On moduli of vector bundles on p-adic curves and attached representations

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Deutschsprachige Zusammenfassung

Über Moduli von Vektorbündeln auf p-adischen Kurven und zugeordneten Darstellungen

In der vorliegenden Arbeit wird das von Deninger/Werner entwickelte p-adische Analogon der klassischen Narasimhan-Seshadri Theorie hinsichtlich der Formulierbarkeit in den Termen der Moduli von Vektorbündeln und entsprechenden Darstellungen untersucht.

Sei X eine glatte, projektive und zusammenhängende Kurve über $\overline{\mathbb{Q}}_p$. Einem Vektorbündel E mit stark semistabiler Reduktion auf $X_{\mathbb{C}_p}$ ordnet das étale Paralleltransport unter anderem eine stetige endlich-dimensionale Darstellung der étalen Fundamentalgruppe von X zu. Andererseits ist jedes Vektorbündel mit stark semistabiler Reduktion ebenfalls semistabil, induziert also einen $\mathbb{C}_p$-wertigen Punkt in dem Modulraum M_X, der semistabile Vektorbündel von entsprechendem Rang und Grad parametrisiert.

In der vorliegenden Arbeit wird gezeigt, dass die Klasse der Vektorbündel auf $X_{\mathbb{C}_p}$ (von festem Rang und Grad), die stark semistabile Reduktion über $\overline{\mathbb{Z}}_p$ haben, im p-adischen Sinne eine offene Teilmenge in $M_X(\overline{\mathbb{Q}}_p)$ induziert. Desweiteren beschreiben wir die obige Zuordnung der Darstellungen in den Termen der Moduli von Vektorbündeln sowie zugeordneten Darstellungen. Wir zeigen, dass diese unter einer technischen Voraussetzung stetig ist. Dafür werden die Methoden aus dem Beweis des ersten Resultats verwendet.

Contents

Introduction

The classical Narasimhan-Seshadri correspondence relates stable vector bundles of degree zero on a Riemann surface to irreducible unitary representations of its fundamental group. In the last years a partial p-adic analogue of the Narasimhan-Seshadri correspondence has been simultaneously developed by Deninger/Werner in [DW05a, DW05b, DW07, Den10, DW10] and Faltings in [Fal05]. In fact, Faltings even developed a p-adic version of Simpson's theory.

Recall that a vector bundle E on a smooth projective curve over a field of characteristic p is called strongly semistable if the pullbacks of E by all non-negative powers of the absolute Frobenius morphism are strongly semistable. A vector bundle on a reduced not necessarily irreducible curve is called strongly semistable if its pullback to the normalization of every irreducible component is strongly semistable. Let X be a smooth projective and connected curve over $\overline{\mathbb{Q}}_p$. Let $\mathbb{C}_p$ be the field of p-adic complex numbers and $\mathfrak{o}$ its ring of integers. Deninger and Werner introduced the following subclass of semistable vector bundles on $X_{\mathbb{C}_p}$: A vector bundle E on $X_{\mathbb{C}_p}$ has strongly semistable reduction (of degree zero) if there exist a certain model $\mathfrak{X}$ over $\overline{\mathbb{Z}}_p$ and a vector bundle $\mathcal{E}$ on $\mathfrak{X}_\mathfrak{o}$ with generic fiber isomorphic to E such that the special fiber of $\mathcal{E}$ is strongly semistable. For this class of vector bundles they established the étale parallel transport, which assigns to E a continuous representation of the étale fundamental groupoid of X. In particular, after restriction to the étale fundamental group $\pi_1(X, x)$ of X, one obtains for each E a p-adic representation of this group.

In the classical theory the correspondence is also formulated in terms of moduli of vector bundles and corresponding representations. The goal of this thesis is to study in what extent this situation can be adapted to the p-adic analogue.

As already mentioned, every vector bundle on $X_{\mathbb{C}_p}$ having strongly semistable reduction is semistable. Let M_X be the moduli space of semistable vector bundles on X of fixed rank and degree. We may consider the latter class of vector bundles (of suitable rank and degree) as a subset of the set of $\mathbb{C}_p$-valued points of M_X. We formulate the following question: Is this subset p-adically open in $M_X(\mathbb{C}_p)$? The answer for the vector bundles of degree zero having (potentially) strongly semistable reduction over $\overline{\mathbb{Z}}_p$ is positive.

Theorem (7.2, 10.5, 10.9). *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ of genus $g \geq 2$. Let E be a vector bundle on X of degree zero, which has potentially strongly semistable reduction. Let M_X be the moduli space of vector bundles on X of rank* $\operatorname{rk} E$ *and degree zero. Then there exists a p-adic neighborhood $U \subset M_X(\overline{\mathbb{Q}}_p)$ of the S-equivalence class of E consisting of S-equivalence classes of vector bundles having potentially strongly semistable reduction.*

By definition, the property of E to have strongly semistable reduction depends on the reduction of some model $\mathcal{E}$ of E over a certain model $\mathfrak{X}$ of X over $\overline{\mathbb{Z}}_p$. The fundamental theorem of Langer/Maruyama [Lan04b, Mar96] states the existence of a certain moduli space M of semistable sheaves in mixed characteristic. Applying this theorem to $\mathfrak{X}$ we could try to use

M as a connection between E and its reduction. However, the difficulty lies in the question whether $\mathcal{E}$ belongs to the class of objects which are parametrized by M. Assuming that $\mathfrak{X}$ is simple enough, we can overcome this problem by using a result of Teixidor i Bigas [TiB95], which establishes a characterization of semistability for vector bundles on semistable curves.

Eventually, we have to eliminate the assumption on the model $\mathfrak{X}$ to be simple enough, in our terms, to be *almost stable*, which means that the special fiber of $\mathfrak{X}$ is a semistable curve, and every rational component of the special fiber intersects the other irreducible components in at least two points.

Theorem (10.10)**.** *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ of genus $g \geq 1$. Let E be a vector bundle on X. Assume that E has a reduction which is either (i) strongly semistable of degree zero or (ii) trivial. Then E has also a reduction with (i) resp. (ii) over an almost stable model.*

The proof of this theorem is based on the birational geometry of arithmetic surfaces over discrete valuation rings and a theorem of Ishimura [Ish83] about the descent of vector bundles on blowing-ups of regular surfaces. These tools force us to consider the reduction of E over $\overline{\mathbb{Z}}_p$.

The last important ingredient in the construction of the p-adic neighborhood is the well-known theorem of Langton. It allows us to extend semistable vector bundles on X to the whole model $\mathfrak{X}$, but it also forces us to work over $\overline{\mathbb{Z}}_p$.

Let $x \in X$ and $\pi = \pi_1(X, x)$ be the étale fundamental group of X. We construct a set-theoretic mapping

$$M_X(\overline{\mathbb{Q}}_p) \to \chi_\pi(\mathbb{C}_p),$$

where the topological space on the right hand side is the character space of p-adic representations of π, which is introduced in the third part. The proof of the first theorem yields a method to show the continuity of this map, however under a technical assumption (cf. Proposition 14.4). I do not know whether this assumption is satisfied or not.

This thesis is organized as follows. The first part is introductory. In the first section we consider the notion of semistability of pure sheaves. We begin with the general definition and then restrict it to the case of curves. In the end we give an example of a curve such that the trivial vector bundle on it is never semistable.

The second section deals with the moduli spaces of semistable sheaves. We give a general definition of a moduli space and recall the main existence theorem. Since we are particularly interested in the case of curves, we collect results for such moduli spaces.

In the third section we state and prove the theorem of Langton. As we could not find any proof of it in the generality we need, we include it here.

In the fourth section we introduce the notion of Jordan-Hölder filtrations in an abelian category. All statements in this section are straightforward to prove. However, since we need them for the category of vector bundles having strongly semistable reduction of degree zero as well as for the category of continuous finite dimensional representations, we decided to include this section to avoid duplication.

In the second part we construct the p-adic neighborhood in the moduli space of semistable vector bundles. In the fifth section we recall the notion of strongly semistable reduction and prepare some auxiliary results.

In the sixth section, in some sense, we test the idea for the construction of the p-adic neighborhood on families of vector bundles without considering the moduli spaces parameterizing them.

In the next section we adapt the results of the previous section to the situation of moduli spaces in the case of *good* strongly semistable reduction, that is, the models we consider are smooth. This assumption completely eliminates the difficulties with different notions of semistability on the special fiber.

Now after the main result is proved in a special case, we need to analyze the semistability of vector bundles on semistable curves. This was done by Teixidor i Bigas. Unfortunately, the original article contains small inaccuracies. Therefore we decided to include the full proof of the main result in the eighth section.

In the ninth section we construct in a purely combinatorial way a weighting of the irreducible components of a semistable curve provided the curve is almost stable. This weighting can then be used to construct an ample divisor such that certain vector bundles are semistable with respect to this divisor. The result is based on the theorem from the previous section.

In the tenth section we prove the main theorem. We proceed in an analogous way to the seventh section. We also include some redundant arguments to make this section self-contained without referring to the proof in the case of good reduction. The weightings from the previous section are used to adjust the different notions of semistability on the special fiber. The theorem is proved under the assumption that we can always reduce the situation to an almost stable model. In the second part of this section we show that this can indeed be achieved.

The eleventh section is more or less unconnected to the rest of this thesis. We describe the jet spaces of the moduli space of semistable vector bundles on a smooth projective and connected curve in terms of such bundles. We included this section since the parallel transport immediately induces a mapping on the jet space using this description. Using the mapping on the level of moduli constructed in the next section one can relate these, provided one has a "good" moduli space of continuous finite dimensional representations, where "good" means that its jet spaces can be defined.

In the third part we consider the parallel transport restricted to the étale fundamental group on the level of moduli. In the twelfth section we revisit the construction and the properties of the parallel transport for the vector bundles with strongly semistable reduction.

In the next section we define two topological spaces describing the isomorphism and Jordan-Hölder equivalence classes of continuous finite dimensional representations. We call the latter the character space of representations. Further, we use the notion of (uniform) physical convergence and convergence in trace introduced by Bellaïche, Chenevier, Khare and Larsen to describe convergence in these spaces.

In the fourteenth section a mapping is defined on the level of moduli of vector bundles to the character space introduced before. It is induced by the parallel transport restricted to the étale fundamental group. First we define it for the families and show the continuity of this mapping. Then we apply a similar approach to the moduli of semistable vector bundles. The continuity of the parallel transport follows, provided a technical assumption is satisfied. The proofs rely on variants of the main results from the second part.

In the last section we give a necessary condition for a p-adic representation, considered there, to be induced by the parallel transport of a vector bundle, which has strongly semistable reduction.

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Part 1

Preliminaries

1. Semistability of pure sheaves

Let E be a coherent sheaf on a Noetherian scheme X. The *dimension* $\dim E$ of E is the dimension of the support $Supp(E)$ of E. We say that E is *pure of dimension* d if $\dim F = d$ for all proper coherent subsheaves $F \subset E$. The sheaf E is pure if it is pure of dimension $d = \dim E$. The purity of a sheaf is a generalization of the property to be torsion free on an integral scheme. Indeed, there exists a unique filtration of a coherent sheaf E

$$0 \subset T_0(E) \subset T_1(E) \subset \ldots \subset T_d(E) = E,$$

where d is the dimension of E and $T_i(E)$ is the maximal subsheaf of E of dimension less or equal i. The sheaf E is then pure of dimension d if and only if $T_{d-1}(E) = 0$. A coherent sheaf E on an integral scheme X is *torsion free* if and only if its torsion subsheaf $T(E) \subset E$ is zero. By definition of the above filtration we have $T(E) = T_{d-1}(E)$.

Proposition 1.1. *A coherent sheaf E on a Noetherian scheme X is pure of dimension d if and only if all associated points of E have the same dimension d.*

Proof. Denote by $Ass(E)$ the set of all associated points of E.

Assume that E is pure of dimension d. Let $x \in Ass(E)$ and $\overline{\{x\}}$ be the closure of $\{x\}$ in X. There exist an open neighborhood $U \subset X$ of x and a quasi-coherent subsheaf $F' \subset E|_U$ on U such that $U \cap \overline{\{x\}}$ is an irreducible component of $Supp(F')$ (cf. [EGAIV$_2$] Proposition 3.1.3). We may extend F' to a quasi-coherent subsheaf F of E, which is a posteriori coherent. By assumption it follows that $\dim E = 0$, in particular

$$d = \dim\left(\overline{\{x\}} \cap U\right) = \dim_x U = \dim_x X = \dim \overline{\{x\}}.$$

Conversely, let $F \subset E$ be a subsheaf on X. Let x be a generic point of an irreducible component of $Supp(F)$. Then $x \in Ass(F)$ (cf. *loc. cit.* 3.1.1.1). Since $Ass(F) \subset Ass(E)$, we have by assumption $\dim \overline{\{x\}} = d$. It follows that $\dim F = d$. □

Example. Let X be a projective reduced (not necessarily irreducible) curve over an algebraically closed field k. Seshadri introduced in [Ses82] septième partie the notion of coherent sheaves of depth one. A module M over a local ring $(A, \mathfrak{m})$ is called to be *of depth one* if there exists an element in $\mathfrak{m}$ which is not a zero-divisor of M. A coherent sheaf E on X is of depth one if E_x is of depth one over $\mathcal{O}_{X,x}$ for all $x \in X$. One can show that the restriction modulo torsion of a sheaf of depth one to an irreducible component of X is either zero or torsion-free. Therefore in this situation the notion of a pure one dimensional sheaf coincides by the above proposition with the notion of a depth one sheaf.

Now let X be a projective scheme over a field k. The cohomology groups $H^i(X, E)$ are finite dimensional vector spaces over k. We put $h^i(E) = h^i(X, E) = \dim_k H^i(X, E)$. Since $h^i(E) = 0$ for $i > d = \dim X$, we may define

$$\chi(E) = \chi(X, E) = \sum_{i=0}^{d} h^i(X, E),$$

which is called the *Euler characteristic* of E. Let H be an ample line bundle on X. There exists a polynomial P_E with rational coefficients such that $P_E(m) = \chi(E \otimes H^m)$ of degree $\dim E$. This polynomial depends only on E and H and is called the *Hilbert polynomial* of E (with respect to H). We write

$$P_E(m) = P(E, m) = \sum_{i=0}^{\dim E} \alpha_i(E) \frac{m^i}{i!}$$

with rational coefficients $\alpha_i(E)$. The *reduced Hilbert polynomial* $p(E) = p_E$ of E is defined as the quotient $P_E/\alpha_d(E)$. There is a natural ordering $\leq$ of polynomials given by the lexicographic order of their coefficients. In other words, $p \leq p'$ if there is an $N \geq 1$ such that for all $n \geq N$ we have $p(n) \leq p'(n)$.

Definition 1.2. Let X be a projective scheme over a field k. A coherent sheaf E on X is called *(Gieseker) semistable* if it is pure and for all proper subsheaves $F \subset E$ we have the inequality

$$p_F \leq p_E.$$

If the inequality is strict, then we call E *(Gieseker) stable*. We say that E is *geometrically stable* if for every finite field extension K of k the inverse image E_K of E along $X_K = X \otimes_k K \to X$ is stable.

Obviously, if k is algebraically closed, the notions of stability and geometrical stability coincide. Analogously, we could define geometrical semistability, however in this case this would be equivalent to the usual notion (cf. [HL10] p. 13).

Proposition 1.3. *Consider a projective scheme X over a field k. The category of semistable sheaves on X with a fixed reduced Hilbert polynomial P is abelian, in which every object is Noetherian and Artinian. Further, it is closed under extensions. In particular, it is a Serre subcategory of the category of coherent sheaves on X.*

Proof. For the first statement cf. [HL10] Remark 1.5.12, for the second cf. [Mar96] Proposition 1.1 (6). □

Recall that a non-empty full subcategory $\mathcal{B}$ of an abelian category $\mathcal{A}$ is called a *Serre subcategory* if for every short exact sequence in $\mathcal{A}$

$$0 \to A' \to A \to A'' \to 0$$

the object A is in $\mathcal{B}$ if and only if A' and A'' are in $\mathcal{B}$.

The next proposition follows from the general framework of Jordan-Hölder filtrations in arbitrary abelian categories (cf. section 4). For a direct argument cf. [HL10] Proposition 1.5.2.

Proposition 1.4. *Consider be a projective scheme X over a field k. Let E be a semistable sheaf on X. There exists a filtration*

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E$$

by semistable sheaves on X such that the quotients $\operatorname{gr}_i E = E_i/E_{i-1}$ are stable with reduced Hilbert polynomial p_E. Up to isomorphism, the sheaf $\operatorname{gr} E = \bigoplus \operatorname{gr}_i E$ is uniquely determined by E.

Proof. The existence follows from the above proposition combined with Corollary 4.6, the uniqueness from Proposition 4.9. □

The above filtration is called *Jordan-Hölder filtration* of E. We say that two semistable sheaves E and E' on X are *S-equivalent* if the associated gradings $\operatorname{gr} E$ and $\operatorname{gr} E'$ are isomorphic. We will see later that S-equivalent sheaves cannot be distinguished in the moduli space. On the other hand, S-equivalence allows us to resolve the problem of the jump phenomenon (cf. Remark 2.5 (2)).

Semistability in case of smooth curves. Let X be a smooth projective and connected curve over an algebraically closed field k. Denote by g the (arithmetic) genus of X, that is, $g = h^1(\mathcal{O}_X)$. It follows either using the torsion filtration or from Proposition 1.1 that a coherent sheaf E on an integral scheme which is pure of dimension one is necessarily torsion-free. Therefore every pure sheaf on X is locally free. We give an argument using the proposition:

Let $T = T(E) \subset E$ be the torsion subsheaf. Then $Supp(T)$ consists of finitely many points. On the other hand, we have $Ass(T) \subset Ass(E)$. By the proposition every point in $Ass(E)$ is one dimensional, hence $Ass(T)$ is empty, since $Ass(T) \subset Supp(T)$. But then $T = 0$.

We denote by r the rank and by d the degree of E (the degree of the line bundle $\bigwedge^r E$). Then by the Riemann-Roch theorem for curves the Hilbert polynomial of E (with respect to an ample line bundle H of degree h) has the following simple form

$$P_E(m) = rhm + d + r(1 - g). \tag{1}$$

We define the *slope* $\mu(E)$ of E by

$$\mu(E) = \frac{d}{r}.$$

It follows that E is semistable if and only if $\mu(F) \leq \mu(E)$ for all proper subbundles $F \subset E$. This condition is obviously independent of the ample line bundle H. It is the original condition for the semistability introduced by Mumford in [Mum63] and successfully used by Seshadri in [Ses67] to solve the moduli problem of vector bundles in the present situation.

Semistability in case of singular curves. In case of a singular curve the situation is more complicated.

Let X be a reduced (not necessarily irreducible) projective curve over an algebraically closed field k. Fix an ample line bundle H on X of degree h and consider a coherent sheaf E on X. The Hilbert polynomial of E is of the form

$$P_E(m) = \alpha_1 m + \alpha_0.$$

If E is pure of dimension one, then α_1 is non-zero. We define the rank $\mathrm{rk}_H(E)$ and the degree $\deg_H(E)$ of E with respect to H by

$$\mathrm{rk}_H(E) = \frac{\alpha_1}{h}, \qquad \deg_H(E) = \alpha_0 - \mathrm{rk}_H(E)(1 - g).$$

In particular, by definition we have the identity (1), hence for $m = 0$

$$\deg_H(E) = \chi(E) - \mathrm{rk}_H(E)\chi(\mathcal{O}_X).$$

Note that rank and degree are not necessarily integers. It may also happen that the rank of E "jumps" on an irreducible component of X.

Example. Let X be the union of two smooth projective and connected curves X_1 and X_2 over k meeting at a unique point P. Let H be an ample divisor on X of degree h and H_1 its restriction to X_1 of degree h_1. Consider a locally free sheaf F of rank 2 and degree d on X_1, hence $\mathrm{rk}_{H_1} F = 2$ since it does not depend on the polarization on a smooth curve. The coherent sheaf $E = i_*F$ on X, where $i : X_1 \to X$ is the closed immersion, is pure of dimension one ($Ass(E)$ consists only of the generic point of X_1, now use Proposition 1.1) with Hilbert polynomial

$$P_E(m) = \chi(E \otimes H^m) = \chi(i^*E \otimes H_1^m) = \chi(F \otimes H_1^m) = 2h_1 m + \chi(F).$$

The second equality follows from the fact that E is supported on the closed subscheme $X_1 \subset X$ and therefore $H^i(X, E) = H^i(X_1, i^*E)$, the third using the isomorphism $i^*i_*F \cong F$. Thus $\mathrm{rk}_H(E) = 2h_1/h$. Since $h = h_1 + h_2$ (h_2 is the degree of the restriction of H to X_2), we have e.g. $\mathrm{rk}_H(E) = 1/2$ if $h_1 = 1$, $h_2 = 3$ and $\mathrm{rk}_H = 1$ if $h_1 = h_2$. On the other hand, $E|_{X_2} = E_P$, hence it is of rank zero.

Now we define as in the case of a smooth curve the *slope* $\mu_H(E)$ of E with respect to H by

$$\mu_H(E) = \frac{\deg_H(E)}{\mathrm{rk}_H(E)}.$$

Here the slope depends on H. The semistability translates in the given situation to the condition that for each proper subsheaf $F \subset E$ we have

$$\mu_H(F) \leq \mu_H(E).$$

Seshadri introduced in [Ses82] septième partie a slight different notion of semistability.

Assume for a moment that X is integral. Let E be a pure one-dimensional sheaf on X, in other words, it is torsion-free. The set of singular points X^{sing} of X is finite, denote by U its complement. Let E_U be the subsheaf of E consisting of sections vanishing on X^{sing}. Then E_U is locally free, say of rank r, and we have $\mathrm{rk}_H(E) = r$. Indeed, there exists an exact sequence

$$0 \to E_U \to E \to T \to 0,$$

where T is a skyscraper sheaf concentrated at X^{sing}. Hence $P_T(m)$ is constant, therefore $\mathrm{rk}_H(T) = 0$ and $\mathrm{rk}_H(E) = \mathrm{rk}_H(E_U)$. But the latter is just r by Riemann-Roch.

The above discussion allows us to define the rank of a torsion free sheaf on an integral curve just as the rank of its restriction to the open dense subset of non-singular points, which is a locally free sheaf.

Now let X be again reduced, not necessarily irreducible. Let $X_1, \dots, X_n$ be the irreducible components of X (we always endow each X_i with the reduced subscheme structure).

For a coherent sheaf E on X we denote by E_i the restriction of E to X_i modulo torsion, i.e. the inverse image of E along the closed immersion $X_i \hookrightarrow X$ modulo its torsion subsheaf. Let $a = (a_1, \dots, a_n)$ be a weighting of irreducible components of X, that is, a_i's are rational numbers with $0 < a_i < 1$ and $\sum_{i=1}^n a_i = 1$. For a pure one-dimensional sheaf E on X we define the *a-rank* by

$$a\text{-}\operatorname{rk}(E) = \sum_{i=1}^{n} a_i \operatorname{rk}(E_i).$$

Definition 1.5. Let X be a reduced projective curve over a field k. We say that a pure one-dimensional sheaf E on X is *a-semistable* if for all proper subsheaves $F \subset E$ we have the inequality

$$\frac{\chi(F)}{a\text{-}\operatorname{rk}(F)} \le \frac{\chi(E)}{a\text{-}\operatorname{rk}(E)}.$$

If the inequality is strict, then we say that E is *a-stable.*

Proposition 1.6. *Consider X and E as in the above definition. Let d_i be the degree of L restricted to X_i. Then we have*

$$\chi(E \otimes L) = \chi(E) + \sum_{i=1}^{n} d_i \operatorname{rk}(E_i).$$

Proof. Cf. [Ses82] septième partie, corolláire 8. □

Let h be the degree of the fixed ample line bundle H on X, h_i the degree of the restriction of H to X_i. Define a weighting a by $a_i = h_i/h$. We use the above proposition to write down the Hilbert polynomial of E in a different form

$$P_E(m) = \chi(E \otimes H^m) = \sum h_i \operatorname{rk}(E_i)m + \chi(E),$$

concluding that $a\text{-}\operatorname{rk}(E) = \operatorname{rk}_H(E)$. One sees immediately that semistability is equivalent to a-semistability for this specific weighting a.

At last, we give an example of a curve X such that the trivial line bundle is never semistable, i.e. it is not semistable with respect to any ample line bundle H on X. The next proposition gives a characterization of semistability for a line bundle. Later we will use a more powerful theorem of Teixidor i Bigas (cf. Theorem 8.2).

Proposition 1.7. *Consider a reduced connected and projective curve X over an algebraically closed field k with a fixed ample line bundle H. Let h_i be the degree of H restricted to X_i. Let L be a line bundle on X. Then L is semistable with respect to H if and only if for every proper and connected curve $D \subset X$ the following inequality is satisfied*

$$\frac{h_D}{h}\chi(L) \le \chi(L_D),$$

where $h_D = \sum_{X_i \subset D} h_i$, and L_D is the restriction of L to D.

Proof. Assume that L is H-semistable. Let $D \subset X$ be a connected proper subcurve. By computing the ranks

$$\operatorname{rk}_H(L) = \sum \frac{h_i}{h} \operatorname{rk}(L_i) = \sum \frac{h_i}{h} = 1,$$
$$\operatorname{rk}_H(L_D) = \sum_{X_i \subset D} \frac{h_i}{h} \operatorname{rk}(L_{D,i}) = \sum_{X_i \subset D} \frac{h_i}{h} \operatorname{rk}(L_i) = \frac{h_D}{h},$$

we obtain the above inequality using the characterization of semistability by Seshadri.

Conversely, let $M \subset L$ be a subsheaf. Let $\overline{D}$ be the support of M and D the closure of its complement in X. We have a surjective morphism $L \to L_D$ (after identifying $i_* L_D = L_D$, where $i : D \hookrightarrow X$ is the closed immersion). Let L_D be its kernel, that is, the following sequence is exact

$$0 \to L^D \to L \to L_D \to 0.$$

The sheaf $M' = \operatorname{im}(M \hookrightarrow L \to L_D)$ is supported on the finite set $D \cap \overline{D}$, therefore it is a torsion subsheaf of L_D, which is torsion-free, hence $M' = 0$ and $M \subset L^D$. It follows that $\chi(M) \leq \chi(L^D)$. Since $\chi(L) = \chi(L^D) + \chi(L_D)$ and $h = h_{\overline{D}} + h_D$, the inequality $h_D/h \cdot \chi(L) \leq \chi(L_D)$ is equivalent to

$$\chi(L^D) \leq \frac{h_{\overline{D}}}{h} \chi(L).$$

Combining both yields

$$\frac{\chi(M)}{\operatorname{rk}_H(M)} \leq \frac{\chi(L)}{\operatorname{rk}_H(L)}$$

since $\operatorname{rk}_H(M) = \sum_{X_i \subset \overline{D}} h_i/h \operatorname{rk}(M_i) = h_{\overline{D}}/h$. Following Seshadri L is semistable with respect to H.

We may restrict ourselves to connected proper subcurves D since for every sheaf E on X we have $E_D = \bigoplus E_{D_i}$, where D_i are the connected components of D. For $D = X$ the inequality is trivial. □

In the above proposition we have implicitly used the fact that every line bundle on an integral curve is semistable. Since the Euler characteristic of the restriction to D can be computed from the Euler characteristic of the restriction to $\overline{D}$, some inequalities may be superfluous. This observations will be examined in section 8.

Example. Let X be as in the previous example. Moreover, assume that the intersection point P of X_1 and X_2 is an *ordinary double point*. For every line bundle L on X such that the degree of the restriction of L to X_1 and X_2 is zero, we have $\chi(L) = \chi(\mathcal{O}_X)$ and $\chi(L_i) = \chi(\mathcal{O}_{X_i})$ $(i = 1, 2)$ since the degree of L is also zero. Then L is semistable with respect to H if and only if

$$\frac{h_i}{h} \chi(\mathcal{O}_X) \leq \chi(\mathcal{O}_{X_i})$$

is satisfied for $i = 1, 2$. There exists an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to T \to 0,$$

where T is the skyscraper sheaf concentrated at P with $\dim_k T_P = 1$ by assumption on P. It follows that

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_1}) + \chi(\mathcal{O}_{X_2}) - 1.$$

Thus the semistability of L with respect to H is equivalent to

$$\frac{h_1}{h} \chi(\mathcal{O}_X) \leq \chi(\mathcal{O}_{X_1}) \leq \frac{h_1}{h} \chi(\mathcal{O}_X) + 1.$$

Assume now that X_1 is isomorphic to $\mathbb{P}^1_k$, hence $\chi(\mathcal{O}_{X_1}) = 1$. The above inequality implies then $\chi(\mathcal{O}_{X_2}) \geq 0$ (h_1 and h are positive). Therefore, if X_2

is of genus greater than one, then there is no polarization H on X such that L is semistable. In particular, in this case the trivial line bundle is never semistable.

2. Moduli spaces of semistable sheaves

Roughly, a moduli problem is a problem of classifying given objects varying in families. Therefore, before we formulate the moduli problem for pure sheaves, we introduce the notion of a family.

Let $f : X \to S$ be a projective morphism of schemes. Fix a relatively ample line bundle on f, we sometimes call it *polarization* of X/S. A *family of coherent sheaves* on the fibers of f is an S-flat coherent sheaf F on X. If S is locally Noetherian, then flatness implies that the Hilbert polynomial $P(F_s)$ is locally constant as a function in $s \in S$. We say that the property P of a coherent sheaf on a scheme is *open* if for every projective morphism $f : X \to S$ of Noetherian schemes and every family of coherent sheaves F on f the set

$$\{s \in S \mid F_s \text{ satisfies } P\} \subset S$$

is open. A family F is said to be a family with P if F_s satisfies the property P for every $s \in S$. For example, we may speak about families of pure, locally free, semistable or geometrically stable sheaves. In fact, we have the following

Proposition 2.1. *The properties of being pure, locally free, semistable or geometrically stable are open in families of coherent sheaves.*

Proof. Cf. [HL10] Lemma 2.1.8 and Proposition 2.3.1. □

Let (Sch/S) be the category of schemes over S. Fix a numerical polynomial P, that is, $P(m) \in \mathbb{N}$ for every integer $m \geq 0$, and let $f : X \to S$ be a projective morphism of Noetherian schemes. For a scheme T in (Sch/S) define the following set as follows

$$\mathcal{M}(T) = \mathcal{M}^P_{X/S}(T) = \left\{ \begin{matrix} \text{families of semistable} \\ \text{sheaves on } X_T = X \times_S T \\ \text{with Hilbert polynomial } P \end{matrix} \right\} / \sim,$$

where $\sim$ is the following equivalence relation: $E \sim E'$ if and only if for some line bundle L on T

(1) $E \cong E' \otimes_{\mathcal{O}_T} L$, or
(2) there exist filtrations

$$0 = E_0 \subset E_1 \subset \ldots \subset E_r = E,$$
$$0 = E'_0 \subset E'_1 \subset \ldots \subset E'_r = E',$$

by coherent subsheaves such that successive gradings $\operatorname{gr}_i E$, $\operatorname{gr}_i E'$ are flat over T, $\operatorname{gr} E \cong \operatorname{gr} E' \otimes_{\mathcal{O}_T} L$ and for every geometric point t in T the above filtrations restricted to t are Jordan-Hölder filtrations of E and E', respectively.

We say that elements of $\mathcal{M}(T)$ are equivalence classes of families E of semistable sheaves on X parametrized by T and write $[E]$ for such an equivalence class.

If $g : T' \to T$ is a morphism in (Sch/S), then the pullback along $id_X \times_S g$ respects families of semistable sheaves and the above equivalence relation, hence induces a mapping $\mathcal{M}(T') \to \mathcal{M}(T)$. This make $\mathcal{M}$ to a (contravariant) functor from the category (Sch/S) to the category of sets.

We can also restrict ourselves only to the families of geometrically stable sheaves. We define

$$\mathcal{M}^s(T) = \mathcal{M}^{P,s}_{X/S} = \left\{ \begin{matrix} \text{families of geometrically stable} \\ \text{sheaves on } X_T = X \times_S T \\ \text{with Hilbert polynomial } P \end{matrix} \right\} / \sim,$$

where second condition in $\sim$ becomes trivial.

Definition 2.2. The defined functor $\mathcal{M} : (Sch/S) \to (Set)$ is called the *moduli problem of semistable sheaves*. The subfunctor $\mathcal{M}^s \subset \mathcal{M}$ is called the *moduli problem of stable sheaves*.

Note that both functors depend on the polarization of X/S. Shortly, we will explain why we restrict ourselves to the semistable sheaves, and why the equivalence relations $\sim$ is defined in such a way.

Before defining the notion of a moduli space, we need a less restrictive concept of the representability of a functor. Let $\mathcal{C}$ be a category with finite fiber products and $\mathcal{C}'$ the category of contravariant functors $\mathcal{C} \to (Set)$ with natural transformations as morphisms. Denote for an object $X \in \mathcal{C}$ by h_X the functor in $\mathcal{C}'$ which sends each object $Y \in \mathcal{C}$ to $Hom_{\mathcal{C}}(Y, X)$.

Definition 2.3. (1) A functor $\mathcal{F} \in \mathcal{C}'$ is *corepresented* by an object $R \in \mathcal{C}$ and a morphism $\alpha : \mathcal{F} \to h_R$ in $\mathcal{C}'$ if α satisfies the following universal property: any morphisms $\beta : \mathcal{F} \to h_T$, where T is an object in $\mathcal{C}$, factors uniquely through α. Whenever clear from the context, we will omit α and say that $\mathcal{F}$ is corepresented by R.

(2) The functor F is *universally corepresented* by R if the corepresentability of F is stable under the arbitrary base change, i.e. for every morphism $f : T \to R$ in $\mathcal{C}$ the induced fiber product functor $\mathcal{T} = h_T \times_{h_R} \mathcal{F}$ is corepresented by T.

(3) In the case, where $\mathcal{C} = (Sch/S)$ is the category of schemes over a fixed scheme S, the functor $\mathcal{F}$ is *uniformly corepresented* by R if the corepresentability of $\mathcal{F}$ by R is stable under the *flat* base change.

Definition 2.4. A *coarse moduli space* of the moduli problem of semistable sheaves is a scheme M over S which corepresents $\mathcal{M}$, and such that the natural transformation $\mathcal{M} \to h_M$ induces a bijection for every geometric point of S. If M represents $\mathcal{M}$, then we say that M is a *fine moduli space*.

Remark 2.5. (1) The reason why one considers only semistable sheaves lies deeply in the geometric invariant theory (GIT) introduced by Mumford in [MFK94]. In case of smooth projective curves over an algebraically closed field one constructs the moduli space as a quotient of a certain *Quot* scheme by an action of an affine group scheme. Taking only semistable[1] points with respect to this actions gives an invariant scheme, whose quotient by the action is a quasi-projective variety. Another fact is that the set of semistable vector bundles with fixed Hilbert polynomial is *bounded*, which is also important for the construction using GIT.

[1] Here we mean semistability introduced in GIT.

(2) One may be tempted to take the equivalence relation defined by the isomorphism classes of families instead of $\sim$. We will explain why we choose to work with $\sim$ as above.

Let F be a family of semistable sheaves parametrized by an S-scheme T. If L is an arbitrary line bundle on S, then the fibers $(F \otimes f_T^* L)_t$ and F_t are isomorphic for every $t \in T$. Thus set-theoretically we cannot distinguish these families. That's why we identify them by (1).

The reason for the second identification lies deeper. We explain this for the case, where S is a point. Assume that $S = \operatorname{Spec} k$, k is an algebraically closed field. Let

$$0 \to F' \to F \to F'' \to 0$$

be an exact sequence of semistable sheaves on X/k. One can construct a family $\mathcal{F}$ parametrized by $\mathbb{A}_k^1$ such that $\mathcal{F}_s \cong F$ for all $s \neq 0$ in $\mathbb{A}_k^1$ and $\mathcal{F}_0 \cong F' \oplus F''$. The construction is done e.g. in the proof of [Sim94] Theorem 1.21, part (3).

Assume that the coarse moduli space M for $\mathcal{M}$ exists. Then the constructed family $\mathcal{F}$ yields by the property of the corepresentability a morphism $f : \mathbb{A}_k^1 \to M$. By construction $f|_{\mathbb{A}_k^1 \setminus \{0\}}$ is constant, hence f is constant. It follows that F and $F' \oplus F''$ define the same point in $M(k)$. Inductively one sees that S-equivalent semistable sheaves on X define the same closed point in M. Since we want that $\mathcal{M}(k) \cong M(k)$ holds, we identify those by (2) in $\sim$.

A projective coarse moduli space always exists (cf. [Lan04b] Theorem 0.2, [Mar96] Theorem 0.6).

Theorem 2.6 (Langer, Maruyama)**.** *Let R be a universally Japanese ring. Let $f : X \to S$ be a projective morphism of R-schemes of finite type over R with geometrically connected fibers. Then for a fixed numerical polynomial P there exists a projective S-scheme $M = M_{X/S}^P$ which uniformly corepresents the functor $\mathcal{M} = \mathcal{M}_{X/S}^P$.*

Moreover, there exists an open subscheme $M_{X/S}^{P,s} \subset M_{X/S}^P$ which universally corepresents the subfunctor $\mathcal{M}^s$.

Recall that an integral domain R is called a *Japanese ring* if for every finite field extension L of its quotient field K the normalization of R in L is a finite R-module. A ring is called *universally Japanese* if every finitely generated integral domain over it is Japanese.

Regression: Moduli of vector bundles on a smooth projective curve. Let X be a smooth projective curve over an algebraically closed field k. As already noticed, pure sheaves on X are locally free sheaves of finite rank or synonymously, vector bundles. The Hilbert polynomial of a vector bundle on X is fully determined by its rank and degree. Fix two integers $r \geq 1$ and d. The moduli problem of vector bundles on X of fixed rank and degree becomes then

$$\mathcal{M}(S) = \mathcal{M}_X^{r,d}(S) = \left\{ \begin{matrix} \text{families of semistable vector bundles} \\ \text{on } X \times S \text{ of rank } r \text{ and degree } d \end{matrix} \right\} / \sim .$$

Contrary to the general moduli problem it does not depend on the polarization.

Theorem 2.7. *Let X be smooth projective and connected curve of genus g over an algebraically closed field k.*

(i) The moduli space M_X^s of stable vector bundles is smooth (if not empty).

(ii) The tangent space $T_{[E]}M_X^s$ of M_X^s at the point $[E]$ induced by a stable vector bundle E on X is canonically isomorphic to

$$\mathrm{Ext}^1_X(E,E).$$

(iii) The dimension of $M_X^{r,d,s}$ is $r^2(g-1)+1$.

Proof. For (i) and (ii) cf. [LP97] Theorem 8.3.2, (iii) follows from (ii) by the following calculation:

We have $\mathrm{Ext}^1_X(E,E) = H^1(X, \mathcal{E}nd(E))$. Now $\mathcal{E}nd(E) \cong E \otimes E^*$. Hence

$$\deg \mathcal{E}nd(E) = \mathrm{rk}\, E \deg E^* + \mathrm{rk}\, E^* \deg E.$$

It follows that $\mathcal{E}nd(E)$ is of rank r^2 and degree zero. Applying Riemann-Roch and using that E is simple, in particular $h^0(X, \mathcal{E}nd(E)) = 1$, we obtain

$$\dim_k T_{[E]}M_X^s = h^1(X, \mathcal{E}nd(E)) = 1 - \chi(E) = r^2(g-1)+1. \qquad \square$$

In fact, except in cases of $g = 2$, $r = 2$ and d is even, the singular points on M_X are exactly those not in M_X^s (cf. [Ses82] première partie, Théorème 45).

Theorem 2.8. *Let X be as in the above theorem. The moduli space $M_X^{r,d}$ is fine if r and d are coprime.*

Proof. Cf. [LP97] Theorem 8.4.2. □

Note that, if r and d are coprime, then every semistable vector bundle is already stable.

Theorem 2.9. *Let X be as in the above theorem. The moduli space M_X is irreducible.*

Proof. Cf. [LP97] Theorem 8.5.2. □

Theorem 2.10. *Let X be as in the above theorem.*

(i) If $g = 0$, there are no stable vector bundles on X of rank $r \geq 2$.

(ii) If $g \geq 2$, then $M_X^{r,d,s}$ is non-empty for all $r \geq 2$, d.

(iii) If $g = 1$, then $M_X^{r,d}$ is non-empty for all $r \geq 2$, d.

Proof. The statement (i) follows from [LP97] Lemma 4.4.1, the remaining statement are exactly *loc. cit.* Theorems 8.6.1 and 8.6.2. □

3. Theorem of Langton

The well-known theorem of Langton in [Lan75] on p. 99 is a statement (in its original form) about the degeneration of semistable sheaves on a smooth projective scheme on X. The valuative criterion of properness together with this theorem implies that the moduli space of semistable sheaves on X is proper. The most general form (known to the author) is stated in [Mar96] Theorem 7.6.

Theorem 3.1. *Let R be a discrete valuation ring with quotient field K and residue field k. Consider a projective and flat scheme $\mathfrak{X}$ over R. Let E be a semistable sheaf on the generic fiber X of $\mathfrak{X}$. Then there exists a family $\mathcal{E}$ of semistable sheaves on $\mathfrak{X}$ with generic fiber $\mathcal{E}_K \cong E$.*

Unfortunately, Maruyama has never given a proof of this theorem. The case, where the special fiber $\mathfrak{X}_k$ of $\mathfrak{X}$ is a smooth projective scheme, is proved in [HL10] Section 2.B (consider the semistability in $Coh_{d,d-1}$ there). Our goal is to give a proof of this theorem for a general $\mathfrak{X}$. The proof was communicated by C. Deninger to the author and is based on the corresponding proof from the lectures "Vector Bundles on curves", held by G. Faltings in Bonn 1995.

Lemma 3.2. *Let M be a non-empty set of polynomials with rational coefficients such that every polynomial in M is a quotient of a polynomial of the form*

$$(*) \qquad P(T) = \sum_{i=0}^{d} a_i \binom{T+i-1}{i}$$

by a_d, where $d > 0$ is fixed, a_i's are integers and $a_d > 0$. Assume that M is bounded below (with respect to lexicographical order) by a polynomial $f(T)$ of the form $(*)$, and the set of a_d's of polynomials in M is bounded above. Then M has a minimum.

Proof. Let α be an upper bound of the a_d's. For $p \in M$ we write P for the polynomial such that $p = P/\alpha_d$. Put $C = d!\alpha!$. Then for all $p \in M$ the polynomial

$$C \cdot p(T) = C \cdot \frac{P(T)}{a_d} = d! \frac{\alpha!}{a_d!} P(T)$$

has integral coefficients. Let p be a polynomial in M. By assumption $f \leq p$ and hence $Cf \leq Cp$. There exist finitely many polynomials with integral coefficients between Cf and Cp. Therefore the set of polynomial in M between f and p is also finite. A finite non-empty set has a minimum. □

For a pure sheaf E on a projective scheme X we write $p(E)$ for the reduced Hilbert polynomial of E (with respect to a fixed polarization). There exists a subsheaf $F \subset E$ such that for all subsheaves $G \subset F$ we have $p(F) \geq E$, and in the case of equality $F \supset G$. This subsheaf is uniquely determined and is semistable. It is called the *maximal destabilizing subsheaf* of E.

Lemma 3.3. *Let*

$$0 \to E/F \to E' \to F \to 0$$

be an exact sequence of pure sheaves on a projective scheme, where F is a maximal destabilizing subsheaf of E. Then $p_{max}(E') \leq p(F)$.

Proof. Since F is the maximal destabilizing subsheaf of E, there exists a Harder-Narasimhan filtration

$$0 \subset F = F_1 \subset \ldots \subset F_n = E,$$

and the filtration

$$0 \subset F_2/F \subset F_3/F \subset \ldots \subset F_n/F = E/F.$$

is a Harder-Narasimhan filtration of E/F. Hence the maximal destabilizing sheaf E/F is isomorphic to F_2/F, and therefore $p_{max}(E/F) = p(F_2/F) < p(F)$.

Let F' be the maximal destabilizing subsheaf of E' and $\varphi : F' \to F$ the composition of the inclusion $F' \subset E'$ and the arrow on the right hand side in the exact sequence. Assume that $p(F') > p(F)$. Then $\varphi = 0$ (cf. [HL10] Proposition 1.2.7). Hence $F' \subset E/F$, and we have $p(F') \leq p_{max}(E/F) < p(F)$, which is a contradiction to the assumption. Thus $p(F') \leq p(F)$. $\square$

Proof of the theorem. Step 0: Minimal family. Let $\mathcal{E}$ be a family of coherent pure sheaves on $\mathfrak{X}$ with $\mathcal{E}_K \cong E$. Since $\mathcal{E}$ is flat over R, we have $p(\mathcal{E}_k) = p(E)$, in particular if $F \subset \mathcal{E}_k$ is the maximal destabilizing subsheaf, we have $p(F) = p_{max}(\mathcal{E}_k) \geq p(\mathcal{E}_k) = p(E)$. Hence the set $\{p_{max}(\mathcal{E}_k) \mid \mathcal{E} \text{ as above}\}$ is bounded below. On the other hand, we have for the multiplicities $\alpha_d(F) \leq \alpha_d(\mathcal{E}_k) = \alpha_d(E)$, where $d = \dim X$. By Lemma 3.2 the above set has a minimum p. Put

$$\alpha = min \left\{ \alpha_d(F) \mid \mathcal{E} \text{ pure with } \mathcal{E}_K \cong E,\ p_{max}(\mathcal{E}_k) = p \right\},$$

and let $\mathcal{E}$ be a family of pure sheaves such that

$$p_{max}(\mathcal{E}) = p, \quad \alpha_d(F) = \alpha \quad \text{and} \quad \mathcal{E}_K \cong E.$$

Step 1: Construction of a sequence $\mathcal{E}^n$. Assume that $F \neq \mathcal{E}_k$, otherwise the theorem follows. Let $\mathcal{E}^1$ be a sheaf with $\pi\mathcal{E} \subset \mathcal{E}^1 \subset \mathcal{E}$ defined as the preimage of F via $f_0 : \mathcal{E} \to i_*\mathcal{E}_k$, where $i : \mathfrak{X}_k \hookrightarrow \mathfrak{X}$ is the canonical inclusion and $\pi \in R$ a uniformizing element. Note that $\mathcal{E}^1_K \cong E$. The morphism f_0 induces an isomorphism

$$\mathcal{E}^1/\pi\mathcal{E} \xrightarrow{\sim} F \subset \mathcal{E}/\pi\mathcal{E}$$

and an exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi\mathcal{E}/\pi\mathcal{E}^1 & \longrightarrow & \mathcal{E}^1/\pi\mathcal{E}^1 & \longrightarrow & F & \longrightarrow & 0 \\ & & & & & \searrow & \| & & \\ & & & & & & \mathcal{E}^1/\pi\mathcal{E} & \hookrightarrow & \mathcal{E}/\pi\mathcal{E}. \end{array}$$

We have $\pi\mathcal{E}/\pi\mathcal{E}^1 \cong \mathcal{E}/\mathcal{E}^1 \cong (\mathcal{E}/\pi\mathcal{E})/(\mathcal{E}^1/\pi\mathcal{E}) \cong (\mathcal{E}/\pi\mathcal{E})/F$. The above sequence transforms into the exact sequence

$$0 \to \mathcal{E}_k/F \to \mathcal{E}^1_k \to F \to 0.$$

By Lemma 3.3 the maximal destabilizing subsheaf F' of $\mathcal{E}^1_k$ satisfies $p(F') \leq p(F)$. Since by choice of $\mathcal{E}$ the polynomial $p_{max}(\mathcal{E}_k)$ is minimal, it follows that $p(F') = p(F)$. Let G be the kernel of $F' \to F$. Assume that $G \neq 0$. Let F''

be the image of $F' \to F$. Since the latter sheaves are semistable, it follows that $p(F'') \leq p(F)$ and $p(F'') \geq p(F')$, therefore $p(F') = p(F'') = p(F)$. From the exact sequence

$$0 \to G \to F' \to F \to 0$$

we conclude that $p(G) = p(F') = p(F)$. On the other hand, $G \subset ker(\mathcal{E}_k^1 \to F) \cong \mathcal{E}_k/F$. It follows that $p(G) \leq p_{max}(\mathcal{E}_k/F) < p_{max}(\mathcal{E}_k) = p(F)$ (compare for that Harder-Narasimhan filtrations as in Lemma 3.3), which is a contradiction. Hence, $G = 0$. Therefore $F' \to F$ is injective, write $F' \subset F$. From the definition of α we conclude $\alpha_d(F') \leq \alpha_d(F) = \alpha$, hence $\alpha_d(F') = \alpha$ since α is minimal. Together with the above identity we obtain $P(F') = P(F)$. It follows that $F/F' = 0$, therefore $F' \cong F$. This gives a splitting of the above exact sequence, whence $\mathcal{E}_k^1 \cong \mathcal{E}_k \oplus F$.

Inductively we define $\mathcal{E}^n$ with $\pi\mathcal{E}^{n-1} \subset \mathcal{E}^n \subset \mathcal{E}^{n-1}$ as the preimage of F via $f_{n-1} : \mathcal{E}^{n-1} \to i_*\mathcal{E}_k^{n-1}$, where F is the maximal destabilizing subsheaf of $\mathcal{E}_k^{n-1}$ (as shown above F is independent of n). The morphism f_{n-1} induces an isomorphism

$$\mathcal{E}^n/\pi\mathcal{E}^{n-1} \xrightarrow{\sim} F \subset \mathcal{E}^{n-1}/\pi\mathcal{E}^{n-1}.$$

We have a split exact sequence

$$(*) \qquad 0 \longrightarrow \mathcal{E}_k^{n-1}/F \longrightarrow \mathcal{E}_k^n \rightleftarrows F \longrightarrow 0,$$

which is obtained from the commutative diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & \pi\mathcal{E}^{n-1}/\pi\mathcal{E}^n & \longrightarrow & \mathcal{E}^n/\pi\mathcal{E}^n & \longrightarrow & F & \longrightarrow 0 \\ & & & \uparrow & \searrow & \| & \\ & & & F & \xrightarrow{\sim} & \mathcal{E}^n/\pi\mathcal{E}^{n-1} & \hookrightarrow \mathcal{E}^{n-1}/\pi\mathcal{E}^{n-1} \end{array}$$

with exact first row. Since $(*)$ splits, we have $\pi\mathcal{E}^{n-1}/\pi\mathcal{E}^n \cap F = 0$ in $\mathcal{E}^n/\pi\mathcal{E}^n = \mathcal{E}_k^n$. Since $F = \mathcal{E}^{n+1}/\pi\mathcal{E}^n$, it follows that $\pi\mathcal{E}^{n-1} \cap \mathcal{E}^{n+1} \subset \pi\mathcal{E}^n$, and since the other inclusion is trivial, we have

$$(2) \qquad \pi\mathcal{E}^{n-1} \cap \mathcal{E}^{n+1} = \pi\mathcal{E}^n.$$

Further, again since $(*)$ splits we have $\pi\mathcal{E}^{n-1}/\pi\mathcal{E}^n + F = \mathcal{E}^n/\pi\mathcal{E}^n$ and again because of $F = \mathcal{E}^{n+1}/\pi\mathcal{E}^n$

$$(3) \qquad \mathcal{E}^n = \pi\mathcal{E}^{n-1} + \mathcal{E}^{n+1}.$$

Step 2: Compatibility of $(\mathcal{E}^n)$*.* We prove the following claim: the inclusion $\mathcal{E}^{n+1} \subset \mathcal{E}^n$ induces an isomorphism

$$(**) \qquad \mathcal{E}^{n+1}/\pi^{n+1}\mathcal{E} \otimes_R R/\pi^n \cong \mathcal{E}^n/\pi^n\mathcal{E}.$$

The identity (3) with $\mathcal{E}^{n-1} \supset \pi^{n-1}\mathcal{E}$, in particular $\pi\mathcal{E}^{n-1} \supset \pi^n\mathcal{E}$, implies $\mathcal{E}^n \supset \pi^n\mathcal{E} + \mathcal{E}^{n+1}$, and since the other inclusion is clear, we have

$$(4) \qquad \mathcal{E}^n = \pi^n\mathcal{E} + \mathcal{E}^{n+1}.$$

It follows that $\mathcal{E}^{n+1} \hookrightarrow \mathcal{E}^n \to \mathcal{E}^n/\pi^n\mathcal{E}$ is surjective. Since $\pi^{n+1}\mathcal{E} \subset \pi^n\mathcal{E}$ in $\mathcal{E}^n$, this map induces a surjective map

$$\mathcal{E}^{n+1}/\pi^{n+1}\mathcal{E} \to \mathcal{E}^n/\pi^n\mathcal{E}.$$

Tensoring with R/π^n yields a map φ

$$\mathcal{E}^{n+1}/\pi^{n+1}\mathcal{E} \otimes_R R/\pi^n \to \mathcal{E}^n/\pi^n\mathcal{E} \otimes_R R/\pi^n = \mathcal{E}^n/\pi^n\mathcal{E},$$

where the domain is isomorphic to $\mathcal{E}^{n+1}/(\pi^{n+1}\mathcal{E}+\pi^n\mathcal{E}^{n+1})$. Thus the kernel of φ is $\pi^n\mathcal{E} \cap \mathcal{E}^{n+1}/(\pi^{n+1}\mathcal{E} + \pi^n\mathcal{E}^{n+1})$. We have to show that

$$\pi^n\mathcal{E} \cap \mathcal{E}^{n+1} = \pi^{n+1}\mathcal{E} + \pi^n\mathcal{E}^{n+1}.$$

It follows then that φ is an isomorphism, and therefore $(**)$ holds.

The inclusion $\supset$ is clear. We proceed by induction. Let $n = 1$. We have $\pi\mathcal{E} \cap \mathcal{E}^2 = \pi\mathcal{E}^1$ by (2). On the other hand, $\pi^2\mathcal{E} + \pi\mathcal{E}^2 = \pi(\pi\mathcal{E} + \mathcal{E}^2) = \pi\mathcal{E}^1$, where the last identity is given by (4).

Assume now that

$$\pi^n\mathcal{E} \cap \mathcal{E}^{n+1} = \pi^{n+1}\mathcal{E} + \pi^n\mathcal{E}^{n+1}$$

holds. Multiplying with π yields

$$\pi^{n+1}\mathcal{E} \cap \pi\mathcal{E}^{n+1} = \pi^{n+2}\mathcal{E} + \pi^{n+1}\mathcal{E}^{n+1}.$$

The left hand side is contained in $\pi^{n+1}\mathcal{E} \cap \mathcal{E}^{n+2}$. Since $\pi^n\mathcal{E} \subset \mathcal{E}^n$ and therefore $\pi^{n+1}\mathcal{E} \subset \pi\mathcal{E}^n$, we have $\pi^{n+1}\mathcal{E} \cap \mathcal{E}^{n+2} = \pi^{n+1}\mathcal{E} \cap (\pi\mathcal{E}^n \cap \mathcal{E}^{n+2}) = \pi^{n+1}\mathcal{E} \cap \pi\mathcal{E}^{n+1}$, where the last identity is (2).

The right hand side contains $\pi^{n+2}\mathcal{E} + \pi^{n+1}\mathcal{E}^{n+2}$. The identity (3) implies $\pi\mathcal{E} + \mathcal{E}^{n+1} = \pi\mathcal{E} + (\pi\mathcal{E}^n + \mathcal{E}^{n+2}) = \pi\mathcal{E} + \mathcal{E}^{n+2}$, and therefore after multiplication with π^{n+1}, we have

$$\pi^{n+2}\mathcal{E} + \pi^{n+1}\mathcal{E}^{n+1} = \pi^{n+2}\mathcal{E} + \pi^{n+1}\mathcal{E}^{n+2}.$$

Thus the induction step follows, and the identity as claimed holds.

Step 3: Algebraization. Put $\mathcal{F}^n = \mathcal{E}^n/\pi^n\mathcal{E}$. It is a coherent sheaf on the scheme $\mathfrak{X}_n = \mathfrak{X} \otimes_R R/\pi^n$. Define a formal sheaf $\hat{\mathcal{F}} = \varprojlim_n \mathcal{F}^n$ on the formal scheme $\hat{\mathfrak{X}} = \varinjlim_n \mathfrak{X}_n$. We have $\hat{\mathcal{F}}_k = \hat{\mathcal{F}}/\pi\hat{\mathcal{F}} = \mathcal{F}^1 = \mathcal{E}^1/\pi\mathcal{E} = F$ on $\mathfrak{X}_k$. The inclusions $\mathcal{F}^n \to \mathcal{E}^n/\pi^n\mathcal{E}$ induce an inclusion $\hat{\mathcal{F}} \to \hat{\mathcal{E}}$. By Grothendieck's existence theorem (cf. [EGAIII$_1$] Théorème 5.1.4) there exists a coherent sheaf $\mathcal{F} \subset \mathcal{E}$ on $\mathfrak{X}$ with $\mathcal{F} \otimes_R R/\pi^n = \mathcal{F}^n$.

The sheaf $\mathcal{F}$ is flat over R as a subsheaf of a flat sheaf and since R is a principal ideal domain. It follows that $p(F) = p(\mathcal{F}_k) = p(\mathcal{F}_K)$ and hence

$$p(E) = p(\mathcal{E}_k) < p(\mathcal{F}_K).$$

In particular, E is not semistable, which is a contradiction to the assumption. Therefore $\mathcal{E}_k$ is semistable, and the family $\mathcal{E}$ is then the family whose existence is claimed in the theorem. □

4. Jordan-Hölder filtrations in abelian categories

A *subobject* of an object A in a category $\mathcal{C}$ is an isomorphism class of monos $B \to A$. Two morphisms $B \to A$ and $B' \to A$ are isomorphic if there exists an isomorphism $B \to B'$ such that the following diagram commutes

$$\begin{array}{ccc} B & \longrightarrow & B' \\ & \searrow \quad \swarrow & \\ & A. & \end{array}$$

The class of all subobjects of an object A is denoted by $Sub(A)$. A category $\mathcal{C}$ is said to be *well-powered* if $Sub(A)$ is a set for every object A in $\mathcal{C}$. Regardless of whether $Sub(A)$ is a set or not, we call it the set of subobjects of A. There is a natural partial order on $Sub(A)$:

$$(b : B \to A) \leq (b' : B' \to A)$$

if and only if there exists a morphism $k : B \to B'$ satisfying $b = b'k$.

Consider a partially ordered set X. Let $x, y \in X$. An element $z \in X$ satisfying the following two properties

(i) $x \leq z$ and $y \leq z$,
(ii) for all $z' \in X$ with $x \leq z'$, $y \leq z'$ follows that $z \leq z'$

is called the *supremum* of x and z. By (ii) it is uniquely determined. Dually, we define the *infimum* of x and y. A *lattice* L is a partially ordered set such that any two elements have a supremum and infimum. The supremum of x and y is denoted by $x \vee y$ and the infimum by $x \wedge y$. A lattice L is called *modular* if the following law is satisfied

$$x \vee (y \wedge z) = (x \vee y) \wedge z \qquad (x, y, z \in L,\ x \leq z).$$

In an abelian category $\mathcal{A}$ we may define naturally the supremum and infimum of two subobjects of a given object A.

Definition 4.1. Let A_i be a family in $Sub(A)$. We define the *intersection* $\bigcap A_i$ of A_i as an object $B \subset A$ satisfying:
(i) $B \leq A_i$ for all i, and
(ii) if $B' \subset A$ with $B' \leq A_i$ for all i, then $B' \leq B$.

The intersection of finitely many subobjects $A_i \subset A$ always exists since in an abelian category finite limits exist, and the intersection of A_i is just the limit of the monos $A_i \to A$. Note that the intersection $\bigcap A_i$ is unique.

Definition 4.2. Let A_i be a family in $Sub(A)$. We define the *union* $\bigcup A_i$ of A_i as an object $B \subset A$ satisfying:
(i) $A_i \leq B$ for all i, and
(ii) if $B' \subset A$ with $A_i \leq B$ for all i, then $B \leq B'$.

For a finite family A_i in $Sub(A)$ take the direct sum $\bigoplus A_i$ and consider the image of $\bigoplus A_i \to A$. This is the union of the A_i's, which is unique.

From now on we assume that all considered categories are well-powered. We conclude directly from the above definitions the following

Proposition 4.3. *Consider an abelian category $\mathcal{A}$. Then for every object A in $\mathcal{A}$ the set of objects $Sub(A)$ is a lattice with union and intersection as supremum and infimum, respectively.*

In fact, we have a stronger property.

Proposition 4.4. *The lattice $Sub(A)$ for A in an abelian category $\mathcal{A}$ is modular.*

An *interval* $I = [a, b]$ in a lattice L is defined as the set of all $x \in L$ with $a \leq x \leq b$. Let $x \in I$. An element $z \in I$ is called a complement of x if $x \wedge z = a$ and $x \vee z = b$. We will use the following characterization of modular lattices to prove the above proposition.

Lemma 4.5. *A lattice L is modular if and only if every interval $I \subset L$ satisfies the following property: any x, $y \in I$, which are comparable and have a common complement z in I, are equal.*

Proof. Cf. [Coh89] §2 Proposition 1.3. □

Proof of the proposition. Let A, B be two objects in $\mathcal{A}$ with $A \leq B$. First note that we have a lattice isomorphism $[A, B] \to [0, B/A] = Sub(B/A)$ defined by $X \mapsto X/A$. Hence we may consider only intervals of the form $Sub(A)$ for an object A in $\mathcal{A}$.

By the above lemma we have to show that for any two comparable objects C, $C' \leq A$ with a common complement $D \leq A$ the objects C and C' are isomorphic. Assume $C \leq C'$, i.e. we have a morphism $\alpha : C \to C'$ with $i'\alpha = i$, where $i : C \to A$ and $i' : C' \to A$ are monos. Since D is a complement of C and C' in $Sub(A)$, we have

$$C \cap D \cong C' \cap D \cong 0,$$
$$C \cup D \cong C' \cup D \cong A.$$

From the exact sequence

$$0 \to C \cap D \to C \oplus D \to C \cup D \to 0$$

and the analogous exact sequence for C' it follows that $A \cong C \oplus D \cong C' \oplus D$. Since in an abelian category every (co)product is a biproduct, we have morphisms $p : A \to C$, $p' : A \to C'$, $q : A \to D$ and $q' : A \to D$ such that (i, j, p, q) and (i', j, p', q') are biproducts, where $j : D \to A$ is a mono. We have

$$ip + jq = id_C + id_D = id_A = id_{C'} + id_D = i'p' + jq'.$$

Multiplying by p' from the left yields

$$p'ip + p'jq = p'i'p' + p'jq'.$$

Using the relations $i = i'\alpha$, $p'j = 0$ and $p'i = id_C$ we obtain

$$\alpha p = p'i'\alpha p = p',$$

hence $\alpha p i' = id_{C'}$. On the other hand, $pi'\alpha = p'i = \alpha_C$. Thus α is an isomorphism. □

Corollary 4.6. *Let A be an object in an abelian category $\mathcal{A}$ such that there exists a maximal filtration of A of length n. Then every filtration of A is finite and can be refined to a filtration of length n.*

Proof. The statement follows from the above proposition and the corresponding statement [Coh89] §2 Proposition 2.4 for modular lattices. □

We define the *length* of an object A in an abelian category $\mathcal{A}$ as the length of the maximal filtration in $Sub(A)$. If the length is finite, we say that A is *of finite length*. In this case a maximal filtration of A is called a *Jordan-Hölder filtration* (short, JH-filtration). It is not necessarily uniquely determined by A. An object A is called *simple* if $Sub(A)$ consists exactly of two elements 0 and A. An object is simple if and only if its length is one. Note that the zero-object is not simple.

Proposition 4.7. *Let A be an object in an abelian category $\mathcal{A}$ of finite length. Then every subquotient of a JH-filtration of A is simple.*

Proof. Let

$$0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$$

be a JH-filtration of A. Assume that there is an $1 \leq i \leq n$ such that A_i/A_{i-1} is not simple. Then there exists a proper subobject $\overline{A} \subset A_i/A_{i-1}$. Taking the preimage of $\overline{A}$ via the canonical morphism $A_i \to A_i/A_{i-1}$ yields a proper object $A_{i-1} \subset A \subset A_i$, and we have found a filtration of length $n+1$, which is a contradiction. □

Proposition 4.8. *Let A, B be simple objects in an abelian category $\mathcal{A}$. Then every morphism $A \to B$ is either 0 or an isomorphism. In particular, $End(A)$ is a skew field.*

Proof. Assume that $f : A \to B$ is non-zero. Then the kernel $C \to A$ of f is non-zero. Since kernels in $\mathcal{A}$ are monos, and A is simple, $C \cong A$ and f is a mono. Let B' be the image of f. It is a non-zero subobject of B, hence $B' \cong B$. Since $B = imf = ker(cokerf)$, it follows that $cokerf = 0$, and f is epi. Hence f is an isomorphism. □

Let

$$0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$$

be a JH-filtration of A. We denote the subquotients of this filtration by $\operatorname{gr}_i A = A_i/A_{i-1}$ for $1 \leq i \leq n$ and call $\operatorname{gr}_i A$ the *i-th grading* of A. The object $\operatorname{gr} A = \bigoplus_{i=1}^n \operatorname{gr}_i A$ is called the *associated grading* to A. The next proposition ensures that these objects are well-defined.

Proposition 4.9. *Let A be an object of finite length n in an abelian category $\mathcal{A}$. Consider two JH-filtrations (A_i) and (A'_i) of A. Then (after a permutation if necessary) the subquotients of (A_i) and (A'_i) are isomorphic. In particular, $\operatorname{gr} A$ is well-defined up to isomorphism.*

Proof. We proceed by induction on n. If $n = 1$, then there is nothing to show. Assume $n > 1$. Let $1 \leq j \leq n$ be the smallest integer such that $A'_1 \subset A_j$, i.e. for all $i < j$ we have $A'_1 \not\subset A_i$. The canonical morphism $A'_1 \to A_j/A_{j-1}$ is then non-zero. Therefore it induces an isomorphism $A'_1 \cong A_j/A_{j-1}$ since both are simple by the above proposition, hence $A_j = A'_1 \oplus A_{j-1}$. We obtain an exact sequence

$$0 \to A_{j-1} \to A/A'_1 \to A/A_j \to 0.$$

Define B_i for $i > j$ as the pullback

$$\begin{array}{ccc} B_i & \longrightarrow & A/A'_1 \\ \downarrow & & \downarrow \\ A_i/A_j & \longrightarrow & A/A_j. \end{array}$$

The canonical morphisms $B_i \to B_{i+1}$ are monos, and we obtain a filtration

$$(*) \qquad 0 \subset A_1 \subset A_2 \subset \ldots \subset A_{j-1} \subset B_{j+1} \subset B_{j+2} \subset \ldots \subset B_n = A/A'_1.$$

From the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & K & \longrightarrow & A_{j-1} & \longrightarrow & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & B_i & \longrightarrow & A/A'_1 & \longrightarrow & A/A_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_i/A_j & \longrightarrow & A/A_j & \longrightarrow & A/A_i & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

we see that the kernel K_i of $B_i \to A_i/A_j$ is isomorphic to A_{j-1} for all $j < i \leq n$. It follows that $B_{j+1}/A_{j-1} \cong A_{i+1}/A_j$ and $B_i/B_{i-1} \cong A_i/A_{i-1}$ for $i > j+1$. Hence the filtration $(*)$ is a JH-filtration of A/A'_1 with gradings A_i/A_{i-1} for $1 \leq i \leq n$, $i \neq j$ of length $n-1$. On the other hand, the filtration

$$0 \subset A'_2/A'_1 \subset A'_3/A'_1 \subset \ldots \subset A/A'_1$$

is also a JH-filtration of A/A'_1 of length $n-1$. By induction hypothesis there exists a permutation $\sigma \in S_n$ with $\sigma(j) = 1$ such that

$$A_i/A_{i-1} \cong A'_{\sigma(i)}/A'_{\sigma(i)-1}$$

for $i \neq j$. Since $A_j/A_{j-1} \cong A'_1$, the proposition follows. □

Consider an exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$. It is easy to see that the grading $\operatorname{gr} A$ is the direct sum of the gradings $\operatorname{gr} A'$ and $\operatorname{gr} A''$. Indeed, we may take a JH-filtration of A' and refine it to the JH-filtration of A. The terms between A' and A modulo A' define a JH-filtration of A''.

Remark 4.10. Let P be a property of objects in $\mathcal{A}$ which is stable under isomorphisms, and such that for a given exact sequence in $\mathcal{A}$

$$0 \to A' \to A \to A'' \to 0$$

A has property P if and only if A' and A'' have property P. In other words, P passes onto subobjects and quotients of A and is stable under extensions. Then it is easy to see that P is compatible with JH-equivalence.

Indeed, let A, B be two objects in $\mathcal{A}$ which are of finite length and are JH-equivalent. Assume that A satisfies P. Let

$$0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$$

be a JH-filtration of A. Then every A_i satisfies P, and therefore also every A_i/A_{i-1}. Now let

$$0 = B_0 \subset B_1 \subset \ldots \subset B_n = B$$

be a JH-filtration of B. Then, since B_i/B_{i-1} is isomorphic to some A_j/A_{j-1} for every $1 \leq i \leq n$, they all satisfy P. It follows inductively that every B_i satisfies P, since

$$0 \to B_{i-1} \to B_i \to B_i/B_{i-1} \to 0$$

is exact.

Lemma 4.11. *Consider an exact functor $F : \mathcal{A} \to \mathcal{B}$ of abelian categories. Let A be an object in $\mathcal{A}$ and $B = F(A)$. Assume that A and B are both of finite length. Then $F(\operatorname{gr} A)$ is JH-equivalent to B.*

Proof. Let

$$0 = A_0 \subset A_1 \subset \ldots \subset A_n = A$$

be a JH-filtration. Put $B_i = F(A_i)$ for $1 \leq i \leq n$. Then we have $F(\operatorname{gr} A) = \bigoplus_{i=1}^n B_i/B_{i-1}$ since F is exact. On the other hand, again by exactness of F we obtain a filtration

$$0 = B_0 \subset B_1 \subset \ldots \subset B_n = B.$$

By Corollary 4.6 we may refine this filtration to a JH-filtration, which consists filtrations

$$(*) \qquad B_{i-1} = B_{i,0} \subset B_{1,1} \subset \ldots \subset B_{i,n_i} = B_i$$

between B_{i-1} and B_i of length n_i for every $1 \leq i \leq n$. Hence we have

$$\operatorname{gr} B = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{n_i} B_{i,j}/B_{i,j-1}.$$

Since gr is compatible with short exact sequences, in particular with direct sums, we have $\operatorname{gr} F(\operatorname{gr} A) = \bigoplus_{i=1}^n \operatorname{gr} B_i/B_{i-1}$. Therefore it is enough to show that $\operatorname{gr} B_i/B_{i-1} = \bigoplus_{j=1}^{n_i} B_{i,j}/B_{i,j-1}$. But this identity follows from the facts that $(*)$ modulo $B_{i,0}$ is a JH-filtration of B_i/B_{i-1} and by using the canonical isomorphisms $(B_{i,j}/B_{i,0})/(B_{i,j-1}/B_{i,0}) \cong B_{i,j}/B_{i,j-1}$. □

Remark. In general, we cannot expect that F is compatible with the grading $F(\operatorname{gr} A) = \operatorname{gr} F(A)$. Indeed, this is equivalent to the fact that F preserves JH-filtrations, and this again to the fact that F sends simple objects to simple objects.

Proposition 4.12. *An exact functor between abelian categories preserves JH-equivalence.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Let A and A' be two objects of finite length in $\mathcal{A}$. We have to show that if $A \sim A'$, then also $F(A) \sim F(A')$, where $\sim$ stands for the JH-equivalence. By the above lemma we have

$$F(A) \sim F(\operatorname{gr} A) \cong F(\operatorname{gr} B) \sim F(B).$$

□

Remarks on literature

Section 1: I learned Definition 1.2 from [Mar96] Definition 0.2. which goes back in such generality to Simpson [Sim94] p. 55, see also [Gie77]. The Jordan-Hölder filtration of semistable vector bundles and S-equivalence was introduced originally by Seshadri in [Ses67] (S stands for Seshadri). The definition of slope and semistability on a smooth projective curve is due to Mumford [Mum63]. The original article does not include the construction of the moduli space of vector bundles of fixed slope. The earliest construction of this moduli space known to me is done in [Ses67]. The example on page 17 is inspired by [LM05] example 2.2. Definition 1.5 is due to Seshadri [Ses82] septième partie, II.9. For the definition of ordinary double point cf. [Liu02] Definition 7.5.13. The idea to test semistability of line bundles by restricting to proper connected subcurves is from [LM05] 3.

Section 2: The general theory about the moduli spaces of pure sheaves I learned from [HL10] Part I. The Hilbert polynomial in families over locally Noetherian schemes is constant by [$EGAIII_2$] Théorème 7.9.4. The exact definition of the moduli functor and in particular of the S-equivalence of *families* is due to Maruyama, cf. [Mar96] Definition 0.5. The definition of corepresentability of a functor is from [Sim94] p. 60 (a comment on p. 60 in [HL10] says that this definition is actually due to Simpson). For boundedness of sheaves cf. [HL10] Definition 1.7.5. For the main existence theorem by Langer and Maruyama besides the given sources in the text cf. also [Lan04a] Theorem 4.1. A good reference on Japanese rings is [Mat80] chapter 12 (a Japanese ring is a N-2 ring in the terminology there) or [$EGAIV_2$] §7. The rest of this section is a summary of the results presented in [LP97] part I and [Ses82] premiére partie.

Section 3: For the notion of the maximal destabilizing sheaf cf. [HL10] Definition 1.3.6.

Section 4: All categorical notions are from [ML98]. The definitions and results concerning modular lattices are from [Coh89] §2. Note that Proposition 4.12 is well-known and may be formulated in terms of Grothendieck groups.

Part 2

p-adic neighborhood

5. Strongly semistable reduction

There are examples of vector bundles on a smooth projective and connected curve C over a field of characteristic p, which are semistable, but their pullback by some power of the (absolute) Frobenius morphism $F: C \to C$ is not. A vector bundle E on C is called *strongly semistable* if $F^{n*}E$ is semistable for all $n \geq 0$.

Definition 5.1. Consider a purely one-dimensional proper scheme Z over a field k of characteristic p. Let E be a vector bundle on Z.

(i) We say that E is *strongly semistable* if for every irreducible component C_i of Z (endowed with its canonical reduced subscheme structure) the pullback $E|_{\tilde{C}_i}$ of E to the normalization $\tilde{C}_i$ of C_i is strongly semistable.

(ii) We say that E is *strongly semistable of degree zero* if E is strongly semistable, and additionally all $E|_{\tilde{C}_i}$ are of degree zero.

Note that a vector bundle E on Z which is strongly semistable of degree zero has degree zero, but E which is strongly semistable with $\deg E = 0$ may have a restriction of degree non-zero.

Proposition 5.2. *Let Z be as above and E a vector bundle on Z. Then E is strongly semistable (of degree zero) if and only if for all smooth projective and connected curves C and all k-morphisms $C \to Z$ the pullback of E to C is semistable (of degree zero).*

Proof. Cf. [DW07] Proposition 12.2.4. □

The category of strongly semistable vector bundles of degree zero on Z is abelian (cf. [Nor82] Lemma 3.6). It is also closed under extensions since the pullback is exact on short exact sequences of vector bundles, and the category of semistable vector bundles on a smooth projective and connected curve is closed under extensions (Proposition 1.3).

Let R be a valuation ring with quotient field K and residue field k. Consider a smooth projective and connected curve X over K. We say that $\mathfrak{X}$ is a *model* of X over R if $\mathfrak{X}$ is a finitely presented, proper and flat scheme over R with generic fiber isomorphic to X. We say a scheme $\mathfrak{X}$ over R is a model (without referring to X) if the generic fiber of $\mathfrak{X}$ is a smooth projective and connected curve over K, and $\mathfrak{X}$ is a model of its generic fiber. A model $\mathfrak{X}'$ of X over R *dominates* $\mathfrak{X}$ if there exists an R-morphism $\varphi: \mathfrak{X}' \to \mathfrak{X}$, and φ induces an isomorphism on generic fibers, in other words, φ is a birational morphism. Since X is integral, it follows that the scheme $\mathfrak{X}$ is also integral (cf. [Liu02] Proposition 4.3.8).

Denote by $\overline{\mathbb{Z}}_p$ and $\mathfrak{o}$ the ring of integers of $\overline{\mathbb{Q}}_p$ and $\mathbb{C}_p$, respectively. Then $k = \overline{\mathbb{F}}_p$ is the residue field of $\overline{\mathbb{Z}}_p$ and $\mathfrak{o}$. By a model $\mathfrak{X}$ of a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ we always mean a model $\mathfrak{X}$ of X over $\overline{\mathbb{Z}}_p$. We denote by $\mathfrak{X}_{\mathfrak{o}}$ the base change to $\mathfrak{o}$ and by $\mathfrak{X}_k$ the special fiber of $\mathfrak{X}_{\mathfrak{o}}$, which coincides with the special fiber of $\mathfrak{X}$.

Before we proceed with the next definition, we recall some well-known results we will frequently use from the theory of Noetherian descent developed in [EGAIV$_3$] §8. The ring $\overline{\mathbb{Z}}_p$ is the inductive limit $\overline{\mathbb{Z}}_p = \varinjlim \mathfrak{o}_K$, where K runs over all finite field extensions of $\mathbb{Q}_p$, and $\mathfrak{o}_K$ is the ring of integers of

K. Since a model $\mathfrak{X}$ over $\overline{\mathbb{Z}}_p$ is of finite presentation over $\overline{\mathbb{Z}}_p$, it is already defined over some $\mathfrak{o}_K$, that is, there exists a scheme $\mathfrak{X}_K$ over $\mathfrak{o}_K$ with

$$\mathfrak{X} = \mathfrak{X}_K \otimes_{\mathfrak{o}_K} \overline{\mathbb{Z}}_p.$$

Sometimes we also say that $\mathfrak{X}$ *descends* to $\mathfrak{X}_K$. The properness and flatness are inherited by $\mathfrak{X}_K$ from $\mathfrak{X}$, hence $\mathfrak{X}_K$ is a model. A coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ is already defined over $\mathfrak{X}_K$ for some K as above. If $\mathcal{F}$ is locally free of finite rank, then the corresponding sheaf is also locally free of the same rank. In an analogous way, we may use the same theory for the fields $\overline{\mathbb{Q}}_p$ and k.

Definition 5.3. Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let E be a vector bundle on $X_{\mathbb{C}_p}$.

(i) We say that E has *strongly semistable reduction* (*of degree zero*) if E is isomorphic to the generic fiber of a vector bundle $\mathcal{E}$ on $\mathfrak{X}_{\mathfrak{o}}$ for some model $\mathfrak{X}$ of X such that the special fiber $\mathcal{E}_k$ of $\mathcal{E}$ is strongly semistable (of degree zero).

(ii) We say that E has *potentially strongly semistable reduction* if there is a finite morphism $\alpha : Y \to X$ of smooth projective and connected curves over $\overline{\mathbb{Q}}_p$ such that $\alpha^*_{\mathbb{C}_p} E$ has strongly semistable reduction.

Let $\mu \in \mathbb{Q}$. We introduce the following categories:

$\mathfrak{B}^s_X$ – the full subcategory of vector bundles on $X_{\mathbb{C}_p}$ which have strongly semistable reduction of degree zero,

$\mathfrak{B}^\mu_X$ – the full subcategory of vector bundles on $X_{\mathbb{C}_p}$ of slope μ which have potentially strongly semistable reduction.

We have the following inclusion of categories

$$\mathfrak{B}^s_X \subset \mathfrak{B}^0_X.$$

Proposition 5.4. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let E be a vector bundle on $X_{\mathbb{C}_p}$ of slope μ. Then E has potentially strongly semistable reduction, i.e. E lies in $\mathfrak{B}^\mu_X$, if and only if there exists a finite morphism $\alpha : Y \to X$ of smooth projective and connected curves over $\overline{\mathbb{Q}}_p$ and a line bundle L on $Y_{\mathbb{C}_p}$ such that $\alpha^*_{\mathbb{C}_p} E \otimes L$ has strongly semistable reduction of degree zero, i.e. it lies in $\mathfrak{B}^s_Y$.*

Proof. This is exactly the statement of [DW10] Theorem 3 i). □

Proposition 5.5. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let E be a vector bundle on $X_{\mathbb{C}_p}$. Assume that E has strongly semistable reduction (of degree zero) over $\mathfrak{X}_{\mathfrak{o}}$, where $\mathfrak{X}$ is a model of X over $\overline{\mathbb{Z}}_p$. Let $\mathcal{Y}$ be a model dominating $\mathfrak{X}$. Then E has also strongly semistable reduction (of degree zero) over $\mathcal{Y}_{\mathfrak{o}}$.*

Proof. By assumption there exists a vector bundle $\mathcal{E}$ on $\mathfrak{X}_{\mathfrak{o}}$ with generic fiber isomorphic to E such that $\mathcal{E}_k$ is strongly semistable (of degree zero). Let $\mathcal{F}$ be the pullback of $\mathcal{E}$ to $\mathcal{Y}_{\mathfrak{o}}$. Then the generic fiber of $\mathcal{F}$ is isomorphic to E since $\mathcal{Y}_{\overline{\mathbb{Q}}_p} \to X$ is an isomorphism. Let C be a smooth projective and connected curve and $\alpha : C \to \mathcal{Y}_k$ a k-morphism. It follows from Proposition 5.2 that the pullback of $\mathcal{E}_k$ via $C \xrightarrow{\alpha} \mathcal{Y}_k \to \mathfrak{X}_k$, which is $\alpha^*\mathcal{F}_k$, is semistable (of degree zero). Again using the same proposition we conclude that $\mathcal{F}_k$ is strongly semistable (of degree zero). □

6. p-ADIC NEIGHBORHOOD FOR FAMILIES

Let $f : X \to S$ be a morphism of schemes and T a scheme over S. As defined in section 2 a family F (of coherent sheaves) on X parametrized by T is a coherent sheaf on $X_T = X \times_S T$ flat over T.

Proposition 6.1. *Consider a morphism $X \to S$ of schemes of finite presentation. Let F be a family on X parametrized by a scheme T over S. Then the set*

$$\{t \in T \mid F_t \text{ is locally free of finite rank}\}$$

is open in T.

Proof. Use the setup explained in [EGAIV$_3$] 8.1.2 a) and apply *loc. cit.* Proposition 8.5.5. □

Let S be a scheme of finite type over a field K. If K is a topological field, then the set $S(K)$ of K-valued points of S naturally inherits its topology. It is the *strong topology* on $S(K)$ induced by the topology on K. When $K = \overline{\mathbb{Q}}_p$ or $K = \mathbb{C}_p$, we refer to it in the following always as to the p-adic topology. More generally, for a scheme Z locally of finite type over a local topological ring R such that R^* is open in R and has continuous inversion there exists a global topology on $Z(R)$ uniquely determined by similar functorial properties as of the strong topology (cf. [Con] Proposition 3.1). When R is a field, it coincides with the usual strong topology. We refer to it as well as to the p-adic topology when $R = \overline{\mathbb{Z}}_p$ or $R = \mathfrak{o}$.

For a continuous homomorphism $R \to R'$ of topological rings the natural map $Z(R) \to Z(R')$ is continuous. If Z is affine of finite type over R and $R \to R'$ is a topological embedding, then so is $Z(R) \to Z(R')$ (cf. *loc. cit.* Example 2.2). Therefore, when Z is not necessarily affine, the latter map is locally a topological embedding. In particular, we may apply this to $\mathfrak{o} \hookrightarrow \mathbb{C}_p$. It follows that if Z is proper scheme over $\mathfrak{o}$, then the bijection $Z(\mathfrak{o}) \cong Z(\mathbb{C}_p)$ is a homeomorphism. The analogous statement is true for $\overline{\mathbb{Z}}_p \hookrightarrow \overline{\mathbb{Q}}_p$.

Notation 6.2. We fix the following notation $\mathcal{E}/\mathfrak{X}_\mathfrak{o}/S$, where $\mathfrak{X}$ is a model over $\overline{\mathbb{Z}}_p$, S is a connected scheme of finite type over $\overline{\mathbb{Z}}_p$, which will play the role of the parameter space, and $\mathcal{E}$ is a family on $\mathfrak{X}_\mathfrak{o}$ parametrized by S. Recall that the generic fiber X of $\mathfrak{X}$ is a smooth projective and connected curve over $\overline{\mathbb{Q}}_p$. By E we denote the family $\mathcal{E}_{\mathbb{C}_p}$ on $X_{\mathbb{C}_p}$ parametrized by S.

The following natural question arises: Is the property of vector bundles to have strongly semistable reduction (of degree zero) p-adically open on the base in families? More exactly, given a family $\mathcal{E}/\mathfrak{X}_\mathfrak{o}/S$ and a $t \in S(\mathfrak{o})$ such that $\mathcal{E}_t$ is a vector bundle having strongly semistable reduction (of degree zero), can we find a p-adic neighborhood of t such that the family $\mathcal{E}$ restricted to this neighborhood consists only of vector bundles having strongly semistable reduction (of degree zero)?

Proposition 6.3. *Consider a family $\mathcal{E}/\mathfrak{X}_\mathfrak{o}/S$ as in 6.2. Let $t \in S(\mathfrak{o})$. Assume that $\mathcal{E}_t$ is a vector bundle on $\mathfrak{X}_\mathfrak{o}$ having strongly semistable reduction (of degree zero). Then there exists a p-adic neighborhood $U \subset S(\mathfrak{o})$ of t such* *that for all $s \in U$ the sheaf $\mathcal{E}_s$ is a vector bundle having strongly semistable reduction (of degree zero).*

Proof. By the above proposition we may assume without loss of generality that $\mathcal{E}_s$ is a vector bundle for all $s \in S(\mathfrak{o})$. Let $t_0 \in S(k)$ be the morphism $\operatorname{Spec} k \to \operatorname{Spec} \mathfrak{o} \xrightarrow{t} S$. Then $\mathcal{E}_{t_0} = \mathcal{E}_{t,k}$ is a strongly semistable vector bundle on the curve $\mathfrak{X}_k$ by assumption. The mapping $S(\mathfrak{o}) \to S(k)$ induced by the reduction $\mathfrak{o} \to k$ is continuous with respect to the p-adic topology on $S(\mathfrak{o})$ and discrete topology on $S(k)$. Therefore the preimage $U \subset S(\mathfrak{o})$ of t_0 is p-adically open. By construction the vector bundle $\mathcal{E}_s$ has the same reduction as $\mathcal{E}_t$ for all $s \in U$. □

Corollary 6.4. *Let $\mathcal{E}/\mathfrak{X}_{\mathfrak{o}}/S$ and t be as in the above proposition. Assume that S is proper over $\overline{\mathbb{Z}}_p$. Then there exists a p-adic neighborhood $U \subset S(\mathbb{C}_p)$ of $t_{\mathbb{C}_p}$ such that for all $s \in U$ the vector bundle E_s on $X_{\mathbb{C}_p}$ has strongly semistable reduction (of degree zero), where $E = \mathcal{E} \otimes \mathbb{C}_p$.*

Proof. By the above proposition there exists a p-adic neighborhood $U' \subset S(\mathfrak{o})$ of t such that $\mathcal{E}_s$ has strongly semistable reduction (of degree zero) for all $s \in U'$. Let U be the image of U' via the natural mapping $S(\mathfrak{o}) \to S(\mathbb{C}_p)$. Since S is proper over $\overline{\mathbb{Z}}_p$, the latter is a p-adic homeomorphism, hence U is p-adically open in $S(\mathbb{C}_p)$. Let $s \in U$. Denote by $s_{\mathfrak{o}} : \operatorname{Spec} \mathfrak{o} \to S$ the $\mathfrak{o}$-section which induces s. Then $\mathcal{E}_{s_{\mathfrak{o}}}$ has strongly semistable reduction (of degree zero) since $s_{\mathfrak{o}} \in U'$. The statement follows then from $\mathcal{E}_{s_{\mathfrak{o}}} \otimes_{\mathfrak{o}} \mathbb{C}_p = \mathcal{E}_s = E_s$. □

Instead of taking the reduction modulo the maximal ideal $\mathfrak{p} \subset \mathfrak{o}$ we also may take the reduction mod p^n for $n \geq 1$. Put $\mathfrak{o}_n = \mathfrak{o}/p^n\mathfrak{o}$. Note that we also have $\mathfrak{o}_n = \overline{\mathbb{Z}}_p/p^n\overline{\mathbb{Z}}_p$.

Corollary 6.5. *With $\mathcal{E}/\mathfrak{X}_{\mathfrak{o}}/S$ and t as in the above proposition there exists for all $n \geq 1$ a p-adic neighborhood $U = U(n) \subset S(\mathfrak{o})$ of t such that for all $s \in U$*

$$\mathcal{E}_{s,n} = \mathcal{E}_{t,n}.$$

Proof. Replace the mapping $S(\mathfrak{o}) \to S(k)$ in the proof of the above proposition by the continuous mapping $S(\mathfrak{o}) \to S(\mathfrak{o}_n)$ induced by the natural homomorphism $\operatorname{mod} p^n : \mathfrak{o} \to \mathfrak{o}_n$, where the topology on $S(\mathfrak{o}_n)$ is also discrete. □

Remark. In the above proofs we have used that $\mathfrak{o}_n$ has the discrete topology. The statement is a special case of the following simple fact: the quotient topology on the quotient of a topological group by an open subgroup is always discrete.

7. p-ADIC NEIGHBORHOOD FOR SMOOTH MODELS

In this section we will prove a similar result to Corollary 6.4. However, instead of considering individual families of vector bundles parametrized by a given scheme we will work with the moduli space parameterizing those. One of the main ingredients of the proof is Theorem 3.1 of Langton, which works for discrete valuation rings. Therefore we restrict ourselves to vector bundles with reduction over $\overline{\mathbb{Z}}_p$.

To be more precise, consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. We say that a vector bundle E on $X_{\mathbb{C}_p}$ has strongly semistable reduction (of degree zero) over $\overline{\mathbb{Z}}_p$ if there exist a model $\mathfrak{X}$ and a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ with $\mathcal{E} \otimes_{\overline{\mathbb{Z}}_p} \mathbb{C}_p \cong E$ such that the special fiber $\mathcal{E}_k$ of $\mathcal{E}$ is strongly semistable (of degree zero). Note that $\mathcal{E}$ is defined on $\mathfrak{X}$ without base change to $\mathfrak{o}$ contrary to Definition 5.3. It follows that E is obtained by base change to $\mathbb{C}_p$ of a vector bundle already defined on X. Thus, by abuse of notations, we regard E as a vector bundle on X and just say that it has strongly semistable reduction (of degree zero).

Proposition 7.1. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$.*

(i) Every vector bundle on $X_{\mathbb{C}_p}$ with potentially strongly semistable reduction is semistable.

(ii) Every vector bundle on X with strongly semistable reduction (of degree zero) is semistable.

Proof. All vector bundles in the category $\mathfrak{B}^s_X$ are semistable by [DW05b] Theorem 13 and Theorem 17. Hence (i) follows from Proposition 5.4.

For (ii) the proofs of Theorem 13 and Theorem 17 work in exactly the same way for vector bundles defined over $\overline{\mathbb{Z}}_p$. In fact the proof of Theorem 13 is even easier since the descent argument there can be made more directly using Noetherian descent for $\overline{\mathbb{Z}}_p$. □

Let M_X be the moduli space of vector bundles of fixed rank r and degree zero. By the above proposition vector bundles of rank r having reduction as in (ii) induce points in $M_X(\overline{\mathbb{Q}}_p)$. Similarly to the previous section we ask the following question: Is the property of vector bundles on X to have strongly semistable reduction of degree zero open in the p-adic topology on $M_X(\overline{\mathbb{Q}}_p)$? In the case of good reduction the answer is positive, as shown below. We say that a vector bundle E has *good strongly semistable reduction (of degree zero)* if E has strongly semistable reduction (of degree zero) over a *smooth* model.

Theorem 7.2. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let E be a vector bundle on X of rank r which has* good *strongly semistable reduction of degree zero. Then there exists a p-adic neighborhood $U \subset M_X(\overline{\mathbb{Q}}_p)$ of the S-equivalence class of E consisting of S-equivalence classes of vector bundles with strongly semistable reduction of degree zero.*

Proof. By assumption there exists a smooth model $\mathfrak{X}$ of X over $\overline{\mathbb{Z}}_p$ and a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ with generic fiber isomorphic to E such that the special fiber $\mathcal{E}_k$ of $\mathcal{E}$ is strongly semistable of degree zero. By Noetherian descent there exists a finite field extension $K/\mathbb{Q}_p$ with ring of integers $\mathfrak{o}_K$ and residue field κ such that $\mathfrak{X}$ descends to a smooth model $\mathfrak{X}_K$ over $\mathfrak{o}_K$. Therefore $\mathfrak{X}_K$ is regular and by Theorem 2.8 of Lichtenbaum [Lic68] it is projective. Fix a relatively ample line bundle $\mathcal{H}$ on $\mathfrak{X} \to \operatorname{Spec} \mathfrak{o}_K$. Further, by Zariski's connectedness principle $\mathfrak{X}_K \to \operatorname{Spec} \mathfrak{o}_K$ has geometrically connected fibers. Finally, $\mathfrak{o}_K$ is universally Japanese (cf. [EGAIV$_2$] Corollaire 7.7.4). Thus by Theorem 2.6 there exists the moduli space $M = M_{\mathfrak{X}_K/\mathfrak{o}_K}$ parameterizing families of semistable sheaves on $\mathfrak{X}_K$ of fixed rank r and degree zero with respect to the line bundle $\mathcal{H}$.

By assumption the vector bundle $\mathcal{E}_k$ is strongly semistable on $\mathfrak{X}_k$. Since $\mathfrak{X}_k$ is smooth, semistability on $\mathfrak{X}_k$ does not depend on $\mathcal{H}$, and $\mathcal{E}_k$ is trivially semistable on $\mathfrak{X}_k$. The vector bundle E on X is semistable of degree zero. Therefore, $\mathcal{E}$ is a semistable family on $\mathfrak{X}$, which induces a point $[\mathcal{E}]$ in $M(\overline{\mathbb{Z}}_p)$.

We have the followings natural maps induced by the homomorphisms $\overline{\mathbb{Z}}_p \to k$ and $\overline{\mathbb{Z}}_p \hookrightarrow \overline{\mathbb{Q}}_p$

$$M(k) \longleftarrow M(\overline{\mathbb{Z}}_p) \longrightarrow M(\overline{\mathbb{Q}}_p),$$

which are continuous with respect to the p-adic topology on $M(\overline{\mathbb{Z}}_p)$ and $M(\overline{\mathbb{Q}}_p)$, and the discrete topology on $M(k)$. The moduli space M is projective, in particular proper. It follows that the map on the right hand side is a homeomorphism. Since M corepresents the corresponding moduli functor uniformly, and $\mathfrak{o}_K \to K \to \overline{\mathbb{Q}}_p$ is flat, $M \otimes_{\mathfrak{o}_K} \overline{\mathbb{Q}}_p$ is canonically isomorphic to M_X by the universal property of the coarse moduli space, where the latter is the moduli space of semistable vector bundles of rank r and degree zero on X.

Let U be the preimage in $M_X(\overline{\mathbb{Q}}_p)$ of the S-equivalence class $[\mathcal{E}_k] \in M(k)$ of $\mathcal{E}_k$ after identifying $M(\overline{\mathbb{Z}}_p) = M_X(\overline{\mathbb{Q}}_p)$. Then U is p-adically open in $M_X(\overline{\mathbb{Q}}_p)$. By construction $[E] \in U$. We show that every vector bundle F on X with $[F] \in U$ has strongly semistable reduction of degree zero.

Let F be such a vector bundle. By Noetherian descent there exists a finite field extension L/K such that F descends to a vector bundle F_L on $X_L = \mathfrak{X}_K \otimes_{\mathfrak{o}_K} L$, which is semistable. By Theorem 3.1 of Langton there exists a family $\mathcal{F}_L$ on $\mathfrak{X}_L = \mathfrak{X}_K \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ of semistable sheaves with generic fiber isomorphic to F_L, where $\mathfrak{o}_L$ is the ring of integers of L. The restriction of $\mathcal{F}_L$ to the special fiber of $\mathfrak{X}_L$ is locally free, since the latter is smooth. Therefore since the restriction of $\mathcal{F}_L$ to X_L is also locally free it follows from [HL10] Lemma 2.1.7 that $\mathcal{F}_L$ is locally free. Put $\mathcal{F} = \mathcal{F}_L \otimes_{\mathfrak{o}_L} \overline{\mathbb{Z}}_p$. By construction of U we have $[\mathcal{F}_k] = [\mathcal{E}_k]$ as points in $M(k)$. Since M is a coarse moduli space, the set $M(k)$ is in bijection with the set of S-equivalence classes of semistable vector bundles on $\mathfrak{X}_k$ of rank r and degree zero. It follows that $\mathcal{F}_k$ and $\mathcal{E}_k$ are S-equivalent as semistable vector bundles on $\mathfrak{X}_k$. Now since the property to be strongly semistable of degree zero is stable under subobjects and quotients (cf. [Nor82] Lemma 3.6 (a)) and is stable under extensions in the category of semistable vector bundles of slope zero on $\mathfrak{X}_k$, it is compatible with the S-equivalence. We conclude that $\mathcal{F}_k$ is strongly semistable of degree zero (cf. Remark 4.10). Hence F has strongly semistable reduction of degree zero as claimed. □

Remark. (i) In order to use Theorem of Langton we have to descend to a model over a discrete valuation ring, which is possible if we consider vector bundles over $\overline{\mathbb{Z}}_p$.

(ii) Restricting to the smooth models of X allows us to consider the vector bundle $\mathcal{E}_k$ as a point in $M(k)$. Strong semistability on a smooth curve implies then trivially semistability, which moreover does not depend on the polarization of the model.

(iii) The last paragraph of the above proof states that a vector bundle on X having a reduction, which is S-equivalent to a strongly semistable vector bundle of degree zero, has already a strongly semistable reduction of degree zero.

In the above proof instead of taking the special fiber we may also take the reduction modulo p^n. With similar argument we obtain the following more precise result. Let K be a finite field extension of $\mathbb{Q}_p$ and $\mathfrak{o}_K$ its ring of integers. Consider a model $\mathfrak{X}_K$ over $\mathfrak{o}_K$. We have seen in the above proof that the moduli space $M_{\mathfrak{X}_K}$ of semistable sheaves on $\mathfrak{X}_K$ of rank r and degree zero exists (smoothness of $\mathfrak{X}_K$ was not necessary for that). We write $[\mathcal{E}]$ for the induced point in $M_{\mathfrak{X}_K}$ by a family $\mathcal{E}$ of semistable sheaves on $\mathfrak{X}_K$.

Proposition 7.3. *Consider a smooth model $\mathfrak{X}_K$ over $\mathfrak{o}_K$. Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$ which has strongly semistable reduction of degree zero with generic fiber E of rank r. Then for every $n \geq 1$ there exists a p-adic neighborhood $U = U(n) \subset M_X(\overline{\mathbb{Q}}_p)$ of the S-equivalence class of E consisting of S-equivalence classes of vector bundles F with the following property:*

There exists a vector bundle $\mathcal{F}$ on $\mathfrak{X}$ with generic fiber isomorphic to F having strongly semistable reduction of degree zero such that

$$[\mathcal{F}_n] = [\mathcal{E}_n] \quad \text{in } M_{\mathfrak{X}_K}(\mathfrak{o}_n),$$

where $\mathcal{F}_n = \mathcal{F} \otimes_{\mathfrak{o}} \mathfrak{o}_n$, $\mathcal{E}_n = \mathcal{E} \otimes_{\mathfrak{o}} \mathfrak{o}_n$.

Proof. Put $M = M_{\mathfrak{X}_K}$. By assumption $\mathcal{E}$ is a family on $\mathfrak{X}$ of semistable sheaves. The reduction mod $p^n : \overline{\mathbb{Z}}_p \to \mathfrak{o}_n$ induces a mapping

$$r_n : M(\overline{\mathbb{Z}}_p) \to M(\mathfrak{o}_n),$$

which is continuous with respect to the p-adic topology on $M(\overline{\mathbb{Z}}_p)$ and the discrete topology on $M(\mathfrak{o}_n)$. Let U be the preimage of the S-equivalence class $[\mathcal{E}_k]$ of $\mathcal{E}_k$ via r_n after identifying $M(\overline{\mathbb{Z}}_p) = M(\overline{\mathbb{Q}}_p) = M_X(\overline{\mathbb{Q}}_p)$ as in the proof of the above theorem. Hence U is p-adically open and contains the S-equivalence class of E.

We claim that U has the desired property. Let $F \in U$. By Noetherian descent and Theorem of Langton 3.1 there exists a family $\mathcal{F}$ on $\mathfrak{X}$ of semistable sheaves with $\mathcal{F}_{\overline{\mathbb{Q}}_p} \cong F$. By construction of U we have $[\mathcal{F}_n] = [\mathcal{E}_n]$ in $M(\mathfrak{o}_n)$. In particular $[\mathcal{F}_k] = [\mathcal{E}_k]$ in $M(k)$. The rest follows in the same way as at the end of the proof of the above theorem. □

8. Characterization of semistability

The main result of this section is due to Teixidor i Bigas (cf. [TiB95] Proposition 1.2). Since the proof in the original article contains certain inaccuracies, we decided to include the full proof here.

Consider a semistable curve X over a field k. Recall that a curve X over an algebraically closed field is *semistable* if it is reduced, connected and has only ordinary double points as singularities. A curve X over a field k (not necessarily algebraically closed) is semistable if the base change of X to the algebraic closure of k is semistable.

Write $X = \bigcup_{i\in I} X_i$, where X_i are the irreducible components of X. Let $\{P_j\}_{j\in J_0}$ be the set consisting of intersection points of the irreducible components of X.

Consider a coherent sheaf F on X. Let F_i be the restriction of F to X_i modulo torsion. Note that for every $i \in I$ we have a canonical surjective morphism[2] $F \to F_i$ which induces an exact sequence

$$0 \to F \to \bigoplus_{i\in I} F_i \to T \to 0,$$

where T is a skyscraper sheaf with support lying in $\{P_j\}_{j\in J_0}$. Since X is semistable, every point P_j lies on exactly two components X_i and $X_{i'}$. Computing the k-dimensions of the stalks in the above sequence at P_j yields

$$\dim_k T_{P_j} = \dim_k F_{i,P_j} + \dim F_{i',P_j} - \dim F_{P_j} = \dim_k F_{P_j}.$$

On the other hand, by taking Euler characteristic we obtain

$$\chi(F) = \sum_{i\in I} \chi(F_i) - h^0(T) = \sum_{i\in I} \chi(F_i) - \sum_{j\in J_0} l(T_{P_j}),$$

where $l(T_{P_j})$ is the length of the $\mathcal{O}_{X,P_j}$-module T_{P_j}. If P_j is rational, then $l(T_{P_j})$ and $\dim_k T_{P_j}$ coincide.

An ordinary double point P on X is called *split* if all points in the normalization of X lying over P are rational over k. In particular, P is rational over k. Assume that F is locally free of rank r and all intersection points of the irreducible components of X are split. In this case, the above exact sequence yields the following identity

$$\chi(F) = \sum_{i\in I} \chi(F_i) - \sum_{j\in J_0} r_j, \tag{5}$$

where $r_j = \dim_k F_{P_j}$.

Let D be a subcurve of X. We define following coherent sheaves on X: F_D is the restriction of F to D, that is, $F_D = i_* i^* F$, where $i : D \hookrightarrow X$ is the closed immersion. The sheaf F^D is the subsheaf of F of all sections which vanish on the "complement" of D. To be more precise, denote by $\overline{D}$ the closure of the complement of D in X, then F^D is the kernel of the canonical surjection $F \to F_D$. We have a canonical exact sequence

$$0 \to F^D \to F \to F_{\overline{D}} \to 0. \tag{6}$$

[2] We denote by F_i as well the direct image via the closed immersion $X_i \hookrightarrow X$ of F_i.

Note that if a split singular point $P \in X$ lies on exactly two irreducible components of X, then these component are smooth at P (cf. [Liu02] Lemma 3.11). In particular, we have $D^{sing} = X^{sing} \setminus (D \cap \overline{D})$.

Proposition 8.1. *Consider a semistable curve X over a field k with split intersection points of its irreducible components. Let E be a locally free sheaf on X of rank r. Then for every subcurve $D \subset X$ we have the following identities*

$$\chi(E_D) = \sum_{X_i \subset D} \chi(E_i) - r\# \left\{ j \in J_0 \mid P_j \in D^{sing} \right\},$$

$$\chi(E^D) = \sum_{X_i \subset D} \chi(E_i) - r\# \left\{ j \in J_0 \mid P_j \in D \right\}.$$

Proof. The first identity follows from (5) applied to E_D, the second from (6). □

Assume that X is projective and fix an ample line bundle H of X. Using the semistability notion of Seshadri (Definition 1.5) with weightings h_i/h, where h is the degree of H and h_i the degree of the restriction of H to the irreducible component $X_i \subset X$, our goal is to prove the following result.

Theorem 8.2 (Teixidor i Bigas)**.** *Consider a projective semistable curve X over a field k with split intersection points of its irreducible components. Fix an ample line bundle H on X. Let E be a vector bundle on X of rank r such that the restriction E_i of E to every irreducible component $X_i \subset X$ is semistable. Further, assume that for each connected subcurve $D \subset X$ the following inequality is satisfied:*

$$\left(\sum_{X_i \subset D} \chi(E_i) - r \cdot k_D \right) \Big/ \left(\sum_{X_i \subset D} \frac{h_i}{h} \right) \leq \chi(E), \tag{7}$$

where $k_D = \# \left\{ j \in J_0 \mid P_j \in D \right\}$. Then E is semistable with respect to H.

Moreover, if all inequalities are strict and additionally at least one E_i is stable, then E is stable with respect to H.

Remark. If we assume that E is H-semistable, then the conditions in the theorem will arise in a natural way. Indeed, for every connected subcurve $D \subset X$ the sheaf $E^{\overline{D}}$ is a subsheaf of E as discussed before. The H-semistability of E implies then

$$\frac{\chi(E^{\overline{D}})}{\sum_i \frac{h_i}{h} r_i} \leq \frac{\chi(E)}{r},$$

where $r_i = \mathrm{rk}(E^{\overline{D}})_i$. But $r_i = r$ if $X_i \subset D$ and zero otherwise. Therefore the above inequality translates into

$$\frac{\chi(E^{\overline{D}})}{\sum_{X_i \subset D} h_i/h} \leq \chi(E).$$

Finally, using the computation of the Euler characteristic of $E^{\overline{D}}$ in the above proposition we obtain the inequality (7) for D.

The proof of the theorem is purely combinatorial. First we need some notations.

For $J \subset J_0$ we define X_J as the subcurve of X consisting of the irreducible components X_i containing at least one of the points of J. By $\mathcal{J}_c$ we denote the following set of subsets of J_0: $J \in \mathcal{J}_c$ if and only if X_J is connected.

Remark. In the original article the definition of X_J is slightly different. The curve X_J consists of the same irreducible components X_i as above, however it is glued only at the points $P_j \in J$. Hence, X_J may not be a subcurve of X. In this case, we consider for each $J \in \mathcal{J}_c$ a similar inequality to (7)

$$\left(\sum_{X_i \subset X_J} \chi(E_i) - r \cdot k_J \right) \Big/ \left(\sum_{X_i \subset X_J} \frac{h_i}{h} \right) \leq \chi(E), \tag{8}$$

where we denote $k_J = k_{X_J}$.

Although it seems that with this definition we obtain more inequalities as considered in the theorem, this is not the case. Indeed, let J be such a subset. We add to J the missing points on X_J, which have to be glued in order to obtain a subcurve. This yields a set $J' = \{j \in J_0 \mid P_J \in X_J\} \supset J$. Note that $X_{J'}$ is now a subcurve of X and is still connected. Even if $X_J \neq X_{J'}$, they both have the same irreducible components and $k_J = k_{J'}$, hence the inequalities (7) for $X_{J'}$ and (8) for X_J are the same.

For sake of simplicity we fix a total ordering of $\mathcal{J}_c$

$$\mathcal{J}_c = \{J(1) = J_0, J(2), \ldots, J(M)\},$$

such that $J(t)$ has at least as many elements as $J(t+1)$.

Fix positive integers r_j for $j \in J_0$ and r_i for $i \in I$ satisfying $r_i \geq r_j$ for all $j \in J_0$ with $P_j \in X_i$. (Later r_j's will be the dimensions of a coherent sheaf at the points P_j and r_i's the ranks of its restriction to X_i's.) We define for $1 \leq t \leq M$ and $j \in J_0$ inductively

$$\begin{aligned} r_{j,0} &= r_j, \\ a_t &= \min_{j \in J(t)} \{r_{j,t-1}\}, \\ r_{j,t} &= \begin{cases} r_{j,t-1} & \text{if } j \notin J(t), \\ r_{j,t-1} - a_t & \text{if } j \in J(t). \end{cases} \end{aligned}$$

Moreover, we define for every $i \in I$

$$r'_i = r_i - \sum_{X_{J(t)} \supset X_i} a_t. \tag{9}$$

Lemma 8.3. *We have for all $j \in J_0$:*

(i) $r_{j,t}$ and a_t are non-negative for all $1 \leq t \leq M$.
(ii) For all $1 \leq t \leq M$

$$r_{j,t} = r_j - \sum_{\substack{k=1 \\ J(k) \ni j}}^{t} a_k.$$

(iii) $r_{j,M} = 0$.

(iv) $r_j = \sum_{J(t)\ni j} a_t$.

Proof. The statements (i) and (ii) follow inductively from the definition. For (iii) note that for every $j \in J_0$ there exists a $1 \le t \le M$ with $J(t) = \{j\}$. Hence, $a_t = r_{j,t-1}$ and $r_{j,M} = r_{j,t-1} - a_t = 0$. The statement (iv) follows from (ii) for $t = M$ and (iii). □

Put for each $1 \le t \le M$

$$j(t) = \text{index} \min_{j \in J(t)} \{r_j\}.$$

Lemma 8.4. *For all $1 \le t \le M$ we have $a_t = r_{j(t),t-1}$.*

Proof. We show the identity by induction on t. For $t = 1$ there is nothing to show. Let $t > 1$. By definition of a_t we have to show that for all $j \in J(t)$

$$r_{j(t),t-1} \le r_{j,t-1}.$$

Fix a $j_0 \in J(t)$. By (ii) in the above lemma this is equivalent to

$$\sum_{\substack{k=1 \\ J(k)\ni j_0}}^{t-1} a_k \le \sum_{\substack{k=1 \\ J(k)\ni j(t)}}^{t-1} a_k.$$

Since all a_k are non-negative, it is enough to show that $a_k = 0$ for all $k < t$ with $j_0 \in J(k)$, $j(t) \notin J(k)$. Since $j_0 \in J(k) \cap J(t)$, the union of $J(k)$ and $J(t)$ defines a connected curve, hence $J(k) \cup J(t) = J(k')$ with $k' \le k$. We have $k' < k$ since otherwise $J(k) \supset J(t)$ and therefore $j(t) \in J(k)$. By the induction hypothesis $a_{k'} = r_{j(k'),k'-1}$. Now $j(k') = \text{index}\min\left\{r_{j(k)}, r_{j(t)}\right\}$.

Assume first that $j(k') = j(t)$. Then a variant of (ii) in the above lemma yields

$$r_{j(t),t-1} = r_{j(t),k'-1} - a_{k'} - \sum_{\substack{l=k'+1 \\ J(l)\ni j(t)}}^{t-1} a_l = - \sum_{\substack{l=k'+1 \\ J(l)\ni j(t)}}^{t-1} a_l.$$

Since all a_l and $r_{j(t),t-1}$ are non-negative, the right-hand side is zero, and we obtain $a_t = r_{j(t),t-1}$ directly.

Assume now that $j(k') = j(k)$. Then by a similar argumentation as in the first case we have $r_{j(k),k-1} = 0 = a_k$ as we have to show. □

From the above lemma we deduce the following

Corollary 8.5. *Let $j_0 \in J(t)$. If r_{j_0} is the minimum of the r_j's for j in some $J(k)$ with $k < t$, then $a_t = 0$.*

Proof. By definition we have

$$a_t = \min_{j \in J(t)} \{r_{j,t-1}\} \le r_{j_0,t-1} \le t_{j_0,k-1} - a_k.$$

The second inequality follows as in the first case of the above proof. The right hand side is zero by the above lemma. □

Lemma 8.6. *For all $i \in I$ the number r'_i is non-negative.*

Proof. Fix an $i \in I$. Let $j_0 \in J_0$ such that $r_{j_0} \geq r_j$ for every $j \in J_0$ with $P_j \in X_i$. We show that

$$r_{j_0} = \sum_{X_{J(t)} \supset X_i} a_t.$$

Since $r_i \geq r_{j_0}$, the statement follows then directly from the definition of r_i'.

From Lemma 8.3 (iv) we know that

$$r_{j_0} = \sum_{J(t) \ni j_0} a_t.$$

We have to show that $a_t = 0$ for all $1 \leq t \leq M$ with $j_0 \notin J(t)$ such that there exists a $j \neq j_0$ with $P_j \in X_i$ and $j \in J(t)$. Since $J(t) \cup \{j_0\}$ defines a connected curve, there exists a $t' < t$ with $J(t') = J(t) \cup \{j_0\}$. Thus we have $j(t') = j(t)$ since $r_j \leq r_{j_0}$. From the above corollary it follows that $a_t = 0$ □

Proof of the theorem. Let $F \subset E$ be a subsheaf. Denote by r_j the k-dimension of the stalk F_{P_j} for every $j \in J_0$. Define an effective Weil divisor on X_i by

$$D = \sum_{P_j \in X_i} P_j.$$

There is a natural exact sequence

$$(*) \qquad 0 \to F_i(-D) \to F_i \to F_{i,D} \to 0,$$

where $F_{i,D}$ is a sheaf supported at D with the stalk at P_j isomorphic to F_{P_j}. Hence we have

$$\chi(F_i) = \chi(F_i(-D)) + \sum_{P_j \in X_i} r_j$$

since all P_j are split. Assume that $r_i = \operatorname{rk} F_i > 0$. The sheaf $E_i(D)$ is H-semistable since the H-semistability is preserved by tensoring with a line bundle. Then since $F_i(D) \subset E_i(D)$ and $r_i = \operatorname{rk} F_i(-D) = \operatorname{rk} F_i$ we have

$$\frac{\chi(F_i(-D))}{r_i} \leq \frac{\chi(E_i(-D))}{r}.$$

For each $i \in I$ let α_i be the number of the points P_j on X_i. It follows that

$$\chi(F_i) \leq \frac{r_i}{r}\chi(E_i(-D)) + \sum_{P_j \in X_i} r_j = \frac{r_i}{r}\left(\chi(E_i) - \alpha_i r\right) + \sum_{P_j \in X_i} r_j,$$

where the equality follows from the exact sequence $(*)$ with F_i replaced by E_i. In the case $r_i = 0$ the above inequality is trivial.

Using (5) we conclude

$$
\begin{aligned}
\frac{\chi(F)}{\sum_{i\in I}\frac{h_i}{h}r_i} &\leq \left(\sum_{i\in I}\chi(F_i) - \sum_{j\in J_0} r_j\right) \Big/ \left(\sum_{i\in I}\frac{h_i}{h}r_i\right) \\
&\leq \left[\frac{1}{r}\sum_{i\in I} r_i\left(\chi(E_i)-\alpha_i r\right) + \underbrace{\sum_{i\in I}\sum_{P_j\in X_i} r_j - \sum_{j\in J_0} r_j}_{(**)}\right] \Big/ \left(\sum_{i\in I}\frac{h_i}{h}r_i\right).
\end{aligned}
$$

Now using Lemma 8.3 (4) we have

$$
\begin{aligned}
(**) &= \sum_{i\in I}\sum_{P_j\in X_i}\sum_{J(t)\ni j} a_t - \sum_{j\in J_0}\sum_{J(t)\ni j} a_t \\
&= \sum_{i\in I}\sum_{t=1}^{M} a_t\cdot \#\left\{j\in J_0 \mid P_j\in X_i, j\in J(t)\right\} - \sum_{t=1}^{M} a_t\cdot \#J(t) \\
&= \sum_{t=1}^{M} a_t\left(\sum_{X_i\subset X_{J(t)}} \#\left\{j\in J_0 \mid P_j\in X_i, j\in J(t)\right\} - \#J(t)\right) \\
&= \sum_{t=1}^{M} a_t\cdot \#J(t).
\end{aligned}
$$

The last equality follows from the fact that X is semistable, in particular every intersection point P_j lies on exactly two irreducible components of X. Hence, from this calculation and the definition (9) of r_i' we get an upper estimate for the numerator

$$
\begin{aligned}
&\frac{1}{r}\sum_{i\in I}\left(r_i' + \sum_{X_{J(t)}\supset X_i}\right)\left(\chi(E_i)-\alpha_i r\right) + \sum_{t=1}^{M} a_t\cdot \#J(t) \\
&= \frac{1}{r}\left[\sum_{i\in I} r_i'\left(\chi(E_i)-\alpha_i r\right) + \sum_{t=1}^{M} a_t\left(\sum_{X_i\subset X_{J(t)}}\left(\chi(E_i)-\alpha_i r\right) + \#J(t) r\right)\right] \\
&= \frac{1}{r}\left[\sum_{i\in I} r_i'\left(\chi(E_i)-\alpha_i r\right) + \sum_{t=1}^{M} a_t\left(\sum_{X_i\subset X_{J(t)}} \chi(E_i) - k_t r\right)\right].
\end{aligned}
$$

The last identity follows from

$$\begin{aligned}\sum_{X_i \subset X_{J(t)}} \alpha_i - \#J(t) &= \sum_{X_i \subset X_{J(t)}} \#\left\{j \in J_0 \mid P_j \in X_i, j \in J(t)\right\} \\ &\quad + \sum_{X_i \subset X_{J(t)}} \#\left\{j \in J_0 \mid P_j \in X_i, j \notin J(t)\right\} - \#J(t) \\ &= 2 \cdot \#J(t) + \sum_{X_i \subset X_{J(t)}} \#\left\{j \in J_0 \mid P_j \in X_i, j \notin J(t)\right\} - \#J(t) \\ &= \#J(t) + \sum_{X_i \subset X_{J(t)}} \#\left\{j \in J_0 \mid P_j \in X_i, j \notin J(t)\right\} = k_t.\end{aligned}$$

Putting all together and using the inequalities (7) (or more precise (7) and (8)) we obtain

$$\begin{aligned}&\frac{\chi(F)}{\sum_{i\in I} \frac{h_i}{h} r_i} \\ &\leq \frac{\frac{1}{r}\left[\sum_{i\in I} r_i' \left(\chi(E_i) - \alpha_i r\right) + \sum_{t=1}^{M} a_t \left(\sum_{X_i \subset X_{J(t)}} \chi(E_i) - k_t r\right)\right]}{\sum_{i\in I} \frac{h_i}{h} r_i' + \sum_{t=1}^{M} a_t \left(\sum_{X_i \subset X_{J(t)}} \frac{h_i}{h}\right)} \\ &\leq \frac{\chi(E)}{r} \cdot \frac{\sum_{i\in I} \frac{h_i}{h} r_i' + \sum_{t=1}^{M} a_t \left(\sum_{X_i \subset X_{J(t)}} \frac{h_i}{h}\right)}{\sum_{i\in I} \frac{h_i}{h} r_i' + \sum_{t=1}^{M} a_t \left(\sum_{X_i \subset X_{J(t)}} \frac{h_i}{h}\right)} = \frac{\chi(E)}{r}.\end{aligned}$$

Note that we have used the positivity of the r_i''s (cf. Lemma 8.6) to obtain an *upper* bound.

If all inequalities in (7) are strict, then the second inequality in the above estimate is strict unless all r_i' are zero and all a_t except a_1 are zero. This implies that all r_i are equal. Assume that there exists an i_0 such that E_{i_0} is stable. If $F_{i_0} \neq E_{i_0}$, then the inequality is strict. If $F_{i_0} = E_{i_0}$, then the rank r_i of F_{i_0} is r and hence F_i is of rank r for all i. It follows that $F = E$. □

9. Combinatorics

Let X be a reduced (not necessarily irreducible) curve over a field k. The *dual graph* Γ of X is the (undirected) graph consisting of irreducible components X_i of X as vertices and $\#(X_i \cap X_j)$ many edges from X_i to X_j.

Consider a graph Γ. For an edge e of Γ we denote by e_1 and e_2 the vertices of e. For two subsets of vertices $I, I' \subset \Gamma$ we define the intersection number of I and I' as

$$\alpha_{I,I'} = \#\left\{e \text{ edge in } \Gamma \mid e_1 \in I, e_2 \in I'\right\}.$$

Since Γ is undirected, we have trivially $\alpha_{I,I'} = \alpha_{I',I}$. Put $\alpha_I = \alpha_{I,I}$. For all disjoint $I, I' \subset \Gamma$ we have the following identity

$$\alpha_{I \cup I'} = \alpha_I + \alpha_{I'} + \alpha_{I,I'}. \tag{10}$$

A *weighting* on Γ is a $\mathbb{Z}$-valued function ω on the vertices of Γ. We will write ω_i instead of $\omega(i)$ for a vertex $i \in \Gamma$. Extend ω to the power set $\mathcal{P}(\Gamma)$ of vertex of Γ by setting

$$\omega_I = \sum_{i \in I} \omega_i - \alpha_I \qquad (I \subset \Gamma). \tag{11}$$

Then from (10) it follows that the extended weighting ω satisfies

$$\omega_{I \cup I'} = \omega_I + \omega_{I'} - \alpha_{I,I'}$$

for disjoint $I, I' \subset \Gamma$.

Consider the dual graph Γ' of X. Remove all edges starting and ending at the same vertex in Γ' and denote by Γ the resulting graph. If we assume that X is projective, the Euler characteristic of irreducible components $X_1, \dots, X_n$ of X defines a weighting χ on Γ by

$$\chi_i = \chi(\mathcal{O}_{X_i}).$$

Here i is the vertex in Γ corresponding to the irreducible component X_i. We refer to the extended weighting of the weighting induced by χ_i as to the *characteristic weighting*.

Proposition 9.1. *Consider a projective semistable curve X over a field k with irreducible components $X_1, \dots, X_n$. Assume that all intersection points of the X_i's are split. Then the characteristic weighting attached to the dual graph of X coincides with the Euler characteristic of subcurves of X.*

Proof. Let $D \subset X$ be a subcurve. Then D induces a subgraph $\Gamma_D \subset \Gamma$. We have to show that

$$\chi_{\Gamma_D} = \chi(\mathcal{O}_D).$$

If we can show the statement for X, it will a posteriori also be true for D. Thus we may assume without loss of generality that $D = X$.

Since X is reduced, we have the following exact sequence

$$0 \to \mathcal{O}_X \to \bigoplus_{i=1}^{n} \mathcal{O}_{X_i} \to T \to 0.$$

The sheaf T is a skyscraper sheaf supported on the set S consisting of the intersection points of $X_1, \dots, X_n$. Since X is semistable and all points in S are split, we have $\dim_k T_P = 1$ for all $P \in S$. Therefore

$$\chi(\mathcal{O}_X) = \sum_{i=1}^{n} \chi(\mathcal{O}_{X_i}) - \#S = \sum_{i=1}^{n} \chi_i - \alpha_\Gamma = \chi_\Gamma. \qquad \square$$

The next proposition is just a reformulation of Theorem 8.2.

Proposition 9.2. *Consider a projective semistable curve X over a field k with split intersection points of its irreducible components. Fix an ample line bundle H on X. Let Γ, χ be attached to X as above. If for every proper $I \subset \Gamma$ the condition*

$$\chi_I \leq \frac{h_I}{h}\chi_\Gamma + \alpha_{I,\Gamma\setminus I}$$

is satisfied, where $h_I = \sum_{i\in I} h_i$, $h_i = \deg H_i$, then every vector bundle E on X with semistable restrictions of degree zero to irreducible components of X is semistable with respect to H.

Moreover, if the above inequalities are all strict, and additionally the restriction of E to at least one irreducible component of X is stable, then E is stable with respect to H.

Note that we do not require that I is connected and therefore we allow superfluous conditions we could deduce from the condition for a connected I. However, this will make it possible to argue in a less technical way.

We fix the situation in the above proposition and let H vary. Our goal is to find an ample line bundle H on X such that the above (strict) inequalities are satisfied. As we will see shortly, the exact values of the characteristic weighting χ are not important, only whether $\chi_i = 1$ or not. Therefore we will call the vertices with $\chi_i = 1$ *marked* and forget χ. We proceed with a purely combinatorial construction of weightings of vertices of Γ, from which the existence of H will easily follow.

Construction. Let Γ be a connected graph with $n \geq 2$ many vertices, without edges starting and ending at the same vertex, and consisting of two types of vertices: marked and non-marked. Further, assume that following conditions are satisfied:

(12) If $i \in \Gamma$ is marked, then i has at least two neighbors;

(13) If all vertices in Γ are marked, then $\alpha_\Gamma > n$.

We will construct rational numbers $\{\delta_i\}_{i\in\Gamma}$ satisfying

(14) $\delta_i > 0, \quad \delta_i > 1$ if i is marked,

(15) $\delta_\Gamma = \alpha_\Gamma$,

(16) $\delta_I < \alpha_I + \alpha_{I,\Gamma\setminus I} \qquad (I \subset \Gamma$ proper),

where $\delta_I = \sum_{i\in I} \delta_i$.

We consider the following non-generic situation. Let C be a subgraph of Γ as in Figure 1. The marked vertices are black, and the vertices i_0 and i_{l+1} may be marked or non-marked, which is indicated by the vertex with a dot. We call C a *chain* of length l. An edge not contained in a chain is called *generic.*

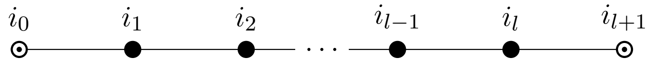

FIGURE 1. Chain of marked vertices of length l

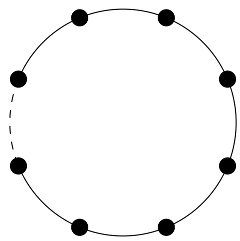

FIGURE 2. Ring of marked vertices

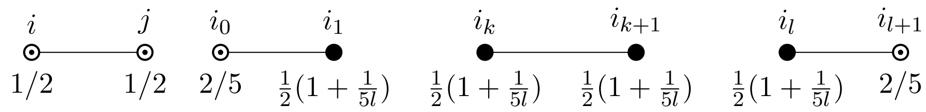

FIGURE 3. Weightings of edges ($1 \leq k \leq l$)

Further, consider a graph as in Figure 2. The graph Γ cannot contain such a subgraph. Indeed, in this case since Γ is connected, it would coincide with this subgraph. It would follow that $\chi_\Gamma = n - \alpha_\Gamma = 1$, which contradicts the assumption (13).

Now assign to each edge e and each vertex i in Γ a number $\delta(e,i)$ as in Figure 3 depending on the case, whether the edge e is generic, a margin of a chain or an inner edge of a chain. For a vertex i which does not lie on e we put $\delta(e,i) = 0$. By definition

$$\sum_{i\in\Gamma} \delta(e,i) = \delta(e,e_1) + \delta(e,e_2) = 1, \qquad \text{if } e \text{ is generic.} \tag{17}$$

For a chain C as in Figure 1 of length $l \geq 1$ we have

$$\sum_{i=0}^{l+1} \sum_{e\subset C} \delta(e,i) = \frac{2}{5} + l(1 + \frac{1}{5l}) + \frac{2}{5} = l + 1 = \alpha_C. \tag{18}$$

Note that if $l = 0$, then C is just a generic edge.

We define

$$\delta_i = \sum_e \delta(e,i) \qquad (i \in \Gamma).$$

Since $n \geq 2$ and Γ is connected, each vertex $i \in \Gamma$ has at least one neighbor, hence $\delta_i > 0$. Let i be marked and not an inner vertex in a chain. Then i has at least three neighbors by condition (12), hence $\delta_i \geq 3 \cdot 2/5 > 1$. If i is an inner point of a chain of length l, then $\delta_i = 1 + 1/5l > 1$ since $l \geq 1$. Therefore we have the condition (14).

We show (15). Let $C_1, \dots, C_m$ be the chains in Γ of lengths $l_1, \dots, l_m$, respectively. Put $C = \bigcup_{\nu=1}^m C_\nu$. We compute using (17) and (18)

$$\delta_\Gamma = \sum_{i\in\Gamma} \delta_i = \sum_{i\in\Gamma}\sum_{e} \delta(e,i) = \sum_{e \text{ generic}} \sum_{i\in\Gamma} \delta(e,i) + \sum_{\nu=1}^{m} \sum_{e\subset C_\nu} \sum_{i\in\Gamma} \delta(e,i)$$

$$= \alpha_{\Gamma\setminus C} + \alpha_{\Gamma\setminus C,C} + \sum_{\nu=1}^{m} (l_\nu + 1) = \alpha_{\Gamma\setminus C} + \alpha_{\Gamma\setminus C,C} + \alpha_C = \alpha_\Gamma.$$

The last identity follows from (10).

We show (16). Let $I \subset \Gamma$ be proper. We have four different types of edges e in Γ depending on I:

E_0**:** $e \subset I$ and is generic,
E_1**:** e is generic and has exactly one vertex lying in I,
E_2**:** e lies in a chain which is contained in I,
E_3**:** e has at least one vertex in I and lies in a chain which is not contained in I.

We write

$$\delta_I = \sum_{i\in I} \delta_i = \sum_{e}\sum_{i\in I} \delta(e,i) = \sum_{e\in E_0}\sum_{i\in I} \delta(e,i) + \sum_{e\in E_1}\sum_{i\in I} \delta(e,i) +$$
$$\sum_{e\in E_2}\sum_{i\in I} \delta(e,i) + \sum_{e\in E_3}\sum_{i\in I} \delta(e,i).$$

and compute the four summands on the right hand side. With (17) we have

$$\sum_{e\in E_0}\sum_{i\in I} \delta(e,i) = \#E_0.$$

Further, we have

$$\sum_{e\in E_1}\sum_{i\in I} \delta(e,i) \le \sum_{e\in E_1}\sum_{i\in \Gamma} \delta(e,i) = \#E_1.$$

Note that the above inequality is strict if $E_1 \neq \emptyset$.

As above we may write E_2 as a disjoint union of chains lying in I and conclude with (18) that

$$\sum_{e\in E_2}\sum_{i\in I} \delta(e,i) = \sum_{\substack{C\subset I \\ \text{chain}}}\sum_{e\subset C} \delta(e,i) = \sum_{\substack{C\subset I \\ \text{chain}}} (l(C)+1) = \#E_2,$$

where $l(C)$ is the length of the chain C. Let $C_1, \dots, C_m$ be the connected components of the subgraph of Γ defined by the union of vertices lying on the edges in E_3. Then C_ν is of the form as one of the graphs in Figure 4, i.e. in the first case it is a segment of a chain, which starts in I and ends in $\Gamma \setminus I$, therefore $i_j \in I$ for all $0 \le j < k$, in the second case it is a segment of a chain starting and ending in $\Gamma \setminus I$, therefore $i_j \in I$ for $0 < j < k$. Let k_ν be the length of C_ν, that is, the number of vertices in C_ν lying in I, and l_ν the length of the full chain in Γ containing C_ν. We have $k_\nu \le l_\nu$.

For the weighting of the first graph in Figure 4 we have

$$2/5 + k(1 + \frac{1}{5l}) \le 2/5 + k(1+\frac{1}{5k}) = k + \frac{3}{5} < k+1,$$

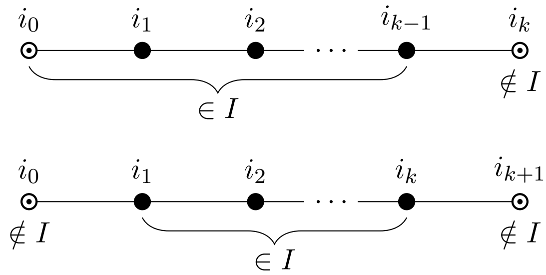

FIGURE 4. Chain segments

of the second

$$k(1+\frac{1}{5l}) \le k(1+\frac{1}{5k}) = k+\frac{1}{5} < k+1,$$

where l is the length of the full chain containing the corresponding segment. Let C be the union of the C_ν's. It follows that

$$\sum_{e\in E_3}\sum_{i\in I}\delta(e,i) = \sum_{\nu=1}^{m}\sum_{e\subset C_\nu}\sum_{i\in I}\delta(e,i) \le \sum_{\nu=1}^{m}(k_\nu+1) = \sum_{\nu=1}^{m}(\alpha_{C_\nu}+\alpha_{C_\nu,\Gamma\setminus I})$$
$$= \alpha_C + \alpha_{C,\Gamma\setminus I} - \sum_{\nu\neq\mu}\alpha_{C_\nu,C_\mu} = \alpha_C+\alpha_{C,\Gamma\setminus I}.$$

Note that the above inequality is strict if $E_3 \neq \emptyset$.

All in all, it follows that

$$\delta_I < \#E_0 + \#E_1 + \#E_2 + \alpha_C + \alpha_{C,\Gamma\setminus I}.$$

The inequality is strict since I is proper, and therefore $E_1 \neq \emptyset$ or $E_3 \neq \emptyset$. Now $\#E_0 + \#E_2 = \alpha_{I\setminus C} + \alpha_{I\setminus C,C}$ and $\#E_1 = \alpha_{I\setminus C,\Gamma\setminus I}$. It follows that

$$\delta_I < \alpha_{I\setminus C} + \alpha_{I\setminus C,C} + \alpha_{I\setminus C,\Gamma\setminus I} + \alpha_C + \alpha_{C,\Gamma\setminus I} = \alpha_I + \alpha_{I,\Gamma\setminus I}.$$

Therefore (16) follows, and the construction is finished.

Remark. The weighting 2/5 of the margin vertices in a chain is ambiguous. Every rational constant $0 < c < 1/2$ would be suitable. We have to guarantee that $\delta_i > 1$ for an inner vertex i of a chain. Further, the identity (18) should also be satisfied. Hence, if we distribute the weightings on the *inner* vertices uniformly, then

$$\delta_i = (l+1-2c)/l.$$

In this case, $\delta_i > 1$ if and only if $c < 1/2$. The inequalities for chain segments in Figure 4 used above yield both that $c > 0$.

Lemma 9.3. *Consider a connected graph Γ with $n \geq 2$ many vertices without edges starting and ending at the same vertex. Let χ be a weighting of Γ extended as in* (11) *with $\chi_i \le 1$ for all $i \in \Gamma$ and $\chi_\Gamma < 0$. Moreover, assume that the following condition is satisfied:*

(S) *If $\chi_i = 1$, then $i \in \Gamma$ has at least two neighbors.*

Then there exist positive rational numbers $\{q_i\}_{i\in\Gamma}$ with $\sum q_i = 1$ satisfying for every proper $I \subset \Gamma$

$$\chi_I < q_I\chi_\Gamma + \alpha_{I,\Gamma\setminus I},$$

where $q_I = \sum_{i\in I} q_i$.

Proof. We consider all vertices in $i \in \Gamma$ with $\chi_i = 1$ as marked. Then the assumption (S) is exactly (12). The condition (13) follows from the assumption that $\chi_i < 0$. Indeed, if all vertices are marked, we have

$$\chi_\Gamma = \sum_{i\in\Gamma} \chi_i - \alpha_\Gamma = n - \alpha_\Gamma.$$

Therefore we are in the situation of the above construction, which yields rational numbers $\{\delta_i\}_{i\in\Gamma}$ with properties (14) – (16). Put $q_i = (\chi_i - \delta_i)/\chi_\Gamma$. Then the inequality $\chi_i - \delta_i < 0$ is satisfied, which follows from the assumptions that $\chi_i \in \mathbb{Z}$ and $\chi_i \leq 1$ combined with (14). Hence $q_i > 0$, since $\chi_\Gamma < 0$. Now (15) yields

$$\sum_{i\in\Gamma} q_i = \frac{1}{\chi_\Gamma}\left(\sum_{i\in\Gamma} \chi_i - \delta_\Gamma\right) = \frac{1}{\chi_\Gamma}\left(\sum_{i\in\Gamma} \chi_i - \alpha_\Gamma\right) = 1.$$

Let $I \subset \Gamma$ be a proper subset. It follows from (16) that

$$\chi_I = \sum_{i\in I} \chi_i - \alpha_I < \sum_{i\in I} \chi_i - \delta_I + \alpha_{I,\Gamma\setminus I} = q_I\chi_\Gamma + \alpha_{I,\Gamma\setminus I}$$

as required. □

Proposition 9.4. *Consider a projective semistable curve X over a field k of genus $g \geq 2$ with irreducible components $X_1, \dots, X_n$ ($n \geq 2$) and split intersection points of the X_i's. Assume that the following condition is satisfied*

(S) *If $g(X_i) = 0$, then X_i meets other components in at least two points.*

Then there exists an ample line bundle H on X such that every vector bundle E on X with semistable restrictions E_i to X_i of degree zero is semistable with respect to H.

Moreover, if at least one E_i is stable, then E is stable with respect to H.

Proof. Let Γ be the graph attached to X as described in the paragraph before Proposition 9.1. Let χ be the characteristic weighting on Γ. By assumption Γ is connected and $\chi_\Gamma = 1 - g < 0$ by Proposition 9.1. Further, (S) implies the corresponding condition in the above lemma for Γ and χ. Hence there exist positive rationals $q_1, \dots, q_n$ with corresponding properties. Write $q_i = d_i/d$ with $d_i, d > 0$ integers.

The subset of singular points $X^{sing} \subset X$ is finite. Hence $X' = X \setminus X^{sing}$ is an open smooth and dense subscheme of X. Therefore every irreducible component X_i contains a closed point $P_i \in X'$. Define a divisor D on X by

$$D = \sum_{i=1}^{n} d_i \cdot P_i.$$

Let $H = \mathcal{O}_X(D)$ be the induced line bundle on X. Then $H_i = H|_{X_i} = \mathcal{O}_{X_i}(d_iP_i)$, and therefore $h_i = \deg H_i = d_i$ by construction of P_i, hence $h_i > 0$. It follows that H is ample (cf. [Liu02] Proposition 7.5.5). Further, we have $h = \deg H = \sum d_i = d$ since $\sum q_i = 1$. The statement follows then from Proposition 9.2. □

Corollary 9.5. *With notations from the above proposition, the S-equivalence class (with respect to H) of the trivial vector bundle on X of rank r consists of locally free sheaves of rank r.*

Proof. By the above proposition $\mathcal{O}_X$ is stable on X with respect to H. It follows that

$$0 \subset \mathcal{O}_X \subset \mathcal{O}_X^2 \subset \ldots \subset \mathcal{O}_X^r$$

is a JH-filtration of $\mathcal{O}_X^r$. Let F be a semistable sheaf on X which is S-equivalent to $\mathcal{O}_X^r$, and consider a JH-filtration of F

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_s = F.$$

Then $s = r$ and $\operatorname{gr}_i F \cong \mathcal{O}_X$ for $1 \leq i \leq r$. We proceed by induction on r. For $r = 1$ there is nothing to show. Let $r > 1$. Then

$$0 \subset F_1 \subset F_2 \subset \ldots \subset F_{r-1}$$

is a JH-filtration of F_{r-1}, hence it is S-equivalent to the trivial bundle on X of rank $r - 1$. By induction hypothesis it follows that F_{r-1} is locally free of rank $r - 1$. The JH-filtration of F induces an exact sequence

$$0 \to F_{r-1} \to F \to \mathcal{O}_X \to 0,$$

from which the statement follows. □

10. p-adic neighborhood for non-smooth models

The goal of this section is to generalize Theorem 7.2 to a bigger class of models than smooth ones. As noted in the remark after Theorem 7.2, we have to guarantee that the reductions of vector bundles to the special fiber of the considered model are semistable with respect to *some* relatively ample line bundle on this model. To obtain such a line bundle, we use the characterization of semistability on semistable curves by Teixidor i Bigas (Theorem 8.2) and results from section 9. To make this section self-contained and independent of section 7, we repeat some arguments already given there.

Recall that a model $\mathfrak{X}$ over a valuation ring R is called *semistable* if its generic $X = \mathfrak{X} \otimes_R K$ and special fiber $\mathfrak{X}_k$ are semistable (K is the quotient field and k the residue field of R). The next proposition allows us to consider the reduction of strongly semistable vector bundles over semistable models.

Proposition 10.1. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let E be a vector bundle on $X_{\mathbb{C}_p}$ with strongly semistable reduction (of degree zero). Then E has strongly semistable reduction (of degree zero) with respect to a projective semistable model of X.*

Proof. By definition there exists a model $\mathfrak{X}$ of X over $\overline{\mathbb{Z}}_p$ such that E has strongly semistable reduction (of degree zero) with respect to $\mathfrak{X}$. We show that there exists a model $\mathfrak{X}'$ dominating $\mathfrak{X}$ with stated properties. The claim follows then from Proposition 5.5.

By Noetherian descent the tuple the model $\mathfrak{X}$ descends to a model $\mathfrak{X}_K$ over the integer ring $\mathfrak{o}_K$ of a finite field extension $K/\mathbb{Q}_p$. Since X_K is integral, the model $\mathfrak{X}_K$ is also integral. By normalizing $\mathfrak{X}_K$ and applying Lipman's resolution of singularities, we may replace $\mathfrak{X}_K$ by a regular model of X_K (cf. [DW05b] **I** in the proof of Theorem 1). By [Liu06] Theorem 2.3 and Remark 4.19 the model $\mathfrak{X}_K$ is dominated by a semistable regular model $\mathfrak{X}'_L$ over the ring of integers $\mathfrak{o}_L$ of a finite field extension L/K after the base change to $\mathfrak{o}_L$. By Theorem 2.8 of Lichtenbaum [Lic68] the model $\mathfrak{X}'_L$ is projective. Let $\mathfrak{X}'$ be the base change of $\mathfrak{X}_L$ to $\overline{\mathbb{Z}}_p$. □

Corollary 10.2. *Consider a model $\mathfrak{X}$ over $\overline{\mathbb{Z}}_p$. Let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}_{\mathfrak{o}}$ with strongly semistable reduction of degree zero. Then there exist a projective semistable model $\mathcal{Y}$ and a morphism $\pi : \mathcal{Y} \to \mathfrak{X}$ inducing a finite morphism $\pi_{\overline{\mathbb{Q}}_p} : Y \to X$ of smooth projective and connected curves such that the vector bundle $\mathcal{F} = \pi_{\mathfrak{o}}^* \mathcal{E}$ has trivial reduction $\mathcal{F}_k$ on the curve $\mathcal{Y}_k$.*

Proof. The existence of $\mathcal{Y}$ follows from [DW05b] Theorem 17 . Now use the above proposition to replace $\mathcal{Y}$ by a projective semistable model. □

Definition 10.3. Let $\mathfrak{X}$ be a projective semistable model over a valuation ring R. We call $\mathfrak{X}$ *almost stable* if its special fiber satisfies the condition (S) in Proposition 9.4.

The next construction and corollary allow us to consider semistable models having a special fiber with only smooth irreducible components.

Construction. Let R be a discrete valuation ring with quotient field K and residue field k. Consider a semistable model $\mathcal{Z}$ over R. Let $C \subset \mathcal{Z}_k$ be an irreducible component of the special fiber of $\mathcal{Z}$ which is geometrically irreducible, $x \in C$ a singular point with only C passing through x. After a finite field extension of K we may assume that $x \in \mathcal{Z}$ is a split ordinary double point (cf. [Liu02] Corollary 10.3.22). The latter result tells us also that

$$\hat{\mathcal{O}}_{\mathcal{Z},x} \cong R[[u,v]]/(uv-c)$$

for some $c \neq 0$ in R of valuation $e_x = \nu_R(c) \geq 1$.

Assume first that $e_x > 1$. Then there exists a model $f : \mathcal{Z}' \to \mathcal{Z}$ dominating $\mathcal{Z}$, which is a sequence of blowing-ups, such that $f^{-1}(x)$ is made of a chain of $(e_x - 1)$-many $\mathbb{P}^1_k$ meeting transversally at rational points and such that f is an isomorphism outside of $f^{-1}(x)$ (cf. [Liu02] Lemma 10.3.21). Moreover, the unique irreducible component of $\mathcal{Z}'_k$ dominating C is smooth.

Assume now that $e_x = 1$. By adjoining a square root of c to K we obtain a finite field extension L of K. Let ν_L be the unique normalized valuation of L extending the valuation of K. Then $\nu_L(c) = 2$. It follows that after the base change to the integral closure R' of R in L, we have $e_x = 2$ for x considered as a point in the model $\mathcal{Z}_{R'}$. Hence, we may proceed as in the first case.

All in all, we have constructed a model $\mathcal{Z}'$ over the integral closure R' of R in some finite field extension of K and a morphism $f : \mathcal{Z}' \to \mathcal{Z}_{R'}$ of models with following properties:

(1) $f^{-1}(x)$ consists of a chain of $\mathbb{P}^1_k$ meeting transversally,
(2) f is an isomorphism outside of $f^{-1}(x)$,
(3) the unique irreducible component of $\mathcal{Z}'_{k'}$ dominating $C_{k'}$ is smooth, where k' is the residue field of R'.

Applying the above construction to each irreducible component of $\mathcal{Z}$ yields the following

Proposition 10.4. *Consider a semistable model $\mathfrak{X}$ over $\overline{\mathbb{Z}}_p$. Then there exists a model $\mathfrak{X}'$ dominating $\mathfrak{X}$ such that all irreducible components of $\mathfrak{X}'_k$ are smooth. If $\mathfrak{X}$ is almost stable, then $\mathfrak{X}'$ is also almost stable.*

Proof. First, by Noetherian descent $\mathfrak{X}$ descends to a model $\mathfrak{X}_K$ over the integer ring $\mathfrak{o}_K$of a finite field extension $K/\mathbb{Q}_p$. Without loss of generality we may assume that all singular points of $\mathfrak{X}_\kappa$ are split (after the base change to a finite extension of K) and its irreducible components are geometrically irreducible, where κ is the residue field of $\mathfrak{o}_K$. Let S be the set consisting of those singular points of $\mathfrak{X}_\kappa$ each of which lies on a unique irreducible component of $\mathfrak{X}_\kappa$. The set S is finite. Inductively we apply the above construction to each point of S and obtain in the end a model $\mathfrak{X}'_L$ over the integer ring $\mathfrak{o}_L$ of some finite field extension L/K dominating $\mathfrak{X} \otimes \mathfrak{o}_L$ such that the special fiber of $\mathfrak{X}'_L$ has only smooth irreducible components. Since the preimage of S in $\mathfrak{X}'_L$ consists of chains of $\mathbb{P}^1_k$'s, and other irreducible components of the special fiber of $\mathfrak{X}'_L$ are the same as of $\mathfrak{X}_\kappa$ (after the base change to the residue field of $\mathfrak{o}_L$), the model $\mathfrak{X}'_L$ is almost stable if $\mathfrak{X}_L$ is almost stable. The base change to $\overline{\mathbb{Z}}_p$ yields the desired model. □

Remark. In the above proposition we may also assume that $\mathfrak{X}'$ is regular. Indeed, the model $\mathfrak{X}'_L$ at the end of the proof may be replaced by its minimal desingularization (cf. [Liu02] Corollary 10.3.25). In fact, the minimal desingularization also remains almost stable if $\mathfrak{X}'_L$ is almost stable, since its fiber over each singular point in the special fiber of $\mathfrak{X}'_L$ consists of a chain of $\mathbb{P}^1_k$'s.

Fix a positive integer r and let X be as above. We denote by M_X the (coarse) moduli space of vector bundles on X of rank r and degree zero. Recall that $M_X(\overline{\mathbb{Q}}_p)$ carries a natural topology induced by the p-adic topology on $\overline{\mathbb{Q}}_p$. Our goal is to prove the following

Theorem 10.5. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ of genus $g \geq 2$. Let E be a vector bundle on X of rank r which has strongly semistable reduction of degree zero, i.e. E lies in $\mathfrak{B}^s_X$. Then there exists a p-adic neighborhood $U \subset M_X(\overline{\mathbb{Q}}_p)$ of the S-equivalence class of E consisting of S-equivalence classes of vector bundles with potentially strongly semistable reduction, i.e. lying in $\mathfrak{B}^0_X$.*

The idea of the proof is the following.

Step 1. Using Corollary 10.2 we obtain a finite cover Y of X such that the pullback F of E to Y has trivial reduction over some semistable model of Y. Further, this cover induces a p-adically continuous mapping

$$M_Y(\overline{\mathbb{Q}}_p) \to M_X(\overline{\mathbb{Q}}_p),$$

where M_Y is the moduli space of vector bundles on Y of rank r and degree zero. Hence it is enough to construct the p-neighborhood for F. We may assume without loss of generality that E has trivial reduction.

Step 2. We assume that E has trivial reduction $\mathcal{E}_k$ over an *almost stable* model $\mathfrak{X}$ of X. Therefore the dual graph Γ attached to the semistable curve $\mathfrak{X}_k$ behaves well, and we are able to produce rational weighting of Γ by combinatorial Lemma 9.3. The next proposition and lemma show the existence of a relatively ample line bundle on $\mathfrak{X}$, which is built from the information given by the weightings, such that the vector bundle E is semistable with respect to that bundle. Finally, we descend to a finite field extension K of $\mathbb{Q}_p$ and consider the moduli space $M = M_{\mathfrak{X}_K}$ of semistable sheaves of rank r and degree zero with respect to the constructed ample line bundle. The trivial bundle $\mathcal{E}_k$ induces then a point in $M(k)$.

Step 3. This is similar to the proof of Theorem 7.2. The moduli space comes with two p-adically continuous maps

$$M(k) \leftarrow M(\overline{\mathbb{Z}}_p) \to M(\overline{\mathbb{Q}}_p).$$

The underlying topological space of the space on the left hand side is a discrete one. Hence taking the preimage of $[\mathcal{E}_k]$ with respect to the first map and noticing that the second map is a homeomorphism yields a neighborhood of $[E]$. To show that it satisfies the stated property we use again Theorem of Langton 3.1.

Step 4. To apply Step 3 we have to show that every vector bundle on X with trivial reduction has trivial reduction over some almost stable model of X. This is one of the statements of Theorem 10.10.

The next proposition is a relative version of Proposition 9.4.

Proposition 10.6. *Let R be a valuation ring with algebraically closed residue field k. Consider a projective almost stable model $\mathfrak{X}$ over R with generic fiber* *of genus $g \geq 2$. Let $C_1, \ldots, C_n$ be the irreducible components of the special fiber $\mathfrak{X}_k$ of $\mathfrak{X}$. Assume that $n \geq 2$. Then there exists a relatively ample line bundle $\mathcal{H}$ on $\mathfrak{X}$ such that every vector bundle E on $\mathfrak{X}_k$ with semistable restrictions E_i to C_i of degree zero is semistable with respect to $\mathcal{H}_k$. Moreover, if additionally at least one of the restrictions E_i is stable, then E is stable with respect to $\mathcal{H}_k$.*

Proof. Let Γ be the dual graph of $\mathfrak{X}_k$ without the edges starting and ending at the same vertex. Let χ be the characteristic weighting on Γ defined in section 9. Since $\mathfrak{X}$ is flat over R, the genus of $\mathfrak{X}_k$ is $g \geq 2$, hence $\chi_\Gamma = \chi(\mathcal{O}_{\mathfrak{X}_k}) < 0$. By assumption Γ is connected, satisfies the condition (S) in Lemma 9.3 and has $n \geq 2$ many vertices. Since k is algebraically closed, all intersection points of the C_i's are split. Thus we may apply Lemma 9.3, which yields positive rational numbers $q_1, \ldots, q_n$ with $\sum q_i = 1$ satisfying for every proper $I \subset \Gamma$

$$(*) \qquad \chi_I < q_I \chi_\Gamma + \alpha_{I, \Gamma \setminus I}.$$

Write $q_i = d_i/d$ with $d_i, d > 0$ integers.

The model $\mathfrak{X}$ is integral since X is. Applying the lemma below we obtain a line bundle $\mathcal{H}$ on $\mathfrak{X}$ with

$$h_i = \deg \mathcal{H}|_{C_i} = d_i, \qquad h = \deg H = \deg \mathcal{H}_k = d,$$

where $H = \mathcal{H}|_X$, $\mathcal{H}_k = \mathcal{H}|_{\mathfrak{X}_k}$. The curves X, $\mathfrak{X}_k$ are without embedded points as reduced curves, thus every line bundle on X and $\mathfrak{X}_k$ is of the form $\mathcal{O}_X(D)$ resp. $\mathcal{O}_{\mathfrak{X}_k}(D)$ for some Cartier divisor D on X resp. $\mathfrak{X}_k$. We deduce from [Liu02] Corollary 7.5.5 that $\mathcal{H}$ is ample as h_i, h are positive. Since $\mathfrak{X}$ is a scheme over an affine base, ampleness of $\mathcal{H}$ is equivalent to relative ampleness of $\mathcal{H}$.

The claim follows from $(*)$ and Proposition 9.2. □

Lemma 10.7. *Let R be a valuation ring with quotient field K and algebraically closed residue field k. Consider a semistable model $\mathcal{Z}$ over R with generic fiber Z and special fiber $\mathcal{Z}_k$. Let $C_1, \ldots, C_n$ be the irreducible components of the special fiber $\mathcal{Z}_k$. Fix some integers $d_1, \ldots, d_n$ and put $d = \sum d_i$. Then there exists a line bundle $\mathcal{L}$ on $\mathcal{Z}$ with*

$$\deg \mathcal{L}_i = d_i, \qquad \deg \mathcal{L}_k = \deg L = d,$$

where $\mathcal{L}_k = \mathcal{L}|_{Z_k}$, $L = \mathcal{L}|_Z$ and $\mathcal{L}_i = \mathcal{L}_k|_{C_i}$.

Proof. Since $\mathcal{Z}$ is semistable, the subset $S = \mathcal{Z}^{sing}$ of singular points of $\mathcal{Z}_k$ consists of finitely many closed points. Therefore the scheme $\mathcal{Z}' = \mathcal{Z} \setminus S$ is an open smooth subscheme of $\mathcal{Z}$ such that $\mathcal{Z}'_k \subset \mathcal{Z}_k$ is dense. It follows that every C_i contains a closed point $P_i \in \mathcal{Z}'_k$. Since k is algebraically closed, we may consider P_i as a k-valued point of $\mathcal{Z}'_k$. By [BLR90] 2.3, Proposition 5

$$\mathcal{Z}'(R) \to \mathcal{Z}'(k) = \mathcal{Z}'_k(k)$$

is surjective, therefore there exists a $\Delta_i \in \mathcal{Z}'(R)$ with $\Delta_{i,k} = P_i$. Identify Δ_i with its image in $\mathcal{Z}'$. In this way we may consider Δ_i as a Weil divisor on $\mathcal{Z}'$. Since $\mathcal{Z}'$ is regular, we have $Div(\mathcal{Z}') \cong Pic(\mathcal{Z}')$. This gives a line bundle $\mathcal{O}(\Delta_i)$ on $\mathcal{Z}'$ which is trivial over $\mathcal{Z}' \setminus \Delta_i$. Thus we can glue $\mathcal{O}(\Delta_i)$ with $\mathcal{O}_{\mathcal{Z}\setminus\Delta_i}$ and obtain a line bundle $\mathcal{L}(i)$ on $\mathcal{Z}$ with

$$\deg \mathcal{L}(i)_k|_{C_j} = \delta_{ij}.$$

Put $\mathcal{L} = \mathcal{L}(1)^{d_1} \otimes \ldots \otimes \mathcal{L}(n)^{d_n}$. We have $\deg \mathcal{L}_k = \sum d_i \deg \mathcal{L}(i) = \sum d_i = d$ and $\deg \mathcal{L}_i = d_i$. The Euler characteristic is constant on the fibers of $\mathcal{L}$ since $\mathcal{L}$ is locally free, therefore by Riemann-Roch

$$\deg L = \chi(L) - \chi(\mathcal{O}_Z) = \chi(\mathcal{L}_k) - \chi(\mathcal{O}_{\mathcal{Z}_k}) = \deg \mathcal{L}_k = d. \qquad \square$$

Lemma 10.8. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ of genus $g \geq 2$. Fix a positive integer r. Assume that X admits an almost stable model over $\overline{\mathbb{Z}}_p$. Then there exists a model*

$$\lambda : \mathfrak{X}_K \to \operatorname{Spec} \mathfrak{o}_K,$$

where K is a finite field extension of $\mathbb{Q}_p$, together with a λ-ample line bundle $\mathcal{H}_K$ on $\mathfrak{X}_K$ with following properties

1. *$\mathfrak{X} = \mathfrak{X}_K \otimes \overline{\mathbb{Z}}_p$ is a model of X,*
2. *$\mathfrak{X}_K$ is almost stable and all irreducible components of the special fiber $\mathfrak{X}_k$ are smooth,*
3. *the moduli space $M = M_{\mathfrak{X}_K/\mathfrak{o}_K}$ of semistable sheaves on $\mathfrak{X}_K$ of rank r and degree zero with respect to $\mathcal{H}$ exists, and*
4. *the trivial line bundle is stable with respect to $\mathcal{H}_k$.*

Proof. Let $\mathfrak{X}$ be an almost stable model of X over $\overline{\mathbb{Z}}_p$. Using Noetherian descent we obtain a model

$$\lambda : \mathfrak{X}_K \to \operatorname{Spec} \mathfrak{o}_K$$

satisfying (1), where K is a finite field extension of $\mathbb{Q}_p$. By Proposition 10.4 we may assume that all irreducible components of $\mathfrak{X}_K$ are smooth and λ is projective. Hence, $\mathfrak{X}_K$ satisfies (2). Since X is connected, $\mathfrak{X}_K$ has geometrically connected fibers. Further, $\mathfrak{o}_K$ is universally Japanese (cf. [EGAIV$_2$] Corollaire 7.7.4). Fix a λ-ample line bundle $\mathcal{H}_K$ on $\mathfrak{X}_K$. By Theorem 2.6 the moduli space $M = M_{\mathfrak{X}_K/Int_K}$ as in (3) exists.

It remains to show that $\mathcal{H}_K$ can be chosen in such a way that every vector bundle on $\mathfrak{X}_k$ as in (4) is semistable with respect to $\mathcal{H}_k = \mathcal{H}_K \otimes k$. If the special fiber of $\mathfrak{X}_K$ is geometrically irreducible, then it is smooth by assumption on $\mathfrak{X}_K$ and (4) follows trivially. Assume that $\mathfrak{X}_k$ is not irreducible. By Proposition 10.6 applied to $\overline{\lambda} = \lambda \otimes \overline{\mathbb{Z}}_p$ there exists a $\overline{\lambda}$-ample line bundle $\mathcal{H}$ on $\mathfrak{X}$ such that the trivial line bundle is stable with respect to $\mathcal{H}_k$. Hence we have (4). $\square$

Proof of the theorem. By assumption there exist a model $\mathfrak{X}$ of X and a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ with generic fiber isomorphic to E. Corollary 10.2 provides us with a model $\mathcal{Y}$ and a morphism $\pi : \mathcal{Y} \to \mathfrak{X}$ such that $\pi_{\overline{\mathbb{Q}}_p}$ is a finite morphism of smooth projective and connected curves, and $\mathcal{F} = \pi^*\mathcal{E}$ has trivial reduction on $\mathcal{Y}_k$. The pullback of a semistable vector bundle on X via $\pi_{\overline{\mathbb{Q}}_p}$ is semistable since we are in the characteristic zero case, hence it induces a natural transformation

$$\mathcal{M}_X \to \mathcal{M}_Y$$

of moduli functors of semistable vector bundles on X resp. Y of rank r and degree zero. By the universal property of the coarse moduli space we obtain a morphism

$$M_X \to M_Y$$

of projective schemes over $\overline{\mathbb{Q}}_p$. It induces a p-adically continuous map

$$M_X(\overline{\mathbb{Q}}_p) \to M_Y(\overline{\mathbb{Q}}_p).$$

Now the vector bundle E having strongly semistable reduction is semistable on X (cf. Proposition 7.1), whence $F = \mathcal{F}_{\overline{\mathbb{Q}}_p}$ is also semistable. Therefore F induces a point $[F]$ in $M_Y(\overline{\mathbb{Q}}_p)$. If we can construct a p-adic neighborhood V of $[F]$ in $M_Y(\overline{\mathbb{Q}}_p)$ such that every vector bundle F' inducing a point in V has a trivial reduction over some model of Y, then taking the preimage of V in $M_Y(\overline{\mathbb{Q}}_p)$ via the above continuous map induces a p-adic neighborhood U in $M_X(\overline{\mathbb{Q}}_p)$ of $[E]$ with stated properties. Indeed, by construction every vector bundle E' inducing a point in U satisfies the following: $\pi^*_{\overline{\mathbb{Q}}_p} E'$ induces a point in V and therefore trivializes over some model of Y. Since every trivial vector bundle on $\mathcal{Y}_k$ is obviously strongly semistable, the vector bundle E' has potentially strongly semistable reduction of degree zero as required.

Note that we may always replace $\mathcal{Y}$ by a dominant model and $\mathcal{F}$ by its pullback to it, since its reduction remains trivial and it still has the generic fiber F. Therefore after replacing $\mathcal{Y}$ first by an almost stable model from Theorem 10.10 and then by a model constructed in the above lemma (using the property (1)), we may assume that $\mathcal{Y}$ satisfies the properties (1) – (4). Consider the moduli space

$$M = M_{\mathcal{Y}_K}$$

provided by (3). By (4) the trivial bundle $\mathcal{F}_k$ defines a point in $\mathcal{M}(k)$. As a locally free sheaf, $\mathcal{F}$ is flat over $\overline{\mathbb{Z}}_p$, hence it is a family of semistable sheaves on $\mathcal{Y}$ and therefore defines a point $[\mathcal{F}] \in M(\overline{\mathbb{Z}}_p)$.

We have the following natural maps induced by the homomorphisms $\overline{\mathbb{Z}}_p \to k$ and $\overline{\mathbb{Z}}_p \hookrightarrow \overline{\mathbb{Q}}_p$

$$M(k) \leftarrow M(\overline{\mathbb{Z}}_p) \to M(\overline{\mathbb{Q}}_p),$$

which are continuous with respect to the p-adic topology on $M(\overline{\mathbb{Z}}_p)$ and $M(\overline{\mathbb{Q}}_p)$, and the discrete topology on $M(k)$. The map on the right hand side is a homeomorphism since the moduli space M is projective, in particular proper. Since $\overline{\mathbb{Q}}_p$ is flat over $\mathfrak{o}_K$ and the moduli space M uniformly corepresents the corresponding moduli functor, $M \otimes_{\mathfrak{o}_K} \overline{\mathbb{Q}}_p$ is canonically isomorphic to M_Y by the universal property of the coarse moduli space.

Let V be the preimage of $M_Y(\overline{\mathbb{Q}}_p)$ of the S-equivalence class $[\mathcal{F}_k] \in M(k)$ of $\mathcal{F}_k$ after identifying $M(\overline{\mathbb{Z}}_p) = M_Y(\overline{\mathbb{Q}}_p)$. Then V is p-adically open in $M_X(\overline{\mathbb{Q}}_p)$ and by construction $[F] \in U$.

The proof is completed by showing that V consists of S-equivalence classes of vector bundles on Y having strongly semistable reduction of degree zero. Let F' be a vector bundle on Y with $[F'] \in V$. By Noetherian descent there exists a finite field extension L/K such that F' descends to a vector bundle F'_L on $Y_L = \mathcal{Y}_K \otimes_{\mathfrak{o}_K} L$ which is semistable. By Theorem 3.1 of Langton there exists a family $\mathcal{F}'_L$ on $\mathcal{Y}_L = \mathcal{Y}_K \otimes_{\mathfrak{o}_K} \mathfrak{o}_L$ of semistable sheaves with generic fiber isomorphic to F'_L, where $\mathfrak{o}_L$ is the integer ring of L. Put $\mathcal{F}' = \mathcal{F}_L \otimes_{\mathfrak{o}_L} \overline{\mathbb{Z}}_p$. By construction $[\mathcal{F}'_k] = [\mathcal{F}_k]$. Since M is the coarse moduli space, it induces a bijection between $M(k)$ and S-equivalence classes of semistable sheaves on $\mathfrak{X}_k$ of rank r and degree zero. Therefore $\mathcal{F}'_k$ is S-equivalent to $\mathcal{F}_k$. Now $\mathcal{F}_k$ is trivial, and by Corollary 9.5 its S-equivalence class consists of locally free sheaves. Hence, $\mathcal{F}'_k$ is locally free. By replacing L with a finite field extension we may assume that the restriction of $\mathcal{F}'_L$ to the special fiber of $\mathcal{Y}_L$ is locally free. Therefore since the restriction of $\mathcal{F}'_L$ to Y_L is also locally free, it follows from [HL10] Lemma 2.1.7 that $\mathcal{F}'_L$, in particular $\mathcal{F}'$, is locally free. Now since the property of being strongly semistable of degree zero on $\mathcal{Y}_k$ is stable under subobjects and quotients (cf. [Nor82] Lemma 3.6 (a)) and is stable under extensions in the category of semistable vector bundles of slope zero on $\mathcal{Y}_k$ it is compatible with the S-equivalence. We conclude that $\mathcal{F}'_k$ is strongly semistable of degree zero (cf. Remark 4.10). Hence F' has strongly semistable reduction of degree zero. □

Remark. The above proof differs slightly from the proof in the smooth case. To use the same technique as in Theorem 7.2 we need to know that if a semistable sheaf on a semistable curve is S-equivalent to a strongly semistable vector bundle, then it is locally free and strongly semistable. Since a JH-filtration of an arbitrary semistable sheaf may contain non-locally free sheaves (e.g. sheaves with support consisting of some but not all irreducible components of the curve in question), it is difficult to establish this. Therefore we trivialize the reduction by passing to an appropriate cover of X. Then we have a canonical JH-filtration consisting of trivial vector bundles and the above statement follows easily by induction over the length.

Corollary 10.9. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ of genus $g \geq 2$. Let E be a vector bundle on X of rank r and degree zero, which has potentially strongly semistable reduction, i.e. E lies in $\mathfrak{B}^0_X$. Then there exists a p-adic neighborhood $U \subset M_X(\overline{\mathbb{Q}}_p)$ of the S-equivalence class of E consisting of S-equivalence classes of vector bundles having potentially strongly semistable reduction, i.e. lying in $\mathfrak{B}^0_X$.*

Proof. There exists a finite morphism $\alpha : Y \to X$ of smooth projective and connected curves over $\overline{\mathbb{Q}}_p$ and a line bundle L on Y such that $F = \alpha^* E \otimes L$ lies in $\mathfrak{B}^s_X$ (cf. Proposition 5.4). The pullback via α and tensoring with L induces a natural transformation

$$\mathcal{M}_X \xrightarrow{\alpha^*} \mathcal{M}_Y \xrightarrow{\otimes L} \mathcal{M}_Y$$

of moduli functors of semistable vector bundles on X resp. Y of rank r and degree zero. By the coarse moduli property we obtain a morphism of corresponding moduli spaces

$$M_X \xrightarrow{\alpha^*} M_Y \xrightarrow{\otimes L} M_Y,$$

which induces a p-adically continuous map

$$M_X(\overline{\mathbb{Q}}_p) \xrightarrow{\alpha^*} M_Y(\overline{\mathbb{Q}}_p) \xrightarrow{\otimes L} M_Y(\overline{\mathbb{Q}}_p).$$

Now by Theorem 10.5 we have a p-adic neighborhood U' of the S-equivalence class $[F]$ of F in $M_Y(\overline{\mathbb{Q}}_p)$ such that every vector bundle on Y defining a point in U' lies in $\mathfrak{B}^0_Y$. Let U be the preimage of U' in $M_X(\overline{\mathbb{Q}}_p)$ via the above mapping. By construction $[E] \in U$. Let E' be a vector bundle on X such that $[E'] \in U$. Then $F' = \alpha^* E' \otimes L$ defines a point in U'. Hence, F' lies in $\mathfrak{B}^0_Y$. Again, using Proposition 5.4 we obtain a finite morphism $\beta : Y' \to Y$ and a line bundle M on Y' such that $\beta^* F' \otimes M$ lies in $\mathfrak{B}^s_{Y'}$. We have

$$\beta^* F' \otimes M = \beta^* \alpha^* E' \otimes \beta^* L \otimes M = (\alpha \circ \beta)^* \mathcal{E} \otimes (\beta^* L \otimes M).$$

Hence by the same proposition E' lies in $\mathfrak{B}^0_X$ as required. □

Almost stable models. To complete the proof of Theorem 10.5 we have to show that every vector bundle (on a curve of genus $g \geq 2$) with trivial reduction has a trivial reduction over an almost stable model. We establish this result also in the case of strongly semistable reduction of degree zero.

Theorem 10.10. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$ of genus $g \geq 1$. Let E be a vector bundle on X. Assume that E has a reduction which is either (i) strongly semistable of degree zero or (ii) trivial. Then E has also a reduction with (i) resp. (ii) over an almost stable model.*

We prove the theorem in two steps. First we construct an almost stable model $\mathcal{Y}$ with a map $\mathfrak{X} \to \mathcal{Y}$, where $\mathfrak{X}$ is a model of X. Then we show that a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ with generic fiber isomorphic to E and satisfying (i) respectively (ii) descends to $\mathcal{Y}$ and still satisfies the corresponding property.

The next proposition is due to Ishimura (cf. [Ish83] Theorem 1) for smooth projective algebraic varieties. The proof below is just an adaption of the original proof to our situation.

Proposition 10.11. *Consider a regular model $\mathfrak{X}$ over a discrete valuation ring R. Let $C \subset \mathfrak{X}_\kappa$ be an irreducible component with $C \cong \mathbb{P}^1_{\kappa'}$ and $C^2 < 0$, where κ' is a finite field extension of the residue field κ of R. Let $f : \mathfrak{X} \to \mathcal{Y}$ be the contraction of C. Then each vector bundle $\mathcal{E}$ on $\mathfrak{X}$ with trivial restriction to C is isomorphic to the pullback $f^*\mathcal{F}$ of a vector bundle $\mathcal{F}$ on $\mathcal{Y}$.*

Note that the contraction of C exists by Castelnuovo's criterion.

Lemma 10.12. *In the situation of the above proposition we have for all $n \geq 0$*

$$H^1(C, \mathcal{O}_{\mathfrak{X}}(-nC)|_C) = 0.$$

Proof. By definition of the intersection number we have

$$C^2 = \left[\kappa' : \kappa\right] \cdot \deg_{\kappa'} \mathcal{O}_{\mathfrak{X}}(C)|_C.$$

Put $d = -\deg_{\kappa'} \mathcal{O}_{\mathfrak{X}}(C)|_C$. Then $\mathcal{O}_{\mathfrak{X}}(-C)|_C \cong \mathcal{O}_C(d)$ since $C \cong \mathbb{P}^1_{\kappa'}$, whence $\mathcal{O}_{\mathfrak{X}}(-nC)|_C \cong \mathcal{O}_C(nd)$. Therefore we have for all $n \geq 0$ by [Liu02] Lemma 5.3.1

$$H^1(C, \mathcal{O}_C(nd)) \cong H^0(C, \mathcal{O}_C(-nd-2))^\vee = 0$$

since $C^2 < 0$. □

Proof of the proposition. Step 1. We show that $f_*\mathcal{E}$ is a locally free sheaf on $\mathcal{Y}$ of rank $r = \operatorname{rk} \mathcal{E}$.

Let $\hat{\mathcal{Y}}$ be the formal completion of $\mathcal{Y}$ along the closed point $s = f(C)$, $\hat{\mathfrak{X}}$ the formal completion of $\mathfrak{X}$ along C. The morphism f induces a morphism

$\hat{f} : \hat{\mathfrak{X}} \to \hat{\mathcal{Y}}$. Let $\mathcal{I}$ be the ideal subsheaf of $\mathcal{O}_{\mathfrak{X}}$ defining the closed (reduced) subscheme $C \subset \mathfrak{X}$. Put $\mathcal{E}_n = \mathcal{E} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1}$ for $n \geq 0$ and $\hat{\mathcal{E}} = \varprojlim_n \mathcal{E}_n$. Since f is proper, the theorem on formal functions (cf. [EGAIII$_1$] Théorème 4.1.5) implies that the natural morphism

$$(f_*\mathcal{E})\hat{\ } \to \hat{f}_*\hat{\mathcal{E}}$$

is an isomorphism. Assume that for every $n \geq 0$ there exists an isomorphism $\varphi_n : \mathcal{E}_n \to (\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})^r$ such that the following diagram

$$\begin{array}{ccc} \mathcal{E}_n & \xrightarrow{\varphi_n} & (\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})^r \\ \downarrow & & \downarrow \\ \mathcal{E}_{n+1} & \xrightarrow[\varphi_{n+1}]{} & (\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+2})^r \end{array}$$

is commutative. Then again applying the theorem on formal functions to the trivial sheaf on $\mathfrak{X}$ of rank r we obtain an isomorphism

$$(f_*\mathcal{E})\hat{\ } \to (f_*\mathcal{O}_{\mathfrak{X}})\hat{\ }^r.$$

Since the generic fiber of $\mathcal{Y}$ is normal, $\mathcal{Y}$ is also normal. Therefore it follows from [Liu02] Corollary 4.4.3 that $f_*\mathcal{O}_{\mathfrak{X}} = \mathcal{O}_{\mathcal{Y}}$ since f is proper and birational. Further, $f_*\mathcal{E}$ is coherent. Now taking the stalks at s yields

$$(f_*\mathcal{E})_s \otimes_{\mathcal{O}_{\mathcal{Y},s}} \hat{\mathcal{O}}_{\mathcal{Y},s} \cong \hat{\mathcal{O}}_{\mathcal{Y},s}^r,$$

where $\hat{\mathcal{O}}_{\mathcal{Y},s}$ is the completion of the local ring $\mathcal{O}_{\mathcal{Y},s}$ at its maximal ideal. It follows that the $\mathcal{O}_{\mathcal{Y},s}$-module $(f_*\mathcal{E})_s$ is free of rank r. Indeed, as we have seen the completion of the module $(f_*\mathcal{E})_s$, which coincides with the scalar extension to $\hat{\mathcal{O}}_{\mathcal{Y},s}$, is free, in particular it is flat. It follows that the finitely generated module $(f_*\mathcal{E})_s$ over the local Noetherian ring $\mathcal{O}_{\mathcal{Y},s}$ is flat (cf. [Bou72] III §5.4 Proposition 4), hence it is already projective, which further implies that it is free.

Since f is an isomorphism from $\mathfrak{X} \setminus C$ to $\mathcal{Y} \setminus \{s\}$, we conclude that $f_*\mathcal{E}$ is locally free of rank r.

It remains to show the existence of the isomorphisms φ_n. We proceed by induction on n. For $n = 0$ there is nothing to show. Let $n > 0$. We have the following exact sequence

$$0 \to \mathcal{I}^{n+1}\mathcal{E}/\mathcal{I}^{n+2}\mathcal{E} \to \mathcal{E}_{n+1} \to \mathcal{E}_n \to 0.$$

Taking the cohomology yields an exact sequence

$$H^0(C, \mathcal{E}_{n+1}) \xrightarrow{\alpha} H^0(C, \mathcal{E}_n) \to H^1(C, \mathcal{I}^{n+1}\mathcal{E}/\mathcal{I}^{n+2}\mathcal{E}).$$

Now we have $\mathcal{I}^{n+1}\mathcal{E}/\mathcal{I}^{n+2}\mathcal{E} \cong \mathcal{O}_{\mathfrak{X}}((-n-1)C)\mathcal{E}|_C$. Indeed, by definition $\mathcal{I}$ is isomorphic to $\mathcal{O}_{\mathfrak{X}}(-C)$. On the other hand, we compute

$$\begin{aligned} \mathcal{I}^{n+1}\mathcal{E}/\mathcal{I}^{n+2}\mathcal{E} &\cong \mathcal{I}^{n+1}/\mathcal{I}^{n+2} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \\ &\cong (\mathcal{I}^{n+1} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_C) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{E} \\ &\cong \mathcal{I}^{n+1}\mathcal{E}|_C. \end{aligned}$$

By assumption $\mathcal{E}|_C$ is trivial, hence

$$H^1(C, \mathcal{I}^{n+1}\mathcal{E}/\mathcal{I}^{n+2}\mathcal{E}) \cong H^1(C, \mathcal{O}_{\mathfrak{X}}((-n-1)C)|_C)^r = 0$$

by the previous lemma. It follows that α is surjective.

Let $f_1, \ldots, f_r$ be a basis of $H^0(C, \mathcal{E}_n)$ over κ, $\tilde{f}_1, \ldots, \tilde{f}_r$ lifts of the f_i's via α and $\mathcal{M} \subset \mathcal{E}^{n+1}$ the subsheaf generated by the sections $\{\tilde{f}_i\}$. We have

$$\mathcal{M} + \mathcal{I}^{n+1}\mathcal{E}_{n+1} = \mathcal{E}_{n+1}.$$

It follows from Nakayama lemma that $\mathcal{M} = \mathcal{E}_{n+1}$. Since $\tilde{f}_1, \ldots, \tilde{f}_r$ are linearly independent, the sheaf $\mathcal{M}$ is isomorphic to $(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})^r$. This gives an isomorphism

$$\varphi_n : \mathcal{E}_{n+1} \to (\mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})^r,$$

which (after a possible composition with an automorphism of the trivial sheaf on the right hand side) is compatible with φ_{n-1}. By induction hypothesis the isomorphisms φ_l for $0 \leq l \leq n$ are compatible. This completes the proof of Step 1.

Step 2. We show that the natural morphism $f^* f_* \mathcal{E} \to \mathcal{E}$ is an isomorphism. The sheaf on the left hand side is locally free of rank r by Step 1. Therefore it is enough to show that this morphism is surjective. This follows from Lemma 2 in [Ish83], which also holds for Noetherian schemes. □

Construction. Let $\mathfrak{X}$ be a semistable model over a discrete valuation ring R with geometrically irreducible components of $\mathfrak{X}_\kappa$ and smooth generic fiber of genus $g \geq 1$, where κ is the residue field of R. We construct an almost stable model $\mathcal{Y}$ from $\mathfrak{X}$ with certain properties stated at the end of the construction.

The minimal desingularization of $f : \mathfrak{X}' \to \mathfrak{X}$ exists (cf. [Liu02] Corollary 10.3.25). Assume that there exists a rational irreducible component C of $\mathfrak{X}'_\kappa$ meeting the other components in exactly one point, that is, C contradicts the condition (S). Using [Liu02] Proposition 9.1.21 we compute

$$0 = C \cdot \mathfrak{X}'_\kappa = \sum C \cdot C_i = C^2 + C \cdot D,$$

where C_i's are the irreducible components of $\mathfrak{X}'_\kappa$ and $D = \sum_{C_i \neq C} C_i$. By assumption $C \cdot D = 1$, hence $C^2 = -1$. Note that since $g \geq 1$, the divisor D is non-zero. By Castelnuovo's criterion there exists a contraction $f : \mathfrak{X}' \to \mathcal{Y}$ of C. In other words, f is a morphism of regular models, $\mathcal{Y}$ is semistable, $f(C)$ is a regular point in $\mathcal{Y}$ and $f : \mathfrak{X}' \setminus C \to \mathcal{Y} \setminus f(C)$ is an isomorphism.

Now, if $\mathcal{Y}$ has again a rational irreducible component which contradicts the condition (S), we proceed as in the last paragraph. Eventually, we obtain a morphism

$$f : \mathfrak{X}' \to \mathcal{Y},$$

which is a sequence of contractions of irreducible components of $\mathfrak{X}'_\kappa$ such that $\mathcal{Y}$ is almost stable. More exactly, let $\mathcal{S}$ be the set of chains of irreducible components of $\mathfrak{X}'_\kappa$ of genus zero, which are attached to other irreducible components of $\mathfrak{X}'_\kappa$ at exactly one point. Then $S = f(\mathcal{S})$ is a set of finitely many closed regular points in $\mathcal{Y}$, and f is an isomorphism outside of $\mathcal{S}$:

$$\begin{array}{ccccccc}
\bigcup \mathcal{S} & \subset & \mathfrak{X}'_\kappa & \longrightarrow & \mathfrak{X}' & \xrightarrow{desing.} & \mathfrak{X} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
S & \subset & \mathcal{Y}_\kappa & \longrightarrow & \mathcal{Y}. & &
\end{array}$$

Lemma 10.13. *With notation as in the above construction let $\mathcal{E}$ be a vector bundle on $\mathfrak{X}$ and $\mathcal{E}'$ the pullback of $\mathcal{E}$ to $\mathfrak{X}'$. Then there exists a vector bundle $\mathcal{F}$ on $\mathcal{Y}$ such that*

$$f^*\mathcal{F} \cong \mathcal{E}'.$$

In particular, $\mathcal{F}_K \cong \mathcal{E}_K$. Moreover,

(i) if $\mathcal{E}$ has strongly semistable reduction of degree zero, then also $\mathcal{F}$,

(ii) if $\mathcal{E}$ has trivial reduction, then also $\mathcal{F}$.

Proof. First, if $\mathcal{E}$ satisfies (i) (resp. (ii)), then $\mathcal{E}'$ also satisfies (i) (resp. (ii)), which follows from Proposition 5.5 for (i) and is trivial for (ii). Further, the morphism f is a sequence of successive contractions of the irreducible components C in $\mathcal{S}$ satisfying $g(C) = 0$ and $C^2 = -1$. In particular, $C \cong \mathbb{P}^1_{\kappa'}$ for some finite field extension κ' of κ. In both cases (i) and (ii) the restriction of $\mathcal{E}$ to a such component C is trivial (note that there are no non-trivial semistable vector bundles on C of degree zero). Thus we may inductively apply Proposition 10.11 beginning with $\mathcal{E}'$. It follows that there exists a vector bundle $\mathcal{F}$ on $\mathcal{Y}$ with $f^*\mathcal{F} \cong \mathcal{E}'$.

Assume that $\mathcal{E}'$ satisfies (i). Let D be an irreducible component of $\mathcal{Y}$. By construction of $\mathcal{Y}$ there is an irreducible component D' of $\mathfrak{X}'$ which is isomorphic to D via f. In particular we may identify $\mathcal{F}|_D$ with $f^*_{D'}\mathcal{F} \cong \mathcal{E}'|_{D'}$. It follows that $\mathcal{F}$ has strongly semistable reduction of degree zero.

Assume that $\mathcal{E}'$ satisfies (ii). The curves $\mathfrak{X}'_\kappa$ and $\mathcal{Y}_\kappa$ are both semistable, and the morphism f_κ is an isomorphism outside of $\mathcal{S}$ and $S = f(\mathcal{S})$. The latter set consists of finitely many regular points in $\mathcal{Y}_\kappa$. Therefore the composition of the inverse morphism of $f_\kappa|_\mathcal{S}$ and the inclusion $\mathfrak{X}'_\kappa \setminus \mathcal{S} \subset \mathfrak{X}'_\kappa$ can be extended to the whole curve $\mathcal{Y}_\kappa$ since the latter is proper. This gives a section $g : \mathcal{Y}_\kappa \to \mathfrak{X}'_\kappa$ of f_κ. From $g^*\mathcal{E}_\kappa \cong g^* f^*_\kappa \mathcal{F}_\kappa \cong \mathcal{F}_\kappa$ it follows that $\mathcal{F}_\kappa$ is trivial. □

Proof of the theorem. Let $\mathfrak{X}$ be a model of X over $\overline{\mathbb{Z}}_p$ and $\mathcal{E}$ a vector bundle on $\mathfrak{X}$ with $\mathcal{E}_{\overline{\mathbb{Q}}_p} \cong E$ satisfying (i) resp. (ii). By Proposition 10.1 we may assume that $\mathfrak{X}$ is semistable. Using Noetherian descent the tuple $(\mathfrak{X}, \mathcal{E})$ descends to a tuple $(\mathfrak{X}_K, \mathcal{E}_K)$, where $\mathfrak{X}_K$ is a model over the integer ring $\mathfrak{o}_K$ of a finite field extension $K/\mathbb{Q}_p$, and $\mathcal{E}_K$ is a vector bundle on $\mathfrak{X}_K$. After replacing K by a finite extension we assume that all irreducible components of $\mathfrak{X}_\kappa$ are geometrically irreducible, where κ is the residue field of $\mathfrak{o}_K$.

Let $\mathcal{Y}_K$ be the almost stable model constructed from $\mathfrak{X}_K$ as in the previous construction. By the above lemma there exists a vector bundle $\mathcal{F}$ on $\mathcal{Y}_K$ with $\mathcal{F}_K \otimes_{\mathfrak{o}_K} K \cong \mathcal{E}_K \otimes_{\mathfrak{o}_K} K$. Put $\mathcal{F} = \mathcal{F}_K \otimes_{\mathfrak{o}_K} \overline{\mathbb{Z}}_p$. Then $\mathcal{F}_{\overline{\mathbb{Q}}_p} \cong E$, and $\mathcal{F}$ satisfies (i) resp. (ii). □

11. Jet spaces

In this section we will describe the jet spaces of the moduli space M_X of semistable vector bundles of fixed rank and degree on a smooth projective and connected curve X over an algebraically closed field k in terms of jet filtrations introduced below. We begin with the description of the jet spaces of its corresponding moduli functor.

Fix a natural number $n \geq 1$. Let A_n be the local Artinian ring $k[t]/t^{n+1}$, $I = (t)$ the maximal ideal in A_n and $\eta : A_n \to k$ the canonical projection defined by $t \mapsto 0$.

Definition 11.1. For a contravariant functor $\mathcal{F} : (Sch/k) \to (Set)$ we define the n-th *jet space* $J^n_x\mathcal{F}$ of $\mathcal{F}$ at $x \in \mathcal{F}(k)$ as the preimage of x via the mapping $F(\eta)$, i.e.

$$J^n_x\mathcal{F} = \mathcal{F}(\eta)^{-1}(x) \subset \mathcal{F}(A_n).$$

Note that A_1 is the algebra of dual numbers, and the first jet is the tangent space T_xF to F at x (cf. [Sch68] Notation 2.6).

Consider a scheme X over a field k. Let E be a coherent sheaf on X. A *deformation* of E over a local Artinian k-algebra A is a pair (F, φ) where F is a coherent sheaf on $X \otimes_k A$, flat over A, and $\varphi : F \otimes_A k \to E$ is an $\mathcal{O}_X$-isomorphism. Here F is tensored with k over the canonical projection $A \to k$. Let (Art/k) be the category of local Artinian k-algebras. Then the *deformation functor* of E is the (covariant) functor

$$\mathcal{D}_E : (Art/k) \to (Set),$$

which assigns to each local Artinian k-algebra A the set of equivalence classes of deformations of E over A. Two deformations (F, φ) and (F', φ') are *equivalent* if there exists an isomorphism $\alpha : F' \to F$ compatible with φ and φ', that is, $\varphi = (\alpha \otimes id_k)\varphi'$.

A *jet filtration* is a filtration of E

$$0 \subset E = F_0 \subset F_1 \subset \ldots \subset F_n = F,$$

where F_i are coherent sheaves on X, with additional datum $\theta \in End_X(F)$, which fits for every $0 \leq i \leq n$ into the exact sequence

$$0 \to F_0 \to F_i \xrightarrow{\theta} F_i \to \operatorname{gr}_i F \to 0 \tag{19}$$

In particular, $\theta|_{F_0} = 0$. We say that two such filtrations (F, θ) and (F', θ) are equivalent if there exists an $\mathcal{O}_X$-isomorphism $\alpha : F' \to F$ which commutes with θ, that is, $\theta\alpha = \alpha\theta'$. Denote by $Filt^n(E)$ the set of equivalence classes of jet filtrations of length n.

Proposition 11.2. *Consider a scheme X over a field k. Let E be a coherent sheaf of X. Then we have a natural bijection*

$$\mathcal{D}_E(A_n) \simeq Filt^n(E).$$

Proof. Let (F, φ) be a deformation of E over A_n. Define $F_i = F \otimes_{A_n} I^{n-i}$ for $0 \leq i \leq n$. It follows from flatness of F over A_n that $F_i \subset F_{i+1}$. Again using flatness the canonical exact sequence

$$0 \to k \xrightarrow{t} A_n \xrightarrow{\pi_n} A_{n-1} \to 0$$

yields an exact sequence

$$0 \to F \otimes_{A_n} k \to F \xrightarrow{t} F \xrightarrow{\pi_n} F \otimes_{A_n} A_{n-1} \to 0.$$

Hence $F \otimes k \cong \ker \pi_n = F \otimes_{A_n} I^n = F_0$, and we identify via φ the sheaves $F_0 = E$. The canonical exact sequence

$$0 \to I \to A_n \to k \to 0$$

is split, hence induces a splitting $\mathcal{O}_X \to \mathcal{O}_{X_n}$, where $X_n = X \otimes_k A_n$. Thus we may consider the sheaves F_i as $\mathcal{O}_X$-modules and obtain a filtration

$$0 \subset E = F_0 \subset F_1 \subset \ldots \subset F_n = F.$$

Define θ as multiplication by t on F. From the exact sequence

$$0 \to I^n \to I^{n-i} \xrightarrow{t} I^{n-i} \to I^{n-i}/I^{n-i+1} \to 0$$

it follows that θ fits into (19) for every $0 \leq i \leq n$. If we take a different deformation (F', φ'), which is equivalent to (F, φ), the corresponding filtrations are obviously equivalent. We have constructed a well-defined map $\Phi : \mathcal{D}_E(A_n) \to Filt^n(E)$. Next, we construct its inverse map.

Consider a jet filtration

$$0 \subset E = F_0 \subset F_1 \subset \ldots \subset F_n = F, \qquad \theta \in End_X(F).$$

To give F an $\mathcal{O}_{X_n}$-module structure we define a multiplication with t on F using θ:

$$t.f = \theta(f), \qquad f \text{ germ of } F.$$

If f is germ of F_i, then it follows from (19) that

$$t.f = 0 \mod F_{i-1},$$

and therefore $t.f$ is a germ of F_{i-1}. Inductively, we deduce that $t^{n+1}.f = 0$, which means that the $\mathcal{O}_{X_n}$-module structure on F is well-defined.

We show that F is flat over A_n. By the lemma below it is enough to show the injectivity of the natural mapping $I \otimes_{A_n} F \to F$. Let f be a germ of F with $t.f = 0$, which means that f is already a germ of F_0. If f' is a germ of F_{i-1}, then using (19) we find a germ g' of F_i with $t.g' = f'$. Thus there exists a germ g of F with $t^n.g = f$. It follows $t \otimes f = t \otimes t^n.g = t^{n+1} \otimes g = 0$.

Tensoring the exact sequence

$$0 \to I^n \to A_n \xrightarrow{t} A_n \to A_n/I \to 0$$

with F over A_n induces by construction of the multiplication with t on F the exact sequence

$$0 \to F \otimes_{A_n} I^n \to F \xrightarrow{t} F \to F \otimes A_n/I \to 0.$$

It follows with (19) that $E = F_0 \cong F \otimes_{A_n} I^n$ and $F_{n-1} \cong F \otimes_{A_n} I$. Hence, the canonical exact sequence

$$0 \to I \to A_n \to k \to 0$$

tensored with F yields the required isomorphism $\varphi : F \otimes_{A_n} k \to E$.

Consider another jet filtration

$$0 \subset E = F'_0 \subset F'_1 \subset \ldots \subset F'_n = F', \qquad \theta' \in End_X(F'),$$

which is equivalent to (F, θ). Let $\alpha : F' \to F$ be an $\mathcal{O}_X$-isomorphism, which defines the equivalence between the filtrations considered. Then it follows that α is already an isomorphism regarding $\mathcal{O}_{X_n}$-module structure on F and F' as constructed above since α commutes with θ and θ'. We claim that $\alpha(F'_0) = F_0$. Indeed, we have $\theta\alpha(F'_0) = \alpha\theta'(F'_0) = 0$, which follows from (19) for F' and $i = n$. Hence, again from (19) for F we have $\alpha(F'_0) \subset \ker\theta = F_0$. The claim follows then from the same arguments for α^{-1}. Consider the following commutative diagram

$$\begin{array}{ccccc} E = F'_0 & \longrightarrow & F' \otimes_{A_n} I^n & \longrightarrow & F' \otimes_{A_n} k \\ \downarrow & & \downarrow{\scriptstyle \alpha\otimes id_{I^n}} & & \downarrow{\scriptstyle \alpha\otimes id_k} \\ E = F_0 & \longrightarrow & F \otimes_{A_n} I^n & \longrightarrow & F \otimes_{A_n} k. \end{array}$$

The compositions of horizontal arrows are by construction φ and φ', respectively. The left vertical arrow is the identity as we have just seen. Thus the constructed deformations are equivalent and we obtain a well-defined mapping $\Psi : Filt^n(E) \to \mathcal{D}_E(A_n)$.

The constructed mappings Φ and Ψ are inverse to each other. □

Lemma 11.3. *Let $A' \to A$ be a surjective homomorphism of Noetherian rings with nilpotent kernel I. Then an A'-module M' is flat if and only if*

(1) $M = M' \otimes_{A'} A$ is flat over A,
(2) the natural mapping $M' \otimes_{A'} I \to M'$ is injective.

Proof. The proof is a slight generalization of [Har10] Proposition 2.2. □

Remark. (i) Using the construction in the proof one sees that for $0 \le i \le n$

$$\mathrm{gr}_i F \cong E,$$

where F is a jet filtration of E. Indeed, as we have seen $F = F_n$ becomes an $\mathcal{O}_{X_n}$-module, flat over A_n. Hence by tensoring the exact sequence

$$0 \to I^{n-i+1} \to I^{n-i} \to k \to 0$$

with F we obtain the exact sequence

$$0 \to F_{i-1} \to F_i \to E \to 0$$

for each $1 \le i \le n$.

(ii) Note that if E is locally free, then all sheaves appearing in a jet filtration of E are also locally free since they all arise as successive extensions of locally free sheaves as explained in (i).

Now we express the n-th jet of the moduli functor $\mathcal{M}$ of semistable vector bundles of fixed rank and degree on a smooth projective and connected curve X over an algebraically closed field k using jet filtrations. For a vector bundle E on X we always assume that E has the appropriate rank and degree to define an S-equivalence class in $\mathcal{M}(k)$, and we denote it by $[E]$.

Proposition 11.4. *Consider a smooth projective and connected curve X over an algebraically closed field k. Let E be a stable vector bundle on X. Then we have a natural bijection of sets*

$$J^n_{[E]}\mathcal{M} \simeq Filt^n(E).$$

for all $n \geq 1$.

Proof. Since E is stable and hence simple, i.e. $End_X(E) = k$, the deformations of E over A_n are just the coherent sheaves F on $X_n = X \otimes_k A_n$ flat over A_n with $F \otimes_{A_n} k \cong E$. Thus $\mathcal{D}_E(A_n) \subset \mathcal{M}(A_n)$, which induces a bijection

$$D_E(A_n) \simeq J^n_{[E]}\mathcal{M}$$

for all $n \geq 1$. Using the above proposition we obtain the required bijection. □

Corollary 11.5. *Consider a smooth projective and connected curve X over an algebraically closed field k. Let E be a stable vector bundle on X. Then there is a natural bijection of sets*

$$T_E\mathcal{M} \simeq \mathrm{Ext}^1_X(E, E).$$

Proof. By the above proposition it is enough to show that $Filt^1(E) = \mathrm{Ext}^1(E, E)$. Let (F, θ) be a jet filtration of length one. Then the homomorphism θ induces an exact sequence

$$0 \to E \to F \to E \to 0.$$

On the other hand, such an extension gives us a filtration $E \subset F$ and a homomorphism θ as the composition of the surjection followed by the inclusion. □

Consider a jet filtration (F, θ) of length n. Reducing mod F_0 induces a filtration

$$\overline{F}_0 \subset \overline{F}_1 \subset \ldots \subset \overline{F}_{n-1} = \overline{F}$$

consisting of coherent modules $\overline{F}_i = F_i/F_0$ on X. By the remark after Proposition 11.2 it follows that $\overline{F}_0 \cong E$. Since by (19) the kernel of θ is F_0, it induces an endomorphism $\bar{\theta}$ of $\overline{F}$ also satisfying (19). Given another jet filtration (F', θ') of length n, which is equivalent to F, the equivalence isomorphism $\alpha : F' \to F$ satisfies $\alpha(F_0') = F_0$ as we have seen in the proof of Proposition 11.2 and hence induces an isomorphism $\bar{\alpha} : \overline{F'} \to \overline{F}$, which commutes with $\bar{\theta}$ and $\bar{\theta}'$. Hence $\overline{F}$ and $\overline{F}'$ are equivalent. We obtain a reduction mapping

$$\mathrm{red} : Filt^n(E) \to Filt^{n-1}(E).$$

Proposition 11.6. *Consider a smooth projective and connected curve X over an algebraically closed field k. Let E be a stable vector bundle on X. The bijections between $J^n_{[E]}\mathcal{M}$ and $Filt^n(E)$ commute with the reduction map constructed above and the map $J^n_{[E]}\mathcal{M} \to J^{n-1}_{[E]}\mathcal{M}$ induced by canonical projection $A_n \xrightarrow{\text{mod } t} A_{n-1}$, that is, for all $n \geq 2$ we have a commutative diagram*

$$\begin{array}{ccc} J^n_{[E]}\mathcal{M} & \longrightarrow & J^{n-1}_{[E]}\mathcal{M} \\ \downarrow \wr & & \downarrow \wr \\ Filt^n(E) & \xrightarrow{\mathrm{red}} & Filt^{n-1}(E). \end{array}$$

Proof. Let $[F] \in J^n_{[E]}\mathcal{M}$. Then $\overline{F} = F \otimes_{A_n} A_{n-1}$ is a representative of the image of $[F]$ in $J^{n-1}_{[E]}\mathcal{M}$. Consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I & \longrightarrow & A_n & \xrightarrow{\text{mod } t} & A_{n-1} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} & & \\
0 & \longrightarrow & I & \longrightarrow & A_n & \xrightarrow{\text{mod } t} & A_{n-1} & \longrightarrow & 0.
\end{array}$$

By construction in Proposition 11.2 and flatness of F over A_n tensoring with F induces a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_0 & \longrightarrow & F & \xrightarrow{\text{mod } t} & \overline{F} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \theta_F} & & \downarrow{\scriptstyle \theta_{\overline{F}}} & & \\
0 & \longrightarrow & F_0 & \longrightarrow & F & \xrightarrow{\text{mod } t} & \overline{F} & \longrightarrow & 0,
\end{array}$$

from which the proposition follows. □

Jet spaces of the moduli space of vector bundles.

Definition 11.7. Let X be a scheme and $x \in X$. Let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_{X,x}$ and $\kappa(x)$ the residue field of x. We call the dual vector space of the vector space $\bigoplus_{i=1}^n m_x^i/m_x^{i+1}$ the n-th *jet space* of X at the point x and denote it by $J^n_x X$.

Note that the first jet is just the Zariski tangent space. We want to show that the n-th jet space of the moduli functor $\mathcal{M}$ at a stable vector bundle E coincides with the n-th jet space of the coarse moduli space M at the point induced by E in M in the sense of the above definition. Intuitively, it means that although M does not represent $\mathcal{M}$ in general, locally at stable points it is not too far from a fine moduli space.

Consider a scheme X over a field k. Let E be a coherent sheaf on X. Let $(R, \mathfrak{m})$ be a local complete Noetherian k-algebra. The ring R is not necessarily in (Art/k), however $R/\mathfrak{m}^n$ is a local Artinian k-algebra for every $n \geq 1$. Consider a $\xi \in \varprojlim_n \mathcal{D}_E(R/m^n) = \mathcal{D}_E(R)$. Then ξ induces a natural transformation

$$\xi : \mathcal{D}_E \to h_R,$$

where h_R is the functor given by $h_R(A) = Hom_{k\text{-alg}}(R, A)$. We say that (R, ξ) *pro-represents* $\mathcal{D}_E$ if the natural transformation ξ is an isomorphism.

Theorem 11.8. *Let E be a stable vector bundle on a smooth projective curve X over an algebraically closed field k. Then the deformation functor $\mathcal{D}_E$ attached to E is pro-represented by the completion $\widehat{\mathcal{O}}_{M,[E]}$ of $\mathcal{O}_{M,[E]}$.*

Proof. Cf. [HL10] Theorem 4.5.1. □

Proposition 11.9. *Let E be a stable sheaf on a smooth projective and connected curve X over an algebraically closed field k. Then for every integer $n \geq 1$ we have a natural bijection*

$$J^n_{[E]}\mathcal{M} \simeq J^n_{[E]}M.$$

Proof. Using the same argumentation as in the proof of Proposition 11.4, we know that $J_E^n\mathcal{M} = \mathcal{D}_E(A_n)$ for all $n \geq 1$. On the other hand, it follows from the above theorem that $\mathcal{D}_E(A_n) \cong Hom_k(\widehat{\mathcal{O}}_{M,E}, A_n)$. Using the fact that $Hom_k(\widehat{\mathcal{O}}_{M,E}, A_n) = Hom_k(\mathcal{O}_{M,E}, A_n)$ the proposition follows from the lemma below. □

Lemma 11.10. *Let $(B, \mathfrak{m})$ be a local k-algebra with $B/\mathfrak{m} = k$. Then for every integer $n \geq 1$ we have a canonical isomorphism*

$$Hom_{k\text{-}alg}(B, A_n) \cong Hom_k(\bigoplus_{i=1}^{n} \mathfrak{m}^i/\mathfrak{m}^{i+1}, k).$$

Proof. We show the statement by induction on n.

Let $n = 1$ and $\varphi : B \to A_n$ be a k-algebra homomorphism. Then φ induces a homomorphism $\mathfrak{m} \to kt \cong k$ of k-vector spaces, which factors through $\mathfrak{m}/\mathfrak{m}^2$. Conversely, a k-vector space homomorphism $\psi : \mathfrak{m}/\mathfrak{m}^2 \to k$ induces a homomorphism $\mathfrak{m} \to kt$. By assumption $B/k = \mathfrak{m}$ we can lift this map to a k-algebra homomorphism $B \to A_n$. These two constructions are inverse to each other.

Let $n > 1$. Consider a k-algebra homomorphism $\varphi : B \to A_n$ and the canonical projection $\pi : A_n \to A_{n-1}$. Then by induction hypothesis $\pi\varphi$ induces a homomorphism $\sum \varphi^{(i)}$ in $Hom_k(\bigoplus_{i=1}^{n-1} \mathfrak{m}^i/\mathfrak{m}^{i+1}, k)$. On the other hand, the restriction of φ to $\mathfrak{m}^n$ induces a homomorphism $\varphi^{(n)} : \mathfrak{m}^n/\mathfrak{m}^{n+1} \to kt^n \simeq k$. Hence, we obtain a $\sum \varphi^{(i)} \in Hom_k(\bigoplus_{i=1}^{n} \mathfrak{m}^i/\mathfrak{m}^{i+1}, k)$.

Conversely, consider a $\psi \in Hom_k(\bigoplus_{i=1}^{n} \mathfrak{m}^i/\mathfrak{m}^{i+1}, k)$. Then by induction hypothesis ψ induces a k-algebra homomorphism $\psi' : B \to A_{n-1}$ and a k-vector space homomorphism $\varphi^{(n)} : \mathfrak{m}^n/\mathfrak{m}^{n+1} \to k$. We have an exact sequence

$$0 \to \mathfrak{m}^n/\mathfrak{m}^{n+1} \to \mathfrak{m}/\mathfrak{m}^{n+1} \to \mathfrak{m}/\mathfrak{m}^n \to 0,$$

which gives us a decomposition $\mathfrak{m}/\mathfrak{m}^{n+1} \simeq \mathfrak{m}^n/\mathfrak{m}^{n+1} \oplus \mathfrak{m}/\mathfrak{m}^{n+1}$ as k-vector spaces. Thus we may consider $\psi' + \varphi^{(n)}$ as a k-algebra homomorphism $\mathfrak{m}/\mathfrak{m}^{n+1} \to A_n$. This defines an element in $Hom_{k\text{-alg}}(B, A_n)$. One checks that both constructions are inverse to each other, and the proposition follows. □

Remarks on literature

Section 5: The whole content of this section is a summary of the definitions in [DW10] §1. For an example of a semistable vector bundle which is not strongly semistable cf. [Gie73] Theorem 1. The theory of Noetherian descent is introduced in [EGAIV$_3$] §8. We list the results implicitly used in the summary: 8.10.5, 11.2.6, 8.5.2, 8.5.5. The result referred to for the proof of Proposition 5.2 is exactly the statement of the proposition, however only for strongly semistable vector bundles of degree zero; the proof remains the same for vector bundles not necessarily of degree zero.

Section 6: For the definition of the strong topology, in addition to the exposé [Con], also cf. [Mum99] I. §10.

Section 7: For Zariski's connectedness principle or more exactly its implication we used [Ill05] Corollary 2.14.

Section 8: For the notion of a semistable curve cf. [Liu02] Definition 10.3.1; for the definition of a split ordinary double point cf. *loc. cit.* Definition 10.3.8.

Section 9: The dual graph of a curve is defined in [Liu02] Section 10.1.4.

Section 10: The Proposition 10.1 is based basically on the proof of Theorem 1 in [DW05b]. The statement for vector bundles with strongly semistable reduction may be concluded also indirectly using Theorem 17 (characterization of the latter property) and Lemma 7 in *loc. cit.* The construction after Definition 10.3 is based on Remark 1 of Coleman in [Lud13]. The construction of the line bundle of given degree in Lemma 10.7 is contained in the proof of [DW10] Theorem 3.

Section 11: For the notions from the deformation theory see [HL10] Appendix 2.A. For a more conceptual approach to jets cf. [Voj07].

Part 3

p-adic representations for vector bundles

12. Construction revisited

In this section we briefly recall the construction of the parallel transport introduced in [DW05b]. Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let $\Pi_1(X)$ be the *fundamental groupoid* of X. The p-adic parallel transport attaches to a vector bundle E in $\mathfrak{B}^s_X$ a continuous representation

$$\rho_E : \Pi_1(X) \to Vec_{\mathbb{C}_p},$$

that is, a continuous functor ρ_E into the category of finite dimensional vector spaces over $\mathbb{C}_p$. Such a representation is called the (p-adic) *parallel transport* for E. The parallel transport is functorial in E, it defines a $\mathbb{C}_p$-linear functor

$$\mathfrak{B}^s_X \to Rep_{\mathbb{C}_p}\Pi_1(X)$$

into the category of continuous representations of $\Pi_1(X)$, that is, the category of continuous functors $\Pi_1(X) \to Vec_{\mathbb{C}_p}$.

For a divisor D on X we write $X \setminus D$ for $X \setminus Supp\ D$.

Theorem 12.1. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let $\mathfrak{X}$ be a model of X and $\mathcal{E}$ a vector bundle on $\mathfrak{X}_{\mathfrak{o}}$ with strongly semistable reduction of degree zero. Then there exists a proper $\mathfrak{o}$-morphism $\pi : \mathcal{Y} \to \mathfrak{X}$ such that the following properties are satisfied:*

(i) $\mathcal{Y}$ is a model of a smooth projective and connected curve Y over $\overline{\mathbb{Q}}_p$,

(ii) $\pi_{\overline{\mathbb{Q}}_p} : Y \to X$ is finite and

$$\pi_{\overline{\mathbb{Q}}_p} : Y \setminus (\pi^{-1}D) \to X \setminus D$$

is étale for a divisor D on X,

(iii) $\lambda_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathrm{Spec}\,\mathfrak{o}}$ is satisfied universally, where $\lambda : \mathcal{Y} \to \mathrm{Spec}\,\mathfrak{o}$ is the structural morphism,

(iv) $\pi_k^*\mathcal{E}_k$ is a trivial vector bundle.

As before we denote by $\mathfrak{o}_n$ the ring $\mathfrak{o}/p^n\mathfrak{o}$ for every $n \geq 1$.

Variant 12.2. *With notations as in the above theorem there exists a morphism π for every $n \geq 1$ satisfying (i), (ii), (iii) and*

(iv') $\pi_n^*\mathcal{E}_n$ is a trivial vector bundle.

Here $\pi_n = \pi \otimes id_{\mathfrak{o}_n}$ and $\mathcal{E}_n = \mathcal{E} \otimes \mathfrak{o}_n$.

Let $\mathfrak{B}_{\mathfrak{X},D}$ be the full subcategory of the category of vector bundles on $\mathfrak{X}_{\mathfrak{o}}$ such that for every object $\mathcal{E}$ in $\mathfrak{B}_{\mathfrak{X},D}$ and every $n \geq 1$ there exists a finitely presented proper $\mathfrak{o}$-morphism $\pi : \mathcal{Y} \to \mathfrak{X}$ of models with (ii) and (iv').

Proof. From [DW05b] Theorem 17 it follows that there exists a divisor D on X such that $\mathcal{E}$ belongs to $\mathfrak{B}_{\mathfrak{X},D}$. Using *loc. cit.* Theorem 1 we may replace $\mathcal{Y}$ by a model additionally satisfying (i) and (iii). □

Remark. Theorem 1 in [DW05b] also states that $\mathcal{Y}$ can be chosen to be a semistable curve over $\mathfrak{o}$.

Fix a model $\mathfrak{X}$ of X. Let $\mathfrak{B}^s_{\mathfrak{X}}$ be the full subcategory of the category of vector bundles on $\mathfrak{X}_{\mathfrak{o}}$ with strongly semistable reduction of degree zero. The functor

$$\mathfrak{B}^s_{\mathfrak{X}} \to \mathfrak{B}^s_X, \qquad \mathcal{E} \mapsto \mathcal{E} \otimes_{\mathfrak{o}} \mathbb{C}_p$$

is essentially surjective by the definition of $\mathfrak{B}^s_X$. The next lemma follows from the above theorem.

Lemma 12.3. *We have*

$$\mathfrak{B}^s_{\mathfrak{X}} = \bigcup_D \mathfrak{B}_{\mathfrak{X},D},$$

where D runs over divisors of X.

Let $\mathfrak{X}$ be a model of X over $\overline{\mathbb{Z}}_p$ and $\mathcal{E}$ a vector bundle on $\mathfrak{X}_{\mathfrak{o}}$ with strongly semistable reduction of degree zero. Fix a divisor D on X and put $U = X \setminus D$. By properness we have $X(\mathbb{C}_p) \cong \mathfrak{X}_{\mathfrak{o}}(\mathfrak{o})$. Consider a geometric point $x \in X(\mathbb{C}_p)$ as a section $x_{\mathfrak{o}} : \operatorname{Spec} \mathfrak{o} \to \mathfrak{X}_{\mathfrak{o}}$. Write $\mathcal{E}_{x_{\mathfrak{o}}} = x^*_{\mathfrak{o}}\mathcal{E}$ viewed as a free $\mathfrak{o}$-module of finite rank r. The reduction mod $p^n : \mathfrak{o} \to \mathfrak{o}_n$ induces a mapping $\mathfrak{X}_{\mathfrak{o}}(\mathfrak{o}) \to \mathfrak{X}_{\mathfrak{o}}(\mathfrak{o}_n)$, which maps $x_{\mathfrak{o}}$ to the morphism

$$x_n : \operatorname{Spec} \mathfrak{o}_n \to \operatorname{Spec} \mathfrak{o} \to \mathfrak{X}_{\mathfrak{o}}.$$

Put $\mathcal{E}_{x_n} = x^*_n\mathcal{E} = \mathcal{E}_{x_{\mathfrak{o}}} \otimes_{\mathfrak{o}} \mathfrak{o}_n$ viewed as a free $\mathfrak{o}_n$-module of rank r. We have

$$\mathcal{E}_{x_{\mathfrak{o}}} = \varprojlim \mathcal{E}_{x_n}$$

as topological modules, where $\mathcal{E}_{x_n}$ is considered as a discrete module.

Construction. For $x \in U(\mathbb{C}_p) = \Pi_1(U)$ put $\rho_{\mathcal{E}} = \mathcal{E}_{x_{\mathfrak{o}}}$. For $x, x' \in U(\mathbb{C}_p)$ we have to construct a continuous mapping on étale paths

$$\rho_{\mathcal{E}} : Hom_{\Pi_1(U)}(x, x') \to Hom_{\mathfrak{o}}(\mathcal{E}_{x_{\mathfrak{o}}}, \mathcal{E}_{x'_{\mathfrak{o}}}).$$

Denote by F_x the fiber functor defined on étale neighborhoods of U by taking the fiber of x. It is enough to define a mapping

$$\rho_{\mathcal{E},n} : Iso(F_x, F_{x'}) \to Hom_{\mathfrak{o}_n}(\mathcal{E}_{x_n}, \mathcal{E}_{x'_n})$$

for each $n \geq 1$, which induces a projective system of mappings. In fact, we define then $\rho_{\mathcal{E}} = \varprojlim \rho_{\mathcal{E},n}$. The continuity of $\rho_{\mathcal{E}}$ follows from construction.

Let γ be an étale path from x to x' in U. Fix an $n \geq 1$. By the above theorem there exists a morphism $\pi : \mathcal{Y} \to \mathfrak{X}$ with properties (i) – (iii) and (iv') such that

$$\pi^*_n\mathcal{E}_n$$

is a trivial bundle on $\mathcal{Y}_n$. Set $Y = \mathcal{Y} \otimes \overline{\mathbb{Q}}_p$, $V = Y \setminus \pi^{-1}(D)$. Then by property (ii) the morphism $\pi : V \to U$ is a finite étale covering. Let $y \in V(\mathbb{C}_p)$ be a point over $x \in U(\mathbb{C}_p)$ and let $y' = \gamma y$ be the image of y under the mapping

$$\gamma_V : F_x(V) \to F_{x'}(V).$$

Hence y' lies over x'. Now property (iii) yields the identity $\lambda_*\mathcal{O}_{\mathcal{Y}_n} = \mathcal{O}_{\operatorname{Spec} \mathfrak{o}_n}$. Therefore the pullback under $y_n : \operatorname{Spec} \mathfrak{o}_n \to \mathcal{Y}_n$ induces an isomorphism

$$y^*_n : \Gamma(\mathcal{Y}_n, \pi^*_n\mathcal{E}_n) \to \Gamma(\operatorname{Spec} \mathfrak{o}_n, y^*_n\pi^*_n\mathcal{E}_n) = \mathcal{E}_{x_n}.$$

We define $\rho_{\mathcal{E},n}(\gamma)$ as

$$\rho_{\mathcal{E},n}(\gamma) = (\gamma y)^*_n \circ (y^*_n)^{-1} = (y')^*_n \circ (y^*_n)^{-1} : \mathcal{E}_{x_n} \to \mathcal{E}_{x'_n}$$

and set

$$\rho_{\mathcal{E}}(\gamma) = \varprojlim \rho_{\mathcal{E},n}(\gamma) : \mathcal{E}_{x_{\mathfrak{o}}} \to \mathcal{E}_{x'_{\mathfrak{o}}}.$$

Proposition 12.4. *The construction of $\rho_{\mathcal{E}}$ is independent of the choice of the model $\mathcal{Y}$ and the point y. It gives a well-defined continuous representation*

$$\rho_{\mathcal{E}} : \Pi_1(U) \to Mod_{\mathfrak{o}}$$

into the category of free finitely generated $\mathfrak{o}$-modules.

Proof. Cf. [DW05b] Theorem 22. □

The next step is to turn the map $\mathcal{E} \to \rho_{\mathcal{E}}$ into a functor

$$\rho : \mathfrak{B}_{\mathfrak{X},D} \to Rep_{\mathfrak{o}}\Pi_1(U)$$

to the category of continuous representations, that is, the category of continuous functors $\Pi_1(X) \to Mod_{\mathfrak{o}}$.

Let $f : \mathcal{E} \to \mathcal{E}'$ be a morphism in $\mathfrak{B}_{\mathfrak{X},D}$. Then the family of $\mathfrak{o}$-module homomorphisms

$$f_{x_{\mathfrak{o}}} = x_{\mathfrak{o}}^* f : \mathcal{E}_{x_{\mathfrak{o}}} \to \mathcal{E}'_{x_{\mathfrak{o}}}, \qquad x \in U(\mathbb{C}_p)$$

defines a natural transformation $\rho_f : \rho_{\mathcal{E}} \to \rho_{\mathcal{E}'}$. Indeed, let γ be an étale path from $x \in U(\mathbb{C}_p)$ to $x' \in U(\mathbb{C}_p)$. Fix an $n \geq 1$. From a slight generalization of Theorem 12.1 (cf. [DW05b] Corollary 3 (3)) there exists a model $\mathcal{Y} \to \mathfrak{X}$ such that both $\pi_n^* \mathcal{E}_n$ and $\pi_n^* \mathcal{E}'_n$ are trivial. Put $f_{x_n} = x_n^* f$, choose a point y over x and set $y' = \gamma y$. From the commutative diagram

$$\begin{array}{ccc}
\mathcal{E}_{x_n} & \xrightarrow{\quad f_{x_n} \quad} & \mathcal{E}'_{x_n} \\
{\scriptstyle y_n^*}\uparrow{\scriptstyle \simeq} & & {\scriptstyle y_n^*}\uparrow{\scriptstyle \simeq} \\
\Gamma(\mathcal{Y}_n, \pi_n^* \mathcal{E}_n) & \xrightarrow{\Gamma(\mathcal{Y}_n, \pi_n^* f_n)} & \Gamma(\mathcal{Y}_n, \pi_n^* \mathcal{E}'_n) \\
{\scriptstyle y'^*_n}\downarrow{\scriptstyle \simeq} & & {\scriptstyle y'^*_n}\downarrow{\scriptstyle \simeq} \\
\mathcal{E}_{x'_n} & \xrightarrow{\quad f_{x'_n} \quad} & \mathcal{E}'_{x'_n}
\end{array}$$

we see that $f_{x'_n} \circ \rho_{\mathcal{E},n}(\gamma) = \rho_{\mathcal{E}',n}(\gamma) \circ f_{x_n}$. Taking the projective limit yields $f_{x'_{\mathfrak{o}}} \circ \rho_{\mathcal{E}}(\gamma) = \rho_{\mathcal{E}'}(\gamma) \circ f_{x_{\mathfrak{o}}}$.

Now, after we have defined the functor ρ, we want to get rid of the divisor D. Theorem 17 in [DW05b] gives us slightly more information about D than stated in Theorem 12.1. Indeed, it also says that for every bundle $\mathcal{E}$ in $\mathfrak{B}^s_{\mathfrak{X}}$ there are trivializing covers $\mathcal{Y}$ and $\mathcal{Y}'$ with respect to divisors D and D', which have *disjoint* support.

Proposition 12.5. *Let U, $U' \subset X$ open, $i : U \cap U' \to U$, $i' : U \cap U' \to U'$ and $j : U \to U \cup U'$, $j' : U' \to U \cup U'$ canonical immersions. This data induces the following commutative diagram*

$$\begin{array}{ccc}
\Pi_1(U \cap U') & \xrightarrow{\quad i_* \quad} & \Pi_1(U) \\
\downarrow{\scriptstyle i'_*} & & \downarrow{\scriptstyle j_*} \\
\Pi_1(U') & \xrightarrow{\quad j_* \quad} & \Pi_1(U \cup U').
\end{array}$$

Let $\mathcal{C}$ be a Hausdorff topological category, $\rho : \Pi_1(U) \to \mathcal{C}$ and $\rho' : \Pi_1(U') \to \mathcal{C}$ two continuous functors satisfying $\rho i_* = \rho' i'_*$. Then there exists a unique continuous functor $\tau : \Pi_1(U \cup U') \to \mathcal{C}$ such that $\tau j_* = \rho$ and $\tau j'_* = \rho'$.

Proof. Cf. [DW05b] Proposition 34. □

From this proposition it is clear that the parallel transport $\rho_{\mathcal{E},1}$ for $\mathcal{E}$, constructed with respect to D, and the parallel transport $\rho_{\mathcal{E},2}$ for $\mathcal{E}$, constructed with respect to D', give a unique continuous representation

$$\rho_{\mathcal{E}} : \Pi_1(X) \to Mod_{\mathfrak{o}}$$

since $D \cap D' = \emptyset$. Using Lemma 12.3 we conclude the following

Theorem 12.6. *There is a well defined functor*

$$\rho : \mathfrak{B}^s_{\mathfrak{X}} \to Rep_{\mathfrak{o}}\Pi_1(X),$$

which extends the constructed functors $\rho : \mathfrak{B}^s_{\mathfrak{X},D} \to Rep_{\mathfrak{o}}\Pi_1(U)$*, where* D *runs over divisors of* $X = \mathfrak{X} \otimes_{\mathfrak{o}} \mathbb{C}_p$ *and* $U = X \setminus D$*.*

Now we are able to construct the functor

$$\mathfrak{B}^s_X \to Rep_{\mathbb{C}_p}\Pi_1(X).$$

For an object E in $\mathfrak{B}^s_X$ we obtain a continuous functor $\rho_E : \Pi_1(X) \to Vec_{\mathbb{C}_p}$ by setting $\rho_E(x) = E_x = x^*E$ for $x \in X(\mathbb{C}_p)$. For $x, x' \in X(\mathbb{C}_p)$ the continuous map

$$\rho_E : Hom_{\Pi_1(X)}(x, x') \to Hom_{\mathbb{C}_p}(E_x, E_{x'})$$

is given by

$$\rho_E(\gamma) = \psi_{x'}^{-1} \circ (\rho_{\mathcal{E}}(\gamma) \otimes_{\mathfrak{o}} \mathbb{C}_p) \circ \psi_x.$$

Here $\mathfrak{X}$ is a model of X, and $\mathcal{E}$ is a vector bundle in $\mathfrak{B}^s_{\mathfrak{X}}$ with an isomorphism $\psi : E \to \mathcal{E} \otimes_{\mathfrak{o}} \mathbb{C}_p$ of vector bundles on $X_{\mathbb{C}_p}$. Moreover ψ_x is the fiber map

$$\psi_x = x^*\psi : E_x \to (\mathcal{E} \otimes_{\mathfrak{o}} \mathbb{C}_p)_x = \mathcal{E}_{x_{\mathfrak{o}}} \otimes_{\mathfrak{o}} \mathbb{C}_p.$$

For a morphism $f : E \to E'$ of vector bundles in $\mathfrak{B}^s_X$ the map $\rho(f)$ is given by the family of linear maps $f_x = x^*(f) : E_x \to E_{x'}$ for each $x \in X(\mathbb{C}_p)$.

From [DW05b] Proposition 27 it follows that this functor does not depend on the choice of the model $\mathfrak{X}$ of X and the vector bundle $\mathcal{E}$. This concludes the construction of the parallel transport for vector bundles in $\mathfrak{B}^s_X$. Although we have only focused on the construction of this functor, we state the properties of ρ proved in *loc.cit.*

Theorem 12.7. *The constructed functor*

$$\mathfrak{B}^s_X \to Rep_{\mathbb{C}_p}\Pi_1(X)$$

is an exact additive functor, which commutes with tensor products and internal homs. It behaves functorially with respect to morphisms of smooth projective and connected curves over $\overline{\mathbb{Q}}_p$ *and automorphisms of* $\overline{\mathbb{Q}}_p$ *over* $\mathbb{Q}_p$*.*

Proof. Cf. [DW05b] Theorem 36. □

Eventually the parallel transport (i.e. the above functor) extends to the category $\mathfrak{B}^0_X$. This is one of the main results in [DW10]. Since we won't use the explicit construction, we only state this result.

Theorem 12.8. *There exists a functor*

$$\mathfrak{B}_X^0 \to Rep_{\mathbb{C}_p}\Pi_1(X), \qquad E \mapsto \rho_E,$$

such that $\rho_E(x) = E_x$ *for every* $x \in X$*, extending the functor in the above theorem. It commutes with tensor products and internal homs, and behaves functorially with respect to morphisms of smooth projective and connected curves over* $\overline{\mathbb{Q}}_p$ *and automorphisms of* $\overline{\mathbb{Q}}_p$ *over* $\mathbb{Q}_p$*.*

Proof. Cf. [DW10] Theorem 10. □

13. Character space of continuous representations

Consider a topological group G. Let K be complete non-archimedean valued field of characteristic zero. A finite dimensional K-vector space V is always equipped with the topology induced by the topology on K. In the following, by a representation $G \to GL(V)$ we mean a continuous homomorphism from G to the K-linear automorphism group of V.

Fix an integer $r \geq 1$ and let

$$Hom_c(G, GL_r(K))$$

be the set of continuous homomorphisms $G \to GL_r(K)$. We regard this set as a subset of the space $C^0(G, K^{r^2})$ of continuous functions on G. The latter is equipped with the topology of the uniform convergence. To be more precise, the topology on $C^0(G, K^{r^2})$ is defined by the neighborhoods

$$B_\varepsilon(f) = \left\{ f' \in C^0(G, K^{r^2}) \mid \sup_{g \in G} \|f'(g) - f(g)\| < \varepsilon \right\},$$

where $\|\cdot\|$ is a fixed norm on K^{r^2} (a different choice of the norm leads to the same topological space). The space $C^0(G, K^{r^2})$ is Hausdorff since the target space is Hausdorff, and is first-countable, which directly follows from the definition of the topology. The set $Hom_c(G, GL_r(K))$ as a subspace inherits this properties.

Consider a representation $\rho : G \to GL(V)$. Let $\varphi : V \to K^r$ be an isomorphism of K-vector spaces. It induces an isomorphism $GL(V) \cong GL_r(K)$, $f \mapsto \varphi f \varphi^{-1}$ and hence by composition with ρ a continuous homomorphism $\varphi^* \rho : G \to GL(V)$. The choice of a different isomorphism $\psi : V \to K^r$ yields an $A \in GL_r(K)$ such that $\varphi = \psi A$. The homomorphism $\psi^* \rho : G \to GL_r(K)$ satisfies

$$\varphi^* \rho(g) = \varphi \rho(g) \varphi^{-1} = \psi A \rho(g) A^{-1} \psi^{-1} = \psi^*(Ad_A \rho)(g) \qquad (g \in G),$$

where $Ad_A : GL_r(K) \to GL_r(K)$, $B \mapsto ABA^{-1}$ is the conjugation with A. Thus ρ defines an equivalence class in the space

$$R_G(K) = R_{G,r}(K) = Hom_c(G, GL_r(K))/GL_r(K)$$

equipped with quotient topology. Here the action of $A \in GL_r(K)$ is given by the composition with Ad_A. On the other hand, two isomorphic representations of G induce the same equivalence class in $R_G(K)$. Therefore we have proved the following

Proposition 13.1. *The set of isomorphism classes of r-dimensional continuous representations of G is in bijection with the topological space $R_G(K)$.*

Remark 13.2. In general, the topology on $R_G(K)$ is not Hausdorff. Here is a well-known example. Let $G = \mathbb{Z}$. Let (a_n) be a sequence in K with $a_n \neq 0$ for all $n \geq 1$, which converges to zero. Define

$$A_n = \begin{pmatrix} 1 & a_n \\ 0 & 1 \end{pmatrix} \in GL_2(K).$$

Let $f_n : \mathbb{Z} \to GL_2(K)$ be the homomorphism defined by $1 \mapsto A_n$. If we consider $\mathbb{Z}$ as a discrete topological group, all f_n are trivially continuous and define the same class c in $R_{\mathbb{Z},2}(K)$, since for $n \neq m$ the matrices A_n and A_m are conjugated. But f_n converges to $f : 1 \mapsto E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence every neighborhood of the conjugation class $[f] = \{f\}$ contains c.

Recall that the category of finite dimensional continuous K-representations is abelian. Further, every object in this category is of finite length, since the condition on the finite K-dimension does not allow infinite filtrations. Therefore, using the abstract formalism from section 4, we have the notion of the Jordan-Hölder filtration and Jordan-Hölder equivalence for every such representation. Define

$$\mathcal{X}_G(K) = \mathcal{X}_{G,r}(K) = Hom_c(G, GL_r(K))/{\sim_{JH}},$$

where $\sim_{JH}$ is the JH-equivalence and equip $\mathcal{X}_G(K)$ with the quotient topology. This space is naturally a quotient space of the space $R_G(K)$ since the JH-equivalence is coarser than the equivalence relation given by taking isomorphism classes.

Let $Hom_c^{ss}(G, GL_r(K)) \subset Hom_c(G, GL_r(K))$ be the subset of semisimple representations $G \to GL_r(K)$. The JH-equivalence descends to this set and is given by taking isomorphism classes, since two semisimple representations are JH-equivalence if they are already isomorphic. Hence, we obtain a canonical inclusion

$$Hom_c^{ss}(G, GL_r(K))/GL_r(K) \subset \mathcal{X}_G(K).$$

On the other hand, every JH-equivalence class c in the set on the right hand side has a unique (up to isomorphism) semistable representation as a representative. This is just the associated grading to some (and hence any) representation in c. Therefore the inclusion is indeed a homeomorphism of topological spaces.

Proposition 13.3. *We have a canonical homeomorphism*

$$\mathcal{X}_G(K) \cong Hom_c^{ss}(G, GL_r(K))/GL_r(K)$$

induced by taking the associated grading.

A quotient of a first-countable space by a topological group, which acts continuously on it, is again a first-countable topological space. Hence, the topologies on the space $R_G(K)$ and $\mathcal{X}_G(K)$ introduced above can be characterized by limit points of convergent series. Basically, this is what we will do using the next definition, which is due to Bellaïche, Chenevier, Khare and Larsen (cf. [BCKL05] Definition 1.1).

Definition 13.4. Let $(\rho_\lambda)_{\lambda \geq 1}$ be a sequence of representations of G on K-vector spaces V^λ.

(i) We say that (ρ_λ) is *(uniformly) physically convergent* if for each λ there exists a K-basis of V_i such that the coefficients $c_{i,j}^\lambda : G \to K$ of ρ_λ with respect to this basis satisfy:

(1) $c_{i,j}^\lambda$ converges uniformly on G, and
(2) $\rho(g) = \lim_{\lambda \to \infty}(c_{i,j}^\lambda(g))_{i,j} \in GL_r(K)$ for all $g \in G$.

The isomorphism class of ρ is called a *physical limit* of (ρ_λ).

(ii) We say that (ρ_λ) is *(uniformly) trace convergent* if the sequence of functions satisfy

(1) $\operatorname{tr}\rho_\lambda : G \to K$ converges uniformly in G, and
(2) $t : G \to K$, $t(g) = \lim_{\lambda\to\infty} \operatorname{tr}\rho_\lambda(g)$ is a trace of some representation ρ of G.

The JH-equivalence class of ρ is called the *trace limit* of (ρ_λ).

There is an ambiguity in the definition of the physical convergence due to the choice of a basis. The sequence (f_n) defined in Remark 13.2 physically converges to f. But after choosing a different basis we may achieve that $f_n(1) = A_1$ for all $n \geq 1$. Thus (f_n) also physically converges to f_1. On the other hand, physical convergence implies trace convergence. But since the topology of uniform convergence on $C^0(G, K)$ is Hausdorff, the trace limit of (ρ_λ) is unique. Therefore two representations whose isomorphism classes are physical limits of (ρ_λ) have the same trace. It follows then from the theorem of Brauer-Nesbitt (cf. [Bou58] §12 1. Proposition 3) that these representations are JH-equivalent. Hence we have proved the following

Proposition 13.5. *Consider a physically convergent sequence (ρ_λ) of representations of G. Let ρ and ρ' be two representations, whose isomorphism classes are both physical limits of (ρ_λ). Then ρ and ρ' are JH-equivalent. In particular, they both define the same point in $\mathcal{X}_G(K)$.*

Proposition 13.6. *Let (ρ_λ) be a sequence of representations of G.*

(i) If (ρ_λ) converges physically to $q \in R_G(K)$, then the induced sequence (q_λ) in $R_G(K)$ converges to q. On the other hand, if a sequence in $R_G(K)$ converges to q, then there exists a sequence of representations (ρ_λ) of G such that ρ_λ represents q_λ and (ρ_λ) converges physically to q.

(ii) If a sequence (q_λ) in $\mathcal{X}_G(K)$ converges to q, then every sequence of representations representing (q_λ) converges in trace to q.

Proof. The statement (i) is just another formulation of the definition of physical convergence. For (ii) note that $tr : M_r(K) \to K$ is continuous. In fact, tr is even Lipschitz. In particular tr is uniformly continuous, which implies that

$$Hom_c(G, GL_r(K)) \xrightarrow{\operatorname{tr}=\operatorname{tr}_*} Hom_c(G, K) \subset C^0(G, K)$$

is continuous. Since JH-equivalent representations have the same trace, the above map factors through a continuous map $\mathcal{X}_G(K) \to Hom_c(G, K)$, which we also denote by tr. Thus $\operatorname{tr} q_\lambda$ converges to $\operatorname{tr} q$, and the statement follows. □

Corollary 13.7. *The topology on $\mathcal{X}_G(K)$ is Hausdorff.*

Proof. From Proposition 13.3 it follows that the space $\mathcal{X}_G(K)$ is first-countable. Hence, its topology is determined by convergent sequences, and we have to check that limits of convergent sequences are unique. Let (q_λ) be a sequence in $\mathcal{X}_G(K)$ converging to q and q'. From (ii) of the above proposition it follows that $(\operatorname{tr} q_\lambda)$ converges uniformly to $\operatorname{tr} q$ and $\operatorname{tr} q'$, hence $\operatorname{tr} q = \operatorname{tr} q'$ since $Hom_c(G, K)$ is Hausdorff. The theorem of Brauer-Nesbitt (cf. [Bou58]

§12 1. Proposition 3) implies then that semisimple representatives of q and q' are JH-equivalent, hence $q = q'$ (cf. Proposition 13.3). $\square$

14. p-ADIC REPRESENTATIONS FOR MODULI OF VECTOR BUNDLES

After introducing the spaces of representation of fixed dimension, we will use the p-adic parallel transport to define set theoretic maps first on the parameter scheme, which parametrizes a family of vector bundles, and secondly on the level of moduli space of vector bundles. From now on let $K = \mathbb{C}_p$.

Consider a smooth projective and connected curve over $\overline{\mathbb{Q}}_p$. Let $\mathfrak{X}$ be a model of X over $\overline{\mathbb{Z}}_p$. Consider a family $\mathcal{E}/\mathfrak{X}/S$ as in 6.2, that is, S is a connected scheme of finite type over $\overline{\mathbb{Z}}_p$, and $\mathcal{E}$ is a family of vector bundles on $\mathfrak{X}_\mathfrak{o}$ parametrized by S. Let E be the family $\mathcal{E} \otimes_\mathfrak{o} \mathbb{C}_p$.

Assume that S is proper. Using the identity $S(\mathbb{C}_p) \cong S(\mathfrak{o})$ we write $s_\mathfrak{o}$ for the induced point in $S(\mathfrak{o})$ by $s \in S(\mathbb{C}_p)$. By Corollary 6.4 the set

$$S^1(\mathbb{C}_p)$$

of points $s \in S(\mathbb{C}_p)$ such that $\mathcal{E}_{s_\mathfrak{o}}$ has strongly semistable reduction of degree zero, which trivially implies that E_s has strongly semistable reduction of degree zero, is p-adically open.

Let $x \in X(\mathbb{C}_p)$ and $\pi = \pi_1(X, x)$ the étale fundamental group of X with base point x. The parallel transport revisited in section 12 for the category $\mathfrak{B}^s_X$ yields an exact functor to the category of finite dimensional continuous $\mathbb{C}_p$-representations of π. Therefore for each $s \in S^1(\mathbb{C}_p)$ the vector bundle E_s induces a continuous representation $\rho_{E_s,x} : \pi \to GL(E_{s,x})$. By taking its isomorphism class we obtain a set theoretic mapping

$$\rho : S^1(\mathbb{C}_p) \to R_\pi(\mathbb{C}_p), \qquad s \mapsto [\rho_{E_s,x}], \tag{20}$$

where $[\cdot]$ on the right hand side denotes the isomorphism class of the representation in question.

Proposition 14.1. *Let $\mathcal{E}/\mathfrak{X}/S$ be as in 6.2, $E = \mathcal{E} \otimes_\mathfrak{o} \mathbb{C}_p$. Assume that S is proper. Then the mapping (20) induced by $\mathcal{E}/\mathfrak{X}/S$*

$$\rho : S^1(\mathbb{C}_p) \to R_\pi(\mathbb{C}_p)$$

is continuous with respect to the p-adic topology on $S^1(\mathbb{C}_p)$.

Note that by the construction of the parallel transport for the category $\mathfrak{B}^s_X$ we have $\rho_{E_s,x} = \rho_{\mathcal{E}_{s_\mathfrak{o}},x} \otimes \mathbb{C}_p$, i.e. the representations are obtained by scalar extension to $\mathbb{C}_p$ of finitely generated free $\mathfrak{o}$-modules. For the proof of the proposition we consider first such representations.

Lemma 14.2. *Let G be a topological group. Let $(L^\lambda)_{\lambda \geq 1}$ be a sequence of free $\mathfrak{o}$-modules of rank r. Consider a sequence of representations $\rho_\lambda : G \to Aut_\mathfrak{o}(L^\lambda)$ and a representation $\rho : G \to GL_r(\mathfrak{o})$. Assume that for each $n \geq 1$ there exists a $\lambda(n) \geq 1$ such that*

$$\rho_{\lambda,n} \cong \rho_n, \tag{$*$}$$

where $\rho_{\lambda,n}$ and ρ_n are the induced representations of G on the $\mathfrak{o}/p^n\mathfrak{o}$-modules L^λ/p^nL^λ resp. $\mathfrak{o}_n^r$. Then the sequence (ρ_λ) is physically convergent to the isomorphism class of ρ.

Proof. Let $n \geq 1$. By $(*)$ after a choice of a different basis of L^λ there exists a $\lambda(n) \geq 1$ such that for all $\lambda \geq \lambda(n)$ we have

$$\rho_{\lambda,n} = \rho_n.$$

It follows that $\rho_\lambda(g) - \rho(g) \in p^n \mathfrak{o}^r$ for all $g \in G$ and $\lambda \geq \lambda(n)$, in particular taking the ij-th component of the corresponding matrices $(1 \leq i, j \leq r)$ yields

$$\sup_{g \in G} |\rho_\lambda(g)_{ij} - \rho(g)_{ij}| \leq \frac{1}{p^n}.$$

Hence $(\rho_\lambda)_{i,j}$ converges uniformly to $\rho_{i,j}$. □

Proof of the proposition. Consider a sequence (s^λ) in $S^1(\mathbb{C}_p)$ converging p-adically to s. Write $\mathcal{E}^\lambda = \mathcal{E}_{s^\lambda_{\mathfrak{o}}}$ and $\mathcal{E}^0 = \mathcal{E}_{s_{\mathfrak{o}}}$. Using Proposition 13.6 (i) we have to show that $\rho_{E_{s^\lambda},x} = \rho_{\mathcal{E}^\lambda,x} \otimes_{\mathfrak{o}} \mathbb{C}_p$ converges physically to $\rho(s)$, which is the isomorphism class of $\rho_{E_s,x} = \rho_{\mathcal{E}^0,x} \otimes_{\mathfrak{o}} \mathbb{C}_p$. By Corollary 6.5 for all $n \geq 1$ there exists a $\lambda(n) \geq 1$ such that for all $\lambda \geq \lambda(n)$ we have

$$\mathcal{E}^\lambda_n = \mathcal{E}^0_n.$$

Hence, $\rho_{\mathcal{E}^\lambda,n} = \rho_{\mathcal{E}^0,n}$. By the above lemma it follows that $\rho_{\mathcal{E}^\lambda,x}$ is physically convergent to the isomorphism class of $\rho_{\mathcal{E}^0,x}$. □

Mapping on the moduli of vector bundles. Instead of considering families, we work with the moduli space of vector bundles parameterizing those.

Let M_X be the moduli space of semistable vector bundles of fixed rank r and degree zero (cf. section 2). By Theorem 10.5 and its Corollary 10.9 the subset

$$M^0_X(\overline{\mathbb{Q}}_p) \subset M_X(\overline{\mathbb{Q}}_p)$$

of S-equivalence classes of vector bundles on X having potentially strongly semistable reduction, i.e. lying in $\mathfrak{B}^0_X$, is p-adically open. On the other hand, the category $\mathfrak{B}^0_X$ is abelian (cf. [DW10] Corollary 5 iii) and is a full subcategory of the abelian category of semistable vector bundles of slope zero (cf. *loc. cit.*). Hence the notion of S-equivalence coincides with the abstract notion of JH-equivalence in the category $\mathfrak{B}^0_X$ (note that all objects in the latter category are of finite length). For a vector bundle E on X in $\mathfrak{B}^0_X$ we write ρ_E for the p-adic representation induced by the parallel transport for the category $\mathfrak{B}^0_X$

$$\rho_{E_{\mathbb{C}_p},x} : \pi \to GL(E_x \otimes \mathbb{C}_p),$$

where $E_{\mathbb{C}_p} = E \otimes_{\overline{\mathbb{Q}}_p} \mathbb{C}_p$. All in all, we have the following

Proposition 14.3. *The parallel transport induces a well-defined mapping*

$$\rho : M^0_X(\overline{\mathbb{Q}}_p) \to \mathcal{X}_\pi(\mathbb{C}_p), \qquad [E] \mapsto [\rho_{E,x}],$$

where $[\cdot]$ *on the right hand side denotes the JH-equivalence class of the representation in question.*

Remark. The locus $M^s_X(\overline{\mathbb{Q}}_p)$ of stable vector bundles in $M_X(\overline{\mathbb{Q}}_p)$ is open, in particular p-adically open. Hence, we may consider the p-adically open subset $M^{s,0}_X(\overline{\mathbb{Q}}_p)$ consisting of stable vector bundles on $X_{\mathbb{C}_p}$ of rank r having strongly semistable reduction. Unfortunately, we do not know whether ρ

preserves simple objects. This would imply that ρ also preserves the length and gradings, and we could consider the mapping

$$\rho : M_X^{s,0}(\overline{\mathbb{Q}}_p) \to R_\pi(\mathbb{C}_p), \qquad [E] \mapsto [\rho_{E,x}].$$

We introduce the following subset of $M_X(\overline{\mathbb{Q}}_p)$: Let E be a vector bundle on X of rank r. Assume that there exists a vector bundle $\mathcal{E}$ on a smooth model $\mathfrak{X}$ of X over $\overline{\mathbb{Z}}_p$ with generic fiber E such that the reduction $\mathcal{E}$ is strongly semistable of degree zero and stable as a vector bundle on the smooth projective curve $\mathfrak{X}_k$. In particular, E has *good* strongly semistable reduction of degree zero and induces a point in $M_X^0(\overline{\mathbb{Q}}_p)$. We denote by $M_X^1(\overline{\mathbb{Q}}_p)$ the subset of $M_X(\overline{\mathbb{Q}}_p)$ parametrized by such vector bundles. Theorem 7.2 yields for every vector bundle E as above a p-adic neighborhood in $M_X(\overline{\mathbb{Q}}_p)$ and by its construction it lies in $M_X^1(\overline{\mathbb{Q}}_p)$. Hence, the latter space is p-adically open in $M_X(\overline{\mathbb{Q}}_p)$.

For a smooth model $\mathfrak{X}$ we denote by $M = M_{\mathfrak{X}_K}$ the moduli space of semistable sheaves on $\mathfrak{X}_K$ of rank r and degree zero, where $\mathfrak{X}_K$ is a model obtained from $\mathfrak{X}$ by descent to a finite field extension K of $\mathbb{Q}_p$. In the following the choice of K is irrelevant.

Proposition 14.4. *Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. Let $M_X^1(\overline{\mathbb{Q}}_p)$ be as above. Under the assumption: for all vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a smooth model $\mathfrak{X}$ we have*

$(*)$ *If $[\mathcal{E}_n] = [\mathcal{F}_n]$ in $M(\mathfrak{o}_n)$, and $\mathcal{E}_k$ or $\mathcal{F}_k$ is stable, then $\rho_{\mathcal{E},n} \cong \rho_{\mathcal{F},n}$;*

the mapping

$$\rho : M_X^1(\overline{\mathbb{Q}}_p) \to \mathcal{X}_\pi(\mathbb{C}_p)$$

induced by the parallel transport is continuous with respect to the p-adic topology on $M_X^1(\overline{\mathbb{Q}}_p)$.

Proof. Consider a sequence (a_λ) in $M_X^1(\overline{\mathbb{Q}}_p)$ converging p-adically to a. Let E^λ and E be vector bundles on X representing a_λ and a, respectively. By definition of $M_X^1(\overline{\mathbb{Q}}_p)$ there exist a smooth model $\mathfrak{X}$ of X over $\overline{\mathbb{Z}}_p$ and a vector bundle $\mathcal{E}$ on $\mathfrak{X}$ with an isomorphism $\psi : \mathcal{E}_{\overline{\mathbb{Q}}_p} \to E$.

It follows from Proposition 7.3 that for each $n \geq 1$ there exists a $\lambda(n) \geq 1$ such that for each E^λ we have a vector bundle $\mathcal{E}^\lambda$ on $\mathfrak{X}$ with an isomorphism $\psi^\lambda : \mathcal{E}^\lambda_{\overline{\mathbb{Q}}_p} \to E^\lambda$ satisfying

$$[\mathcal{E}_n^\lambda] = [\mathcal{E}_n] \text{ in } M(\mathfrak{o}_n)$$

for all $\lambda \geq \lambda(n)$. By assumption $(*)$ this implies that representations $\rho_{\mathcal{E}^\lambda,n}$ and $\rho_{\mathcal{E},n}$ are isomorphic. By Lemma 14.2 we conclude that $\rho_{\mathcal{E}^\lambda,x}$ physically converges to the isomorphism class of $\rho_{\mathcal{E},x}$.

Now by construction of the parallel transport we have

$$\rho_{E^\lambda,x} = Ad_{\psi_x^\lambda}\rho_{\mathcal{E}^\lambda,x}, \qquad \rho_{E,x} = Ad_{\psi_x}\rho_{\mathcal{E},x}.$$

In particular, the representations $\rho_{\mathcal{E}^\lambda,x}$ (resp. $\rho_{\mathcal{E},x}$) considered as $\mathbb{C}_p$-representations (by extending scalars to $\mathbb{C}_p$) define the same point in $R_\pi(\mathbb{C}_p)$ as $\rho_{E^\lambda,x}$ (resp. $\rho_{E,x}$). By Proposition 13.6 (i) the sequence $[\rho_{\mathcal{E}^\lambda,x}]$ in $R_\pi(\mathbb{C}_p)$ converges then to $[\rho_{\mathcal{E},x}]$. Let $\rho_\lambda = \rho(a_\lambda) = [\rho_{E^\lambda}] = [\rho_{\mathcal{E}^\lambda,x} \otimes_{\mathfrak{o}} \mathbb{C}_p]$ in $\mathcal{X}_\pi(\mathbb{C}_p)$. It follows that the sequence (ρ_λ) converges to $\rho(a) = [\rho_E] = [\rho_{\mathcal{E}} \otimes_{\mathfrak{o}} \mathbb{C}_p]$. □

15. Image of ρ

We follow the idea of [DW05a] Definition 18 and define a similar notion to the rank one case of a representation of ∞-type. Every representation induced by the p-adic parallel transport on a vector bundle defined over $\overline{\mathbb{Q}}_p$ is then of a such type.

Consider a smooth projective and connected curve X over $\overline{\mathbb{Q}}_p$. By Noetherian descent there exist a finite field extension K of $\mathbb{Q}_p$ and a smooth projective and geometrically connected curve X_K over K with $X_K \otimes_K \overline{\mathbb{Q}}_p = X$. Let $x \in X(K)$ and denote by G_K the absolute Galois group of $\overline{\mathbb{Q}}_p/K$. By functoriality of the étale fundamental group $\pi_1(X, x)$ for every $\sigma \in G_K$ there exists an isomorphism

$$\pi_1(X, x) \xrightarrow{a_\sigma} \pi_1(X, x).$$

satisfying $a_e = id$ and $a_{\sigma\tau} = a_\sigma a_\tau$ for σ, $\tau \in G_K$. Note that $\sigma(x) = x$. In other words the group G_K acts on $\pi = \pi_1(X, x)$.

Let V be a finite dimensional $\mathbb{C}_p$-vector space. Recall that $Aut_K(\mathbb{C}_p) = Aut_K(\overline{\mathbb{Q}}_p)$. We define $\sigma_* V = {}^\sigma V$ as the abelian group V with twisted $\mathbb{C}_p$-vector space structure

$$\alpha \cdot v = \sigma^{-1}(\alpha) v \qquad (\alpha \in \mathbb{C}_p, v \in V). \tag{1}$$

More conceptually, ${}^\sigma V = V \otimes_{\mathbb{C}_p, \sigma} \mathbb{C}_p$ tensored over the automorphism $\sigma : \mathbb{C}_p \to \mathbb{C}_p$. We have

$${}^e V = V, \qquad {}^{\sigma\tau}(V) = {}^\tau({}^\sigma V).$$

We denote the identity $V \to {}^\sigma V$ by σ since it is σ-linear. Let $d_\sigma : GL(V) \to GL({}^\sigma V)$ be the homomorphism induced by σ, i.e.

$$d_\sigma(f) = \sigma f \sigma^{-1}, \qquad f \in GL(V).$$

We have

$$d_{\sigma\tau} = \sigma_*(d_\tau), \tag{2}$$

where $\sigma_* : GL({}^\tau V) \to GL({}^{\sigma\tau} V)$ is again the induced homomorphism by $\sigma : {}^\tau V \to {}^{\sigma\tau} V$, but now it is defined on a different vector space.

As before, a representation of π is always a continuous representation of π on a finite dimensional $\mathbb{C}_p$-vector space. With the above notations we define an "action" of G_K on representations of π.

Let $\rho : \pi \to GL(V)$ be a representation. We define ${}^\sigma\rho$ for $\sigma \in G_K$ by the following commutative diagram

$$\begin{array}{ccc} \pi & \xrightarrow{\rho} & GL(V) \\ \downarrow{\scriptstyle a_\sigma} & & \downarrow{\scriptstyle d_\sigma} \\ \pi & \xrightarrow{{}^\sigma\rho} & GL({}^\sigma V). \end{array}$$

Because of (1) and (2) we have

$${}^e\rho = \rho, \qquad {}^{\sigma\tau}\rho = {}^\sigma({}^\tau\rho). \tag{3}$$

Lemma 15.1. *Consider two isomorphic representations $\rho : \pi \to GL(V)$ and $\rho' : \pi \to GL(W)$. Then ${}^\sigma\rho$ is isomorphic to ${}^\sigma\rho'$ for all $\sigma \in G_K$.*

Proof. Let $\varphi : V \to V'$ be a $\mathbb{C}_p$-linear isomorphism inducing an isomorphism of representations ρ and ρ', i.e. $\varphi\rho = \rho'$. We have a commutative diagram

$$\begin{array}{ccccc} \pi & \xrightarrow{\rho} & GL(V) & \xrightarrow{\varphi} & GL(W) \\ \downarrow{\scriptstyle a_\sigma} & & \downarrow{\scriptstyle d_\sigma} & & \downarrow{\scriptstyle d_\sigma} \\ \pi & \xrightarrow{{}^\sigma\rho} & GL({}^\sigma V) & \xrightarrow{\sigma_*\varphi} & GL({}^\sigma W). \end{array}$$

It follows that ${}^\sigma\rho' = {}^\sigma(\varphi\rho) = \sigma_*\varphi{}^\sigma\rho$. Therefore ${}^\sigma\rho$ and ${}^\sigma\rho'$ are isomorphic via $\sigma_*\varphi$. □

Let G_ρ be the following subset of G_K

$$G_\rho = \{\sigma \in G_K \mid {}^\sigma\rho \cong \rho \text{ as representations}\}.$$

Proposition 15.2. *The set G_ρ is a subgroup of G_K.*

Proof. Let σ, $\rho \in G_\rho$. Applying (3) and the above lemma yields

$${}^{\sigma\tau}\rho ={}^\sigma ({}^\tau\rho) \cong {}^\sigma\rho \cong \rho.$$

Hence, $\sigma\tau \in G_\rho$. The identity ${}^e\rho = \rho$ is trivial. It remains to show that $\sigma^{-1} \in G_\rho$ if $\sigma \in G_\rho$. Let $\varphi : V \to^\sigma V$ be a $\mathbb{C}_p$-linear isomorphism inducing an isomorphism between ρ and ${}^\sigma\rho$. From (3) it follows that

$$\rho = {}^e\rho = {}^{\sigma^{-1}\sigma}\rho = {}^{\sigma^{-1}}(\varphi\rho) = \sigma_*(d_{\sigma^{-1}})\varphi\rho a_{\sigma^{-1}}^{-1},$$

which is equivalent to $\varphi^{-1}\sigma_*(d_{\sigma^{-1}})^{-1}\rho = \rho a_\sigma$ since $a_{\sigma^{-1}}^{-1} = a_\sigma$. Define $\psi = d_{\sigma^{-1}}\varphi^{-1}\sigma_*(d_{\sigma^{-1}})^{-1}$, i.e. ψ is the composition

$$V = {}^{\sigma\sigma^{-1}}V \xleftarrow{\sigma_*(d_{\sigma^{-1}})} {}^\sigma V \xleftarrow{\varphi} V \xrightarrow{d_{\sigma^{-1}}} {}^{\sigma^{-1}}V.$$

Then we have $\psi\rho = {}^{\sigma^{-1}}\rho$, hence $\sigma^{-1} \in G_\rho$, and the proposition follows. □

Definition 15.3. Consider a smooth projective and geometrically connected curve X_K over a finite field extension K of $\mathbb{Q}_p$. Denote by X its base change to $\overline{\mathbb{Q}}_p$. Fix an $x \in X(K)$. A continuous representation $\rho : \pi_1(X, x) \to GL(V)$ on a finite dimensional $\mathbb{C}_p$-vector space V is called to be *of ∞-type* if the subgroup $G_\rho \subset G_K$ is open.

Proposition 15.4. *Consider a curve X as in the above definition. Let E be a vector bundle on $X_{\mathbb{C}_p}$ with strongly semistable reduction of degree zero. Assume that E is defined over $\overline{\mathbb{Q}}_p$. Then the representation*

$$\rho_{E,x} : \pi_1(X, x) \to GL(E_x)$$

induced by the parallel transport is of ∞-type.

Proof. By assumption E is defined over $\overline{\mathbb{Q}}_p$, hence by Noetherian descent there exist a finite field extension L/K and a vector bundle E_L on $X_L = X_K \otimes_K L$ such that $E_L \otimes_L \mathbb{C}_p = E$. We compute for $\sigma \in G_L$

$$\begin{aligned} ({}^\sigma E)_x &= E_x \otimes_{\mathbb{C}_p,\sigma} \mathbb{C}_p = (E_{L,x} \otimes_L \mathbb{C}_p) \otimes_{\mathbb{C}_p,\sigma} \mathbb{C}_p \\ &= E_{L,x} \otimes_{L,\sigma|_L} \mathbb{C}_p = E_{L,x} \otimes_L \mathbb{C}_p = E_x. \end{aligned}$$

The last identity holds since $\sigma|_L = id$. Therefore we have a commutative diagram with $\pi = \pi_1(X, x)$

$$\begin{array}{ccc} \pi & \xrightarrow{\rho} & GL(E_x) \\ \downarrow{\scriptstyle a_\sigma} & & \| \\ \pi & \xrightarrow{{}^\sigma\rho} & GL(E_x), \end{array}$$

which shows that ${}^\sigma\rho$ is equal to ρ. Thus $G_L \subset G_\rho$, in other words, the group G_ρ contains an open subgroup of G_K, hence is itself open in G_K. $\square$

Remarks on literature

Section 12: The whole section is based on [DW05b] §3. For the definition of the étale fundamental groupoid cf. *loc. cit.* p. 577.

Section 13: The topology of the uniform convergence may be defined in a more elegant or, in some sense, more conceptual way using the notion of uniformity, cf. [Bou89] Chapter X, I.

Section 14: –

Section 15: The definition of the G_K-"action" on representations is from [DW05b] p. 583.

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