,\, \mathcal{H}' \oplus \mathcal{H}',\, \begin{bmatrix} 0 & (PUWU^*P)^{-1} \\ (PUWU^*P) & 0 \end{bmatrix}\right)$$

is a representative of $\mathbf{x}$ satisfying $[\varphi(y), PUWU^*P] \in \mathcal{L}^p$. Applying Proposition 1.5.2 for $\mathcal{D}_{\varphi(\bigoplus_{\mathbb{N}} \mathbb{C})} \cap \mathcal{D}^p_{\varphi(y)}$, we see that we can replace $PUWU^*P$ by its unitary part while preserving these properties. However, the proof of Theorem 5.1.1 shows that such a module cannot exist. □

6.2.4 Corollary. *Let M be a smooth, connected, compact manifold of dimension greater than zero. If φ is a faithful representation of $C(M)$, then*

$$\bigcup_p \mathcal{D}^p_{\varphi(C^\infty(M))} \quad \textit{is not dense in} \quad \mathcal{D}_{\varphi(C(M))}.$$

Proof. $C(M)$ is generated by finitely many self-adjoint smooth coordinate functions x_i. Since M is connected, every non-constant continuous function f on M has an infinite, connected range. Thus, $\operatorname{sp}(\varphi(f)) \cup \{0\} = \operatorname{Ran} f \cup \{0\}$ cannot be the spectrum of a compact operator. Moreover, note that $\mathcal{D}_{\{\varphi(x_i)\}} = \mathcal{D}_{\varphi(C(M))}$. □

* * *

In his recent work on operators with trace-class self-commutator [Voi11], Voiculescu introduces Banach algebras $E\Lambda(\Omega)$ for Borel sets $\Omega \subseteq \mathbb{C}$. If Ω is bounded, $E\Lambda(\Omega)$ agrees with $\mathcal{D}^2_Z \subseteq \mathcal{B}(L^2(\Omega))$ where Z denotes the operator on $L^2(\Omega)$ of multiplication by the identity function on Ω.

If P is a projection in $\mathcal{D}^2_Z$, then the commutator

$$[PZ^*P, PZP] = -(PZ^*P^\perp ZP - PZP^\perp Z^*P)$$

lies in $\mathcal{L}^1$ since $PZ^*P^\perp ZP, PZP^\perp Z^*P \in \mathcal{L}^2 \cdot \mathcal{L}^2 \subseteq \mathcal{L}^1$. Therefore, PZP has a Helton-Howe measure with density function $\delta_P \in L^1(\overline{\Omega})$ assigned to it [HH73],[Pin72]. This means that if $f, g \in \mathbb{C}[X, Y]$ are polynomials (in commuting variables X,Y), then

$$\operatorname{Tr}\, [f(\operatorname{Re} PZP, \operatorname{Im} PZP), g(\operatorname{Re} PZP, \operatorname{Im} PZP)] = \int_{\mathbb{R}^2} h(x, y)\delta_P(x, y)\, \mathrm{d}x\mathrm{d}y$$

where h is the polynomial given by

$$h(X, Y) = \frac{\partial f}{\partial X}\frac{\partial g}{\partial Y} - \frac{\partial f}{\partial Y}\frac{\partial g}{\partial X}.$$

Using Theorem 6.2.1, Voiculescu proves that the assignment $P \mapsto \delta_P$ descends to a map

$$\begin{aligned} K_0(E\Lambda(\Omega)) &\longrightarrow L^1(\overline{\Omega}) \\ [P] &\longmapsto \delta_P \end{aligned}$$

(each K-theory class of $E\Lambda(\Omega)$ is represented by a projection in $E\Lambda(\Omega)$). In particular, $\operatorname{Tr}[P(\operatorname{Re} Z)P, P(\operatorname{Im} Z)P]$ does not depend on the K-theory class of P.

We translate this statement into the language of finitely summable K-homology and cyclic cohomology and reproduce his proof in this setting:

6.2.5 Theorem. *If $\mathscr{F} = (\varphi, \mathcal{H}, F)$ is a 2-summable Fredholm module over a commutative topological $*$-algebra $\mathcal{A}$ with $F^* = F$, $F^2 = I$, then its one-dimensional character*

$$\tau_{\mathscr{F}} := \frac{1}{4} \operatorname{Tr} \Big(F[F, \varphi(\cdot)][F, \varphi(\cdot)] \Big)$$

depends (as a bilinear form on $\mathcal{A}$) only on the class of $\mathscr{F}$ in $K_2^1(\mathcal{A})$.

Proof. Let $x, y \in \mathcal{A}$ be given. By the linearity of $\tau_{\mathscr{F}}$, we can assume that x and y are self-adjoint. $\tau_{\mathscr{F}}$ is obviously invariant under unitary equivalence and addition of degenerate modules. Thus, by Theorem 2.4.1 and Corollary 2.4.4, it suffices to show that $\tau_{\mathscr{F}}(x, y) = \tau_{\mathscr{F}'}(x, y)$ if $\mathscr{F}'$ is given as $\mathscr{F}' = (\varphi, \mathcal{H}, UFU^*)$ where U is a unitary that commutes with φ up to 2-summable operators.

Since F is a symmetry, we have for $P := 1/2(F + I)$

$$P[F, \varphi(x)] = P\varphi(x) - P\varphi(x)F = 2P\varphi(x)P^{\perp}$$

and

$$P^{\perp}[F, \varphi(x)] = -P^{\perp}\varphi(x) - P^{\perp}\varphi(x)F = -2P^{\perp}\varphi(x)P,$$

thus

$$\begin{aligned} F[F, \varphi(x)][F, \varphi(y)] &= P[F, \varphi(x)][F, \varphi(y)] - P^{\perp}[F, \varphi(x)][F, \varphi(y)] \\ &= -4P\varphi(x)P^{\perp}\varphi(y)P + 4P^{\perp}\varphi(x)P\varphi(y)P^{\perp}. \end{aligned}$$

By assumption, $P^{\perp}\varphi(x)P$ and $P\varphi(y)P^{\perp}$ lie in $\mathcal{L}^2$, so

$$\begin{aligned} \tau_{\mathscr{F}}(x, y) &= -\operatorname{Tr} \Big(P\varphi(x)P^{\perp}\varphi(y)P - P\varphi(y)P^{\perp}\varphi(x)P \Big) \\ &= \operatorname{Tr}\, [P\varphi(x)P, P\varphi(y)P], \end{aligned} \tag{6.4}$$

where we have used Proposition 1.1.10 and the commutativity of $\mathcal{A}$. For $\tau_{\mathscr{F}'}$ we obtain

$$\begin{aligned} \tau_{\mathscr{F}'}(x, y) &= \operatorname{Tr}\, [UPU^*\varphi(x)UPU^*, UPU^*\varphi(y)UPU^*] \\ &= \operatorname{Tr}\, [PU^*\varphi(x)UP, PU^*\varphi(y)UP]. \end{aligned}$$

By Theorem 6.2.1, there exists an increasing sequence $\{A_n\}$ of positive, contractive finite-rank operators strongly converging to I that is quasi-central for $\varphi(x)$ and $\varphi(y)$ in the $\mathcal{L}^2$-norm. In particular, we have for increasing n:

$$\begin{aligned} &\big\|[PA_nP, P\varphi(x)P]\big\|_2 = \big\|[P(I - A_n)P, P\varphi(x)P]\big\|_2 \\ &\quad = \big\|P[P(I - A_n)P, \varphi(x)]P\big\|_2 \\ &\quad \le \big\|[P, \varphi(x)](I - A_n)\big\|_2 + \big\|[I - A_n, \varphi(x)]\big\|_2 + \big\|(I - A_n)[P, \varphi(x)]\big\|_2 \to 0. \end{aligned}$$

The first and third summand tend to zero since $[P, \varphi(x)] \in \mathcal{L}^2$ and A_n is an approximate unit for $\mathcal{L}^2$. The same holds true for $\varphi(y)$, so $B_n := PA_nP$ is quasi-central for $P\varphi(x)P$, $P\varphi(y)P$ and converges strongly to P.

Abbreviating $P\varphi(x)P$, $P\varphi(y)P$, $PU^*\varphi(x)UP$, $PU^*\varphi(y)UP$ by X, Y, X', and Y', we thus obtain

$$\begin{aligned}
&|(\tau_{\mathscr{F}} - \tau_{\mathscr{F}'})(\varphi(x), \varphi(y))| \\
&\qquad = \lim_{n\to\infty} \left| \operatorname{Tr}\Big(B_n([X,Y] - [X',Y'])\Big)\right| \\
&\qquad \leq \limsup_{n\to\infty} \left| \operatorname{Tr}\Big(B_n[(X-X'),Y]\Big)\right| + \limsup_{n\to\infty} \left| \operatorname{Tr}\Big(B_n[X',(Y-Y')]\Big)\right| .
\end{aligned}$$

For the first summand we have

$$\begin{aligned}
\left| \operatorname{Tr}\Big(B_n[(X-X'),Y]\Big)\right| &= \left| \operatorname{Tr}\Big([B_n(X-X'),Y]\Big) - \operatorname{Tr}\Big([B_n,Y](X-X')\Big)\right| \\
&= \left| \operatorname{Tr}\Big([B_n,Y](X-X')\Big)\right| \\
&\leq \|[B_n,Y]\|_2 \cdot \|X-X'\|_2 \xrightarrow{n\to\infty} 0 .
\end{aligned}$$

Here we have used that $B_n(X-X')$ is, as a finite-rank operator, trace-class and that $X - X' \in \mathcal{L}^2$ since $[U, \varphi(x)] \in \mathcal{L}^2$. For the second summand we obtain in the same way

$$\begin{aligned}
\left| \operatorname{Tr}\Big(B_n[X',Y-Y']\Big)\right| &\leq \|[B_n,X']\|_2 \cdot \|Y-Y'\|_2 \\
&\leq (\|[B_n,X]\|_2 + \|[B_n,X'-X]\|_2)\|Y-Y'\|_2 \xrightarrow{n\to\infty} 0 .
\end{aligned}$$

$\|[B_n, X'-X]\|_2$ converges to 0 since $X' - X \in \mathcal{L}^2(P\mathcal{H})$. □

We can use this result to prove that $K_2^1(C^\infty(M))$ never vanishes as long as M is at least 2-dimensional:

6.2.6 Theorem. *If M is a smooth manifold of dimension greater than or equal 2, then*

$$K_2^1(C_0^\infty(M)) \neq 0.$$

Proof. First, consider the Fredholm module $\mathscr{F} := (\varphi, L^2(S^1), F)$ over $C^\infty(S^1)$ where φ is the representation of $C^\infty(S^1)$ on $L^2(S^1)$ by multiplication and F is given with respect to the standard orthonormal basis $\{z^n\}$ $(n \in \mathbb{Z})$ as

$$Fz^n := \begin{cases} z^n & n \geq 0 \\ -z^n & n < 0 \end{cases} .$$

By direct computation one sees that $[F, \varphi(z)]$ is a rank-one operator. In particular, it lies in $\mathcal{L}^1$. If $f \in C^\infty(S^1)$ is a smooth function, then we can write f as a (uniformly convergent) series $f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$ with rapidly decreasing Fourier coefficients c_n. Thus

$$\begin{aligned}
\|[F,\varphi(f)]\|_1 &\leq \sum_{n\in\mathbb{Z}} |c_n| \cdot \|[F,\varphi(z)^n]\|_1 \\
&\leq \sum_{n\in\mathbb{Z}} |n| \cdot |c_n| \cdot \|[F,\varphi(z)]\|_1 < \infty.
\end{aligned}$$

Hence, $\mathscr{F}$ is a 1-summable Fredholm module over $C^\infty(M)$. Moreover, using equation (6.4) we get for $P := 1/2(F+I)$

$$\begin{aligned}\tau_{\mathscr{F}}(z^{-1}, z) &= -\operatorname{Tr}\left(P\varphi(z^{-1})P^\perp\varphi(z)P - P\varphi(z)P^\perp\varphi(z^{-1})P\right)\\ &= \operatorname{Tr}\left(P\varphi(z)P^\perp\varphi(z^{-1})P\right) = 1.\end{aligned}$$

Since M is at least 2-dimensional, we can find a smooth embedding of S^1 into M, which induces a surjective map $\alpha : C^\infty(M) \longrightarrow C^\infty(S^1)$. If $f, g \in C^\infty(M)$ are pre-images of z^{-1} and z, then

$$\tau_{\alpha^*\mathscr{F}}(f, g) = \tau_{\mathscr{F}}(z^{-1}, z) = 1 \neq 0.$$

Thus, we conclude from Theorem 6.2.5 that $K^1_2(C^\infty(M))$ cannot vanish. □

Using Bott periodicity, one easily checks that $K^1(C(S^2)) = K^1(\mathbb{C}) = 0$. Thus, we have proven that the comparison map $K^1_2(C^\infty(S^2)) \longrightarrow K^1(C(S^2))$ is not injective.

By considering different disjoint embeddings of S^1 into M and evaluating the Chern character with pre-images of z^{-1} and z that vanish on the respective other embeddings of S^1, it is easy to see that $K^1_2(C^\infty(M))$ is in fact very large.

Using similar methods, Salinas shows in [Sal83] that if $\mathscr{F} = (\varphi, \mathcal{H}, F)$ $(F^* = F, F^2 = I)$ is a $2n$-summable Fredholm module over a commutative topological $*$-algebra $\mathcal{A}$ and $P := 1/2(F+I)$, then the fundamental trace form given by

$$L_{\mathscr{F}}(x_1, \ldots, x_{2n}) := \operatorname{Tr} \sum_{\sigma \in S_{2n}} \operatorname{sgn}(\sigma) P\varphi(x_{\sigma(1)})P \cdots P\varphi(x_{\sigma(2n)})P$$

only depends on the class of $\mathscr{F}$ in $K^1_{2n}(\mathcal{A})$ (the author shows this for smooth extensions of compact subsets of $\mathbb{C}^n$, but the proof carries over to K^1_{2n}). From this it should be possible to obtain higher-dimensional analogues of Theorem 6.2.6.

Note that by [Con85, Proposition I.7.7], $L_{\mathscr{F}}$ is in fact the complete anti-symmetrization of the $(2n-1)$-dimensional Chern character of $\mathscr{F}$. Thus, while both agree for $n = 1$, $L_{\mathscr{F}}$ is a coarser invariant than the Chern character for $n > 1$. In particular, it is unclear whether Theorem 6.2.5 also extends in some way to higher dimensions.

Chapter 7

Degenerate Cases

In this final chapter we consider several cases in which K^*_{fin} degenerates. We start by proving that if $\mathcal{A}$ contains a closed barreled ideal $\mathcal{J}$ with a bounded approximate unit, then the associated sequence

$$0 \longleftarrow K^*_{fin}(\mathcal{J}) \longleftarrow K^*_{fin}(\mathcal{A}) \longleftarrow K^*_{fin}(\mathcal{A}/\mathcal{J}) \longleftarrow 0.$$

is always exact.

The rest of this chapter contains examples of algebras $\mathcal{A}$ over which every p-summable Fredholm module is a p-summable perturbation of a nearly degenerate module. By Theorem 3.1.5, these algebras have degenerate finitely summable K-homology groups

$$K^0_{fin}(\mathcal{A}) = \mathrm{FR}(\mathcal{A}) \quad \text{and} \quad K^1_{fin}(\mathcal{A}) = 0.$$

The classes of algebras we study are amenable Banach algebras (Section 7.2), group algebras of lattices in higher rank Lie groups (Section 7.3) and algebras which are generated by a bounded group of unitaries like C^*-algebras or $\ell^1(\Gamma)$ for any discrete group Γ (Section 7.4).

7.1 Algebras with bounded approximate units

7.1.1 Theorem. *Let $\mathcal{A}$ be a topological $*$-algebra and $\mathcal{J}$ a closed, barreled $*$-ideal of $\mathcal{A}$ with a bounded left-approximate unit. If ι denotes the embedding of $\mathcal{J}$ into $\mathcal{A}$ and π the quotient map from $\mathcal{A}$ to $\mathcal{A}/\mathcal{J}$, then the sequence*

$$0 \longleftarrow K^*_p(\mathcal{J}) \underset{\iota^*}{\overset{e}{\rightleftarrows}} K^*_p(\mathcal{A}) \overset{\pi^*}{\longleftarrow} K^*_p(\mathcal{A}/\mathcal{J}) \longleftarrow 0$$

is split-exact with a natural split e.

The same is true for K^*_{fin}.

Proof. We first define the split e. Let $\mathbf{x}$ be a class in $K_p^*(\mathcal{J})$ represented by a p-summable Fredholm module $\mathscr{F} = (\varphi, \mathcal{H}, F)$.

If $\{u_\lambda\}$ is a bounded left-approximate unit for $\mathcal{J}$, then the set $\{xu_\lambda\}$ is bounded in $\mathcal{J}$ for any fixed $x \in \mathcal{A}$ since multiplication in $\mathcal{A}$ is continuous. $\mathcal{J}$ is barreled, so Proposition 3.3.1 implies that there is a $C < \infty$ such that

$$\big\|[F, \varphi(xu_\lambda)]\big\|_p < C \quad \text{and} \quad \|\varphi(xu_\lambda)(F^2 - I)\|_p < C \qquad \text{for all } \lambda.$$

Extend φ to a representation of $\mathcal{A}$ with the same support (Proposition 1.2.7). By Proposition 1.2.5, the net $\{u_\lambda\}$ converges strongly to the support projection of φ. As left and right multiplication are strongly continuous, $\varphi(xu_\lambda)(F^2 - I)$ converges strongly to $\varphi(x)(F^2 - I)$ and $[F, \varphi(xu_\lambda)]$ converges strongly to $[F, \varphi(x)]$. The non-commutative Fatou lemma (Proposition 1.1.13) hence implies that

$$[F, \varphi(x)] \in \mathcal{L}^p \quad \text{and} \quad \varphi(x)(F^2 - I) \in \mathcal{L}^p.$$

Thus, by extending φ to $\mathcal{A}$, $\mathscr{F}$ extends to a p-summable module over $\mathcal{A}$, and we define $e(\mathbf{x})$ to be its class in $K_p^*(\mathcal{A})$.

This assignment is obviously compatible with direct sums, unitary equivalence, and operator homotopies, so e is indeed a well-defined group homomorphism. By definition, ι^* is a left-inverse to e, so ι^* is surjective.

Next, we prove exactness in the middle. To this end, let $\mathscr{F} = (\varphi, \mathcal{H}, F)$ now be a p-summable module over $\mathcal{A}$. Since $\{u_\lambda\}$ is bounded and $\varphi(u_\lambda)$ converges strongly to the support projection P of φ restricted to $\mathcal{J}$, by the same line of argument as above we see that $[P, F] \in \mathcal{L}^p$. Moreover, P commutes exactly with φ (Proposition 1.2.3), so

$$\mathscr{F}^P \oplus \mathscr{F}^{P^\perp} := (P\varphi, P\mathcal{H}, PFP) \oplus (P^\perp\varphi, P^\perp\mathcal{H}, P^\perp F P^\perp)$$

is a p-summable perturbation of $\mathscr{F}$.

We have $P^\perp\varphi \circ \iota = 0$, so $\iota^*[\mathscr{F}^{P^\perp}] = \mathbf{0}$, whereas $e \circ \iota^*([\mathscr{F}^P]) = [\mathscr{F}^P]$. Thus, $\iota^*[\mathscr{F}] = \mathbf{0}$ implies $[\mathscr{F}^P] = \mathbf{0} \in K_p^*(\mathcal{A})$ and it remains to show that $[\mathscr{F}^{P^\perp}]$ lies in the image of π^*.

Since $P^\perp\varphi$ vanishes on $\mathcal{J}$, $P^\perp\varphi$ factors through π, and we can write $P^\perp\varphi = \tilde{\varphi} \circ \pi$. Note that $\tilde{\varphi}$ is continuous by definition of the quotient topology. Thus,

$$(\tilde{\varphi}, P^\perp\mathcal{H}, P^\perp F P^\perp)$$

is a p-summable module over $\mathcal{A}/\mathcal{J}$ and $\mathscr{F}^{P^\perp} = \pi^*\,(\tilde{\varphi}, P^\perp\mathcal{H}, P^\perp F P^\perp)$.

Finally, we have to prove the injectivity of π^*. Thus, let $\mathscr{F} = (\varphi, \mathcal{H}, F)$ be a p-summable module over $\mathcal{A}/\mathcal{J}$ and assume that $\pi^*[\mathscr{F}] = \mathbf{0}$. This means that there is a degenerate module $\mathscr{D} = (\psi, \mathcal{N}, G)$ over $\mathcal{A}$ such that $\pi^*\mathscr{F} \oplus \mathscr{D}$ is operator-homotopic to a degenerate module via an operator homotopy of the form

$$(\varphi \circ \pi \oplus \psi, \mathcal{H} \oplus \mathcal{N}, F_t).$$

Let P be the support projection of $\varphi \circ \pi \oplus \psi$ restricted to $\mathcal{J}$. By the same arguments as above we obtain an operator homotopy

$$\Big(P^\perp(\varphi \circ \pi \oplus \psi), P^\perp(\mathcal{H} \oplus \mathcal{N}), P^\perp F_t P^\perp\Big). \tag{7.1}$$

If Q denotes the support projection of ψ restricted to $\mathcal{J}$, then $P^\perp = I \oplus Q^\perp$ since $\varphi \circ \pi$ vanishes identically on $\mathcal{J}$. Moreover, $Q^\perp\psi$ again factors as $\tilde{\psi} \circ \pi$. Therefore, we can rewrite (7.1) as

$$\Big(\varphi \circ \pi \oplus \tilde{\psi} \circ \pi, \mathcal{H} \oplus Q^\perp\mathcal{N}, (I \oplus Q^\perp)F_t(I \oplus Q^\perp)\Big).$$

Thus,

$$\Big(\varphi \oplus \tilde{\psi}, \mathcal{H} \oplus Q^\perp\mathcal{N}, (I \oplus Q^\perp)F_t(I \oplus Q^\perp)\Big)$$

is an operator homotopy between the modules

$$\mathscr{F} \oplus (\tilde{\psi}, Q^\perp\mathcal{N}, Q^\perp G Q^\perp) \text{ and } \Big(\varphi \oplus \tilde{\psi}, \mathcal{H} \oplus Q^\perp\mathcal{N}, (I \oplus Q^\perp)F_1(I \oplus Q^\perp)\Big).$$

Since $\mathscr{D}$ and $(\varphi \circ \pi, \mathcal{H} \oplus \mathcal{N}, F_1)$ are degenerate, $Q^\perp$ and $I \oplus Q^\perp$ commute exactly with G and F_1 (they are strong limits of operators which exactly commute with G and F_1). This implies that

$$(\tilde{\psi}, Q^\perp\mathcal{N}, Q^\perp G Q^\perp) \text{ and } (\varphi \oplus \tilde{\psi}, \mathcal{H} \oplus Q^\perp\mathcal{N}, (I \oplus Q^\perp)F_1(I \oplus Q^\perp))$$

are degenerate as well, so $[\mathscr{F}] = \mathbf{0} \in K_p^*(\mathcal{A}/\mathcal{J})$.

The analogous statement for K_{fin}^* follows immediately from the statement for K_p^*. □

7.1.2 Remark. Closed subspaces of barreled spaces are not barreled in general. But closed subspaces of Fréchet spaces are, of course, again Fréchet spaces. Thus, the conditions of Theorem 7.1.1 are fulfilled if $\mathcal{A}$ is a Fréchet $*$-algebra and $\mathcal{J}$ a closed ideal in $\mathcal{A}$ with a bounded left-approximate unit.

7.1.3 Example. Consider the cone $\mathcal{C}\mathbb{R}$ of smooth functions on $\mathbb{R}$ given by

$$\mathcal{C}\mathbb{R} := \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \begin{array}{l} \lim_{x\to\infty} f(x) \text{ exists, } \lim_{x\to-\infty} f(x) = 0, \\ \lim_{x\to\pm\infty} f^{(n)}(x) = 0 \text{ for all } n \in \mathbb{N} \end{array} \right\}.$$

Endowed with the topology given by the semi-norms $|f|_n := \sup_{x\in\mathbb{R}} |f^{(n)}(x)|$ $(n \in \mathbb{N}_0)$, $\mathcal{C}\mathbb{C}$ becomes a Fréchet $*$-algebra. We want to show that $K^0_{fin}(\mathcal{C}\mathbb{R})$ does not vanish.

To this end, note that $\mathcal{C}\mathbb{R}$ contains as an ideal the suspension algebra

$$\mathcal{S}\mathbb{R} := \Big\{ f \in C^\infty(\mathbb{R}) \;\Big|\; \lim_{x\to\pm\infty} f^{(n)}(x) = 0 \text{ for all } n \in \mathbb{N}_0 \Big\} = C_0^\infty(\mathbb{R}).$$

$\mathcal{S}\mathbb{R}$ is the kernel of the evaluation map sending $f \in \mathcal{C}\mathbb{R}$ to $\lim_{x\to\infty} f(x)$, so we obtain the short exact sequence

$$0 \longrightarrow \mathcal{S}\mathbb{R} \longrightarrow \mathcal{C}\mathbb{R} \longrightarrow \mathbb{C} \longrightarrow 0.$$

Let $g : [0,1] \longrightarrow \mathbb{R}$ be a smooth function such that $g(0) = 1, g(1) = 0$ and $\lim_{x\to 0} g^{(n)}(x) = \lim_{x\to 1} g^{(n)}(x) = 0$ for all $n \in \mathbb{N}$. Then we can define an approximate unit $\{u_n\}$ of $\mathcal{S}\mathbb{R}$ as

$$u_n(x) := \begin{cases} 0 & x \notin [-n-1, n+1] \\ g(-x-n) & x \in [-n-1, -n] \\ 1 & x \in [-n, n] \\ g(x-n) & x \in [n, n+1] \end{cases}.$$

Obviously, all u_n lie in $\mathcal{S}\mathbb{R}$, and it is easy to check that $\{u_n\}$ is indeed a bounded approximate unit for $\mathcal{S}\mathbb{R}$.

Thus, Theorem 7.1.1 applies and we obtain the exact sequence

$$0 \longleftarrow K^0_{fin}(\mathcal{S}\mathbb{R}) \longleftarrow K^0_{fin}(\mathcal{C}\mathbb{R}) \longleftarrow K^0_{fin}(\mathbb{C}) \longleftarrow 0.$$

As $K^0_{fin}(\mathbb{C}) = \mathbb{Z}$, the claim follows.

Note that this argument does not work if we define $\mathcal{C}\mathbb{R}$ in terms of functions whose derivatives are of rapid decay since then $\{u_n\}$ would not be bounded in the appropriate Fréchet algebra topology on $\mathcal{C}\mathbb{R}$.

7.2 Amenable Banach algebras

Let $\mathcal{A}$ be a Banach algebra. A bimodule M over $\mathcal{A}$ is called a Banach bimodule if M is a Banach space and there is a $C > 0$ such that

$$\|a\xi b\| \le C\|a\|\|\xi\|\|b\| \qquad \text{for all } a, b \in \mathcal{A},\ \xi \in M.$$

If M is a Banach bimodule, its topological dual M^* also carries the structure of a Banach bimodule defined by

$$\begin{aligned} \langle xa, \xi\rangle &:= \langle x, a\xi\rangle \\ \langle ax, \xi\rangle &:= \langle x, \xi a\rangle \end{aligned} \qquad \text{for all } a \in \mathcal{A},\ \xi \in M,\ x \in M^*. \tag{7.2}$$

A bounded derivation of $\mathcal{A}$ into M is a continuous map $\varphi : \mathcal{A} \longrightarrow M$ such that

$$\varphi(ab) = \varphi(a)b + a\varphi(b) \qquad \text{for all } a, b \in \mathcal{A}.$$

It is called inner, if there is a $\xi \in M$ such that

$$\varphi(a) = [\xi, a] \qquad \text{for all } a \in \mathcal{A}.$$

One readily checks that such a map is indeed a bounded derivation for any $\xi \in M$.

7.2.1 Definition ([Joh72])**.** A Banach algebra $\mathcal{A}$ is called amenable if every bounded derivation of $\mathcal{A}$ into any dual Banach bimodule M^* is inner.

7.2.2 Theorem. *If $\mathcal{A}$ is an amenable Banach $*$-algebra, then*

$$K_p^1(\mathcal{A}) = 0$$

and there is a natural isomorphism

$$K_p^0(\mathcal{A}) \cong \mathrm{FR}(\mathcal{A}).$$

The same is true for K_{fin}^*.

Proof. First assume that $\mathcal{A}$ is unital. To prove the theorem for $K_p^*(\mathcal{A})$, it suffices by Theorem 3.1.5 to show that every p-summable module in $\widetilde{\mathbb{F}}_p^*(A)$ is a p-summable perturbation of a nearly degenerate module.

Remember that all Schatten classes $\mathcal{L}^p(\mathcal{H})$ $(p \geq 1)$ are dual spaces with predual $\mathcal{L}^q(\mathcal{H})$ $(q := (1-1/p)^{-1})$ for $p > 1$ and with predual $\mathcal{K}(\mathcal{H})$ for $p = 1$ (Proposition 1.1.12). The pairing between these spaces is given by

$$\langle T,\, S\rangle = \operatorname{Tr} TS$$

for $T \in \mathcal{L}^p(\mathcal{H})$ and $S \in \mathcal{L}^q(\mathcal{H})$ (resp. $S \in \mathcal{K}(\mathcal{H})$).

Left and right multiplication by bounded operators turn each $\mathcal{L}^p(\mathcal{H})$ into a Banach bimodule over $\mathcal{B}(\mathcal{H})$. We have

$$\begin{aligned} \langle TU,\, S\rangle &= \operatorname{Tr} TUS = \langle T,\, US\rangle \\ \langle UT,\, S\rangle &= \operatorname{Tr} UTS = \operatorname{Tr} TSU = \langle T,\, SU\rangle \end{aligned}$$

for all $T \in \mathcal{L}^p(\mathcal{H})$, $S \in \mathcal{L}^q(\mathcal{H})$ (resp. $S \in \mathcal{K}(\mathcal{H})$), and $U \in \mathcal{B}(\mathcal{H})$. Thus, each $\mathcal{L}^p(\mathcal{H})$ $(1 \leq p < \infty)$ is a dual Banach bimodule over $\mathcal{B}(\mathcal{H})$ in the sense of (7.2). Note that this is not the case for the compact operators.

To prove our claim in the odd case, let $\mathscr{F} = (\varphi, \mathcal{H}, F)$ be a p-summable odd Fredholm module over $\mathcal{A}$. Pull back $\mathcal{L}^p(\mathcal{H})$ via φ to a dual bimodule over $\mathcal{A}$ (i.e. the left and right actions of $\mathcal{A}$ on $\mathcal{L}^p(\mathcal{H})$ are given by $\varphi(x) \cdot T$ and $T \cdot \varphi(x)$).

By Proposition 3.3.1, the map $D_F : \mathcal{A} \longrightarrow \mathcal{L}^p(\mathcal{H}),\ x \longmapsto [F, \varphi(x)]$ is continuous. Hence, it is a bounded derivation of $\mathcal{A}$ into $\mathcal{L}^p(\mathcal{H})$. The amenability of $\mathcal{A}$ implies that there is a $T \in \mathcal{L}^p(\mathcal{H})$ such that

$$[F, \varphi(x)] = D_F(x) = [T, \varphi(x)] \qquad \text{for all } x \in \mathcal{A}.$$

It follows that $\mathscr{F}$ is a p-summable perturbation of the nearly degenerate module $(\varphi, \mathcal{H}, F - T)$.

For the even case let

$$\mathscr{F} = \left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\right) \in \widetilde{\mathbb{F}}_p^0(\mathcal{A})$$

be an even p-summable module over $\mathcal{A}$ with unitary U.

As above, the map $D_U : \mathcal{A} \longrightarrow \mathcal{L}^p(\mathcal{H}),\ x \longmapsto [U, \varphi(x)]$ is a bounded derivation of $\mathcal{A}$ into $\mathcal{L}^p(\mathcal{H})$, so there exists a $T \in \mathcal{L}^p(\mathcal{H})$ such that

$$[U, \varphi(x)] = [T, \varphi(x)] \qquad \text{for all } x \in \mathcal{A}.$$

Hence, $\mathscr{F}$ is a p-summable perturbation of the nearly degenerate module

$$\left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & U^* - T^* \\ U - T & 0 \end{bmatrix}\right).$$

Next, let $\mathcal{A}$ be non-unital. Since $\mathbb{C}$ is amenable and extensions of amenable Banach algebras by amenable Banach algebras are again amenable [Joh72, Proposition 5.1], $\mathcal{A}^+$ is amenable as well.

Thus, $K_p^1(\mathcal{A}) = 0$ immediately follows from the unital case and Proposition 3.2.1. For the even case, consider the following diagram.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & K_p^0(\mathcal{A}) & \xleftarrow{\iota^*} & K_p^0(\mathcal{A}^+) & \xleftarrow{\pi^*} & K_p^0(\mathbb{C}) & \longleftarrow & 0 \\
 & & \Phi \Big\uparrow & & \Phi^+ \Big\uparrow & & \Phi^{\mathbb{C}} \Big\uparrow & & \\
0 & \longleftarrow & \mathrm{FR}(\mathcal{A}) & \xleftarrow{r} & \mathrm{FR}(\mathcal{A}^+) & \xleftarrow{\pi^*} & \mathrm{FR}(\mathbb{C}) & \longleftarrow & 0
\end{array}$$

In this diagram Φ, Φ^+ and $\Phi^{\mathbb{C}}$ denote the natural maps from Theorem 3.1.5, $\pi : \mathcal{A}^+ \longrightarrow \mathbb{C}$ is the quotient map, and r is obtained by restricting representations of $\mathcal{A}^+$ to $\mathcal{A}$ and then taking their non-degenerate part.

The commutativity of the square to the right is immediate from the definitions, and the square to the left is commutative by Proposition 2.4.7. The upper row is exact by Proposition 3.2.1. Moreover, by definition we have $r \circ \pi^* = 0$, and we can extend every representation of $\mathcal{A}$ to a unital representation of $\mathcal{A}^+$, so r is surjective. The injectivity of π^* and the exactness in the middle of the lower row is given by Proposition 1.2.3. Thus, the lower row is exact as well.

We already know that Φ^+ and $\Phi^{\mathbb{C}}$ are isomorphisms. Therefore, Φ must also be an isomorphism by the Five Lemma.

Finally, since the isomorphisms between $\mathrm{FR}(A)$ and $K_p^0(\mathcal{A})$ are compatible with the connecting maps $K_p^0(\mathcal{A}) \longrightarrow K_q^0(\mathcal{A})$ $(p < q)$, we obtain the same isomorphism for the inductive limit $K_{fin}^0(\mathcal{A})$. □

7.3 Higher rank Lie groups

In [Pus11] Puschnigg discusses the existence of finitely summable modules over higher rank Lie groups. A higher rank Lie group is a product of simple real Lie groups (with finite center) which are of real rank at least 2.

If Γ is a discrete group, denote by $\mathrm{FR}(\Gamma)$ the Grothendieck group of the monoid of unitary equivalence classes of finite-dimensional unitary representations of Γ with addition given by the direct sum.

The author proves:

7.3.1 Theorem. *Let* Γ *be a lattice in a higher rank Lie group and* $\mathbb{C}\Gamma$ *the group algebra of* Γ *endowed with the* $C^*(\Gamma)$ *topology. Then for* $p > 1$

$$K_p^1(\mathbb{C}\Gamma) = 0$$

and there is a natural isomorphism

$$K_p^0(\mathbb{C}\Gamma) \cong \mathrm{FR}(\Gamma).$$

The same is true for K_{fin}^*.

The starting point for the proof of this theorem is the work of Bader, Furman, Gelander and Monod on rigidity theory in [BFGM07] where the authors show that for a standard measure space (X, μ), any action of Γ on $L^p(X, \mu)$ $(1 < p < \infty)$ by affine isometries has a global fixed point.

Puschnigg transfers this result to the non-commutative L^p-spaces $\mathcal{L}^p(\mathcal{H})$ [Pus11, Corollary 5.10]. In particular, he obtains that if $\mathscr{F} = (\varphi, \mathcal{H}, F)$ is a p-summable Fredholm module over $\mathbb{C}\Gamma$ with unital φ, then the action of Γ on $\mathcal{L}^p(\mathcal{H})$ which for $\gamma \in \Gamma$ is given by

$$T \mapsto \varphi(\gamma)T\varphi(\gamma^{-1}) + \Big(\varphi(\gamma)F\varphi(\gamma^{-1}) - F\Big)$$

has a global fixed point $T_0 \in \mathcal{L}^p(\mathcal{H})$. This implies that $(\varphi, \mathcal{H}, F + T_0)$ is a nearly degenerate module. By a similar construction for even modules, one sees that the conditions of Theorem 3.1.5 are met (we use the same line of argument in Section 7.4, where we provide more details). The theorem now follows by noting that there is a one-to-one correspondence between the continuous, non-degenerate, finite-dimensional representations of $\mathbb{C}\Gamma$ (with the $C^*(\Gamma)$ topology) and the finite-dimensional unitary representations of Γ.[†]

By the same argument the author obtains:

7.3.2 Theorem. *Let* Γ *be a lattice in a higher rank Lie group and* $\mathbb{C}\Gamma$ *the group algebra of* Γ *endowed with the* $C^*_{red}(\Gamma)$ *topology. Then for* $p > 1$

$$K_p^*(\mathbb{C}\Gamma) = 0.$$

The same is true for K_{fin}^*.

The proof carries over verbatim from Theorem 7.3.1. The only thing to note is that $\mathrm{FR}(\mathbb{C}\Gamma)$ contains fewer representations in this case, as not every unitary representation of Γ induces a continuous representation of $\mathbb{C}\Gamma$ with respect to the $C^*_{red}(\Gamma)$ topology. In fact, Puschnigg argues that Γ, being non-compact and enjoying Kazhdan's Property (T), is not amenable from which it follows that no finite-dimensional unitary representation of Γ extends to a continuous representation of $C^*_{red}(\Gamma)$. Therefore, $\mathrm{FR}(\mathbb{C}\Gamma) = 0$.

[†]Puschnigg proves the injectivity of the map from $\mathrm{FR}(\Gamma)$ to $K_p^0(\mathbb{C}\Gamma)$ differently by showing that the composition

$$\mathrm{FR}(\Gamma) \longrightarrow K_p^0(\mathbb{C}\Gamma) \longrightarrow K^0(C^*(\Gamma)) \xrightarrow{\mathrm{Ind}} \mathrm{Hom}(K_0(C^*(\Gamma)), \mathbb{C})$$

is injective for every group enjoying Kazhdan's property (T) [Pus11, Lemma 3.7].

7.4 Algebras with bounded groups of unitaries

A Banach space V is called L-embedded if its bidual can be decomposed as

$$V^{**} = V \oplus V_0,$$

where V is identified with its canonical embedding in V^{**}, such that

$$\|v + v_0\| = \|v\| + \|v_0\| \qquad \text{for all } v \in V,\ v_0 \in V_0.$$

In particular, reflexive Banach spaces and preduals of von Neumann algebras[†] are L-embedded. Since the Schatten classes $\mathcal{L}^p(\mathcal{H})$ $(1 < p < \infty)$ are reflexive and $\mathcal{L}^1(\mathcal{H})$ is the predual of $\mathcal{B}(\mathcal{H})$, we see that every $\mathcal{L}^p(\mathcal{H})$ $(1 \leq p < \infty)$ is L-embedded.

In [BGM11] the authors prove the following fixed point theorem:

7.4.1 Theorem. *Let X be a non-empty bounded subset of an L-embedded Banach space V. Then there is a common fixed point in V for all affine isometries S of V satisfying $S(X) = X$.*

The proof proceeds as follows. Let $\rho_V(X)$ denote the circumradius of X in V given by

$$\rho_V(X) := \inf\{r \geq 0 \,|\, \exists x \in V : X \subseteq \overline{B}(x, r)\}.$$

The Chebyshev center of X in V is then defined as

$$C_V(X) := \{c \in V \,|\, X \subseteq \overline{B}(c, \rho_V(X))\}.$$

The authors show that if V is L-embedded, then $C_V(X)$ is convex, weakly compact and non-empty. Since $C_V(X)$ is preserved by any affine isometry of V preserving X and since the set of all these maps forms a non-contracting semigroup (Definition A.1.8), the Ryll-Nardzewski fixed point theorem (Theorem A.1.9) applies and provides the desired fixed point. Note that this fixed point, while lying in $C_V(X)$, does not need to lie in X.

Similarly to Corollary C in [BGM11], we can conclude:

7.4.2 Theorem. *If $\mathcal{A}$ is a barreled topological $*$-algebra such that $\tilde{\mathcal{A}}$ contains a bounded group $\mathcal{U}$ of unitaries generating a dense subalgebra of $\tilde{\mathcal{A}}$, then*

$$K_p^1(\mathcal{A}) = 0$$

and there is a natural isomorphism

$$K_p^0(\mathcal{A}) \cong \mathrm{FR}(\mathcal{A}).$$

The same is true for K^*_{fin}.

[†] If M is a von Neumann algebra, then M^* can be decomposed into M_* and a so-called singular part $M^\perp$ [Tak02, Theorem III.2.14].

Proof. First, let $\mathcal{A}$ be unital and let $\mathscr{F} = (\varphi, \mathcal{H}, F)$ be an odd Fredholm module over $\mathcal{A}$. We want to show that $\mathscr{F}$ is a p-summable perturbation of a nearly degenerate module so that we can apply Theorem 3.1.5 again.

Note that $\mathscr{F}$ is a p-summable perturbation of the direct sum

$$\Big(\varphi(1_{\mathcal{A}})\varphi, \varphi(1_{\mathcal{A}})\mathcal{H}, \varphi(1_{\mathcal{A}})F\varphi(1_{\mathcal{A}})\Big) \oplus \Big(0, \varphi(1_{\mathcal{A}})^{\perp}\mathcal{H}, \varphi(1_{\mathcal{A}})^{\perp}F\varphi(1_{\mathcal{A}})^{\perp}\Big).$$

Since the second summand is nearly degenerate, we can assume that φ is unital.

For each $u \in \mathcal{U}$ define α_u to be the map $\alpha_u : \mathcal{L}^p(\mathcal{H}) \longrightarrow \mathcal{L}^p(\mathcal{H})$ given by

$$\alpha_u(T) = \varphi(u)T\varphi(u^*) + \Big(\varphi(u)F\varphi(u^*) - F\Big) \qquad \text{for all } T \in \mathcal{L}^p(\mathcal{H}).$$

Since each $u \in \mathcal{U}$ is unitary and φ is a unital $*$-homomorphism the α_u are affine isometries. Moreover,

$$\begin{aligned} &\alpha_u(\alpha_v(T)) \\ &\quad = \varphi(u)\Big(\varphi(v)T\varphi(v^*) + \varphi(v)F\varphi(v^*) - F\Big)\varphi(u^*) + \varphi(u)F\varphi(u^*) - F \\ &\quad = \varphi(uv)T\varphi(v^*u^*) + \varphi(uv)F\varphi(v^*u^*) - \varphi(u)F\varphi(u^*) + \varphi(u)F\varphi(u^*) - F \\ &\quad = \alpha_{uv}(T). \end{aligned}$$

Thus, α is a group action.

$\mathcal{A}$ is barreled and $\mathcal{U}$ is bounded, so we know by Proposition 3.3.1 that

$$\{\varphi(u)F\varphi(u^*) - F \mid u \in \mathcal{U}\}$$

is bounded in $\mathcal{L}^p(\mathcal{H})$. Since this is the orbit of $0 \in \mathcal{L}^p(\mathcal{H})$, it is preserved by α. Therefore, Theorem 7.4.1 implies that there is a $T_0 \in \mathcal{L}^p(\mathcal{H})$ such that

$$\varphi(u)T_0\varphi(u^*) + \varphi(u)F\varphi(u^*) - F = T_0 \qquad \text{for all } u \in \mathcal{U},$$

thus

$$[F + T_0, \varphi(u)] = 0 \qquad \text{for all } u \in \mathcal{U}.$$

Since this relation extends to the C^*-algebra generated by $\{\varphi(u) \mid u \in \mathcal{U}\}$, it holds for all $x \in \mathcal{A}$. Hence, $\mathscr{F}$ is a p-summable perturbation of the nearly degenerate module $(\varphi, \mathcal{H}, F + T_0)$.

For the even case, let

$$\mathscr{F} = \left(\varphi \oplus \varphi,\, \mathcal{H} \oplus \mathcal{H},\, \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix}\right) \in \widetilde{\mathbb{F}}^0_p(\mathcal{A})$$

be given. By the same argument as for the odd case, it suffices to consider instead modules $\mathscr{F}'$ of the form

$$\mathscr{F}' = \left(\varphi' \oplus \varphi',\, \mathcal{H}' \oplus \mathcal{H}',\, \begin{bmatrix} 0 & U'^* \\ U' & 0 \end{bmatrix}\right)$$

where φ' is unital but U' is no longer necessarily unitary.

Let $\mathcal{U}$ act on $\mathcal{L}^p(\mathcal{H})$ by the affine isometries

$$\alpha_u(T) = \varphi'(u)T\varphi'(u^*) + \Big(\varphi'(u)U'\varphi'(u^*) - U'\Big).$$

Proposition 3.3.1 implies again that $\{\varphi'(u)U'\varphi'(u^*) - U' \mid u \in \mathcal{U}\}$ is bounded. If $T_0 \in \mathcal{L}^p(\mathcal{H})$ is a fixed point of the action, then

$$\left(\varphi' \oplus \varphi', \mathcal{H}' \oplus \mathcal{H}' \begin{bmatrix} 0 & U'^* + T_0^* \\ U' + T_0 & 0 \end{bmatrix}\right)$$

is nearly degenerate and a p-summable perturbation of $\mathscr{F}'$.

Finally, if $\mathcal{A}$ is non-unital, note that $\mathcal{A}^+$ is barreled if $\mathcal{A}$ is because $\mathcal{A}^+$ is (as a topological vector space) the direct sum of $\mathcal{A}$ and $\mathbb{C}$, and direct sums of barreled spaces are barreled again (Proposition A.1.5). Therefore, what we have proven applies to $\mathcal{A}^+$ and we can reduce the non-unital to the unital case as in the proof of Theorem 7.2.2.

The analogous statement for K^*_{fin} again follows immediately from the statement for K^*_p. □

7.4.3 Corollary. *If A is a C^*-algebra, then*

$$K^0_p(A) \cong \mathrm{FR}(A) \quad \textit{and} \quad K^1_p(A) = 0$$

as well as

$$K^0_{fin}(A) \cong \mathrm{FR}(A) \quad \textit{and} \quad K^1_{fin}(A) = 0.$$

7.4.4 Corollary. *If Γ is a discrete group, then*

$$K^0_p(\ell^1(\Gamma)) \cong \mathrm{FR}(\ell^1(\Gamma)) \quad \textit{and} \quad K^1_p(\ell^1(\Gamma)) = 0$$

as well as

$$K^0_{fin}(\ell^1(\Gamma)) \cong \mathrm{FR}(\ell^1(\Gamma)) \quad \textit{and} \quad K^1_{fin}(\ell^1(\Gamma)) = 0.$$

Appendix

A.1 Functional analysis

A.1.1 Definition. A topological vector space is a vector space endowed with a topology making addition and scalar multiplication jointly continuous.

A.1.2 Definition. A subset X of a topological vector space V is called bounded if for every neighbourhood U of 0 there exists a $\lambda \in \mathbb{R}$ such that

$$X \subseteq \lambda U.$$

A.1.3 Definition. Let V be a topological vector space. A barrel in V is an absorbing, balanced, closed, and convex set. V is called barreled if every barrel in V is a neighbourhood of 0.

A.1.4 Examples. The class of barreled spaces includes:

— Fréchet spaces, in particular all Banach spaces

— all locally convex spaces which are Baire spaces

— LF-spaces (e.g. the test spaces $C_c^\infty(\mathbb{R}^n)$ in distribution theory)

[SW99, Chapter II, 7.1 and 7.2].

A.1.5 Proposition. *If V_1 and V_2 are barreled topological vector spaces, then $V_1 \oplus V_2$ is a barreled topological vector space.*

Proof. Let X be a barrel in $V_1 \oplus V_2$. If $\iota_i : V_i \longrightarrow V_1 \oplus V_2$ $(i = 1, 2)$ denote the embeddings of V_i into $V_1 \oplus V_2$, then $\iota_i^{-1}(X)$ are barrels in V_i containing open neighbourhoods U_i of $0 \in V_i$. Thus, X contains $U_1 \oplus 0 \cup 0 \oplus U_2$ and, since X is convex, the open set $\frac{1}{2}U_1 \oplus \frac{1}{2}U_2$. □

A.1.6 Proposition. *If V_1 is a barreled normed space and V_2 is a barreled locally convex space, then their projective tensor product $V_1 \hat{\otimes} V_2$ is barreled as well.*

Proof. [PCB87, Proposition 11.2.2]. □

A.1.7 Theorem (Closed Graph Theorem)**.** *Let $f : V \longrightarrow W$ be a linear map between a barreled space V and a Fréchet space W. If*

$$\mathrm{Graph}(f) := \{(v, f(v)) \mid v \in V\} \subseteq V \times W$$

is closed, then f is continuous.

Proof. [SW99, Chapter IV, 8.5]. □

A.1.8 Definition. Let $Q \subset V$ be a subset of a locally convex vector space. A set $\mathcal{S}$ of maps from Q into itself is called non-contracting if for all $\xi, \eta \in Q$ with $\xi \neq \eta$ there is a continuous semi-norm ν on V and an $\varepsilon > 0$ such that

$$\nu(S\xi - S\eta) > \varepsilon \qquad \text{for all } S \in \mathcal{S}.$$

A.1.9 Theorem (Ryll-Nardzewski)**.** *Let Q be a non-empty, weakly compact, convex subset of a Hausdorff locally convex vector space V. If $\mathcal{S}$ is a non-contracting semigroup of weakly continuous affine maps of Q into itself, then all maps in $\mathcal{S}$ share a common fixed point.*

Proof. [NA67]. □

A.2 Finite-dimensional C^*-algebras and matrix units

A.2.1 Definition. Let $\mathcal{A}$ be a $*$-algebra and $n_k \in \mathbb{N}$ $(1 \leq k \leq N)$. A set $\{e_{ij}^{(k)} \mid 1 \leq i, j \leq n_k\} \subseteq A$ is called a system of matrix units if its elements satisfy the conditions

$$(e_{ij}^{(k)})^* = e_{ji}^{(k)} \quad \text{and} \quad e_{i'j'}^{(k')} e_{ij}^{(k)} = \delta_{kk'} \delta_{j'i} e_{i'j}^{(k)}$$

for all $1 \leq i, j \leq n_k$ and $1 \leq i', j' \leq n_{k'}$. We also write $e_i^{(k)}$ for the projections $e_{ii}^{(k)}$.

A.2.2 Theorem. *If A is a finite-dimensional C^*-algebra, then A is isomorphic to the direct sum*

$$A \cong \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{C}),$$

for some natural numbers N and n_i $(1 \leq i \leq N)$.

Proof. [Dav96, Theorem III.1.1]. □

A.2.3 Definition. Let A be a finite-dimensional C^*-algebra. We call a system of matrix units $\{e_{ij}^{(k)}\} \subseteq A$ a system of matrix units for A (short: matrix units for A) if it forms a linear basis of A.

By Theorem A.2.2, one can find a system of matrix units for each finite-dimensional C^*-algebra.

A.2.4 Proposition. *Let A be a C^*-subalgebra of a finite-dimensional C^*-algebra B and $\{e_{ij}^{(k)}\}$ a system of matrix units for A. Then there is a compatible system of matrix units $\{f_{rs}^{(l)}\}$ for B in the sense that each $e_{ij}^{(k)}$ can be uniquely written as*

$$e_{ij}^{(k)} = \sum_{n=1}^{m} f_{r_n s_n}^{(l_n)}$$

where the $f_{r_n s_n}^{(l_n)}$ have pairwise orthogonal support and range projections.

Proof. Without loss of generality we can assume that A and B share the same unit (otherwise replace A by $A \oplus \mathbb{C}(1_B - 1_A)$). Furthermore, by considering every direct summand of B separately, we can assume that $B = M_n(\mathbb{C})$.

Decompose each $e_1^{(k)}$ as an orthogonal sum of minimal projections

$$e_1^{(k)} = f_{k11} + f_{k12} + \ldots + f_{k1m_k},$$

and choose arbitrary partial isometries $f_{k1r,111}$ between f_{k1r} and f_{k11}. Then define

$$f_{kir,111} := e_{i1}^{(k)} \cdot f_{k1r,111} \quad \text{and} \quad f_{kir,ljs} := f_{kir,111} \cdot f_{ljs,111}^*.$$

Since A and B share the same unit, this defines matrix units for B.

If $e_{ij}^{(k)}$ belongs to the matrix units for A, then

$$\begin{aligned}
\sum_{r=1}^{m_k} f_{kir,kjr} &= \sum_{r=1}^{m_k} e_{i1}^{(k)} \cdot f_{k1r,111} \cdot \left(e_{j1}^{(k)} \cdot f_{k1r,111}\right)^* \\
&= e_{i1}^{(k)} \Big(\sum_{r=1}^{m_k} f_{k1r,111} \cdot f_{k1r,111}^*\Big) e_{1j}^{(k)} \\
&= e_{i1}^{(k)} \cdot e_1^{(k)} \cdot e_{1j}^{(k)} \\
&= e_{ij}^{(k)}.
\end{aligned}$$

□

A.3 Completely positive maps

A.3.1 Definition. Let A and B be C^*-algebras. A bounded linear map $\varphi : A \longrightarrow B$ is called positive if it maps positive elements to positive elements. φ is called completely positive if $\varphi^{(n)} : M_n(A) \longrightarrow M_n(B)$ is positive for all $n \in \mathbb{N}$, where $\varphi^{(n)}$ is defined as

$$\varphi^{(n)}([a_{ij}]) := [\varphi(a_{ij})] \qquad \text{for } [a_{ij}] \in M_n(A).$$

We call φ a completely positive contraction (c.p.c.) if φ is completely positive and $\|\varphi\| \leq 1$.

A.3.2 Example. If $\varphi : A \longrightarrow B$ is a $*$-homomorphism between C^*-algebras and $x \in B$ with $\|x\| \leq 1$, then $x\varphi(\cdot)x^*$ is a completely positive contraction.

A.3.3 Theorem (Stinespring)**.** *Let A be a C^*-algebra, $\mathcal{H}$ a Hilbert space and $\varphi : A \longrightarrow \mathcal{B}(\mathcal{H})$ a completely positive contraction. Then there exists a Hilbert space $\mathcal{H}'$ and a $*$-representation*

$$\tilde{\varphi} : A \longrightarrow \mathcal{B}(\mathcal{H} \oplus \mathcal{H}')$$

such that $\tilde{\varphi}$ is of the form

$$\tilde{\varphi}(x) = \begin{bmatrix} \varphi(x) & \tilde{\varphi}_{12}(x) \\ \tilde{\varphi}_{21}(x) & \tilde{\varphi}_{22}(x) \end{bmatrix} \qquad \textit{for all } x \in A.$$

$\tilde{\varphi}$ is called a Stinespring dilation of φ.

Proof. [BO08, Theorem 1.5.3 and Remark 1.5.4]. □

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