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› The diffeomorphism group of an exotic sphere

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Abstract

This thesis is concerned with the diffeomorphism groups of exotic spheres. A smooth homotopy sphere is a smooth manifold, which is homotopy equivalent to a sphere, but not necessarily diffeomorphic. If it is not diffeomorphic, it is referred to as an exotic sphere. There is an evaluation map from the diffeomorphism group of an exotic sphere to the exotic sphere itself, given by evaluation at a basepoint. Methods from homotopy theory and orthogonal calculus are used to investigate the existence question of a section of this evaluation map. The obstruction to the existence of such a section is given by an element in a certain homotopy group. The main result is the non-existence of such a section in dimension seven in the case of the generator of the Kervaire-Milnor group of homotopy spheres.

Zusammenfassung

Diese Doktorarbeit befasst sich mit den Diffeomorphismengruppen exotischer Sphären. Eine glatte Homotopiesphäre ist eine glatte Mannigfaltigkeit, die homotopieäquivalent aber nicht notwendigerweise diffeomorph zu einer Standardsphäre ist. Falls sie nicht diffeomorph ist, so wird sie als exotische Sphäre bezeichnet. Es gibt eine Auswertungsabbildung von der Diffeomorphismengruppe einer exotischen Sphäre zur exotischen Sphäre selbst, gegeben durch Auswertung an einem Basispunkt. Die Existenzfrage eines Schnittes zu dieser Auswertungsabbildung wird unter Zuhilfenahme von Methoden aus Homotopietheorie und Orthogonalkalkül untersucht. Das Hindernis zur Existenz eines solchen Schnittes ist durch ein Element in einer bestimmten Homotopiegruppe gegeben. Das Hauptresultat ist die Nichtexistenz eines solchen Schnittes in Dimension sieben im Fall des Erzeugers der Kervaire-Milnor Gruppe der Homotopiesphären.

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Contents

Notation

All notations presented in this section are likewise introduced during the line of argument. This table is rather meant as a consultation possibility while reading. The indications below only describe the predominant use of the symbols. Note that spectra X are typeset differently than spaces X.

n	A natural number $n \geq 5$, unless stated otherwise;
S^n	The topological n-dimensional unit sphere in $\mathbb{R}^{n+1}$; In some contexts S^n is the standard smooth sphere;
D^n	The standard n-dimensional closed unit ball in $\mathbb{R}^n$;
Σ	A pseudo smooth structure on S^n;
$\mathfrak{S}(S^n)$	The space of pseudo smooth structures on S^n;
S^n_Σ	S^n endowed with a pseudo smooth structure Σ;
$\mathrm{Diff}(S^n_\Sigma)$	The topological monoid of pseudo diffeomorphisms of S^n_Σ;
$\mathrm{TOP}(n)$	The topological group of homeomorphisms of $\mathbb{R}^n$;
$\mathrm{TOP}(S^n)$	The topological group of homeomorphisms of S^n;
$\mathrm{O}(n)$	The topological group of orthogonal transformations of $\mathbb{R}^n$;
$\mathrm{G}(n)$	The topological monoid of homotopy self-equivalences of S^{n-1};
ev_1	The evaluation map on a mapping space at the basepoint 1;
μ	A multiplicative structure $S^n \times S^n \to S^n$;
δ, d	A map $\delta : S^n \to S^n$ of degree $\deg(\delta) = d$;
s	A lift $S^n \to \mathrm{TOP}(S^n)$ of δ along $\mathrm{ev}_1 : \mathrm{TOP}(S^n) \to S^n$;
$\overline{s}$	A lift $S^n \to \mathrm{Diff}(S^n_\Sigma)$ of δ or s along $\mathrm{ev}_1 : \mathrm{Diff}(S^n_\Sigma) \to S^n$;
ξ	A lift of Σ to $\pi_{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$;
HO	A natural transformation of functors $\pi_n \Rightarrow \pi_{2n}$ as $\mathbf{TopMon} \to \mathbf{Set}$ or the associated $\pi_{n+1} \Rightarrow \pi_{2n+1}$ as $\mathbf{Top}_* \to \mathbf{Set}$;
σ	A map $S^{2n+1} \to S^{n+1}$; of Hopf invariant 1 in the case $n = 7$;
$\mathrm{Obs}(s, \Sigma)$	The obstruction element in $\pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$ to finding a lift $\overline{s}$;
S	The sphere spectrum;
$\mathsf{X}_{h\mathbb{Z}_2}, \mathsf{X}^{h\mathbb{Z}_2}, \mathsf{X}^{rh\mathbb{Z}_2}, \mathsf{X}^{th\mathbb{Z}_2}$	The homotopy orbit, fixed point, reduced fixed point and Tate spectra of a spectrum X with involution;
$\mathsf{L}_\bullet(R), \mathsf{L}^\bullet(R)$	The quadratic and symmetric L-theory spectra of a ring R with involution;
$\mathsf{K}(R)$	The algebraic K-theory spectrum of a ring R with involution;
$\mathsf{A}(*)$	The Waldhausen A-theory spectrum of a point;
$\mathsf{Wh}(*)$	The smooth Whitehead spectrum of a point;
Ξ	The Weiss-Williams map $\mathsf{L}^\bullet(R) \to \mathsf{K}(R)^{th\mathbb{Z}_2}$;
$\mathsf{bo}, \mathrm{BO} \times \mathbb{Z}$	The connective real topological K-theory spectrum and its associated infinite loop space;
$\mathbb{Z}_2^\wedge, \mathbb{Q}_2^\wedge$	The integers or rationals completed at 2;
θ_F	The derivative spectrum associated to $F : \mathcal{J} \to \mathbf{Top}_*$;
$T_1 F$	The first orthogonal calculus Taylor approximation of F;
ϕ_F	The approximation map $\phi_F : F \to T_1 F$, called η in [Wei95];

Introduction

This introduction is organized into several parts. The foreword provides a general idea of topology to the novice. The state of the art concerning diffeomorphism groups of exotic spheres is summarized in the literature review. Thereafter, the research question is presented together with the most important research results found in this thesis. The methodology part aims at informing the reader what kind of methods are used to examine the main research problem. Finally, the outline of argument contains a successive list of the most important ideas employed in each of the sections of this thesis. The latter is rather technical, yet hopefully provides a good reference point for the thesis composition.

Foreword. The field of topology arose, as many other mathematical disciplines, out of the necessity for describing natural phenomena. Similar to geometry, it is concerned with the description and analysis of geometric structures and spaces. In contrast to classical geometry, the aim in topology is to consider these spaces up to continuous deformations. This allows us to gather their properties more flexibly.

Algebraic topology, a branch of topology, analyzes what is referred to as topological spaces by associating algebraic objects to them. The assignment is generally invariant up to continuous deformations, and it allows the classification of geometric objects using methods from algebra. Numerous modern applications arose, unanticipated a century ago. Examples include the study of the human DNA using knot theory and the ability to cope with vast statistical data employing persistent homology, among many others.

In Euclidean space $\mathbb{R}^{n+1}$, the points of unit norm form the standard n-dimensional sphere S^n. The 2-sphere S^2 can be thought of as a good modeling example for the surface of a 3-dimensional ball such as our planet Earth. Its rotation over time about its own axis abstractly is nothing but a smooth bijective assignment sending any given point on the surface to another one. This is what topologists more generally call a diffeomorphism on S^2. The study of the topological group of all diffeomorphisms is inherently necessary.

However, with its peaks and gorges, Earth can roughly be thought of as not having a smooth surface. In topology, there is a formal notion of smooth structure describing this situation and allowing to differentiate between essentially distinct smooth structures. As a crude classification, any smooth sphere S^n_Σ is called exotic if it is homeomorphic but not diffeomorphic to the standard sphere S^n. We attempt to distinguish the associated diffeomorphism groups $\mathrm{Diff}\,(S^n_\Sigma)$ and $\mathrm{Diff}\,(S^n)$ of an exotic and the standard smooth sphere, respectively.

As an application, exotic smooth structures and diffeomorphism groups of exotic spheres may also influence the field of general relativity and astrophysics, as discussed in [AMB07, Chapter 10]. As an illustrative example, in the setting of an exotic $\mathbb{R}^4$ as space-time, Einstein's general relativity gravity might become highly non-trivial also in areas with no matter present, as explained in [AMB07]. The discussion on quantum field theories also includes higher dimensional cases.

Literature review. Let S^n denote the standard topological unit n-sphere in $\mathbb{R}^{n+1}$, and throughout let $n \geq 5$ unless explicitly stated otherwise. First and foremost, differential topology is concerned with the study of smooth structures on manifolds and smooth mappings between these. All topological homotopy n-spheres, $n \geq 5$, are homeomorphic to S^n. This is due to Smale's proof [Sma61, Theorem A] of the generalized topological Poincaré conjecture in these dimensions. The question of different diffeomorphism types of S^n remains.

Exotic spheres were discovered by Milnor [Mil56] in 1956, and the existence and classification questions of smooth structures have been intensively studied since. In a celebrated result, Kervaire and Milnor [KM63] presented results about the finite group θ_n of oriented diffeomorphism classes of oriented smooth homotopy n-spheres. They also defined and analyzed the subgroup bP^{n+1} of homotopy spheres bounding parallelizable manifolds. Their ideas fit into the format of a surgery exact sequence as formulated later. Up to now, all advances in the classification of smooth homotopy spheres rely on these methods and the close interplay $\theta_n/bP^{n+1} \hookrightarrow \pi_n^s/\operatorname{im} J$ [Ada66], [Bro69], [Lüc02, Theorem 6.46] with the stable homotopy groups of spheres and the J-homomorphism. It is the lack of knowledge of π_n^s that prevents a complete answer at this point.

In contrast, to this day it seems that the study of the associated diffeomorphism groups $\mathrm{Diff}\,(S^n_\Sigma)$ has not nearly attracted as much attention. Here, let Σ denote an exotic smooth structure on S^n, and S^n_Σ the sphere endowed with this structure. The study of $\pi_0\,(\mathrm{Diff}\,(S^n))$ was stimulated by the concept of twisted spheres. These are exotic spheres $S^n_\Sigma = D^n \cup_f D^n$ constructed with the help of an orientation-preserving diffeomorphism $f : S^{n-1} \to S^{n-1}$. There is an isomorphism $\theta_n \cong \pi_0\left(\mathrm{Diff}^+(S^{n-1})\right)$ for $n \geq 6$, by [Cer70] and [Sma62]. In other words, every homotopy sphere is a twisted sphere in these dimensions. Leading to this result, the investigation of pseudo isotopies, a weaker notion than isotopy, has received considerable attention. One of the most notable results is considered to be the pseudo isotopy theorem [Cer70]. A few major results carry over to the diffeomorphism groups $\mathrm{Diff}\,(S^n_\Sigma)$ of exotic spheres as follows. For a smooth sphere S^n_Σ, the group $\mathrm{Diff}\,(S^n_\Sigma)$ has the homotopy type of a CW-complex, as it can be embedded as an open subset of the separable infinite-dimensional Hilbert space H [Hen70]. In fact, $\mathrm{Diff}\,(S^n_\Sigma)$ is a Fréchet manifold, i. e. locally homeomorphic to H. As such it is determined up to homeomorphism by its homotopy type [Hen70]. However, at least for $n \geq 7$, the group $\mathrm{Diff}_0(S^n)$ does not have the homotopy type of a finite-dimensional Lie group [ABK72].

In an attempt to distinguish the homotopy groups of diffeomorphism groups of exotic spheres from those of standard spheres, Schultz [Sch71] investigated the degree of symmetry $N(S^n_\Sigma)$. It can be used as an indication for the difference of the diffeomorphism groups. Amongst other ideas, he employed the fibration sequence $\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}_*(S^n_\Sigma) \to \mathrm{SO}\,(n)$. Note that in contrast to spheres, the diffeomorphism groups of disks $\mathrm{Diff}_\partial(D^n) \cong \mathrm{Diff}_\partial(D^n_\Sigma)$ are isomorphic for any smooth structure Σ. This is essentially due to the existence of only one smooth structure on D^n for $n \geq 5$ [Mil65], see also Lemma 2.1.

A variant of the fibration sequence mentioned above has later also been studied by Pedersen and Ray [RP80, Diagram 1.3], namely its generalization to unpointed diffeomorphisms as $\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}\,(S^n_\Sigma) \to \mathrm{SO}\,(n+1)$. Pedersen and Ray define a subset $X_k(S^n_\Sigma) \subset \pi_k\,(\mathrm{SO}\,(n+1))$ whose elements satisfy certain properties that may help to distinguish the exotic case from the standard case. After all, $X_k(S^n)$ is empty for the standard sphere. Most notably, an element $X_k(S^n_\Sigma)$ would imply an interplay between $\mathrm{SO}\,(n+1)$ and $\mathrm{Diff}\,(S^n_\Sigma)$ in the exotic case, which is non-existent for S^n. However, by [RP80, (2.2) Theorem] an element $x \in X_k(S^n_\Sigma)$ presumes the existence of a very specific non-zero element in the cokernel $\pi^s_{n+k}/\operatorname{im} J$ of the J-homomorphism. The authors do not give examples, and to date the problem seems difficult to access. Also, the exact implications on the homotopy groups of $\mathrm{Diff}\,(S^n_\Sigma)$ remain unclear from [RP80].

Note that after a general setup, this thesis specializes in smooth structures in θ_n lifting to $L_{n+1}\,(\mathbb{Z})$, i. e. the subgroup bP^{n+1} of spheres bounding parallelizable manifolds. Since $\theta_n/bP^{n+1} \hookrightarrow \pi^s_n/\operatorname{im} J$, the aim of [RP80] and its approach are somewhat distinct from the present one.

One of the most general questions that comes to mind in this context is whether it is possible to sensibly relate two versions of the [RP80]-type fiber sequences for standard and exotic smooth structures, for example, by homotopy equivalences. This is discussed in more detail in Section 2. As this attempt seems rather demanding, one can resort to a related question which was raised by Lytchak and Wilking in [LW16]. Their main theorem is concerned with the classification of Riemannian foliations on topological spheres. In the case of a smooth homotopy 15-sphere, such a foliation is given by the fibers of a Riemannian submersion $S^{15}_\Sigma \to S^8_\Sigma$ to a smooth homotopy 8-sphere. It is unclear whether the fibers may admit exotic structures. As explained in [LW16], the fibration is determined by its coordinate changes and ultimately by a certain homotopy class of map $\overline{s} : S^7 \to \mathrm{Diff}\,(S^7_\Sigma)$, with S^7_Σ possibly endowed with an exotic structure. Therefore, the existence question of such maps $\overline{s}$ satisfying $\mathrm{ev}_1 \circ \overline{s} \simeq \mathrm{id}_{S^7}$ becomes important. Here $\mathrm{ev}_1 : \mathrm{Diff}\,(S^7_\Sigma) \to S^7$ denotes the evaluation map at the basepoint. According to Wilking, the conjectured result in this thesis may rule out the existence of certain foliations as described above.

Research aim and results. In this thesis we develop methods which work towards distinguishing the diffeomorphism groups $\mathrm{Diff}\,(S^n_\Sigma)$ and $\mathrm{Diff}\,(S^n)$ for certain exotic structures Σ. A reformulation of the question in [LW16] reads as follows. In dimension $n = 7$, does the evaluation map $\mathrm{ev}_1 : \mathrm{Diff}\,(S^7_\Sigma) \to S^7$ admit a section for an exotic smooth structure $\Sigma \in \theta_7$? In the case of the standard sphere we may define a section using the Cayley multiplication (Example 2.4). This contrasts the following conjecture for one of the exotic 7-spheres, as discussed in Section 2. Recall the surjection $L_8\,(\mathbb{Z}) \to \theta_7$ (Lemma 1.42), i. e. every exotic 7-sphere bounds a parallelizable manifold.

Conjecture 2.6. *Let the smooth structure $\Sigma \in \theta_7$ be the image of $1 \in L_8\,(\mathbb{Z})$. Then there exists no homotopy section $\overline{s} : S^7 \to \mathrm{Diff}\,(S^7_\Sigma)$ to the evaluation map ev_1.*

Thus the conjecture would distinguish the case of the standard smooth sphere from the case of the exotic smooth sphere $\Sigma \in \theta_7$. More illustrative consequences would include Corollaries 2.8 to 2.11.

In the general case $n \geq 5$ outside the Hopf dimensions $1, 3$ and 7, one cannot hope to find a section of the evaluation map as described above, for reasons as laid out in Example 1.50, Lemma 3.6 and in [MT91, Theorem 6.13]. The most sensible generalization of the Lytchak-Wilking existence question of $\overline{s} : S^7 \to \mathrm{Diff}\,(S^7_\Sigma)$ is the following: Does a given self-map $\delta : S^n \to S^n$ admit a lift to $\mathrm{Diff}\,(S^n_\Sigma)$ along the evaluation map $\mathrm{ev}_1 : \mathrm{Diff}\,(S^n_\Sigma) \to S^n$ for an exotic smooth structure $\Sigma \in \theta_n$? By [MT91, Theorem 6.13], this is the case for any map δ of even degree for the standard sphere. The more general Conjecture 2.7 would allow to distinguish the diffeomorphism group of the standard sphere from the diffeomorphism groups of exotic spheres $\Sigma \in bP^{n+1} \subset \theta_n$ for $n \equiv 3 \mod 4$.

This thesis is concerned foremost with the analysis and reformulation of the lifting question into a homotopy theoretical problem. The simplification of this problem into a more approachable formulation takes up the second half of this thesis. However, the proof of Conjecture 2.6 is unfinished at this point despite several attempts. Further ideas are currently in development, allowing the missing computational part to be completed. Already existing ideas and most necessary arguments for the theoretical part are explained in the course of this thesis and outlined in the sequel below. By examining certain lifting obstructions, the conjecture can be narrowed down to an explicit computational problem, summarized as the following theorem. It constitutes the most important research result of this thesis. Let $\sigma : S^{15} \to S^8$ denote a map of Hopf invariant one. As we will see later, smooth structures $\Sigma \in \theta_7$ are classified by maps $S^7 \to {}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}$. Let $\xi : S^8 \to {}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}$ denote an L-theoretic lift of such a smooth structure along $\Omega\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \to {}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}$. Let $\phi : {}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)} \to T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$ denote the approximation map to the first orthogonal calculus Taylor approximation $T_1({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)})$.

Theorem 2.5. *If $\phi \circ \xi \circ \sigma \neq 0 \in \pi_{15}\left(T_1({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)})\right)$, then there exists no homotopy section $\overline{s} : S^7 \longrightarrow \mathrm{Diff}\,(S^7_\Sigma)$ to the evaluation map* ev_1.

The discourse and, in particular, the assertions are held as general as possible in view of a more general Conjecture 2.7. However, a proof of the latter is far less within reach than for Conjecture 2.6. Details and explanations as well as obstacles towards the more general claim are laid out during the entire line of argument. A detailed account of the main results in this thesis is given in Section 2.

At this point the attention of the reader also shall be drawn to some results gathered of possible independent interest. The homotopy theoretical description of smooth structures as described in Section 1.2 may be an interesting alternative to those provided by [Mor69b] and [KS77]. Its flexibility is underlined by the simple, yet surprising result of Proposition 1.12. A spectrum level version of a special case of the Atiyah-Segal completion theorem is proved in Theorem 1.61. In short, with bo denoting the real connective topological K-theory spectrum endowed with the trivial $\mathbb{Z}_2$-action, $\mathsf{bo} \vee \mathsf{bo}_2^\wedge \simeq \mathsf{bo}^{h\mathbb{Z}_2}$ for the connective cover of the homotopy fixed points $\mathsf{bo}^{h\mathbb{Z}_2}$, reminiscent of $\mathrm{RO}(\mathbb{Z}_2)^\wedge \otimes \mathrm{KO}^*(*) \cong \mathrm{KO}^*(\mathrm{B}\mathbb{Z}_2)$. Section 1.6 also contains an overview of the homotopy groups of the homotopy orbit, fixed point and Tate spectra associated with bo. Theorem 1.80 almost completely describes the long exact sequence on homotopy groups associated with the norm map for bo. The appendix contains a few selected results on the evaluation map and homotopy pullbacks, of which most are known but perhaps not written up in detail.

Last but not least, the approximation theory in Sections 5 and 6 of this thesis relies heavily on orthogonal calculus and the Weiss-Williams theory of automorphisms of manifolds relating L-theory to algebraic K-theory. In this sense the present thesis shall hopefully provide an application of their theory. Perhaps this allows to indicate the advantages as well as possible limitations of the Weiss-Williams theory for a wider audience.

Methodology. Despite the fact that the results in this thesis aim to answer a question from differential geometry, the methods employed towards its investigation are of an algebraic topological nature. For example, it is only the homotopy theoretic properties of exotic spheres that appear in this treatise. Indeed, [Mor69a] and [KS77] allow smooth structures on S^n to be understood roughly as maps $S^n \to {}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}$. The Kervaire-Milnor group θ_n of oriented diffeomorphism classes of oriented homotopy n-spheres [KM63] can be identified with $\pi_n\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$, which is isomorphic to $\pi_n\left({}^{\mathrm{TOP}}/_{\mathrm{O}}\right)$. In the case $n = 7$ of the main research question, all of the elements of θ_n can be lifted to $\pi_{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$ by a kind of surgery exact sequence [Ran92]. In all other dimensions, we only follow through with the arguments for smooth structures admitting such a lift.

As explained more precisely in the outline below, we develop obstructions to finding a lift of δ to $\mathrm{Diff}\left(S^n_\Sigma\right)$. Their concise justification relies most notably on the optimized adaptation of the homotopy theoretic formulation of smooth structures as developed in Section 1.2. Also, results in [MT91] from classical homotopy theory help to reformulate some of the assertions. The ultimate obstruction is reformulated using a homotopy operation $\pi_{n+1} \Rightarrow \pi_{2n+1}$. Its properties are studied using the looping-delooping equivalence obtained from the group completion theorem [MS76], as well as Hopf maps and Whitehead products. An additivity formula [Hil55] in homotopy groups and the James construction [Jam55] assist in finding its exact shape. All that remains of the research question is to show that a certain value of this homotopy operation does not vanish.

Orthogonal calculus [Wei95] provides an approximation for functors on inner product spaces like $F(V) = {}^{\mathrm{G}(V)}/_{\mathrm{TOP}(V)}$ and $F(V) = {}^{\mathrm{TOP}(V)}/_{\mathrm{O}(V)}$ in terms of the first Taylor approximation T_1F. The analysis of ${}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$ was also initiated in [WW88]. The map $\Xi : \mathsf{L}_\bullet(R) \to \mathsf{K}(R)^{th\mathbb{Z}_2}$ from quadratic L-theory to the Tate construction on K-theory is studied in depth in the emerging paper series in progress. Since both quadratic L-theory and the need for an approximating space show up naturally in our line of argument, the properties of Ξ play a significant role.

Most notably, this thesis relies on the unpublished results announced for the continuation [WW18] of the Weiss-Williams paper series on *Automorphisms of manifolds and algebraic K-theory*, [WW88], [WW89] and [WW14]. As a synopsis, suffice it to say that [WW18] is in part about understanding the first derivative spectra θ_F algebraically, including their involution. More generally, it is concerned with the algebraic aspects of the entire map $(T_1F)(V) \to F(\mathbb{R}^\infty)$. As an example, in some cases this allows to draw conclusions on the inherent action of the structure groups of such fibrations in algebraic terms. Another benefit is the description of a duality respecting map ${}^{\mathrm{G}}/_{\mathrm{O}} \to \Omega_!^{\infty+1}\mathsf{Wh}(*)$ in K-theoretic terms, which is strongly expected to coincide with the Hatcher-Waldhausen map [Wal82]. Facts from orthogonal calculus are reformulated by [WW18] in a manner which turns out to be convenient for this thesis. The necessary results are cited in detail where appropriate.

In Section 7, an attempted calculation of the approximated obstruction is presented. The methods are instructive and indicate also how to go about the problem with a different model. The preliminaries include the Atiyah-Segal completion theorem [AS69] for the real topological K-theory spectrum bo and other L- and K-theory. In Section 7, a sort of functional cohomology operation for bo is inspected. Its properties are linked to the obstructing element using Adams' calculations [Ada73] of the image of $\pi_*(\mathsf{bu})$ in the rational homology of bu, together with an analysis of the Steenrod algebra.

Outline of argument. Preliminarily, we review the definitions of the orthogonal group $\mathrm{O}(n)$, the group of homeomorphisms $\mathrm{TOP}(n)$ and $\mathrm{TOP}(S^n)$ of $\mathbb{R}^n$ respectively S^n, and the monoid $\mathrm{G}(n)$ of homotopy automorphisms of S^{n-1} in Section 1.1. These are presented alongside with a convenient identification for ${}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}$ and a summary of the connectivity of the stable inclusions. Also, in this thesis we employ what we will refer to as pseudo smooth structures, the space of which being an adequate, homotopy theoretic substitute for the classical notion of smooth structures. These turn out to be maps $\Sigma : S^n \to {}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}$. Their properties are developed in Section 1.2. This concept is not related to pseudo isotopies. A quick review of the surgery exact sequence in the case of the sphere as well as the fundamentals of L-theory are presented in Section 1.3. These are followed in Section 1.4 by a short account on multiplicative structures and their relation to lifts of a given self-map $\delta : S^n \to S^n$ to $\mathrm{TOP}(S^n)$. An even shorter summary of the homotopy fixed point, homotopy orbit and Tate constructions as well as the definition of the norm map are given in Section 1.5. The preliminaries are concluded by Section 1.6, which presents facts related to the real connective K-theory spectrum. A generalized spectrum-level Atiyah-Segal completion map provides insight into the structure $\mathsf{bo} \vee \mathsf{bo}_2^\wedge \simeq \mathsf{bo}^{h\mathbb{Z}_2}$ of the connective homotopy fixed point spectrum $\mathsf{bo}^{h\mathbb{Z}_2}$, compare Theorem 1.61. A detailed description of the norm map allows an almost complete description of its associated long exact sequence on homotopy groups in Theorem 1.80. Section 1.6 provides the basis for Section 7.

The introductory motivation via $\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}\,(S^n_\Sigma) \to \mathrm{SO}\,(n+1)$ from [RP80] is followed up on and more thoroughly discussed in Section 2, alongside with more precise statements and conjectures.

For any smooth structure Σ, a first obstruction to finding a lift of $\delta : S^n \to S^n$ to $\mathrm{Diff}\,(S^n_\Sigma)$ is given by finding a lift s to $\mathrm{TOP}\,(S^n)$. According to Obstruction 3.2, such a lift exists if and only if the pullback bundle $\delta^*TS^n_\Sigma$ of the tangent bundle TS^n_Σ along δ is trivial. In fact, this is the case if and only if n is odd and $\deg(\delta)$ is even, or $n = 7$, as developed in Section 3.1. We continue to pin down the exact Obstruction 3.16 for the existence of a lift to Diff (S^n_Σ) in Section 3.2. It vanishes if and only if a certain diagram (3.4) commutes up to homotopy. The shape of (3.4) is reminiscent of the defining property for a pseudo diffeomorphism as laid out in Definition 1.25.

The subsequent Section 4 is all about understanding and translating the aforementioned obstruction into a homotopy theoretical statement. To this end, we define a natural transformation of sets $\mathrm{HO} : \pi_n \Rightarrow \pi_{2n}$ on the category of topological monoids. This homotopy operation is given by a general construction based on diagrams of similar shape as (3.4). Using the delooping machinery, this definition is extended to the category of spaces, on which it is represented by composition with a map $\sigma : S^{2n+1} \to S^{n+1}$. The obstruction from Section 3.2 can be restated using this homotopy operation, adapted to the setting of (3.4). Concretely, there exists a lift of δ to Diff (S^n_Σ) if and only if $\mathrm{Obs}(s,\Sigma) = 0 \in \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$. If $n = 7$ and $\deg(\delta) = 1$, the lift $s \circ \sigma \in \pi_{14}\left(\mathrm{TOP}\left(S^7\right)\right)$ of the obstruction is given roughly by the composition of the lift s to $\mathrm{TOP}\left(S^7\right)$ with σ of Hopf invariant one. Using that $\Sigma \in \pi_7\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ lifts to $\xi \in \pi_8\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$, the obstructing element $\mathrm{Obs}(s,\Sigma)$ can be considered as the image of $\xi \circ \sigma$ with a correction term applied. The latter mostly consists of a Whitehead product involving ξ, as developed in Section 4.5 and stated precisely in Obstruction 4.26. In order to show the main conjecture, it is necessary to show that $\mathrm{Obs}(s,\Sigma) \neq 0 \in \pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ for non-standard $\Sigma \neq 0$ in the relevant cases. Our attempt is to show that $\xi \circ \sigma \neq 0 \in \pi_{15}\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$ for lifts ξ of $\Sigma \neq 0$, and to show that the obstruction is still non-zero upon adding the correction term.

The main object of Section 5 is to develop an approximating model for the space ${}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$. Here, by an approximating model we mean a spectrum P together with a map $\phi : \Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right) \to \Omega^\infty \mathsf{P}$. The model P is similar to and derived from the algebraic description of the first orthogonal calculus Taylor approximation $T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$. The expectation is that it is easier to show that $\phi \circ \xi \circ \sigma$ does not vanish in $\pi_n(\mathsf{P})$, which would imply $\xi \circ \sigma \neq 0$. The approximating model P is defined as a mixture of quadratic L-theory and Waldhausen A-theory of a point. The two are linked using the Weiss-Williams map Ξ from quadratic L-theory $\mathsf{L}_\bullet(R)$ to the Tate fixed point spectrum of K-theory $\mathsf{K}(R)^{th\mathbb{Z}_2}$. The map Ξ is one of two arrows used to define a homotopy pullback square. The homotopy pullback then constitutes the approximating model P. The convenient shape of the main Proposition 5.9 is inferred from the reformulation of the fundamentals of orthogonal calculus in algebraic terms as announced for the continuation [WW18] of the Weiss-Williams paper series on *Automorphisms of manifolds and algebraic K-theory*.

The theoretical part is finished by Section 6. We show that in the case $n = 7$ the image of $\xi \circ \sigma$ under $\partial : \pi_{15}\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \to \pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ is also non-zero if this was the case for $\xi \circ \sigma$ itself. We do this by using an analogous orthogonal calculus approximation compatible with the one in the previous section. This consistency manifests itself in terms of a certain commutative square (6.7) given by the naturality of the first Taylor approximation in orthogonal calculus. Fortunately, the correction term vanishes in the approximating model. This is shown in the case of general n by splitting the two entries in the Whitehead product into separate factors of a product space. The vanishing implies that $\phi \circ \xi \circ \sigma$ is transferred via the appropriate map between the approximations to the image $\phi(\mathrm{Obs}(s,\Sigma))$ of the obstruction in the approximating model. The injectivity of this map on π_{15} is shown using knowledge of the stable homotopy groups of spheres and the low dimensional properties of the smooth Whitehead spectrum $\mathsf{Wh}(*)$. Overall this would entail the non-vanishing of $\mathrm{Obs}(s,\Sigma)$ in the appropriate cases, as desired for the conjectures and for Theorem 2.5.

The discussion in Section 7.5 sheds some light on possible methods to approach the calculation problem $\phi \circ \xi \circ \sigma \neq 0$ in the simplified model P. No tangible non-vanishing statement is included in this thesis, as this is still the subject of ongoing research. Further promising ideas are currently in development, allowing the missing computational part to be completed and published in the near future. It is only the attempt via real topological K-theory that cannot succeed, and the pertaining details are presented in Section 7. In particular, the calculational term is equal to zero, and we analyze why. In Section 7.1 we develop the simplification P_{bo} of the approximating model using the map $\mathsf{A}(*) \to \mathsf{K}(\mathbb{Z}) \to \mathsf{bo}$. It is shown to admit a simple description in the case $n = 7$ in Section 7.2. As a result, we deduce in Sections 7.3 and 7.4 that any existing nullhomotopy for $\phi \circ \xi \circ \sigma$ is actually also detected in a certain homology group. In Section 7.5 we see that already the effect of $\Xi : \mathsf{L}_\bullet(\mathbb{Z}) \to \mathsf{Wh}(*)^{th\mathbb{Z}_2}_{[3]}$ on π_8 inhibits successful calculations with bo. Here we are using the Postnikov truncation of $\mathsf{Wh}(*)$. This fact allows important conclusions for the calculation involving $\mathsf{Wh}(*)$ and sets the groundwork for current ongoing research.

1. Preliminaries

Throughout, we denote by $n \geq 5$ the dimension of the standard n-dimensional topological sphere S^n in $\mathbb{R}^{n+1}$.

Notation 1.1. (i) For any topological space X, we denote by $X_+ = X \coprod \{*\}$ the space X with an additional disjoint point. The new point is taken as basepoint, turning X_+ into a pointed space.
(ii) Let G be a group and $k \in \mathbb{Z}$. Denote by $\mathsf{H}(G,k)$ the Eilenberg-MacLane spectrum with homotopy concentrated in dimension k and given by G.

The standard notation for the latter is rather $\Sigma^k \mathsf{H}G$, but Σ shall be reserved as best possible for pseudo smooth structures.

1.1. The spaces O, TOP and G

Definition 1.2. Let M and N be manifolds. Denote the space of homeomorphisms $M \to N$ by $\mathrm{TOP}(M,N)$, and endow it with the compact-open topology. Introduce the short notation $\mathrm{TOP}(M) := \mathrm{TOP}(M,M)$ and $\mathrm{TOP}(n) := \mathrm{TOP}(\mathbb{R}^n)$ for the respective topological groups. We also especially point out the group $\mathrm{TOP}(S^n)$. Denote by $\mathrm{O}(n)$ the topological group of orthogonal transformations of $\mathbb{R}^n$, and by $\mathrm{G}(n)$ the topological monoid of homotopy self-equivalences of S^{n-1}, i. e. $\mathrm{G}(n) := \{f : S^{n-1} \to S^{n-1} \mid \deg(f) = \pm 1\}$. Define the corresponding stable objects via their respective homotopy colimits, which in these cases are colimits, in the sense of

$$\mathrm{O} := \operatorname{colim}\left[\mathrm{O}(1) \to \mathrm{O}(2) \to \mathrm{O}(3) \to \dots\right].$$

A comprehensive introduction to homotopy limits and colimits can be found in [Dug08].

The spaces $\mathrm{O}(n)$, $\mathrm{TOP}(n)$, $\mathrm{G}(n)$ etc. are all at least A_∞-spaces. We may consider their classifying spaces $\mathrm{BO}(n)$, $\mathrm{BTOP}(n)$, $\mathrm{BG}(n)$ etc. These are sometimes also more generally referred to as deloopings for topological monoids. The idea of constructing the more general deloopings for grouplike topological monoids goes back for example to [Seg74].

Example 1.3. The space BO can be interpreted as the classifying space for real stable vector bundles. The space BG can be seen as the classifying space for stable spherical fibrations.

Notation 1.4. The notation G is standard, yet somewhat unfortunate, since it clashes with the notation of a group G. Note the difference in typesetting. The standard maps relating the main objects defined in Definition 1.2 are

$$\mathrm{O}(n) \xrightarrow{\text{incl}} \mathrm{TOP}(n) \xrightarrow{\simeq} \mathrm{TOP}_*(n) \xrightarrow{\text{restrict}} \mathrm{G}(n), \tag{1.1}$$

compare [Rud16, Remark 2.10]. The inclusion $\mathrm{O}(n) \to \mathrm{TOP}(n)$ is quite clear. $\mathrm{TOP}_*(n)$ denotes the origin-preserving homeomorphisms of $\mathbb{R}^n$, and a deformation retract $\mathrm{TOP}(n) \to \mathrm{TOP}_*(n)$ is induced by a deformation retract $\mathbb{R}^n \simeq *$. Any $f \in \mathrm{TOP}_*(n)$ can be restricted to a homeomorphism $\mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$ and thus yields a homotopy equivalence $S^{n-1} \to S^{n-1}$ given by $x \mapsto f(x)/|f(x)|$. The composite map $\mathrm{O}(n) \to \mathrm{G}(n)$ is in fact simply the restriction of the orthogonal mapping to S^{n-1}. Note at this point that we have a map $(-)^c : \mathrm{TOP}(n) \to \mathrm{TOP}(S^n)$ denoting the one-point compactification of a homeomorphism. It is induced by the stereographic inclusion $\mathbb{R}^n \to S^n$, $0 \mapsto 1 = (1, 0, \dots, 0) \in S^n$ of $\mathbb{R}^n$ into its one-point compactification S^n.

Definition 1.5. For any pair of A_∞-spaces $H \to G$, we introduce the notation

$$^{G}/_{H} := \mathrm{hofiber}\,[\mathrm{B}H \to \mathrm{B}G]\,, \quad \text{e.g. } ^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)} = \mathrm{hofiber}\,[\mathrm{BO}(n) \to \mathrm{BTOP}(n)]\,.$$

We are particularly interested in instances induced by maps between the stable or unstable groups and monoids from Definition 1.2 in the sense of Notation 1.4.

According to [Rud16, 2.7], the spaces BO, BTOP, BG, $^{\mathrm{TOP}}/_{\mathrm{O}}$ etc. are homogeneous spaces and have the homotopy type of CW-complexes. Compare also [Mil59].

Example 1.6. We think of the space $^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$ as the homotopy fiber of the map $\mathrm{BTOP}(n) \to \mathrm{BG}(n)$. It is induced by the restriction map $\mathrm{TOP}(n) \to \mathrm{G}(n)$. In short, the classifying space $^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$ carries the universal fiber bundle γ with fiber $\mathbb{R}^n$ and structure group $\mathrm{TOP}_*(n) \simeq \mathrm{TOP}(n)$. The induced spherical bundle $S(\gamma)$ comes with a fiber homotopy trivialization $S(\gamma) \xrightarrow{\simeq} {}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)} \times S^{n-1}$.

Remark 1.7. As described in Definition 1.2, TOP is the colimit of the $\mathrm{TOP}(n)$. One can also define the colimit TOPS as the colimit of the $\mathrm{TOP}(S^n)$. These can be related as follows. Consider the one-point compactification $c : \mathrm{TOP}(n) \to \mathrm{TOP}(S^n)$ and the map $u : \mathrm{TOP}(S^n) \to \mathrm{TOP}(n+1)$ given by radially extending a homeomorphism of the unit sphere to the entire $\mathbb{R}^{n+1}$. We relate these via

$$\begin{tikzcd}
\mathrm{TOP}(n) \arrow[r, "\mathrm{incl}"] \arrow[d, "c"'] & \mathrm{TOP}(n+1) \arrow[d, "c"] \\
\mathrm{TOP}(S^n) \arrow[r, "\mathrm{incl}"] \arrow[ur, "u"] & \mathrm{TOP}(S^{n+1}).
\end{tikzcd}$$

The lower triangle commutes, whereas the upper triangle only commutes up to homotopy. Hence up to homotopy, the inclusion maps factor via u and c, meaning we obtain an induced homotopy equivalence $c_\infty : \mathrm{TOP} \overset{\simeq}{\longleftrightarrow} \mathrm{TOPS} : u_\infty$. We implicitly identify $\mathrm{TOP} = \mathrm{TOPS}$.

1.1.1. An identification for $\mathrm{TOP}(n)/\mathrm{O}(n)$

The notation $^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}$ as quotient is actually suggestive for left coset spaces. We make the following alternative definitions in two special cases.

Definition 1.8. The group $\mathrm{O}(n)$ acts from the right on both $\mathrm{TOP}(n)$ as well as $\mathrm{TOP}(S^{n-1})$ by precomposition. We may then form the orbit sets

$$\begin{aligned} {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)} &= \{f \circ \mathrm{O}(n) \mid f \in \mathrm{TOP}(n)\} \\ \text{and}\quad {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} &= \{f \circ \mathrm{O}(n+1) \mid f \in \mathrm{TOP}(S^n)\}. \end{aligned}$$

One can also think of $\mathrm{O}(n)$ as being a subgroup, then the sets above simply contain precisely all left cosets.

Remark 1.9. Consider the projection maps, exemplified through $p : \mathrm{TOP}(n) \to {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$, $f \mapsto f \circ \mathrm{O}(n)$. In the alternative definition, we topologize the sets above using the quotient topology, in the sense that we endow them with the finest topologies which make the projection maps continuous. In this sense one can also think of p as a quotient map, yet the present situation should not be confused with the common quotient by the subspace $\mathrm{O}(n)$. We justify our choice of topology and link it back to the original definition with the following lemma.

Lemma 1.10. *The spaces from Definition 1.8 fit into long exact sequences of the type*

$$\ldots \to \pi_k(\mathrm{O}(n)) \xrightarrow{i_k} \pi_k(\mathrm{TOP}(n)) \xrightarrow{p_k} \pi_k\left({}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}\right) \xrightarrow{\partial_k} \pi_{k-1}(\mathrm{O}(n)) \to \ldots$$

This ensures that we are dealing with the correct weak homotopy types.

Proof. We only check the case depicted above. We show that $\mathrm{TOP}(n) \to {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$ is in fact a fiber bundle. According to [Sep07, 1.1.4.1] the orthogonal group $\mathrm{O}(n)$ is a compact Lie group. It acts on the group of homeomorphisms $\mathrm{TOP}(n)$, which is a completely regular space since every topological group is. By [Gle50, Theorem 3.6] or as constructed in [Kar58], we obtain a local cross section of $\mathrm{TOP}(n) \to {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$ as defined in Section A.1. Lemma A.1 guarantees that $\mathrm{TOP}(n) \to {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$ is a fiber bundle with fiber $\mathrm{O}(n)$. As such it induces a long exact sequence as desired. □

Remark 1.11. The orbit or coset spaces of Definition 1.8 above are not canonically endowed with a group structure. However, we have a natural left $\mathrm{O}(n)$-action on ${}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$ via $A \cdot [f]_{\mathrm{O}(n)} = [A \cdot f]_{\mathrm{O}(n)}$. Analogously, we get a natural left $\mathrm{O}(n+1)$-action on ${}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$. Both actions come from restricting the natural actions of $\mathrm{TOP}(n)$ and $\mathrm{TOP}(S^n)$, respectively.

We relate these left actions as follows.

Proposition 1.12. *The stereographic inclusion $\mathbb{R}^n \to S^n$, $0 \mapsto 1 = (1, 0, \ldots, 0) \in S^n$ of $\mathbb{R}^n$ into its one-point compactification S^n induces an inclusion $(-)^c : \mathrm{TOP}(n) \to \mathrm{TOP}(S^n)$. This in turn induces a homeomorphism*

$${}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)} \longrightarrow {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$$

which is $\mathrm{O}(n)$-equivariant. For similar reasons, ${}^{\mathrm{TOP}_*(S^n)}\big/_{\mathrm{O}(n)} \to {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ is a homeomorphism.

Note that we use the point 1 as basepoint in S^n, as opposed to the choice -1 in S^n coming from ∞ outside $\mathbb{R}^n$.

Example 1.13. According to [Hat83, Appendix (2)], ${}^{\mathrm{TOP}(3)}\big/_{\mathrm{O}(3)} \simeq *$. Using Proposition 1.12, we have $\mathrm{TOP}\,(S^3) \simeq \mathrm{O}\,(4)$, which was left to the reader in [Hat83].

Example 1.14. The natural left $\mathrm{TOP}\,(S^n)$-action on ${}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ restricts to a $\mathrm{TOP}\,(S^n)$-action on ${}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$, and extends the given $\mathrm{TOP}\,(n)$-action.

Proof of Proposition 1.12. Exceptionally in this proof, we consider the non-standard inclusion $\mathrm{O}\,(n) \subset \mathrm{O}\,(n+1)$ by

$$A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

This way the natural $\mathrm{O}\,(n+1)$-action on S^n extends the $\mathrm{O}\,(n)$-action on $\mathbb{R}^n \subset S^n$. Consequently the inclusion $\mathrm{incl} : \mathrm{TOP}\,(n) \to \mathrm{TOP}\,(S^n)$ descends to a map

$$i : {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)} \longrightarrow {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \twoheadrightarrow {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}.$$

Then i is $\mathrm{O}\,(n)$-equivariant, since the $\mathrm{O}\,(n)$-action on S^n is compatible with the one on $\mathbb{R}^n \subset S^n$ in our setting. It remains to construct an inverse map $j : {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \longrightarrow {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$. Let $[f] \in {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$. Choose an $A_f \in \mathrm{O}\,(n+1)$ such that

$$A_f(\infty) = f^{-1}(\infty). \tag{1.2}$$

Here $\infty \in S^n$ is the only point $(-1, 0, \ldots, 0) \in S^n \setminus \mathbb{R}^n$, opposite the basepoint $1 \in S^n$. Then we have $[f] = [f \circ A_f] \in {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ and $f \circ A_f$ fixes ∞, it is the extension of $f \circ A_f|_{\mathbb{R}^n} \in \mathrm{TOP}\,(n)$. We define $j([f]) := [f \circ A_f]$. This is independent of the choice of A_f since condition (1.2) reduces the ambiguity to $\mathrm{O}\,(n) \subset \mathrm{O}\,(n+1)$. The choice of $[A_f] \in \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)}$, which corresponds to $A_f(\infty) = f^{-1}(\infty) \in S^n$ via $\frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} \cong S^n$, depends continuously on f, whence making j continuous. It is not hard to see that i and j are mutual inverses. Proceed analogously for the second inclusion using (1.2) modified to $A_f(1) = f^{-1}(1)$. $\square$

1.1.2. Connectivity of the stable inclusions

In this section we study the connectivity of some of the inclusion maps. This allows to conclude on the connectivity of the stable inclusions. As always, $n \geq 5$.

Lemma 1.15 ([BL74, Theorem 5.1])**.** *The inclusion-induced map*

$${}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)} \to {}^{\mathrm{TOP}(n+1)}\big/_{\mathrm{O}(n+1)}$$

is $(n+3)$-connected.

Lemma 1.16. *The following inclusion-induced maps are $(2n-3)$-connected,*

$${}^{\mathrm{O}(n+1)}\big/_{\mathrm{O}(n)} \to {}^{\mathrm{G}(n+1)}\big/_{\mathrm{G}(n)}, \qquad {}^{\mathrm{G}(n)}\big/_{\mathrm{O}(n)} \to {}^{\mathrm{G}(n+1)}\big/_{\mathrm{O}(n+1)}.$$

Proof. For the first map this is due to [BL74, Proposition 5.3 (6)]. For the second, consider the diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & {}^{\mathrm{G}(n)}/_{\mathrm{O}(n)} \arrow[r] \arrow[d] & {}^{\mathrm{G}(n+1)}/_{\mathrm{O}(n+1)} \arrow[d] \\
{}^{\mathrm{O}(n+1)}/_{\mathrm{O}(n)} \arrow[r] \arrow[d] & \mathrm{BO}\,(n) \arrow[r] \arrow[d] & \mathrm{BO}\,(n+1) \arrow[d] \\
{}^{\mathrm{G}(n+1)}/_{\mathrm{G}(n)} \arrow[r] & \mathrm{BG}\,(n) \arrow[r] & \mathrm{BG}\,(n+1),
\end{tikzcd}$$

where T denotes the homotopy fiber of the upper horizontal map. It is also referred to as the total homotopy fiber of the bottom right square diagram. Since all of the quotient spaces are defined by taking the correct homotopy fibers, this diagram commutes. Also, it induces a long lattice on homotopy groups which is exact in its rows and columns. In particular, [BL74, Proposition 5.3 (6)] implies the vanishing of $\pi_i(T)$ for $i \leq 2n-4$. Thus the map in question is $(2n-3)$-connected. $\square$

Lemma 1.17. *The following inclusion-induced maps are* $(n+2)$*-connected,*

$$ {}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)} \to {}^{\mathrm{G}(n+1)}/_{\mathrm{TOP}(n+1)}, \qquad {}^{\mathrm{TOP}(n+1)}/_{\mathrm{TOP}(n)} \to {}^{\mathrm{G}(n+1)}/_{\mathrm{G}(n)}. $$

Proof. Consider the diagram made up of horizontal fiber sequences,

$$\begin{tikzcd}
{}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)} \arrow[r] \arrow[d, "f"] & {}^{\mathrm{G}(n)}/_{\mathrm{O}(n)} \arrow[r] \arrow[d, "g"] & {}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)} \arrow[d, "h"] \\
{}^{\mathrm{TOP}(n+1)}/_{\mathrm{O}(n+1)} \arrow[r] & {}^{\mathrm{G}(n+1)}/_{\mathrm{O}(n+1)} \arrow[r] & {}^{\mathrm{G}(n+1)}/_{\mathrm{TOP}(n+1)}.
\end{tikzcd}$$

For $n \geq 5$, the map f is $(n+3)$-connected by Lemma 1.15, and g is $(2n-3)$-connected by Lemma 1.16. The 5-Lemma on the long exact sequence on homotopy groups implies that h is at least $(n+2)$-connected. The connectivity statement for the second map is derived analogously as in the proof of Lemma 1.16. $\square$

1.2. Pseudo smooth structures on spheres

In this section we develop some basic statements about smooth structures on spheres. We always assume $n \geq 5$.

Definition 1.18. Smooth atlases on the given topological sphere S^n are said to be *compatible* or *equivalent* if the combined atlas is smooth. A *maximal atlas* is the union of all atlases in an equivalence class. A *smooth structure* on S^n is a maximal atlas.

We wish to forgo this definition and introduce a more convenient equivalent model. In order to tell the new definition apart from the standard one, we will refer to these objects as pseudo smooth structures. Recall the stereographic identification $\mathbb{R}^n \cup \infty \to S^n$ sending $0 \mapsto 1$ and $\infty \mapsto -1$. Denote by $\mathrm{ev}_1 : \mathrm{TOP}\,(S^n) \to S^n$ the evaluation map at the basepoint $1 \in S^n$. See Section A.1 for details.

Motivation 1.19. Smoothing theory as introduced by [Mor69a] and [KS77] explains that smooth structures can be given by local data on a topological tangent bundle. Roughly, a smooth structure is locally determined by putting a vector space structure on the topological tangent space $\mathcal{T}_p S^n$ at all $p \in S^n$. In other words, one must specify a pointed homeomorphism $\mathbb{R}^n \to \mathcal{T}_p S^n$ modulo $\mathrm{O}(n)$. Thus globally a smooth structure on S^n is determined by a section of the bundle over S^n with fiber ${}^{\mathrm{TOP}_*(\mathbb{R}^n, \mathcal{T}_p S^n)}\big/_{\mathrm{O}(n)}$. Using one-point compactification yields a canonical identification

$$\begin{aligned} {}^{\mathrm{TOP}_*(\mathbb{R}^n,\mathcal{T}_pS^n)}\big/_{\mathrm{O}(n)} &\simeq {}^{\mathrm{TOP}_*(\mathbb{R}^n\cup\infty,\mathcal{T}_pS^n\cup\infty)}\big/_{\mathrm{O}(n)} \\ &\cong {}^{\mathrm{TOP}_*(S^n)}\big/_{\mathrm{O}(n)} \cong {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \cong {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)} \end{aligned} \tag{1.3}$$

for each of the fibers, using also Proposition 1.12. Seen this way, a smooth structure on S^n is given simply by a map $S^n \to {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}$ to the fiber of the trivial bundle. Since $[f] \in {}^{\mathrm{TOP}_*(\mathbb{R}^n\cup\infty,\mathcal{T}_pS^n\cup\infty)}\big/_{\mathrm{O}(n)}$ may not fix ∞, the first identification is only a strong deformation retract, induced from a deformation retract $S^n \setminus 1 \simeq \infty$.
The fiber ${}^{\mathrm{TOP}_*(S^n)}\big/_{\mathrm{O}(n)}$ over $p \in S^n$ consists of the maps sending $1 \mapsto p$. All fibers include disjointly into and make up ${}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)}$. The bundle projection is given by $\mathrm{ev}_1 : {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \to S^n$. Thus one can think of a smooth structure as a section $\Sigma : S^n \to {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)}$, satisfying the well-defined condition $\Sigma(p)(1) = p$. Proposition 1.21 shows that the bundle with fibers (1.3) is actually trivial.

Definition 1.20. A section Σ to the bundle

$$\mathrm{ev}_1 : {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \to S^n$$

is called a **pointed pseudo smooth structure** if $\Sigma(1) = [\mathrm{id}] \in {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)}$. The space of pointed sections to this bundle is denoted by $\mathfrak{S}(S^n)$ and called the *space of pseudo smooth structures* on S^n. It is endowed with the compact-open topology.

An *unpointed pseudo smooth structure* is defined correspondingly. For technical reasons all pseudo smooth structures in this thesis are required to be pointed.

Proposition 1.21. *The fiber bundle* $\mathrm{ev}_1 : {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \to S^n$ *is a trivial bundle with fiber* ${}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$. *A trivializing homeomorphism is given by*

$$\mathrm{ev}_1 \times \mathrm{proj} : {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \xrightarrow{\cong} S^n \times {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}. \tag{1.4}$$

Proof. Consider the ladder with horizontal fiber sequences, compare Lemma A.5,

$$\begin{tikzcd} \mathrm{O}(n) \arrow[r] \arrow[d] & \mathrm{TOP}(S^n) \arrow[r, "\mathrm{proj}"] \arrow[d] & {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \arrow[d, "\mathrm{ev}_1 \times \mathrm{proj}"] \\ \mathrm{O}(n) \arrow[r] & \mathrm{TOP}(S^n) \arrow[r, "\mathrm{ev}_1 \times \mathrm{proj}"] & S^n \times {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}. \end{tikzcd} \tag{1.5}$$

It is straight forward to check that it commutes. Thus we obtain a homotopy equivalence of the two right-hand spaces. Additionally, we check injectivity. Let

$[f], [g] \in \mathrm{TOP}(S^n)/\mathrm{O}(n)$, equal under $\mathrm{ev}_1 \times \mathrm{proj}$. Then $f = g \circ A$ with $A \in \mathrm{O}(n+1)$. Since $f(1) = g(1)$, we have $A(1) = 1$ and thus $A \in \mathrm{O}(n)$. To see surjectivity, let $(p, [f]) \in S^n \times \mathrm{TOP}(S^n)/\mathrm{O}(n+1)$. Lift $f^{-1}(p) \in S^n$ to A along the fibration $\mathrm{ev}_1 : \mathrm{O}(n+1) \to S^n$. Then $[f \circ A]_{\mathrm{O}(n)}$ is a preimage as desired. $\square$

Notation 1.22. In view of Proposition 1.21 we may not always distinguish between the space $\mathfrak{S}(S^n)$ and the space of based maps $\mathrm{map}_*\left(S^n, \mathrm{TOP}(S^n)/\mathrm{O}(n+1)\right)$ to the fiber $\mathrm{TOP}(S^n)/\mathrm{O}(n+1)$ of the trivial bundle (1.4). In view of Proposition 1.12, we may also sometimes drop the distinction between $\mathrm{map}_*\left(S^n, \mathrm{TOP}(n)/\mathrm{O}(n)\right)$ and the space $\mathrm{map}_*\left(S^n, \mathrm{TOP}(S^n)/\mathrm{O}(n+1)\right)$, all endowed with the compact-open topology.
Throughout, S^n_Σ denotes the smooth manifold obtained by endowing the topological sphere S^n with the smooth structure Σ.

Motivation 1.23. Consider the fiber sequence $\mathrm{TOP}(S^n)/\mathrm{O}(n) \to \mathrm{BO}(n) \to \mathrm{BTOP}(S^n)$. In the sense of Definition 1.5, a map $S^n \to \mathrm{TOP}(S^n)/\mathrm{O}(n)$ classifies a vector bundle over S^n with preferred trivialization of the induced, one-point compactified S^n-bundle. Each image point $[f] \in \mathrm{TOP}(S^n)/\mathrm{O}(n)$ determines a vector space $f(\mathbb{R}^n)$ with vector space structure pushed from $\mathbb{R}^n$ via f. Here $\mathbb{R}^n = S^n \setminus -1$.

At a point $p \in S^n$, the pseudo tangent space is given as the image $T_pS^n_\Sigma := \Sigma(p)(S^n \setminus -1)$, with vector space structure pushed forward from $\mathbb{R}^n \cong S^n \setminus -1$ via $\Sigma(p)$.

Definition 1.24. Let $\Sigma : S^n \to \mathrm{TOP}(S^n)/\mathrm{O}(n)$ be a section denoting a pseudo smooth structure. Then we call the map $\iota \circ \Sigma : S^n \to \mathrm{TOP}(S^n)/\mathrm{O}(n) \to \mathrm{BO}(n)$ the classifying map for the **pseudo tangent bundle** of S^n_Σ. Write $TS^n_\Sigma := \bigcup_{p \in S^n} T_pS^n_\Sigma$ for the pseudo tangent bundle.

Definition 1.25. Let $\Sigma : S^n \to \mathrm{TOP}(S^n)/\mathrm{O}(n)$ denote a pseudo smooth structure. Let $f : S^n \to S^n$ be a homeomorphism. We define a **pseudo diffeomorphism** $f : S^n_\Sigma \to S^n_\Sigma$ as follows. It is a pair (f, H) with f being the underlying homeomorphism and H being a homotopy making (1.6) commute up to homotopy.

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "H"] & S^n \arrow[d, "\Sigma"] \\ \mathrm{TOP}(S^n)/\mathrm{O}(n) \arrow[r, "f_*"] & \mathrm{TOP}(S^n)/\mathrm{O}(n) \end{tikzcd} \tag{1.6}$$

Here f_* denotes the induced map $\mathrm{TOP}(S^n)/\mathrm{O}(n) \to \mathrm{TOP}(S^n)/\mathrm{O}(n)$ given by composition with f. We require H to respect the map f in the sense $\mathrm{ev}_1 \circ H = f$ for all $t \in I$. The composition involving concatenation of homotopies turns the set $\mathrm{Diff}(S^n_\Sigma)$ into the **grouplike topological monoid of pseudo diffeomorphisms**. It is endowed with the compact-open topology, i. e. subspace topology of a product of mapping spaces.

Note that the diffeomorphisms are not required to be pointed maps. We sometimes also project H to the fiber ${}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ and obtain a condition analogous to (1.6). In that case there is no additional ev_1-condition on H.

Remark 1.26 (Details on the composition)**.** Pseudo diffeomorphisms (f, H) and (f', H') can be composed as follows. The maps f and f' are composed as usual. The homotopies are composed accordingly, meaning that $H' \circ H := (H' \circ f) * (f'_* \circ H)$, where $*$ denotes concatenation of homotopies, and $(f'_* \circ H)$ is to be understood as composition $(f'_* \circ H(-,t))$ for each t. This composition has the identity element given by $\mathrm{id}_{S^n_\Sigma} = (\mathrm{id}_{S^n}, \text{stationary})$. It is associative if we use Moore composition of homotopies. However, there are no inverse elements in general. An inverse up to homotopy is given by (f^{-1}, H^{-1}). Here H^{-1} is short for $\overline{f_*^{-1} \circ H \circ f^{-1}}$, where the bar denotes reverse direction. It is not difficult to check that there are canonical homotopies $(f, H) \circ (f^{-1}, H^{-1}) \simeq \mathrm{id}_{S^n_\Sigma} \simeq (f^{-1}, H^{-1}) \circ (f, H)$, with the homotopies being homotopic rel ∂I to the stationary homotopy.

Notation 1.27. A pseudo smooth structure $\Sigma \in \mathfrak{S}(S^n)$ is called **exotic** if it is not pseudo diffeomorphic to the standard Euclidean pseudo smooth structure on S^n. The latter is given by the canonical section of $\mathrm{ev}_1 : {}^{\mathrm{O}(n+1)}\big/_{\mathrm{O}(n)} \to S^n$, i. e. it takes values in ${}^{\mathrm{O}(n+1)}\big/_{\mathrm{O}(n)} \subset {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)}$.

Remark 1.28. There is a forgetful map of topological monoids

$$\begin{aligned} \mathrm{Diff}\,(S^n_\Sigma) &\to \mathrm{TOP}\,(S^n) \\ (f, H) &\mapsto f. \end{aligned}$$

Example 1.29. For any unpointed pseudo smooth structure Σ and $f \in \mathrm{TOP}\,(S^n)$ one can define its *pushforward* $f_*\Sigma$ as $f_* \circ \Sigma \circ f^{-1}$. This induces a canonical $\mathrm{TOP}\,(S^n)$-action on the space of unpointed pseudo smooth structures, which in particular recovers the canonical $\mathrm{TOP}\,(S^n)$-action on $\mathfrak{S}_{\text{classical}}(S^n)$, the space of classical smooth structures. In other words, $(f, \text{const}) : S^n_\Sigma \to S^n_{f_*\Sigma}$ is a pseudo diffeomorphism.

Note that the pushforward only works for unpointed pseudo smooth structures. For pointed pseudo smooth structures, one could only allow $f \in \mathrm{TOP}_*(S^n)$. Note also the fundamental difference between the simple composition $f_* \circ \Sigma$ as illustrated by the left lower composition in (1.6), and the construction $f_*\Sigma$. Only the former is used in this thesis. We justify the definition of pseudo smooth structures.

Remark 1.30 ([Mor69b], [KS77, Essay V, p. 240])**.** The canonical map

$$\mathfrak{S}_{\text{classical}}(S^n) \xrightarrow{\simeq} \left\{ \text{Lifts} \quad \begin{tikzcd} & \mathrm{BO}\,(n) \arrow[d] \\ S^n \arrow[ur, dashed] \arrow[r, "\tau"] & \mathrm{BTOP}\,(n) \end{tikzcd} \right\}$$

is a homotopy equivalence and induces $\mathfrak{S}_{\text{classical}}(S^n) \simeq \Omega^n \left({}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}\right)$. Consequently,

$$\mathfrak{S}_{\text{classical}}(S^n) \simeq \mathrm{map}_*\left(S^n, {}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}\right) \cong \mathrm{map}_*\left(S^n, {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}\right) \cong \mathfrak{S}(S^n)$$

are linked using Proposition 1.12 and Proposition 1.21.

Following [KS77], we may use the fibration sequence $\mathrm{Diff}_\partial(D^n) \to \mathrm{Homeo}_\partial(D^n) \to \mathfrak{S}_{\text{classical}}(S^n)$. Knowing that the middle term is contractible [KS77], one obtains

Remark 1.31 (Morlet's Theorem [KS77, Essay V, Theorem 3.4])**.** There is a homotopy equivalence

$$\mathrm{Diff}_\partial(D^n) \simeq \Omega^{n+1}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right).$$

Again we may link

$$\mathrm{Diff}_\partial(D^n) \simeq \mathrm{map}_*\left(S^{n+1}, {}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right) \cong \mathrm{map}_*\left(S^{n+1}, {}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)}\right)$$

using Proposition 1.12. We use this to motivate Definition 1.25. Consider the case of (1.6) with a homeomorphism $f : D^n \to D^n$ fixing the boundary. As seen above, $f \simeq \mathrm{id}_{D^n}$. Any homotopy H for (1.6) then extends via $\Sigma \simeq f_* \circ \Sigma \simeq_H \Sigma \circ f \simeq \Sigma$ to a loop in $\mathfrak{S}(S^n)$. In other words, it is a loop in $\mathrm{map}_*\left(S^n, {}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)}\right)$, or an element in $\mathrm{map}_*\left(S^{n+1}, {}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)}\right)$.

Let us finish this section with a result on tangent bundles of spheres.

Lemma 1.32 ([RP80, Lemma 1.1])**.** *Consider the standard smooth sphere S^n and some exotic sphere S^n_Σ. Their tangent bundles are isomorphic, $TS^n \cong TS^n_\Sigma$.*

1.3. Exotic spheres and the surgery exact sequence

Notation 1.33. Let R denote an associative ring with unit and involution. Then denote by $L_*(R)$ the *quadratic L-theory groups of* R as defined in [Lüc02, Definition 4.42]. For a group π or a topological space X one can consider the associated L-theory groups $L_*(\mathbb{Z}[\pi])$ and $L_*(\mathbb{Z}[\pi_1(X)])$. The quadratic L-theory groups are the homotopy groups of the underlying *quadratic L-theory spectrum* $\mathsf{L}_\bullet(R)$ [Ran92, page 8],

$$L_n(R) = \pi_n(\mathsf{L}_\bullet(R)).$$

The L-theory spectrum can also be generalized for ring spectra with involution as input, see [Ran92]. The exact definition of $L_*(R)$ is rather lengthy and not relevant to the present thesis. One can think of quadratic L-theory of R as a bordism theory of free left modules over R on specific generators, equipped with non-singular quadratic forms. There is an Ω-spectrum $\mathsf{L}^\bullet(\mathbb{Z})$ whose homotopy groups are the symmetric L-theory groups. Symmetric L-theory analogously employs symmetric forms. Moreover, the connective symmetric L-theory spectrum is a ring spectrum, and the quadratic L-theory spectrum is a module spectrum over this ring spectrum. Compare [Ran92, page 12]. Note that for a connective ring spectrum R with involution we have invariance of the corresponding quadratic L-theory spectra, $\mathsf{L}_\bullet(\mathsf{R}) \simeq \mathsf{L}_\bullet(\pi_0(\mathsf{R}))$. Consequently, we obtain

Lemma 1.34 ([Ran92, page 8])**.** *The space ${}^{\mathrm{G}}/_{\mathrm{TOP}}$ relates to the L-theory spectra via*

$${}^{\mathrm{G}}/_{\mathrm{TOP}} \simeq \Omega^\infty_0 \mathsf{L}_\bullet(\mathbb{Z}) \simeq \Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}).$$

Here Ω_0^∞ denotes the Ω^∞ functor combined with taking the basepoint component. Equivalently, this amounts to taking the infinite loop space Ω^∞ of the 1-connective cover of $\mathsf{L}_\bullet(\mathbb{Z})$. Denote by S the sphere spectrum.

Definition 1.35 (Kervaire-Milnor [KM63])**.** Denote by θ_n the set of oriented h-cobordism classes of oriented homotopy n-spheres. By [KM63, Theorem 1.1], θ_n is an abelian group under the connected sum operation.

See [Lüc02, Chapter 6.1] for details on h-cobordism classes.

Remark 1.36. By [Sma61, Theorem A], all topological homotopy n-spheres, $n \geq 5$, are homeomorphic to S^n. In other words, the generalized topological Poincaré conjecture holds in these dimensions.

Lemma 1.37 ([Lüc02, Lemma 6.2])**.** *For $n \geq 5$, the set θ_n is naturally bijective to the set of oriented diffeomorphism classes of oriented homotopy n-spheres.*

This follows from the s-cobordism theorem. We recall the adapted surgery exact sequence and link it to our needs in Lemma 1.39 below.

Lemma 1.38 ([Lüc02, Theorem 6.11])**.** *The surgery exact sequence [Ran92] can be adapted to a surgery exact sequence for homotopy spheres as follows. The following long sequence of abelian groups which extends infinitely to the left is exact,*

$$\cdots \to \Omega_{k+1}^{\mathrm{alm}} \to L_{k+1}(\mathbb{Z}) \xrightarrow{\partial} \theta_k \to \Omega_k^{\mathrm{alm}} \to \cdots \to \Omega_5^{\mathrm{alm}} \to L_5(\mathbb{Z}).$$

Here Ω_k^{alm} [Lüc02, Definition 6.8] denotes the abelian group of almost stably framed bordism classes of almost stably framed closed oriented manifolds of dimension k. It suffices for now that it is linked to ${}^{\mathrm{G}}/_{\mathrm{O}}$ as follows.

Lemma 1.39. *The exact sequence of Lemma 1.38 above is equivalent to the long exact sequence on homotopy groups obtained from the fiber sequence ${}^{\mathrm{TOP}}/_{\mathrm{O}} \to {}^{\mathrm{G}}/_{\mathrm{O}} \to {}^{\mathrm{G}}/_{\mathrm{TOP}}$, i. e. we have*

$$\begin{tikzcd} \cdots \arrow[r] & \pi_{n+1}({}^{\mathrm{G}}/_{\mathrm{O}}) \arrow[r] \arrow[d, "\cong"'] & \pi_{n+1}({}^{\mathrm{G}}/_{\mathrm{TOP}}) \arrow[r] \arrow[d, "\cong"'] & \pi_n({}^{\mathrm{TOP}}/_{\mathrm{O}}) \arrow[r] \arrow[d, "\cong"'] & \pi_n({}^{\mathrm{G}}/_{\mathrm{O}}) \arrow[r] \arrow[d, "\cong"'] & \cdots \\ \cdots \arrow[r] & \Omega_{n+1}^{\mathrm{alm}} \arrow[r] & L_{n+1}(\mathbb{Z}) \arrow[r] & \theta_n \arrow[r] & \Omega_n^{\mathrm{alm}} \arrow[r] & \cdots \end{tikzcd}$$

Proof. According to [KS77, Essay V, Theorem 5.5 I], the space ${}^{\mathrm{TOP}}/_{\mathrm{PL}}$ is of type $K(\mathbb{Z}_2, 3)$. Using this, [Lüc02, Theorem 6.48] turns into a statement about the TOP-case. Comparing the appropriate arrows yields the result. □

Lemma 1.40. *Consider the long exact sequence on homotopy groups associated to the fibration sequence ${}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)} \to {}^{\mathrm{G}(n)}/_{\mathrm{O}(n)} \to {}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$. It relates to the fiber* *sequence* ${}^{\mathrm{TOP}}/_{\mathrm{O}} \to {}^{\mathrm{G}}/_{\mathrm{O}} \to {}^{\mathrm{G}}/_{\mathrm{TOP}}$ *by inclusion. We obtain a commutative diagram:*

$$\begin{tikzcd}
\pi_{n+1}({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}) \arrow[r] \arrow[d, hook, "\cong"'] & \pi_n({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}) \arrow[d, hook, "\cong"'] \\
\pi_{n+1}({}^{\mathrm{G}}/_{\mathrm{TOP}}) \arrow[r] \arrow[d, "\cong"'] & \pi_n({}^{\mathrm{TOP}}/_{\mathrm{O}}) \arrow[d, "\cong"'] \\
\pi_{n+1}(\mathsf{L}_\bullet(\mathbb{Z})) \arrow[r] & \theta_n
\end{tikzcd}$$

The left lower vertical map is induced by ${}^{\mathrm{G}}/_{\mathrm{TOP}} \simeq \Omega_0^\infty \mathsf{L}_\bullet(\mathbb{Z})$ *from Lemma 1.34.*

Proof. We need only worry about the upper square. It is clear that it commutes. The vertical inclusion induced maps are isomorphisms by Lemma 1.15 and Lemma 1.17. $\square$

At this point we give a quick overview of the L-groups of $\mathbb{Z}$.

Lemma 1.41. *The quadratic and symmetric L-groups of $\mathbb{Z}$ are 4-periodic and are given as follows, compare [Ran92, page 8] and [Ran92, page 12].*

$$L_k(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \equiv 0 \text{ mod } 4 \\ \mathbb{Z}_2 & \text{if } k \equiv 2 \text{ mod } 4 \\ 0 & \text{otherwise} \end{cases} \qquad L^k(\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k \equiv 0 \text{ mod } 4 \\ \mathbb{Z}_2 & \text{if } k \equiv 1 \text{ mod } 4 \\ 0 & \text{otherwise.} \end{cases}$$

The quadratic L-groups are given by the signature and the Arf-invariant of quadratic forms

$$\frac{1}{8} \cdot \mathrm{sign} : L_0(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z} \qquad \mathrm{Arf} : L_2(\mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}_2,$$

compare [Lüc02, Theorem 4.30 and 4.32]. There is a symmetrization map

$$\mathsf{L}_\bullet(\mathbb{Z}) \xrightarrow{\mathrm{sym}} \mathsf{L}^\bullet(\mathbb{Z}),$$

which induces multiplication by 8 *on* π_{4k}.

The latter can be deduced from the long exact sequence including $\hat{L}^*$ and knowledge of its values, compare [Ran92, page 12 and 13]. We state two results for $n = 7$.

Lemma 1.42. *The boundary map* $L_8(\mathbb{Z}) \to \theta_7$ *is surjective.*

Proof. We show that $\Omega_7^{\mathrm{alm}} = 0$, then Lemma 1.38 finishes the proof. According to [Lüc02, Lemma 6.16], there is a long exact sequence of the form

$$\cdots \to \pi_k(\mathrm{SO}) \to \Omega_k^{\mathrm{fr}} \to \Omega_k^{\mathrm{alm}} \to \pi_{k-1}(\mathrm{SO}) \to \Omega_{k-1}^{\mathrm{fr}} \to \dots$$

where Ω_k^{fr} [Lüc02, Definition 6.8] denotes the abelian group of stably framed bordism classes of stably framed closed oriented manifolds of dimension k. By [Lüc02, p. 132], $\pi_6(\mathrm{SO}) = \pi_6(\mathrm{O}) = 0$. By [Lüc02, Lemma 6.24 and 6.26], the map $\pi_7(\mathrm{SO}) \to \Omega_7^{\mathrm{fr}}$ is surjective. Therefore, $\Omega_7^{\mathrm{alm}} = 0$. $\square$

In particular, we also obtain the known result $\pi_7({}^{\mathrm{G}}/_{\mathrm{O}}) = 0$.

Corollary 1.43. *Any exotic pseudo smooth structure* $\Sigma \in \pi_7({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)})$ *lifts along the boundary map* $\partial : \pi_8({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}) \to \pi_7({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)})$.

1.4. Multiplicative structures on the sphere

Let $\Sigma \in \mathfrak{S}(S^n)$ be a pseudo smooth structure on S^n. Consider the evaluation maps

$$\mathrm{ev}_1 : \mathrm{TOP}(S^n) \longrightarrow S^n, \qquad\qquad \mathrm{ev}_1 : \mathrm{Diff}(S^n_\Sigma) \longrightarrow S^n,$$

at the basepoint $1 \in S^n$. According to Lemma A.2 and Corollary A.4, these evaluation maps are fibrations. Let $\delta : S^n \to S^n$ with $\delta(1) = 1$ denote an arbitrary, yet fixed basepoint preserving self-map of S^n of degree $d \geq 1$. We are interested in homotopy lifts s and even more in homotopy lifts $\overline{s}$ as indicated by

$$\begin{tikzcd}
 & & \mathrm{Diff}(S^n_\Sigma) \arrow[d, "\text{forget}"] \arrow[dd, bend left, "\mathrm{ev}_1"] \\
 & & \mathrm{TOP}(S^n) \arrow[d, two heads, "\mathrm{ev}_1"] \\
S^n \arrow[rruu, dashed, "\overline{s}"] \arrow[rru, dashed, "s"] \arrow[rr, "\delta"] & & S^n
\end{tikzcd} \tag{1.7}$$

Note that any homotopy lift s or $\overline{s}$ is homotopic to a strict lift of δ using the fibration property of the evaluation maps. In this section we will briefly establish some general properties of such lifts.

Definition 1.44. A **(k,l)-bidegree (non-unital) multiplicative structure** μ on S^n is a continuous map $\mu : S^n \times S^n \to S^n$ together with a distinguished basepoint $* \in S^n$ such that $\mu(-,*)$ is a degree k map and $\mu(*,-)$ is a degree l map.

Note that a $(1,1)$-bidegree multiplicative structure is simply an H-space structure. We will only be interested in bidegrees $(k,1)$. We always use $* = 1 \in S^n$ as basepoint.

Lemma 1.45. *Let $s : S^n \to \mathrm{TOP}(S^n)$ be a homotopy lift as in (1.7), i. e. s satisfies $\mathrm{ev}_1 \circ s \simeq \delta$. Assume that $s(1) \in \mathrm{TOP}_0(S^n)$ lies inside the identity component of $\mathrm{TOP}(S^n)$. Then the adjoint map of s induces a $(d,1)$-bidegree multiplicative structure μ on S^n via*

$$\begin{aligned} \mu : S^n \times S^n &\to S^n \\ (p,q) &\mapsto s(p)(q). \end{aligned}$$

Proof. We must show the properties in Definition 1.44 with $1 \in S^n$. For the left unit, we have $\mu(1,-) = s(1) \simeq \mathrm{id}_{S^n}$. For the right unit, $\mu(-,1) = s(-)(1) = \mathrm{ev}_1 \circ s \simeq \delta$. □

Lemma 1.46. *Let $s_1, s_2 \in \pi_n(\mathrm{TOP}(S^n))$, i. e. the homotopy classes of two maps $s_1, s_2 : S^n \to \mathrm{TOP}(S^n)$ with $s_1(1) = s_2(1) = \mathrm{id}_{S^n} \in \mathrm{TOP}_0(S^n)$. Assume that their adjoints are of $(k_1, 1)$ and $(k_2, 1)$-bidegree. Then the adjoint of $s_1 + s_2$ is of bidegree $(k_1 + k_2, 1)$.*

Proof. Recall the addition of elements $s_1, s_2 \in \pi_n(\mathrm{TOP}(S^n))$ is given by the class

$$S^n \xrightarrow{\text{pinch}} S^n \vee S^n \xrightarrow{s_1 \vee s_2} S^n.$$

Consider the mapping degree $\deg : \pi_n(S^n) \to \mathbb{Z}$, $[g] \mapsto \deg([g])$. This map is a ring isomorphism from $([S^n, S^n], +, \circ)$ to $\mathbb{Z}$. We know already that the (left) degrees of s_1 and s_2 are given by $k_1 = \deg(\mathrm{ev}_1 \circ s_1)$ and $k_2 = \deg(\mathrm{ev}_1 \circ s_2)$. We wish to compute the left degree of $s_1 + s_2$. The right degree always remains 1 because this is fixed by the component of $\mathrm{TOP}\,(S^n)$ that both s_1 and s_2 map to. Compute

$$\begin{aligned}\deg(\mathrm{ev}_1 \circ(s_1+s_2)) &= \deg(\mathrm{ev}_1 \circ(s_1 \vee s_2) \circ \mathrm{pinch}) = \deg((\mathrm{ev}_1 \circ s_1 \vee \mathrm{ev}_1 \circ s_2) \circ \mathrm{pinch}) \\ &= \deg(\mathrm{ev}_1 \circ s_1 + \mathrm{ev}_1 \circ s_2) = k_1 + k_2. \end{aligned} \qquad \square$$

Example 1.47 (Compare with Lemma 3.8)**.** Consider an odd $n \geq 5$. By Lemma 3.8, there is a map $s : S^n \to \mathrm{SO}\,(n+1) \subset \mathrm{TOP}\,(S^n)$ such that the composition $\mathrm{ev}_1 \circ s : S^n \to S^n$ has degree 2. In other words, s is a lift of a map of degree two as in (1.7). We may assume that s satisfies $s(1) = \mathrm{id}_{S^n}$ by passing to $s'(p) := s(p) \circ s(1)^{-1}$. It is not hard to check that s' is also a strict lift of δ. By Lemma 1.45, its adjoint $s^{\mathrm{ad}} : S^n \times S^n \to S^n$ is a bidegree $(2,1)$-map. In other words, odd spheres admit a bidegree $(2,1)$ multiplicative structure.
We may also precompose s with a degree k map $\rho : S^n \to S^n$ and think of this as multiple of s in the sense $s \circ \rho = \sum_k s$. Its adjoint $(s \circ \rho)^{\mathrm{ad}}$ is then nothing but a bidegree $(2k, 1)$-map $S^n \times S^n \to S^n$ by Lemma 1.46.

Remark 1.48. Consider the special case where δ has degree $d = 1$. Without loss of generality, $\delta = \mathrm{id}_{S^n}$. A homotopy lift s as in (1.7) is then nothing but a homotopy section. By Lemma 1.45, the resulting μ is an actual H-space structure.

Proposition 1.49 (James, [Jam57a, Thm. 1.1] and [Jam57b, Thm. 1.4])**.** *Any H-space structure on S^n, $n \geq 5$, is neither homotopy-associative nor homotopy-commutative.*

Remark 1.50. By [Ada60], for $n \geq 5$, only S^7 is an H-space, and it is neither homotopy-associative nor homotopy-commutative. The only S^n with $n \geq 5$ that admits a trivial tangent bundle is S^7, compare [BM58].

1.5. Homotopy orbits, fixed points and the Tate construction

In this section we define the homotopy fixed point and homotopy orbit spectra, along with the norm map and the Tate spectrum. Recall that in general, given a pointed space X with an action of a finite group G, the *orbit space* is given by the quotient space $X/G = (\{*\}_+ \wedge X)/G =: \{*\}_+ \wedge_G X$, where one uses the diagonal action on $\{*\}_+ \wedge X$. The smash product $\wedge_G$ is sometimes referred to as *balanced smash product*. The definition above is generalized to homotopy orbits by taking a contractible substitute for $\{*\}$ on which G acts freely, namely $\mathrm{E}G$. The *homotopy orbit space* is given by $X_{hG} := \mathrm{E}G_+ \wedge_G X$. This can be generalized further to spectra by using the naive smash product of a space with a spectrum. The case of a spectrum X with involution specializes as below.

Definition 1.51. Define the $\mathbb{Z}_2$-**homotopy orbit spectrum** of X by

$$\mathsf{X}_{h\mathbb{Z}_2} := \mathrm{E}\mathbb{Z}_{2+} \wedge_{\mathbb{Z}_2} \mathsf{X}.$$

We use the infinite-dimensional sphere S^∞ with the antipodal $\mathbb{Z}_2$-action as a model for $\mathrm{E}\mathbb{Z}_2$. The corresponding model for $\mathrm{B}\mathbb{Z}_2$ is the infinite-dimensional real projective space $\mathbb{R}P^\infty$.
We continue to define the homotopy fixed point spectrum. Recall that in general, given a pointed space X with an action of a finite group G, the *fixed point space* is given by the space $\mathrm{map}_G(\{*\}_+, X)$ of basepoint preserving G-equivariant maps. Recall the replacement $\mathrm{E}G$ for $\{*\}$. The *homotopy fixed point space* is given by $X^{hG} := \mathrm{map}_G(\mathrm{E}G_+, X)$. Note that it comes with a canonical map to X. It is given by precomposing a G-map $\mathrm{E}G_+ \to X$ with the inclusion-induced map $\mathrm{E}\{1\}_+ \to \mathrm{E}G_+$ and identifying $\mathrm{map}_{\{1\}}(\mathrm{E}\{1\}_+, X) \cong \mathrm{map}(\{*\}_+, X) \cong X$. Here we use the one-point space $*$ as a model for $\mathrm{E}\{1\}$. We use this to define the *reduced homotopy fixed point space* by

$$X^{rhG} := \mathrm{hofiber}\left[X^{hG} \to X\right].$$

This can be generalized further to spectra by considering the mapping spectra in the category of spectra. The case of a spectrum X with involution specializes as below.

Definition 1.52. Define the $\mathbb{Z}_2$-**homotopy fixed point spectrum** of X and the **reduced** $\mathbb{Z}_2$-**homotopy fixed point spectrum** of X, respectively, by

$$\mathsf{X}^{h\mathbb{Z}_2} := \mathrm{map}_{\mathbb{Z}_2}(\mathrm{E}\mathbb{Z}_{2+}, \mathsf{X}), \qquad \mathsf{X}^{rh\mathbb{Z}_2} := \mathrm{hofiber}\left[\mathsf{X}^{h\mathbb{Z}_2} \to \mathsf{X}\right].$$

Definition 1.53. There is a *norm map* τ_{X} linking the two constructions,

$$\tau_{\mathsf{X}} : \mathsf{X}_{h\mathbb{Z}_2} \longrightarrow \mathsf{X}^{h\mathbb{Z}_2}.$$

A good definition can be found in [ACD89, Section 2]. As it is somewhat involved, we do not reproduce it here. Adem, Cohen and Dwyer [ACD89] go on to define the Tate spectrum $\mathsf{X}^{th\mathbb{Z}_2}$ as the homotopy fiber of the map τ_{X}. However, we adhere to [WW00].

Definition 1.54. Define the $\mathbb{Z}_2$-**Tate spectrum** of X as the homotopy *cofiber*

$$\mathsf{X}^{th\mathbb{Z}_2} := \mathrm{hocofiber}\left[\tau_{\mathsf{X}} : \mathsf{X}_{h\mathbb{Z}_2} \to \mathsf{X}^{h\mathbb{Z}_2}\right]$$

of the norm map $\tau_{\mathsf{X}} : \mathsf{X}_{h\mathbb{Z}_2} \to \mathsf{X}^{h\mathbb{Z}_2}$.

Remark 1.55. The Tate spectrum fits into the norm fiber sequence

$$\mathsf{X}_{h\mathbb{Z}_2} \xrightarrow{\tau_{\mathsf{X}}} \mathsf{X}^{h\mathbb{Z}_2} \to \mathsf{X}^{th\mathbb{Z}_2}.$$

1.6. The real K-theory spectrum

In this section we introduce some facts related to the real connective K-theory spectrum. These include reviewing the associated homotopy fixed point and homotopy orbit spectra, along with the norm map, the Tate spectrum and various other notions. A generalized spectrum-level Atiyah-Segal completion map provides insight into the structure of the homotopy fixed point spectrum, compare Theorem 1.61. A detailed description of the norm map allows an almost complete description of its associated long exact sequence on homotopy groups in Theorem 1.80. The reader should note that the facts presented in this section are relevant exclusively to the arguments employed in the computational attempt in Section 7.

Notation 1.56. Denote by bo the real connective K-theory spectrum, in the following sense. The real topological K-theory of isomorphism classes of $\mathbb{R}$-vector bundles over a space X is represented by homotopy classes of maps $[X, \mathrm{BO} \times \mathbb{Z}]$ if X is paracompact. Bott has shown that $\mathrm{BO} \times \mathbb{Z} \simeq \Omega^8(\mathrm{BO} \times \mathbb{Z})$, meaning $\mathrm{BO} \times \mathbb{Z}$ is in fact an infinite loop space. We denote the associated Ω-spectrum by bo, which some people also denote by ko. In contrast, KO usually stands for the periodic real K-theory spectrum. It comes with negative homotopy as imposed by Bott periodicity, whereas bo does not by definition. A basepoint in $\mathrm{BO} \times \mathbb{Z}$ can be defined via the trivial 0-dimensional vector space. We endow bo with the trivial $\mathbb{Z}_2$-action.

If necessary one can think of bo as being a CW-spectrum, for example by using a CW-replacement. For our purposes the weak notion suffices. Next, Definition 1.52 specializes as follows in the case of a trivial action. The *$\mathbb{Z}_2$-homotopy fixed point spectrum of* bo with respect to the (trivial) $\mathbb{Z}_2$-action can be written as

$$\mathsf{bo}^{h\mathbb{Z}_2} := \mathrm{map}_{\mathbb{Z}_2}(\mathrm{E}\mathbb{Z}_{2+}, \mathsf{bo}) \cong \mathrm{map}(\mathrm{B}\mathbb{Z}_{2+}, \mathsf{bo}).$$

For simplicity we work with a connective version of $\mathsf{bo}^{h\mathbb{Z}_2}$, for example by considering its *connective cover*, and continue to denote it by $\mathsf{bo}^{h\mathbb{Z}_2}$. An explanation can be found in [Sch12, Chapter II.8]. Define the *reduced $\mathbb{Z}_2$-homotopy fixed point spectrum of* bo by

$$\mathsf{bo}^{rh\mathbb{Z}_2} := \mathrm{hofiber}\left[\mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}\right].$$

1.6.1. A spectrum level Atiyah-Segal completion map

Definition 1.57. For a commutative ring R and proper ideal I, define the *I-adic completion $R_I^\wedge$ of R* as the inverse limit $R_I^\wedge := \lim\left[\ldots \to {}^{R}/_{I^3R} \to {}^{R}/_{I^2R} \to {}^{R}/_{IR}\right]$ under the canonical projections ${}^{R}/_{I^kR} \to {}^{R}/_{I^{k-1}R}$. As a specialization, we obtain the *p-adic completion $\mathbb{Z}_p^\wedge$ of* $\mathbb{Z}$ as the inverse limit $\mathbb{Z}_p^\wedge := \lim\left[\ldots \to \mathbb{Z}_{p^3} \to \mathbb{Z}_{p^2} \to \mathbb{Z}_p\right]$ under the system of canonical projections $\mathbb{Z}_{p^j} \to \mathbb{Z}_{p^{j-1}}$.

Notice that there is a canonical inclusion $\mathbb{Z} \to \mathbb{Z}_p^\wedge$. For an introduction to p-adic numbers, see [Gou97]. Note that the p-adic completion of $\mathbb{Z}$ is a localization at p, yet not the universal p-localization in general. It is usually coarser.

Definition 1.58. Define the *real representation ring* $\mathrm{RO}(G)$ of a finite group G as follows. Following [Seg68b], a (finite-dimensional linear) representation of a group G on a (finite-dimensional) $\mathbb{R}$-vector space V is a group homomorphism $\rho : G \to \mathrm{GL}(V)$. The representation ring $\mathrm{RO}(G)$ is then made up of formal differences of isomorphism classes of such representations. Addition on $\mathrm{RO}(G)$ is given by direct sum, and multiplication by tensoring.

Denote by $I_G = IO_G = \ker[\mathrm{RO}(G) \to \mathrm{RO}(1) = \mathbb{Z}]$ the *augmentation ideal* of G. The map sends any irreducible representation to its dimension.

Lemma 1.59. *We have* $\mathrm{RO}(\mathbb{Z}_2) \cong {}^{\mathbb{Z}[x]}\big/_{(x^2-1)}$, *where* $1 \in \mathrm{RO}(\mathbb{Z}_2)$ *corresponds to the trivial representation, and* $x \in \mathrm{RO}(\mathbb{Z}_2)$ *corresponds to the reflection representation. These representations are the only irreducible representations. We deduce an additive isomorphism* $\mathrm{RO}(\mathbb{Z}_2) \cong \mathbb{Z}{<}1{>} \oplus \mathbb{Z}{<}x-1{>}$. *Upon* I*-adic completion of* $\mathrm{RO}(\mathbb{Z}_2)$ *at the augmentation ideal* $I_{\mathbb{Z}_2} = (x-1)$, *one obtains* $\mathrm{RO}(\mathbb{Z}_2)^\wedge \cong \mathbb{Z} \oplus \mathbb{Z}_2^\wedge$ *as groups.*

Proof. We begin by proving $\mathrm{RO}(\mathbb{Z}_2) \cong {}^{\mathbb{Z}[x]}\big/_{(x^2-1)}$. Consider any one-dimensional representation given by $\rho(1) = f$. Since we are only interested in isomorphism classes, we may choose $V = \mathbb{R}$. Then by $\mathbb{Z}_2$-linearity of ρ, we have $f^2 = \mathrm{id}_{\mathbb{R}}$, i.e. f is idempotent. This leaves only the choice between the two representations $\rho_1(1) = \mathrm{id}$ and $\rho_2(1) = \text{reflection}$. It is not hard to see that any higher-dimensional representation splits as a sum of ρ_1's and ρ_2's. Thus we have $\mathrm{RO}(\mathbb{Z}_2) \cong \mathbb{Z}{<}\rho_1{>} \oplus \mathbb{Z}{<}\rho_2{>}$ as groups. Suggestively, we write 1 for ρ_1 and x for ρ_2. Indeed, the tensor product of vector spaces with their corresponding representations behaves as expected. In particular $x^2 = 1$, from which we obtain the ring isomorphism as claimed.
We now perform the I-adic completion of $\mathbb{Z}[x]/(x^2-1)$ with respect to $I_{\mathbb{Z}_2}$. We compute the appropriate kernel $I_{\mathbb{Z}_2} \cong (x-1)$, and compute further that $I^j = 2^{j-1} \cdot (x-1)$. Consider the group isomorphism $\mathrm{RO}(\mathbb{Z}_2) \cong \mathbb{Z}{<}1{>} \oplus \mathbb{Z}{<}x-1{>}$, using $x-1$ as second generator instead of x. Then I-adic completion yields

$$\mathrm{RO}(\mathbb{Z}_2)^\wedge = \lim_j \frac{\mathrm{RO}(\mathbb{Z}_2)}{I^j \mathrm{RO}(\mathbb{Z}_2)} \cong \lim_j \left[\mathbb{Z}{<}1{>} \oplus \frac{\mathbb{Z}{<}x-1{>}}{2^{j-1}{<}x-1{>}}\right] \cong \mathbb{Z} \oplus \mathbb{Z}_2^\wedge. \qquad \square$$

Next we define the p-completion of a spectrum. The more general idea of a localization is as follows. Given a ring R and a spectrum X, one constructs a functor $\mathsf{X} \to \mathsf{X}_R$. The spectrum X_R is called an R-localization of X. It satisfies $\pi_*(\mathsf{X}_R) = \pi_*(\mathsf{X})_R$, i.e. the homotopy groups of X_R are given by localizing the homotopy groups of X. Bousfield and Kan [BK71] describe all details and prerequisites in a very readable fashion. Another classical reference is [Sul74]. Anyhow, we specialize directly to the following case.

Proposition 1.60. *Let* X *denote a CW-spectrum and* p *a prime. There exists a functorial construction [Bou79, Proposition 2.5]* $\mathsf{X} \to \mathsf{X}_p^\wedge$ *with the property*

$$\mathbb{Z}_p^\wedge \otimes \pi_*(\mathsf{X}) \cong \pi_*(\mathsf{X})_p^\wedge \cong \pi_*\left(\mathsf{X}_p^\wedge\right)$$

if the groups $\pi_*(\mathsf{X})$ *are finitely generated.*

We apply this construction to the CW-replacement of bo to obtain $\mathsf{bo}_2^\wedge$. The Atiyah-Segal completion theorem [AS69, Theorem 7.1] for $G = \mathbb{Z}_2$ and a space X with trivial $\mathbb{Z}_2$-action can be generalized to a statement about the underlying spectra as follows. More precisely, we generalize Corollary 1.67, which itself is only implicit in [AS69]. Some details below will become clearer only in the sequel.

Theorem 1.61 (Generalization of an Atiyah-Segal special case)**.** *Recall that* bo *denotes the connective real K-theory spectrum,* $\mathsf{bo}_2^\wedge$ *its 2-adic completion and* $\mathsf{bo}^{h\mathbb{Z}_2}$ *the connective cover of its homotopy fixed points,* $\mathrm{RO}(\mathbb{Z}_2)$ *the real representation ring of* $\mathbb{Z}_2$ *and* $\mathrm{RO}(\mathbb{Z}_2)^\wedge \cong \mathbb{Z} \oplus \mathbb{Z}_2^\wedge$ *its* $I_{\mathbb{Z}_2}$*-adic completion. There is a weak homotopy equivalence of spectra*

$$\mathsf{bo} \vee \mathsf{bo}_2^\wedge \xrightarrow{\sim} \mathsf{bo}^{h\mathbb{Z}_2} \tag{1.8}$$

which induces the Atiyah-Segal completion isomorphism on its homotopy groups,

$$\pi_*\left(\mathsf{bo} \vee \mathsf{bo}_2^\wedge\right) \cong \mathrm{RO}(\mathbb{Z}_2)^\wedge \otimes \mathrm{KO}^*(*) \xrightarrow{\cong} \mathrm{KO}^*(\mathrm{B}\mathbb{Z}_2) \cong \pi_*\left(\mathsf{bo}^{h\mathbb{Z}_2}\right). \tag{1.9}$$

Remark 1.73 provides a basic insight into the nature of (1.8). The implications of (1.9) on homotopy groups are explored in Corollary 1.75. As in [AS69], we use the representable definition $\mathrm{KO}^*(X) = [X, \mathrm{BO} \times \mathbb{Z}]$. We draw the following conclusion.

Corollary 1.62. *There is a weak homotopy equivalence*

$$\mathsf{bo}_2^\wedge \xrightarrow{\sim} \mathsf{bo}^{rh\mathbb{Z}_2}$$

compatible with (1.8) *via inclusion.*

We devote the rest of this section to the proof of Theorem 1.61 and Corollary 1.62.

Remark 1.63. In the proofs of Theorem 1.61 and Lemma 1.79 below we will proceed by analyzing and proving the statements on the level of infinite loop spaces. Adams [Ada78, Pretheorem 2.3.2] provides mutually inverse functors

$$\Omega^\infty : \mathbf{connSp} \longleftrightarrow \mathbf{Top}_{\Omega^\infty} : \mathrm{B}^\infty$$

between the category of connective spectra and the category of based infinite loop spaces. In the case of bo the corresponding infinite loop space is $\mathrm{BO} \times \mathbb{Z}$. In the proofs maps are often described in terms of their action on stable vector bundles represented by maps $X \to \mathrm{BO} \times \mathbb{Z}$. A point in $\mathrm{BO} \times \mathbb{Z}$ is represented by a map $* \to \mathrm{BO} \times \mathbb{Z}$ which must factor through some finite $\mathrm{BO}(i) \times \mathbb{Z}$. In other words, it is a finite-dimensional vector space.

A general spectrum Y comes with an intrinsic addition as follows. One can (weakly) identify the product $\mathsf{Y} \times \mathsf{Y}$ with the coproduct $\mathsf{Y} \vee \mathsf{Y}$. On any spectrum, we have the usual fold map $\mathsf{Y} \vee \mathsf{Y} \to \mathsf{Y}$. Addition is then the composite

$$+ : \mathsf{Y} \times \mathsf{Y} \xleftarrow{\sim} \mathsf{Y} \vee \mathsf{Y} \xrightarrow{\text{fold}} \mathsf{Y}.$$

It is defined up to higher homotopies. Note that on the corresponding infinite loop space $\Omega^\infty \mathsf{Y}$, the addition $+$ is simply given via the infinite loop structure, i. e. concatenation of loops.

For general spectra X, Y, consider the diagonal map $\mathsf{X} \to \mathsf{X} \times \mathsf{X}$. For any two maps $g_1, g_2 : \mathsf{X} \to \mathsf{Y}$ we may define the homotopy class

$$[g_1 + g_2] : \mathsf{X} \xrightarrow{\text{diag}} \mathsf{X} \times \mathsf{X} \xrightarrow{g_1 \times g_2} \mathsf{Y} \times \mathsf{Y} \xrightarrow{+} \mathsf{Y}.$$

We wish now to introduce another, essentially equal addition. Consider the disjoint union $\coprod_{n\geq 0} \mathrm{BO}(n)$. This can be thought of as classifying space for vector bundles of arbitrary dimension. There is a sum operation given by the Whitney sum $\oplus$ of vector bundles. It is well-defined up to canonical isomorphism. Higher associativity and commutativity properties (up to homotopy) are derived from these canonical isomorphisms correspondingly. However, $\pi_0\left(\coprod_{n\geq 0} \mathrm{BO}(n)\right) \cong \mathbb{N}_0$ is not a group. This can be rectified by completing it to $\mathbb{Z}$. The above process leads to $\mathrm{BO}\times\mathbb{Z}$, which is turned into an E_∞-space by the considerations above. Using the Adams' recognition principle [Ada78], the Whitney sum $\oplus$ actually provides an infinite loop space structure on $\mathrm{BO}\times\mathbb{Z}$. Recall that the associated infinite loop space of bo is given by $\Omega^\infty \mathsf{bo} = \mathrm{BO}\times\mathbb{Z}$. The former infinite loop structure $+$ is induced by Bott periodicity by construction. We recall that Bott periodicity can be expressed as tensoring with the Bott element, $-\otimes\beta : \mathrm{BO}\times\mathbb{Z} \xrightarrow{\simeq} \Omega^8(\mathrm{BO}\times\mathbb{Z})$. This map is in fact additive on homotopy groups, i. e. it is compatible with the Whitney sum $\oplus$. In other words, up to homotopy one can pass the Whitney sum $\oplus$ pointwise through the Bott loop structure on $\mathrm{BO}\times\mathbb{Z}$. This guarantees the prerequisite interchange formula for a standard Eckmann-Hilton argument. Consequently, the addition $+$ coming from bo given by concatenation of loops is the same as the Whitney sum $\oplus$ up to homotopy. This ensures compatibility when switching between bo, $\mathrm{BO}\times\mathbb{Z}$ and $\mathrm{KO}^*(-)$ as well as between the loop structure addition and Whitney sum towards the end of this section.

We start with a brief overview of Atiyah Real K-theory. Note that this is *Real* with an uppercase R, not to be confused with the usual lowercase real (topological) K-theory. Consider spaces X with an action of $\mathbb{Z}_2$, i. e. an involution. These are what Atiyah [Ati66] calls Real spaces. We may consider $\mathbb{C}$-vector bundles over X together with involutions on their total spaces, coinciding on X with the aforementioned $\mathbb{Z}_2$-action. This can be thought of as modeling a generalization of the conjugation action on complex vector bundles. We require this involution to be antilinear on fibers, i. e. $f(\lambda\cdot v) = \overline{\lambda}\cdot f(v)$ for $\lambda\in\mathbb{C}$. Then $\mathrm{KR}(X)$ is defined as the Grothendiek group of isomorphism classes of such bundles with involution.

Atiyah Real K-theory is a generalization of real topological K-theory. Assume the $\mathbb{Z}_2$-action is trivial, i. e. it fixes the base X pointwise. In this case an involution f on a vector bundle as above fixes each fiber. Also, it induces a splitting into two real vector bundles of equal dimension. The one given by $\bigcup_{x\in X}\{v\in p^{-1}(x) \mid v = f(v)\}$ can be thought of as underlying real vector bundle, the other as underlying imaginary part. Consequently $\mathrm{KR}^*(X)\cong\mathrm{KO}^*(X)$ if the $\mathbb{Z}_2$-action is trivial, see also [Ati66, page 371]. For more details on the subject, see [Ati66].

We continue with a brief overview of G-equivariant K-theory. Let G denote a finite discrete group. A G-vector bundle over a G-space X is a vector bundle equipped with an action of G on its total space E. The projection to X is required to be a G-map. The action is also required to be linear on the fibers. For more details, see [Seg68a]. The G-equivariant (complex) K-theory $\mathrm{K}_G(X)$ of a space X is the Grothendiek group of isomorphism classes of G-vector bundles over X. By letting G act trivially on the suspension coordinates, one obtains an equivariant cohomology theory $\mathrm{K}^*_G(X)$. There

is the analogous notion of G-equivariant real K-theory $\mathrm{KO}^*_G(X)$.
Furthermore, there is a generalization to Atiyah Real K-theory. It is defined using a group G equipped with an action of $\mathbb{Z}_2$, and what is known as Real G-spaces. These are spaces X with an action of $G \rtimes \mathbb{Z}_2$. Since we will only require the real case, it suffices to note that the analogue $\mathrm{KR}^*_G(X) \cong \mathrm{KO}^*_G(X)$ also holds in the G-equivariant setting if the $\mathbb{Z}_2$-action on G and X is trivial. For more details, we refer the reader to the introduction in [AS69, Chapter 6].
We turn to the representation ring. The Real analogue $\mathrm{RR}(G)$ is defined just as in Definition 1.58, but with Real vector spaces. It also comes with an isomorphism $\mathrm{RR}(G) \cong \mathrm{KR}_G(*)$. If the involution of G is trivial, $\mathrm{RR}(G) \cong \mathrm{RO}(G)$ is the usual real representation ring, according to [AS69, page 13]. The augmentation ideal is given by $IR_G = \ker[\mathrm{RR}(G) \to \mathbb{Z}]$ as in the real case.

Definition 1.64. Let X be a space. We define the *completed real K-theory* of X as the 2-adic completion of real K-theory groups, $(\mathrm{KO}^\wedge)^*(X) := \mathrm{KO}^*(X)^\wedge_2$.

Lemma 1.65. *The representing spectrum to completed real K-theory is $\mathsf{bo}^\wedge_2$.*

Proof. Here we take X to be a compact CW-space. The above is immediate from the construction of $(\mathrm{KO}^\wedge)^*(X)$ and the property of $\mathsf{bo}^\wedge_2$ via Proposition 1.60. □

We recall a conclusion of the Atiyah-Segal theorem in the Real case. It is announced by Remark (i) after [AS69, Theorem 7.1]. It is the Real analogue of [AS69, Proposition 4.2]. For any pointed G-space X, recall that X_{hG} denotes the homotopy orbit space $X \wedge_G \mathrm{E}G_+$, in contrast to the unpointed definition in [AS69]. The map α will be explained below in the simplified setting.

Proposition 1.66 (Atiyah-Segal [AS69])**.** *Let X be a compact Real G-space such that $\mathrm{KR}^*_G(X)$ is finite over $\mathrm{RR}(G)$. Then the homomorphism*

$$\alpha : \mathrm{KR}^*_G(X) \longrightarrow \mathrm{KR}^*(X_{hG})$$

induces an isomorphism of the IR_G-adic completion of $\mathrm{KR}^*_G(X)$ with $\mathrm{KR}^*(X_{hG})$.

The previous introduction and discussion immediately imply

Corollary 1.67. *Let X be a compact G-space such that $\mathrm{KO}^*_G(X)$ is finite over $\mathrm{RO}(G)$. Then the homomorphism*

$$\alpha : \mathrm{KO}^*_G(X) \longrightarrow \mathrm{KO}^*(X_{hG})$$

induces an isomorphism of the IO_G-adic completion of $\mathrm{KO}^*_G(X)$ with $\mathrm{KO}^*(X_{hG})$.

Construction 1.68. Henceforth we specialize in $G = \mathbb{Z}_2$ and trivial $\mathbb{Z}_2$-spaces X, i. e. spaces X with trivial $\mathbb{Z}_2$-action. In this case we note $X_{h\mathbb{Z}_2} \cong X \wedge \mathrm{B}\mathbb{Z}_{2+}$. Let F denote a $\mathbb{Z}_2$-equivariant real vector bundle over X, or $\mathbb{Z}_2$*-vector bundle* for short. Then one may form the bundle ${}^{(F\wedge \mathrm{E}\mathbb{Z}_{2+})}/_{\mathbb{Z}_2}$ over $X_{h\mathbb{Z}_2}$. This is the construction employed in [AS69]. They also note on [AS69, page 2] that the assignment $F \mapsto {}^{(F\wedge \mathrm{E}\mathbb{Z}_{2+})}/_{\mathbb{Z}_2}$ is additive and thus induces the homomorphism subject of the Atiyah-Segal theorem,

$$\alpha : \mathrm{KO}^*_{\mathbb{Z}_2}(X) \to \mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+}).$$

Let X be a trivial $\mathbb{Z}_2$-space. Then there is the natural inclusion

$$\mathrm{KO}^*(X) \to \mathrm{KO}^*_{\mathbb{Z}_2}(X) \tag{1.10}$$

induced by sending an ordinary vector bundle over X to the same vector bundle with trivial $\mathbb{Z}_2$-action. Using $\mathrm{RO}(\mathbb{Z}_2) \cong \mathrm{KO}_{\mathbb{Z}_2}(*)$ according to [AS69, page 17], we also consider the inclusion

$$\mathrm{RO}(\mathbb{Z}_2) \cong \mathrm{KO}_{\mathbb{Z}_2}(*) \to \mathrm{KO}^*_{\mathbb{Z}_2}(X) \tag{1.11}$$

given by pulling back a $\mathbb{Z}_2$-vector space to a $\mathbb{Z}_2$-vector bundle over X. Recall that the irreducible $\mathbb{Z}_2$-representations are characterized by $\mathrm{RO}(\mathbb{Z}_2)$. Lemma 1.59 specified these as the 1-dimensional vector space with trivial $\mathbb{Z}_2$-action and the one with reflection action by $\mathbb{Z}_2$. We will call these the trivial and flip vector spaces ε and ϕ, respectively. These are elements of $\mathrm{KO}_{\mathbb{Z}_2}(*)$. In other words, the map (1.11) sends $1 \mapsto \varepsilon \mapsto \mathrm{pullback}(\varepsilon)$ and $x \mapsto \phi \mapsto \mathrm{pullback}(\phi)$. We will cease to point out the difference between a vector bundle and its pullback unless it is unclear from context. The natural maps (1.10) and (1.11) combine to a map

$$\begin{aligned} \mathrm{RO}(\mathbb{Z}_2) \otimes_{\mathbb{Z}} \mathrm{KO}^*(X) &\longrightarrow \mathrm{KO}^*_{\mathbb{Z}_2}(X), \\ 1 \otimes \xi &\longmapsto \varepsilon \otimes \xi, \\ x \otimes \xi &\longmapsto \phi \otimes \xi. \end{aligned} \tag{1.12}$$

An abstract tensor product of a representation and a bundle on the left is sent to the internal tensor product of bundles over X on the right.

Lemma 1.69 (Splitting)**.** *Let X be a trivial $\mathbb{Z}_2$-space. The natural homomorphism* (1.12) *of graded rings is an isomorphism. The inverse can be informally thought of as* splitting *the $\mathbb{Z}_2$-bundles,*

$$F \mapsto [\varepsilon \otimes F_{1\text{-eigenspace}}] \oplus \left[\phi \otimes F_{(-1)\text{-eigenspace}}\right].$$

Upon completion, it becomes

$$\mathrm{RO}(\mathbb{Z}_2)^\wedge \otimes_{\mathbb{Z}} \mathrm{KO}^*(X) \xrightarrow{\cong} \mathrm{KO}^*_{\mathbb{Z}_2}(X) \Big/ {IO_{\mathbb{Z}_2} \cdot \mathrm{KO}^*_{\mathbb{Z}_2}(X)}.$$

Proof. A $\mathbb{Z}_2$-vector bundle F over X decomposes into two parts. The $\mathbb{Z}_2$-action on the bundle F can be thought of as an invertible matrix bundle acting on and respecting each fiber of F, since the action on the base X is trivial. Recall that the irreducible $\mathbb{Z}_2$-representations are given by the trivial and flip vector spaces ε and ϕ, respectively. These are elements of $\mathrm{KO}_{\mathbb{Z}_2}(*)$. Returning to the bundle F, we see that it decomposes into the eigenspace 1-bundle and the eigenspace (-1)-bundle of the matrix bundle. Alternatively, this can be achieved by applying the action on F. This yields a continuous splitting into the fixed points (corresponds to the eigenspace 1-bundle) and their orthogonal complement (corresponds to the eigenspace (-1)-bundle). Anyhow, one obtains a natural splitting map

$$F \mapsto F_{\text{trivial}} \oplus F_{\text{flip}},$$

where $F_{\text{trivial}} = F_1 \otimes \varepsilon$ and $F_{\text{flip}} = F_2 \otimes \phi$ for some ordinary vector bundles F_1 and F_2. This map is in fact both additive and multiplicative using the multiplication on $\mathrm{RO}(\mathbb{Z}_2)$, as well as the inverse we sought.
A more formal, yet less illustrative argument goes as follows. Consider X as a trivial Real $\mathbb{Z}_2$-space, in both senses of the word trivial. By the discussion preceding [AS69, Proposition 8.1] and the proposition itself, we see that $\mathrm{KR}_{\mathbb{Z}_2}(X) \cong \mathrm{RR}(\mathbb{Z}_2) \otimes \mathrm{KR}(X)$. If the Real involution on the group $G = \mathbb{Z}_2$ is trivial, then $\mathrm{RR}(\mathbb{Z}_2)$ is the usual real representation ring, by [AS69, page 13]. Therefore, in the case of a trivial Real $\mathbb{Z}_2$-space, the expression above specializes as usual to the real version $\mathrm{KO}^*_{\mathbb{Z}_2}(X) \cong \mathrm{RO}(\mathbb{Z}_2) \otimes_{\mathbb{Z}} \mathrm{KO}^*(X)$. As pointed out below [Seg68a, Proposition 2.2], the isomorphism is given via an inverse to the natural map $\mathrm{RO}(\mathbb{Z}_2) \otimes_{\mathbb{Z}} \mathrm{KO}^*(X) \to \mathrm{KO}^*_{\mathbb{Z}_2}(X)$. The completion assertion follows formally using the powers of the ideal IO_G in $\mathrm{RO}(\mathbb{Z}_2)$. $\square$

We need not stop here, however. Lemma 1.59 provides an additive isomorphism $\mathrm{RO}(\mathbb{Z}_2) \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle x-1\rangle$. The decomposition is not chosen in the obvious way, it is rather chosen compatible with the isomorphism for the completed representation ring $\mathrm{RO}(\mathbb{Z}_2)^\wedge \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}_2^\wedge\langle x-1\rangle$. We obtain the following identification.

Lemma 1.70. *Let X be a trivial $\mathbb{Z}_2$-space. There is an isomorphism*

$$\begin{aligned} \mathrm{RO}(\mathbb{Z}_2) \otimes_{\mathbb{Z}} \mathrm{KO}^*(X) &\cong \mathrm{KO}^*(X) \oplus \mathrm{KO}^*(X) \\ 1 \otimes \xi &\leftarrow (\xi, 0) \\ (x-1) \otimes \xi &\leftarrow (0, \xi) \end{aligned}$$

of graded modules. Upon completion, it becomes

$$\mathrm{RO}(\mathbb{Z}_2)^\wedge \otimes_{\mathbb{Z}} \mathrm{KO}^*(X) \cong \mathrm{KO}^*(X) \oplus (\mathrm{KO}^\wedge)^*(X).$$

We obtain the following as an immediate corollary of Corollary 1.67 and Lemma 1.69.

Corollary 1.71. *Let X be a compact trivial $\mathbb{Z}_2$-space. Assume $\mathrm{KO}^*_{\mathbb{Z}_2}(X)$ is finite over $\mathrm{RO}(\mathbb{Z}_2)$. Then the map*

$$\alpha : \mathrm{RO}(\mathbb{Z}_2) \otimes \mathrm{KO}^*(X) \longrightarrow \mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+}) \tag{1.13}$$

induced by the vector bundle Construction 1.68 is an isomorphism upon completion of $\mathrm{RO}(\mathbb{Z}_2)$ to $\mathrm{RO}(\mathbb{Z}_2)^\wedge$.

Remark 1.72. We may also wish to see what the map (1.13) does on generators. To begin, consider a bundle ξ over X with trivial $\mathbb{Z}_2$-action and the element $1 \in \mathrm{RO}(\mathbb{Z}_2)$. We may consider $1 \otimes \xi$ as an element in $\mathrm{KO}^*_{\mathbb{Z}_2}(X)$. Since $\mathbb{Z}_2$ acts neither on base nor total space of ξ, Construction 1.68 yields $\xi \wedge \mathrm{B}\mathbb{Z}_{2+}$. The latter denotes the pullback of ξ along the canonical projection map $X \wedge \mathrm{B}\mathbb{Z}_{2+} \to X$. We obtain $1 \otimes \xi \mapsto \xi = \xi \otimes \varepsilon \in \mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+})$. Here we think of $\varepsilon \in \mathrm{KO}_{\mathbb{Z}_2}(*)$ as $\varepsilon \in \mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+})$ via Construction 1.68.
We turn to the element $x \otimes \xi \in \mathrm{RO}(\mathbb{Z}_2) \otimes \mathrm{KO}^*(X)$. Consider $\phi \in \mathrm{KO}_{\mathbb{Z}_2}(*)$. When applying Construction 1.68 to ϕ with $X = \{*\}$, we obtain the bundle ${}^{(\phi \wedge \mathrm{E}\mathbb{Z}_{2+})}/_{\mathbb{Z}_2}$. This

time $\mathbb{Z}_2$ acts on ϕ. Recall the convenient models $\mathrm{E}\mathbb{Z}_2 = S^\infty$ and $\mathrm{B}\mathbb{Z}_2 = \mathbb{R}P^\infty$. It is not hard to see that $\gamma \cong {}^{(\phi\wedge \mathrm{E}\mathbb{Z}_{2+})}/_{\mathbb{Z}_2}$ is in fact the tautological line bundle over $\mathbb{R}P^\infty$. We first take the pullback of ϕ to X and then apply Construction 1.68. We abuse notation and say that the pullback of ϕ in $\mathrm{KO}^*_{\mathbb{Z}_2}(X)$ is mapped to $\gamma \in \mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+})$. We can see that it does not make a difference if we tensor first with ξ, or whether we map both via α first. In short,

$$\begin{aligned} \alpha : \mathrm{RO}(\mathbb{Z}_2) \otimes \mathrm{KO}^*(X) &\longrightarrow \mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+}) \\ 1 \otimes \xi &\longmapsto \xi \otimes \varepsilon \\ x \otimes \xi &\longmapsto \xi \otimes \gamma, \end{aligned}$$

where the tensor product on the right denotes tensor product of the pullback of ξ along $X \wedge \mathrm{B}\mathbb{Z}_{2+} \to X$ with the bundles ε and γ as induced via Construction 1.68, respectively.

The functor $\mathrm{RO}(\mathbb{Z}_2) \otimes \mathrm{KO}^*(X)$ is represented by $\mathsf{bo} \vee \mathsf{bo}$ by Lemma 1.70. Likewise, the functor $\mathrm{RO}(\mathbb{Z}_2)^\wedge \otimes_{\mathbb{Z}} \mathrm{KO}^*(X)$ is represented by $\mathsf{bo} \vee \mathsf{bo}_2^\wedge$, by Lemma 1.65 and Proposition 1.60. In other words, the natural transformation (1.13) can be rewritten as

$$\alpha : [X, \mathsf{bo}]_{-*} \oplus [X, \mathsf{bo}]_{-*} \to [X \wedge \mathrm{B}\mathbb{Z}_{2+}, \mathsf{bo}]_{-*}. \tag{1.14}$$

The left-hand side simplifies as $[X, \mathsf{bo}]_{-*} \oplus [X, \mathsf{bo}]_{-*} \cong [X, \mathsf{bo} \times \mathsf{bo}]_{-*} \cong [X, \mathsf{bo} \vee \mathsf{bo}]_{-*}$, and the right-hand side as $[X \wedge \mathrm{B}\mathbb{Z}_{2+}, \mathsf{bo}]_{-*} \cong [X, \mathrm{map}\,(\mathrm{B}\mathbb{Z}_{2+}, \mathsf{bo})]_{-*} = [X, \mathsf{bo}^{h\mathbb{Z}_2}]_{-*}$. In other words, α induces a natural transformation

$$\alpha' : [X, \mathsf{bo} \vee \mathsf{bo}]_{-*} \to [X, \mathsf{bo}^{h\mathbb{Z}_2}]_{-*}. \tag{1.15}$$

Employing the Yoneda lemma, we see that α' is represented by a map

$$f : \mathsf{bo} \vee \mathsf{bo} \to \mathsf{bo}^{h\mathbb{Z}_2}. \tag{1.16}$$

We consider the completion of α',

$$[X, \mathsf{bo} \vee \mathsf{bo}_2^\wedge]_{-*} \to [X, \mathsf{bo}^{h\mathbb{Z}_2}]_{-*}.$$

By Corollary 1.71, this is an isomorphism for any compact space X for which $\mathrm{KO}^*_{\mathbb{Z}_2}(X)$ is finite over $\mathrm{RO}(\mathbb{Z}_2)$. According to [AS69, page 5], this is the case for a sphere $X = S^k$ with trivial $\mathbb{Z}_2$-action. We obtain an isomorphism

$$f_* : \pi_k\left(\mathsf{bo} \vee \mathsf{bo}_2^\wedge\right) \xrightarrow{\cong} \pi_k\left(\mathsf{bo}^{h\mathbb{Z}_2}\right). \tag{1.17}$$

In negative homotopy degrees, the expressions on both sides are zero by our choice of connective spectra. In other words, f is a weak homotopy equivalence. This completes the proof of Theorem 1.61.

Remark 1.73. The representing map f is not easily understood. It is a priori only an element in the homotopy class $\alpha([\mathrm{id}_{\mathsf{bo}\vee\mathsf{bo}}]) \in \left[\mathsf{bo}\vee\mathsf{bo}, \mathsf{bo}^{h\mathbb{Z}_2}\right]$. It is far easier to describe the natural transformation α in (1.14) instead. Indeed, the left-hand side provides two vector bundles ξ_1 and ξ_2 over X which are mapped via

$$\begin{aligned}\alpha : [X, \mathsf{bo}] \oplus [X, \mathsf{bo}] &\to [\mathrm{B}\mathbb{Z}_{2+} \wedge X, \mathsf{bo}],\\ (\xi_1, \xi_2) &\mapsto (\varepsilon \otimes \xi_1) \oplus ((\gamma - \varepsilon) \otimes \xi_2).\end{aligned}$$

Here ε and γ denote the trivial and tautological line bundles over $\mathrm{B}\mathbb{Z}_2 = \mathbb{R}P^\infty$, respectively. This follows immediately from Lemma 1.70 and Remark 1.72.

Proof of Corollary 1.62. We argue that $\mathsf{bo} \vee \mathsf{bo}_2^\wedge \xrightarrow{\sim} \mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}$ is simply the projection onto the first factor. The lemma is then obvious. We may take the singleton $\{*\}$ as a model for $\mathrm{B}\{1\}$. The map $\mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}$ is identified with $\mathrm{map}\,(\mathrm{B}\mathbb{Z}_{2+}, \mathsf{bo}) \to \mathrm{map}\,(\{*\}_+, \mathsf{bo})$, induced by the inclusion $\mathrm{B}\{1\} \to \mathrm{B}\mathbb{Z}_2$. If we think of this map in terms of the representing functors as in Remark 1.72, we have $\mathrm{KO}^*(X \wedge \mathrm{B}\mathbb{Z}_{2+}) \to \mathrm{KO}^*(X \wedge \{*\}_+)$. It is obvious that with $\mathbb{Z} \cong \mathrm{RO}(1)$ we have that $\mathrm{RO}(1) \otimes \mathrm{KO}^*(X) \to \mathrm{KO}^*(X \wedge \{*\}_+)$ is an isomorphism fitting into a naturality diagram with the one in Remark 1.72. Therefore precisely terms of both forms $1 \otimes \xi$ and $x \otimes \xi$ in $\mathrm{RO}(\mathbb{Z}_2) \otimes \mathrm{KO}^*(X)$ are reduced to $1 \otimes \xi \in \mathrm{RO}(1) \otimes \mathrm{KO}^*(X)$ and correspond to $\xi \in \mathrm{KO}^*(X)$. In particular, an element $(x-1) \otimes \xi$ is reduced to 0. Following the reasoning leading to the notation $\mathsf{bo} \vee \mathsf{bo}_2^\wedge$, it is not difficult to deduce the claim. □

1.6.2. The norm map for the real K-theory spectrum

Definition 1.51 specializes for a trivial action as below. The $\mathbb{Z}_2$*-homotopy orbit spectrum of* bo can be written as

$$\mathsf{bo}_{h\mathbb{Z}_2} := \mathrm{E}\mathbb{Z}_{2+} \wedge_{\mathbb{Z}_2} \mathsf{bo} = \mathrm{B}\mathbb{Z}_{2+} \wedge \mathsf{bo}.$$

Note that this is a connective spectrum by construction. Define the $\mathbb{Z}_2$*-Tate spectrum of* bo as the homotopy cofiber

$$\mathsf{bo}^{th\mathbb{Z}_2} := \text{hocofiber}\left[\tau_{\mathsf{bo}} : \mathsf{bo}_{h\mathbb{Z}_2} \to \mathsf{bo}^{h\mathbb{Z}_2}\right]$$

of the norm map $\tau_{\mathsf{bo}} : \mathsf{bo}_{h\mathbb{Z}_2} \to \mathsf{bo}^{h\mathbb{Z}_2}$. The way we defined it here, the Tate spectrum of bo is in fact a connective spectrum. One can also define the Tate spectrum as the homotopy cofiber of τ_{bo} with the original non-connective $\mathsf{bo}^{h\mathbb{Z}_2}$. It is not hard to see that one produces the same spectrum as above once taking its connective cover. We turn to a collection of facts on the different homotopy groups involved.

Lemma 1.74 (Compare [Ada78, Section 5.1])**.** *The positive homotopy groups of the spectrum* bo *are 8-periodic and are given as follows.*

$$\pi_i(\mathsf{bo}) \cong \begin{cases} \mathbb{Z} & \text{if } i \equiv 0, 4 \text{ mod } 8 \text{ and } i \geq 0 \\ \mathbb{Z}_2 & \text{if } i \equiv 1, 2 \text{ mod } 8 \text{ and } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 1.75. *The homotopy groups of* $\mathsf{bo}^{h\mathbb{Z}_2}$ *are given as follows.*

$$\pi_i\left(\mathsf{bo}^{h\mathbb{Z}_2}\right) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2^\wedge & \text{if } i \equiv 0,4 \mod 8 \text{ and } i \geq 0 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } i \equiv 1,2 \mod 8 \text{ and } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 1.76 ([BG10, Section 12.2.D page 238])**.** *The homotopy groups of* $\mathsf{bo}_{h\mathbb{Z}_2}$ *are given as follows for* $i \geq 0$.

$$\pi_{8i+j}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right) \cong \begin{cases} \mathbb{Z} & \text{if } j = 0,4 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } j = 1,2 \\ \mathbb{Z}_{2^{4i+3}} & \text{if } j = 3 \\ \mathbb{Z}_{2^{4i+4}} & \text{if } j = 7 \\ 0 & \text{if } j = 5,6 \end{cases}$$

Lemma 1.77 ([DM84, Theorem 1.4])**.** *There is a homotopy equivalence of spectra as indicated below, turning* $\mathsf{bo}^{th\mathbb{Z}_2}$ *into a generalized Eilenberg-MacLane spectrum. The homotopy groups of* $\mathsf{bo}^{th\mathbb{Z}_2}$ *are consequently given as follows.*

$$\mathsf{bo}^{th\mathbb{Z}_2} \simeq \bigvee_{i\geq 0} \mathsf{H}\left(\mathbb{Z}_2^\wedge, 4i\right), \qquad \pi_i\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \cong \begin{cases} \mathbb{Z}_2^\wedge & \text{if } i \equiv 0 \mod 4 \text{ and } i \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

It is worth pointing out that both $\mathsf{bo}^{h\mathbb{Z}_2}$ and $\mathsf{bo}^{th\mathbb{Z}_2}$ are 8-periodic in positive degrees, whereas $\mathsf{bo}_{h\mathbb{Z}_2}$ is not periodic at all. In particular, the norm map τ_{bo} is not periodic either. We examine this map further in the sequel. Consider the trivial group $\{1\}$. A valid model for both $\mathrm{E}\{1\}$ and $\mathrm{B}\{1\}$ is a one-point space. One may think of the spectrum bo as the homotopy orbit spectrum of $\{1\}$, namely as $\mathrm{B}\{1\}_+ \wedge \mathsf{bo} \cong S^0 \wedge \mathsf{bo} \cong \mathsf{bo}$. The inclusion of the basepoint $\mathrm{B}\{1\} \to \mathrm{B}\mathbb{Z}_2$ induces a map $\mathsf{bo}_{h\{1\}} \to \mathsf{bo}_{h\mathbb{Z}_2}$. This can be thought of as a map $\mathrm{incl} : \mathsf{bo} \to \mathsf{bo}_{h\mathbb{Z}_2}$. There is a similar construction to obtain a map $\mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}$ as follows. We have $\mathrm{map}_*\left(\mathrm{B}\{1\}_+, \mathsf{bo}\right) \cong \mathrm{map}_*\left(S^0, \mathsf{bo}\right) \cong \mathsf{bo}$. Again the inclusion of the basepoint $\mathrm{B}\{1\} \to \mathrm{B}\mathbb{Z}_2$ induces a map $\mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}^{h\{1\}}$. This can be thought of as a map $\mathrm{restrict} : \mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}$.

Lemma 1.78. *The homotopy orbit spectrum* $\mathsf{bo}_{h\mathbb{Z}_2}$ *admits a split embedding*

$$\mathsf{bo} \hookrightarrow \left(S^0 \wedge \mathsf{bo}\right) \vee \left(\mathrm{B}\mathbb{Z}_2 \wedge \mathsf{bo}\right) \simeq \mathsf{bo}_{h\mathbb{Z}_2},$$

which is compatible with the natural map $\mathsf{bo} \to \mathsf{bo}_{h\mathbb{Z}_2}$*. In particular, this map induces an identification* $\pi_{4k}\left(\mathsf{bo}\right) \cong \pi_{4k}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)$*. Also, we obtain the inclusion into the first summand* $\pi_{8k+1}\left(\mathsf{bo}\right) \cong \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \pi_{8k+1}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)$*, and likewise for* π_{8k+2}.

Proof. Recall that $\mathsf{bo}_{h\mathbb{Z}_2} = \mathrm{B}\mathbb{Z}_{2+} \wedge \mathsf{bo}$ by definition. Already $\mathrm{B}\mathbb{Z}_2$ comes with a canonical basepoint, namely the one given by the one-point space $\mathrm{B}\{1\}$. In other words, we may write $\mathrm{B}\mathbb{Z}_{2+} \cong S^0 \vee \mathrm{B}\mathbb{Z}_2$. We apply distributivity of the smash product and obtain $\mathrm{B}\mathbb{Z}_{2+} \wedge \mathsf{bo} \cong \left(S^0 \wedge \mathsf{bo}\right) \vee \left(\mathrm{B}\mathbb{Z}_2 \wedge \mathsf{bo}\right)$. By construction of the summand

$(S^0 \wedge \mathsf{bo})$, it is clear that it corresponds to the image of bo in $\mathsf{bo}_{h\mathbb{Z}_2}$. A projection back to bo is obviously given by projecting onto the summand $(S^0 \wedge \mathsf{bo})$.
On π_{4k} the split embedding induces a split monomorphism $\mathbb{Z} \cong \pi_{4k}(\mathsf{bo}) \hookrightarrow \pi_{4k}(\mathsf{bo}_{h\mathbb{Z}_2}) \cong \mathbb{Z}$. It is obviously an isomorphism. This means that $\pi_{4k}(\mathsf{bo}_{h\mathbb{Z}_2})$ is given by the summand $\pi_{4k}(S^0 \wedge \mathsf{bo})$. Since $\mathsf{bo} \to S^0 \wedge \mathsf{bo}$ induces the identity on homotopy groups, the split monomorphism is in fact the identity.
On π_{8k+1} the split embedding induces a split monomorphism $\pi_{8k+1}(\mathsf{bo}) \cong \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \cong \pi_{8k+1}(\mathsf{bo}_{h\mathbb{Z}_2})$. We identify the first $\mathbb{Z}_2$-summand with $\pi_{8k+1}(S^0 \wedge \mathsf{bo})$. Then the split monomorphism is simply the inclusion into the first summand. $\square$

Lemma 1.79. *The composition*

$$\tau_{\mathsf{bo}} \circ \mathrm{incl} : \mathsf{bo} \hookrightarrow \mathsf{bo}_{h\mathbb{Z}_2} \to \mathsf{bo}^{h\mathbb{Z}_2}$$

is given on vector bundles as $\xi \mapsto (\varepsilon \oplus \gamma) \otimes \xi$. The latter denotes the tensor product of $\varepsilon \oplus \gamma$ with the pullback of ξ to $\mathrm{B}\mathbb{Z}_{2+} \wedge X$, compare Remark 1.72. Here we think of $\xi \in [X, \mathsf{bo}]_*$ as a stable vector bundle trivialized at the basepoint with dimension zero. Using the identification given by Theorem 1.61 and Corollary 1.75, whenever $\pi_k(\mathsf{bo}) \neq 0$,

$$\alpha_*^{-1} \circ (\tau_{\mathsf{bo}})_* \circ \mathrm{incl}_* : \pi_k(\mathsf{bo}) \to \pi_k(\mathsf{bo}) \oplus \pi_k(\mathsf{bo}_2^\wedge)$$
$$1 \mapsto (2, 1).$$

It is worth pointing out that the description of the norm map as given above employs the ring structure on the spectrum bo. Originally, the definition of the norm map does not require a ring structure on the underlying spectrum.

Proof. Consider the composition

$$\mathrm{restrict} \circ \tau_{\mathsf{bo}} \circ \mathrm{incl} : \mathsf{bo} \hookrightarrow \mathsf{bo}_{h\mathbb{Z}_2} \to \mathsf{bo}^{h\mathbb{Z}_2} \to \mathsf{bo}.$$

Weiss and Williams [WW89, Proposition 2.4] show that this map is in the homotopy class of $\sum_{g\in\mathbb{Z}_2} g : \mathsf{bo} \to \mathsf{bo}$. We interpret its meaning. Each element $g \in \mathbb{Z}_2$ induces a self-map of bo. Since the category **Sp** of spectra is additive, it comes equipped with an addition on maps. It is given by an intrinsic pointwise addition in the target spectrum, compare Remark 1.63. By that same remark, we may work with $\oplus$ instead of $+$ when examining the extended norm map $\sum_{g\in\mathbb{Z}_2} g : \mathsf{bo} \to \mathsf{bo}$.
The action of $\mathbb{Z}_2$ on bo is trivial. Thus the map $\sum_{g\in\mathbb{Z}_2} g$ is simply $\mathrm{id}_{\mathsf{bo}} \oplus \mathrm{id}_{\mathsf{bo}}$. Again we think of points $x \in \mathsf{bo}$ as vector spaces V. In these terms, the map is given by $x \mapsto x \oplus x$ or $V \mapsto V \oplus V$, respectively. In terms of a (stable) vector bundle given by $X \to \mathrm{BO} \times \mathbb{Z}$, we have $\xi \mapsto \xi \oplus \xi$.
We consider specifically the map $\{*\} \to \mathrm{BO} \times \mathbb{Z}$ pointing to the vector space $\varepsilon = \mathbb{R}$. The extended norm map $\sum_{g\in\mathbb{Z}_2} g$ is given on the infinite loop space $\mathrm{BO} \times \mathbb{Z}$ by $\oplus \circ \mathrm{diag} : \mathrm{BO} \times \mathbb{Z} \to \mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z} \to \mathrm{BO} \times \mathbb{Z}$. Consider the $\mathbb{Z}_2$-action on $\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z}$ which interchanges the factors. The addition $\oplus$ is not commutative, but it is commutative up to higher homotopies. In other words, the addition $\oplus$ factors

via the homotopy orbit space $(\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z})_{h\mathbb{Z}_2}$. We set up the diagram (1.18) below, preliminarily without the dashed arrows. We notice that the map $(\varepsilon, \varepsilon)$ is a $\mathbb{Z}_2$-map, with $\{*\}_+$ endowed with the obvious trivial $\mathbb{Z}_2$-action. Consequently, the composite into $(\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z})_{h\mathbb{Z}_2}$ factors via $(\{*\}_+)_{h\mathbb{Z}_2}$. This is the same as $\mathrm{B}\mathbb{Z}_{2+}$. The diagonal composite in (1.18) denotes the map restrict $\circ \tau_{\mathsf{bo}} \circ$ incl at the level of $\Omega^\infty \mathsf{bo} = \mathrm{BO} \times \mathbb{Z}$. Denote the lower horizontal composite map $\mathrm{B}\mathbb{Z}_{2+} \to \mathrm{BO} \times \mathbb{Z}$ by ξ_0. It can be thought of as representing a vector bundle over $\mathrm{B}\mathbb{Z}_{2+}$. However, it can also be viewed as an element of $(\mathrm{BO} \times \mathbb{Z})^{h\mathbb{Z}_2}$ if $\mathrm{BO} \times \mathbb{Z}$ is given the trivial $\mathbb{Z}_2$-action. Applying the map restrict to it is the same as precomposing the map $\xi_0 : \mathrm{B}\mathbb{Z}_{2+} \to \mathrm{BO} \times \mathbb{Z}$ with incl : $\{*\}_+ \to \mathrm{B}\mathbb{Z}_2$. It turns out by Weiss and Williams' work [WW00] that this element ξ_0 is in fact the image of $\varepsilon \in \mathrm{BO} \times \mathbb{Z}$ under the map $\tau_{\mathrm{BO}\times\mathbb{Z}} \circ$ incl.

$$\begin{tikzcd}
 & \mathrm{BO} \times \mathbb{Z} \arrow[dr, "\mathrm{diag}"] \arrow[ddrr, bend left, "\mathrm{restrict} \circ \tau_{\mathrm{BO}\times\mathbb{Z}} \circ \mathrm{incl}"] & & \\
\{*\}_+ \arrow[ur, "\varepsilon"] \arrow[rr, "{(\varepsilon,\varepsilon)}"] \arrow[d, dashed, "\mathrm{incl}"] & & \mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z} \arrow[d] \arrow[dr, "\oplus"] & \\
\mathrm{B}\mathbb{Z}_{2+} \arrow[rr, dashed] & & (\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z})_{h\mathbb{Z}_2} \arrow[r] & \mathrm{BO} \times \mathbb{Z}
\end{tikzcd} \tag{1.18}$$

We proceed to examine ξ_0. It is clear from the considerations at the beginning of the proof that ξ_0 is a bundle of rank two, yet that is all we know presently. The $\mathbb{Z}_2$-action on $\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z}$ is non-trivial, meaning that its associated homotopy orbit space is not simply given by smashing with $\mathrm{B}\mathbb{Z}_{2+}$. It is given by ${}^{\mathrm{E}\mathbb{Z}_{2+} \wedge (\mathrm{BO}\times\mathbb{Z}\times\mathrm{BO}\times\mathbb{Z})}/_{\mathbb{Z}_2}$. A two-fold covering of the latter is provided by $\mathrm{E}\mathbb{Z}_{2+} \wedge (\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z})$. It is compatible with the covering $\mathrm{E}\mathbb{Z}_2 \to \mathrm{B}\mathbb{Z}_2$ and thus fits into

$$\begin{tikzcd}
\mathrm{E}\mathbb{Z}_{2+} \arrow[r] \arrow[d] & \mathrm{E}\mathbb{Z}_{2+} \wedge (\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z}) \arrow[d] \arrow[dr, "\oplus"] & \\
\mathrm{B}\mathbb{Z}_{2+} \arrow[r] & (\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z})_{h\mathbb{Z}_2} \arrow[r, "\oplus"'] & \mathrm{BO} \times \mathbb{Z}.
\end{tikzcd} \tag{1.19}$$

The diagonal addition map is simply given by the Whitney sum of the two elements in $\mathrm{BO} \times \mathbb{Z}$. The right-hand vertical map in (1.18) factors via $\mathrm{E}\mathbb{Z}_{2+} \wedge (\mathrm{BO} \times \mathbb{Z} \times \mathrm{BO} \times \mathbb{Z})$. Thus in our setting, the two vector bundles are given by ε. Since the upper horizontal map in (1.19) is induced from the upper horizontal map in (1.18), the composite map $\mathrm{E}\mathbb{Z}_{2+} \to \mathrm{BO} \times \mathbb{Z}$ represents the trivial bundle $\varepsilon \oplus \varepsilon$ over $\mathrm{E}\mathbb{Z}_2$. Upon taking the quotient by the $\mathbb{Z}_2$-action, we obtain the desired bundle ξ_0. The total space of ξ_0 can be identified with ${}^{\mathrm{E}\mathbb{Z}_2 \times \mathbb{R}^2}/_{\mathbb{Z}_2}$, where the $\mathbb{Z}_2$-action on $\mathbb{R}^2$ permutes the axes. Over any $x \in \mathrm{B}\mathbb{Z}_2$, we may identify the fiber as $\mathrm{map}\,(p^{-1}(x), \mathbb{R})$, where $p : \mathrm{E}\mathbb{Z}_2 \to \mathrm{B}\mathbb{Z}_2$ is the projection. We analyze the bundle $\xi_0 = \bigcup_{x \in \mathrm{B}\mathbb{Z}_2} \mathrm{map}\,(p^{-1}(x), \mathbb{R})$.
The twofold covering p allows to choose two open contractible sets U_1 and U_2 in $\mathrm{B}\mathbb{Z}_2$ covering $\mathrm{B}\mathbb{Z}_2$. The covering p is trivial over both U_1 and U_2. The transition map on $U_1 \cap U_2$ simply interchanges the fibers. Consider the splitting of ξ_0 given by the constant maps $\xi_c = \bigcup_{x \in \mathrm{B}\mathbb{Z}_2} \{f : p^{-1}(x) \to \mathbb{R} \mid f(y_1) = f(y_2)\}$ and the anti-constant

maps $\xi_a = \bigcup_{x \in \mathrm{B}\mathbb{Z}_2} \{f : p^{-1}(x) \to \mathbb{R} \mid f(y_1) = -f(y_2)\}$, with $y_1, y_2 \in p^{-1}(x)$ denoting the two appropriate preimages. Both of these bundles are trivial over U_1 and U_2. The transition map on $U_1 \cap U_2$ interchanges $y_1 \leftrightarrow y_2$. On the bundle ξ_c of constant maps, this induces the identity transition map. Therefore ξ_c is a trivial line bundle over $\mathrm{B}\mathbb{Z}_2$. On the bundle ξ_a of anti-constant maps, the condition on f forces the transition to be $f \leftrightarrow -f$, since $f(y_1) = -f(y_2) \leftrightarrow -f(y_1)$. The identification $U_1 \ni f \leftrightarrow -f \in U_2$ on the overlap is precisely what happens in the case of the tautological line bundle over $\mathrm{B}\mathbb{Z}_2$. Thus we identify $\xi_a \cong \gamma$. Altogether, we have $\xi_0 \cong \varepsilon \oplus \gamma$, and we obtain

$$\mathrm{BO} \times \mathbb{Z} \to (\mathrm{BO} \times \mathbb{Z})^{h\mathbb{Z}_2}, \quad \varepsilon \mapsto \varepsilon \oplus \gamma.$$

It is not difficult to generalize this argument to an arbitrary vector space V determined by some $x \in \mathrm{BO} \times \mathbb{Z}$. One simply needs to change $\mathbb{R}$ for the associated V. The argument follows using $\mathrm{map}\,(p^{-1}(x), V) \cong \mathrm{map}\,(p^{-1}(x), \mathbb{R}) \otimes V$, which leads to $\xi_0 \otimes V$. Consequently, a vector bundle ξ over X determined by $X \to \mathrm{BO} \times \mathbb{Z}$ is mapped to the bundle $(\varepsilon \oplus \gamma) \otimes \xi$ over $\mathrm{B}\mathbb{Z}_{2+} \wedge X$.
Theorem 1.61 identifies the homotopy groups of $\mathsf{bo}^{h\mathbb{Z}_2}$ as those of $\mathsf{bo} \vee \mathsf{bo}_2^\wedge$. We use Remark 1.73 to understand the effect of this identification on homotopy groups. Under α', we have $(\xi_1 \oplus \xi_2, \xi_2) \mapsto (\varepsilon \otimes \xi_1) \oplus (\gamma \otimes \xi_2)$. In the case of vector bundles over the sphere, α' is in fact an isomorphism, and we can write $(\varepsilon \oplus \gamma) \otimes \xi \mapsto (\xi \oplus \xi, \xi)$ under $[S^k, \mathsf{bo}^{h\mathbb{Z}_2}] \to [S^k, \mathsf{bo} \vee \mathsf{bo}_2^\wedge]$. Altogether, we obtain $\xi \mapsto (\xi \oplus \xi, \xi)$. Under the chosen identifications for the homotopy groups, this corresponds to $1 \mapsto (2, 1)$. $\square$

As a consequence, we understand the Tate cofiber sequence associated to bo.

Theorem 1.80. *The long exact sequence on π_* associated to the cofiber sequence*

$$\mathsf{bo}_{h\mathbb{Z}_2} \xrightarrow{\tau_{\mathsf{bo}}} \mathsf{bo}^{h\mathbb{Z}_2} \longrightarrow \mathsf{bo}^{th\mathbb{Z}_2}$$

is described as follows. All homotopy groups that are not shown are equal to zero. In particular, the long exact sequence is broken up into short bits. We assume $i \geq 0$.

$$0 \to \underbrace{\pi_{8i+8}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}} \xrightarrow{\cdot(2,1)} \underbrace{\pi_{8i+8}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}\oplus\mathbb{Z}_2^\wedge} \xrightarrow{\cdot\begin{pmatrix} 2^{4i+4} \\ -2^{4i+5} \end{pmatrix}} \underbrace{\pi_{8i+8}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2^\wedge} \xrightarrow{\text{proj}} \underbrace{\pi_{8i+7}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_{2^{4i+4}}} \to 0$$

$$0 \to \underbrace{\pi_{8i+4}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}} \xrightarrow{\cdot(2,1)} \underbrace{\pi_{8i+4}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}\oplus\mathbb{Z}_2^\wedge} \xrightarrow{\cdot\begin{pmatrix} 2^{4i+3} \\ -2^{4i+4} \end{pmatrix}} \underbrace{\pi_{8i+4}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2^\wedge} \xrightarrow{\text{proj}} \underbrace{\pi_{8i+3}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_{2^{4i+3}}} \to 0$$

$$0 \to \underbrace{\pi_{8i+2}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2\oplus\mathbb{Z}_2} \xrightarrow{\cdot\left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda \end{smallmatrix}\right)} \underbrace{\pi_{8i+2}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2\oplus\mathbb{Z}_2} \to 0$$

$$0 \to \underbrace{\pi_{8i+1}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2\oplus\mathbb{Z}_2} \xrightarrow{\cdot\left(\begin{smallmatrix} 0 & 1 \\ 1 & \lambda \end{smallmatrix}\right)} \underbrace{\pi_{8i+1}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2\oplus\mathbb{Z}_2} \to 0$$

$$0 \to \underbrace{\pi_0\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}} \xrightarrow{\cdot(2,1)} \underbrace{\pi_0\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}\oplus\mathbb{Z}_2^\wedge} \xrightarrow{\cdot\left(\begin{smallmatrix}1\\-2\end{smallmatrix}\right)} \underbrace{\pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)}_{\cong\mathbb{Z}_2^\wedge} \to 0$$

The case of π_0 is special since $\pi_{-1}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right) = 0$. The λ showing up above is unknown to the author at this point and possibly different on a case by case basis.

Remark 1.81. For better readability in Section 7.2, we summarize that the map $\mathsf{bo}^{rh\mathbb{Z}_2} \to \mathsf{bo}^{th\mathbb{Z}_2}$ is given by $b \mapsto -2b$ on π_0. On π_8, it is given by $b \mapsto -32b$.

Remark 1.82. With the connective versions of $\mathsf{bo}^{h\mathbb{Z}_2}$ and $\mathsf{bo}^{th\mathbb{Z}_2}$, all negative homotopy groups vanish in the long exact sequence. The case of the original non-connective $\mathsf{bo}^{h\mathbb{Z}_2}$ and $\mathsf{bo}^{th\mathbb{Z}_2}$ is not very complicated. By [DM84, Theorem 1.4], there is a homotopy equivalence of spectra $\mathsf{bo}^{th\mathbb{Z}_2} \simeq \bigvee_{i\in\mathbb{Z}} \mathsf{H}\left(\mathbb{Z}_2^\wedge, 4i\right)$ turning $\mathsf{bo}^{th\mathbb{Z}_2}$ into a fully 4-periodic generalized Eilenberg-MacLane spectrum. The homotopy groups of $\mathsf{bo}^{th\mathbb{Z}_2}$ are thus given by the non-zero $\pi_{4i}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \cong \mathbb{Z}_2^\wedge$ in every degree $4i$. Consequently, the inclusion into the cofiber yields the isomorphisms $\pi_{4i}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right) \to \pi_{4i}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ in negative degrees. These are the only non-zero maps.

Proof of Theorem 1.80. We go through all the cases of π_*. Corollary 1.75, Lemma 1.76 and Lemma 1.77 were already employed implicitly in the statement to identify all the homotopy groups.
We start with the case of the first line. By Lemma 1.78, the embedding $\mathsf{bo} \hookrightarrow \mathsf{bo}_{h\mathbb{Z}_2}$ induces an identification $\pi_{4k}(\mathsf{bo}) \cong \pi_{4k}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)$. Consequently, Lemma 1.79 reveals information on the map $(\tau_{\mathsf{bo}})_* : \pi_{4k}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right) \to \pi_{4k}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)$ itself. Namely, it is given by $1 \mapsto (2,1)$. On the other hand, the map $\pi_{8i+8}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \twoheadrightarrow \pi_{8i+7}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)$ must be the projection by exactness. Consequently, the image of $\pi_{8i+8}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)$ in $\pi_{8i+8}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ must be $2^{4i+4}\mathbb{Z}_2^\wedge$. It is not hard to deduce the exact shape of the map in question.
The second sequence is completely analogous, with slightly different integers. Likewise, the last sequence does not differ substantially. The main difference is that $\pi_{-1}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right) = 0$. This is due to both $\mathrm{B}\mathbb{Z}_2$ as well as bo being connective. Consequently, $\pi_0\left(\mathsf{bo}^{h\mathbb{Z}_2}\right) \to \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ must be surjective. It is given by $(a,b) \mapsto a - 2b$.
The remaining cases are different from the above, yet analogous to one another. By Lemma 1.78, the embedding $\mathsf{bo} \hookrightarrow \mathsf{bo}_{h\mathbb{Z}_2}$ induces a monomorphism $\pi_{8i+2}(\mathsf{bo}) \hookrightarrow \pi_{8i+2}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)$ given by inclusion $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$ into the first summand. Consequently, Lemma 1.79 reveals information on the map $(\tau_{\mathsf{bo}})_* : \pi_{8i+2}\left(\mathsf{bo}_{h\mathbb{Z}_2}\right) \to \pi_{8i+2}\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)$ itself. Namely, we know $(1,0) \mapsto (2,1) = (0,1)$. Exactness forces $(\tau_{\mathsf{bo}})_*$ to be an isomorphism, i. e. the associated matrix must be invertible. Thus it is of proclaimed shape. Finding the exact values of λ is left for future research. □

2. Statement of results

We begin by motivating the main research question 2.3.

Lemma 2.1. *For any smooth structure* Σ *we have* $\mathrm{Diff}_\partial(D^n) \cong \mathrm{Diff}_\partial(D^n_\Sigma)$ *as topological groups.*

Proof. For simplicity, we think of $\mathrm{Diff}_\partial(D^n_\Sigma)$ as the classical group of diffeomorphisms. Up to diffeomorphism, there exists only one smooth structure on D^n for $n \geq 5$ [Mil65]. Let $\alpha : D^n \to D^n_\Sigma$ denote a diffeomorphism. It might not fix the boundary pointwise. However, for $f \in \mathrm{Diff}_\partial(D^n_\Sigma)$ the map $\alpha^{-1} \circ f \circ \alpha : D^n \to D^n$ is a diffeomorphism fixing the boundary pointwise. It is not hard to see that we get an isomorphism of topological groups. $\square$

Let $\Sigma \in \mathfrak{S}(S^n)$ denote a smooth structure on S^n. For the sake of argument in this section we may assume Σ is standard on some embedded disk $D^n \subset S^n$. This can always be assumed since every smooth sphere is diffeomorphic to a twisted sphere, which follows from [Cer70].

Proposition 2.2 ([RP80, Diagram 1.3])**.** *There is a fibration sequence*

$$\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}\,(S^n_\Sigma) \to \mathrm{Fr}_\mathrm{O}\,(TS^n_\Sigma) \tag{2.1}$$

with the second map being over the respective evaluation maps to S^n*.*

Here $\mathrm{Fr}_\mathrm{O}\,(TS^n_\Sigma)$ denotes the orthogonal frame bundle associated to S^n_Σ. The evaluation fibration on $\mathrm{Diff}\,(S^n_\Sigma)$ is given by evaluation at a basepoint and also introduced in detail below, whereas the map $\mathrm{Fr}_\mathrm{O}\,(TS^n_\Sigma) \to S^n$ is the usual bundle projection. The frame bundle comes with a homeomorphism $\mathrm{Fr}_\mathrm{O}\,(TS^n_\Sigma) \cong \mathrm{SO}\,(n+1)$ over the evaluation maps, by [RP80, (1.1) Lemma]. The map $\mathrm{SO}\,(n+1) \to S^n$ is the usual fibration with fiber $\mathrm{SO}\,(n)$. The similar fibration sequence $\mathrm{Diff}_\partial(D^n) \to \mathrm{Diff}_*(S^n_\Sigma) \to \mathrm{SO}\,(n)$ is also studied in [RP80] as well as [Sch71]. The left-hand map in (2.1) is a group homomorphism, whereas the right-hand map is not. We may ask whether there is a homotopy equivalence $\mathrm{Diff}\,(S^n) \to \mathrm{Diff}\,(S^n_\Sigma)$ fitting into a homotopy commutative diagram in the spirit of

$$\begin{tikzcd}
\mathrm{Diff}_\partial(D^n) \arrow[r] \arrow[d, "\cong"] & \mathrm{Diff}\,(S^n) \arrow[r] \arrow[d, dashed, "\simeq"] & \mathrm{Fr}_\mathrm{O}\,(TS^n) \arrow[d, "\cong"] \\
\mathrm{Diff}_\partial(D^n_\Sigma) \arrow[r] & \mathrm{Diff}\,(S^n_\Sigma) \arrow[r] & \mathrm{Fr}_\mathrm{O}\,(TS^n_\Sigma)
\end{tikzcd} \tag{2.2}$$

relating the diffeomorphism groups for different smooth structures. Even more restrictive would be to ask that the map in question be multiplicative. In other words, to ask for a homotopy equivalence fitting into

$$\begin{tikzcd}
\mathrm{Fr}_\mathrm{O}\,(TS^n) \arrow[r] \arrow[d, "\cong"] & \mathrm{BDiff}_\partial(D^n) \arrow[r] \arrow[d, "\cong"] & \mathrm{BDiff}\,(S^n) \arrow[d, dashed, "\simeq"] \\
\mathrm{Fr}_\mathrm{O}\,(TS^n_\Sigma) \arrow[r] & \mathrm{BDiff}_\partial(D^n_\Sigma) \arrow[r] & \mathrm{BDiff}\,(S^n_\Sigma).
\end{tikzcd}$$

This question is unfortunately hard to answer. Instead we turn to a different viewpoint. We know that the fibration sequence (2.1) is over S^n via the respective evaluation maps. Therefore one could also ask for (2.2) to be over S^n, which is in fact an approachable question. The non-existence result announced by Conjecture 2.6 implies the non-existence of such a homotopy equivalence as described above, compare Corollary 2.10.
Let S^n denote the standard n-dimensional topological sphere in $\mathbb{R}^{n+1}$ and S^n_Σ the sphere endowed with a pseudo smooth structure $\Sigma \in \mathfrak{S}(S^n)$, compare Section 1.2. Consider the evaluation maps at the basepoint $1 \in S^n$,

$$\mathrm{ev}_1 : \mathrm{TOP}(S^n) \longrightarrow S^n, \qquad\qquad \mathrm{ev}_1 : \mathrm{Diff}(S^n_\Sigma) \longrightarrow S^n.$$

These are fibrations, by Lemma A.2 and Corollary A.4. The discussion above leads to the question whether the latter map admits a section. However, it turns out that this is more than one can hope for even in the general case of $\mathrm{TOP}(S^n)$, compare Obstruction 3.2. We relax the question somewhat. Let $\delta : S^n \to S^n$ with $\delta(1) = 1$ denote an arbitrary, yet fixed basepoint preserving self-map of S^n of degree $d \geq 1$.

Question 2.3. Does there exist a lift $\overline{s}$ of δ to $\mathrm{Diff}(S^n_\Sigma)$ as indicated in (2.3) in the case of an exotic smooth structure Σ?

$$\begin{tikzcd}
 & \mathrm{Diff}(S^n_\Sigma) \arrow[d, "\text{forget}"] \arrow[dd, bend left=60, two heads, "\mathrm{ev}_1"] \\
 & \mathrm{TOP}(S^n) \arrow[d, two heads, "\mathrm{ev}_1"] \\
S^n \arrow[uur, dashed, "\overline{s}"] \arrow[ur, dashed, "s"'] \arrow[r, "\delta"'] & S^n
\end{tikzcd} \tag{2.3}$$

Note that the cases of negative $d < 0$ are similar, just by precomposing δ with a degree -1 self-map of S^n. The case of $d = 0$ is not very interesting. Also, any homotopy lift s or $\overline{s}$ as in (2.3) is homotopic to a strict lift, since the evaluation maps are fibrations. We thus concentrate on the existence question for strict lifts. In this case we have the strict equalities $\mathrm{ev}_1 \circ s = \delta$ and $\mathrm{ev}_1 \circ \overline{s} = \delta$, respectively.

Example 2.4. For $n = 7$ and the standard sphere, a lift $\overline{s}$ of $\mathrm{id} : S^n \to S^n$ exists. It uses the Cayley multiplication on $\mathbb{R}^8$ and is given by

$$\overline{s} : S^7 \longrightarrow \mathrm{SO}(8) \subset \mathrm{Diff}\left(S^7\right), \text{ with } p \longmapsto \overline{s}(p), \text{ where } \overline{s}(p)(q) := p \cdot q,$$

compare [MT91, Theorem IV.6.5(1)]. For $d > 1$, we have the obvious lift $\overline{s} \circ \delta$ of δ.

It seems that the case of the standard smooth sphere is somewhat special in this respect, compare Conjectures 2.6 and 2.7. The following theorem is the main statement of this thesis. Let $\sigma : S^{15} \to S^8$ denote a map of Hopf invariant one. Let $\Sigma \in \mathfrak{S}(S^7)$ denote a pseudo smooth structure on S^7. Let $\xi : S^8 \to {}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}$ denote a lift of Σ to $\pi_8\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right)$, compare Lemmata 1.40 and 1.42. Let $\phi : {}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)} \to T_1\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right)$ denote the approximation map to the first orthogonal calculus Taylor approximation $T_1\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right)$, compare Section 5.1 for details.

Theorem 2.5. *If $\phi \circ \xi \circ \sigma \neq 0 \in \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right)$, then there exists no homotopy section*

$$\overline{s} : S^7 \longrightarrow \mathrm{Diff}\left(S^7_\Sigma\right)$$

to the evaluation map ev_1 *as depicted in* (2.3).

We postpone the proof until the end of Section 6. Denote by $\Sigma_{\mathrm{KM}} : S^7 \to {}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}$ the image of the distinguished generator $1 \in L_8(\mathbb{Z})$ in $\theta_7 \cong \mathbb{Z}_{28}$. This pseudo smooth structure generates the group θ_7 of homotopy 7-spheres, see Definition 1.35. We call it the *pseudo smooth Kervaire Milnor structure*. The exotic sphere S^7_Σ for $\Sigma = \Sigma_{\mathrm{KM}}$ can be thought of as the boundary of a stably framed smooth 8-dimensional manifold M with signature 8. The following conjecture is the main research effort of this thesis. Currently there is sufficient evidence to suggest that it holds true.

Conjecture 2.6. *Let $\Sigma = \Sigma_{\mathrm{KM}} \in \mathfrak{S}(S^7)$ be the pseudo smooth Kervaire Milnor structure. Then there exists no homotopy section*

$$\overline{s} : S^7 \longrightarrow \mathrm{Diff}\left(S^7_\Sigma\right)$$

to the evaluation map ev_1 *as depicted in* (2.3).

The theorem and the conjecture above are deduced by identifying certain obstructions to the lifting problem (2.3) before proving that these do not vanish. For example, a lift to $\mathrm{TOP}(S^n)$ exists if and only if the bundle $\delta^* TS^n_\Sigma$ is trivial, by Obstruction 3.2. As Oscar Randal-Williams pointed out, such a trivialization yields a lift $s : S^n \to \mathrm{Fr}_{\mathrm{O}}(TS^n_\Sigma)$ of δ. Lifting this map further to $\mathrm{Diff}(S^n_\Sigma)$ along (2.1) is equivalent to lifting the associated element in $\pi_n(\mathrm{Fr}_{\mathrm{O}}(TS^n_\Sigma))$ to $\pi_n(\mathrm{Diff}(S^n_\Sigma))$ in the associated long exact sequence. In other words, the image of $[s]$ in $\pi_{n-1}(\mathrm{Diff}_\partial(D^n))$ is the obstruction to the existence of such a lift. Using Remark 1.31 we identify $\pi_{n-1}(\mathrm{Diff}_\partial(D^n)) \cong \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$. Denote by $\mathrm{Obs}(s, \Sigma)$ the corresponding element to the image of $[s]$ in $\pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$.

Obstruction 4.14 (Idea)**.** *There exists a homotopy lift $\overline{s} : S^n \to \mathrm{Diff}(S^n_\Sigma)$ of s as in* (2.3) *if and only if the obstruction element* $\mathrm{Obs}(s, \Sigma) \in \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$ *vanishes.*

This obstruction is developed in detail via Obstruction 3.16. The latter states that a lift to $\mathrm{Diff}(S^n_\Sigma)$ exists if and only if a certain diagram (3.4) commutes. Section 4 provides a detailed description of $\mathrm{Obs}(s, \Sigma)$ which will allow further computations in the sequel. The methods developed in this thesis suggest that the following conjecture might hold. It is most likely for 2-torsion elements $\Sigma \in \theta_n$.

Conjecture 2.7. *Let $n \geq 7$ with $n \equiv 3 \mod 4$. Let $\delta : S^n \to S^n$ denote a self-map of S^n of degree $d = 1$ or $d = 2$. Let $\Sigma \in \mathfrak{S}(S^n)$ be a pseudo smooth structure in the image of $L_{n+1}(\mathbb{Z})$, compare Lemma 1.39. Then there exists a homotopy lift $\overline{s} : S^n \longrightarrow \mathrm{Diff}(S^n_\Sigma)$ of δ along the evaluation map* ev_1 *as depicted in* (2.3) *if and only if both Σ is the standard pseudo smooth structure on S^n as well as $[d = 2$ or $n = 7]$.*

Although far more details to the approach of Conjecture 2.7 are given in this thesis, at least the existence assertion for the standard sphere is immediate, see Remark 3.10. A non-existence result for all even dimensions n is also immediate from Section 3.1. However, the non-existence part for general n in the case of an exotic pseudo smooth sphere is currently beyond the scope of the methods employed in this thesis. All reasonings are kept as general as possible, and aspects relevant to higher-dimensional cases $n > 7$ are discussed during the line of argument.
We draw conclusions should Conjecture 2.6 hold true. They have corresponding analogues in dimensions $n \geq 7$ should Conjecture 2.7 hold true. Let $\Sigma = \Sigma_{\mathrm{KM}} \in \mathfrak{S}\left(S^7\right)$ be the pseudo smooth Kervaire Milnor structure.

Corollary 2.8 (of Conjecture 2.6)**.** *The induced map*

$$(\mathrm{ev}_1)_* : \pi_7\left(\mathrm{Diff}\left(S^7_\Sigma\right)\right) \to \pi_7\left(S^7\right)$$

is not surjective, whereas its counterpart for the standard 7-sphere is.

Corollary 2.9 (of Conjecture 2.6)**.** *The map* $\mathrm{Diff}\left(S^7_\Sigma\right) \to \mathrm{Fr}_{\mathrm{O}}\left(TS^7_\Sigma\right)$ *from* (2.1) *does not admit a section.*

Corollary 2.10 (of Conjecture 2.6)**.** *There exists no homotopy equivalence* $\mathrm{Diff}\left(S^7\right) \to \mathrm{Diff}\left(S^7_\Sigma\right)$ *over* S^7*, i. e. respecting the fibers of the evaluation maps.*

Proof. Let h denote a hypothetical homotopy equivalence as above. Choose a homotopy section $s : S^7 \to \mathrm{Diff}\left(S^7\right)$. Then $h \circ s$ yields a homotopy section to $\mathrm{Diff}\left(S^7_\Sigma\right)$. However, Conjecture 2.6 precludes such an existence. □

Corollary 2.11 (of Conjecture 2.6)**.** *There exists no map* $\alpha : \mathrm{Diff}\left(S^7\right) \to \mathrm{Diff}\left(S^7_\Sigma\right)$ *fitting into a homotopy commutative diagram*

$$\begin{tikzcd}
\mathrm{Diff}\left(S^7\right) \times S^7 \arrow[r, "\alpha\times\mathrm{id}"] \arrow[d, "\mathrm{ev}"] & \mathrm{Diff}\left(S^7_\Sigma\right) \times S^7 \arrow[d, "\mathrm{ev}"] \\
S^7 \arrow[r, "\mathrm{id}"] & S^7,
\end{tikzcd}$$

where ev *denotes the action of* $\mathrm{Diff}\left(S^7\right)$ *respectively* $\mathrm{Diff}\left(S^7_\Sigma\right)$ *on* S^7 *by* $(f, p) \mapsto f(p)$*.*

Proof. Assume there were an α as outlined. Restrict the diagram to $\mathrm{Diff}\left(S^7\right) \times 1 \xrightarrow{\alpha\times\mathrm{id}} \mathrm{Diff}\left(S^7_\Sigma\right) \times 1$. Then the Cayley section s_C is a right inverse to the left-hand vertical evaluation map ev, which is simply ev_1. Then $\alpha \circ s_C$ is a section of the right-hand vertical map $\mathrm{Diff}\left(S^7_\Sigma\right) \to S^7$, a contradiction. Indeed, $\mathrm{ev}_1 \circ \alpha \circ s_C \simeq \mathrm{id} \circ \mathrm{ev}_1 \circ s \simeq \mathrm{id}$ using that the diagram is homotopy commutative. □

3. Obstructions to the lifting question

3.1. First obstruction: Lifting to the group of homeomorphisms

We wish to examine an obstruction to finding a lift s or $\overline{s}$ as shown in (2.3). We prove a series of lemmata which will patch together to yield Obstruction 3.2. Recall that $n \geq 5$ is the dimension of S^n and $d \geq 1$ is the degree of the self-map δ of S^n. Consider

$$\begin{tikzcd}
 & \mathrm{TOP}\left(S^n\right) \arrow[d, two heads, "\mathrm{proj}"] \\
 & {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)} \arrow[d, "\cong"', "\mathrm{proj} \times \mathrm{ev}_1"] \\
S^n \arrow[uur, dashed, "s"] \arrow[ur, "\Sigma\circ\delta"] \arrow[r, "(\Sigma\circ\delta)\times\delta"'] & {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \times S^n.
\end{tikzcd} \tag{3.1}$$

Recall that TS^n_Σ is the associated pseudo tangent bundle to Σ, and $\delta^* TS^n_\Sigma$ denotes its pullback along δ.

Proposition 3.1. *Let $\Sigma \in \mathfrak{S}\left(S^n\right)$ denote a pseudo smooth structure, and $\delta : S^n \to S^n$ a self-map of S^n. There is a homotopy equivalence*

$$\left\{\begin{matrix} \textit{lifts } s : S^n \longrightarrow \mathrm{TOP}\left(S^n\right) \\ \textit{as depicted in } (3.1) \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{Trivializations } \chi : \delta^* TS^n_\Sigma \cong T_1 S^n_\Sigma \times S^n \textit{ of} \\ \textit{the pullback of the pseudo tangent bundle} \end{matrix}\right\}.$$

The left-pointing arrow is illustrated by Construction 3.4. We can restrict the homotopy equivalence to those lifts satisfying $s(1) = \mathrm{id}_{S^n}$ and trivializations satisfying $\chi : \delta^* TS^n_\Sigma \cong T_1 S^n_\Sigma \times S^n$ being a bundle isomorphism with $\chi_1 = \mathrm{id}_{T_1 S^n_\Sigma}$. The left-hand side is topologized as subset of $\mathrm{map}\left(S^n, \mathrm{TOP}\left(S^n\right)\right)$ with the compact-open topology. The right-hand side is topologized as the space of nullhomotopies of $\iota \circ \Sigma \circ \delta : S^n \to \mathrm{BO}\left(n\right)$. Details can be found in the proof. We first formulate the following obstruction.

Obstruction 3.2. *Let $\Sigma : S^n \to {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ be a pseudo smooth structure. In order to possibly find a homotopy lift $\overline{s}$ of δ to $\mathrm{Diff}\left(S^n_\Sigma\right)$ as in (2.3), the following equivalent conditions must be satisfied.*

- *There exists a lift $s : S^n \to \mathrm{TOP}\left(S^n\right)$ as in (3.1),*
- *the pullback bundle $\delta^* TS^n_\Sigma$ of the pseudo tangent bundle TS^n_Σ along δ is trivial,*
- *n is odd and d is even, or $n = 7$.*

Note that all these conditions are independent of the chosen pseudo smooth structure. We postpone the proof. Construction 3.4 below tells us how to use a trivialization of $\delta^* TS^n_\Sigma$ to create a lift s to $\mathrm{TOP}\left(S^n\right)$, a first step towards $\mathrm{Diff}\left(S^n_\Sigma\right)$. The following property will become important in Section 3.2 and Section 4.3.

Remark 3.3. Lifts $s : S^n \longrightarrow \mathrm{TOP}\,(S^n)$ as depicted in (3.1) are lifts as in (2.3) which additionally satisfy $\Sigma \circ \delta = \mathrm{proj} \circ s$. This way the lifts s are directly linked to Σ, ready to potentially be easily upgraded to lifts to $\mathrm{Diff}\,(S^n_\Sigma)$ in Section 3.2.

Construction 3.4. Let $\Sigma \in \mathfrak{S}\,(S^n)$ be a pseudo smooth structure. Recall that the pseudo tangent bundle is given by $T_pS^n_\Sigma := \Sigma(p)(S^n \setminus -1)$ with vector space structure pushed from $\mathbb{R}^n$ via $\Sigma(p)$. Assume given a trivialization χ_p, $p \in S^n$, of the pullback bundle $\delta^*TS^n_\Sigma$. Then one can associate to $\Sigma \circ \delta$ and δ a constructed lift $s : S^n \longrightarrow \mathrm{TOP}\,(S^n)$ as depicted in (3.1) as follows.
The bundle $\delta^*TS^n_\Sigma$ is given by $\delta^*TS^n_\Sigma = \{(p, v) \mid p \in S^n,\ v \in T_{\delta(p)}S^n_\Sigma,\ \delta(p) = \mathrm{proj}(v)\}$. Given a trivialization of $\delta^*TS^n_\Sigma$, we may express this as a continuous family $\chi_p : T_1S^n_\Sigma \to T_{\delta(p)}S^n_\Sigma$, $p \in S^n$, of vector space isomorphisms. From this we can construct a family of homeomorphisms $s(p) : S^n \to S^n$, $p \in S^n$, as one-point compactifications of the dashed mapping in (3.2). Here $q_p := \Sigma(p)(-1)$ is *the* point in $S^n \setminus T_pS^n_\Sigma$. Recall the convention $\Sigma(1) \in \mathrm{O}\,(n)$, i. e. $q_1 = -1$.

$$\begin{tikzcd} S^n \setminus -1 \arrow[r, equal] \arrow[rrr, dashed, bend right, "s(p)"'] & T_1S^n_\Sigma \arrow[r, "\chi_p", "\cong"'] & T_{\delta(p)}S^n_\Sigma \arrow[r, equal] & S^n \setminus q_{\delta(p)} \end{tikzcd} \tag{3.2}$$

The resulting map $s : S^n \to \mathrm{TOP}\,(S^n)$ now satisfies $s(p)(1) = \delta(p)$ and thus constitutes a lift of δ. Also, it is not difficult to see that $s(1) = \mathrm{id}_{S^n}$ if $\chi_1 = \mathrm{id}_{T_1S^n_\Sigma}$.
We check $\mathrm{proj} \circ s = \Sigma \circ \delta$ pointwise. Indeed, a representative $\Sigma(\delta(p)) : S^n \to S^n$ restricts by definition to a vector space isomorphism $\Sigma(\delta(p)) : S^n \setminus -1 \to S^n \setminus q_{\delta(p)}$, i. e. a linear map. Recall that $\Sigma(1)$ is the identity by definition. We also have the linear map $\chi_p : T_1S^n_\Sigma = S^n \setminus -1 \to S^n \setminus q_{\delta(p)} = T_{\delta(p)}S^n_\Sigma$. Thus $\chi_p \circ \Sigma(1) : S^n \setminus -1 \to S^n \setminus q_{\delta(p)}$ is a linear map, just like $\Sigma(\delta(p))$. Hence $s(p) = \chi_p^c : S^n \to S^n$ and $\Sigma(\delta(p)) : S^n \to S^n$ differ only by $A \in \mathrm{O}\,(n)$. We have shown $[s(p) \circ \Sigma(1)] = [s(p)] = \Sigma(\delta(p))$.

Proof of Proposition 3.1. We modify (3.1) to suit our needs as follows.

$$\begin{tikzcd} & \mathrm{TOP}\,(S^n) \arrow[d, two heads, "\mathrm{proj}"] \arrow[dr, two heads, "\mathrm{proj} \times \mathrm{ev}_1"] & \\ S^n \arrow[ur, dashed, "s"] \arrow[r, "\Sigma\circ\delta"'] & \mathrm{TOP}(S^n)\big/\mathrm{O}(n) \arrow[r, "\mathrm{proj} \times \mathrm{ev}_1", "\cong"'] \arrow[d, two heads, "\iota"] & \mathrm{TOP}(S^n)\big/\mathrm{O}(n+1) \times S^n \\ & \mathrm{BO}\,(n) & \end{tikzcd}$$

By (1.5) the right-hand triangle commutes. The horizontal composition yields the simpler map $(\Sigma \circ \delta) \times \delta$. The composition $\iota \circ \Sigma \circ \delta$ is by Definition 1.24 the classifying map of the pullback $\delta^*TS^n_\Sigma$ of the pseudo tangent bundle TS^n_Σ of S^n_Σ. Since ι is a fibration with homotopy fiber $\mathrm{TOP}\,(S^n)$, the space of nullhomotopies of $\iota \circ \Sigma \circ \delta$ is homotopy equivalent to the space of lifts of $\Sigma \circ \delta$ to s. Using the right-hand horizontal homeomorphism provided by Proposition 1.21, it is homotopy equivalent to the space of lifts of $(\Sigma \circ \delta) \times \delta$ to s as depicted in (3.1). $\square$

Remark 3.5. Any hypothetical lift $\overline{s} : S^n \to \mathrm{Diff}\,(S^n_\Sigma)$ of δ entails a preferred trivialization of $\delta^* T S^n_\Sigma$ as follows. Recall the defining property (1.6) for a pseudo diffeomorphism. We consider it at $1 \in S^n$ with $f \mathrel{\widehat{=}} \overline{s}(p)$ and $H \mathrel{\widehat{=}} H^p$. Then the left lower composition is $[\overline{s}(p)_* \circ \Sigma(1)]_{\mathrm{O}(n)} = [\overline{s}(p)]_{\mathrm{O}(n)}$, since $\Sigma(1) \in \mathrm{O}\,(n)$ by convention. On the other hand, the upper right composition is $\Sigma(\overline{s}(p)(1)) = \Sigma(\delta(p))$. The two maps $\overline{s}(p) \simeq \Sigma(\delta(p))$ are homotopic via $H^p(1,-)$ in ${}^{\mathrm{TOP}(S^n)}\!\big/_{\mathrm{O}(n)}$ by (1.6). Consequently we obtain a homotopy of the compositions into $\mathrm{BO}\,(n)$. This amounts to an isotopy and thus a vector bundle isomorphism $\overline{s}(p)_* T_1 S^n_\Sigma \cong T_{\delta(p)} S^n_\Sigma$, $p \in S^n$, also induced by $H^p(1,-)$. Since $\overline{s}(-)_*(T_1 S^n_\Sigma \times S^n)$ is a trivial bundle, so is $\delta^* T S^n_\Sigma$. In other words, we obtain a continuous choice in $p \in S^n$ of representatives for the restrictions $\Sigma(\delta(p))|_{S^n \setminus -1}$. We call these $\chi_p : T_1 S^n_\Sigma = S^n \setminus -1 \to S^n \setminus q_p = T_{\delta(p)} S^n_\Sigma$. Since $\Sigma(1) \sim \mathrm{id}$, we may impose $\chi_1 = \mathrm{id}$ without loss of generality.

We now prove a series of lemmata for the proof of Obstruction 3.2. Perhaps some of these are of independent interest. Let $f : S^n \longrightarrow S^n$ be a self-map of S^n. Consider the evaluation map $\mathrm{ev}_1 : \mathrm{SO}\,(n+1) \to S^n$ at the basepoint $1 \in S^n$. This is the standard projection with fiber $\mathrm{SO}\,(n)$. It is a fibration, as stated in [MT91, Theorem 6.5(2)]. We may consider lifts of f as shown in

$$
\begin{tikzcd}
 & \mathrm{SO}\,(n+1) \arrow[d, two heads, "\mathrm{ev}_1"] \\
S^n \arrow[ur, dashed, "j"] \arrow[r, "f"'] & S^n .
\end{tikzcd}
\tag{3.3}
$$

Lemma 3.6. *Let f be given as above. A lift j of f as depicted in* (3.3) *exists if and only if the pullback bundle $f^* T S^n$ is a trivial bundle.*

Proof. ($\Rightarrow$) Assume we have given a lift j of f as in (3.3). We wish to find a family of isomorphisms $\psi_p : \mathbb{R}^n \to (f^* T S^n)_p = T_{f(p)} S^n$. Write the matrix $j(p) = (j_1(p), j_2(p), \ldots, j_{n+1}(p))$ as a tuple of its column vectors. We know that $\mathrm{ev}_1 \circ j = f$, thus $j_1(p) = \mathrm{ev}_1(j(p)) = f(p)$, meaning that the first column vector $j_1(p)$ in the matrix $j(p)$ is precisely the point $f(p)$, written as vector $f(p) \in \mathbb{R}^{n+1}$. Since $j(p) \in \mathrm{SO}\,(n+1)$, the vectors $j_2(p), \ldots, j_{n+1}(p)$ span the subspace of $\mathbb{R}^{n+1}$ orthogonal to $j_1(p)$, which is precisely $T_{f(p)} S^n$. It has become evident that the family $\psi_p : (a_1, \ldots, a_n) \mapsto (a_1 j_2(p), \ldots, a_n j_{n+1}(p))$ is a family of isomorphisms trivializing $f^* T S^n$.
($\Leftarrow$) Conversely, assume we are given a family of isomorphisms $\psi_p : \mathbb{R}^n \to T_{f(p)} S^n$ trivializing $f^* T S^n$. Without loss of generality, we may assume these isomorphisms to be orientation-preserving. We provide a lift j of f as in (3.3). Motivated by the argument above, we simply define $j_i(p) := \psi_p(e_{i-1})$, for $i = 2, \ldots n+1$, where e_i is the i-th standard basis vector of $\mathbb{R}^n$. Define further $j_1(p) := f(p)$. It is easily checked that $j := (j_1, \ldots, j_{n+1}) : S^n \longrightarrow \mathrm{SO}\,(n+1)$ is well-defined since we assumed ψ_p is a linear map and because $T_{f(p)} S^n$ is orthogonal to $f(p)$. Obviously, f factors as $\mathrm{ev}_1 \circ j$. $\square$

Corollary 3.7. *Let $\Sigma \in \mathfrak{S}\,(S^n)$ be a pseudo smooth structure. Let f be given as above. A lift of f as depicted in* (3.3) *exists if and only if the pullback bundle $f^* T S^n_\Sigma$ is a trivial bundle.*

Proof. We know that $f^*TS^n_\Sigma$ is trivial if and only if f^*TS^n is trivial, by Lemma 1.32. Lemma 3.6 then finishes the proof. □

Lemma 3.8 (Mimura-Toda [MT91])**.** *Consider an odd* $n \geq 5$. *There is a map* $j : S^n \to \mathrm{SO}\,(n+1)$ *such that the composition* $\mathrm{ev}_1 \circ j : S^n \to S^n$ *has degree* 2.

Proof. In [MT91, Theorem 6.13] is stated that $(\mathrm{ev}_1)_*(\pi_n(\mathrm{SO}\,(n+1))) = 2\pi_n(S^n)$ for $n \neq 7$ and $(\mathrm{ev}_1)_*(\pi_n(\mathrm{SO}\,(n+1))) = \pi_n(S^n)$ for $n = 7$. It remains only to choose a preimage $j \in \pi_n(\mathrm{SO}\,(n+1))$ of $2 \in \pi_n(S^n)$. □

In essence, this is due to the tangent bundle classified by $S^{n+1} \to \mathrm{BSO}\,(n+1)$ having Euler class two. Recall that $d \geq 1$ is the degree of the self-map δ of S^n.

Lemma 3.9. *The pullback bundle* δ^*TS^n *of the tangent bundle* TS^n *along* δ *is trivial if and only if* n *is odd and* d *is even, or* $n = 7$.

Proof. ($\Rightarrow$) We show the negation of this implication. In other words, if $n \neq 7$ and [n is even or d is odd], then δ^*TS^n is not trivial. As first case, assume n was even. Then we have $e(\delta^*TS^n) = \delta^*e(TS^n) = \deg(\delta) \cdot e(TS^n) = 2d$, since n is even. Here e denotes the Euler class of a given bundle. Since the Euler class of δ^*TS^n is non-zero, the bundle δ^*TS^n cannot posses a nowhere-zero section, hence it cannot be trivial either.
As second, complementary case we assume d is odd and $n \neq 7$ is odd. Assume δ^*TS^n was trivial under these circumstances. Then by Lemma 3.6, δ lifts to $j : S^n \to \mathrm{SO}\,(n+1)$. Since $d = \deg(\delta)$ is odd, this contradicts the statement in [MT91] as well as Lemma 3.8, being that $(\mathrm{ev}_1)_*(\pi_n(\mathrm{SO}\,(n+1))) = 2\pi_n(S^n)$ for $n \neq 7$.
($\Leftarrow$) Assuming n is odd and d is even, or $n = 7$, we wish to show that δ^*TS^n is trivial. If $n = 7$, then δ^*TS^7 is trivial by Remark 1.50. Assume next we have given an odd $n \geq 5$ and an even $d \geq 1$. Lemma 3.8 provides a map $j : S^n \to \mathrm{SO}\,(n+1)$ such that the composition $\mathrm{ev}_1 \circ j$ has degree 2. Since δ has even degree, it factors via the given degree 2 map $\mathrm{ev}_1 \circ j$, i. e. $\delta \simeq \mathrm{ev}_1 \circ j \circ f$ for some $f : S^n \to S^n$. By Lemma 3.6, $(\mathrm{ev}_1 \circ j \circ f)^*TS^n$ is a trivial bundle. Since pullbacks via homotopic maps yield isomorphic bundles, δ^*TS^n is also a trivial bundle. □

Proof of Obstruction 3.2. Lemma 3.9 together with Lemma 1.32 guarantee that the pullback bundle $\delta^*TS^n_\Sigma$ of the pseudo tangent bundle TS^n_Σ along δ is trivial if and only if n is odd and d is even, or $n = 7$. By Proposition 3.1, these are equivalent to finding a lift s as desired. By Remark 3.5, these equivalent conditions are in fact an obstruction to finding a lift $\overline{s}$ to $\mathrm{Diff}\,(S^n_\Sigma)$. □

Remark 3.10 (Existence of a lift for the standard sphere)**.** Let $n \geq 5$ odd and $d = 2$ or $n = 7$. Then Lemma 3.9 guarantees that the pullback bundle δ^*TS^n of the tangent bundle TS^n along δ is trivial. By Lemma 3.6, δ factors via the evaluation map $\mathrm{ev}_1 : \mathrm{SO}\,(n+1) \to S^n$. Since $\mathrm{SO}\,(n+1) \subset \mathrm{Diff}\,(S^n)$, we obtain an induced lift as desired.

3.2. Second obstruction: Lifting to the group of diffeomorphisms

In this section we investigate the possibility of lifting the given self-map δ of S^n to a map $\overline{s} : S^n \to \operatorname{Diff}(S^n_\Sigma)$. In view of Obstruction 3.2, we must meet specific conditions as given in Setup 3.11, which shall apply throughout the rest of this thesis. Although we could employ any admissible s, we use Construction 3.4 for technical reasons.

Setup 3.11. Let $\Sigma \in \mathfrak{S}(S^n)$ denote a pseudo smooth structure. The construction of a hypothetical lift $\overline{s} : S^n \to \operatorname{Diff}(S^n_\Sigma)$ as in (2.3) requires (i), from which (ii) and (iii) follow.

(i) A trivialization $\chi : T_1 S^n_\Sigma \times S^n \to \delta^* T S^n_\Sigma$ of the pullback $\delta^* T S^n_\Sigma$ of the associated pseudo tangent bundle $T S^n_\Sigma$, satisfying $\chi_1 = \mathrm{id} : T_1 S^n_\Sigma \to T_1 S^n_\Sigma$,

(ii) an odd dimension n of S^n,

(iii) an even degree $\deg(\delta) = d$, unless perhaps in the special case $n = 7$.

From these conditions one obtains a lift $s : S^n \longrightarrow \operatorname{TOP}(S^n)$ of $\delta^*\Sigma$ and δ, in the sense of (3.1), satisfying $s(1) = \mathrm{id}_{S^n}$ via Construction 3.4. It is adjoint to a degree d multiplicative structure μ on S^n given by $\mu(p, q) = s(p)(q)$, by Lemma 1.45.

Not all of the choices and conditions above are obviously justified as necessary assumptions. Obstruction 3.2 requires all but the normalization of the trivialization of $\delta^* T S^n_\Sigma$ via $\chi_1 = \mathrm{id} : T_1 S^n_\Sigma \to T_1 S^n_\Sigma$, i. e. the choice of a reduced trivialization. We check that there is no loss of generality, i. e. that we may still obtain any hypothetical homotopy lift $\overline{s}$.

Lemma 3.12. *Any hypothetical homotopy lift $\overline{s} : S^n \to \operatorname{Diff}_0(S^n_\Sigma)$ of δ is isotopic to a strict lift $\overline{s}'$ of some map $s : S^n \longrightarrow \operatorname{TOP}(S^n)$ as given via Construction 3.4.*

Here $\operatorname{Diff}_0(S^n_\Sigma)$ denotes the identity component of $\operatorname{Diff}(S^n_\Sigma)$. For better readability, we postpone proofs until after the main statement. The other connected components of $\operatorname{Diff}(S^n_\Sigma)$ are treated by

Remark 3.13. Let $f \in \operatorname{Diff}(S^n_\Sigma)$. Then multiplication with f induces a homotopy equivalence $\operatorname{Diff}(S^n_\Sigma) \to \operatorname{Diff}(S^n_\Sigma)$, $g \mapsto f \circ g$, by Lemma A.10. This can be restricted to $\operatorname{Diff}_0(S^n_\Sigma)$ and thus yields a homotopy equivalence between $\operatorname{Diff}_0(S^n_\Sigma)$ and the component of f in $\operatorname{Diff}(S^n_\Sigma)$. Therefore the theory of liftings $\overline{s} : S^n \to \operatorname{Diff}_0(S^n_\Sigma)$ carries over to any other component of $\operatorname{Diff}(S^n_\Sigma)$.

Using the setup we created above, we wish to come up with a good description of an obstruction to finding a homotopy lift $\overline{s} : S^n \to \operatorname{Diff}(S^n_\Sigma)$ of δ as in (2.3). Recall the canonical left $\operatorname{TOP}(S^n)$-action on ${}^{\operatorname{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ given by composition with a representative. We assemble the information from Setup 3.11 into the following diagram.

$$\begin{tikzcd} S^n \times S^n \arrow[r, "s\times\Sigma"] \arrow[d, "\mu"] & \mathrm{TOP}\,(S^n) \times {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \arrow[d, "\circ"] \\ S^n \arrow[r, "\Sigma"] & {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \end{tikzcd} \tag{3.4}$$

Here μ denotes the multiplicative structure induced by s as in Setup 3.11. It is very important to understand that we may set up this square without the existence of a lift $\overline{s}$ to the pseudo diffeomorphisms $\mathrm{Diff}\,(S^n_\Sigma)$, but already when Obstruction 3.2 vanishes. In contrast, the homotopy commutativity of this square will be the subject of Obstruction 3.16 below. In general, there is no reason to expect (3.4) to commute up to homotopy. We have the following positive partial result.

Lemma 3.14. *With Setup 3.11,* (3.4) *commutes strictly when restricted to $S^n \vee S^n$.*

Remark 3.15. For a general homotopy lift $s : S^n \to \mathrm{TOP}\,(S^n)$ of δ, in general (3.4) does not commute strictly when restricted to $S^n \vee S^n$. This is why the specific choice of lift s via Construction 3.4 is important.

The following obstruction is the main result of this section and in a way determines whether or not we may find the desired homotopy lift $\overline{s} : S^n \to \mathrm{Diff}\,(S^n_\Sigma)$ of δ.

Obstruction 3.16. *There exists a homotopy lift $\overline{s}$ of s in* (2.3) *if and only if* (3.4) *commutes up to homotopy.*

Equivalently, one can require the homotopy to be relative to $S^n \vee S^n$. However, the different homotopies will *not* be homotopic via a higher homotopy in general.

Proof of Obstruction 3.16. Assume there exists a (homotopy) lift $\overline{s}$ of s in (2.3). By Lemma 3.12, $\overline{s}$ is isotopic to a (strict) lift of some map $s : S^n \longrightarrow \mathrm{TOP}\,(S^n)$ as given via Construction 3.4. We continue to denote this strict lift by $\overline{s}$. It now makes sense to relate $\overline{s} = (s, H)$ to (3.4). Then for each $p \in S^n$, the pairs $(s(p), H^p)$ fit into the defining diagram (1.6) for pseudo diffeomorphisms,

$$\begin{tikzcd} S^n \arrow[r, "\mu(p,-)"] \arrow[d, "\Sigma"] \arrow[dr, phantom, "H^p"] & S^n \arrow[d, "\Sigma"] \\ {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \arrow[r, "s(p)_*"] & {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} . \end{tikzcd}$$

This is simply (3.4) restricted to $\{p\} \times S^n$ and flipped diagonally. Since s is a continuous map $S^n \to \mathrm{Diff}\,(S^n_\Sigma)$, H becomes a proper homotopy for the homotopy commutativity of (3.4).
Conversely, given a homotopy H making (3.4) commute up to homotopy, we may restrict the diagram to $\{p\} \times S^n$ for each $p \in S^n$ to see that $s(p)$ can be upgraded to a pseudo diffeomorphism using H^p. Since the latter homotopies match up nicely, we obtain a continuous map $\overline{s} : S^n \to \mathrm{Diff}\,(S^n_\Sigma)$. It is a lift of δ because it is a lift of s by construction. $\square$

Before we turn to the omitted proofs, let us state a technical lemma, needed later for a proof in Section 4.4.

Lemma 3.17. *The lift $\overline{s}' : S^n \to \mathrm{Diff}\,(S^n_\Sigma)$ constructed in Lemma 3.12 induces a homotopy H for* (3.4) *as given by Obstruction 3.16. When restricting the diagram to $S^n \vee S^n$, this homotopy satisfies $H|_{S^n \vee S^n \times I} \simeq \mathrm{stationary}\,\mathrm{rel}\,\partial I$, i. e. it is homotopic to a stationary homotopy relative to ∂I.*

Proof of Lemma 3.12. Let $\Sigma : S^n \to {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)}$ be a pseudo smooth structure. Let $\overline{s} : S^n \longrightarrow \mathrm{Diff}\,(S^n_\Sigma)$ be a hypothetical homotopy lift as in (2.3), i. e. $\mathrm{ev}_1 \circ \overline{s} \simeq \delta$. By Corollary A.4 we may assume that $\overline{s}$ is a strict lift $\mathrm{ev}_1 \circ \overline{s} = \delta$. In the following, we construct lifts s and $\overline{s}''$ as outlined in the lemma. In (i), we use the existence of inverses in $\mathrm{Diff}\,(S^n_\Sigma)$ up to homotopy. In (ii), we use the fibration property of $\mathrm{Diff}\,(S^n_\Sigma) \to \mathrm{TOP}\,(S^n)$. Recall the notation $\overline{s}(p) = (\overline{s}(p), H^p) \in \mathrm{Diff}\,(S^n_\Sigma)$ for a pseudo diffeomorphism.
(i) In a first step, we define an $\overline{s}'$ with $\overline{s}'(1)$ having the identity as underlying homeomorphism $S^n \to S^n$. Denote by $\overline{s}(1)^{-1}$ the inverse up to homotopy of $\overline{s}(1)$ in $\mathrm{Diff}\,(S^n_\Sigma)$, compare Remark 1.26. Define $\overline{s}'(p) := \overline{s}(p) \circ \overline{s}(1)^{-1}$. Since $\overline{s}(1) \in \mathrm{Diff}_0(S^n_\Sigma)$, we have $\overline{s}(1) \simeq \mathrm{id}_{S^n_\Sigma}$ and thus $\overline{s}' \simeq \overline{s}$. Note for the end of (ii) that $\overline{s}'(1) \simeq (\mathrm{id}_{S^n}, \mathrm{stationary})$ via a canonical homotopy satisfying $(H')^1 \simeq \mathrm{stationary}\,\mathrm{rel}\,\partial I$, compare Remark 1.26. Also, $\mathrm{ev}_1 \circ \overline{s}' = \delta$ is immediate using $\overline{s}(1)^{-1}(1) = 1$.
(ii) In this step, we identify an explicit s as claimed together with a strict lift $\overline{s}''$ of s and show that $\overline{s}' \simeq \overline{s}''$ as maps to $\mathrm{Diff}\,(S^n_\Sigma)$. By Remark 3.5, $\overline{s}' : S^n \to \mathrm{Diff}\,(S^n_\Sigma)$ entails a trivialization of $\delta^* T S^n_\Sigma$ via $\chi'_p : T_1 S^n_\Sigma \to T_{\delta(p)} S^n_\Sigma$, $p \in S^n$. With $s(p) := (\chi')^c_p$ via Construction 3.4, we have $s(p) \simeq \overline{s}'(p)$ in $\mathrm{TOP}\,(S^n)$ as in (3.5), by the arguments in Remark 3.5.

$$\begin{tikzcd}
S^n \arrow[r, bend left, "s(p)=(\chi')^c_p"] \arrow[r, bend right, "\overline{s}'(p)"'] \arrow[r, phantom, "(H')^p(1,-)"] & S^n
\end{tikzcd} \tag{3.5}$$

Use the fibration property of $\mathrm{Diff}\,(S^n_\Sigma) \to \mathrm{TOP}\,(S^n)$ as seen in Lemma A.3 to obtain a strict lift $\overline{s}''$ of s, with $\overline{s}'' \simeq \overline{s}'$ isotopic in $\mathrm{Diff}\,(S^n_\Sigma)$. This proves Lemma 3.12.
Note for the proof of Lemma 3.17, that by the construction of $\overline{s}''$ via (A.1), $\overline{s}''(1) \simeq (\mathrm{id}_{S^n}, \mathrm{stationary})$ via a canonical homotopy satisfying $(H'')^1 \simeq \mathrm{stationary}\,\mathrm{rel}\,\partial I$, since it is the concatenation of homotopies satisfying this property, see (i). $\square$

Proof of Lemma 3.14. Let $\Sigma \in \mathfrak{S}(S^n)$ be a pseudo smooth structure. Recall Setup 3.11 with $s : S^n \to \mathrm{TOP}(S^n)$ via Construction 3.4. On $\{1\} \times S^n$ we have $\Sigma(\mu(1,q)) = \Sigma(q) = s(1)_* \circ \Sigma(q)$, using that we forced $s(1) = \mathrm{id}$. For $(p,1) \in S^n \times \{1\}$ the two compositions $\Sigma(\mu(p,1)) = \Sigma(s(p)(1)) = \Sigma(\delta(p))$ and $s(p)_* \circ \Sigma(1)$ are equal by the calculations in Construction 3.4. $\square$

Proof of Lemma 3.17. By Obstruction 3.16, the lift $\overline{s}''$ induces a homotopy K for the commutativity of (3.4). We show that upon restricting the diagram to $S^n \vee S^n$, the restriction satisfies $K|_{S^n \vee S^n \times I} \simeq \text{stationary rel}\,\partial I$, i. e. it is homotopic to a stationary homotopy relative to ∂I.
Take a look at (3.4) restricted to $\{1\} \times S^n$. Recall that $\overline{s}''(1) = (\mathrm{id}_{S^n}, (H'')^1)$. We have seen above that $(H'')^1 \simeq \text{stationary rel}\,\partial I$. The same is then true for the induced homotopy K on $\{1\} \times S^n$.
On the other hand, for $p \in S^n$ the homotopy $(H'')^p(1,-)$ restricted to $q = 1$ is likewise homotopic to the stationary homotopy rel ∂I. This is due to it being a concatenation of $(H')^p(1,-)$ and its reverse, as given via (A.1) and (3.5). This translates analogously to $K|_{S^n \times \{1\} \times I} \simeq \text{stationary rel}\,\partial I$. $\square$

The problem behind the technicalities is the following. When reducing to the case $s(1) = \mathrm{id}$ and a strict lift $\overline{s}$ of s, we can obtain $\overline{s}(1) = (\mathrm{id}, H^1)$ with $H^1 \simeq \text{stationary rel}\,\partial I$ only, since $\mathrm{Diff}\,(S^n_\Sigma)$ is not a group. The homotopy H^1 cannot be forced stationary canonically.

4. A homotopy operation

In this section we introduce a certain homotopy operation, i. e. a natural transformation (of sets) between the homotopy group functors

$$\pi_n \Rightarrow \pi_{2n}.$$

In Section 4.3 we will see how it facilitates the description of Obstruction 3.16 in terms of elements in certain homotopy groups. Denote by **TopMon** the category of path-connected topological monoids together with continuous monoidal maps. Most importantly, these are topological spaces A that come equipped with a multiplicative structure $A \times A \to A$ with unit, which also acts as basepoint. Furthermore, the delooping functor B can be applied to these spaces. Fix a multiplication map $\mu : S^n \times S^n \to S^n$. It is appropriate to think of the multiplication μ that was used in Section 3.2. Assume for now that the basepoint of S^n is a strict unit for μ, such as is the case if S^n is an H-space. As Adams has shown, this is actually quite restrictive. An adapted alternative in view of Lemma 1.45 is discussed in Section 4.3 below. We perform a general construction in this section.

Construction 4.1. Let A be a topological monoid, and $f : S^n \to A$ represent an element in $\pi_n(A)$. Using the multiplications μ on S^n and ν on A, we may set up

$$\begin{tikzcd} S^n \times S^n \arrow[r, "f\times f"] \arrow[d, "\mu"'] & A \times A \arrow[d, "\nu"] \\ S^n \arrow[r, "f"] & A. \end{tikzcd} \tag{4.1}$$

From this we obtain two different maps $f_1, f_2 : S^n \times S^n \to A$, namely as the upper right composition and lower left composition, respectively.

$$\begin{tikzcd} S^n \times S^n \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & A \end{tikzcd}$$

Consider the cell decomposition $S^n \times S^n = (S^n \vee S^n) \bigcup_\psi D^{2n}$, where the disk D^{2n} is attached to $S^n \vee S^n$ via the attaching map $\psi|_{\partial D^{2n}} : \partial D^{2n} \to S^n \vee S^n$, the restriction of a map $\psi : D^{2n} \to S^n \times S^n$. We know that on the wedge μ and ν simply restrict to the fold maps $S^n \vee S^n \to S^n$ and $A \vee A \to A$. Also, f is a pointed map and thus preserves the unit. Therefore the two compositions f_1 and f_2 agree on $S^n \vee S^n$. Define a map

$$F : S^{2n} \to A$$

on the upper and lower hemispheres D^{2n}_+ and D^{2n}_- of S^{2n} via $F|_{D^{2n}_+} := f_1 \circ \psi$ and $F|_{D^{2n}_-} := f_2 \circ \psi$. This is well-defined on the equator since $f_1 \circ \psi|_{S^{2n-1}}$ and $f_2 \circ \psi|_{S^{2n-1}}$ agree.

There is a short explanation in Remark 4.27 why it is necessary to require a strict two-sided unit for μ in the present setup. The specialized setup of Section 4.3, however, works also in the more general setting of $(d,1)$-multiplications μ.

Definition 4.2. We denote the resulting homotopy class of F by $\mathrm{HO}([f]) \in \pi_{2n}(A)$. Although the notation does not specify, HO is highly dependent on the choices of μ and ν. The latter generally shows up naturally, and the choice of μ is implied in the cases we consider.

Lemma 4.3. *The element* $\mathrm{HO}(f) \in \pi_{2n}(A)$ *is independent of the choice of representative f of $[f] \in \pi_n(A)$.*

Proof. Assume given $f \simeq f'$ via a homotopy H. This induces homotopies $H_1 : f_1 \simeq f_1'$ and $H_2 : f_2 \simeq f_2'$ on the upper right compositions and lower left compositions $S^n \times S^n \to A$, respectively. Both homotopies agree on $(S^n \vee S^n) \times I$ for the same reason f_1 and f_2 did. Therefore we may apply the same patching argument once more to obtain a map $K : S^{2n} \times I \to A$, which provides a homotopy $F \simeq F'$. □

Lemma 4.4. *The homotopy operation* $\mathrm{HO} : \pi_n \Rightarrow \pi_{2n}$ *is a natural transformation of functors* **TopMon** $\to$ **Set**.

However, it is *not* a natural transformation of functors **TopMon** $\to$ **Gr**. We will see details regarding this fact in Lemma 4.9.

Proof. Let A and A' be topological monoids and $m : A \to A'$ a morphism. We must show that HO is natural with respect to maps of topological monoids. In other words, given any $[f] \in \pi_n(A)$, we need to show $m_*\mathrm{HO}([f]) = \mathrm{HO}(m_*[f])$. Consider

$$\begin{tikzcd} S^n \times S^n \arrow[r, "f\times f"] \arrow[d, "\mu"'] & A \times A \arrow[r, "m\times m"] \arrow[d, "\nu"] & A' \times A' \arrow[d, "\nu'"] \\ S^n \arrow[r, "f"] & A \arrow[r, "m"] & A'. \end{tikzcd} \tag{4.2}$$

The right-hand square in (4.2) commutes. A straight forward analysis employing the definition of F shows that the two maps $m \circ F_f$ and $F_{m\circ f}$ agree. □

4.1. Extension to the category of spaces

In this section we define a related homotopy operation $\mathrm{HO} : \pi_{n+1} \Rightarrow \pi_{2n+1}$ on the category of spaces. Recall the functors $\Omega : \mathbf{Top}_* \to \mathbf{TopMon}$ and $\mathrm{B} : \mathbf{TopMon} \to \mathbf{Top}_*$, which come with a natural homotopy equivalence $\Omega \mathrm{B} A \simeq A$ for any grouplike topological monoid A. This is a special case of the group completion theorem [MS76] as explained in [Hat14, Proposition D.2]. Also, we have the natural equivalence $\pi_{n+1}(X) \cong \pi_n(\Omega X)$ for any pointed space.

Definition 4.5. Define a natural homotopy operation $\mathrm{HO} : \pi_{n+1} \Rightarrow \pi_{2n+1}$ on $\mathbf{Top}_*$,

$$\pi_{n+1}(-) \cong \pi_n(\Omega -) \Longrightarrow \pi_{2n}(\Omega -) \cong \pi_{2n+1}(-). \tag{4.3}$$

This naturally extends the definition of HO to $\mathbf{Top}_*$. Indeed, given a topological monoid A, we have $\Omega \mathrm{B}A \simeq A$. Then $\mathrm{HO}_{\mathbf{TopMon}}$ is defined on $\pi_n(A) \cong \pi_n(\Omega \mathrm{B}A) \cong \pi_{n+1}(\mathrm{B}A)$ and agrees on the latter with $\mathrm{HO}_{\mathbf{Top}_*}$. In this sense, $\mathrm{HO}_{\mathbf{Top}_*}$ naturally extends $\mathrm{HO}_{\mathbf{TopMon}}$, thought of as defined on the image of B in $\mathbf{Top}_*$.

Observation 4.6. The two functors $\pi_{n+1}, \pi_{2n+1} : \mathbf{Top}_* \to \mathbf{Set}$ are represented by $[S^{n+1}, -]$ and $[S^{2n+1}, -]$. The Yoneda lemma consequently identifies a homotopy class of map $\sigma : S^{2n+1} \to S^{n+1}$ as representing the natural transformation HO. Note that the homotopy class of σ depends on the choice of multiplication map μ, see Section 4.4.

Recall that $\Omega \mathrm{B}A \simeq A$ naturally. Consequently, an element $\alpha \in \pi_n(A) \cong [S^n, \Omega \mathrm{B}A]$ has a corresponding adjoint $S^1 \wedge S^n \to \mathrm{B}A$, which we will denote by $\alpha^{\mathrm{ad}} \in \pi_{n+1}(\mathrm{B}A)$ for $n \geq 0$. Let $f : A \to A'$ denote a topological monoid map. Let $\mathrm{B}f$ denote the delooping of f. We deduce some naturality results. These allow us to freely interchange and cease to distinguish α and α^{ad} and their images $\mathrm{HO}(\alpha)$ and $\mathrm{HO}\left(\alpha^{\mathrm{ad}}\right) = \alpha^{\mathrm{ad}} \circ \sigma$. Henceforth we drop $(-)^{\mathrm{ad}}$ from notation.

Lemma 4.7. *Let A be a topological monoid, and $\alpha, \alpha_1, \alpha_2 \in \pi_n(A)$ and let $f : A \to A'$ be a topological monoid map. Then we have the following two naturality statements.*

- *Naturality with respect to pushforwards:* $(f_*\alpha)^{\mathrm{ad}} = (\mathrm{B}f)_*\left(\alpha^{\mathrm{ad}}\right)$,
- *Naturality with respect to sums:* $(\alpha_1 + \alpha_2)^{\mathrm{ad}} = \alpha_1^{\mathrm{ad}} + \alpha_2^{\mathrm{ad}}$.

Lemma 4.8. *Let A be a topological monoid and $\alpha \in \pi_n(A)$. Using naturality with respect to a given monoid map $f : A \to A'$, one obtains*

$$\mathrm{HO}\left(\alpha^{\mathrm{ad}}\right) = \mathrm{HO}(\alpha)^{\mathrm{ad}}, \qquad \mathrm{HO}\left((f_*\alpha)^{\mathrm{ad}}\right) = (f_*\mathrm{HO}(\alpha))^{\mathrm{ad}} = (\mathrm{B}f)_*\left(\mathrm{HO}(\alpha)^{\mathrm{ad}}\right),$$

amongst other equations. In more general words, HO, $(-)^{\mathrm{ad}}$ *and* f_* *interchange freely, as long as one interchanges* $f_* \leftrightarrow (\mathrm{B}f)_*$ *correctly.*

Proof of Lemma 4.7. As for the first part, the formula $(f_*\alpha)^{\mathrm{ad}} = (\mathrm{B}f)_*\left(\alpha^{\mathrm{ad}}\right)$ is fairly straightforward. By the naturality of $\Omega \mathrm{B}A \simeq A$, the map $f_*\alpha : S^n \to \Omega \mathrm{B}A'$ factors via $\Omega \mathrm{B}A$ along $\Omega \mathrm{B}f$. Since it factors via a loop map, the adjoint $(f_*\alpha)^{\mathrm{ad}} : S^1 \wedge S^n \to \mathrm{B}A'$ also factors via $\mathrm{B}A$ along $\mathrm{B}f$.
The second part seems even more natural, yet requires some explanation. In the definition of α^{ad}, we use the characterization of the adjoint of $\alpha \in [S^n, \Omega \mathrm{B}A] \cong [S^1 \wedge S^n, \mathrm{B}A]$ via

$$\begin{tikzcd} S^1 \wedge S^n \arrow[r, "\mathrm{id} \wedge \alpha"] \arrow[dr, "\alpha^{\mathrm{ad}}"'] & S^1 \wedge \Omega \mathrm{B}A \arrow[d, "\mathrm{ev}"] \\ & \mathrm{B}A. \end{tikzcd} \tag{4.4}$$

Recall that smash products are distributive over the wedge sum up to natural homeomorphism. Since the pinch map only affects one of the coordinates, we obtain the left commutative triangle in (4.5) as long as $n \geq 1$. We set up

$$\begin{tikzcd} S^1 \wedge S^n \arrow[r, "\mathrm{id} \wedge \mathrm{pinch}"] \arrow[dr, "\mathrm{pinch}"'] & S^1 \wedge (S^n \vee S^n) \arrow[r, "\mathrm{id} \wedge (\alpha_1 \vee \alpha_2)"] \arrow[d, "\cong"] & S^1 \wedge A \arrow[d, "\simeq"] \\ & S^{n+1} \vee S^{n+1} \arrow[dr, "\alpha_1^{\mathrm{ad}} \vee \alpha_2^{\mathrm{ad}}"] & S^1 \wedge \Omega \mathrm{B}A. \arrow[d, "\mathrm{ev}"] \\ & & \mathrm{B}A \end{tikzcd} \tag{4.5}$$

The upper horizontal composition is in the homotopy class $\mathrm{id} \wedge (\alpha_1 + \alpha_2)$, whereas the lower left diagonal composition is in the class $\alpha_1^{\mathrm{ad}} + \alpha_2^{\mathrm{ad}}$. In view of the characterization via (4.4), we see that $\alpha_1^{\mathrm{ad}} + \alpha_2^{\mathrm{ad}}$ represents the same homotopy class as $(\alpha_1 + \alpha_2)^{\mathrm{ad}}$. □

Proof of Lemma 4.8. We compute $\mathrm{HO}\left(\alpha^{\mathrm{ad}}\right)$. Since $\alpha^{\mathrm{ad}} \in \pi_{n+1}(\mathrm{B}A)$, we apply $\mathrm{HO}_{\mathbf{Top}_*}$ via (4.3),

$$\begin{aligned} [S^{n+1} \xrightarrow{\alpha^{\mathrm{ad}}} \mathrm{B}A] &\mapsto [S^n \xrightarrow{\alpha} \Omega \mathrm{B}A] \,\widehat{=}\, [S^n \xrightarrow{\alpha} A] \\ &\mapsto [S^{2n} \xrightarrow{\mathrm{HO}(\alpha)} A] \,\widehat{=}\, [S^{2n} \xrightarrow{\mathrm{HO}(\alpha)} \Omega \mathrm{B}A] \\ &\mapsto [S^{2n+1} \xrightarrow{\mathrm{HO}(\alpha)^{\mathrm{ad}}} \mathrm{B}A], \end{aligned}$$

thanks to the naturality of $\Omega \mathrm{B}A \simeq A$. The naturality with respect to a given topological monoid map $f : A \to A'$, or in other words, the interchangeability of HO, B, $(-)^{\mathrm{ad}}$ and f_*, is a conclusion of the equations above and of Lemma 4.4 and 4.7. □

We show a summation formula. Baues [Bau81] studied similar notions of additivity of homotopy classes of maps, but he does not cover the case below.

Lemma 4.9. *Let X be a based topological space. Let $\alpha, \beta \in \pi_{n+1}(X)$. Then*

$$(\alpha + \beta) \circ \sigma = \alpha \circ \sigma + \beta \circ \sigma + \mathrm{Hopf}(\sigma) \cdot [\alpha, \beta],$$

where $[\alpha, \beta]$ denotes the Whitehead product of α and β, and $\mathrm{Hopf}(\sigma)$ *denotes the Hopf invariant of the map $\sigma : S^{2n+1} \to S^{n+1}$.*

Proof. Let ι_1 and ι_2 denote the two inclusions $S^{n+1} \to S^{n+1} \vee S^{n+1}$ and denote by pinch : $S^{n+1} \to S^{n+1} \vee S^{n+1}$ the pinch map. Recall that the pinch map is the comultiplication on S^{n+1} which induces the addition on homotopy groups, i. e. $\iota_1 + \iota_2 = \mathrm{pinch}$. We do not distinguish between representatives for the classes α and β and the classes themselves. A representative of the sum $\alpha + \beta$ is given by $(\alpha \vee \beta) \circ \mathrm{pinch} = (\alpha \vee \beta) \circ (\iota_1 + \iota_2)$. In the case $S^{n+1} \vee S^{n+1}$, [Hil55, Theorem A] specializes to (4.6),

$$\pi_{2n+1}(S^{n+1} \vee S^{n+1}) \cong \pi_{2n+1}(S^{n+1}) \oplus \pi_{2n+1}(S^{n+1}) \oplus \pi_{2n+1}(S^{2n+1}). \tag{4.6}$$

$$(\iota_1 + \iota_2) \circ \sigma = \iota_1 \circ \sigma + \iota_2 \circ \sigma + [\iota_1, \iota_2] \circ (\mathrm{Hopf}(\sigma) \cdot \mathrm{id}_{S^{2n+1}}) \tag{4.7}$$

Equation (4.7) is a specialization of (4.6) using [Hil55, Equation (6.1)]. We continue now by adapting (4.7) to our situation. Composing (4.7) with $\alpha \vee \beta$ yields

$$\begin{aligned}(\alpha+\beta)\circ\sigma &= (\alpha\vee\beta)\circ\big(\iota_1\circ\sigma+\iota_2\circ\sigma+\mathrm{Hopf}(\sigma)\cdot[\iota_1,\iota_2]\big)\\ &= (\alpha\vee\beta)\circ\iota_1\circ\sigma+(\alpha\vee\beta)\circ\iota_2\circ\sigma+\mathrm{Hopf}(\sigma)\cdot(\alpha\vee\beta)\circ[\iota_1,\iota_2]\\ &= \alpha\circ\sigma+\beta\circ\sigma+\mathrm{Hopf}(\sigma)\cdot[\alpha,\beta],\end{aligned}$$

using distributivity of $(\alpha\vee\beta)\circ$ and the definition of ι_1 and ι_2. Furthermore, $[\iota_1,\iota_2]$ is by definition the Whitehead map $w : S^{2n+1}\to S^{n+1}\vee S^{n+1}$, the attaching map of the $(2n+2)$-cell in $S^{n+1}\times S^{n+1}$. Thus $(\alpha\vee\beta)\circ[\iota_1,\iota_2]=[\alpha,\beta]$. □

4.2. Extension to the category of spectra

Recall that the homotopy operation HO on $\mathbf{Top}_*$ is simply given by precomposition with $\sigma\in\pi_{2n+1}(S^{n+1})$. In order to define a stable version of the homotopy operation for the category $\mathbf{Sp}$ of spectra, we first need to stabilize σ. Denote by $\sigma_{\mathrm{s}}\in\pi_n^s$ the element in the stable homotopy groups of spheres obtained by stabilizing σ via suspension.

Definition 4.10. Let X be a spectrum. Define a natural stable homotopy operation $\mathrm{HO}:\pi_0\Rightarrow\pi_n$ on $\mathbf{Sp}$ via precomposition of stable maps with σ_{s},

$$\begin{aligned}\pi_0(\mathsf{X}) &\Rightarrow \pi_n(\mathsf{X})\\ [\alpha:\mathsf{S}\to\mathsf{X}] &\mapsto [\alpha\circ\sigma_{\mathrm{s}}:S^n\wedge\mathsf{S}\to\mathsf{X}].\end{aligned}$$

By Yoneda's lemma, we indeed obtain a natural transformation of functors $\mathbf{Sp}\Rightarrow\mathbf{Set}$. Lemma 4.12 will in fact upgrade this to a natural transformation of functors $\mathbf{Sp}\Rightarrow\mathbf{Gr}$.

Remark 4.11. The stable homotopy operation defined above is a generalization of the homotopy operation HO on spaces as in Definition 4.5. This is an easy fact since we simply replace σ by its stabilization σ_{s}. See also Lemma 4.20 and Corollary 4.21. From now on we cease to distinguish between σ and σ_{s} if the meaning is clear from context.

Lemma 4.12. *Let X be an Ω-spectrum. The stable homotopy operation given by Definition 4.10 is in fact a bilinear operation*

$$\pi_n^s\times\pi_0(\mathsf{X})\to\pi_n(\mathsf{X})\quad\textit{given by}\quad(\nu,\alpha)\mapsto\alpha\circ\nu.$$

If one fixes a stable map in π_n^s, one is left with a natural transformation $\pi_0\Rightarrow\pi_n$ of group-valued functors on Ω-spectra.

Proof. We show that the operation is linear in α. Recall the additivity formula for HO from Lemma 4.9. By [Whi78, Corollary X.7.8], Whitehead brackets vanish in an H-space. Therefore, HO is linear on the associated infinite loop space. For linearity in ν, recall that a representative of $\alpha\circ(\nu_1+\nu_2)$ on homotopy groups is given by

$$S^{n+k}\xrightarrow{\text{pinch}}S^{n+k}\vee S^{n+k}\xrightarrow{\nu_1\vee\nu_2}S^k\xrightarrow{\alpha}\mathsf{X},$$

for some $k\geq 0$. The latter composition can be identified with $\alpha\circ\nu_1\vee\alpha\circ\nu_2$, meaning the above composition is actually also a representative for $\alpha\circ\nu_1+\alpha\circ\nu_2$. □

4.3. Relation to the second obstruction

In this section we link the newly developed homotopy operation $\mathrm{HO}_{\mathbf{TopMon}} : \pi_n \Rightarrow \pi_{2n}$ to our findings in Section 3.2. Recall that we denote by $\Sigma \in \mathfrak{S}(S^n)$ a pseudo smooth structure. We will frequently identify it with a map $\Sigma : S^n \to {}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)} \cong {}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}$. Recall (3.4). The upper horizontal map is given by $s \times \Sigma$, the lower horizontal map by Σ. The resemblance of (3.4) with (4.1) is inarguable, yet not perfect. Unfortunately, ${}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)}$ is not a topological monoid. However, we may adapt Construction 4.1 to our situation as follows. The upper right and lower left compositions once more yield two maps

$$\begin{tikzcd} S^n \times S^n \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & {}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)}. \end{tikzcd}$$

The important fact derived from the properties of an element in $\pi_n(A)$ was that f_1 and f_2 agree on the wedge $S^n \vee S^n$. Since we chose $s : S^n \longrightarrow \mathrm{TOP}(S^n)$ via Construction 3.4, this is guaranteed in this situation by Lemma 3.14. In particular, we no longer require $\mu : S^n \times S^n \to S^n$ to be an H-space structure, achieving far greater generality in that respect. The rest of the construction follows through analogously.

Definition 4.13. Denote the resulting element by $\mathrm{Obs}(s, \Sigma) \in \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$.

We introduce this notation to distinguish it from the slightly different homotopy operation HO. It is not hard to see that the class $\mathrm{Obs}(s, \Sigma)$ is independent under homotopies of both s and Σ. Moreover, this class constitutes the obstruction to the original lift problem in (2.3).

Obstruction 4.14. *Let $\Sigma \in \mathfrak{S}(S^n)$ denote a pseudo smooth structure. Let $s : S^n \to \mathrm{TOP}(S^n)$ be given as in Setup 3.11. There exists a homotopy lift $\overline{s} : S^n \to \mathrm{Diff}(S^n_\Sigma)$ of s as in* (2.3) *if and only if* $\mathrm{Obs}(s, \Sigma) = 0 \in \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$.

We conclude this section with an auxiliary lemma and the proof of the obstruction.

Lemma 4.15. *When using Construction 4.1 starting with maps $f_1, f_2 : S^n \times S^n \to A$ as above, the following are equivalent:*

- *$f_1 \simeq_H f_2$ with $H|_{S^n \vee S^n}$ homotopic to the stationary homotopy rel ∂I,*
- $\mathrm{HO}(f_1, f_2) = [F] = 0 \in \pi_{2n}(A)$.

Proof of Obstruction 4.14. ($\Rightarrow$) Assume there exists a lift $\overline{s} : S^n \to \mathrm{Diff}(S^n_\Sigma)$ of $s : S^n \to \mathrm{TOP}(S^n)$ as given in Setup 3.11. Lemma 3.17 guarantees that the homotopy H of (3.4) provided by Obstruction 3.16 fulfills the requirements of Lemma 4.15. Consequently, $\mathrm{HO}(f_1, f_2) = \mathrm{Obs}(s, \Sigma) = 0 \in \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$.
($\Leftarrow$) Assume $\mathrm{Obs}(s, \Sigma) = 0 \in \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$. Then by Lemma 4.15, the two different compositions in (3.4) are homotopic. In other words, (3.4) commutes up to homotopy. By Obstruction 3.16, there exists a lift $\overline{s} : S^n \to \mathrm{Diff}(S^n_\Sigma)$ of s as in (2.3). $\square$

Proof of Lemma 4.15. ($\Rightarrow$) Let $H : f_1 \simeq f_2$ be a homotopy $H : S^n \times S^n \times I \to A$. Recall that $f_1 = f_2$ on $S^n \vee S^n$ by requirement, and that H is homotopic to a stationary homotopy on $(S^n \vee S^n) \times I$ rel ∂I. Since $(S^n \vee S^n) \times I \to S^n \times S^n \times I$ is a cofibration, we may assume that H is already stationary on $S^n \vee S^n$. We can then think of it as a homotopy $H : \left(S^n \vee S^n \bigcup_\psi D^{2n}\right) \times I \to A$. By restricting it, we have a homotopy $H : D^{2n} \times I \to A$ stationary on $\partial D^{2n} \times I$. Upon taking the quotient by the latter subset, one obtains an induced map $L : D^{2n+1} \to A$ on the quotient. The restriction to the boundary is $\mathrm{HO}(f_1, f_2)$ by construction, and L is a nullhomotopy for it.
($\Leftarrow$) Assume $F \simeq$ const via some homotopy L. We proceed backwards. View L as a map $D^{2n+1} \to A$. Extend it to $H : D^{2n} \times I \to A$ by precomposing with the quotient map $D^{2n} \times I \to {D^{2n}\times I}/{\partial D^{2n} \times I} \cong D^{2n+1}$. The map H can be considered as a homotopy stationary on ∂D^{2n} by construction, i. e. $H(-,t) \equiv f_1 \circ \psi \equiv f_2 \circ \psi$ on $\partial D^{2n} \times I$. Therefore, the extension to $\left(S^n \vee S^n \bigcup_\psi D^{2n}\right) \times I$ via the stationary homotopy $f_1|_{S^n \vee S^n} = f_2|_{S^n \vee S^n}$ on $S^n \vee S^n$ yields the desired homotopy $f_1 \simeq f_2 \,\mathrm{rel}\, S^n \vee S^n$. $\square$

4.4. The formulation in dimension seven

For the rest of Section 4 we specialize in and fix dimension $n = 7$ and $\delta = \mathrm{id}_{S^n}$. Explanations as to why things get more complicated for $n \neq 7$ and how one could possibly resolve them will be presented in Remark 4.27. We *do not* specialize in the pseudo smooth Kervaire Milnor structure $\Sigma_{\mathrm{KM}} \in \mathfrak{S}(S^7)$.
Recall that by Proposition 3.1 and Remark 3.3, the map $\Sigma : S^7 \to {\mathrm{TOP}(S^7)}/{\mathrm{O}(8)}$ lifts to the map $s : S^7 \to \mathrm{TOP}(S^7)$ along the standard projection $\mathrm{TOP}(S^7) \to {\mathrm{TOP}(S^7)}/{\mathrm{O}(8)}$, as depicted in (3.1). In other words, $\Sigma \simeq \mathrm{proj} \circ s$. This means that the upper horizontal composition in (4.8) set up below is simply $s \times \Sigma$ and that the lower horizontal composition is Σ. The right-hand square in (4.8) commutes by construction.

$$\begin{tikzcd}
S^7 \times S^7 \arrow[r, "s\times s"] \arrow[d, "\mu"'] & \mathrm{TOP}(S^7) \times \mathrm{TOP}(S^7) \arrow[r, "\mathrm{id}\times\mathrm{proj}"] \arrow[d, "\circ"] & \mathrm{TOP}(S^7) \times {\mathrm{TOP}(S^7)}/{\mathrm{O}(8)} \arrow[d, "\circ\text{-action}"] \\
S^7 \arrow[r, "s"] & \mathrm{TOP}(S^7) \arrow[r, "\mathrm{proj}"] & {\mathrm{TOP}(S^7)}/{\mathrm{O}(8)}
\end{tikzcd} \tag{4.8}$$

Note already the most important property of s for Lemma 4.19: it is adjoint to a $(1,1)$-multiplication on S^7. In analogy to Lemma 4.4 and its proof, including (4.2), it is not difficult to deduce the following fact.

Lemma 4.16. *The obstruction* $\mathrm{Obs}(s, \Sigma) \in \pi_{14}\left({\mathrm{TOP}(S^7)}/{\mathrm{O}(8)}\right)$ *lifts to the element* $s \circ \sigma \in \pi_{14}(\mathrm{TOP}(S^7))$*, i. e.* $\mathrm{proj} \circ s \circ \sigma = \mathrm{Obs}(s, \Sigma)$.

Remark 4.17. The notation $\mathrm{proj} \circ s \circ \sigma$ may be somewhat misleading in one respect. Namely, one cannot compose these maps as is. Technically, one needs to write $\mathrm{proj} \circ \left(s^{\mathrm{ad}} \circ \sigma\right)_{\mathrm{ad}}$ using the appropriate adjoints, since $s^{\mathrm{ad}} \circ \sigma \in \pi_{15}(\mathrm{BTOP}(S^7))$. In other words, the projection map preserves the homotopy operation only in the confined setting of (4.8) and Lemma 4.16. In particular, one *cannot* conclude that $\Sigma \circ \delta \circ \sigma = \mathrm{Obs}(s, \Sigma) \in \pi_{2n}\left({\mathrm{TOP}(n)}/{\mathrm{O}(n)}\right)$ using $\mathrm{proj} \circ s = \Sigma \circ \delta$ from Proposition 3.1.

Example 4.18. In view of Obstruction 4.14, one could ask whether the lift $s \circ \sigma \in \pi_{15}(\mathrm{BTOP}\,(S^7))$ must be trivial in order for a lift map $\overline{s} : S^7 \to \mathrm{Diff}\,(S^7_\Sigma)$ of s to exist. This is actually not the case. For example, the Cayley multiplication is not homotopy associative, by Proposition 1.49. This prevents $s_{\mathrm{Cayley}} \circ \sigma = 0$ although $s_{\mathrm{Cayley}} : S^7 \to \mathrm{SO}\,(8) \subset \mathrm{Diff}\,(S^7)$ is a lift, see Example 2.4.

In spite of this perhaps disappointing fact, the lift $s \circ \sigma \in \pi_{14}(\mathrm{TOP}\,(S^7))$ comes with the advantages associated to a topological monoid $\mathrm{TOP}\,(S^7)$. For example, the operation HO is defined in the standard way introduced in Definition 4.2. In particular, one can now take a closer look at the representing element σ for HO on $\mathrm{BTOP}\,(S^7)$ associated to $\mu : S^7 \times S^7 \to S^7$. Note that s maps to the identity component $\mathrm{TOP}_0(S^7)$.

Lemma 4.19. *We have seen in Observation 4.6 that* $\mathrm{HO} : \pi_8 \Rightarrow \pi_{15}$ *is represented by a map* $\sigma \in \pi_{15}(S^8) \cong \mathbb{Z} \oplus \mathbb{Z}_{120}$. *Assume the multiplication* μ *on* S^7 *was chosen according to Setup 3.11 with* $\delta \simeq \mathrm{id}_{S^7}$. *Then* σ *has Hopf invariant* $\mathrm{Hopf}\,(\sigma) = 1$.

The proof can be found below. We proceed first to examine the effects of stabilizing the problem.

Lemma 4.20 ([MT68, page 204])**.** *The Hopf map* $\sigma : S^{15} \to S^8$ *of Hopf invariant one stabilizes to the generator* $1 \in \pi_7^s \cong \mathbb{Z}_{240}$.

In fact, already its simple suspension $\Sigma\sigma : S^{16} \to S^9$ is a stable map. A more geometric proof was given by Grant [Gra15]. We draw the following conclusion for Section 5.

Corollary 4.21. *By Definition 4.10, the stable* $\mathrm{HO} : \pi_0 \Rightarrow \pi_7$ *is represented by a map* $\sigma \in \pi_7^s$. *Assume the multiplication* μ *on* S^7 *was chosen according to Setup 3.11 with* $\delta \simeq \mathrm{id}_{S^n}$. *Then* σ *has stable Hopf invariant one and is a generator of* $\pi_7^s \cong \mathbb{Z}_{240}$.

We may now identify the stable composition $f_s \circ \sigma_{\mathrm{s}}$ as seen in Definition 4.10 with the stabilization of the composition $f \circ \sigma$.

Proof of Lemma 4.19. We prove the lemma in greater generality and do not necessarily assume $n = 7$. This helps seeing what could possibly happen in higher dimensions, as will be discussed towards the end of this section. Recall the definition of the homotopy operation $\mathrm{HO} : \pi_{n+1} \Longrightarrow \pi_{2n+1}$ as given in Definition 4.5 on $\mathbf{Top}_*$ via $\pi_{n+1}(-) \cong \pi_n(\Omega -) \Longrightarrow \pi_{2n}(\Omega -) \cong \pi_{2n+1}(-)$. As pointed out by Observation 4.6 we may obtain a representing map $\sigma \in \pi_{2n+1}(S^{n+1})$ by evaluating $\mathrm{HO}_{\mathbf{Top}_*}$ on the identity $\mathrm{id} \in \pi_{n+1}(S^{n+1})$. Owing to the definition above, this means evaluating the adjoint to the identity, the canonical inclusion $S^n \to \Omega S^{n+1}$, on $\mathrm{HO}_{\mathbf{TopMon}} : \pi_n(\Omega S^{n+1}) \Longrightarrow \pi_{2n}(\Omega S^{n+1})$.
We may choose the *James reduced product* or *James construction* $J(S^n) \simeq \Omega S^{n+1}$ as a substitute for ΩS^{n+1}, see [Jam55]. It is made up by the disjoint union of S^n, $(S^n)^2$, $(S^n)^3$, $\ldots$, and its quotient taken by the relation $(x, 1, y) \sim (x, y)$ etc. It consists of one cell each in all dimensions $n \cdot l$, $l \geq 0$. The product $* : J(S^n) \times J(S^n) \to J(S^n)$ itself is given formally on the n-cells as $(x, y) \mapsto (x, y) \in 2n$-cell. We establish a fact for later. Let $J(S^n)^{(n)}$ and $J(S^n)^{(2n)}$ denote the n- and $2n$-skeletons of $J(S^n)$,

respectively. Due to the simple form of the product, the image of $J(S^n)^{(n)} \times J(S^n)^{(n)}$ is contained in $J(S^n)^{(2n)}$ and the image of $J(S^n)^{(n)} \vee J(S^n)^{(n)} \to J(S^n)$ is actually contained in the n-cell $S^n \subset J(S^n)$. Note that the $2n$-cell is attached to the n-cell via the composition $S^{2n-1} \xrightarrow{w} S^n \vee S^n \xrightarrow{\text{fold}} S^n$, where w denotes the Whitehead map. The $2n$-cell in $J(S^n)^{(n)} \times J(S^n)^{(n)} \cong S^n \times S^n$ is also attached to $S^n \vee S^n$ via the Whitehead map. It now follows formally that the induced map

$$* : J(S^n)^{(n)} \wedge J(S^n)^{(n)} \to {}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}} \tag{4.9}$$

is simply the identity $\mathrm{id} : S^{2n} \to S^{2n}$. We attempt to pin down the element obtained using Construction 4.1. Consider the following diagram.

$$\begin{tikzcd} S^n \times S^n \arrow[r, "\mathrm{incl} \times \mathrm{incl}"] \arrow[d, "\mu"'] & J(S^n) \times J(S^n) \arrow[d, "*"] \\ S^n \arrow[r, "\mathrm{incl}"] & J(S^n). \end{tikzcd} \tag{4.10}$$

The upper horizontal map actually lands in $J(S^n)^{(n)} \times J(S^n)^{(n)}$, and the upper right composition inside $J(S^n)^{(2n)} \subset J(S^n)$. The image of the lower left composition even lies inside $J(S^n)^{(n)}$. Consider the projection from $J(S^n)^{(2n)}$ to ${}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}}$. We use this to determine the value of the image of

$$\sigma \in \pi_{2n+1}(S^{n+1}) \cong \pi_{2n}(\Omega S^{n+1}) \cong \pi_{2n}(J(S^n)) \to \pi_{2n}\big({}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}}\big) \cong \pi_{2n}(S^{2n}).$$

One can call the value in $\pi_{2n}(S^{2n})$ the Hopf invariant. Indeed, if we can show that the image of σ is $1 \in \pi_{2n}(S^{2n})$, then this means that σ is indivisible in $\pi_{2n+1}(S^{n+1})$. Consequently, it would have Hopf invariant 1.

So far, we have only reformulated the problem and have not capitalized on the actual definition of HO. Since the lower left composition in (4.10) lands in the n-skeleton of $J(S^n)^{(2n)}$, composing it with the projection to ${}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}}$ yields a constant map. We simplify (4.10) to

$$\begin{tikzcd} S^n \times S^n \arrow[r, bend left, "* \circ (\mathrm{incl} \times \mathrm{incl})"] \arrow[r, bend right, "\mathrm{incl} \circ \mu"'] & J(S^n)^{(2n)} \arrow[r, "\mathrm{proj}"] & {}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}}. \end{tikzcd} \tag{4.11}$$

Recall that the two maps agree on the wedge $S^n \vee S^n$. The homotopy operation constructs the element $\mathrm{HO}\,(\mathrm{incl}) \in \pi_{2n}\big(J(S^n)^{(2n)}\big)$ as the gluing of two associated maps $D^{2n} \to J(S^n)^{(2n)}$ along their boundary ∂D^{2n}, see Construction 4.1. As we have argued above, the lower composition in (4.11) is actually constant when composed with the projection to ${}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}}$. Thus $\mathrm{proj}(\mathrm{HO}\,(\mathrm{incl})) \in \pi_{2n}\left({}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}}\right)$ is simply given by the upper map, namely

$${}^{D^{2n}} \big/ {}_{\partial D^{2n}} \cong {}^{S^n \times S^n} \big/ {}_{S^n \vee S^n} \xrightarrow{\mathrm{proj} \circ * \circ (\mathrm{incl} \times \mathrm{incl})} {}^{J(S^n)^{(2n)}} \big/ {}_{J(S^n)^{(n)}} \cong S^{2n}.$$

By construction, this map factors via ${}^{J(S^n)^{(n)}\times J(S^n)^{(n)}}\big/_{J(S^n)^{(n)}\vee J(S^n)^{(n)}}$, because the image of $S^n \vee S^n$ in $J(S^n)^{(n)} \times J(S^n)^{(n)}$ under $\mathrm{incl}\times\mathrm{incl}$ is contained in $J(S^n)^{(n)} \vee J(S^n)^{(n)}$. It is not hard to see that it factors as

$$S^n \wedge S^n \xrightarrow{\mathrm{incl}\wedge\mathrm{incl}} J(S^n)^{(n)} \wedge J(S^n)^{(n)} \xrightarrow{*} {}^{J(S^n)^{(2n)}}\big/_{J(S^n)^{(n)}} \cong S^{2n},$$

where $*$ denotes the induced map (4.9) which was seen to be the identity. The map $\mathrm{incl}\wedge\mathrm{incl}$ is of course also the identity. Therefore the composition is simply the identity $S^{2n} \to S^{2n}$ and σ has Hopf invariant 1. □

Remark 4.22. (i) Assume one were to set up a generalized homotopy operation with corresponding generalized (4.10) with a (k,l)-multiplication μ. Then of course one would have to adapt the upper horizontal map to $(\mathrm{incl}\circ k, \mathrm{incl}\circ l)$ in the general case. Here $\mathrm{incl}\circ k$ is short for incl composed with some degree k self-map of S^n. The proof of Lemma 4.19 goes through with minor adaptations. In this general setting, the representing map σ has Hopf invariant $k\cdot l$.
(ii) A generalization as above is actually an alternative way of constructing Hopf maps starting with multiplications $\mu : S^n \times S^n \to S^n$ of bidegree (k,l). It should *not* be confused with the more standard construction via

$$S^n * S^n \to \Sigma(S^n \times S^n) \xrightarrow{\Sigma\mu} \Sigma S^n,$$

see [MT68, Chapter 4, Exercises] for details. Although both constructions yield the same Hopf invariant, it is not clear whether they yield the same homotopy class of map $\sigma \in \pi_{2n+1}(S^{n+1})$.

4.5. A reformulation employing a lift to quadratic L-theory

We now turn to examining the lift $s\circ\sigma \in \pi_{15}(\mathrm{BTOP}\,(S^7))$ of our main obstruction. To this end, we will first examine the element $s \in \pi_7(\mathrm{TOP}\,(S^7))$ and split it into the sum of two other interesting elements. Consider the right-hand square

$$\begin{tikzcd}
\Omega\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right) \arrow[r, dashed, "\iota"] \arrow[d, dashed, "\simeq"] & \mathrm{TOP}\,(7) \arrow[r] \arrow[d] & \mathrm{G}\,(7) \arrow[d] \\
\Omega\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right) \arrow[r, dashed, "\iota'"] & {}^{\mathrm{TOP}(7)}\big/_{\mathrm{O}(7)} \arrow[r] & {}^{\mathrm{G}(7)}\big/_{\mathrm{O}(7)}.
\end{tikzcd}$$

Its vertical homotopy fibers are identified as $\mathrm{O}\,(7)$, making it a homotopy pullback square. Thus one can choose $\Omega\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right)$ as model for the horizontal homotopy fibers. The inclusion $\iota' : \Omega\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right) \to {}^{\mathrm{TOP}(7)}\big/_{\mathrm{O}(7)}$ factors via the inclusion to and projection from $\mathrm{TOP}\,(7)$. Consequently, we obtain the commutative left-hand triangle in (4.12). Upon adding a commutative square on the right induced by the one-point compactification, we obtain

$$\begin{tikzcd}
\Omega\left({}^{\mathrm{G}(7)}\big/_{\mathrm{TOP}(7)}\right) \arrow[r, "\iota"] \arrow[dr, "\iota'"'] & \mathrm{TOP}\,(7) \arrow[r, "c"] \arrow[d] & \mathrm{TOP}\,(S^7) \arrow[d] \\
 & {}^{\mathrm{TOP}(7)}\big/_{\mathrm{O}(7)} \arrow[r, "\cong"] & {}^{\mathrm{TOP}(S^7)}\big/_{\mathrm{O}(8)}.
\end{tikzcd} \tag{4.12}$$

We add the homotopy fiber $\mathrm{O}(8)$ of the right-hand vertical map into the picture and consider the triangle on homotopy groups,

$$\begin{tikzcd}
\pi_7\left(\Omega\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right) \arrow[r, "c_*\circ\iota_*"] \arrow[dr, "\iota'_*"'] & \pi_7\left(\mathrm{TOP}\left(S^7\right)\right) \arrow[d, "\mathrm{proj}_*"] & \pi_7\left(\mathrm{O}\left(8\right)\right) \arrow[l] \\
 & \pi_7\left({}^{\mathrm{TOP}(S^7)}/_{\mathrm{O}(8)}\right). &
\end{tikzcd} \tag{4.13}$$

Recall our chosen pseudo smooth structure $\Sigma \in \pi_7\left({}^{\mathrm{TOP}(S^7)}/_{\mathrm{O}(8)}\right)$. Recall the lift $s \in \pi_7\left(\mathrm{TOP}\left(S^7\right)\right)$ of Σ by Remark 3.3, as depicted in (3.1). Furthermore, recall that Corollary 1.43 guarantees a lift $\xi \in \pi_8\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \cong \pi_7\left(\Omega\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right)$ of Σ along the inclusion ι'_*. It is mapped to $c_*\iota_*(\xi)$ in $\pi_7\left(\mathrm{TOP}\left(S^7\right)\right)$ via $c \circ \iota$.
Summarizing, we have two lifts $c_*\iota_*(\xi)$ and s of Σ in $\pi_7\left(\mathrm{TOP}\left(S^7\right)\right)$. By exactness at $\pi_7\left(\mathrm{TOP}\left(S^7\right)\right)$, the element $s - c_*\iota_*(\xi)$ must come from $\pi_7\left(\mathrm{O}\left(8\right)\right)$. This proves

Lemma 4.23. *Recall the lift $s \in \pi_7\left(\mathrm{TOP}\left(S^7\right)\right)$ as described by (3.1) and the lift $c_*\iota_*(\xi) \in \pi_7\left(\mathrm{TOP}\left(S^7\right)\right)$ induced from Corollary 1.43 as above. Then for some $s_0 \in \pi_7\left(\mathrm{O}\left(8\right)\right)$,*

$$s = c_*\iota_*(\xi) + \mathrm{incl}_*(s_0). \tag{4.14}$$

Remark 4.24. (i) ξ will be useful in Section 5. However, in contrast to s it is neither a lift of δ as in (2.3) nor could it sensibly describe the action of a smooth map as in (3.4). Namely the adjoint of any representative of $c_*\iota_*(\xi)$ is a $(0,1)$-bidegree multiplication. In particular, $(\mathrm{ev}_1)_* \circ c_* \circ \iota_*(\xi)$ is the zero class in $\pi_7(S^7)$.
(ii) We also have the following related observation. Lemma 4.16 states $\mathrm{proj} \circ s \circ \sigma = \mathrm{Obs}(s, \Sigma)$. However, $\iota'_* \circ \xi \circ \sigma \neq \mathrm{Obs}(s, \Sigma)$.
(iii) The adjoint to s_0 is a $(1,1)$-bidegree multiplicative structure by Lemma 1.46. The element s_0 corresponds to an element in $\pi_8(\mathrm{BO}\left(8\right))$. The Euler class of the bundle defined via $S^8 \to \mathrm{BO}\left(8\right)$ is one, meaning its top Stiefel Whitney class is non-zero. In particular, s_0 is stably non-trivial.

Proof. (i) To see that the adjoint of $c_*\iota_*(\xi)$ is a $(0,1)$-bidegree multiplicative structure, recall that $\iota_*(\xi) \in \pi_7(\mathrm{TOP}\left(7\right))$. Therefore, its adjoint $S^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ evaluated at $0 \in \mathbb{R}^7$ is a nullhomotopic map. Thus the same is true for the adjoint $S^7 \times S^7 \to S^7$.
(ii) This is evident. After all, ι' is not a monoid map and the multiplication on $\Omega\left({}^{\mathrm{G}(7)}/_{\mathrm{O}(7)}\right)$ is incompatible with the action of $s \in \pi_7(\mathrm{TOP}\left(S^7\right))$ on ${}^{\mathrm{TOP}(S^7)}/_{\mathrm{O}(8)}$. □

Lemma 4.25. *There is a description of $s \circ \sigma \in \pi_{15}\left(\mathrm{BTOP}\left(S^7\right)\right)$ as*

$$s \circ \sigma = c \circ \iota \circ \xi \circ \sigma + \mathrm{incl} \circ s_0 \circ \sigma + \left[c \circ \iota \circ \xi, \mathrm{incl} \circ s_0\right].$$

Proof. This is proven using (4.14) with the sum formula from Lemma 4.9. Lemma 4.7 helps by getting the technicalities right, and Lemma 4.19 guarantees $\mathrm{Hopf}\left(\sigma\right) = 1$. □

As a fairly immediate corollary we obtain the specialized version of Obstruction 4.14 in dimension $n = 7$. Recall that $\Sigma \in \pi_7\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ denotes a pseudo smooth structure, and $\xi \in \pi_8\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$ denotes a lift of Σ along the boundary homomorphism $\partial : \pi_8\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \to \pi_7\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$. Recall $s : S^7 \to \mathrm{TOP}\left(S^7\right)$ as given in Setup 3.11. Recall the correction $s_0 = s - \xi \in \pi_7\left(\mathrm{O}\left(8\right)\right)$. Let $\sigma : S^{15} \to S^8$ be a map of Hopf invariant one.

Obstruction 4.26 (in dimension 7)**.** *There exists a homotopy lift* $\overline{s} : S^7 \to \mathrm{Diff}\left(S^7_\Sigma\right)$ *of* s *as in* (2.3) *if and only if the element*

$$\mathrm{Obs}(s, \Sigma) = \partial(\xi \circ \sigma) + \mathrm{proj}_*\left[c \circ \iota \circ \xi, \mathrm{incl} \circ s_0\right] \in \pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right) \tag{4.15}$$

vanishes.

Note that the formula $\partial(\xi \circ \sigma)$ is technically correct, in contrast to $\mathrm{proj} \circ s \circ \sigma$ in Lemma 4.16. This is due to the degree shift occurring via the boundary homomorphism. Notice also that the Whitehead bracket is evaluated in $\mathrm{BTOP}\left(S^7\right)$, and that only its adjoint is projected to $\pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$. One can only carefully identify the latter with $\pi_{15}\left(\mathrm{BTOP}\left(7\right), \mathrm{BO}\left(7\right)\right)$.

Proof. Note $\mathrm{Obs}(s, \Sigma) = \mathrm{proj} \circ s \circ \sigma = \mathrm{proj}_*((s^{\mathrm{ad}} \circ \sigma)_{\mathrm{ad}}) \in \pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ by Lemma 4.16. We use Lemma 4.25 to deduce (4.15). Note $\mathrm{proj} \circ \mathrm{incl} \circ s_0 = 0$ using the appropriate long exact sequence as in (4.13). Also by (4.13) and by definition, $\mathrm{proj}_*\, c_* \iota_* = \iota'_* = \partial$. By Obstruction 4.14 the equal terms in (4.15) vanish if and only if there is a lift $\overline{s}$ as desired. To get the details right, one needs to pay attention as to whether one uses HO on **TopMon** or $\mathbf{Top}_*$. However, this is not problematic when using Lemma 4.8 and the Ω-B-adjunction. □

We conclude the section with a discussion of some of the issues one currently might run into when attempting to adapt all methods to the case of general $n > 7$ and also $\deg \delta > 1$. The remarks are separated and labeled according to their topic.

Remark 4.27 (Generalization for $n > 7$)**.** *The homotopy operation for* $(2, 1)$*-multiplications.* We come back to the prerequisites in Construction 4.1. We assumed to start with a multiplication μ with strict two-sided unit. One can adapt the construction to a setting with a $(k, 1)$-multiplication μ for example via

$$\begin{tikzcd} S^n \times S^n \arrow[r, "(f \circ \delta) \times f"] \arrow[d, "\mu"'] & A \times A \arrow[d, "\nu"] \\ S^n \arrow[r, "f"] & A. \end{tikzcd} \tag{4.16}$$

This is commutative on the axes $S^n \vee S^n$ as well as reflects the setup (3.4) naturally occurring via the Definition 1.25 of a pseudo diffeomorphism. One can also apply the machinery and generalizations as provided by Sections 4.1 and 4.2. Thus one obtains a representing element $\sigma : S^{2n+1} \to S^{n+1}$. Section 4.3 also goes through unchanged.

Unfortunately, Proposition 3.1 provides only a lift $s : S^n \to \mathrm{TOP}\,(S^n)$ of $\Sigma \circ \delta$, where $\Sigma : S^n \to {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n)}$ is a pseudo smooth structure. In view of Obstruction 3.2, there is no lift of Σ for $n > 7$. Consequently, there is no sensible lift of the obstruction in the sense of (4.8) or $\mathrm{proj} \circ s \circ \sigma = \mathrm{Obs}(s, \Sigma)$. Unfortunately neither can one substitute $\Sigma \circ \delta$ for $\mathrm{proj} \circ s$, since the triple composition is not technically a composition, as explained in Remark 4.17. In other words, the obstruction $\mathrm{Obs}(s, \Sigma)$ is *not* linked to a homotopy operation represented by a Hopf map $\sigma : S^{2n+1} \to S^{n+1}$ in the sense discussed in this thesis.
The argument above is of course assuming we are using the A_∞-space $\mathrm{TOP}\,(S^n)$. One may also ask whether lifting $\Sigma \in \pi_n \left({}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)}\right)$ to $\pi_n\,(\mathrm{TOP}\,(n))$ might be possible. It turns out that it is not, for many similar reasons. This approach additionally shows problems with naturality. Seemingly no other A_∞-space sensibly relates to ${}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$ and Obstruction 3.16. However, a completely different method may still allow to relate $\mathrm{Obs}(s, \Sigma)$ to the approximating setup of Section 5.

Remark 4.28 (Generalization for $n > 7$)**.** *Even Hopf invariant.* The Hopf invariant of the representing map $\sigma : S^{2n+1} \to S^{n+1}$ for $\mathrm{HO} : \pi_{n+1} \Longrightarrow \pi_{2n+1}$ is computed in Lemma 4.19. The proof reveals that $\mathrm{Hopf}\,(\sigma) = 2$ if $\deg(\delta) = 2$. An even Hopf invariant element might be a Whitehead element, i. e. given via Whitehead product $[\iota, \iota]$. This is not a stable element, meaning it would vanish in $T_1({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)})$ in Section 5. However, we do not expect the element σ to be a Whitehead element. Of course the increase in Hopf invariant entails the necessity for more sensitive computational models.

Remark 4.29 (Generalization for $n > 7$)**.** *Lifting to* ${}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}$. Corollary 6.4 shows that the Whitehead product in (4.15) vanishes in the approximating model $T_1({}^{\mathrm{TOP}(n)}\big/_{\mathrm{O}(n)})$. Corollary 6.5 concludes that $\mathrm{Obs}(s, \Sigma) = \partial(\xi \circ \sigma)$ in the approximation $T_1({}^{\mathrm{TOP}(7)}\big/_{\mathrm{O}(7)})$. Let $n > 7$. Assume $\Sigma \in bP^{n+1}$, i. e. it lifts to a $\xi \in L_{n+1}\,(\mathbb{Z})$. A connection as in Section 4.4 is needed to relate $\mathrm{Obs}(s, \Sigma)$ and $\partial(\xi \circ \sigma)$. This is not obvious, since the boundary map does not respect the homotopy operation, as already pointed out in Remark 4.24(ii). The author is optimistic that future research will allow linking $\mathrm{Obs}(s, \Sigma)$ and $\partial(\xi \circ \sigma)$ as desired.

5. An approximation of $\mathrm{G}(n)/\mathrm{TOP}(n)$

Recall the homotopy operation developed in Section 4. The main object now is to develop an approximating model for the space ${}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$. In this section, by an approximating model we mean a spectrum P together with an approximating map $\phi : \Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right) \to \Omega^{\infty}\mathsf{P}$. The expectation is that the calculation of the relevant homotopy groups π_0 and π_n of P is easier than the corresponding one for ${}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$. This should facilitate showing $\phi \circ \xi \circ \sigma \neq 0$ in $\pi_n(\mathsf{P})$, which would imply the non-vanishing of $\xi \circ \sigma \in \pi_{2n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$.
The approximating model P is developed using the first orthogonal calculus Taylor approximation $T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$. It can be written as a mixture of quadratic L-theory and Waldhausen A-theory using descriptions provided by [WW18]. The two are linked using the Weiss-Williams map Ξ from quadratic L-theory $\mathsf{L}_{\bullet}(R)$ to the Tate fixed point spectrum of K-theory $\mathsf{K}(R)^{th\mathbb{Z}_2}$. Detailed results can be found in the main Proposition 5.9.
We recall the most important aspects of Waldhausen K-theory for Waldhausen categories and duality notions. Fundamentals of orthogonal calculus are reviewed and summarized as needed. These are accompanied by several results announced for the continuation [WW18] of the Weiss-Williams paper series on *Automorphisms of manifolds and algebraic K-theory*. The section is concluded with Proposition 5.9.

5.1. K-theory and duality

Definition 5.1. Let X be a space. Denote by $\mathcal{R}_{\mathrm{f}}(X)$ the category of finite retractive relative CW-spaces over X. Define the **Waldhausen A-theory** $\mathsf{A}(X)$ of a space X to be the Waldhausen K-theory $\mathsf{K}(\mathcal{R}_{\mathrm{f}}(X))$ of this category, compare [Wal85]. Denote by $\mathsf{Wh}^{\mathrm{Diff}}(X)$ the smooth Whitehead spectrum, in short $\mathsf{Wh}(X)$. It fits into the splitting of $\mathsf{A}(X)$ as $\mathsf{Wh}(X) \times \Sigma^{\infty}X_{+}$, see [Ros05, Section 6]. In particular, we may identify $\mathsf{Wh}(*) = \mathrm{hofiber}[\mathsf{A}(*) \to \mathsf{S}]$. In fact, the associated infinite loop space to $\Omega^2\mathsf{Wh}(M)$ is the *stable smooth pseudoisotopy (or concordance) space* of a smooth compact manifold M, see [Ros05, Section 7] for details.

In the definition above, we use *Waldhausen K-theory* to define A-theory. Its definition can also be found in [Wal85]. Most importantly, it generalizes usual algebraic K-theory by $\mathsf{K}^{\mathrm{Wald}}(\mathcal{C}) \simeq \mathsf{K}^{\mathrm{alg}}(R)$ if R is a ring and $\mathcal{C}$ is the category of bounded chain complexes of finitely generated projective R-modules, compare [DS04, Lemma 3.4]. As for A-theory, it suffices to think of it via its splitting as mentioned above. A-theory can also be expressed using Quillen K-theory $\mathsf{K}(\mathsf{S}) \simeq \mathsf{A}(*)$, see [Wal79].
There is also a generalization of L-theory for ring spectra with involution. If R is a connective ring spectrum with involution, then $\mathsf{L}_{\bullet}(\pi_0(\mathsf{R})) \simeq \mathsf{L}_{\bullet}(\mathsf{R})$, as quadratic L-theory is quite insensitive to the input. A discrete connective ring spectrum R with involution generalizes symmetric L-theory via $\mathsf{L}^{\bullet}(\pi_0(\mathsf{R})) \simeq \mathsf{L}^{\bullet}(\mathsf{R})$. More generally, L-theory can also be defined on Waldhausen categories. Definitions can be found in [WW98]. K-theory can be equipped with a duality notion. Dualities are constructed

using what is known as Spanier-Whitehead dualities on Waldhausen categories. For $\mathsf{A}(*)$ and $\mathsf{Wh}(*)$ one can think of them as being induced from a spherical fibration over a point. Generally, the natural map $\mathsf{Wh}(*) \to \mathsf{A}(*)$ respects this duality. In our case $\mathsf{A}(*) \to \mathsf{Wh}(*)$ also respects the duality since any spherical fibration over a point can be trivialized. The usual duality defined on these spectra is referred to as 0-duality. One can also define versions $\mathsf{K}(\mathsf{R}; n)$ and $\mathsf{A}(X; n)$ of K-theory and A-theory with n-duality. For definitions and details, see [WW98] and [WW00]. All duality involutions can be thought of as $\mathbb{Z}_2$-actions on the corresponding spectra, allowing to define homotopy orbit, fixed point and Tate spectra. We use the notation $\mathsf{L}_\bullet(\mathsf{R}; n)$ for an analogue for quadratic L-theory summarized as follows. Lemma 5.2 also characterizes the n-duality on $\mathsf{A}(*)$.

Lemma 5.2. *In general,* 1*-duality relates to* 0*-duality via the equations* $\mathsf{A}(*; 1) \simeq_{\mathbb{Z}_2} \Omega\left(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{A}(*; 0)\right)$ *for A-theory. There is a similar equation* $\mathsf{L}_\bullet(\mathbb{Z}; 1) \simeq \Omega \mathsf{L}_\bullet(\mathbb{Z}; 0)$ *for quadratic L-theory, although there is no duality notion on L-theory.*

The equation for L-theory is reminiscent of $\mathsf{A}(*; 1)^{th\mathbb{Z}_2} \simeq \Omega \mathsf{A}(*; 0)^{th\mathbb{Z}_2}$ (Lemma 5.10). Here $S^1_!$ denotes the circle equipped with the reflection $\mathbb{Z}_2$-action, as opposed to the usual S^1 with trivial $\mathbb{Z}_2$-action. Note also the difference to the infinite-dimensional sphere $S^\infty = \mathrm{E}\mathbb{Z}_2$ equipped with the *antipodal* $\mathbb{Z}_2$-action.

Proof. Let $(\mathcal{C}, \odot)$ be a Waldhausen category with duality notion, see [WW98] for definitions. Consider the category $(\mathcal{C}^2, \odot^2)$ of pairs of objects $X \hookrightarrow X'$ in $\mathcal{C}$ linked by a cofibration. The induced duality notion is given as $(X', X) \odot^2 (Y', Y) :=$ hofiber$\,[X \odot Y \to X' \odot Y']$. Note that $(X', X) \odot^2 (Y', Y)$ comes with natural maps to both $\Omega((X'/X) \odot Y')$ and $\Omega(X' \odot (Y'/Y))$. The non-degeneracy condition for $\odot^2$ is given by imposing the non-degeneracy condition on these corresponding pairings. There is a sequence of Waldhausen categories with duality

$$(\mathcal{C}, \odot_1) \to (\mathcal{C}^2, \odot^2) \to (\mathcal{C}, \odot).$$

Indeed, the right-hand map is the (surjective) projection $(X', X) \mapsto X$. The duality is preserved, as the homotopy fiber $(X', X) \odot^2 (Y', Y)$ includes naturally into the domain $X \odot Y$. The kernel of the right-hand map is made up of pairs $(X', *)$, where $*$ denotes a contractible object in $\mathcal{C}$. The duality notion is given via $(X', *) \odot^2 (Y', *) =$ hofiber$\,[* \odot * \to X' \odot Y']$, which is simply $\Omega(X' \odot Y')$. This is what we call 1-duality $X' \odot_1 Y'$. We analyze the short exact sequence obtained upon applying K-theory,

$$\mathsf{K}(\mathcal{C}, \odot_1) \to \mathsf{K}(\mathcal{C}^2, \odot^2) \to \mathsf{K}(\mathcal{C}, \odot).$$

Using the additivity theorem in K-theory, the middle term splits as $\mathsf{K}(\mathcal{C}^2, \odot^2) \simeq \mathsf{K}(\mathcal{C}) \vee \mathsf{K}(\mathcal{C})$, disregarding the involution. To understand the involution $\odot^2$, we think of the first summand as the objects $(*, X)$ which are projected to X. Their duals are in fact the objects (X, X), and we may think of the other summand as given via these objects. In particular, the involution is free, sending one summand to the other.

We appeal to a general fact to identify the kernel $\mathsf{K}(\mathcal{C}, \odot_1)$. Let X be a spectrum with involution. Take a look at the cofiber sequence $S^0 \to S^1_! \to {}^{S^1_!}/_{S^0}$. The latter term can be identified as ${}^{S^1_!}/_{S^0} \cong_{\mathbb{Z}_2} S^1 \vee S^1$, with the $\mathbb{Z}_2$-action on $S^1 \vee S^1$ given by permuting the summands. Upon smashing with X we obtain $(S^1 \vee S^1) \wedge_{\mathbb{Z}_2} \mathsf{X} \cong (S^1 \wedge \mathsf{X}) \vee_{\mathbb{Z}_2} (S^1 \wedge \mathsf{X})$, with the $\mathbb{Z}_2$-action given by sending one summand homeomorphically to the other. There arises a cofiber sequence $\mathsf{X} \to S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X} \to (S^1 \wedge \mathsf{X}) \vee_{\mathbb{Z}_2} (S^1 \wedge \mathsf{X})$, equivalent by shift to $\Omega(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X}) \to \mathsf{X} \vee_{\mathbb{Z}_2} \mathsf{X} \to \mathsf{X}$. The case $\mathsf{K}(\mathsf{S}) = \mathsf{A}(*)$ proves the assertion.
The result for quadratic L-theory is actually analogous to the above. It behaves like the Tate construction on K-theory. The L-theory of pairs is contractible, and also fits into a fiber sequence of the type above. This already shows $\mathsf{L}_\bullet(\mathbb{Z}; 1) \simeq \Omega \mathsf{L}_\bullet(\mathbb{Z}; 0)$. However, we also present a convenient model for quadratic L-theory at the infinite loop space level. For the usual version with 0-duality, think of it as the realization of a simplicial set. A 0-simplex is a chain complex over the base ring equipped with 0-duality. A 1-simplex is then a bordism between such objects, and so on. This yields a model for $\Omega \mathsf{L}_\bullet(\mathbb{Z}; 0)$, with a 0-simplex here being a 1-simplex as above, but with trivial faces. This means it is simply a chain complex, but the duality notion still takes into account the dimension one higher. In other words, it is a chain complex with 1-duality. This way the model for $\Omega \mathsf{L}_\bullet(\mathbb{Z}; 0)$ equally well works as a model for $\mathsf{L}_\bullet(\mathbb{Z}; 1)$. $\square$

5.2. Orthogonal calculus and the approximating model

We review the fundamentals of orthogonal calculus, as explained in [Wei95] and [Wei98]. Denote by $\mathcal{J}$ the category with objects being the finite-dimensional real vector spaces with a positive definite inner product. Morphisms are linear maps respecting the inner products. A functor $F : \mathcal{J} \to \mathbf{Top}_*$ is called continuous if the evaluation map $\mathrm{mor}(V, W) \times F(V) \to F(W)$ is continuous for all objects $V, W \in \mathcal{J}$. In short, the first Taylor approximation aspect of orthogonal calculus is concerned with describing a sort of right approximation $F \to T_1 F$ to F with special properties. Write $F(\mathbb{R}^\infty) = \mathrm{hocolim}\, F(\mathbb{R}^n)$.

Proposition 5.3 ([Wei95]). *The functor F determines a fibered spectrum θ^F on $F(\mathbb{R}^\infty)$ with fiberwise involution on its fibers θ^F_x together with a map ω fitting into the commutative diagram*

$$\begin{tikzcd} F(\mathbb{R}^n) \arrow[r, dashed] \arrow[d, "\mathrm{incl}"] & \coprod_{x \in F(\mathbb{R}^\infty)} \Omega^{\infty-1}\left(\theta^F_x \wedge_{\mathbb{Z}_2} S^{n-1}_+\right) \arrow[d, "\mathrm{incl}"] \\ F(\mathbb{R}^\infty) \arrow[r, "\omega"] & \coprod_{x \in F(\mathbb{R}^\infty)} \Omega^{\infty-1}\left(\theta^F_x \wedge_{\mathbb{Z}_2} S^{\infty}_+\right). \end{tikzcd} \tag{5.1}$$

The class $[\omega] \in H^1\left(F(\mathbb{R}^\infty), \theta^F_{h\mathbb{Z}_2}\right)$ is the total Stiefel-Whitney class *associated to F.*

Here the coproduct notation is supposed to denote the total space with fibers as specified.

Proposition 5.4 ([Wei95]). *The first approximation stage T_1F is the homotopy pullback of* (5.1) *with the upper left corner removed,*

$$\begin{array}{ccc} & & \coprod_{x\in F(\mathbb{R}^\infty)} \Omega^{\infty-1}\left(\theta_x^F \wedge_{\mathbb{Z}_2} S_+^{n-1}\right) \\ & & \big\downarrow{\scriptstyle \mathrm{incl}} \\ F(\mathbb{R}^\infty) & \xrightarrow{\omega} & \coprod_{x\in F(\mathbb{R}^\infty)} \Omega^{\infty-1}\left(\theta_x^F \wedge_{\mathbb{Z}_2} S_+^{\infty}\right), \end{array} \tag{5.2}$$

and $F(\mathbb{R}^n) \to T_1F(\mathbb{R}^n)$ is the induced map.

We summarize recent research results restating the fundamentals in a convenient manner.

Corollary 5.5 ([WW18]). *Let F be the functor $V \mapsto {}^{\mathrm{G}(V)}/_{\mathrm{TOP}(V)}$. In this case the spectrum $\theta_x^F = \mathsf{Wh}(*) = \mathrm{hofiber}[\mathsf{A}(*) \to \mathsf{S}]$ is trivially fibered, with standard involution. Then $T_1F(\mathbb{R}^n)$ is the homotopy pullback of the modified* (5.2),

$$\begin{array}{ccc} & & \Omega^{\infty-1}\left(\mathsf{Wh}(*) \wedge_{\mathbb{Z}_2} S_+^{n-1}\right) \\ & & \big\downarrow \\ {}^{\mathrm{G}}/_{\mathrm{TOP}} & \xrightarrow{\omega} & \Omega^{\infty-1}\left(\mathsf{Wh}(*) \wedge_{\mathbb{Z}_2} S_+^{\infty}\right), \end{array}$$

and $F(\mathbb{R}^n) \to T_1F(\mathbb{R}^n)$ is the induced map.

Proposition 5.6. *There is a map of spectra*

$$\Xi : \mathsf{L}^\bullet(R) \to \mathsf{K}(R)^{th\mathbb{Z}_2}$$

for any ring R with involution, compare [WW89]. It can be generalized to a map Ξ on $\mathsf{L}^\bullet(\mathcal{C})$ using Waldhausen categories $\mathcal{C}$, and to a map Ξ on $\mathsf{L}^\bullet(\mathsf{R})$ using ring spectra R, see [WW00].

Proposition 5.7 ([WW18]). *The composition*

$${}^{\mathrm{G}}/_{\mathrm{TOP}} \xrightarrow{\omega} \Omega^{\infty-1}\left(\mathsf{Wh}(*) \wedge_{\mathbb{Z}_2} S_+^{\infty}\right) \to \Omega^{\infty-1}\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^{\infty}\right)$$

induced by the split forgetful map $\mathsf{Wh}(*) \to \mathsf{A}(*)$ can be identified with Ω_0^∞ of

$$\mathsf{L}_\bullet(\mathsf{S}) \xrightarrow{\mathrm{sym}} \mathsf{L}^\bullet(\mathsf{S}) \xrightarrow{\Xi} \mathsf{K}(\mathsf{S})^{th\mathbb{Z}_2} \to S^1 \wedge \mathsf{K}(\mathsf{S})_{h\mathbb{Z}_2}. \tag{5.3}$$

The map $\mathsf{L}_\bullet(\mathsf{S}) \to \mathsf{L}^\bullet(\mathsf{S})$ is the symmetrization map on L-theory, compare Lemma 1.41. The map $\mathsf{K}(\mathsf{S})^{th\mathbb{Z}_2} \to S^1 \wedge \mathsf{K}(\mathsf{S})_{h\mathbb{Z}_2}$ is induced by the norm cofiber sequence, compare Remark 1.55.

Note that $\Omega^{\infty-1}\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^{\infty}\right) = \Omega^\infty S^1 \wedge \mathsf{K}(\mathsf{S})_{h\mathbb{Z}_2}$, compare Definition 1.51. In addition, we identify $\Omega_0^\infty \mathsf{L}_\bullet(\mathsf{S})$ with ${}^{\mathrm{G}}/_{\mathrm{TOP}}$ via Lemma 1.34. One notable aspect of Ξ is that on $\mathsf{L}^\bullet(\mathsf{S})$ it maps to $\mathsf{A}(*)^{th\mathbb{Z}_2}$, but restricted to $\mathsf{L}_\bullet(\mathsf{S})$ it actually only maps into $\mathsf{Wh}(*)^{th\mathbb{Z}_2}$. We use the proposition above to approximate $T_1F(\mathbb{R}^n)$ by a similar model which involves (5.3). This way we may analyze the approximating space using our knowledge of the map Ξ.

Remark 5.8. Again, let F be the functor $V \mapsto {}^{\mathrm{G}(V)}\big/_{\mathrm{TOP}(V)}$. Then $T_1F(\mathbb{R}^n)$ and by consequence $F(\mathbb{R}^n) = {}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}$ can be approximated by the homotopy pullback of

$$\begin{tikzcd} & \Omega^{\infty-1}\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^{n-1}\right) \arrow[d, "\iota"] \\ {}^{\mathrm{G}}\big/_{\mathrm{TOP}} \arrow[r, "\omega"] & \Omega^{\infty-1}\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^{\infty}\right). \end{tikzcd} \tag{5.4}$$

Proposition 5.9. *The space $\Omega^{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right)$ can be approximated by Ω^∞ of the homotopy pullback P of*

$$\begin{tikzcd} & \mathsf{A}(*; n+1)^{rh\mathbb{Z}_2} \arrow[d] \\ \mathsf{L}_\bullet(\mathsf{S}; n+1) \arrow[r] & \mathsf{A}(*; n+1)^{th\mathbb{Z}_2}. \end{tikzcd} \tag{5.5}$$

Also, the approximating map $\Omega^{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right) \to \Omega^\infty \mathsf{P}$ is an $(n+1)$-fold loop map. The composition $\Omega^{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right) \to \Omega^{\infty+n+1}\mathsf{L}_\bullet(\mathsf{S}) \simeq \Omega^\infty \mathsf{L}_\bullet(\mathsf{S}; n+1)$ is the stabilization map as described in Lemma 1.40. The map $\mathsf{L}_\bullet(\mathsf{S}; n+1) \to \mathsf{A}(*; n+1)^{th\mathbb{Z}_2}$ is the usual symmetrization map composed with Ξ, and $\mathsf{A}(*; n+1)^{rh\mathbb{Z}_2} \to \mathsf{A}(*; n+1)^{th\mathbb{Z}_2}$ is the obvious composition via $\mathsf{A}(*; n+1)^{h\mathbb{Z}_2}$.

Stated this way, we see that the approximation of ${}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}$ is given by a mixture of L-theory and K-theory. It can be thought of as a spectrum containing a little of both signatures usually treated by L-theory and (Euler) characteristics usually treated by A-theory. We prove two lemmata and conclude with the proof of Proposition 5.9.

Lemma 5.10. *Let X be a spectrum with involution. The homotopy cofibers of the inclusion induced maps*

$$\mathsf{X}^{h\mathbb{Z}_2} \to \left(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X}\right)^{h\mathbb{Z}_2} \tag{5.6}$$

$$\mathsf{X}_{h\mathbb{Z}_2} \to \left(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X}\right)_{h\mathbb{Z}_2} \tag{5.7}$$

can both be identified as $S^1 \wedge \mathsf{X}$. As a consequence, $\mathsf{X}^{h\mathbb{Z}_2}$ can be identified with $(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X})^{rh\mathbb{Z}_2}$, with (5.6) being the inclusion of the homotopy fiber. Also, the following inclusion induced map is a homotopy equivalence,

$$\mathsf{X}^{th\mathbb{Z}_2} \xrightarrow{\simeq} \left(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X}\right)^{th\mathbb{Z}_2}.$$

The inclusions are induced by the inclusion $S^0 \to S^1_!$ of the fixed points of $S^1_!$.

Proof. As in the proof of Lemma 5.2, we have the underlying cofiber sequence $\mathsf{X} \to S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X} \to (S^1 \wedge \mathsf{X}) \vee_{\mathbb{Z}_2} (S^1 \wedge \mathsf{X})$, with the $\mathbb{Z}_2$-action on the cofiber given by sending one summand homeomorphically to the other. As a consequence,

$$\begin{aligned} \mathrm{map}_{\mathbb{Z}_2}\left(\mathrm{E}\mathbb{Z}_{2+}, (S^1 \wedge \mathsf{X}) \vee (S^1 \wedge \mathsf{X})\right) &\cong \mathrm{map}\left(\mathrm{E}\mathbb{Z}_{2+}, S^1 \wedge \mathsf{X}\right) \\ &\simeq \mathrm{map}\left(\{*\}_+, S^1 \wedge \mathsf{X}\right) \cong S^1 \wedge \mathsf{X}, \end{aligned}$$

$$\begin{aligned}
\mathrm{E}\mathbb{Z}_{2+} \wedge_{\mathbb{Z}_2} \left[(S^1 \wedge \mathsf{X}) \vee (S^1 \wedge \mathsf{X})\right] &\cong \left[(\mathrm{E}\mathbb{Z}_{2+} \wedge S^1 \wedge \mathsf{X}) \vee (\mathrm{E}\mathbb{Z}_{2+} \wedge S^1 \wedge \mathsf{X})\right] /_{\mathbb{Z}_2} \\
&\cong \mathrm{E}\mathbb{Z}_{2+} \wedge S^1 \wedge \mathsf{X} \\
&\simeq \{*\}_+ \wedge S^1 \wedge \mathsf{X} \cong S^1 \wedge \mathsf{X}.
\end{aligned}$$

Hence we may identify the homotopy cofibers of both (5.6) and (5.7) as $S^1 \wedge \mathsf{X}$. It is not difficult to see that the map $(S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X})^{h\mathbb{Z}_2} \to S^1 \wedge \mathsf{X}$ to the cofiber as given above is actually the same as the canonical map defined above Definition 1.52. The rest of the assertion follows.
We turn to proving the second statement. We insert the two cofiber sequences into the diagram and complete them vertically by their associated norm cofiber sequences,

$$\begin{tikzcd}
\mathsf{X}_{h\mathbb{Z}_2} \arrow[r] \arrow[d] & (S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X})_{h\mathbb{Z}_2} \arrow[r] \arrow[d] & S^1 \wedge \mathsf{X} \arrow[d, "\mathrm{id}"] \\
\mathsf{X}^{h\mathbb{Z}_2} \arrow[r] \arrow[d] & (S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X})^{h\mathbb{Z}_2} \arrow[r] \arrow[d] & S^1 \wedge \mathsf{X} \arrow[d] \\
\mathsf{X}^{th\mathbb{Z}_2} \arrow[r] & (S^1_! \wedge_{\mathbb{Z}_2} \mathsf{X})^{th\mathbb{Z}_2} \arrow[r] & *.
\end{tikzcd} \tag{5.8}$$

It is not difficult to see that the norm map $S^1 \wedge \mathsf{X} \to S^1 \wedge \mathsf{X}$ is actually the identity. As consequence, the Tate spectrum is contractible and the left lower horizontal map becomes a homotopy equivalence. □

Lemma 5.11. *There is a* $\mathbb{Z}_2$*-equivariant homeomorphism* ${}^{S^\infty}/_{S^{i-1}} \cong_{\mathbb{Z}_2} S^i_! \wedge_{\mathbb{Z}_2} S^\infty_+$*, and the two following canonical maps are homotopic,*

$$\mathrm{proj} : S^\infty_+ \to {}^{S^\infty}/_{S^{i-1}} \qquad\qquad \mathrm{incl} : S^\infty_+ \cong S^0 \wedge S^\infty_+ \to S^i_! \wedge_{\mathbb{Z}_2} S^\infty_+.$$

Proof. Consider S^l and $\mathbb{R}P^l$ for large l, together with corresponding subspaces S^{l-i} and $\mathbb{R}P^{l-i}$. The canonical normal bundle of $\mathbb{R}P^{l-i}$ in $\mathbb{R}P^l$ is given by the Whitney sum $\oplus\gamma$ of i tautological line bundles over $\mathbb{R}P^{l-i}$. If chosen carefully, this normal bundle fills the entire $\mathbb{R}P^l$ except for the complementary $\mathbb{R}P^{i-1}$. The Thom space of the normal bundle can then be identified with ${}^{\mathbb{R}P^l}/_{\mathbb{R}P^{i-1}}$. Analogously, the same holds for S^{l-i} inside S^l. The normal bundle is made up of canonical $\mathbb{Z}_2$-line bundles, with the $\mathbb{Z}_2$-action being reminiscent of the $\mathbb{Z}_2$-action on S^l. Likewise we may write ${}^{S^l}/_{S^{i-1}}$ as a sort of Thom space of the normal bundle with $\mathbb{Z}_2$-action. Since the Thom space is again a Whitney sum of line bundles, we obtain ${}^{S^l}/_{S^{i-1}} \cong_{\mathbb{Z}_2} \mathrm{Th}\left(NS^{l-i}\right) \cong_{\mathbb{Z}_2} S^i_! \wedge_{\mathbb{Z}_2} S^{l-i}_+$.
Consider the maps

$$\begin{tikzcd}
S^{l-i}_+ \arrow[r, hook, "\mathrm{incl}"] \arrow[drr, hook, bend right, "\mathrm{incl}"'] & S^l_+ \arrow[r, two heads, "\mathrm{proj}"] & {}^{S^l}/_{S^{i-1}} \arrow[d, "\cong"] \\
& & S^i_! \wedge_{\mathbb{Z}_2} S^{l-i}_+,
\end{tikzcd}$$

with the diagonal inclusion map using the inclusion $S^0 \to S^i_!$ of the fixed points in $S^i_!$. This diagram commutes by construction. Upon passing to the colimit as $l \to \infty$ we obtain the claimed homeomorphism. The inclusion $S_+^{\infty-i} \hookrightarrow S_+^\infty$ becomes a homotopy equivalence. Thus the two maps as outlined agree up to homotopy. $\square$

Proof of Proposition 5.9. We use Lemma A.8 several times to modify (5.4) into the desired shape. First, we pass to the version in the category of spectra by exchanging $\Omega_0^\infty \mathsf{L}_\bullet(\mathsf{S})$ for ${}^{\mathrm{G}}/_{\mathrm{TOP}}$ via Lemma 1.34 and then forgetting about Ω^∞. To obtain the desired result, we loop everything $(n+1)$-times and obtain an analogous approximation result for $\Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$ as in Remark 5.8. Diagram (5.4) translates to

$$\begin{tikzcd} & \Omega^{n+1}\left(S^1 \wedge \mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^{n-1}\right) \arrow[d, "\iota"] \\ \Omega^{n+1}\mathsf{L}_\bullet(\mathsf{S}) \arrow[r] & \Omega^{n+1}\left(S^1 \wedge \mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^\infty\right). \end{tikzcd} \tag{5.9}$$

By Proposition 5.7, the horizontal arrow in (5.9) is identified with the $(n+1)$-fold loop of the composition $\mathsf{L}_\bullet(\mathsf{S}) \to \mathsf{L}^\bullet(\mathsf{S}) \to \mathsf{A}(*)^{th\mathbb{Z}_2} \to S^1 \wedge \mathsf{A}(*)_{h\mathbb{Z}_2}$. Identify the homotopy cofiber of ι in (5.9) as $\Omega^{n+1}\left(S^1 \wedge \mathsf{A}(*) \wedge_{\mathbb{Z}_2} {}^{S^\infty}/_{S^{n-1}}\right)$. Then by Lemma A.8 we may identify the homotopy pullback of (5.9) with the homotopy fiber of the composition

$$\begin{tikzcd} \Omega^{n+1}\mathsf{L}_\bullet(\mathsf{S}) \arrow[r] & \Omega^{n+1}\mathsf{A}(*)^{th\mathbb{Z}_2} \arrow[r] & \Omega^{n+1}\left(S^1 \wedge \mathsf{A}(*)_{h\mathbb{Z}_2}\right) \arrow[d] \\ & & \Omega^{n+1}\left(S^1 \wedge \mathsf{A}(*) \wedge_{\mathbb{Z}_2} {}^{S^\infty}/_{S^{n-1}}\right). \end{tikzcd} \tag{5.10}$$

Consider

$$\begin{tikzcd} \mathsf{A}(*)^{th\mathbb{Z}_2} \arrow[r] \arrow[d, "\simeq"] & S^1 \wedge \mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_+^\infty \arrow[d] \\ \left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n\right)^{th\mathbb{Z}_2} \arrow[r] & S^1 \wedge \mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n \wedge_{\mathbb{Z}_2} S_+^\infty. \end{tikzcd}$$

It commutes by naturality. By Lemma 5.10 the left vertical map is a homotopy equivalence. By Lemma 5.11, Ω^{n+1} of the right vertical map can be identified with the vertical map in (5.10). Thus we may exchange (5.10) by

$$\begin{tikzcd} \Omega^{n+1}\mathsf{L}_\bullet(\mathsf{S}) \arrow[r] & \Omega^{n+1}\mathsf{A}(*)^{th\mathbb{Z}_2} \arrow[d, "\simeq"] & \\ & \Omega^{n+1}\left(\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n\right)^{th\mathbb{Z}_2}\right) \arrow[r] & \Omega^n\left(\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n\right)_{h\mathbb{Z}_2}\right), \end{tikzcd} \tag{5.11}$$

noting that $\Omega^{n+1}S^1 \wedge (-) \mathrel{\widehat{=}} \Omega^n(-)$. Using Lemma A.8 and the norm fiber sequence for $\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n$ we exchange the homotopy fiber of (5.11) by the homotopy pullback of

$$\begin{tikzcd} & & \Omega^{n+1}\left(\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n\right)^{h\mathbb{Z}_2}\right) \arrow[d] \\ \Omega^{n+1}\mathsf{L}_\bullet(\mathsf{S}) \arrow[r] & \Omega^{n+1}\mathsf{A}(*)^{th\mathbb{Z}_2} \arrow[r, "\simeq"] & \Omega^{n+1}\left(\left(\mathsf{A}(*) \wedge_{\mathbb{Z}_2} S_!^n\right)^{th\mathbb{Z}_2}\right). \end{tikzcd} \tag{5.12}$$

We extend (5.12) to the right using a homotopy equivalence and thereby exchange the homotopy pullback with the homotopy pullback of the solid arrow diagram in

$$\begin{tikzcd} & \Omega^{n+1}\left(\mathsf{A}(*)\wedge_{\mathbb{Z}_2} S^n_!\right)^{h\mathbb{Z}_2} \arrow[r] \arrow[d, dashed] & \Omega^{n+1}\left(\mathsf{A}(*)\wedge_{\mathbb{Z}_2} S^{n+1}_!\right)^{h\mathbb{Z}_2} \arrow[d] \\ \Omega^{n+1}\mathsf{L}_\bullet(\mathsf{S}) \arrow[r] & \Omega^{n+1}\left(\mathsf{A}(*)\wedge_{\mathbb{Z}_2} S^n_!\right)^{th\mathbb{Z}_2} \arrow[r, "\simeq"] & \Omega^{n+1}\left(\mathsf{A}(*)\wedge_{\mathbb{Z}_2} S^{n+1}_!\right)^{th\mathbb{Z}_2}. \end{tikzcd} \tag{5.13}$$

This is an equivalent model. We remove the superfluous left vertical arrow. By Lemma 5.10 the upper horizontal map can be identified with the inclusion of the reduced homotopy fixed points $\Omega^{n+1}\left(\mathsf{A}(*)\wedge_{\mathbb{Z}_2} S^{n+1}_!\right)^{rh\mathbb{Z}_2}$ into the usual homotopy fixed points. The composition to the lower right is the usual map to the Tate fixed points. It is not difficult to see that the loops functor Ω commutes with the homotopy orbit, fixed point and Tate constructions.
Use Lemma 5.2 to identify $\Omega^{n+1}\mathsf{L}_\bullet(\mathsf{S}) \simeq \mathsf{L}_\bullet(\mathsf{S}; n+1)$ and $\Omega^{n+1}(S^{n+1}_! \wedge_{\mathbb{Z}_2} \mathsf{A}(*;0)) \simeq_{\mathbb{Z}_2} \mathsf{A}(*;n{+}1)$ as the spectrum $\mathsf{A}(*)$ with $(n{+}1)$-duality instead of the usual 0-duality. In the proof of Lemma 5.2 we established a short exact sequence with free duality involution on the middle term. Thus when passing to the Tate fixed points we immediately obtain $\Omega^{n+1}\mathsf{A}(*;0)^{th\mathbb{Z}_2} \simeq \mathsf{A}(*;n+1)^{th\mathbb{Z}_2}$. We use this to obtain a similar model for the Tate fixed points of $\mathsf{A}(*)$ as for L-theory. Analogously, consider $\mathsf{A}(*;0)$ as the realization of a simplicial spectrum. A k-simplex in $\mathsf{A}(*;0)$ is a diagram of retractive spaces over the usual k-simplex. Using these models it follows more easily by inspection that we may substitute Ξ_{n+1} for $\Omega^{n+1}\Xi_0$. Diagram (5.5) follows.
The assertion of having a loop map is straight forward. Until the end of the proof, it is clear that all maps from $\Omega^{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right)$ actually stem from maps defined on ${}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}$. In particular the maps $\Omega^{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right) \to \Omega^\infty\mathsf{P}$ and the composition $\Omega^{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right) \to \Omega^{\infty+n+1}\mathsf{L}_\bullet(\mathsf{S})$ are $(n+1)$-fold loop maps. The latter map is identified back in Corollary 5.5 via (5.1) as the stabilization ${}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)} \to {}^{\mathrm{G}}\big/_{\mathrm{TOP}}$, composed with the equivalence to $\Omega^\infty_0\mathsf{L}_\bullet(\mathsf{S})$ and looped $(n+1)$-times. □

Remark 5.12. The map Ξ reads $\mathsf{L}_\bullet(\mathbb{Z}; n+1) \to \mathsf{A}(*;n+1)^{th\mathbb{Z}_2}$. One can map further to $S^1 \wedge \mathsf{A}(*;n+1)_{h\mathbb{Z}_2}$ to obtain the class ω_{n+1} in the $(n+1)$-duality case. Here, it actually factors via $\Omega^{n+1}S^1 \wedge (\mathsf{A}(*)\wedge_{\mathbb{Z}_2} S^n_!)_{h\mathbb{Z}_2} \simeq \mathsf{A}(*;n)_{h\mathbb{Z}_2}$, the homotopy cofiber of $\mathsf{A}(*;n+1)^{rh\mathbb{Z}_2} \to \mathsf{A}(*;n+1)^{th\mathbb{Z}_2}$. This can be seen by extending (5.13) by the vertical homotopy cofibers and using (5.8). The homotopy pullback P of (5.5) can thus equivalently be written as

$$\mathsf{P} \simeq \mathrm{hofiber}\left[\mathsf{L}_\bullet(\mathbb{Z}; n+1) \to \mathsf{A}(*;n)_{h\mathbb{Z}_2}\right], \tag{5.14}$$

and *not* as the homotopy fiber associated to $S^1 \wedge \mathsf{A}(*;n)_{h\mathbb{Z}_2}$ nor $S^1 \wedge \mathsf{A}(*;n+1)_{h\mathbb{Z}_2}$.

6. An approximation of $\mathrm{TOP}(n)/\mathrm{O}(n)$

Assume the resulting element $\xi \circ \sigma \in \pi_{15}\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$ from the homotopy operation does not vanish in the approximation, i. e. $\phi_{\mathrm{G/TOP}} \circ \xi \circ \sigma \neq 0 \in \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right)$. In this section we show that this implies that the main obstruction $\mathrm{Obs}(s, \Sigma) \neq 0 \in \pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ does not vanish. We do this by showing that its approximation $\phi_{\mathrm{TOP/O}}(\mathrm{Obs}(s,\Sigma)) \in \pi_{14}\left(T_1\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)\right)$ does not vanish. From now on forth, we may suppress the subscripts from ϕ.
Recall (4.15), i. e. $\mathrm{Obs}(s,\Sigma) = \partial(\xi \circ \sigma) + \mathrm{proj}_*\left[c \circ \iota \circ \xi, \mathrm{incl} \circ s_0\right]$. In Section 6.1 we show that the Whitehead product correction term vanishes in $\pi_{14}\left(T_1\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)\right)$. In Section 6.2 we show that in the long exact sequence the boundary homomorphism $\partial : \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right) \to \pi_{14}\left(T_1\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)\right)$ is injective, compare Proposition 6.14. In other words, if $\phi \circ \xi \circ \sigma \neq 0$, then $\mathrm{Obs}(s,\Sigma) \neq 0$.

6.1. The vanishing correction term

In this section we show that the Whitehead product term $[c \circ \iota \circ \xi, \mathrm{incl} \circ s_0]$ already vanishes in the first orthogonal calculus Taylor approximation $T_1\mathrm{BTOP}(S^n)$. This is valid for general $n \geq 5$. Recall that $\phi_{\mathrm{BTOP}(S^n)} : \mathrm{BTOP}(S^n) \to T_1\mathrm{BTOP}(S^n)$ denotes the orthogonal calculus approximation map into the first Taylor approximation of $\mathrm{BTOP}(S^n)$.

Lemma 6.1. *The Whitehead product* $(\phi_{\mathrm{BTOP}(S^n)})_*[c \circ \iota \circ \xi, \mathrm{incl} \circ s_0]$ *vanishes in the first Taylor approximation* $T_1\mathrm{BTOP}(S^n)$*.*

Lemma 6.2. *The derivative spectrum of the functor* $V \mapsto \mathrm{BTOP}(S(V))$ *is given by* $\theta_{\mathrm{BTOP}(S(V))} \simeq \mathsf{S} \times \Omega^1_! \mathsf{Wh}(*)$*.*

Proof of Lemma 6.2. By [Wei95], the derivative spectrum $\theta_{\mathrm{BTOP}(V)}$ of $V \mapsto \mathrm{BTOP}(V)$ is given as $\theta_{\mathrm{BTOP}(V)} \simeq \mathsf{A}(*) \simeq \mathsf{S} \times \mathsf{Wh}(*)$. Here S is endowed with the trivial involution, and $\mathsf{Wh}(*)$ comes with the usual duality involution. To prove the lemma, we identify a certain splitting in the case of the shifted functors on $V \oplus \mathbb{R}$. The shift is reminiscent of the dimension shift in $\mathbb{R}^n \mapsto \mathrm{BTOP}(S^n) = \mathrm{BTOP}(S(\mathbb{R}^n \oplus \mathbb{R}))$, which we need later. It is not difficult to deduce $\theta_{\mathrm{BO}(V \oplus \mathbb{R})} \simeq S^1_! \wedge \mathsf{S}$ as well as $\theta_{\mathrm{BG}(V \oplus \mathbb{R})} \simeq S^1_! \wedge \mathsf{S}$ for the shifted functors from [Wei95]. The canonical map between these is an equivalence, see also Lemma 6.6 below. We use the factorization

$$\mathrm{BO}(V \oplus \mathbb{R}) \to \mathrm{BTOP}(S(V \oplus \mathbb{R})) \to \mathrm{BG}(V \oplus \mathbb{R}) \tag{6.1}$$

to see that a factor $S^1_! \wedge \mathsf{S}$ of $\theta_{\mathrm{BTOP}(S(V \oplus \mathbb{R}))}$ splits off. We continue to put $\theta_{\mathrm{BTOP}(S(V \oplus \mathbb{R}))}$ into context. Recall $\theta_{\mathrm{G}(V)/\mathrm{TOP}(V)} \simeq \mathsf{Wh}(*)$, compare Corollary 5.5. It maps via

$$\theta_{\mathrm{G}(V)/\mathrm{TOP}(V)} \to \theta_{\mathrm{BTOP}(V)} \to \theta_{\mathrm{BTOP}(S(V \oplus \mathbb{R}))} \to \theta_{\mathrm{BG}(V \oplus \mathbb{R})} \tag{6.2}$$

nullhomotopically to $S^1_! \wedge \mathsf{S}$, because ${}^{\mathrm{G}(V)}/_{\mathrm{TOP}(V)} \to \mathrm{BTOP}(V) \to \mathrm{BG}(V)$ is already nullhomotopic. We argue in the following that the resulting

$$\theta_{\mathrm{G}(V)/\mathrm{TOP}(V)} \to \theta_{\mathrm{BTOP}(S(V \oplus \mathbb{R}))} \to \theta_{\mathrm{BG}(V \oplus \mathbb{R})} \tag{6.3}$$

is a split fiber sequence, from which $\theta_{\mathrm{BTOP}(S(V))} \simeq \Omega^1_! \left(\mathsf{Wh}(*) \times (S^1_! \wedge \mathsf{S})\right)$ follows. The splitting is induced from (6.1) via $\theta_{\mathrm{BG}(V\oplus\mathbb{R})} \simeq \theta_{\mathrm{BO}(V\oplus\mathbb{R})} \to \theta_{\mathrm{BTOP}(V\oplus\mathbb{R})}$. For the fiber sequence property, consider

$$\begin{tikzcd} \mathrm{BTOP}\,(V) \arrow[r] \arrow[d] & \mathrm{BG}\,(V) \arrow[d] \\ \mathrm{BTOP}\,(S(V\oplus\mathbb{R})) \arrow[r] & \mathrm{BG}\,(V\oplus\mathbb{R}). \end{tikzcd} \tag{6.4}$$

Note that in the version of (6.4) associated to BO $(V) \to$ BTOP (V), the vertical homotopy fibers coincide. Using $\theta_{\mathrm{BG}(V\oplus\mathbb{R})} \simeq \theta_{\mathrm{BO}(V\oplus\mathbb{R})}$, (6.4) is likewise a homotopy pullback diagram on the derivative spectrum level. This yields the necessary identification of the homotopy fiber of $\theta_{\mathrm{BTOP}(S(V\oplus\mathbb{R}))} \to \theta_{\mathrm{BG}(V\oplus\mathbb{R})}$ with $\theta_{\mathrm{G}(V)/\mathrm{TOP}(V)}$. In particular, note for the proof below that $\theta_{\mathrm{G}(V)/\mathrm{TOP}(V)}$ maps to the $\mathsf{Wh}(*)$-factor of $\theta_{\mathrm{BTOP}(V\oplus\mathbb{R})}$. ☐

Proof of Lemma 6.1. We prove the first statement by separating $c \circ \iota \circ \xi$ and $\mathrm{incl} \circ s_0$ step by step into two different factors of a product space, as illustrated by (6.5). The Whitehead bracket then vanishes.

We identify BTOPS $\simeq$ BTOP as described in Remark 1.7. Using [Wei95, Theorem 9.1] we may identify the homotopy fiber $F_{\mathrm{BTOP}(S^n)}$ of the map $T_1\mathrm{BTOP}\,(S^n) \to \mathrm{BTOP}$ as $\Omega^\infty(\theta_{\mathrm{BTOP}(S(V\oplus\mathbb{R}))})_{h\mathbb{Z}_2}$, see also (6.5) below. Write $F_{\mathrm{TOP}} = F_{\mathrm{BTOP}(S^n)}$ for short. Recall that in this thesis $\Omega^\infty\theta_{h\mathbb{Z}_2}$ always denotes first taking orbits, then Ω^∞.

Introduce the abbreviation $\alpha = (\phi_{\mathrm{BTOP}(S^n)})_* c_* \iota_* (\xi)$ and $\beta = (\phi_{\mathrm{BTOP}(S^n)})_* \,\mathrm{incl}_* (s_0)$ for the two elements in $\pi_{n+1}(T_1(\mathrm{BTOP}\,(S^n)))$. Since BTOP is an infinite loop space, the image of $[\alpha, \beta]$ in BTOP vanishes. In other words, we may view the Whitehead bracket as an element in $\pi_{2n+1}(F_{\mathrm{TOP}})$ of the fiber F_{TOP}. By [WW18], we may identify $T_1\mathrm{BTOP}\,(S^n)$ as the homotopy pullback of $\mathrm{BTOP} \to \mathrm{BG} \leftarrow E_{\mathrm{G}}$ for some E_{G} as illustrated by (6.5). This is roughly due to the fact that the TOP-action on $\theta_{\mathrm{BTOP}(V)}$ with involution can be extended to a G-action. As a consequence, $E_{\mathrm{G}} \to \mathrm{BG}$ also has homotopy fiber $F_{\mathrm{G}} \simeq F_{\mathrm{TOP}}$, and we identify $[\alpha, \beta]$ with $[g_*\alpha, g_*\beta]$.

Recall that $\xi \in \pi_{n+1}\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right)$. Its image in $\pi_{n+1}(\mathrm{BG}\,(n))$ is zero. Upon viewing F_{TOP} as a fiber over BG, we can thus conclude that $g_*\alpha \in \pi_{n+1}(F_{\mathrm{G}})$ itself also lies in the fiber. By (6.2), $T_1\left({}^{\mathrm{G}(n)}\big/_{\mathrm{TOP}(n)}\right)$ maps only into the $\Omega^\infty\mathsf{Wh}(*)_{h\mathbb{Z}_2}$-subspaces of $T_1\mathrm{BTOP}\,(S^n)$, and further to E_{G}. Note here that the two left arrows in (6.2) reflect precisely ι and c from (4.12) on the derivative spectrum level. In other words, $g_*\alpha \in \pi_{n+1}(\Omega^\infty\mathsf{Wh}(*)_{h\mathbb{Z}_2}) \subset \pi_{n+1}(F_{\mathrm{G}})$. On the other hand, $\theta_{\mathrm{BO}(V\oplus\mathbb{R})} \simeq S^1_! \wedge \mathsf{S}$ as seen above. Thus $g_*\beta$ occupies only the $\Omega^\infty(S^1_! \wedge \mathsf{S})_{h\mathbb{Z}_2}$-subspaces $\pi_{n+1}(T_1\mathrm{BTOP}\,(S^n))$ before being mapped to $\pi_{n+1}(E_{\mathrm{G}})$, yet it is also non-zero in $\pi_{n+1}(\mathrm{BO})$ by Remark 4.24, and likewise non-zero in $\pi_{n+1}(\mathrm{BG})$. Beware that the factors $\Omega^\infty\mathsf{Wh}(*)_{h\mathbb{Z}_2}$ and $\Omega^\infty(S^1_! \wedge \mathsf{S})_{h\mathbb{Z}_2}$ can only be split in the homotopy fiber in a duality-respecting manner, yet the homotopy fiber sequence of fibered spaces over BG does not split.

We attempt to amend this deficit and thereby separate α and β further. Consider the following diagram as a roadmap for the arguments in the sequel. The two lower left squares are homotopy pullback squares. The vertical homotopy fibers $F_{\mathrm{TOP}} \simeq F_{\mathrm{G}} \simeq F_{\mathrm{O}}$

can all be identified with each other, so we may simply write F for any of these versions.

$$\begin{tikzcd}
F_{\mathrm{TOP}} \arrow[r, "\simeq"] \arrow[d] & F_{\mathrm{G}} \arrow[d] & F_{\mathrm{O}} \arrow[l, "\simeq"'] \arrow[d] & F^1_{\mathrm{O}} \times F^2_{\mathrm{O}} \arrow[l, "\simeq"'] \arrow[d] & \Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}) \times F^2_{\mathrm{O}} \arrow[l] \arrow[d] \\
T_1\mathrm{BTOP}\,(S^n) \arrow[r] \arrow[d] & E_{\mathrm{G}} \arrow[d] & E_{\mathrm{O}} \arrow[l] \arrow[d] & E^1_{\mathrm{O}} \times_{\mathrm{BO}} E^2_{\mathrm{O}} \arrow[l, "\cong"'] \arrow[d] & \Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}) \times E^2_{\mathrm{O}} \arrow[l] \arrow[d] \\
\mathrm{BTOP} \arrow[r, "g"] & \mathrm{BG} & \mathrm{BO} \arrow[l, "h"'] & \mathrm{BO} \arrow[l, "="'] & \mathrm{BO} \arrow[l, "="']
\end{tikzcd} \tag{6.5}$$

We pull back E_{G} along $h : \mathrm{BO} \to \mathrm{BG}$ to E_{O}. Map and identify $[h_*^{-1} g_*\alpha, h_*^{-1} g_*\beta]$ in our new copy F_{O} of F. As earlier, $h_*^{-1} g_*\alpha$ is mapped and identified using the identification $F_{\mathrm{O}} \simeq F_{\mathrm{G}}$. The element $h_*^{-1} g_*\beta$ canonically lies in E_{O} since s_0 is mapped to E_{G} via $T_1\mathrm{BTOP}\,(S^n)$ and to BO by stabilization. Recall that the homotopy fiber F splits as $\Omega^\infty \mathsf{Wh}(*)_{h\mathbb{Z}_2} \times \Omega^\infty (S^1_! \wedge \mathsf{S})_{h\mathbb{Z}_2} =: F^1_{\mathrm{O}} \times F^2_{\mathrm{O}}$. The key idea for this proof is the following. Over BO there is a canonical splitting of F, i. e. over BO the splitting of F (over the basepoint) extends to a fiberwise splitting respecting the involution. This is explained in Remark 6.3 below. Thus we may write $E_{\mathrm{O}} \to \mathrm{BO}$ as the fiberwise product of two homotopy fiber sequences $F^j_{\mathrm{O}} \to E^j_{\mathrm{O}} \to \mathrm{BO}$, $j = 1, 2$, of fibered spaces over BO. In other words, $E_{\mathrm{O}} \cong E^1_{\mathrm{O}} \times_{\mathrm{BO}} E^2_{\mathrm{O}}$. This is not a product yet. We introduce a further modification to rectify this problem.

Recall diagram (5.3). Note that Ξ is in fact a G-map. The action of G on $\mathsf{A}(*)$ is induced via $\mathsf{S} \wedge \mathsf{A}(*)$ from the canonical G-action on S. This action is non-trivial, yet only the action on the duality is interesting. It is a byproduct of the description of the duality via spherical fibrations, and analyzed in [WW18]. We construct a (fiberwise) fiber sequence

$$\Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}) \times \Omega^\infty (S^1_! \wedge \mathsf{S})_{h\mathbb{Z}_2} \to E'_{\mathrm{O}} \to \mathrm{BO}.$$

Recall that Ξ maps quadratic L-theory into the $\mathsf{Wh}(*)_{h\mathbb{Z}_2}$-factor. Define a map fiberwise to E_{O} using

$$\Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}) \to \Omega^\infty \mathsf{Wh}(*)_{h\mathbb{Z}_2} = F^1_{\mathrm{O}} \to F \to E_{\mathrm{O}}$$

on the first factor, and the inclusion of the fibers $\Omega^\infty (S^1_! \wedge \mathsf{S})_{h\mathbb{Z}_2} = F^2_{\mathrm{O}} \to F \to E_{\mathrm{O}}$ on the second factor. Define $E'_{\mathrm{O}} \to E_{\mathrm{O}} \to \mathrm{BO}$ as the projection. Since Ξ is a G- and in particular an O-map, this construction is well-defined. We obtain a fiber-respecting map $E'_{\mathrm{O}} \to E_{\mathrm{O}}$ over BO as expected intuitively.

The fact that we use $\mathsf{Wh}(*)_{h\mathbb{Z}_2}$ instead of $S^1 \wedge \mathsf{Wh}(*)_{h\mathbb{Z}_2}$ is explained by Remark 5.12. The action of G on the domain is best seen by viewing $\mathsf{L}_\bullet(\mathsf{S})$ via retractive spectra over a point. However, using its usual definition via chain complexes, the equivalent $\mathsf{L}_\bullet(\mathsf{H}\mathbb{Z}) = \mathsf{L}_\bullet(\mathbb{Z})$ has trivial SG-action. In particular, $(E'_{\mathrm{O}})^1 = \mathrm{BO} \times \Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S})$ is a product. Thus $E'_{\mathrm{O}} = \Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}) \times E^2_{\mathrm{O}}$. Note that the only important properties of Ξ at this point are its O-equivariance, that the SO-action on the domain is trivial, and that α lifts along Ξ as explained in the sequel.

Recall $\alpha = \phi_* c_* \iota_*(\xi) \in \pi_{n+1}\,(T_1\mathrm{BTOP}\,(S^n))$. We had lifted α into the fiber F. Since ξ comes from L-theory, $h_*^{-1} g_*\alpha$ lifts to a corresponding $\alpha' \in \pi_{n+1}\,(\mathsf{L}_\bullet(\mathsf{S}))$. Recall that

$\beta' = h_*^{-1} g_* \beta \in \pi_{n+1}(E^2_{\mathrm{O}})$. In summary, α' and β' are separated into the two factors of $\Omega^\infty_0 \mathsf{L}_\bullet(\mathsf{S}) \times E^2_{\mathrm{O}}$.
The Samelson product $\langle \alpha', \beta' \rangle \in \pi_{2n}(\Omega E'_{\mathrm{O}})$ [Adh16, Definition 9.5.1] is related to the Whitehead product by taking a commutator of the adjoints on the loop space [Adh16, Theorem 9.6.2]. Since α' and β' are in separate factors, the commutator vanishes. Thus $[\alpha', \beta'] = 0$ and also $[\alpha, \beta] = 0 \in \pi_{2n+1}(F)$. □

Remark 6.3. We come back to the duality splitting over BO. Recall that the homotopy fiber F splits as $\Omega^\infty(S^1_! \wedge \mathsf{S})_{h\mathbb{Z}_2} \times \Omega^\infty \mathsf{Wh}(*)_{h\mathbb{Z}_2}$. We argue that over BO there is a canonical splitting of F, i. e. over BO the splitting of F (over the basepoint) extends to a fiberwise splitting respecting the involution.
Recall that orthogonal calculus [Wei95] supplies us with fibered derivative spectra θ_F over $F(\mathbb{R}^\infty)$ associated to each of the functors $F : V \mapsto \mathrm{BO}(V \oplus \mathbb{R})$, $\mathrm{BTOP}(S(V \oplus \mathbb{R}))$ and $\mathrm{BG}(V \oplus \mathbb{R})$. We already know [WW18] that the fibered spectrum $\theta_{\mathrm{BTOP}(S(V \oplus \mathbb{R}))}$ can actually be pulled back from BG. We may pull back both fibered spectra to BO to obtain maps

$$\theta_{\mathrm{BO}(V \oplus \mathbb{R})} \to \theta_{\mathrm{BTOP}(S(V \oplus \mathbb{R}))} \to \theta_{\mathrm{BG}(V \oplus \mathbb{R})}$$

of fibered spectra over BO, reminiscent of (6.1). Again the composition is the identity on $S^1_! \wedge \mathsf{S}$, from which the canonical splitting over BO follows. Note that a similar statement also holds for example for $\theta_{\mathrm{BTOP}(V)} = \mathsf{Wh}(*) \times \mathsf{S}$.

Corollary 6.4. *The Whitehead product* $(\phi_{\mathrm{TOP}(n)/\mathrm{O}(n)})_* \operatorname{proj}_* [c \circ \iota \circ \xi, \operatorname{incl} \circ s_0]$ *vanishes in the first Taylor approximation* $T_1\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)$.

Proof. The claim follows easily by using the naturality of the first orthogonal calculus Taylor approximation as indicated by

$$\begin{tikzcd}
\pi_{2n+1}(\mathrm{BTOP}(S^n)) \arrow[r] \arrow[d, "\phi"] & \pi_{2n+1}(\mathrm{BTOP}(S^n), \mathrm{BO}(n+1)) \arrow[d, "\phi"] \\
\pi_{2n+1}(T_1\mathrm{BTOP}(S^n)) \arrow[r] & \pi_{2n+1}(T_1\mathrm{BTOP}(S^n), T_1\mathrm{BO}(n+1)).
\end{tikzcd}$$

The final polish is applied by employing a natural identification $\pi_{2n}\left({}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)}\right) \cong \pi_{2n+1}(\mathrm{BTOP}(S^n), \mathrm{BO}(n+1))$, together with the usual identification ${}^{\mathrm{TOP}(S^n)}/_{\mathrm{O}(n+1)} \cong {}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}$. The analogue of the latter holds also for their T_1-approximations since also ${}^{\mathrm{TOPS}}/_{\mathrm{O}} \cong {}^{\mathrm{TOP}}/_{\mathrm{O}}$. □

Corollary 6.5. *As a consequence, we obtain a description of the element*

$$\phi_* \mathrm{Obs}(s, \Sigma) = \partial(\phi \circ \xi \circ \sigma) \in \pi_{14}\left(T_1\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)\right), \tag{6.6}$$

where $\partial : \pi_{15}\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \to \pi_{14}\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)$ *is the boundary homomorphism.*

This is immediate from Obstruction 4.26 and Corollary 6.4, using also (6.7).

6.2. The injectivity of the boundary homomorphism

Consider the commutative diagram induced from the naturality of the first orthogonal calculus Taylor approximation functor T_1,

$$\begin{array}{ccc} \pi_{2n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right) & \xrightarrow{\partial} & \pi_{2n}\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right) \\ \Big\downarrow{\scriptstyle \phi_{\mathrm{G/TOP}}} & & \Big\downarrow{\scriptstyle \phi_{\mathrm{TOP/O}}} \\ \pi_{2n+1}\left(T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)\right) & \xrightarrow{\partial} & \pi_{2n}\left(T_1\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)\right). \end{array} \tag{6.7}$$

We show injectivity of ∂ for $n = 7$ on the T_1-level by proving that the preceding map in the long exact sequence $\pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{O}(7)}\right)\right) \to \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right)$ is zero. For the general case $n > 7$, see Remark 6.15.

The proof can be briefly outlined as follows. We identify $1 \in \mathbb{Z}_2 \cong \pi_{15}\left({}^{\mathrm{G}}/_{\mathrm{O}}\right) \cong \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{O}(7)}\right)\right)$, which can be lifted to $\kappa \cdot \eta \in \pi_{15}^s \cong \pi_{15}(\mathrm{G})$ and further to $\pi_{15}(\mathrm{TOP})$. Since the composition $\mathrm{TOP} \to \mathrm{G} \to {}^{\mathrm{G}}/_{\mathrm{TOP}}$ is nullhomotopic, we obtain a map $\lambda : \mathrm{TOP} \to \Omega^{\infty-7}\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}$, i.e. to the homotopy fiber of $T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \to {}^{\mathrm{G}}/_{\mathrm{TOP}}$. We show that $\lambda \circ \kappa$ vanishes at the Postnikov stage $\mathsf{Wh}(*;7)_{[3,5]}$, meaning it lifts to $\mathsf{Wh}(*)_{[7],h\mathbb{Z}_2}$. This is an Eilenberg-MacLane spectrum, meaning the composition $\lambda \circ \kappa \circ \eta$ vanishes as desired. We begin by identifying $T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{O}(n)}\right)$.

Lemma 6.6. $T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{O}(n)}\right) \simeq {}^{\mathrm{G}}/_{\mathrm{O}}$.

Proof. By [Wei95], the two derivative spectra $\theta_{\mathrm{BO}(V)} \simeq \mathsf{S}$ and $\theta_{\mathrm{BG}(V)} \simeq \mathsf{S}$ are equivalent. This is due to the high connectivity of ${}^{\mathrm{O}(n+1)}/_{\mathrm{O}(n)} \to {}^{\mathrm{G}(n+1)}/_{\mathrm{G}(n)}$ as seen in Lemma 1.16. Thus we can identify the vertical homotopy fibers of (6.8),

$$\begin{array}{ccc} T_1\mathrm{BO}(n) & \longrightarrow & T_1\mathrm{BG}(n) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{BO} & \longrightarrow & \mathrm{BG}. \end{array} \tag{6.8}$$

Consequently, (6.8) is a homotopy pullback diagram, and we may identify the horizontal homotopy fibers $T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{O}(n)}\right) \simeq {}^{\mathrm{G}}/_{\mathrm{O}}$. □

Lemma 6.7. $\pi_{15}\left({}^{\mathrm{G}}/_{\mathrm{O}}\right) \cong \mathbb{Z}_2$*, and there is a commutative diagram*

$$\begin{array}{ccc} \pi_{15}^s & \xrightarrow{\cong} & \pi_{15}(\mathrm{G}) \\ \Big\downarrow & & \Big\downarrow \\ {}^{\pi_{15}^s}/_J & \xrightarrow{\cong} & \pi_{15}\left({}^{\mathrm{G}}/_{\mathrm{O}}\right). \end{array} \tag{6.9}$$

Proof. By [KM63], $|\theta_{15}| = 16256$ and $|bP_{16}| = 8128$. Therefore, by Lemma 1.39, the cokernel of $\pi_{16}\left({}^{\mathrm{G}}/_{\mathrm{TOP}}\right) \to \pi_{15}\left({}^{\mathrm{TOP}}/_{\mathrm{O}}\right)$ is $\mathbb{Z}_2$. Since $\pi_{15}\left({}^{\mathrm{G}}/_{\mathrm{TOP}}\right) = 0$ by Lemma 1.41, we can identify $\pi_{15}\left({}^{\mathrm{G}}/_{\mathrm{O}}\right)$ with the cokernel $\mathbb{Z}_2$ above. We proceed to argue towards (6.9). By [Lüc02, Theorem 6.48], for $k \geq 5$ we may identify the sequences

$$\ldots \to \pi_k(\mathrm{SO}) \to \Omega_k^{\mathrm{fr}} \to \Omega_k^{\mathrm{alm}} \to \ldots \text{ and } \ldots \to \pi_k(\mathrm{O}) \to \pi_k(\mathrm{G}) \to \pi_k\left({}^{\mathrm{G}}/_{\mathrm{O}}\right) \to \ldots$$

By [Lüc02, Lemma 6.24], the composition $\pi_k\,(\mathrm{SO}) \to \Omega_k^{\mathrm{fr}} \to \pi_k^s$ with the Pontrjagin-Thom isomorphism can be identified with the J-homomorphism. Since $\pi_{14}\,(\mathrm{O}) = 0$ [Lüc02], $\pi_{15}\,({}^{\mathrm{G}}/_{\mathrm{O}})$ is indeed identified via $\Omega_{15}^{\mathrm{alm}}$ with $\operatorname{coker} J$ using the identification above. $\square$

Denote by $\kappa \in \pi_{14}^s \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ [Rav86, Table A.3.3] the non-zero element of Kervaire invariant zero [Bro69] and by $\eta \in \pi_1^s$ a generating first stable Hopf map. These are notations as given by Toda [Tod71].

Corollary 6.8. *The generator of* $\pi_{15}\,({}^{\mathrm{G}}/_{\mathrm{O}})$ *lifts to the element* $\kappa \cdot \eta \in \pi_{15}^s$.

Proof. By [Rav86, Table A.3.3], the cokernel of the J-homomorphism contains precisely one non-zero element. It can be written as claimed. $\square$

Proposition 6.9. *The morphism* $\pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{O}(7)}\right)\right) \to \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right)$ *is zero.*

Proof. Recall that the splitting in Definition 5.1 induces a map of spectra $\mathsf{A}(*) \to \mathsf{Wh}(*)$ respecting the duality. Recall also that Ξ actually takes values in the Tate construction $\mathsf{Wh}(*)^{th\mathbb{Z}_2}$ if defined on quadratic L-theory. Therefore, by (5.14),

$$\Omega^8 T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \simeq \operatorname{hofiber}\left[\Omega^8({}^{\mathrm{G}}/_{\mathrm{TOP}}) \to \Omega^\infty \mathsf{Wh}(*;7)_{h\mathbb{Z}_2}\right].$$

In other words, $\Omega^{\infty-7}\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}$ is the homotopy fiber of the right lower vertical map in (6.10). In comparison to (5.14), here we use $\Omega^8({}^{\mathrm{G}}/_{\mathrm{TOP}}) \simeq \mathbf{L}_\bullet\,(\mathbb{Z};8)$. In order to show that $\pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{O}(7)}\right)\right) \to \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right)$ is zero, we must show that the lift $\kappa \cdot \eta \in \pi_{15}^s \cong \pi_{15}\,(\mathrm{G})$ is mapped to zero under the composition $\mathrm{G} \to T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$ as shown in

$$\begin{tikzcd}
 & & \mathrm{TOP} \arrow[rr, dashed, "\lambda"] \arrow[d] & & \Omega^{\infty-7}\mathsf{Wh}(*;7)_{h\mathbb{Z}_2} \arrow[d] \\
S^{15} \arrow[r, "\eta"] & S^{14} \arrow[r, "\kappa"] \arrow[ur, dashed, "\kappa'"] & \mathrm{G} \arrow[r, dashed] \arrow[dr, bend right] & T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{O}(7)}\right) \arrow[r] \arrow[d, "\simeq"] & T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \arrow[d] \\
 & & & {}^{\mathrm{G}}/_{\mathrm{O}} \arrow[r] & {}^{\mathrm{G}}/_{\mathrm{TOP}}\,.
\end{tikzcd} \tag{6.10}$$

By the discussion around [Lüc02, (6.40)] and the methods of the proof of (6.9), $\kappa \in \pi_{14}^s$ is mapped to zero via $\pi_{14}^s \to \pi_{14}\,(\mathrm{G}) \to \pi_{14}\,({}^{\mathrm{G}}/_{\mathrm{TOP}})$. This is essentially because its Kervaire invariant is zero. As a consequence, κ can be lifted uniquely to $\kappa' \in \pi_{14}\,(\mathrm{TOP}) \cong \mathbb{Z}_2$. The same is then true for $\kappa \circ \eta$, which lifts to $\kappa' \circ \eta$. Note also that the composition $\mathrm{TOP} \to {}^{\mathrm{G}}/_{\mathrm{TOP}}$ is nullhomotopic, meaning there is a lift λ to $\Omega^{\infty-7}\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}$. As a consequence, $[\lambda \circ \kappa'] \in \pi_7\left(\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}\right)$ is a lift of the image of κ in $T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$. In the sequel we prove that $[\lambda \circ \kappa' \circ \eta] = 0$, which entails the statement.

We begin by establishing some facts about $\mathsf{Wh}(*)$. It is 2-connected [Rog02]. In fact, according to [Rog02, Theorem 5.8] and [Rog03, Corollary 4.9] the only non-zero homotopy groups up to and including degree 8 are $\pi_3 \cong \mathbb{Z}_2$, $\pi_5 \cong \mathbb{Z}$ and $\pi_7 \cong \mathbb{Z}_2$. Denote by $\mathsf{X}_{[k_1,k_2]}$ the Whitehead-Postnikov truncation of X with $\pi_i = 0$ for $i > k_2$ and $i < k_1$. It is clear that the involution on $\mathsf{H}(\mathbb{Z}_2, 3) \simeq_{\mathbb{Z}_2} S^1_! \wedge \mathsf{H}(\mathbb{Z}_2, 2)$ is trivial, and analogously for $\mathsf{H}(\mathbb{Z}_2, 7)$. As already insinuated, the duality is the same on $\Omega^7 S^7_! \wedge \mathsf{H}(\mathbb{Z}_2, 3)$. However, we show in the sequel that the involution is not trivial on π_5 and given by $\mathsf{Wh}(*)_{[5]} = S^1_! \wedge \mathsf{H}(\mathbb{Z}, 4)$. We begin with the following calculations.

Lemma 6.10. *Let $k \geq 0$. The homotopy orbit, fixed point and Tate spectra associated to the Eilenberg-MacLane spectrum $\mathsf{H}(\mathbb{Z}_2, k)$ are given by*

$$\mathsf{H}(\mathbb{Z}_2, k)_{h\mathbb{Z}_2} \simeq \prod_{i \geq k} \mathsf{H}(\mathbb{Z}_2, i), \qquad \mathsf{H}(\mathbb{Z}_2, k)^{th\mathbb{Z}_2} \simeq \prod_{i \in \mathbb{Z}} \mathsf{H}(\mathbb{Z}_2, i),$$
$$\mathsf{H}(\mathbb{Z}_2, k)^{h\mathbb{Z}_2} \simeq \prod_{i \leq k} \mathsf{H}(\mathbb{Z}_2, i), \qquad \mathsf{H}(\mathbb{Z}_2, k)^{rh\mathbb{Z}_2} \simeq \prod_{i \leq k-1} \mathsf{H}(\mathbb{Z}_2, i).$$

All isomorphisms from and to the Tate spectrum in the long exact sequence on homotopy groups associated to the norm cofiber sequence of $\mathsf{H}(\mathbb{Z}_2, k)$ are induced by identity mappings from one $\mathsf{H}(\mathbb{Z}_2, i)$-summand to another in each respective spectrum.

Proof. Recall that $\mathrm{B}\mathbb{Z}_{2+}$ is $\mathbb{R}P^\infty_+$ and has one cell in each positive dimension. Using the naive smash product of a space with a spectrum, we obtain $\pi_i\left(\mathsf{H}(\mathbb{Z}_2, k)_{h\mathbb{Z}_2}\right) \cong \pi_{i-k}(\mathrm{B}\mathbb{Z}_{2+} \wedge \mathsf{H}\mathbb{Z}_2) \cong H_{i-k}(\mathbb{R}P^\infty, \mathbb{Z}_2)$. The claim follows since a smash product with an Eilenberg-MacLane spectrum is itself (non-canonically) equivalent to a generalized Eilenberg-MacLane spectrum, see [Rav86, Proposition 2.1.2]. For the fixed points, we may write $\pi_i\big(\mathsf{H}(\mathbb{Z}_2, k)^{h\mathbb{Z}_2}\big) \cong [S^i \wedge \mathrm{B}\mathbb{Z}_{2+}, \mathsf{H}(\mathbb{Z}_2, k)] \cong H^{k-i}(\mathbb{R}P^\infty, \mathbb{Z}_2)$. Notice that the mapping spectrum of $\mathrm{B}\mathbb{Z}_{2+}$ to $\mathsf{H}(\mathbb{Z}_2, k)$ is a module spectrum over $\mathsf{H}\mathbb{Z}_2$, which turns it into a generalized Eilenberg-MacLane spectrum, from which the claim follows. The reduced homotopy fixed points $\mathsf{H}(\mathbb{Z}_2, k)^{rh\mathbb{Z}_2}$ are simply given as homotopy fiber of $\mathsf{H}(\mathbb{Z}_2, k)^{h\mathbb{Z}_2} \to \mathsf{H}(\mathbb{Z}_2, k)$. The norm map is the zero map on homotopy groups, since it is multiplication by two. From this one can deduce the homotopy groups of the Tate spectrum using the appropriate long exact sequence. Furthermore, it is an Eilenberg-MacLane spectrum since, as in [WW89], one can rewrite

$$\mathsf{H}(\mathbb{Z}_2, k)^{th\mathbb{Z}_2} \simeq \mathrm{hocolim}_i \left[S^i_! \wedge \mathsf{H}(\mathbb{Z}_2, k)^{h\mathbb{Z}_2}\right]. \qquad \square$$

Analogously, we find

Lemma 6.11. *Let $k \geq 0$. The homotopy orbit, fixed point and Tate spectra associated to the Eilenberg-MacLane spectrum $\mathsf{H}(\mathbb{Z}, k)$ with trivial involution are given by*

$$\mathsf{H}(\mathbb{Z}, k)_{h\mathbb{Z}_2} \simeq \mathsf{H}(\mathbb{Z}, k) \vee \prod_{i > 0} \mathsf{H}(\mathbb{Z}_2, k + 2i - 1), \qquad \mathsf{H}(\mathbb{Z}, k)^{th\mathbb{Z}_2} \simeq \prod_{i \in \mathbb{Z}} \mathsf{H}(\mathbb{Z}_2, k + 2i),$$
$$\mathsf{H}(\mathbb{Z}, k)^{h\mathbb{Z}_2} \simeq \mathsf{H}(\mathbb{Z}, k) \vee \prod_{i < 0} \mathsf{H}(\mathbb{Z}_2, k + 2i), \qquad \mathsf{H}(\mathbb{Z}, k)^{rh\mathbb{Z}_2} \simeq \prod_{i < 0} \mathsf{H}(\mathbb{Z}_2, k + 2i).$$

Lemma 6.12. *Let $k \geq 0$. The homotopy orbit and fixed point spectra associated to $\mathsf{H}(\mathbb{Z}^-, k+1) := S^1_! \wedge \mathsf{H}(\mathbb{Z}, k)$ are given by*

$$\mathsf{H}\left(\mathbb{Z}^-, k+1\right)_{h\mathbb{Z}_2} \simeq \prod_{i\geq 0} \mathsf{H}(\mathbb{Z}_2, k+1+2i), \quad \mathsf{H}\left(\mathbb{Z}^-, k+1\right)^{h\mathbb{Z}_2} \simeq \prod_{i\leq 0} \mathsf{H}(\mathbb{Z}_2, k+2i).$$

By Lemma 5.10, $\mathsf{H}(\mathbb{Z}^-, k+1)^{th\mathbb{Z}_2} \simeq \mathsf{H}(\mathbb{Z}, k)^{th\mathbb{Z}_2}$, and the reduced homotopy fixed points are identified with the homotopy fixed points in Lemma 6.11. The proof of Lemma 6.12 employs these facts, Lemma 6.11 and a straight forward analysis of the long lattice on homotopy groups associated to (5.8).
We analyze $\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}$. The calculations above also show that any homotopy groups $\pi_k(\mathsf{Wh}(*;7))$, $k \geq 9$, cannot contribute to $\pi_7\big(\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}\big)$ for dimensional reasons. In other words, the Postnikov truncation induces an isomorphism

$$\pi_7\big(\mathsf{Wh}(*;7)_{h\mathbb{Z}_2}\big) \to \pi_7\big(\mathsf{Wh}(*;7)_{[3,7],h\mathbb{Z}_2}\big).$$

We must obtain a better understanding of the duality on $\mathsf{Wh}(*;7)_{[3,7]}$. According to [Rog02, p. 923], the (non-involutory) Postnikov or k-invariant for π_7 factors as $k^8 = Sq^5 \circ p^5_3 : \mathsf{Wh}(*)_{[3,5]} \to \mathsf{Wh}(*)_{[3]} \simeq \mathsf{H}(\mathbb{Z}_2, 3) \to \mathsf{H}(\mathbb{Z}_2, 8)$. In simple words, π_7 is attached to $\mathsf{Wh}(*)_{[3]}$ independently of π_5. In the following, we show that this is also the case for the duality-respecting Postnikov invariant.
Recall the Hatcher-Waldhausen map ${}^{\mathrm{G}}/_{\mathrm{O}} \to \Omega^{\infty+1}\mathsf{Wh}(*)$ [Wal82, Corollary 3.4]. By [Rog02, Theorem 7.5] it is 8-connected 2-locally. A duality respecting map ${}^{\mathrm{G}}/_{\mathrm{O}} \to \Omega^{\infty}\Omega^1_!\mathsf{Wh}(*)$ is defined in K-theoretic terms in [WW18]. Here ${}^{\mathrm{G}}/_{\mathrm{O}}$ is endowed with the trivial involution. This allows to think of the map as ${}^{\mathrm{G}}/_{\mathrm{O}} \to \Omega^{\infty}\big(\Omega^1_!\mathsf{Wh}(*)\big)^{h\mathbb{Z}_2}$. It is not proven yet, but expected to fit into a commutative diagram of the form

$$\begin{tikzcd} & \Omega^{\infty}\left(\Omega^1_!\mathsf{Wh}(*)\right)^{h\mathbb{Z}_2} \arrow[d, "\text{forget}"] \\ {}^{\mathrm{G}}/_{\mathrm{O}} \arrow[ur] \arrow[r] & \Omega^{\infty+1}\mathsf{Wh}(*). \end{tikzcd} \tag{6.11}$$

We can conclude from (6.11) that the involution on $\Omega^{\infty}\Omega^1_!\mathsf{Wh}(*)_{[0,7]}$ is trivial. Unfortunately, this a priori *does not* allow to conclude that the involution on $\Omega^1_!\mathsf{Wh}(*)_{[0,7]}$ is also trivial. However, if we consider the 3-connected cover of the Hatcher-Waldhausen map, then the stable map

$$\Sigma^{\infty}\left(({}^{\mathrm{G}}/_{\mathrm{O}})_{[4,\infty]}\right) \to \left(\Omega^1_!\mathsf{Wh}(*)\right)_{[4,\infty]}$$

is guaranteed to coincide with the Hatcher-Waldhausen map between dimensions 4 and 6. In particular, the involution on $\mathsf{Wh}(*)_{[5]}$ is given by the one on $S^1_! \wedge \mathsf{H}(\mathbb{Z}, 4)$, as claimed.
As consequence, the involution-respecting Postnikov invariant k^7 between π_6 and π_4 of $(\Omega^1_!\mathsf{Wh}(*))_{[4,\infty]}$ vanishes, since it does on $({}^{\mathrm{G}}/_{\mathrm{O}})_{[4,\infty]}$. Thus also the involutory Postnikov

invariant for π_7 factors as $k^8 : \mathsf{Wh}(*)_{[3,5]} \to \mathsf{Wh}(*)_{[3]} \simeq \mathsf{H}(\mathbb{Z}_2, 3) \to \mathsf{H}(\mathbb{Z}_2, 8)$. This can also be deduced in the case of 7-duality on $\mathsf{Wh}(*)$. In other words, there is a non-standard Postnikov-type duality-respecting projection

$$\mathsf{Wh}(*;7)_{[3,7]} \to \mathsf{Wh}(*;7)_{[3],[7]}. \tag{6.12}$$

This map induces a monomorphism on $\pi_7((-)_{h\mathbb{Z}_2})$. This can be verified using the long exact sequence on homotopy groups induced by (6.12) on $\pi_7((-)_{h\mathbb{Z}_2})$, together with the fact $\pi_7\big(\mathsf{Wh}(*;7)_{[5],h\mathbb{Z}_2}\big) = 0$. The latter is verified using

$$\mathsf{Wh}(*;7)_{[5]} \simeq \Omega^7 S^7_! \wedge \mathsf{Wh}(*)_{[5]} \simeq \Omega^7 S^8_! \wedge \mathsf{H}(\mathbb{Z}, 4).$$

A similar analysis as for Lemma 6.12 then reveals that the homotopy of its homotopy orbits is concentrated in even dimensions. As a next step, consider the Postnikov projection

$$\mathsf{Wh}(*;7)_{[3],[7],h\mathbb{Z}_2} \to \mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}.$$

Note that Lemma 6.10 reveals that 7-duality is the same as 0-duality on $\mathsf{Wh}(*)_{[3]}$ and likewise on $\mathsf{Wh}(*)_{[7]}$. We show that $[\lambda \circ \kappa'] = 0 \in \pi_7\big(\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}\big)$. To this end, consider the Hurewicz homomorphism composed with coefficient change to $\mathbb{Z}_2$,

$$h : \pi_7\left(\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}\right) \to H_7\big(\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}, \mathbb{Z}_2\big).$$

It is injective, because $\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}$ is a generalized Eilenberg-MacLane spectrum made up of $\mathsf{H}\mathbb{Z}_2$-factors only. If $[\lambda \circ \kappa'] \neq 0 \in \pi_7\big(\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}\big)$, then $\lambda_* h(\kappa') = h(\lambda \circ \kappa') \neq 0 \in H_7\big(\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}, \mathbb{Z}_2\big)$. This contradicts Lemma 6.13 below.
We finish the proof first. Since $[\lambda \circ \kappa'] = 0 \in \pi_7\big(\mathsf{Wh}(*)_{[3],h\mathbb{Z}_2}\big)$, the element $\lambda \circ \kappa'$ actually lifts along $\mathsf{Wh}(*)_{[7],h\mathbb{Z}_2} \to \mathsf{Wh}(*;7)_{[3],[7],h\mathbb{Z}_2}$ to a map $\rho : S^{14} \to \Omega^{\infty-7}\mathsf{H}(\mathbb{Z}_2, 7)_{h\mathbb{Z}_2}$. Since $\mathsf{H}(\mathbb{Z}_2, 7)_{h\mathbb{Z}_2}$ is an Eilenberg-MacLane spectrum, the composition $\rho \circ \eta$ automatically vanishes. Consequently, also $[\lambda \circ \kappa' \circ \eta] = 0$, which proves Proposition 6.9. □

Lemma 6.13 (Zare [Zar15], [Zar17a])**.** *Denote by $h : \pi_{14}(\mathrm{TOP}) \to H_{14}(\mathrm{TOP}, \mathbb{Z}_2)$ the Hurewicz homomorphism. The image $h(\kappa')$ of κ' in $H_{14}(\mathrm{TOP}, \mathbb{Z}_2)$ vanishes.*

Proof. This proof is due to Hadi Zare [Zar17a] and simply reproduced here. It builds upon [Zar15, Lemma 4.3], i. e. that the image of κ in $H_{14}(\mathrm{G}, \mathbb{Z}_2)$ vanishes. Similar computations can also be found in [Zar17b]. Consider $\kappa \in \pi^s_{14}$ as an element in $\pi_{14}(Q_0 S^0)$ using $\mathrm{SG} \simeq Q_1 S^0 \simeq Q_0 S^0$. Denote by $\nu \in \pi^s_3$ a generating Hopf invariant one map. By [Tod62], κ can be written as the triple Toda bracket $\langle \beta, \alpha, \nu \rangle$, with

$$S^{13} \xrightarrow{\beta} \Omega^6\Sigma^6(\Sigma^4 K) \xrightarrow{\Omega^6\Sigma^6\alpha} \Omega^6\Sigma^6 S^3 \xrightarrow{\nu} Q_0 S^0,$$

adhering to the notation and definitions in [Zar15]. Suffice it to say that K denotes a $\mathbb{Z}_2$-Moore spectrum. The map $\pi_3(\mathrm{TOP}) \to \pi_3(\mathrm{G})$ is the non-standard epimorphism $\mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_{24}$ [Mil88, Lemma 9]. Therefore ν lifts to $\nu' \in \pi_3(\mathrm{TOP})$ and we can use it to construct an analogous triple Toda bracket for κ' with trivial indeterminacy. It is represented by a composition of the form

$$S^{14} \xrightarrow{\beta^\flat} C_{\Omega^6\Sigma^6\alpha} \xrightarrow{(\nu')^\sharp} \mathrm{TOP}.$$

$\beta^\flat_* = 0$ on homology by the arguments for [Zar15, Lemma 4.3], i. e. $h(\kappa') = 0$. $\square$

We obtain the following proposition as a corollary of Proposition 6.9.

Proposition 6.14. *The morphism* $\partial : \pi_{15}\left(T_1\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)\right) \to \pi_{14}\left(T_1\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)\right)$ *is injective.*

We conclude with the

Proof of Theorem 2.5. If $\phi \circ \xi \circ \sigma \neq 0 \in \pi_{15}\left(T_1({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)})\right)$, then by Proposition 6.14, $\partial(\phi \circ \xi \circ \sigma) \neq 0 \in \pi_{14}\left(T_1\left({}^{\mathrm{TOP}(7)}/_{\mathrm{O}(7)}\right)\right)$. By Corollary 6.5, $\phi_*\mathrm{Obs}(s, \Sigma) \neq 0$, meaning already $\mathrm{Obs}(s, \Sigma) \neq 0$. The theorem follows using Obstruction 4.14. $\square$

Remark 6.15 (Generalization for $n > 7$)**.** In general one cannot say that the boundary homomorphism $\partial : \pi_{2n+1}\left(T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)\right) \to \pi_{2n}\left(T_1\left({}^{\mathrm{TOP}(n)}/_{\mathrm{O}(n)}\right)\right)$ is injective. However, it is likely that one can show that the elements of $\pi_{2n+1}\left({}^{\mathrm{G}}/_{\mathrm{O}}\right)$ do not intervene with $\phi \circ \xi \circ \sigma \in \pi_{2n+1}\left(T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)\right)$. This will probably require an argument adapted to the exact type of approximating model used for $T_1\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$.

7. An attempted calculation using bo

Recall the homotopy operation HO developed in Section 4 and the approximating model $\Omega^\infty \mathsf{P}_{\mathsf{A}(*)}$ for the space ${}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}$ developed in Section 5. The main object in this section is to provide a further simplification $\Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right) \to \Omega^\infty \mathsf{P}_{\mathsf{A}(*)} \to \Omega^\infty \mathsf{P}_{\mathsf{bo}} \to \Omega^\infty \mathsf{F}$, where F is a simplification of P_{bo}. This was supposed to be used to show that applying the stable homotopy operation to the distinguished generator $\phi \circ \xi \in \pi_0(\mathsf{F})$ yields a non-vanishing element $\phi \circ \xi \circ \sigma \in \pi_7(\mathsf{F})$. Unfortunately, this is not possible, as pointed out in Problem 7.24. We deal with the case of general n as far as possible.
The reader is assumed to be familiar with the contents of Section 1.6. In Section 7.1 we develop the simplification P_{bo} of the approximating model as Proposition 7.6 using the map $\mathsf{A}(*) \to \mathsf{K}(\mathbb{Z}) \to \mathsf{bo}$. This is approximated further to a certain homotopy fiber F in Section 7.2. The value of $\phi \circ \xi \circ \sigma \in \pi_7(\mathsf{F})$ is then computed in Sections 7.3 and 7.4. The discussion in Section 7.5 concisely explores the reasons why this approximation does not work and how it might be amended.

7.1. The approximation using topological K-theory

Lemma 7.1. *There is a natural induced map of spectra* $\mathsf{A}(*) \to \mathsf{K}(\mathbb{Z}) \to \mathsf{bo}$ *respecting the standard involutions. This holds analogously for* $\mathsf{A}(*; n+1)$.

On bo, $4k$-duality is trivial, but on $\mathsf{K}(\mathbb{Z})$ it is not.

Proof. We think of the K-theory spectrum $\mathsf{A}(*) \simeq \mathsf{K}(\mathsf{S}) = \mathrm{BGL}(\mathsf{S})^+ \times \mathbb{Z}$ via the Quillen $+$-construction. Upon mapping S to its ring of components $\mathbb{Z}$, we obtain an induced map $\mathrm{BGL}(\mathsf{S})^+ \times \mathbb{Z} \to \mathrm{BGL}(\mathbb{Z})^+ \times \mathbb{Z}$, which yields $\mathsf{A}(*) \to \mathsf{K}(\mathbb{Z})$. It respects the duality by [WW98]. By [WW00, Proof of 7.6], the standard induced map $\mathsf{K}(\mathbb{Z}) \to \mathsf{K}(\mathbb{R}^{\mathrm{top}}) = \mathsf{bo}$ can be viewed as a map of spectra with involution. The action of $\mathbb{Z}_2$ on bo is the trivial one. $\square$

Lemma 7.2 ([WW00])**.** *In the linear case* $\mathsf{K}(\mathbb{Z})$*,* 4*-duality is the same as* 0*-duality. In other words,* $\mathsf{K}(\mathbb{Z}; 0) \simeq_{\mathbb{Z}_2} \mathsf{K}(\mathbb{Z}; 4)$*. There is a similar formula for quadratic* L*-theory,* $\mathsf{L}_\bullet(\mathbb{Z}; 0) \simeq \mathsf{L}_\bullet(\mathbb{Z}; 4)$*.*

The same is then also true for bo. The periodicity of the L-theory spectrum in the linear case is shown in [WW00], as well as the periodicity of $\mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2}$ and Ξ. This can be shown by providing an automorphism of the category of chain complexes over $\mathbb{Z}$ with duality, with a chain complex sent to its 2-fold suspension. This corresponds to suspending the tensor products four times. Using Waldhausen K-theory, this technique also provides the argument for $\mathsf{K}(\mathbb{Z})$.

Lemma 7.3 ([WW00, Corollary 7.7 and 7.10])**.** *The composition* $\Xi : \mathsf{L}^\bullet(\mathbb{Z}) \to \mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2} \to \mathsf{bo}^{th\mathbb{Z}_2}$ *induces the standard inclusion* $\mathbb{Z} \to \mathbb{Z}_2^\wedge$ *on* π_{4k}*.*

Proof sketch. [WW00, Corollary 7.10] states that the spectrum $\mathsf{K}(R)^{th\mathbb{Z}_2}$ is a generalized Eilenberg-MacLane spectrum for every discrete ring R with involution. It is also 4-periodic, $\mathsf{K}(R)^{th\mathbb{Z}_2} \simeq \Omega^4 \mathsf{K}(R)^{th\mathbb{Z}_2}$. The 4-periodicity of $\mathsf{bo}^{th\mathbb{Z}_2}$ mentioned in Lemma 1.77 also applies to the map $\pi_{4k}\left(\mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2}\right) \to \pi_{4k}(\mathsf{bo}^{th\mathbb{Z}_2})$ by the arguments in [WW00, 7.5 and 7.6]. Analogous to [WW00, Proposition 7.6], $1 \in \pi_{4k}\left(\mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2}\right)$ is sent to an infinite cyclic element in $\pi_{4k}(\mathsf{bo}^{th\mathbb{Z}_2})$. By [WW00, Corollary 7.7 and 7.10], the homomorphism $\Xi_* : \pi_{4k}(\mathsf{L}^\bullet(\mathbb{Z})) \to \pi_{4k}\left(\mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2}\right)$ is injective for any $k \in \mathbb{Z}$. Since Ξ is a map of ring spectra, it follows $1 \in \pi_0(\mathsf{L}^\bullet(\mathbb{Z})) \to 1 \in \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$, i. e. the map is simply the inclusion of $\mathbb{Z}$ in $\mathbb{Z}_2^\wedge$. Due to periodicity, the assertion also follows for π_{4k}. Compare also with the statements in [WW00, Proof of Calculation 8.3]. □

Notation 7.4. Write Ξ_{bo} for the composition

$$\Xi_{\mathsf{bo}} : \mathsf{L}_\bullet(\mathbb{Z}) \xrightarrow{\text{sym}} \mathsf{L}^\bullet(\mathbb{Z}) \xrightarrow{\Xi} \mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2} \to \mathsf{bo}^{th\mathbb{Z}_2}.$$

Corollary 7.5. *The map* Ξ_{bo} *induces multiplication by eight* $\mathbb{Z} \to \mathbb{Z}_2^\wedge$ *on* π_{4k}.

Proposition 7.6. *Let* $n \equiv 3 \mod 4$. *There exists a map* $\Omega^{n+1}\left(\mathrm{G}(n)/\mathrm{TOP}(n)\right) \to \Omega^\infty \mathsf{P}$ *from* $\Omega^{n+1}\left(\mathrm{G}(n)/\mathrm{TOP}(n)\right)$ *to* Ω^∞ *of the homotopy pullback* P *of*

$$\mathsf{L}_\bullet(\mathbb{Z}) \xrightarrow{\Xi_{\mathsf{bo}}} \mathsf{bo}^{th\mathbb{Z}_2} \longleftarrow \mathsf{bo}^{rh\mathbb{Z}_2}. \tag{7.1}$$

The right-hand map is the norm cofiber induced map. The composition to L-theory, $\Omega^{n+1}\left(\mathrm{G}(n)/\mathrm{TOP}(n)\right) \to \Omega^\infty \mathsf{P} \to \Omega^\infty \mathsf{L}_\bullet(\mathbb{Z})$ *is precisely the stabilization to* $\Omega^{n+1}\left(\mathrm{G}/\mathrm{TOP}\right)$ *followed by the appropriate periodicity equivalence on L-theory from Lemma 7.2. As overview,*

$$\begin{tikzcd}
\Omega^{n+1}\left(\mathrm{G}(n)/\mathrm{TOP}(n)\right) \arrow[dr, dashed] \arrow[drr] \arrow[ddr, "\text{stab.}"'] & & \\
& \Omega^\infty \mathsf{P} \arrow[r] \arrow[d] & \Omega^\infty \mathsf{bo}^{rh\mathbb{Z}_2} \arrow[d] \\
& \Omega^\infty \mathsf{L}_\bullet(\mathbb{Z}) \arrow[r, "\Xi_{\mathsf{bo}}"] & \Omega^\infty \mathsf{bo}^{th\mathbb{Z}_2}.
\end{tikzcd} \tag{7.2}$$

Proof. We may approximate (5.5) to the right using $\mathsf{A}(*) \to \mathsf{K}(\mathbb{Z})$ from Lemma 7.1. Use Lemma 7.2 to switch from $(n+1)$- to 0-duality for $\mathsf{L}_\bullet(\mathbb{Z})$, $\mathsf{K}(\mathbb{Z})$ and Ξ. We exchange Ξ defined with $(n+1)$-duality by Ξ defined with 0-duality for similar reasons as employed in the proof of Proposition 5.9. Most notably, because Ξ respects equivalences preserving the duality notion. We may substitute $\Xi \circ \text{sym}$ defined on $\mathsf{L}_\bullet(\mathsf{S})$ by Ξ directly defined for the ring $\mathbb{Z}$ using the following diagram induced by restriction to the component ring,

$$\begin{tikzcd}
\mathsf{L}_\bullet(\mathbb{Z}) \arrow[r, "\Xi\circ\text{sym}"] \arrow[d, "\simeq"] & \mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2} \\
\mathsf{L}_\bullet(\mathsf{S}) \arrow[r, "\Xi\circ\text{sym}"] & \mathsf{A}(*)^{th\mathbb{Z}_2}. \arrow[u]
\end{tikzcd}$$

We approximate further using $\mathsf{K}(\mathbb{Z}) \to \mathsf{bo}$ from Lemma 7.1. □

Lemma 7.7. *Let $n \equiv 3 \mod 4$. The map to the homotopy fixed points induces $1 \in \pi_0\left(\Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)\right) \mapsto 4 \in \mathbb{Z}_2^{\wedge} \cong \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right)$ up to sign.*

A word of caution is necessary. The inclusion $1 \mapsto 4$ is perhaps only correct up to a sign. This depends on the exact sign convention used in Theorem 1.80 and Lemma A.6. Since the sign is of no importance whatsoever in our calculation, we prefer not to get distracted by the tedious tracking of this sign, at the expense of a technical inconsistency. This applies also in several situations in the sequel, where certain isomorphisms may actually switch signs.

Proof. According to Corollary 7.5, Ξ_{bo} sends the generator $1 \in \pi_{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right) \cong \pi_0\left(\mathsf{L}_\bullet\left(\mathbb{Z}\right)\right)$ to $8 \in \mathbb{Z}_2^{\wedge} \cong \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$. By Remark 1.81, the map $\pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \to \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ is multiplication by 2 on $\mathbb{Z}_2^{\wedge}$. Therefore, the element $4 \in \mathbb{Z}_2^{\wedge}$ is the unique preimage of $8 \in \mathbb{Z}_2^{\wedge} \cong \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$. $\square$

We use the descriptions above to compute two relevant homotopy groups of the homotopy pullback P as follows.

Lemma 7.8. *Let $n \equiv 3 \mod 4$. Then $\pi_0\left(\mathsf{P}\right) \cong \mathbb{Z}$ and the approximating map $\Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right) \to \Omega^\infty \mathsf{P}$ is the identity $\mathbb{Z} \to \mathbb{Z}$ on π_0. Also, $\pi_n\left(\mathsf{P}\right) \cong \mathbb{Z}_8$.*

Proof. (i) We apply Lemma A.6 to the homotopy pullback diagram in (7.2),

$$\cdots \to \pi_1\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \to \pi_0\left(\mathsf{P}\right) \to \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_0\left(\mathsf{L}_\bullet\left(\mathbb{Z}\right)\right) \to \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \to \dots \tag{7.3}$$

By Lemma 1.77, $\pi_1\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) = 0$. Consequently, $\pi_0\left(\mathsf{P}\right)$ is simply isomorphic to the kernel of the latter map in (7.3). Recall $\pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \cong \mathbb{Z}_2^{\wedge}$. Theorem 1.80, Corollary 7.5 and Lemma 7.7 identify the latter map in (7.3) as $(a, c) \mapsto -2a + 8c$ on

$$\mathbb{Z}_2^{\wedge} \oplus \mathbb{Z} \cong \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_0\left(\mathsf{L}_\bullet\left(\mathbb{Z}\right)\right) \longrightarrow \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \cong \mathbb{Z}_2^{\wedge}.$$

The kernel is then given by the $\mathbb{Z}$-span of $(4, 1)$. In other words, we have $\pi_0\left(\mathsf{P}\right) \cong \mathbb{Z}$ with $1 \in \pi_0\left(\mathsf{P}\right) \mapsto 4 \in \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right)$ and $1 \in \pi_0\left(\mathsf{P}\right) \mapsto 1 \in \pi_0\left(\mathsf{L}_\bullet\left(\mathbb{Z}\right)\right)$.
(ii) We proceed completely analogously. Since $n \geq 7$, we may show that the image of $\pi_{n+1}\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_{n+1}\left(\mathsf{L}_\bullet\left(\mathbb{Z}\right)\right) \to \pi_{n+1}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ is given by $8\mathbb{Z}_2^{\wedge}$. One may also identify $\pi_n\left(\mathsf{P}\right)$ as the cofiber of this map, which is $\mathbb{Z}_8$ as claimed. $\square$

7.2. The simplification as homotopy fiber in dimension seven

From now on forth, we specialize completely to the setting of Conjecture 2.6. In other words, we fix dimension $n = 7$ and more importantly $\delta = \mathrm{id}_{S^n}$. Also, let $\Sigma_{\mathrm{KM}} \in \mathfrak{S}\left(S^7\right)$ be the pseudo smooth Kervaire Milnor structure lifting to the distinguished generator $1 \in L_8\left(\mathbb{Z}\right)$, compare Lemmata 1.40 and 1.42.
In this section we construct a further approximation $\mathsf{P} \to \mathsf{F}$ for $\Omega^8\left({}^{\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right)$. The main benefit of F is that it admits a map from an F', which in turn has a very simple description $\pi_0\left(\mathsf{F}'\right) \cong \pi_0\left(\mathsf{bo}_2^{\wedge}\right)$ and a canonical identification $\pi_7\left(\mathsf{F}'\right) \cong {}^{\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^{\wedge}, 8\right)\right)}/_{\pi_8\left(\mathsf{bo}_2^{\wedge}\right)}$.

This will allow hands-on computations of the stable homotopy operation HO in Sections 7.3 and 7.4. The main findings of this section are summarized in Proposition 7.14. We proceed by modifying the homotopy pullback diagram step by step to a simplified formulation.

Remark 7.9. We norm $\mathsf{H}(\mathbb{Q}_2^\wedge, 8)$ via an isomorphism $\pi_8(\mathsf{H}(\mathbb{Q}_2^\wedge, 8)) \cong \mathbb{Q}_2^\wedge$. Consider $\pi_8(\mathsf{H}(\mathbb{Z}, 8)) \to \pi_8(\mathsf{H}(\mathbb{Q}_2^\wedge, 8))$. For a choice of generator $1 \in \pi_8(\mathsf{H}(\mathbb{Z}, 8))$, we take its image $1 \in \pi_8(\mathsf{H}(\mathbb{Q}_2^\wedge, 8))$ as a distinguished generator.

Lemma 7.10. *Consider the diagram*

$$\begin{tikzcd}
\mathsf{P} \arrow[r] \arrow[d] & \mathsf{bo}^{rh\mathbb{Z}_2} \arrow[d] \arrow[dr, dashed] & \\
\mathsf{L}_\bullet(\mathbb{Z}) \arrow[r, "\Xi_{\mathsf{bo}}"] & \mathsf{bo}^{th\mathbb{Z}_2} \arrow[r, "\mathrm{incl} \circ \mathrm{proj}"] & \mathsf{H}(\mathbb{Q}_2^\wedge, 8).
\end{tikzcd}$$

The lower right-hand horizontal map denotes the projection onto the wedge summand $\mathsf{H}(\mathbb{Z}_2^\wedge, 8)$ *followed by the inclusion to* $\mathsf{H}(\mathbb{Q}_2^\wedge, 8)$. *Then we can consider the new homotopy pullback* P' *of* $\mathsf{L}_\bullet(\mathbb{Z}) \to \mathsf{H}(\mathbb{Q}_2^\wedge, 8) \leftarrow \mathsf{bo}^{rh\mathbb{Z}_2}$ *and obtain the inclusion* $\pi_7(\mathsf{P}) \hookrightarrow \pi_7(\mathsf{P}')$ *via* $\mathbb{Z}_8 \hookrightarrow {}^{\mathbb{Q}}/_{8\mathbb{Z}}$ *on* π_7 *as well as the inclusion* $1 \mapsto (4,1)$ *under*

$$\mathbb{Z} \cong \pi_0(\mathsf{P}) \hookrightarrow \mathbb{Z}_2^\wedge \oplus \mathbb{Z} \cong \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_0(\mathsf{L}_\bullet(\mathbb{Z})) \cong \pi_0(\mathsf{P}').$$

Proof. The diagonal map is of course the two arrow composition. By Lemma A.7,

$$\begin{tikzcd}
0 \arrow[r] & \pi_0(\mathsf{P}) \arrow[r] \arrow[d] & \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_0(\mathsf{L}_\bullet(\mathbb{Z})) \arrow[r] \arrow[d, "\mathrm{id}"] & \pi_0\left(\mathsf{bo}^{th\mathbb{Z}_2}\right) \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & \pi_0(\mathsf{P}') \arrow[r] & \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_0(\mathsf{L}_\bullet(\mathbb{Z})) \arrow[r] & \pi_0(\mathsf{H}(\mathbb{Q}_2^\wedge, 8)) \arrow[r] & \dots
\end{tikzcd}$$

Again, $\pi_0(\mathsf{P})$ and likewise $\pi_0(\mathsf{P}')$ can be seen as the kernels of the right-hand maps. Also, $\pi_0(\mathsf{H}(\mathbb{Q}_2^\wedge, 8)) = 0$, hence we get an isomorphism $\pi_0(\mathsf{P}') \to \mathbb{Z}_2^\wedge \oplus \mathbb{Z}$. Consequently, $\pi_0(\mathsf{P}) \hookrightarrow \pi_0(\mathsf{P}')$ is the inclusion of the kernel $\pi_0(\mathsf{P})$ in $\pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_0(\mathsf{L}_\bullet(\mathbb{Z}))$.
For π_7, use Lemma A.7 analogously. We particularly pay attention to the norming of $\mathbb{Q}_2^\wedge$ via Remark 7.9. Again, the image of $\pi_8\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \oplus \pi_8(\mathsf{L}_\bullet(\mathbb{Z}))$ in both rows is given by $8\mathbb{Z}_2^\wedge$. Analogously, $\pi_7(\mathsf{P}') \cong {}^{\mathbb{Q}}/_{8\mathbb{Z}}$ and $\pi_7(\mathsf{P}) \hookrightarrow \pi_7(\mathsf{P}')$ is as desired. ☐

Lemma 7.11. *Define the homotopy fiber*

$$\mathsf{F} := \mathrm{hofiber}\left[\rho : \mathsf{bo}^{rh\mathbb{Z}_2} \to \mathsf{bo}^{th\mathbb{Z}_2} \to \mathsf{H}(\mathbb{Q}_2^\wedge, 8) \to \mathsf{H}({}^{\mathbb{Q}}/_{8\mathbb{Z}}, 8)\right].$$

Then there is a natural map $\mathsf{P}' \to \mathsf{F}$ *inducing an isomorphism on* π_7 *and an epimorphism* $\pi_0(\mathsf{P}') \to \pi_0(\mathsf{F})$ *given by the projection* $\pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \times \pi_0(\mathsf{L}_\bullet(\mathbb{Z})) \to \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right)$.

The proof is analogous to the proof of Lemma 7.10 by using

$$\begin{tikzcd}
\mathsf{P}' \arrow[r] \arrow[d] & \mathsf{L}_\bullet(\mathbb{Z}) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathsf{bo}^{rh\mathbb{Z}_2} \arrow[r] & \mathsf{H}(\mathbb{Q}_2^\wedge, 8) \arrow[r] & \mathsf{H}({}^{\mathbb{Q}}/_{8\mathbb{Z}}, 8).
\end{tikzcd}$$

Notice that the map ρ factors via $\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$ by construction. We use this fact to enlarge $\pi_7\left(\mathsf{F}\right)$. Due to the norming of $\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$ in Remark 7.9, the generator $1 \in \pi_8\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ is sent precisely to the chosen generator in $\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)$. Recall the weak equivalence $\mathsf{bo}_2^\wedge \xrightarrow{\sim} \mathsf{bo}^{rh\mathbb{Z}_2}$ from Corollary 1.62. Denote the overall composition $\mathsf{bo}_2^\wedge \to \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$ by β.

Remark 7.12. The map $\beta : \mathsf{bo}_2^\wedge \xrightarrow{\sim} \mathsf{bo}^{rh\mathbb{Z}_2} \to \mathsf{bo}^{th\mathbb{Z}_2} \to \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$ is given on π_8 by $\mathbb{Z}_2^\wedge \to \mathbb{Q}_2^\wedge$, $1 \mapsto 32$. Write $\mathsf{F}' := \operatorname{hofiber} \beta$ for the homotopy fiber of β.

Lemma 7.13. *There is a natural map* $\mathsf{F}' \to \mathsf{F}$ *which induces the projection map* $\mathbb{Q}/_{32\mathbb{Z}} \to \mathbb{Q}/_{8\mathbb{Z}}$ *on* π_7 *and the identity on* π_0. *While* $\pi_8\left(\mathsf{F}\right) \cong \pi_8\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right)$, *we have* $\pi_8\left(\mathsf{F}'\right) = 0$ *and thus canonically* $\pi_7\left(\mathsf{F}'\right) \cong {\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge,8\right)\right)}/_{\pi_8\left(\mathsf{bo}_2^\wedge\right)}$.

In the statement above we implicitly identify $\pi_8\left(\mathsf{bo}_2^\wedge\right)$ with its image in $\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)$. Again, the proof is simple using the following diagram,

$$\begin{tikzcd}
\mathsf{F}' \arrow[r] \arrow[d] & \mathsf{bo}_2^\wedge \arrow[r, "\sim"] \arrow[d, "\beta"] & \mathsf{bo}^{rh\mathbb{Z}_2} \arrow[d, "\rho"] \\
* \arrow[r] & \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right) \arrow[r, "\mathrm{proj}"] & \mathsf{H}\left(\mathbb{Q}/_{8\mathbb{Z}}, 8\right).
\end{tikzcd}$$

The following summary is now straight forward.

Proposition 7.14. *Recall the definitions of* β *and* F' *via Remark 7.12. There is a spectrum* F *together with natural maps* $\mathsf{P} \to \mathsf{F} \leftarrow \mathsf{F}'$ *inducing*

$$\begin{tikzcd}
\pi_7\left(\mathsf{P}\right) \arrow[r] \arrow[d, equal] & \pi_7\left(\mathsf{F}\right) \arrow[d, equal] & \pi_7\left(\mathsf{F}'\right) \arrow[l] \arrow[d, equal] \\
\mathbb{Z}_8 \arrow[r, hook] & \mathbb{Q}/_{8\mathbb{Z}} & \mathbb{Q}/_{32\mathbb{Z}} \arrow[l, two heads]
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
\pi_0\left(\mathsf{P}\right) \arrow[r] \arrow[d, equal] & \pi_0\left(\mathsf{F}\right) \arrow[d, equal] & \pi_0\left(\mathsf{F}'\right) \arrow[l, "\cong"'] \arrow[d, equal] \\
\mathbb{Z} \arrow[r, "\cdot 4"] & \mathbb{Z}_2^\wedge & \mathbb{Z}_2^\wedge. \arrow[l, leftrightarrow, "\mathrm{id}"']
\end{tikzcd}$$

We note in particular that $\pi_0\left(\mathsf{F}'\right) \cong \pi_0\left(\mathsf{bo}_2^\wedge\right)$ *canonically and*

$$\pi_7\left(\mathsf{F}'\right) \cong {\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge,8\right)\right)}/_{\pi_8\left(\mathsf{bo}_2^\wedge\right)} \tag{7.4}$$

are naturally identified. The distinguished generator is mapped by

$$1 \in \pi_8\left({\mathrm{G}(7)}/_{\mathrm{TOP}(7)}\right) \cong \mathbb{Z} \mapsto 4 \in \pi_0\left(\mathsf{F}\right) \cong \pi_0\left(\mathsf{bo}_2^\wedge\right) \cong \mathbb{Z}_2^\wedge.$$

7.3. A functional cohomology operation for $\mathsf{bo}_2^\wedge$

Let C_σ denote the cofiber of the stable map $\sigma \in \pi_7^s$. In a first step, this section is about establishing a map $H_8(\mathsf{C}_\sigma) \to {H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}/_{\pi_8(\mathsf{bo}_2^\wedge)}$. The latter quotient comes with an epimorphism to $\pi_7\left(\mathsf{F}'\right) \cong {\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge,8\right)\right)}/_{\pi_8(\mathsf{bo}_2^\wedge)}$. Let $\alpha : \mathsf{S} \to \mathsf{bo}_2^\wedge$ represent an element in $\pi_0\left(\mathsf{bo}_2^\wedge\right)$. By Proposition 7.14, this α lifts to a unique $[\overline{\alpha}] \in \pi_0\left(\mathsf{F}'\right)$, i. e. $\alpha \simeq \iota \circ \overline{\alpha}$. The image of $1 \in H_8(\mathsf{C}_\sigma)$ is shown to correspond to $\overline{\alpha} \circ \sigma$ in Proposition 7.15. In other words, Proposition 7.15 relates the homotopy operation to the algebraic properties of σ. Later in Section 7.4, we will calculate the order of the image of $1 \in H_8(\mathsf{C}_\sigma)$ in ${}^{H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}\big/_{\pi_8(\mathsf{bo}_2^\wedge)}$, using in particular the Hopf invariant of σ. The order is closely related to the order of the corresponding element in ${}^{\pi_8(\mathsf{H}(\mathbb{Q}_2^\wedge,8))}\big/_{\pi_8(\mathsf{bo}_2^\wedge)}$, and thus provides details on the value of $\overline{\alpha} \circ \sigma$.

The general setup in this section is similar to, albeit more general in nature than the one employed for functional cohomology operations as seen in [MT68, Chapter 16]. Recall the stable Hopf map $\sigma \in \pi_7^s$. We think of it here as a map of spectra $\mathsf{S}^7 \to \mathsf{S}$, where S^7 is short for $S^7 \wedge \mathsf{S}$. Consider its homotopy cofiber $\mathsf{C}_\sigma := \mathsf{S} \cup_\sigma e^8$, where e^8 denotes a stable 8-cell. Since $\pi_7(\mathsf{bo}_2^\wedge) = 0$, the composition $\alpha \circ \sigma$ is nullhomotopic. A choice of nullhomotopy N allows to extend α over C_σ,

$$\begin{tikzcd} \mathsf{S} \arrow[r, "\alpha"] \arrow[d] & \mathsf{bo}_2^\wedge \\ \mathsf{C}_\sigma \arrow[ur, dashed, "\alpha'"] & \end{tikzcd} \tag{7.5}$$

Recall the Hurewicz homomorphism $\pi_8(\mathsf{bo}_2^\wedge) \to H_8(\mathsf{bo}_2^\wedge, \mathbb{Z}) \to H_8(\mathsf{bo}_2^\wedge, \mathbb{Q})$. We may consider $\pi_8(\mathsf{bo}_2^\wedge)$ as a subgroup of $H_8(\mathsf{bo}_2^\wedge, \mathbb{Q})$ via this map. Recall the map $\beta : \mathsf{bo}_2^\wedge \to \mathsf{H}(\mathbb{Q}_2^\wedge, 8)$ as defined in Proposition 7.14.

Proposition 7.15. *The map*

$$[\alpha'_*] = \mathrm{proj} \circ \alpha'_* : H_8(\mathsf{C}_\sigma) \longrightarrow H_8(\mathsf{bo}_2^\wedge, \mathbb{Q}) \longrightarrow {}^{H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}\big/_{\pi_8(\mathsf{bo}_2^\wedge)} \tag{7.6}$$

induced by α'_* is independent of the choice of nullhomotopy $N : \alpha \circ \sigma \simeq \mathrm{const}$ and the induced choice of map α'. Moreover, there is an epimorphism using (7.4),

$$\beta_{*,\#} : {}^{H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}\big/_{\pi_8(\mathsf{bo}_2^\wedge)} \longrightarrow \pi_7(\mathsf{F}') \tag{7.7}$$

under which the image of $1 \in H_8(\mathsf{C}_\sigma)$ corresponds to $\overline{\alpha} \circ \sigma$, i. e.

$$\beta_{*,\#} \circ [\alpha'_*] : H_8(\mathsf{C}_\sigma) \longrightarrow \pi_7(\mathsf{F}'), \ \textit{satisfies} \ 1 \longmapsto \overline{\alpha} \circ \sigma. \tag{7.8}$$

Proof. Part I: The map $[\alpha'_*]$ (7.6) is well-defined.

The definition of α' is ambiguous and non-canonical. Since $\pi_7(\mathsf{bo}_2^\wedge) = 0$, the composition $\alpha \circ \sigma$ is nullhomotopic. Let $N : \alpha \circ \sigma \simeq \mathrm{const}$ be a choice of nullhomotopy inducing a map α'_N. Any two nullhomotopies N, N' are represented by maps $e^8 \to \mathsf{bo}_2^\wedge$ that coincide on ∂e^8 as $\alpha \circ \sigma$. We glue these to a stable map $\Delta_{N,N'} : S^8 \wedge \mathsf{S} \to \mathsf{bo}_2^\wedge$. It can correspond to any element in $\pi_8(\mathsf{bo}_2^\wedge)$. Consider the generator $1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z})$. It is represented by the simplex $e^8 \subset \mathsf{C}_\sigma$, i. e. by the stable 8-cell attached to the sphere spectrum S. This simplex is sent to $N = \alpha'|_{e^8} : e^8 \to \mathsf{bo}_2^\wedge$ by construction of α'.

$$\begin{tikzcd} 1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z}) \arrow[r, "\alpha'_*"] & N \in H_8(\mathsf{bo}_2^\wedge, \mathbb{Z}) \\ & \pi_8(\mathsf{bo}_2^\wedge) \arrow[u, "\text{Hurewicz}"'] \end{tikzcd} \tag{7.9}$$

The group $\pi_8\left(\mathsf{bo}_2^\wedge\right)$ maps to $H_8(\mathsf{bo}_2^\wedge, \mathbb{Z})$ by the Hurewicz homomorphism. In this way, any indeterminacy $\Delta_{N,N'}$ (if oriented correctly) acts on the cycle N by sending it to N'. As a result, $[\alpha'_*]$ is well-defined.

Part II: The epimorphism (7.7) $H_8(\mathsf{bo}_2^\wedge,\mathbb{Q}) \big/ \pi_8(\mathsf{bo}_2^\wedge) \to \pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right) \big/ \pi_8(\mathsf{bo}_2^\wedge) \cong \pi_7\left(\mathsf{F}'\right)$.
Consider the diagram induced by β and the Hurewicz homomorphisms,

$$\begin{tikzcd} \pi_8\left(\mathsf{bo}_2^\wedge\right) \arrow[r] \arrow[d, "\beta_\#"] & H_8(\mathsf{bo}_2^\wedge, \mathbb{Q}) \arrow[d, "\beta_*"] \\ \pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right) \arrow[r] & H_8(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right), \mathbb{Q}). \end{tikzcd} \tag{7.10}$$

The lower map is the first non-zero Hurewicz isomorphism. This identification is natural and respects our norming convention in Remark 7.9. Consequently, we may view β_* as a map $H_8(\mathsf{bo}_2^\wedge, \mathbb{Q}) \to \pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)$. Since (7.10) commutes, the image of $\pi_8\left(\mathsf{bo}_2^\wedge\right)$ in $H_8(\mathsf{bo}_2^\wedge, \mathbb{Q})$ is sent via β_* to the image of $\pi_8\left(\mathsf{bo}_2^\wedge\right)$ in $\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)$ under $\beta_\#$. Since $\beta_\#$ is non-zero, so is β_*. This turns it into a linear surjective $\mathbb{Q}$-vector space map. It is then not hard to deduce that the induced (7.7) is also an epimorphism.

Part IIIa: A reformulation for $\beta_{*,\#} \circ [\alpha'_*]$.
We construct an auxiliary map $\gamma : \pi_7^s \times \pi_0\left(\mathsf{F}'\right) \to \pi_7\left(\mathsf{F}'\right)$ that will link the image of $1 \in H_8(\mathsf{C}_\sigma)$ with $\overline{\alpha} \circ \sigma$. In other words, we rewrite HO in a convenient manner. Set up the problem as follows.

$$\begin{tikzcd} \mathsf{S}^7 \arrow[d, "\sigma"] \arrow[dr, "\alpha\circ\sigma"] & & \\ \mathsf{S} \arrow[r, "\alpha"] & \mathsf{bo}_2^\wedge \arrow[r, "\beta"] & \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right). \end{tikzcd}$$

With $\beta : \mathsf{bo}_2^\wedge \to \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$ and $\iota : \mathsf{F}' \to \mathsf{bo}_2^\wedge$ we have $N_0 : \beta \circ \iota \simeq \text{const}$ by a fixed nullhomotopy. Using $\iota \circ \overline{\alpha} \simeq \alpha$ we obtain a canonical choice of homotopy $N_0 \circ \overline{\alpha} \circ \sigma : \beta \circ \alpha \circ \sigma \simeq \text{const}$. We can think of the two choices $N_0 \circ \overline{\alpha} \circ \sigma$, $\beta \circ N : \beta \circ \alpha \circ \sigma \simeq \text{const}$ as maps $e^8 \to \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$ and glue them to obtain a map $\gamma(N, \overline{\alpha}, \sigma) : \mathsf{S}^8 \to \mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$, where S^8 is short for $S^8 \wedge \mathsf{S}$.
Therefore, the choice of nullhomotopy N yields an element in $\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right) \cong \mathbb{Q}_2^\wedge$. Any other choice N' yields an indeterminacy $\Delta_{N,N'} : \mathsf{S}^8 \to \mathsf{bo}_2^\wedge$ as seen in Part I, which can be viewed via $\beta \circ \Delta_{N,N'}$ inside $\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)$. Thus we obtain a well-defined map

$$(\sigma, \overline{\alpha}) \in \pi_7^s \times \pi_0\left(\mathsf{F}'\right) \mapsto \gamma_{\overline{\alpha},\sigma} = [\gamma(N, \overline{\alpha}, \sigma)] \in \pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right) \big/ \pi_8(\mathsf{bo}_2^\wedge) \cong \pi_7\left(\mathsf{F}'\right),$$

independent of the choice of nullhomotopy N for $\alpha \circ \sigma \simeq \text{const}$.

Part IIIb: Identifying $\gamma_{\overline{\alpha},\sigma}$ with the image of $1 \in H_8(\mathsf{C}_\sigma)$ under $\beta_{*,\#} \circ [\alpha'_*]$
Use (7.9) and extend it to the right by (7.10) in the $\mathbb{Z}$-coefficient homology,

$$\begin{tikzcd}
1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z}) \arrow[r, mapsto, "\alpha'_*"] & N \in H_8(\mathsf{bo}_2^\wedge, \mathbb{Z}) \arrow[r, mapsto, "\beta_*"] & \beta \circ N \in H_8(\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8), \mathbb{Z}) \\
& \pi_8\left(\mathsf{bo}_2^\wedge\right) \arrow[u, "\text{Hurewicz}"'] \arrow[r] & \pi_8\left(\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8)\right). \arrow[u, "\cong", "\text{Hurewicz}"']
\end{tikzcd}$$

The Hurewicz isomorphism $\pi_8\left(\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8)\right) \to H_8(\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8), \mathbb{Z})$ sends the previously constructed element $\gamma(N, \overline{\alpha}, \sigma)$ to $\beta \circ N + N_0 \circ \overline{\alpha} \circ \sigma$. Since $N_0 \circ \overline{\alpha} \circ \sigma$ is a boundary, this image is the same as $\beta \circ N$. Via either way in the square, the element $\Delta_{N,N'} \in \pi_8\left(\mathsf{bo}_2^\wedge\right)$ exchanges $\beta \circ N$ by $\beta \circ N'$.

Part IIIc: Identifying $\gamma_{\overline{\alpha},\sigma}$ with $[\overline{\alpha} \circ \sigma]$.
In other words, we show that the map set up in Part IIIa is in fact the well-known HO : $\pi_0 \Rightarrow \pi_7$ given by precomposition with σ. Interpret the homotopy fiber of β as $\mathsf{F}' := \{(x, \omega) \mid x \in \mathsf{bo}_2^\wedge,\ \omega \text{ path } \beta(x) \to *\}$. With this model, $\overline{\alpha}$ is given by $\overline{\alpha} : p \mapsto (\alpha(p), N_0(\overline{\alpha}(p), -))$. We iterate this construction for hofiber$[\iota : \mathsf{F}' \to \mathsf{bo}_2^\wedge]$ and can identify

$$\left\{(x, \omega, \nu) \mid x \in \mathsf{bo}_2^\wedge,\ \omega \text{ path } \beta(x) \rightsquigarrow *,\ \nu \text{ path } (x, \omega) \rightsquigarrow (*, \mathrm{const})\right\} \xrightarrow{\simeq} \Omega\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8)$$

using a classical result. The choice of nullhomotopy $N : \alpha \circ \sigma \simeq \mathrm{const}$ was used in Part IIIa to construct an extension of α to the cofiber of σ. Here we may interpret it as nullhomotopy $N : \iota \circ \overline{\alpha} \circ \sigma \simeq \mathrm{const}$. Thus it provides a lift $\tilde{\alpha}$ of $\overline{\alpha} \circ \sigma$ to the homotopy fiber of $\iota : \mathsf{F}' \to \mathsf{bo}_2^\wedge$. We may write it as follows, see also (7.11),

$$\tilde{\alpha} : p \mapsto (\alpha \circ \sigma(p), N_0(\overline{\alpha} \circ \sigma(p), -), N(p, -)) \ \widehat{=}\ (N_0 \circ \overline{\alpha} \circ \sigma) * \overline{(\beta \circ N)}(p, -).$$

It is not hard to see that the adjoint of $\tilde{\alpha}$ is given by the gluing $\mathsf{S}^8 \to (N_0 \circ \overline{\alpha} \circ \sigma) \cup_{\mathsf{S}^7} (\beta \circ N)$ previously defined. An analogous consideration shows that the difference $\tilde{\alpha}_1 - \tilde{\alpha}_2$, corresponding to two choices N_1 and N_2 for N, lifts to an element in $\pi_7\left(\Omega\mathsf{bo}_2^\wedge\right)$ adjoint to $\Delta_{N_1,N_2} \in \pi_8\left(\mathsf{bo}_2^\wedge\right)$. Hence we have shown

$$\gamma_{\overline{\alpha},\sigma} = [\gamma(N, \overline{\alpha}, \sigma)] \in {}^{\pi_8\left(\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)}\big/_{\pi_8\left(\mathsf{bo}_2^\wedge\right)} \leftrightarrow [\tilde{\alpha}]_{\pi_7\left(\Omega\mathsf{bo}_2^\wedge\right)} \in {}^{\pi_7\left(\Omega\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)\right)}\big/_{\pi_7\left(\Omega\mathsf{bo}_2^\wedge\right)}.$$

The latter corresponds precisely to the element $[\overline{\alpha} \circ \sigma] \in \pi_7\left(\mathsf{F}'\right)$ in the cofiber $\pi_7\left(\mathsf{F}'\right)$ of the map $\pi_8\left(\mathsf{bo}_2^\wedge\right) \to \pi_8\left(\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8)\right)$. □

$$\begin{tikzcd}
& \Omega\mathsf{H}\,(\mathbb{Q}_2^\wedge, 8) \arrow[d] & \\
\mathsf{S}^7 \arrow[ur, dashed, "\tilde{\alpha}"] \arrow[r, "\overline{\alpha}\circ\sigma"] \arrow[d, "\sigma"'] & \mathsf{F}' \arrow[d, "\iota"] & \\
\mathsf{S} \arrow[ur, "\overline{\alpha}"] \arrow[r, "\alpha"] & \mathsf{bo}_2^\wedge \arrow[r, "\beta"] & \mathsf{H}\,(\mathbb{Q}_2^\wedge, 8)
\end{tikzcd} \tag{7.11}$$

7.4. The image of the homology of $H_8(\mathsf{C}_\sigma, \mathbb{Z})$

In this section we will calculate the order of the image of $1 \in H_8(\mathsf{C}_\sigma)$ in ${}^{H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}\big/_{\pi_8(\mathsf{bo}_2^\wedge)}$. Proposition 7.16 sums up the most important result that can be deduced. The proof involves some of Adams' findings in [Ada73] and [Ada74]. Lemma 7.18 provides a convenient analysis of the $\mathbb{Z}_2$-homology $H_8(\mathsf{bo}, \mathbb{Z}_2)$ of bo. Roughly, the generator $u \in H^0(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2$ yields $Sq^8(u) \in H^8(\mathsf{bo}, \mathbb{Z}_2)$, which lifts to an integral cohomology class. Consequently, the image of $1 \in H_8(\mathsf{C}_\sigma)$ in $H_8(\mathsf{bo}_2^\wedge, \mathbb{Z})$ is not divisible by 2, see Lemma 7.19. Lemma 7.20 identifies the image of $H_8(\mathsf{bo}, \mathbb{Z})$ in $H_8(\mathsf{bo}, \mathbb{Q})$ and allows for a distinguished identification of $H_8(\mathsf{bo}, \mathbb{Q})$ with $\mathbb{Q}$, in analogy to Adams' way of norming $H_8(\mathsf{bu}, \mathbb{Q})$ in [Ada73]. Also, $1 \in \pi_8\,(\mathsf{bo})$ is $k \cdot Sq^8(u)$ in $H_8(\mathsf{bo}, \mathbb{Q})$, with $k = 16$ 2-locally. The present findings help us see that the 2-torsion part of π_7^s is embedded in $\pi_7\,(\mathsf{F}') \cong {}^{\mathbb{Q}}\big/_{32\mathbb{Z}}$ in Section 7.5. We now fix the choice of $\alpha : \mathsf{S} \to \mathsf{bo}_2^\wedge$ to represent a generator in $\pi_0\,(\mathsf{bo}_2^\wedge)$. Recall the extension of α to the cofiber C_σ, see (7.5).

Proposition 7.16. *The following image* 2*-locally has order* 16*, i. e.* 16 *divides* k *in*

$$1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z}) \cong \mathbb{Z} \longmapsto 1 \in {}^{H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}\big/_{\pi_8(\mathsf{bo}_2^\wedge)} \cong {}^{\mathbb{Q}}\big/_{k\mathbb{Z}}.$$

We reduce the assertion as follows. We may use the universality of the localization and see that α factors via $\mathsf{S} \to \mathsf{bo} \to \mathsf{bo}_2^\wedge$, since $[\alpha] = 1 \in \pi_0\,(\mathsf{bo}) \subset \pi_0\,(\mathsf{bo}_2^\wedge)$. The same is true for the corresponding extension α' on the cofiber C_σ, by choosing an adequate nullhomotopy. The theory of Section 7.3 carries through analogously for these elements of $\pi_0\,(\mathsf{bo})$ and $[\mathsf{C}_\sigma, \mathsf{bo}]$. This allows to sensibly state the following proposition.

Proposition 7.17. *The following image* 2*-locally has order* 16*, i. e.* 16 *divides* k *in*

$$1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z}) \cong \mathbb{Z} \longmapsto 1 \in {}^{H_8(\mathsf{bo},\mathbb{Q})}\big/_{\pi_8(\mathsf{bo})} \cong {}^{\mathbb{Q}}\big/_{k\mathbb{Z}}.$$

Proof of Proposition 7.16 from 7.17. Fix factorizations $\alpha : \mathsf{S} \to \mathsf{bo} \to \mathsf{bo}_2^\wedge$ for a generator in $\pi_0\,(\mathsf{bo}_2^\wedge)$, and correspondingly for α'. Consider the commutative diagram

$$\begin{tikzcd}
H_8(\mathsf{C}_\sigma) \arrow[r, "{[\alpha'_*]}"] \arrow[dr, "{[\mathrm{loc}\circ\alpha'_*]}"'] & H_8(\mathsf{bo}, \mathbb{Q}) \arrow[r] \arrow[d] & {}^{H_8(\mathsf{bo},\mathbb{Q})}\big/_{\pi_8(\mathsf{bo})} \arrow[d, "\cong"] \\
 & H_8(\mathsf{bo}_2^\wedge, \mathbb{Q}) \arrow[r] & {}^{H_8(\mathsf{bo}_2^\wedge,\mathbb{Q})}\big/_{\pi_8(\mathsf{bo}_2^\wedge)}.
\end{tikzcd}$$

The natural right-hand vertical map turns out to be an isomorphism, using the universality property of localization on both homotopy and homology. □

The proof of Proposition 7.17 is divided into several lemmata and will take up the rest of this section.

Lemma 7.18. $H_8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2$ *and is generated by the element* Sq^8*.*

Proof. Adams [Ada74, Proposition III.16.6] shows that $H^*(\mathsf{bo}, \mathbb{Z}_2) \cong {}^{A}\big/_{(ASq^1+ASq^2)}$, where A denotes the Steenrod algebra and Sq^1 is considered in notation via Serre-Cartan basis. One checks that the degree 8 basis elements are $Sq^5Sq^2Sq^1$, Sq^6Sq^2, Sq^7Sq^1 and Sq^8. Except for Sq^8, these are contained in the subalgebra $ASq^1 + ASq^2$. Consequently, $H^8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2{<}Sq^8{>}$. The universal coefficient theorem with $\mathbb{Z}_2$-coefficients implies $H^8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathrm{Hom}\,(H_8(\mathsf{bo}, \mathbb{Z}_2), \mathbb{Z}_2)$. Thus $H_8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2{<}Sq^8{>}$. □

Lemma 7.19. *The image of* $1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z}) \cong \mathbb{Z} \mapsto 1 \in H_8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2$ *is non-zero. Consequently, the image of* $1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z})$ *in* $H_8(\mathsf{bo}, \mathbb{Z})$ *is not divisible by* 2.

Proof. Recall that the Steenrod squares as operation $Sq^i : H^i \to H^{2i}$ coincide with the unstable cup product. Recall $\sigma : S^{15} \to S^8$. Denote the homotopy cofiber of σ by C, i. e. $C = S^8 \bigcup_\sigma D^{16}$. It is not hard to see that $H^i(C, \mathbb{Z}) \cong \mathbb{Z}$ for $i = 0, 8, 16$, and vanishes otherwise. One can define the Hopf invariant of σ via

$$\smile^2 : 1 \in H^8(C, \mathbb{Z}) \mapsto \mathrm{Hopf}\,(\sigma) \in H^{16}(C, \mathbb{Z}).$$

By Lemma 4.19, the map σ we are considering has Hopf invariant 1. Consequently, the Steenrod square $Sq^8 : H^8(C, \mathbb{Z}_2) \to H^{16}(C, \mathbb{Z}_2)$ is non-zero. Upon passing to the stable cofiber and Steenrod operations, $Sq^8 : H^0(\mathsf{C}_\sigma, \mathbb{Z}_2) \to H^8(\mathsf{C}_\sigma, \mathbb{Z}_2)$ is non-zero on the spectrum C_σ. We set up a commutative diagram as follows.

$$\begin{tikzcd}
\mathrm{Hom}\,(H_8(\mathsf{C}_\sigma, \mathbb{Z}_2), \mathbb{Z}_2) & \mathrm{Hom}\,(H_8(\mathsf{bo}, \mathbb{Z}_2), \mathbb{Z}_2) \arrow[l, "(\alpha')^*"'] \\
H^8(\mathsf{C}_\sigma, \mathbb{Z}_2) \arrow[u, "\mathrm{proj}", "\cong"'] & H^8(\mathsf{bo}, \mathbb{Z}_2) \arrow[l, "(\alpha')^*"'] \arrow[u, "\mathrm{proj}", "\cong"'] \\
H^0(\mathsf{C}_\sigma, \mathbb{Z}_2) \arrow[u, "Sq^8", "\neq 0"'] & H^0(\mathsf{bo}, \mathbb{Z}_2) \arrow[l, "(\alpha')^*"', "\cong"] \arrow[u, "Sq^8"]
\end{tikzcd} \tag{7.12}$$

The universal coefficient theorem A.9 in the case $\mathbb{Z}_2$ has vanishing Ext-term. Thus the projection to $\mathrm{Hom}\,(H_i(\mathsf{C}_\sigma, \mathbb{Z}_2), \mathbb{Z}_2)$ is in fact an isomorphism. The map α' induces an isomorphism on the lower horizontal of (7.12) because α was chosen a generator in $\pi_0\,(\mathsf{bo})$. Since the lower left composition of maps is non-zero, the upper right composition must be so too. The map $(\alpha')^* : \mathrm{Hom}\,(H_8(\mathsf{C}_\sigma, \mathbb{Z}_2), \mathbb{Z}_2) \leftarrow \mathrm{Hom}\,(H_8(\mathsf{bo}, \mathbb{Z}_2), \mathbb{Z}_2)$ being non-zero entails that the underlying map $\alpha'_* : H_8(\mathsf{C}_\sigma, \mathbb{Z}_2) \to H_8(\mathsf{bo}, \mathbb{Z}_2)$ is non-zero. The universal coefficient theorem A.9 in the homology case implies that precomposing with coefficient change, we obtain a non-zero map $\alpha'_* : H_8(\mathsf{C}_\sigma, \mathbb{Z}) \to H_8(\mathsf{bo}, \mathbb{Z}_2)$. The claim follows with the identification from Lemma 7.18. □

Let $u \in \pi_8\,(\mathsf{bo})$ denote a generator. We identify it with its images in $H_8(\mathsf{bo}, \mathbb{Z})$ and $H_8(\mathsf{bo}, \mathbb{Q})$.

Lemma 7.20. *The homology* $H_8(\mathsf{bo}, \mathbb{Q})$ *is a one-dimensional* $\mathbb{Q}$*-vector space. The image of* $H_8(\mathsf{bo}, \mathbb{Z})$ *in* $H_8(\mathsf{bo}, \mathbb{Q})$ *is the* $\mathbb{Z}$*-submodule generated by the element* u/k*, where* k *is divisible by* 16.

Remark 7.21. In analogy to Remark 7.9 and Adams' way of norming $H_8(\mathsf{bu}, \mathbb{Q})$ in [Ada73], we fix the element u/k as a distinguished generator for $H_8(\mathsf{bo}, \mathbb{Q})$. From now on forth, set the convention of a distinguished isomorphism $H_8(\mathsf{bo}, \mathbb{Q}) \cong \mathbb{Q}$ sending $u/k \mapsto 1$. It induces a distinguished isomorphism ${}^{H_8(\mathsf{bo},\mathbb{Q})}\big/_{\pi_8(\mathsf{bo})} \cong {}^{\mathbb{Q}}\big/_{k\mathbb{Z}}$.

Note how the norming is not as easily stated as in the case of $\mathsf{H}\left(\mathbb{Q}_2^\wedge, 8\right)$. After all, $H_8(\mathsf{bo}, \mathbb{Z})$ is most likely not isomorphic to $\mathbb{Z}$, hence does not allow a distinguished generator in contrast to its $\mathbb{Q}$-linearization.

Proof of Lemma 7.20. (i) Consider the complexification at the spectrum level $\mathsf{bo} \to \mathsf{bu}$. We may consider the diagram coming from the natural Hurewicz homomorphism

$$\begin{tikzcd} \pi_8\left(\mathsf{bo}\right) \arrow[r] \arrow[d, "\cong"'] & H_8(\mathsf{bo}, \mathbb{Z}) \arrow[r] \arrow[d] & H_8(\mathsf{bo}, \mathbb{Q}) \arrow[d] \\ \pi_8\left(\mathsf{bu}\right) \arrow[r] & H_8(\mathsf{bu}, \mathbb{Z}) \arrow[r] & H_8(\mathsf{bu}, \mathbb{Q}). \end{tikzcd} \tag{7.13}$$

Bott periodicity is compatible with the complexification, which accounts for the isomorphism on π_8. Adams [Ada73] identifies $H_8(\mathsf{bu}, \mathbb{Q})$ as a one-dimensional $\mathbb{Q}$-vector space. He also sets the convention of identifying the generator $v \in \pi_8\left(\mathsf{bu}\right)$ with its images in $H_8(\mathsf{bu}, \mathbb{Z})$ and $H_8(\mathsf{bu}, \mathbb{Q})$. By [Ada73, Theorem 1] and [Ada61], the image of $H_8(\mathsf{bu}, \mathbb{Z})$ in $H_8(\mathsf{bu}, \mathbb{Q})$ is the $\mathbb{Z}$-submodule generated by the element $v/720$. The distinguished generator $v/720$ for $H_8(\mathsf{bu}, \mathbb{Q})$ induces a distinguished isomorphism $H_8(\mathsf{bu}, \mathbb{Q}) \cong \mathbb{Q}$. In other words, the lower horizontal composition is $\cdot 720 : \pi_8\left(\mathsf{bu}\right) \to H_8(\mathsf{bu}, \mathbb{Q})$. The universal coefficient theorem specializes to

$$0 \to H_8(\mathsf{bo}, \mathbb{Z}) \otimes \mathbb{Z}_2 \to H_8(\mathsf{bo}, \mathbb{Z}_2) \to \operatorname{Tor}(H_7(\mathsf{bo}, \mathbb{Z}), \mathbb{Z}_2) \to 0.$$

Since $H_8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2$ has dimension 1, the group $H_8(\mathsf{bo}, \mathbb{Z})$ can contain at most one (non-canonical) infinite cyclic summand. Since the lower left composition in (7.13) is non-zero, so is the upper right composition. Consequently, $H_8(\mathsf{bo}, \mathbb{Q})$ is of dimension at least one. Thus $H_8(\mathsf{bo}, \mathbb{Z})$ must contain precisely one (non-canonical) infinite cyclic summand. This proves the first part of the Proposition. As another consequence,

$$H_8(\mathsf{bo}, \mathbb{Z}) \otimes \mathbb{Z}_2 \xrightarrow{\cong} H_8(\mathsf{bo}, \mathbb{Z}_2) \tag{7.14}$$

is an isomorphism turning the infinite cyclic summand into $H_8(\mathsf{bo}, \mathbb{Z}_2)$. In addition, $H_8(\mathsf{bo}, \mathbb{Z})$ cannot contain any 2-torsion.

(ii) By the considerations above, we know that the image of $H_8(\mathsf{bo}, \mathbb{Z})$ in $H_8(\mathsf{bo}, \mathbb{Q})$ must be a $\mathbb{Z}$-submodule generated by an element u/k for some $k \in \mathbb{N}$, which we fix as a distinguished generator for $H_8(\mathsf{bo}, \mathbb{Q})$. By commutativity of (7.13), $k \mapsto 720$ under $H_8(\mathsf{bo}, \mathbb{Q}) \to H_8(\mathsf{bu}, \mathbb{Q})$. Since both elements come from elements in $H_8(\mathsf{bo}, \mathbb{Z})$ respectively $H_8(\mathsf{bu}, \mathbb{Z})$, the multiplication must be given by $720/k \in \mathbb{Z}$. In other words, k divides 720. On the other hand, we show that 16 divides k. Extend (7.12) to the right with the bu-version of the right-hand column in (7.12). The complexification induces a lower horizontal isomorphism $H^0(\mathsf{bo}, \mathbb{Z}_2) \leftarrow H^0(\mathsf{bu}, \mathbb{Z}_2)$. Previous reasonings apply to the bu-case as well. Here we also obtain a non-zero map $H_8(\mathsf{bo}, \mathbb{Z}_2) \to H_8(\mathsf{bu}, \mathbb{Z}_2)$ *induced by complexification*. Consequently, the left vertical composition in (7.15) is non-zero.

$$\begin{tikzcd} H_8(\mathsf{C}_\sigma, \mathbb{Z}_2) \arrow[d] & H_8(\mathsf{C}_\sigma, \mathbb{Z}) \arrow[l] \arrow[d] & \\ H_8(\mathsf{bo}, \mathbb{Z}_2) \arrow[d] & H_8(\mathsf{bo}, \mathbb{Z}) \arrow[l] \arrow[r] \arrow[d] & H_8(\mathsf{bo}, \mathbb{Q}) \arrow[d] \\ H_8(\mathsf{bu}, \mathbb{Z}_2) & H_8(\mathsf{bu}, \mathbb{Z}) \arrow[l] \arrow[r] & H_8(\mathsf{bu}, \mathbb{Q}) \end{tikzcd} \tag{7.15}$$

By Lemma 7.19 the generator $1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z})$ is sent to $1 \in H_8(\mathsf{bo}, \mathbb{Z}_2) \cong \mathbb{Z}_2$. By (7.14), $1 \in H_8(\mathsf{bo}, \mathbb{Z}_2)$ comes from the (non-canonical) infinite cyclic summand in $H_8(\mathsf{bo}, \mathbb{Z})$. A valid preimage can be identified as $u/k \in H_8(\mathsf{bo}, \mathbb{Z})$. We have seen that $u/k \mapsto 1 \in H_8(\mathsf{bu}, \mathbb{Z}_2)$ under the composition via $H_8(\mathsf{bo}, \mathbb{Z}_2)$. Thus the image of $u/k \in H_8(\mathsf{bo}, \mathbb{Z})$ in $H_8(\mathsf{bu}, \mathbb{Z})$ cannot be divisible by 2. It must be a multiple of $v/720$, a choice of generator for the infinite cyclic summand in $H_8(\mathsf{bu}, \mathbb{Q})$ by a similar argument as above. Consequently, the image of u/k in $H_8(\mathsf{bu}, \mathbb{Q}) \cong \mathbb{Q}$ is an odd multiple of the distinguished generator 1. This odd multiple is precisely the integer $720/k \in \mathbb{Z}$ by our arguments at the beginning of (ii), compare also with (7.13). □

We can consider the quotient group ${}^{H_8(\mathsf{bo},\mathbb{Q})}\big/_{\pi_8(\mathsf{bo})}$. The distinguished generator $1 \in \mathbb{Q} \cong H_8(\mathsf{bo}, \mathbb{Q})$ then has order divisible by 16 in this quotient group. Moreover, we have seen in the second part of the proof above that

$$1 \in H_8(\mathsf{C}_\sigma, \mathbb{Z}) \cong \mathbb{Z} \mapsto 1 \in {}^{\mathbb{Q}}\big/_{k\mathbb{Z}} \cong {}^{H_8(\mathsf{bo},\mathbb{Q})}\big/_{\pi_8(\mathsf{bo})}.$$

Consequently, this image has order divisible by 16, which proves Proposition 7.17.

7.5. Discussion

The discussion in this section shows that the current attempted calculation does not reveal a non-vanishing result as desired. Most interestingly, the reason lies a lot earlier in the approximation process. The essence of the previous calculations can be expressed using a Postnikov truncation $\mathsf{Wh}(*)_{[3]}$ of $\mathsf{Wh}(*)$, see Lemma 7.26. It appears that it should be possible to obtain a non-vanishing result by working with a less truncated version. Currently, ideas are in development to use $\mathsf{Wh}(*)_{[3,7]}$ and its properties as described in [Rog02]. This attempt will be pursued in the future. In this section arguments and proofs are kept deliberately short since it ought only to convey principal ideas rather than stringent results or methods.

Lemma 7.22. *The epimorphism $\beta_{*,\#}$ (7.7) is in fact an isomorphism. With our norming conventions, it is the non-standard isomorphism ${}^{\mathbb{Q}}\big/_{k\mathbb{Z}} \to {}^{\mathbb{Q}}\big/_{32\mathbb{Z}}$ sending $[1] \mapsto [32/k]$.*

Proof. By Lemma 7.20, $H_8(\mathsf{bo}_2^\wedge, \mathbb{Q})$ is a one-dimensional $\mathbb{Q}_2^\wedge$-vector space, rendering the right vertical map β_* in (7.10) an isomorphism. It is not hard to deduce that (7.7) is an isomorphism using that (7.10) commutes. $\square$

Proposition 7.23. *Applying the stable homotopy operation to* $1 \in \pi_0(\mathsf{F}') \cong \pi_0(\mathsf{F}) \cong \mathbb{Z}_2^\wedge$ *yields an element* $1 \cdot \sigma \in \pi_7(\mathsf{F}') \cong \mathbb{Q}/_{32\mathbb{Z}}$ *of order* 16.

This follows from Proposition 7.15 and 7.16 and Lemma 7.22.

Problem 7.24. Upon passing to $\pi_7(\mathsf{F})$ via $\mathbb{Q}/_{32\mathbb{Z}} \to \mathbb{Q}/_{8\mathbb{Z}}$, applying the stable homotopy operation to $4 \in \pi_0(\mathsf{F}) \cong \mathbb{Z}_2^\wedge$ yields a vanishing element $4 \cdot \sigma = 0 \in \pi_7(\mathsf{F}) \cong \mathbb{Q}/_{8\mathbb{Z}}$. This way it is *not possible* to show that applying the homotopy operation to $1 \in \pi_8\left(\mathrm{G}(7)/_{\mathrm{TOP}(7)}\right) \cong \mathbb{Z}$ yields a non-zero element $1 \cdot \sigma \neq 0 \in \pi_{15}\left(\mathrm{G}(7)/_{\mathrm{TOP}(7)}\right)$.

While looking for reasons one notices a crucial fact. Recall that the splitting in Definition 5.1 induces a map of spectra $\mathsf{A}(*) \to \mathsf{Wh}(*)$ respecting the duality. Despite being suppressed from notation, we use 8-duality. Recall also that Ξ actually takes values in the Tate construction $\mathsf{Wh}(*)^{th\mathbb{Z}_2}$ if defined on quadratic L-theory. Write Ξ_{Wh} for the composition $\mathsf{L}_\bullet(\mathbb{Z}) \simeq \mathsf{L}_\bullet(\mathsf{S}) \xrightarrow{\mathrm{sym}} \mathsf{L}^\bullet(\mathsf{S}) \xrightarrow{\Xi} \mathsf{Wh}(*)^{th\mathbb{Z}_2}$.
Since $\mathsf{Wh}(*)$ is 2-connected, the approximating map $\mathsf{Wh}(*) \to \mathsf{bo}$ canonically lifts to a map $\mathsf{Wh}(*) = \mathsf{Wh}(*)_{[3,\infty]} \to \mathsf{bo}_{[4,\infty]}$. It is possible to see that the Whitehead-tower-induced map $\pi_{4k}\big(\mathsf{bo}_{[4,\infty]}^{th\mathbb{Z}_2}\big) \to \pi_{4k}\left(\mathsf{bo}^{th\mathbb{Z}_2}\right)$ is the inclusion $8\mathbb{Z}_2^\wedge \to \mathbb{Z}_2^\wedge$ of an index 8 subgroup. As a consequence, $\mathsf{L}_\bullet(\mathbb{Z}) \to \mathsf{bo}_{[4,\infty]}^{th\mathbb{Z}_2}$ induces isomorphisms on π_{4k}, rendering $\pi_7\left(\mathsf{P}_{\mathsf{bo},[4,\infty]}\right) = 0$ for the homotopy pullback of the analogous (7.1). This explains why approximating further via $\mathsf{bo}_{[4,\infty]} \to \mathsf{bo}$ leads nowhere and we run into Problem 7.24. Although this might seem foolish at this point, for instructional reasons we also take a look at the approximation $\mathsf{Wh}(*) \to \mathsf{bo}_{[4,\infty]} \to \mathsf{bo}_{[4]}$ using a Postnikov projection. Of course this approximation also leads nowhere, but we notice that we may rewrite the map as $\mathsf{Wh}(*)_{[3,\infty]} \to \mathsf{Wh}(*)_{[3,4]} \to \mathsf{bo}_{[3,4]}$. The two truncated spectra are Eilenberg-MacLane spectra, and the map $\mathsf{Wh}(*)_{[3]} \to \mathsf{bo}_{[4]}$ turns out to be a Bockstein. The approximations with $\mathsf{Wh}(*)_{[3]}$ and $\mathsf{bo}_{[4]}$ lead to very similar results. In essence, this is due to $\mathsf{Wh}(*)_{[3]} \to \mathsf{bo}_{[4]}$ inducing isomorphisms on π_{2k} of the Tate construction and in relevant dimensions for the fixed points. We take a closer look at $\mathsf{Wh}(*)_{[3]}$ for inspirational reasons and begin by establishing some basics. Recall in particular Lemma 6.10.

Lemma 7.25. *The map* $\mathrm{proj} \circ \Xi_{\mathrm{Wh}} : \mathsf{L}_\bullet(\mathbb{Z}) \to \mathsf{Wh}(*)_{[3]}^{th\mathbb{Z}_2}$ *on* π_{4k} *sends* $1 \in \mathbb{Z} \longmapsto 1 \in \mathbb{Z}_2$.

Proof sketch. We argue only for π_0. Recall (7.2). The analogous diagram can be set up for $\mathsf{Wh}(*)$ using Proposition 5.9 and the duality respecting map $\mathsf{A}(*) \to \mathsf{Wh}(*)$. By definition and Corollary 5.5, Ξ_{bo} factors via Ξ_{Wh}. Similarly, $\Omega^{n+1}\left(\mathrm{G}(n)/_{\mathrm{TOP}(n)}\right) \to \mathsf{bo}^{rh\mathbb{Z}_2}$ also factors via $\mathsf{Wh}(*)^{rh\mathbb{Z}_2}$ for general $n \geq 7$.
By Lemma 7.7, this map to $\mathsf{bo}^{rh\mathbb{Z}_2}$ is multiplication by 4 on π_0. Recall Theorem 1.61 and the spectral sequence computations of $\mathrm{KO}^*(\mathrm{B}\mathbb{Z}_2) \cong \pi_*\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)$ and $\pi_*\left(\mathsf{bo}_{h\mathbb{Z}_2}\right)$

in [BG10]. Each of the non-zero homotopy groups of $\pi_*(\mathsf{bo})$ contribute a non-trivial $\mathbb{Z}_2$-extension to $\pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right)$. Recall that $1 \in \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right) \subset \pi_0\left(\mathsf{bo}^{h\mathbb{Z}_2}\right)$ corresponds in $\mathrm{KO}^*(\mathrm{B}\mathbb{Z}_2)$ to the formal difference $\gamma - \varepsilon$ of the tautological and trivial line bundles over $\mathbb{R}P^\infty$ by Remark 1.73. The multiple $4 \cdot (\gamma - \varepsilon)$ can be trivialized on the skeleton $(\mathbb{R}P^\infty)^{(3)}$ since $4 \in \pi_0\left(\mathsf{bo}^{rh\mathbb{Z}_2}\right)$ vanishes in $\mathbb{Z}_4$, the cumulative extensions up to $\pi_3(\mathsf{bo})$. However, it cannot be trivialized on $(\mathbb{R}P^\infty)^{(4)}$ because of the obstruction for analogous reasons.
We inspect the image of $1 \in \pi_0\left(\Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)\right)$ in $\pi_0\mathsf{Wh}(*)^{rh\mathbb{Z}_2}_{[3]}$. The image in $\pi_0\mathsf{Wh}(*)^{rh\mathbb{Z}_2}$ is a lift of $4 \cdot (\gamma - \varepsilon)$ by previous arguments. In other words, the map $\mathbb{R}P^\infty \to \mathsf{bo}$ factors via $\mathsf{Wh}(*)$. The corresponding obstruction cannot vanish for $\mathsf{Wh}(*)$ either. Since it has only $\pi_3(\mathsf{Wh}(*)) \cong \mathbb{Z}_2 \neq 0$ in these dimensions, the element $1 \in \pi_3(\mathsf{Wh}(*))$ constitutes the obstruction. Consequently the image of $1 \in \pi_0\left(\Omega^{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)\right)$ in $\pi_0\mathsf{Wh}(*)^{rh\mathbb{Z}_2}_{[3]}$ is the non-zero element.
By Lemma 6.10, the latter is sent to $1 \in \mathbb{Z}_2 \cong \pi_0\mathsf{Wh}(*)^{th\mathbb{Z}_2}_{[3]}$. Since $\pi_0(\mathsf{L}_\bullet(\mathsf{S})) \cong \pi_{n+1}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$, the assertion follows on π_0. For the general case one can consider a modification of Proposition 5.9 and (7.2) involving $\Omega^{n+1-4k}\left({}^{\mathrm{G}(n)}/_{\mathrm{TOP}(n)}\right)$ and $(n+1-4k)$-duality instead of $(n+1)$-duality. The π_{4k}-case is then analogous to the π_0-case, and the map Ξ_{Wh} does not see the change in loop degree. Of course one needs to ensure that $n+1 \geq 4k$. □

We revert back to $n = 7$. Consider now the resulting homotopy pullback $\mathsf{P}_{\mathsf{Wh}(*),[3]}$ in analogy to (7.1). Again, $\pi_7\left(\mathsf{P}_{\mathsf{Wh}(*),[3]}\right) = 0$ because Ξ_{Wh} is an isomorphism on π_8 by Lemma 7.25. Of course this was to be expected since this approximation is very similar to the one for $\mathsf{bo}_{[4]}$. However, in this case it is not possible to construct F and F' as in Section 7.2 for reasons becoming apparent in the sequel. Instead we pass to

$$\text{hofiber}\left[\mathsf{L}_\bullet(\mathbb{Z};8) \to \text{cone}\left[\mathsf{H}(\mathbb{Z}_2,3)^{rh\mathbb{Z}_2} \to \mathsf{H}(\mathbb{Z}_2,3)^{th\mathbb{Z}_2}\right]\right],$$

by Lemma A.8. In dimension 8 the cone is $\mathsf{H}(\mathbb{Z}_2,8)$, so we can simplify further to

$$\mathsf{F}_{\mathsf{Wh}(*),[3]} := \text{hofiber}\left[\mathsf{L}_\bullet(\mathbb{Z};8) \to \mathsf{H}(\mathbb{Z}_2,8)\right],$$

using the projection $\mathsf{H}(\mathbb{Z}_2,3)^{th\mathbb{Z}_2} \to \mathsf{H}(\mathbb{Z}_2,8)$. However, the latter implicitly uses the naive $\mathsf{H}\mathbb{Z}_2$-module structure on $\mathsf{H}(\mathbb{Z}_2,3)^{th\mathbb{Z}_2}$. Note that in the linear case $\Xi : \mathsf{L}^\bullet(\mathbb{Z}) \to \mathsf{K}(\mathbb{Z})^{th\mathbb{Z}_2}$ is a map of module spectra over the ring spectrum $\mathsf{L}^\bullet(\mathbb{Z})$ [WW00]. 2-locally this turns into an $\mathsf{H}\mathbb{Z}$-module map. The same is true for the composition to $\mathsf{bo}^{th\mathbb{Z}_2}_{[4]}$ by the arguments in [WW00]. It turns out that the structure on $\mathsf{bo}^{th\mathbb{Z}_2}_{[4]}$ induced from $\mathsf{L}^\bullet(\mathbb{Z})$ is most likely *not* the naive structure. This statement carries over to the case of $\mathsf{Wh}(*)^{th\mathbb{Z}_2}_{[3]}$. Expressed using the naive structure on $\mathsf{H}(\mathbb{Z}_2,3)^{th\mathbb{Z}_2}$, the summands $\mathsf{L}_\bullet(\mathbb{Z})_{[0]}$ and $\mathsf{L}_\bullet(\mathbb{Z})_{[8]}$ map to $\mathsf{H}(\mathbb{Z}_2,8)$ via $(Sq^8 + Sq^D) \circ \text{proj}$ and $\text{proj} : \mathsf{H}\mathbb{Z} \to \mathsf{H}\mathbb{Z}_2$, respectively. Here Sq^D denotes a possibly non-vanishing sum of decomposable Steenrod squares. To no surprise, the Sq^8 already showed up in Section 7.4. In essence, for the cohomology generators z_0 and z_8 of $\mathsf{L}_\bullet(\mathbb{Z})$ in dimension 0 and 8 we have sketched

Lemma 7.26. $(Sq^8 + Sq^D)z_0 + z_8 = 0$ *in* $\mathrm{hofiber}\left[\mathsf{L}_\bullet(\mathbb{Z}) \to \mathsf{H}(\mathbb{Z}_2, 8)\right]$.

One can also see that $(Sq^8 + Sq^D)z_0 = 0$ indeed turns into a contradiction. Otherwise, Ξ on $\mathsf{L}_\bullet(\mathbb{Z})_{[0]} \vee \mathsf{L}_\bullet(\mathbb{Z})_{[8]}$ would be an $\mathsf{H}\mathbb{Z}$-module map with respect to the naive module structure, at least 2-locally. Using the 2-local module map $\mathsf{H}\mathbb{Z} \to \mathsf{L}_\bullet(\mathbb{Z})$, the resulting modified homotopy fiber

$$\mathsf{F}'_{\mathsf{Wh}(*),[3]} := \mathrm{hofiber}\left[\mathsf{H}\mathbb{Z} \to \mathsf{H}(\mathbb{Z}_2, 8)\right],$$

would also be a module over $\mathsf{H}\mathbb{Z}$. As a consequence, $\mathrm{HO}(1) = 1 \cdot \sigma$ would vanish. This is not the case by similar considerations as in Proposition 7.23.

Remark 7.27. As a future prospect, it is possible to imagine working with the truncated spectrum $\mathsf{Wh}(*)_{[3,7]}$. It is not a wedge of Eilenberg-MacLane spectra by [Rog02] and [Rog03], and the involution on $\pi_5(\mathsf{Wh}(*)) \cong \mathbb{Z}$ is non-trivial, already because it is non-trivial rationally. Further properties are described in [Rog02]. However, for $n = 7$ one can only hope to detect the obstruction in this manner for Σ_{KM}. For higher dimensions $n > 7$, other non-trivial homotopy groups of $\mathsf{Wh}(*)$ might contribute sufficiently to possibly detect the obstruction in the case of $d = 2$ and a generator $1 \in \pi_n\left(\mathrm{G}(n) / \mathrm{TOP}(n)\right)$. Perhaps one could also achieve results using topological cyclic homology.

A. Appendix

A.1. The evaluation map

Consider a topological group G and a closed subgroup H of G. We may consider the projection map $p : G \to G/H$, $g \mapsto gH$, where $X := G/H$ denotes the coset space of G by H as usual. The subgroup H corresponds via p to a basepoint $x_0 := p(H)$ of X. A *local cross-section* of H in G is a neighborhood V of x_0 together with a continuous map $s : V \to G$, such that $p \circ s = \mathrm{id}_V$. We may now state

Lemma A.1 ([Ste99, Corollary p. 31]). *If H has a local cross-section in G, then G is a fiber bundle over G/H relative to p. The fiber of the bundle is H and the group is H acting on the fiber by left translations.*

From this we will deduce

Lemma A.2. *Let G be the topological group* $\mathrm{TOP}\,(S^n)$*. Then the evaluation map*

$$\begin{aligned} \mathrm{ev}_1 : G &\to S^n \\ g &\mapsto g(1), \end{aligned}$$

given by evaluating the function g at the basepoint $1 \in S^n$, is a fiber bundle with fiber G_1, the subgroup of those functions fixing 1.

We may speak of the evaluation maps also for other mapping spaces than $\mathrm{TOP}\,(S^n)$. For a map $f : S^n \to \mathrm{TOP}\,(S^n)$, note the fundamental difference between $\mathrm{ev}_1 \circ f : p \mapsto f(p)(1)$ and $\mathrm{ev}_1(f) : p \mapsto f(1)(p)$. The latter is never used in this thesis.

Proof. At first we define $H := G_1$ to be the stabilizer subgroup of $1 \in S^n$. We apply Lemma A.1. As a first step, we provide a local cross-section to ev_1. Consider the fiber bundle $\mathrm{SO}\,(n) \to \mathrm{SO}\,(n+1) \to S^n$. This comes with a local cross-section s on some neighborhood V of 1. Since $\mathrm{SO}\,(n+1) \subset \mathrm{TOP}\,(S^n)$, we automatically obtain our cross-section in this case.
As an intermediate step we identify G/H. In our setting, [Ste99, Theorem p. 30] yields a natural homeomorphism $q : G/H \to S^n$, provided the action-induced map $G \to S^n, g \mapsto g(1)$ is open. We proceed to show this. Let $V(K,U) \subset G$ be the open set $\{f \in G \mid f(K) \subset U\}$, where K is compact and U is open. These sets form a subbase for the compact-open topology. We contend ourselves with showing that the images of the $V(K,U)$ under ev_1 are open in S^n. Fix K and U. Let $W := \mathrm{ev}_1(V) = \{f(1) \in S^n \mid f \in V\}$. Choose $f \in V$. We show that $f(1)$ has an open neighborhood in W. Recall the construction of the local cross-section above. To any point $x \in S^n$ it provides a local cross-section s_x near x. Let s be such a cross-section on a small disk D around $f(1)$. Since f is a homeomorphism, it takes K to a closed set inside U. Thus as long as we wiggle f only a little, the resulting f' will still take K into U. Hence if we choose D sufficiently small then $s(D) \subset V$, i.e. $D \subset W$ is a neighborhood of $f(1)$ inside W.

Hence we have $G/H \cong S^n$ naturally, so we obtain a local cross-section $G/H \to G$. Therefore, by applying Lemma A.1, $p : G \to G/H$ is a bundle with fiber H. Using the natural identification, the same also holds for $\mathrm{ev}_1 : G \to S^n$. $\square$

Lemma A.3. *Let $\Sigma \in \mathfrak{S}(S^n)$ be a pseudo smooth structure on S^n. Consider the grouplike topological monoid* $\mathrm{Diff}\,(S^n_\Sigma)$ *of pseudo diffeomorphisms of S^n_Σ. Then the forgetful map*

$$\begin{aligned} \mathrm{forget} : \mathrm{Diff}\,(S^n_\Sigma) &\to \mathrm{TOP}\,(S^n) \\ (f, H) &\mapsto f \end{aligned}$$

is a Serre fibration.

Proof. Let $(f, H) : X \to \mathrm{Diff}\,(S^n_\Sigma)$ be a map and $F : X \times I \to \mathrm{TOP}\,(S^n)$ be a homotopy starting at $F(-, 0) = f$. Write $(f, H)(x) = (f^x, H^x)$, compare Definition 1.25. Denote by F_t the induced partial homotopy from $F(-, 0) = f$ to $F(-, t)$, i. e. $F_t(x, s) = F(x, s \cdot t)$. We define a lift $G : X \times I \to \mathrm{Diff}\,(S^n_\Sigma)$ of F via

$$G(x, t) := \Big(F(x, t), (F_t(x, -)_* \circ \Sigma) * H^x * \overline{(\Sigma \circ F_t(x, -))}\Big),$$

as illustrated by (A.1). Here $K_1 * K_2$ denotes concatenation of homotopies. $\square$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
S^n \arrow[r, "f^x"] \arrow[r, bend left=60, "F(x,t)"] \arrow[r, phantom, bend left=35, "F_t(x,-)"'] \arrow[d, "\Sigma"'] \arrow[dr, phantom, "H^x"] & S^n \arrow[d, "\Sigma"] \\
\mathrm{TOP}(S^n)\big/\mathrm{O}(n+1) \arrow[r, "f^x_*"'] \arrow[r, bend right=60, "F(x,t)_*"'] \arrow[r, phantom, bend right=35, "F_t(x,-)_*"] & \mathrm{TOP}(S^n)\big/\mathrm{O}(n+1)
\end{tikzcd} \tag{A.1}$$

From the lemmata above we may deduce

Corollary A.4. *Let $\Sigma \in \mathfrak{S}(S^n)$ be a pseudo smooth structure on S^n. Then*

$$\begin{aligned} \mathrm{ev}_1 : \mathrm{Diff}\,(S^n_\Sigma) &\to S^n \\ (f, H) &\mapsto f(1), \end{aligned}$$

given by evaluating f at the basepoint, is a Serre fibration.

Note that by [MT91, Theorem 6.5(2)] the evaluation map $\mathrm{O}(n+1) \to S^n$ is also a fibration with fiber $\mathrm{O}(n)$.

Lemma A.5. *The following natural map is a fibration with fiber* $\mathrm{O}(n)$*,*

$$\mathrm{TOP}(S^n) \xrightarrow{\mathrm{proj} \times \mathrm{ev}_1} {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)} \times S^n.$$

Proof. The proof is technical but straight forward. We check the homotopy lifting property. Let (h_1, h_2) be a homotopy from a space X to the base space. Let H_1 be a lift of h_1 along the fibration $\mathrm{TOP}(S^n) \to {}^{\mathrm{TOP}(S^n)}\big/_{\mathrm{O}(n+1)}$. Let $h_3(x,t) := H_1(x,t)^{-1}(h_2(x,t))$ be the preimages of the values of h_2 under the homeomorphisms given by H_1. Let H_3 be a lift of h_3 along the evaluation fibration $\mathrm{O}(n+1) \to S^n$. Then the pointwise composition $H_1 \circ H_3$ satisfies $\mathrm{ev}_1 \circ (H_1 \circ H_3) = h_2$ and $\mathrm{proj} \circ (H_1 \circ H_3) = h_1$. As to the fiber, consider the basepoint $([\mathrm{id}_{S^n}], 1)$ in the base space. Its preimages are maps in $\mathrm{O}(n+1)$ fixing $1 \mapsto 1$. These are orthogonal maps in $\mathrm{O}(n)$. $\square$

A.2. Homotopy pullbacks

Here is a result on homotopy pullbacks following [Mal14, Proposition 3.2].

Lemma A.6. *Consider a homotopy pullback square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D. \end{tikzcd}$$

Then for any choice of basepoint in A and its images in B, C and D, there is a natural fiber sequence

$$\Omega D \to A \to B \times C$$

yielding a natural long exact sequence on homotopy groups

$$\cdots \to \pi_i(A) \to \pi_i(B) \times \pi_i(C) \to \pi_i(D) \to \pi_{i-1}(A) \to \dots$$

which ends with $\pi_0(B) \times \pi_0(C)$. If A, B, C and D are spectra, then this sequence extends infinitely to the right. The maps in the long exact sequence are induced from the maps on spaces $A \to B \times C$ and $\Omega D \to A$. Only $\pi_i(B) \times \pi_i(C) \to \pi_i(D)$ is given by $(f_*, -g_*)$ with a sign.

When using connective spectra B, C and D, it is not clear whether we obtain a connective spectrum A as the homotopy pullback of the others in the category of spectra. In particular, A might come with an interesting $\pi_{-1}(\mathsf{A})$. However, in all cases where we apply the lemma above, it turns out that $\pi_{-1}(\mathsf{A}) = 0$.

Proof. The setup for this proof is taken from [Mal14, Proposition 3.2]. Consider the model $A = \big\{(b, \omega, c) \in B \times D^I \times C \mid f(b) = \omega(0),\ g(c) = \omega(1)\big\}$. Whenever $(b, \omega, c) \in A$ is sent to the basepoint in $B \times C$, the path ω must be a loop in D. Consequently, ΩD is the fiber of the map $A \to B \times C$. On the other hand, we show that it is a fibration, from which the main claim follows. Indeed, consider a diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d] & A \arrow[d] \\ X \times I \arrow[r, "H"] \arrow[ur, dashed, "H'"] & B \times C. \end{tikzcd} \tag{A.2}$$

We wish to provide a homotopy H'. Fix $H' = H_B$ on B and $H' = H_C$ on C. Set the part $H'|_{D^I}$ on D^I as

$$H'|_{D^I}(x,t) := \Big[f \circ H_B^t(x,-)\Big] * h_\omega(x) * \Big[g \circ \overline{H}_C^t(x,-)\Big], \tag{A.3}$$

where $H_B^t(x,-)$ is the path from $H_B(x,0)$ to $H_B(x,t)$ in B provided by H_B. Here $h_\omega(x)$ denotes the D^I-component of $h : X \to A$. H_C^t is defined analogously to H_B^t, although we require it in reverse direction. Basically $H'|_{D^I}$ changes the loops by appending paths to the starting point and from the endpoint.
Next we must verify that the maps in the long exact sequence are indeed induced from the obvious ones. For $\pi_i(A) \to \pi_i(B) \times \pi_i(C)$ this is obvious. For $\pi_i(D) \to \pi_{i-1}(A)$ this is fairly straight forward using the adjunction $\pi_i(D) \cong \pi_{i-1}(\Omega D)$. The case $\pi_i(B) \times \pi_i(C) \to \pi_i(D)$ needs more attention. We only sketch the ideas and omit the technical details. Recall that the long exact sequence associated to a fibration $F \to E \to B'$ is given as

$$\dots \to \pi_i(F) \to \pi_i(E) \to \pi_i(E,F) \to \pi_{i-1}(F) \to \dots$$

The boundary homomorphism is given by restricting a representative $(I^i, \partial I^i, J^{i-1}) \to (E, F, *)$ to the I^{i-1}-part of ∂I^i not contained in J^{i-1}. Recall the usual isomorphism $\pi_i(B') \to \pi_i(E,F)$ turning the sequence above into the more common version with $\pi_i(B')$. Its inverse is famously easily described by composing $(I^i, \partial I^i, J^{i-1}) \to (E, F, *) \to (B', *, *)$. Our direction above, however, is described by lifting a map $(I^i, \partial I^i) \to (B', *)$ using the lifting property

$$\begin{tikzcd} J^{i-1} \arrow[r, "*"] \arrow[d, "\mathrm{incl}"'] & E \arrow[d] \\ I^i \arrow[r] \arrow[ur, dashed] & B'. \end{tikzcd} \tag{A.4}$$

This indeed yields a map $(I^i, \partial I^i, J^{i-1}) \to (E, F, *)$. Translated to our specific setting, we have

$$\begin{tikzcd} \dots \arrow[r] & \pi_i(\Omega D) \arrow[r] & \pi_i(A) \arrow[r] \arrow[dr] & \pi_i(A, \Omega D) \arrow[r] & \pi_{i-1}(\Omega D) \arrow[r] & \dots \\ & & & \pi_i(B) \times \pi_i(C). \arrow[u, "\cong"] \arrow[ur] & & \end{tikzcd}$$

It is a general fact that the left-hand triangle commutes, using the induced maps on spaces on the diagonal map. We wish to see that the right-hand triangle commutes using the natural map $\pi_i(B) \times \pi_i(C) \to \pi_i(D) \cong \pi_{i-1}(\Omega D)$ for $i \geq 1$. We compute the composition via $\pi_i(A, \Omega D)$ with the naturally given boundary homomorphism. Start with an element $\beta + \gamma \in \pi_i(B) \oplus \pi_i(C)$. We find a lifting $(I^i, \partial I^i) \to (A, \Omega D)$ as in (A.4) using the lifting property (A.2). Applying the boundary homomorphism means restricting to $\partial I^i \to \Omega D$. Here $\Omega D \subset A$ sits inside A as the D^I-component, knowing that the endpoints are basepoints. In other words, the restricted lifting $\partial I^i \to \Omega D$ is given via (A.3), with slight modifications to adhere to the setting of (A.4). In this case, $h_\omega(x) \equiv *$ is constant, and H_B^t and H_C^t run through the i-dimension of β and γ, respectively. It is thus not hard to see that $\partial I^i \to \Omega D$ is adjoint to $f_*\beta - g_*\gamma$. The negative sign is reminiscent of reversing the homotopy H_C^t in (A.3). $\square$

Lemma A.7. *Consider a homotopy pullback square together with an extension to the right by a commutative square,*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r, dashed] \arrow[d, "f"] & B' \arrow[d, dashed, "f'"] \\ C \arrow[r, "g"] & D \arrow[r, dashed, "g'"] & D'. \end{tikzcd}$$

Then for any choice of basepoint in A and its images in B, C, D, B' and D', there is a natural long ladder with exact rows,

$$\begin{tikzcd} \dots \arrow[r] & \pi_i(A) \arrow[r] \arrow[d] & \pi_i(B) \times \pi_i(C) \arrow[r] \arrow[d] & \pi_i(D) \arrow[r] \arrow[d] & \pi_{i-1}(A) \arrow[r] \arrow[d] & \dots \\ \dots \arrow[r] & \pi_i(A') \arrow[r] & \pi_i(B') \times \pi_i(C) \arrow[r] & \pi_i(D') \arrow[r] & \pi_{i-1}(A') \arrow[r] & \dots \end{tikzcd}$$

Here A' denotes the homotopy pullback of $C \to D' \leftarrow B'$. All arrows are induced by

$$\begin{tikzcd} A \arrow[rr] \arrow[ddd] \arrow[dr, dashed] & & B \arrow[rr] \arrow[ddd, "f"] & & B' \arrow[ddd, "f'"] \\ & A' \arrow[urrr] \arrow[ddl] & & & \\ & & & & \\ C \arrow[rr, "g"] & & D \arrow[rr, "g'"] & & D'. \end{tikzcd} \tag{A.5}$$

The ladder ends with $\pi_0(B) \times \pi_0(C)$, unless we are in the category of spectra.

Proof. We obtain the two long exact horizontal sequences from Lemma A.6. The dashed arrow in (A.5) is given by the universal property of the homotopy pullback A'. (A.5) commutes by construction. Consequently the two associated fiber sequences as in Lemma A.6 are connected via a commutative ladder. The claim follows immediately. $\square$

We have the following special situation in the category of spectra.

Lemma A.8. *Let* B, C *and* D *be spectra. Let* A *denote the homotopy pullback fitting into the homotopy pullback square*

$$\begin{tikzcd} \mathsf{A} \arrow[r] \arrow[d] & \mathsf{B} \arrow[d, "f"] \\ \mathsf{C} \arrow[r, "g"] & \mathsf{D}. \end{tikzcd}$$

Denote by E *the homotopy cofiber of* f, *and write* $\mathsf{A}' := \mathrm{hofiber}\,[\mathsf{C} \to \mathsf{D} \to \mathsf{E}]$. *Then there is a canonical homotopy equivalence* $\mathsf{A} \xrightarrow{\simeq} \mathsf{A}'$.

Proof. Consider the diagram

$$\begin{tikzcd} \mathsf{A} \arrow[r] \arrow[d] & \mathsf{B} \arrow[r] \arrow[d, "f"] & * \arrow[d] \\ \mathsf{C} \arrow[r, "g"] & \mathsf{D} \arrow[r] & \mathsf{E} \end{tikzcd}$$

The left-hand square is a homotopy pullback square by assumption. The right-hand square is a homotopy pushout square, which is the same as a homotopy pullback square in the category of spectra. Thus the outer rectangle is also a homotopy pullback square, meaning that the canonical dashed arrow $\mathsf{A} \to \mathsf{A}'$ in the analogous (A.5) must be a homotopy equivalence. □

A.3. Miscellaneous

Theorem A.9 (Universal coefficient theorem)**.** *Homology case. For any ring* R *there is a natural short exact sequence*

$$0 \to H_i(X, \mathbb{Z}) \otimes R \to H_i(X, R) \to \mathrm{Tor}(H_{i-1}(X, \mathbb{Z}), R) \to 0.$$

Cohomology case. For any principal ideal domain R *and* R*-module* M *there is a natural short exact sequence*

$$0 \to \mathrm{Ext}^1_R(H_{i-1}(X, R), M) \to H^i(X, M) \to \mathrm{Hom}_R(H_i(X, R), M) \to 0.$$

We will be considering $R = \mathbb{Z}$ *or* $R = \mathbb{Z}_2$ *and mostly* $M = \mathbb{Z}_2$.

Lemma A.10. *Let* M *be a topological monoid. Assume it is grouplike, i. e.* $\pi_0(M)$ *is a group. Fix* $n \in M$. *Denote by* $\mu_n : M \to M$, $m \mapsto n \cdot m$ *left multiplication by* n. *Then* μ_n *is a homotopy equivalence.*

Proof. First, M admits inverses up to homotopy. Indeed, for any $[n] \in \pi_0(M)$ there exists $[n^{-1}] \in \pi_0(M)$ such that $[n] \cdot [n^{-1}] = [n^{-1}] \cdot [n] = [1] \in \pi_0(M)$. Here $1 \in M$ denotes the unit. In other words, there are paths $n^{-1} \cdot n \rightsquigarrow 1$ and $n \cdot n^{-1} \rightsquigarrow 1$. These translate into homotopies for the compositions $\mu_{n^{-1}} \circ \mu_n \simeq \mathrm{id}_M$ and $\mu_n \circ \mu_{n^{-1}} \simeq \mathrm{id}_M$. This makes $\mu_{n^{-1}}$ a homotopy inverse to μ_n. □

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