Münster J. of Math. **10** (2017), 59–74
DOI 10.17879/33249452641
urn:nbn:de:hbz:6-33249452996

Münster Journal of Mathematics

Equivariant correspondences and the Borel–Bott–Weil theorem

Heath Emerson and Robert Yuncken

(Communicated by Siegfried Echterhoff)

Abstract. We prove an analog of the Borel–Bott–Weil theorem in equivariant KK-theory by constructing certain canonical equivariant correspondences between minimal flag varieties G/B, with G a complex semisimple Lie group.

1. Introduction

Let G be a complex semisimple Lie group and $B \subset G$ a minimal parabolic subgroup. Let μ be a weight for G and E_μ the corresponding induced holomorphic line bundle on the flag manifold $X = G/B$. The Dolbeault cohomology group $H^*(X, E_\mu)$ with its canonical action of G, is a graded-finite-dimensional representation of G, and, more relevantly for us, of its maximal compact subgroup $K \subset G$. The Borel–Bott–Weil theorem computes this representation [4].

Bott's key observation was that there is a Weyl-group symmetry in the solution to the problem: if the weights μ and μ' are in the same orbit of the shifted Weyl group action, then $H^*(X, E_\mu)$ and $H^*(X, E_{\mu'})$ are equal, up to a shift in degree. In this paper, we will look at this symmetry from the point of view of correspondences in geometric equivariant K-theory.

The bridge between Dolbeault cohomology and K-theory is provided by index theory of elliptic operators: $H^*(G/K, E_\mu)$, as a virtual K-representation, is the same as the K-index $\mathrm{Index}_K[\bar\partial]_\mu \in \mathrm{R}(K)$, in the sense of Atiyah and Singer [1], of the Dolbeault operator twisted by E_μ. From the point of view of Kasparov theory, the class $[\bar\partial]_\mu$ is an element of the K-equivariant K-homology $\mathrm{KK}^K(G/B, \mathbb{C})$ of G/B. This $\mathrm{R}(K)$-module is acted on by the bivariant group $\mathrm{KK}^K(G/B, G/B)$, for which a topological model was developed in [6] using

Heath Emerson was supported by a Natural Science and Engineering Council of Canada (NSERC) Discovery grant. Part of this research was carried out while Robert Yuncken was supported by a PIMS post-doctoral fellowship at the University of Victoria, and part was supported by the project SINGSTAR of the Agence Nationale de la Recherche, ANR-14-CE25-0012-01.

the theory of equivariant correspondences. The correspondence theory is the main tool used in this work. We consider certain canonical correspondences $\Lambda(w)$, parameterized by the elements of the Weyl group W, compute how these correspondences act on equivariant K-homology, and relate it to the Borel–Bott–Weil theorem.

Let $\mathfrak{h}$ be a Cartan subalgebra of the Lie algebra $\mathfrak{g}$ of G and $\Gamma_W \subset \mathfrak{h}^*$ the lattice of weights. Let Δ^+ be a set of positive roots for G, which brings with it a generating set of simple reflections for the Weyl group W and a corresponding word length function $l : W \to \mathbb{N}$. Up to conjugacy, the minimal parabolic subgroup $B \subset G$ is the subgroup with Lie algebra $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the α-root space of the Lie algebra of G.

Let

$$[G/B]_\mu \in \mathrm{KK}^K(C(G/B), \mathbb{C}) =: \mathrm{K}_0^K(G/B)$$

be the class of the Dolbeault operator on G/B twisted by the K-equivariant line bundle E_μ. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$ be half the sum of the positive roots. The following theorem is essentially due to Bott.

Theorem 1.1. *In the above notation, for any weight μ of G and any $w \in W$, the identity*

$$\mathrm{Index}_K[G/B]_\mu = (-1)^{l(w)}\mathrm{Index}_K[G/B]_{w(\mu+\rho)-\rho}$$

holds in $\mathrm{R}(K) = \mathrm{KK}^K(\mathbb{C}, \mathbb{C})$.

The focus of this article is the KK-theory which lies behind Theorem 1.1. We show how to prove Theorem 1.1 using the theory [6] of equivariant correspondences. For a verification of the Weyl character formula using similar techniques, see the paper [3].

The Weyl group element w conjugates the subgroup B to another minimal parabolic subgroup B_w. The homogeneous space $G/B \cap B_w$ admits a pair of natural K-equivariant holomorphic fibrations to G/B and G/B_w. Since the latter space is K-equivariantly biholomorphic to G/B, we have realized $G/B \cap B_w$ as a holomorphic fibered space over G/B in two different ways. In fact, in each case $G/B \cap B_w$ is K-equivariantly biholomorphic to the total space of a complex vector bundle over G/B. Using the Thom class $\tau(\mathrm{q}_w)$ associated to the latter of these fibrations, we get a K-equivariant holomorphic correspondence

$$G/B \xleftarrow{\mathrm{p}_w} (G/B \cap B_w, \tau(\mathrm{q}_w)) \xrightarrow{\mathrm{q}_w} G/B_w \cong G/B$$

from G/B to itself. This yields an element of $\widehat{\mathrm{KK}}^K(G/B, G/B)$ which we denote by $\Lambda(w)$ and call the *Borel–Bott–Weil morphism with parameter $w \in W$.*

The main result of the paper is the following.

Theorem 1.2 (Borel–Bott–Weil product formula)**.** *For any weight μ and $w \in W$, the identity*

$$\Lambda(w) \otimes_{G/B} [G/B]_\mu = (-1)^{l(w)}[G/B]_{w(\mu+\rho)-\rho} \in \mathrm{KK}^K(G/B, \star)$$

holds, where $\Lambda(w)$ is the Borel–Bott–Weil morphism with parameter w.

This easily implies the analog Theorem 1.1 of the Borel–Bott–Weil theorem above.

Remark 1.3. The ring $\mathrm{KK}^K(G/B, G/B)$ is computed explicitly in [3], see also [9]. We will see that the class $\Lambda(w)$ above corresponds to the class which is referred to as the "intertwiner" I_w in [3].

We close by noting that we can replace K-equivariance by G-equivariance in Theorem 1.1, using the Baum–Connes conjecture. Classically, the Borel–Bott–Weil theorem is a statement about holomorphic (non-unitary) representations of noncompact groups. Kasparov theory does not admit such representations. Instead, equivariant Kasparov theory for noncompact groups uses unitary, but possibly infinite-dimensional representations, and almost-equivariant Fredholm operators; these are the cycles for the Kasparov representation ring $\mathrm{KK}^G(\mathbb{C}, \mathbb{C})$. There is a restriction map

$$\mathrm{KK}^G(A, B) \to \mathrm{KK}^K(A, B)$$

when $K \subset G$ is a maximal compact subgroup as above, by forgetting G-equivariance to K-equivariance on cycles. The Baum–Connes apparatus shows that this map is an isomorphism when A has the form $A = \mathcal{C}(G/B) \otimes A'$ for some G-C*-algebra A'; this follows from a theorem of Tu [10]. Since all the analytic Kasparov classes defined by us have this form, Theorems 1.1 and 1.2 have their counterparts with K replaced by G.

2. Preliminaries

2.1. **Equivariant correspondences.** The environment in which the calculations of this paper will take place is the topological model for equivariant Kasparov theory developed in [6]. We refer the reader to this article for details on the framework. All correspondences used in this paper will be smooth, which simplifies the definitions. Let K be a compact Lie group and let X and Y be smooth K-manifolds, i.e., smooth manifolds with smooth actions of K. A smooth correspondence is given by a quadruple (M, f, b, ξ) where

- M is a smooth K-manifold,
- $f : M \to Y$ is a smooth K-equivariantly K-oriented map,
- $b : M \to X$ is a smooth K-equivariant map, and
- $\xi \in \mathrm{RK}^*_{K,X}(M)$ is a smooth K-equivariant K-theory class with compact support along the fibers of b (in the terminology of [6], a K-theory class with M-compact support).

We usually use the notation

$$X \xleftarrow{b} (M, \xi) \xrightarrow{f} Y,$$

as in [5] (the origin of the theory) to denote the quadruple above.

Note that if X is compact (the case throughout in this article), then

$$\mathrm{RK}^*_{K,X}(M) = \mathrm{K}^*_K(M),$$

the ordinary, compactly supported, K-equivariant K-theory of M.

The *degree* of the correspondence is the sum of the degrees of ξ and of f.

Equivalence classes of equivariant correspondences make up the morphisms in the additive category $\widehat{\mathrm{KK}}^K$ explained in [6]; there is a natural transformation $\widehat{\mathrm{KK}}^K \to \mathrm{KK}^K$ to the usual analytic equivariant Kasparov category, inducing an *isomorphism* $\widehat{\mathrm{KK}}^K(X, Y) \to \mathrm{KK}^K(X, Y)$ if X is a *normally non-singular* K-manifold, that is, if X admits a smooth, K-equivariant embedding into a finite-dimensional representation of K. A smooth K-manifold of finite orbit type is automatically normally non-singular, and in particular, all smooth, compact K-manifolds are normally non-singular. All concrete K-manifolds we meet in this paper are normally non-singular.

We generally operate in the category $\widehat{\mathrm{KK}}^K$ in this paper.

For any pair of K-spaces X and Y, $\widehat{\mathrm{KK}}^K_*(X, Y)$ denotes the abelian group of equivalence classes of equivariant correspondences from X to Y, graded by parity of degree.

Two standard examples of $\widehat{\mathrm{KK}}^K$-classes are important; to fix notation, we recall them.

Example 2.2. If $b : Y \to X$ is a *proper* K-equivariant map, we define

$$b^* := \big[X \xleftarrow{b} (Y, \mathbf{1}_Y) \xrightarrow{\mathrm{id}} Y\big],$$

where $\mathbf{1}_Y$ is the class of the trivial line bundle $Y \times \mathbb{C}$, the unit in $\mathrm{RK}^*_{K,X}(Y) = \mathrm{RK}^*_K(Y)$.

Example 2.3. If Φ is an equivariantly K-oriented smooth map from X to Y, where X and Y are smooth K-manifolds, we define the *wrong-way class* of Φ as

$$\Phi_! := \big[X \xleftarrow{\mathrm{id}} (X, \mathbf{1}_X) \xrightarrow{\Phi} Y\big],$$

where $\mathbf{1}_X$ is the class of the trivial line bundle $E \times \mathbb{C}$ in $\mathrm{RK}^*_K(X)$.

By a *complex K-manifold* we shall mean a smooth complex manifold X equipped with a holomorphic action of K. The tangent bundle $\mathrm{T}X$ has a canonical K-equivariant complex structure and a corresponding K-equivariant K-orientation. This supplies an equivariant K-orientation on the map from X to a point. The corresponding wrong-way class is called the *(topological) fundamental class of* X, and denoted by $[X]$. Its image in $\mathrm{KK}^K_0(C_0(X), \mathbb{C})$ is the class of the K-equivariant Dolbeault operator on X.

Next, let M_1, M_2, Y be complex K-manifolds. Assume that both M_1 and M_2 are normally non-singular K-manifolds.

Two smooth maps $f_1 : M_1 \to Y$ and $b_2 : M_2 \to Y$ are *transverse* if for every pair of points $m_1 \in M_1$ and $m_2 \in M_2$ with $f_1(m_1) = b_2(m_2)$, the map

$$T_{m_1} M_1 \oplus T_{m_2} M_2 \to T_{f_1(m_1)} Y, \quad (\xi_1, \xi_2) \mapsto D_{m_1} f_1(\xi_1) + D_{m_2} b_2(\xi_2)$$

is surjective. It is shown in [6] that when transversality holds, the fiber product

$$M_1 \times_Y M_2 := \big\{(m_1, m_2) \mid f_1(m_1) = b_2(m_2)\big\}$$

is itself a smooth K-manifold (of finite orbit type) and the projection

$$\mathrm{pr}_2 : M_1 \times_Y M_2 \to M_2$$

inherits a canonical equivariant K-orientation from the K-orientation on f_1.

If f_1 and b_1 are *holomorphic* maps, the fiber product $M_1 \times_Y M_2$ will be a complex manifold, and the projection pr_2 will be a holomorphic map; the corresponding K-orientation agrees with the one described in the previous paragraph.

2.4. **Complex semisimple Lie groups.** Here we review some standard structure theory for semisimple groups and fix notation for the remainder of the paper. For details, see, for example, [8].

Let G be a complex connected semisimple Lie group and $\mathfrak{g}$ its Lie algebra. Denote by $B(\,)$ its Killing form. Let θ be a Cartan involution on $\mathfrak{g}$, so that

$$\langle v, w\rangle := -B(\theta(v), w), \quad v, w \in \mathfrak{g}$$

is a positive definite inner product on $\mathfrak{g}$; the archetypal example is the operation of negative-conjugate-transpose on $\mathfrak{sl}_n(\mathbb{C})$. The $+1$-eigenspace of θ is the Lie algebra $\mathfrak{k}$ of a maximal compact subgroup K of G.

Fix $\mathfrak{h}$, a θ-stable Cartan subalgebra. Let $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{k}$, which is the Lie algebra of a maximal torus T in K. We have $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$, where $\mathfrak{a} = i\mathfrak{t}$, and we let A denote the subgroup of G with Lie algebra $\mathfrak{a}$.

The set of roots will be denoted by Δ, with $\mathfrak{g}_\alpha$ denoting the root space of $\alpha \in \Delta$. We fix a choice of positive roots Δ^+, and recall that every positive root is a nonnegative integral combination of simple roots. The lattice of weights will be denoted by Γ_W, and the dominant weights are those $\lambda \in \Gamma_W$ for which $\langle \lambda, \alpha\rangle \geq 0$ for every positive root α. We will frequently abuse notation by blurring the distinction between a weight $\mu \in \Gamma_W$, the corresponding representation of T, and the corresponding holomorphic representation of $H = T \cdot A$.

The Weyl group is $W = N_G(H)/Z_G(H)$. We will frequently identify elements $w \in W$ with a lift to an element of $N_G(H) \subseteq G$, at least when the choice of lift makes no difference. The usual action of the Weyl group on weights will be denoted by $\mu \mapsto w(\mu)$. Let $\rho := \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ be the half-sum of positive roots. We will often refer to the *shifted action* of the Weyl group, which is the action

$$w : \lambda \mapsto w(\lambda + \rho) - \rho.$$

We fix the standard Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the nilpotent subalgebra $\mathfrak{n} := \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. The associated subgroups are denoted by B and N. For each element w of the Weyl group, there are conjugate subgroups

$$B_w := wBw^{-1}, \quad N_w := wNw^{-1}$$

with corresponding Lie algebras $\mathfrak{b}_w$ and $\mathfrak{n}_w$. We also define the Lie algebra $\bar{\mathfrak{n}} := \theta\mathfrak{n} = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$, as well as its conjugates $\bar{\mathfrak{n}}_w := \mathrm{Ad}(w)\bar{\mathfrak{n}}$ for each $w \in W$.

The flag variety of G is the complex homogeneous space G/B. It is K-equivariantly diffeomorphic to K/T via the map

$$\iota : K/T \xrightarrow{\cong} G/B, \quad kT \mapsto kB.$$

However, we shall try to distinguish the two spaces as much as possible. The difference is technical but important: G/B, having a natural complex structure, is canonically K-oriented, while K/T only inherits a K-orientation once it is identified with G/B. Moreover, for any $w \in W$, there is a K-equivariant diffeomorphism

$$\iota_w : K/T \xrightarrow{\cong} G/B_w, \quad kT \mapsto kB_w,$$

each inducing a different K-orientation on K/T. This technicality is of course absolutely central to what follows.

3. The Borel–Bott–Weil theorem

3.1. **Twisted fundamental classes.** Let μ be a weight of G. As mentioned above, it corresponds to a holomorphic representation of H, and one can extend it to a holomorphic character of B which is trivial on N. We denote the one-dimensional representation space by $\mathbb{C}_\mu$.

We shall use the notation E_μ throughout to denote the induced G-equivariant line bundle

$$E_\mu := G \times_B \mathbb{C}_\mu.$$

We also have $E_\mu \cong K \times_T \mathbb{C}_\mu$ by restriction.

Recall (see, e.g., Antony Wassermann's Frobenius reciprocity theorem [2, Thm. 20.5.5]) that $\mathrm{K}^*_K(K/T)$ is isomorphic to $\mathrm{K}^*_T(\mathbb{C}) = R(T)$, the representation ring of T, as a $\mathbb{Z}$-module. The representation ring is just $\mathbb{Z}[\Gamma_W]$, the group ring of the weight lattice, and the isomorphism is given by induction:

$$\mathrm{Ind}_T^K : R(T) \xrightarrow{\cong} \mathrm{K}^*_K(K/T), \quad [\mu] \mapsto [E_\mu].$$

Definition 3.2. Given $\mu \in \Gamma_W$, we define the *μ-twisting class* to be the element $[[\mu]] \in \widehat{\mathrm{KK}}^K(G/B, G/B)$ given by the following correspondence:

$$G/B \xleftarrow{\mathrm{id}} (G/B, [E_\mu]) \xrightarrow{\mathrm{id}} G/B.$$

The *μ-twisted fundamental class of* G/B, denoted by $[G/B]_\mu$, is the class of the K-equivariant correspondence

$$G/B \xleftarrow{\mathrm{id}} (G/B, [E_\mu]) \to \star$$

in $\widehat{\mathrm{KK}}^K(G/B, \star)$.

Thus, $[G/B]_\mu = [[\mu]] \otimes_{G/B} [G/B]$, where $[G/B] := [G/B]_0$ is the (untwisted) fundamental class of G/B. The reason for the terminology is that $[G/B]$ institutes a duality isomorphism (see [6])

$$\widehat{\mathrm{KK}}^K_*(G/B \times X, Y) \cong \widehat{\mathrm{KK}}^K_*(X, G/B \times Y)$$

valid for arbitrary K-spaces X and Y. For example if $X = Y = \star$ then duality gives an isomorphism

$$\mathrm{R}(T) = \mathbb{Z}[\Gamma_W] \cong \widehat{\mathrm{KK}}^K(G/B, \star).$$

This duality can easily be verified to send the point mass at a weight $\mu \in \Gamma_W$ to the class $[G/B]_\mu$.

3.3. **Borel–Bott–Weil correspondences.** Let w be an element of the Weyl group $W = N_G(H)/Z_G(H)$. Recall that the subgroup $B_w := wBw^{-1}$ is independent of the choice of lift of w to $N_G(H) \subseteq G$. It is another minimal parabolic subgroup of G.

Consider the homogeneous space $G/(B \cap B_w)$. This admits two G-equivariant fibrations, given by the natural maps

$$p_w : (G/B \cap B_w) \to G/B, \quad \mathrm{q}_w : (G/B \cap B_w) \to G/B_w.$$

Viewing $G/(B \cap B_w)$ as a K-space by restriction, both of these fibrations can be realized as K-equivariant vector bundle projections, as we now describe.

Recall that we define $N_w := wNw^{-1}$, $\bar{N}_w := w\bar{N}w^{-1}$, with Lie algebras $\mathfrak{n}_w$ and $\bar{\mathfrak{n}}_w$ respectively. Then $\mathfrak{n} = (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) \oplus (\mathfrak{n} \cap \mathfrak{n}_w)$ is a decomposition of $\mathfrak{n}$ into Lie subalgebras. Since N is a connected simply-connected nilpotent Lie group, there is a corresponding factorization $N = (N \cap \bar{N}_w)(N \cap N_w)$.

Lemma 3.4. *Let $w \in W$. One can define a K-equivariant diffeomorphism*

$$\varphi_w : |K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)| \xrightarrow{\cong} G/(B \cap B_w)$$

by the formula $\varphi_w : [k, X] \mapsto k\exp(X).(B \cap B_w)$ such that the diagram

(1)
$$\begin{array}{ccc} K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) & \xrightarrow[\cong]{\varphi_w} & G/(B \cap B_w) \\ {\scriptstyle \pi_w}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{p}_w} \\ K/T & \xrightarrow[\cong]{\qquad\qquad} & G/B \end{array}$$

commutes. Moreover, φ_w is fiberwise holomorphic (with respect to the fibrations π_w and p_w).

In other words, the K-equivariant fibrations $K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w) \to K/T$ and $G/(B \cap B_w) \to G/B$ are equivalent in the category of K-equivariant fibrations with holomorphic fibers.

Proof. To see that the map φ_w is well-defined we compute, for any $k \in K$, $t \in T$, $X \in \mathfrak{n} \cap \bar{\mathfrak{n}}_w$:

$$\begin{aligned} \varphi_w([kt, \mathrm{Ad}(t^{-1})X]) &= kt.t^{-1}\exp(X)t(B \cap B_w) \\ &= k\exp(X).(B \cap B_w) = \varphi_w([k, X]). \end{aligned}$$

Next we show surjectivity. Let $g \in G$ be arbitrary. There is a decomposition

$$G = KNA = K(N \cap \bar{N}_w)(N \cap N_w)A,$$

and we decompose g as $g = kn_1n_2a$ accordingly. Since $(N \cap N_w)A \subseteq B \cap B_w$, we have $\varphi_w([k, \log(n_1)]) = g(B \cap B_w)$.

Next suppose $[k, X]$ and $[k', X'] \in K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)$ have the same image under φ_w. Since $B \cap B_w = T \cdot (N \cap N_w)A$, there exist $t \in T$ and $n_2a \in (N \cap N_w)A$ such that

$$k' \exp(X') = k \exp(X)tn_2a = kt \exp(\mathrm{Ad}(t^{-1}X))n_2a.$$

By the uniqueness of the $K(N \cap \bar{N}_w)(N \cap N_w)A$-decomposition, we have $k' = kt$ and $X' = \mathrm{Ad}(t^{-1})X$, which is to say $[k', X'] = [k, X]$.

Next we show that φ_w is a diffeomorphism. By K-equivariance it suffices to show that it is a local diffeomorphism at each $[e, X]$ where $X \in \mathfrak{n} \cap \bar{\mathfrak{n}}_w$ and $e \in K$ is the identity. The derivative of the diagram (1) at $[e, X]$ is

$$\begin{array}{ccc} T_{[e,X]}\big(K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)\big) & \xrightarrow{D\varphi_w} & \mathfrak{g}/(\mathfrak{b} \cap \mathfrak{b}_w) \\ {\scriptstyle D\pi_w}\downarrow & & \downarrow{\scriptstyle D\mathrm{p}_w} \\ \mathfrak{k}/\mathfrak{t} & \xrightarrow[\cong]{} & \mathfrak{g}/\mathfrak{b}. \end{array}$$

The left and bottom maps are surjective. But also, since the exponential on $\mathfrak{n} \cap \bar{\mathfrak{n}}_w$ is a diffeomorphism onto its image, $D\varphi_w$ maps the vertical tangent space $\mathfrak{n} \cap \bar{\mathfrak{n}}_w \subset T_{[e,X]}(K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w))$ onto $(\mathfrak{n} \cap \bar{\mathfrak{n}}_w)/(\mathfrak{b} \cap \mathfrak{b}_w) = \ker(D\mathrm{p}_w)$. Therefore $D\varphi_w$ is surjective.

That the map is K-equivariant is straight-forward, as is the commutativity of the diagram of bundle maps. Fiberwise holomorphicity follows from the holomorphicity of the exponential map. □

Remark 3.5. There is an alternative realization of the space $G/(B \cap B_w)$ as a K-equivariant vector bundle via the diagram

$$\begin{array}{ccc} K \times_T (\mathfrak{n}_w \cap \bar{\mathfrak{n}}) & \xrightarrow[\cong]{\varphi'_w} & G/(B \cap B_w) \\ {\scriptstyle \pi'_w}\downarrow & & \downarrow{\scriptstyle \mathrm{q}_w} \\ K/T & \xrightarrow[\cong]{} & G/B_w, \end{array} \tag{2}$$

where the top map has essentially the same defining formula:

$$\varphi'_w : [k, X'] \mapsto k \exp(X').(B \cap B_w).$$

The proof is basically identical. Thus, the holomorphic manifold $G/(B \cap B_w)$ admits two distinct structures as a complex K-equivariant vector bundle over K/T, via the maps π_w and π'_w. This point will be of crucial importance later.

Definition 3.6. Using the diagrams (1) and (2), we may consider the zero sections of the two complex vector bundles $K \times_T (\mathfrak{n}_w \cap \bar{\mathfrak{n}})$ and $K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)$ as K-equivariant maps $\zeta_w : G/B \to G/(B \cap B_w)$ and $\zeta'_w : G/B_w \to G/(B \cap B_w)$. They are given simply by

$$\zeta_w : kB \mapsto k(B \cap B_w), \quad \zeta'_w : kB_w \mapsto k(B \cap B_w)$$

for $k \in K$, where we stress that in applying these formulas, we are obliged to choose coset representatives k belonging to the compact subgroup K.

The importance of realizing $G/(B \cap B_w)$ as a complex K-vector bundle over G/B is that there is a Thom class

$$\tau(\mathrm{p}_w) \in \mathrm{K}^*_K(G/(B \cap B_w)),$$

obtained by pushing forward the Thom class from $\mathrm{K}^*(|K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)|)$.

Note that this Thom class is dependent upon the fibration map

$$\mathrm{p}_w : G/(B \cap B_w) \to G/B.$$

The alternative fibration $\mathrm{q}_w : G/(B \cap B_w) \to G/B_w$ defines a different class $\tau(\mathrm{q}_w)$, pushed forward from $\mathrm{K}^*_K(|K \times_T (\mathfrak{n}_w \cap \bar{\mathfrak{n}})|)$.

Let $w \in W$. The spaces G/B_w and G/B are G-equivariantly diffeomorphic, even biholomorphic, via the right multiplication map $R_w : g.(wBw^{-1}) \mapsto gw.B$. We can now define one of our main objects of study.

Definition 3.7. The *Borel–Bott–Weil morphism* $\Lambda(w) \in \widehat{\mathrm{KK}}^K(G/B, G/B)$ *with parameter* $w \in W$ is the class of the K-equivariant holomorphic correspondence

$$G/B \xleftarrow{\mathrm{p}_w} (G/B \cap B_w, \tau(\mathrm{q}_w)) \xrightarrow{\mathrm{q}_w} G/B_w \xrightarrow[\simeq]{R_w} G/B.$$

Example 3.8. If $w = e$ is the identity element, then $B \cap B_e = B$, $\mathfrak{n} \cap \bar{\mathfrak{n}}_w$ is the zero Lie subalgebra, inducing to the zero vector bundle on K/T, and $\tau(\mathrm{p}_e)$ is the Thom class [1] of the zero vector bundle. Thus $\Lambda(e) = 1$ is represented by the correspondence

$$G/B \xleftarrow{\mathrm{id}} (G/B, [1]) \xrightarrow{\mathrm{id}} G/B$$

which is the identity correspondence. Thus $\Lambda(e) = 1 \in \widehat{\mathrm{KK}}^K(G/B, G/B)$.

3.9. **Product structure.** For $w \in W$, we denote by $\iota_w : K/T \xrightarrow{\simeq} G/B_w$ the K-equivariant diffeomorphism defined by $kT \mapsto kB_w$ for $k \in K$.

Each of these maps identifies K/T with a complex manifold, with K acting by a holomorphic action, and this complex structure induces a corresponding K-equivariant Spin^c-structure on K/T. All of these Spin^c-structures will be different. To keep track of them, we use the complex picture whenever possible.

Definition 3.10. We denote by $I_w : G/B \to G/B_w$ the (non-holomorphic) K-equivariant diffeomorphism defined by the commuting diagram

$$\begin{array}{ccc} G/B & \xrightarrow{I_w} & G/B_w \\ \iota_e \uparrow \simeq & & \iota_w \uparrow \simeq \\ K/T & \xrightarrow{\mathrm{id}} & K/T. \end{array}$$

Thus, I_w corresponds to the identity map on K/T but with an unusual K-orientation.

If $w \in W$, then right translation $R_w : G/B_w \to G/B$ is a K-equivariant map yielding an element $R_w^* \in \widehat{\mathrm{KK}}^K(G/B, G/B_w)$. The following proposition asserts, roughly, that after twisting R_w by the change of equivariant K-orientation induced by I_w, we get exactly the Borel–Bott–Weil correspondence $\Lambda(w)$.

Proposition 3.11. *The identity*

$$\Lambda(w) = (I_w^{-1} \circ R_w^{-1})^*$$

holds in $\widehat{\mathrm{KK}}^K(G/B, G/B)$.

Proof. The map R_w is biholomorphic, so $(R_w^{-1})^* = R_{w!}$. Using the realization of $G/(B \cap B_w)$ as a K-equivariant vector bundle over G/B_w, we can perform a Thom modification to get

$$(I_w^{-1} \circ R_w^{-1})^* = \big[G/B \xleftarrow{I_w^{-1}} G/B_w \xrightarrow{R_w} G/B\big] \tag{3}$$
$$= \big[G/B \xleftarrow{I_w^{-1}\circ \mathrm{q}_w} (G/(B \cap B_w), \tau(\mathrm{q}_w)) \xrightarrow{R_w \circ \mathrm{q}_w} G/B\big].$$

We claim that this is equivalent, via a bordism, to the correspondence

$$\Lambda(w) = \big[G/B \xleftarrow{\mathrm{p}_w} (G/(B \cap B_w), \tau(\mathrm{q}_w)) \xrightarrow{R_w \circ \mathrm{q}_w} G/B\big]. \tag{4}$$

To see this, consider first the linear retraction γ_t of

$$G/(B \cap B_w) \cong K \times_T (\mathfrak{n} \cap \overline{\mathfrak{n}}_w)$$

onto its zero section:

$$\gamma_t : G/(B \cap B_w) \xrightarrow{\varphi_w^{-1}} K \times_T (\mathfrak{n} \cap \overline{\mathfrak{n}}_w) \to K \times_T (\mathfrak{n} \cap \overline{\mathfrak{n}}_w) \xrightarrow{\varphi_w} G/(B \cap B_w),$$
$$(k, X) \mapsto (k, tX).$$

We use this to define the smooth K-equivariant homotopy

$$h_t := I_w^{-1} \circ \mathrm{q}_w \circ \gamma_t : G/(B \cap B_w) \to G/B$$

between $h_0 = \mathrm{p}_w$ and $h_1 = I_w^{-1} \circ \mathrm{q}_w$.

We want to show that this homotopy yields a bordism of correspondences

$$\big[G/B \xleftarrow{h} (G/(B \cap B_w) \times [0,1], \mathrm{pr}_1^* \tau(\mathrm{q}_w)) \xrightarrow{R_w \circ \mathrm{q}_w \circ \mathrm{pr}_1} G/B\big] \tag{5}$$

between (3) and (4). Here pr_1 denotes the projection

$$\mathrm{pr}_1 : G/(B \cap B_w) \times [0,1] \to G/(B \cap B_w),$$

i.e., the right-hand map and the K-theory class in (5) are constant in t. To verify that (5) is a well-defined correspondence we need to check that the K-theory class $\mathrm{pr}_1^* \tau(\mathrm{q}_w)$ has compact support along the fibers of h.

Let $kB \in G/B$; note that we may take $k \in K$. Suppose $g(B \cap B_w) \in \mathrm{supp}(\tau(\mathrm{q}_w)) \cap h_t^{-1}(kB)$. The support of the Thom class $\tau(\mathrm{q}_w)$ is the zero section $\zeta_w'(G/B_w) = K.(B \cap B_w) \subseteq G/B \cap B_w$, so we may take $g = k' \in K$. Then

$$kB = h_t(k'(B \cap B_w)) = I_w \circ \mathrm{q}_w \circ \gamma_t(k'(B \cap B_w)) = I_w(k'B_w) = k'B.$$

Therefore, the support of $\mathrm{pr}_1^*\tau(\mathrm{q}_w)$ in the fiber $h^{-1}(kB)$ is $\{kB\}\times[0,1]$. Hence (5) is indeed a bordism between the correspondences (3) and (4). This completes the proof. □

Corollary 3.12. *The map $w \mapsto \Lambda(w)$ is a group homomorphism from the Weyl group into the invertible elements of the ring $\widehat{\mathrm{KK}}^K(G/B, G/B)$.*

Proof. One just needs to check that $R_{w_1} \circ I_{w_1} \circ R_{w_2} \circ I_{w_2} = R_{w_1w_2} \circ I_{w_1w_2}$. This is immediate if one represents elements of G/B as kB with $k \in K$. □

3.13. **Commutation relations in $\widehat{\mathbf{KK}}^{\boldsymbol{K}}(\boldsymbol{G/B, G/B})$.** We begin with some generalities on pullbacks of induced bundles.

Let $H_2 \leq H_1 \leq G$ be a nested sequence of closed Lie subgroups, and let V be a vector space with a representation of H_1. If $\mathrm{p} : G/H_2 \twoheadrightarrow G/H_1$ denotes the canonical fibration map, then there is an equivariant bundle isomorphism

$$\mathrm{p}^*(G \times_{H_1} V) \cong G \times_{H_2} V, \tag{6}$$

given by the following pullback diagram:

$$\begin{array}{ccc} G\times_{H_2} V & \longrightarrow & G\times_{H_1} V \\ \downarrow & & \downarrow \\ G/H_2 & \xrightarrow{p} & G/H_1, \end{array} \qquad \begin{array}{ccc} [g,v] & \longmapsto & [g,v] \\ \downarrow & & \downarrow \\ gH_2 & \longmapsto & gH_1. \end{array}$$

Recall that each weight μ defines a one-dimensional holomorphic representation of B. It will be convenient to use an explicit notation for this in the next few paragraphs, so we denote it by $\sigma_\mu : B \to \mathrm{End}(\mathbb{C}_\mu)$. We shall denote by σ_μ^w the representation of B_w defined by conjugating by $w \in W$:

$$\sigma_\mu^w(wbw^{-1}) := \sigma_\mu(b).$$

Then there is a G-equivariant bundle isomorphism

$$R_w^*(G \times_B \mathbb{C}_\mu) = G \times_{B_w} \mathbb{C}_\mu,$$

where the representation of B_w on the right-hand side is σ_μ^w. The appropriate pullback diagram is:

$$\begin{array}{ccc} G\times_{B_w} \mathbb{C}_{w(\mu)} & \longrightarrow & G\times_{B} \mathbb{C}_\mu \\ \downarrow & & \downarrow \\ G/B_w & \xrightarrow{R_w} & G/B, \end{array} \qquad \begin{array}{ccc} [g,v] & \longmapsto & [gw,v] \\ \downarrow & & \downarrow \\ gB_w & \longmapsto & gwB. \end{array}$$

Lemma 3.14. *For any $\mu \in \Gamma_W$ and $w \in W$ we have $\mathrm{q}_w^* R_w^* E_\mu \cong \mathrm{p}_w^* E_{w(\mu)}$ as G-equivariant complex line bundles over $G/(B \cap B_w)$.*

Proof. As described above, $\mathrm{q}_w^* R_w^* E_\mu \cong \mathrm{q}_w^*(G \times_{B_w} \mathbb{C}_\mu)$. Restricting the conjugated representation σ_μ^w to $B \cap B_w$ yields a representation which is trivial on $N \cap N_w$ and given by $e^{w(\mu)}$ on $T \cdot A$. Thus (6) gives $\mathrm{q}_w^* R_w^* E_\mu \cong G \times_{B \cap B_w} \mathbb{C}_{w(\mu)}$. This is isomorphic to $\mathrm{p}_w^* E_{w(\mu)}$ by (6) again. □

Proposition 3.15. *For any $\mu \in \Gamma_W$ and $w \in W$,*

$$\Lambda(w) \otimes_{G/B} [[\mu]] = [[w(\mu)]] \otimes_{G/B} \Lambda(w).$$

Proof. We calculate

$$\begin{aligned}
&\Lambda(w) \otimes_{G/B} [[\mu]] \\
&= G/B \xleftarrow{\mathrm{p}_w} (G/B \cap B_w, \tau(\mathrm{q}_w)) \xrightarrow{R_w \circ \mathrm{q}_w} G/B \xleftarrow{\mathrm{id}} (G/B, [E_\mu]) \xrightarrow{\mathrm{id}} G/B \\
&= G/B \xleftarrow{\mathrm{p}_w} (G/B \cap B_w, \tau(\mathrm{q}_w).\mathrm{q}_w^* R_w^*[E_\mu]) \xrightarrow{R_w \circ \mathrm{q}_w} G/B \\
&= G/B \xleftarrow{\mathrm{p}_w} (G/B \cap B_w, \tau(\mathrm{q}_w).\mathrm{p}_w^*[E_{w(\mu)}]) \xrightarrow{R_w \circ \mathrm{q}_w} G/B \\
&= G/B \xleftarrow{\mathrm{id}} (G/B, [E_{w(\mu]}]) \xrightarrow{\mathrm{id}} G/B \xleftarrow{\mathrm{p}_w} (G/B \cap B_w, \tau(\mathrm{q}_w)) \xrightarrow{R_w \circ \mathrm{q}_w} G/B \\
&= [[w(\mu)]] \otimes_{G/B} \Lambda(w).
\end{aligned}$$

This completes the proof. ☐

3.16. **Comparing Thom classes.** We begin this section by comparing the two Thom classes $\tau(\mathrm{p}_w)$ and $\tau(\mathrm{q}_w)$ on the space $G/B \cap B_w$ (see Section 3.3). It will suffice to consider the case where w is the reflection in a simple root α. In that case we have $\mathfrak{n} \cap \bar{\mathfrak{n}}_w = \mathfrak{g}_\alpha$ and $\mathfrak{n}_w \cap \bar{\mathfrak{n}} = \mathfrak{g}_{-\alpha}$.

Recall that $\tau(\mathrm{p}_w)$ is the pushforward of the Thom class of

$$|K \times_T (\mathfrak{n} \cap \bar{\mathfrak{n}}_w)| = |K \times_T \mathfrak{g}_\alpha|$$

via the bundle isomorphism of (1). Taking advantage of the complex structure on the fibers, the corresponding spinor bundle is $K \times_T \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_\alpha$. There is an $\mathrm{Ad}(T)$-invariant inner product on $\mathfrak{g}_\alpha$ via the Killing form. Letting λ_X denote the exterior product by $X \in \mathfrak{g}_\alpha$, we have a Clifford algebra representation

$$\mathsf{c} : \mathfrak{g}_\alpha \to \mathrm{End}(\textstyle\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_\alpha), \quad \mathsf{c}(X) := \lambda_X - \lambda_X^*.$$

The Thom class of $|K \times_T \mathfrak{g}_\alpha|$ is the pullback of the spinor bundle along the bundle projection $\pi_w : K \times_T \mathfrak{g}_\alpha \to K/T$, equipped with the bundle endomorphism which at each point is the Clifford representation of that point.

Since $K \times_T \bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_\alpha \cong \mathbb{C}_0 \oplus \mathbb{C}_\alpha$, we can identify the spinor bundle over K/T with $G \times_B (\mathbb{C}_0 \oplus \mathbb{C}_\alpha)$. The space $\mathbb{C}_\alpha$ here identifies naturally with $\mathfrak{g}_\alpha$ as a T-space, but not as a B-space: we have made an arbitrary extension to a B-representation.

Using equation (6), the push-forward of the Thom class by φ_w is then

$$\tau(\mathrm{p}_w) = (G \times_{B \cap B_w} (\mathbb{C}_0 \oplus \mathbb{C}_\alpha),\ \mathsf{C}_w),$$

where C_w is the bundle endomorphism defined at each point of $G/(B \cap B_w)$ by

$$\mathsf{C}_w(k \exp(X)(B \cap B_w)) = \mathsf{c}(X) \quad \text{for } k \in K,\ X \in \mathfrak{g}_\alpha.$$

A similar calculation shows that the Thom class $\tau(\mathrm{q}_w)$ associated to the other projection is

$$\tau(\mathrm{q}_w) = (G \times_{B \cap B_w} (\mathbb{C}_0 \oplus \mathbb{C}_{-\alpha}),\ \mathsf{C}'_w),$$

where

$$\mathsf{C}'_w(k\exp(X')(B\cap B_w)) = \mathsf{c}(X') \quad \text{for } k\in K,\ X'\in\mathfrak{g}_{-\alpha}.$$

To compare these two classes, we define a homotopy. For $t\in[0,1]$, define a map

$$\begin{aligned}\gamma_t : G/(B\cap B_w) &\to G/(B\cap B_w),\\ k\exp(X)(B\cap B_w) &\mapsto k\exp(tX)(B\cap B_w) \quad \text{for } k\in K,\ X\in\mathfrak{g}_\alpha.\end{aligned}$$

This is just the pushforward by φ_w of the retraction of the bundle $K\times_T\mathfrak{g}_\alpha$ to the zero section.

Consider the smooth family Φ_t of bundle endomorphisms of the vector bundle $G\times_{B\cap B_w}(\mathbb{C}_0\oplus\mathbb{C}_{-\alpha})$ defined by

$$\Phi_t(z) := \mathsf{C}'(\gamma_t(x)), \quad z\in G/B\cap B_w.$$

Since γ_0 has image the zero section, Φ_0 is the zero endomorphism. By smoothness and the compactness of G/B, the family

$$\Psi_t := \frac{1}{t}\Phi_t \quad (t\neq 0)$$

has a well-defined limit at $t=0$, which we denote by Ψ_0.

Lemma 3.17. *Let θ denote the Cartan involution on $\mathfrak{g}$. At a point $k\exp(X)(B\cap B_w)$ of $G/(B\cap B_w)$, where $k\in K$ and $X\in\mathfrak{g}_\alpha$, the limit $\Psi_0(k\exp(X)B\cap B_w)$ is the endomorphism of the fiber $\bigwedge^\bullet_{\mathbb{C}}\mathfrak{g}_{-\alpha}$ defined by*

$$\Psi_0(k\exp(X)B\cap B_w) = \mathsf{c}(-\theta X).$$

Proof. We have $\gamma_t(k\exp(X)(B\cap B_w)) = k\exp(tX)(B\cap B_w)$. By the Campbell–Baker–Hausdorff formula,

$$\exp(tX) = \exp(t(X+\theta X))\exp(-t\theta X)\exp(o(t)).$$

Since $\exp(t(X+\theta X))\in K$, we have that Ψ_t acts on the fiber at $k\exp(X)B\cap B_w$ by

$$\Psi_t(k\exp(X)B\cap B_w) = \frac{1}{t}\mathsf{c}(-t\theta X + o(t)),$$

which has limit $\mathsf{c}(-\theta X)$ as $t\to 0$. □

In the next lemma, we fix identifications of $\mathfrak{g}_{\pm\alpha}$ with $\mathbb{C}$ by identifying some arbitrary unit vector $Y\in\mathfrak{g}_{-\alpha}$ with 1, and likewise with $\theta Y\in\mathfrak{g}_\alpha$. Ultimately the choice of this Y makes no difference.

Lemma 3.18. *Fix $Y\in\mathfrak{g}_{-\alpha}$ with $\|Y\|=1$. Define a grading-reversing map $\beta:\bigwedge^\bullet_{\mathbb{C}}\mathfrak{g}_{-\alpha}\to\bigwedge^\bullet_{\mathbb{C}}\mathfrak{g}_\alpha$ by*

$$\begin{aligned}\beta &: \omega\mapsto\omega.\theta Y && \text{for } \omega\in{\textstyle\bigwedge^0_{\mathbb{C}}}\,\mathfrak{g}_{-\alpha}=\mathbb{C},\\ \beta &: X'\mapsto\langle Y,X'\rangle && \text{for } X'\in{\textstyle\bigwedge^1_{\mathbb{C}}}\,\mathfrak{g}_{-\alpha}=\mathfrak{g}_{-\alpha}.\end{aligned}$$

Then for any $X\in\mathfrak{g}_\alpha$,

$$\beta^{-1}\mathsf{c}(X)\beta = \mathsf{c}(-\theta X).$$

Remark 3.19. Equivalently, $\beta = \theta \circ \circledast$, where $\circledast$ is the (anti-linear) Hodge $*$-operator on $\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha}$.

Proof. We calculate

$$\begin{aligned} &\beta^{-1}\lambda_X\beta : \omega \mapsto 0 && \text{for } \omega \in \mathbb{C}, \\ &\beta^{-1}\lambda_X\beta : \omega Y \overset{\beta}{\longmapsto} \omega \overset{\lambda_X}{\longmapsto} \omega X \overset{\beta^{-1}}{\longmapsto} \langle \theta Y, \omega X\rangle && \text{for } \omega Y \in \mathfrak{g}_{-\alpha} \end{aligned}$$

and

$$\begin{aligned} &\lambda_{\theta X}^* : \omega \mapsto 0 && \text{for } \omega \in \mathbb{C}, \\ &\lambda_{\theta X}^* : \omega Y \mapsto \langle \theta X, \omega Y\rangle && \text{for } \omega Y \in \mathfrak{g}_{-\alpha}. \end{aligned}$$

These maps are equal since θ is anti-unitary. Also $\beta^{-1}\lambda_X^*\beta = \lambda_{\theta X}$, by the unitarity of β. The result now follows from the definition $\mathsf{c}(X) := \lambda_X - \lambda_X^*$. ☐

The map β is not T-equivariant—it alters the weights, since it maps $\mathfrak{g}_{-\alpha}$ to $\mathbb{C}_0$ and $\mathbb{C}_0$ to $\mathfrak{g}_\alpha$. But if we alter it by defining

$$\beta' : \textstyle\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha} \to (\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}) \otimes \mathfrak{g}_{-\alpha}, \quad Z \mapsto \beta Z \otimes Y,$$

then it is weight-preserving, and hence T-equivariant. It induces a grading-reversing bundle isomorphism

$$\begin{aligned} \mathrm{id} \times_{B\cap B_w} \beta' : G \times_{B\cap B_w} \textstyle\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{-\alpha} &\to G \times_{B\cap B_w} ((\textstyle\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}) \otimes \mathfrak{g}_{-\alpha}) \\ &\cong (G \times_{B\cap B_w} \textstyle\bigwedge_{\mathbb{C}}^{\bullet} \mathfrak{g}_{\alpha}) \otimes_{G/B\cap B_w} \mathrm{p}^* E_{-\alpha}, \end{aligned}$$

which intertwines the bundle endomorphisms Ψ_0 and $\mathsf{C}_w \otimes \mathrm{id}$. Combining this with the fact that $\Psi_1 = \mathsf{C}_w$, we have proven the following fact.

Proposition 3.20. *If $w \in W$ is the reflection in the simple root α, then $\tau(\mathrm{q}_w) = -\tau(\mathrm{p}_w) \otimes \mathrm{p}_w^*[E_{-\alpha}]$ in $\mathrm{K}_K^*(G/(B \cap B_w))$.*

Remark 3.21. There is a more general formula: for any $w \in W$,

$$\tau(\mathrm{q}_w) = (-1)^{l(w)} \tau(\mathrm{p}_w) \otimes \mathrm{p}_w^*[E_{w(\rho)-\rho}],$$

where ρ is the half-sum of the positive roots. This can be proven along the same lines as above with significantly more work, or deduced from results to follow. We shall not need it.

3.22. **The Borel–Bott–Weil theorem: Action on K-homology and indices.**

Proof of Theorem 1.2. We wish to show

$$\Lambda(w) \otimes_{G/B} [G/B]_\mu = (-1)^{l(w)} [G/B]_{w(\mu+\rho)-\rho}.$$

By the multiplicativity of the map $w \mapsto \Lambda(w)$, it suffices to take a reflection w in a simple root α.

Let $\mu \in \Gamma_W$. Using the fact that $[G/B]_\mu = [[\mu]] \otimes_{G/B} [G/B]$, Proposition 3.15 gives

$$\Lambda(w) \otimes_{G/B} [G/B]_\mu = [[w(\mu)]] \otimes_{G/B} \Lambda(w) \otimes_{G/B} [G/B].$$

From Proposition 3.20,

$$\Lambda(w) \otimes_{G/B} [G/B] = \big[G/B \xleftarrow{\mathrm{p}_w} (\, G/(B \cap B_w),\, -\tau(\mathrm{p}_w) \otimes \mathrm{p}_w^*[E_{-\alpha}]\,) \to \star\big].$$

Since $G/(B \cap B_w) \xrightarrow{\mathrm{p}_w} G/B$ is K-equivariantly diffeomorphic to a vector bundle with Thom class $\tau(\mathrm{p}_w)$, the latter correspondence is precisely the Thom modification of

$$\big[G/B \xleftarrow{\mathrm{id}} (G/B,\, -[E_{-\alpha}]) \to \star\big] = -[G/B]_{-\alpha}.$$

So we get

$$\Lambda(w) \otimes_{G/B} [G/B]_\mu = -[G/B]_{w(\mu)-\alpha}.$$

Since w is the reflection in α, we have $\alpha = w(\rho)-\rho$, which proves the result. □

We now pass to the index-theoretic application. Let $\mathrm{pt} : G/B \to \star$ denote the map of G/B to a point and $\mathrm{pt}^* \in \widehat{\mathrm{KK}}^K(\mathbb{C}, G/B)$ its topological KK-theory class.

For a weight μ, the topological K-index of the twisted fundamental class $[G/B]_\mu \in \widehat{\mathrm{KK}}^G(G/B, \star)$ is defined by

$$\mathrm{Index}_K[G/B]_\mu := \mathrm{pt}^* \otimes_{G/B} [G/B]_\mu \in \widehat{\mathrm{KK}}^K(\mathbb{C}, \mathbb{C}).$$

We do not bother to use different notation for the *analytic* index

$$\mathrm{Index}_K[G/B]_\mu \in \mathrm{KK}^K(\mathbb{C}, \mathbb{C}) \cong \mathrm{R}(K);$$

which one we are talking about will be made clear by the context. The analytic index, as a graded representation of K, is the same as the cohomology group $H^*(G/B, E_\mu)$ figuring in the classical Borel–Bott–Weil theorem, and it equals the image of the topological index under the map

$$\widehat{\mathrm{KK}}^K(G/B, \star) \to \mathrm{KK}^K(\mathcal{C}(G/B), \mathbb{C})$$

(for a proof see [7].)

Proof of Theorem 1.1. We note that a Thom modification yields

$$\begin{aligned} \text{(7)} \qquad \mathrm{pt}^* \otimes_{G/B} \Lambda(w) &= \big[\star \leftarrow (G/(B \cap B_w),\, \tau(\mathrm{q}_w)) \xrightarrow{R_w \circ \mathrm{q}_w} G/B\big] \\ &= \big[\star \leftarrow G/B_w \xrightarrow{R_w} G/B\big] \\ &= \mathrm{pt}^*. \end{aligned}$$

Composing with the μ-twisted fundamental class on the right and applying Theorem 1.2 gives

$$(-1)^{l(w)} \mathrm{Index}_K[G/B]_{w(\mu+\rho)-\rho} = \mathrm{Index}_K[G/B]_\mu.$$ □

Remark 3.23. Let us also record the action of the Borel–Bott–Weil classes on equivariant K-theory. The induction isomorphism $\mathrm{R}(T) \xrightarrow{\cong} \mathrm{K}_K(K/T)$ associates to $[\mu]$ the correspondence

$$[E_\mu] := \big[\star \leftarrow (K/T,\, [E_\mu]) \xrightarrow{\mathrm{id}} K/T\big] = \mathrm{pt}^*[[\mu]].$$

Thus, if we compose the commutation relation of Proposition 3.15 on the left by pt^* and use equation (7), we get the right action:

$$[E_{w(\mu)}] \otimes_{G/B} \Lambda(w) = [E_\mu].$$

References

[1] M. F. Atiyah and I. M. Singer, The index of elliptic operators. I, Ann. of Math. (2) **87** (1968), 484–530. MR0236950

[2] B. Blackadar, *K-theory for operator algebras*, second edition, Math. Sci. Res. Inst. Publ., 5, Cambridge Univ. Press, Cambridge, 1998. MR1656031

[3] J. Block and N. Higson, Weyl character formula in KK-theory, in *Noncommutative geometry and physics. 3*, 299–334, Keio COE Lect. Ser. Math. Sci., 1, World Sci. Publ., Hackensack, NJ, 2013. MR3098596

[4] R. Bott, Homogeneous vector bundles, Ann. of Math. (2) **66** (1957), 203–248. MR0089473

[5] A. Connes and G. Skandalis, The longitudinal index theorem for foliations, Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183. MR0775126

[6] H. Emerson and R. Meyer, Bivariant K-theory via correspondences, Adv. Math. **225** (2010), no. 5, 2883–2919. MR2680187

[7] H. Emerson and R. Meyer, Equivariant embedding theorems and topological index maps, Adv. Math. **225** (2010), no. 5, 2840–2882. MR2680186

[8] A. W. Knapp, *Lie groups beyond an introduction*, second edition, Progr. Math., 140, Birkhäuser Boston, Boston, MA, 2002. MR1920389

[9] H.-H. Leung, Divided difference operators in equivariant KK-theory, J. Topol. Anal. **6** (2014), no. 2, 237–261. MR3191651

[10] J.-L. Tu, La conjecture de Baum–Connes pour les feuilletages moyennables, K-Theory **17** (1999), no. 3, 215–264. MR1703305

Received August 8, 2015; accepted June 11, 2016

Heath Emerson
Department of Mathematics and Statistics, University of Victoria,
PO Box 3045 STN CSC, Victoria, B.C., Canada V8W 3P4
E-mail: hemerson@math.uvic.ca

Robert Yuncken
Laboratoire de Mathématiques,
Université Blaise Pascal, Clermont-Ferrand II,
Campus des Cézeaux, 63177 Aubière cedex, France
E-mail: yuncken@math.univ-bpclermont.fr