Lower bound estimates related to primes in short intervals

Inaugural–Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch–Naturwisssenschaftlichen Fakultät
der Westfälischen Wilhelms–Universität Münster
vorgelegt von

Arne Lutz Juhas
aus Münster
–2016–

Dekan:	Prof. Dr. Xiaoyi Jiang
Erste Gutachterin:	PD Dr. Karin Halupczok
Zweiter Gutachter:	Prof. Dr. Jörn Steuding
Dritter Gutachter:	Prof. Dr. Zakhar Kabluchko
Tag der mündlichen Prüfung:	28.02.2017
Tag der Promotion:	28.02.2017

Acknowledgements

I would like to thank my advisor, PD Dr. Karin Halupczok, for her extremely helpful support and help during the work on my thesis. She was always there when I needed her advice and good judgement.
Last but not least, a word of grateful thanks to my family for their unwavering moral support.

Summary

We consider lower bounds for the second moment of prime numbers in short intervals as well as the pair correlation function, whose asymptotic behavior is predicted by Montgomery's Pair Correlation Conjecture. D. Goldston obtained such lower bounds by employing a truncated version of the von Mangoldt function.

Using a modified approach based upon his method, we improve on former results in the conditional as well as in the unconditional case. Our method can also be applied to obtain lower bounds for the variance of primes over residue classes as well as to gain further information about the distribution of primes in short intervals.

Contents

List of Notations

$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
x	sufficiently large positive number
ε	sufficiently small positive number,
	not necessarily the same in each occurrence
δ, κ, A	positive real numbers
h, y, B R, Q	real numbers ≥ 1
p	prime number
s	complex number
n, r, ℓ, q, N, M	natural numbers
a, b, k, m	integers
$[x]$	the largest integer less than or equal to x
(a, b)	greatest common divisor of a and b
$\sum_{m\sim M}$	$\sum_{M<m\leq 2M}$
$\sum_{a(m)}^{*}$	$\sum_{\substack{a=1 \\ (a,m)=1}}^{m}$
$f(a, b) = f((a, b))$	for an arithmetic function f
$\varphi(n) = n\prod_{p\mid n}\big(1 - \frac{1}{p}\big)$	Euler's totient function

$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^k, \\ 0, & \text{otherwise} \end{cases}$		von Mangoldt function
$\psi(x) = \sum_{n \le x} \Lambda(n)$		
$\psi(x; m, a) = \sum_{\substack{n \le x \\ n \equiv a \bmod m}} \Lambda(n)$		
$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n = p_1 p_2 \dots p_k, \\ 0, & \text{otherwise} \end{cases}$		Möbius function
$\omega(n)$		number of distinct prime divisors of n
$\tau(n)$		number of divisors of n
$\sigma(n)$		the sum of positive divisors of n
$\varphi_2(n)$	>	the function defined by $\varphi_2(p) = p - 2$, extended to squarefree n by multiplicativity
$f(x) = O(g(x))$		Landau notation:
		there exists some $C > 0$, such that $\lvert f(x)\rvert \le C\lvert g(x)\rvert$, $x > x_0(C)$
$f(x) = o(g(x))$		Landau notation:
		for every $\varepsilon > 0$ there exists some $x_0(\varepsilon)$, such that $\lvert f(x)\rvert \le \varepsilon\lvert g(x)\rvert$, $x > x_0(\varepsilon)$
$f(x) \sim g(x)$ for $x \to \infty$		$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1$
$f(x) \ll g(x)$		$f(x) = O(g(x))$
$\sum_{m \sim M}$		$\sum_{M < m \le 2M}$
$\mathfrak{S}(k) = \begin{cases} \mathfrak{S} \prod_{\substack{p \mid k \\ p > 2}} \frac{p-1}{p-2}, & \text{if } 2 \mid k, \\ 0, & \text{if } 2 \nmid k \end{cases}$		
with $\mathfrak{S} := 2 \prod_{p > 2} \big(1 - \frac{1}{(p-1)^2}\big)$		singular series
$\mathfrak{S}_R(k) := \sum_{r \le R} \frac{\mu(r)\mu(r,k)\varphi(r,k)}{\varphi^2(r)}$		truncated singular series

$\varrho(t) = t - [t] + \frac{1}{2}$	sawtooth function
$\chi \bmod m$, $\chi(m)$	Dirichlet character to modulus m
$\delta_\chi = \begin{cases} 1, & \text{if } \chi \text{ is principal,} \\ 0, & \text{otherwise} \end{cases}$	
$\psi(x;\chi) = \sum_{n \le x} \Lambda(n)\chi(n)$	
$\sum_{\chi(m)}^{*}$	a sum restricted to primitive characters $\chi \bmod m$
$\mathrm{E}_{m,a} = \begin{cases} 1, & \text{if } (a,m) = 1, \\ 0, & \text{if } (a,m) > 1 \end{cases}$	
$E(x;m,a) = \psi(x;m,a) - E_{m,a}\dfrac{x}{\varphi(m)}$	
$\zeta(s)$	Riemann zeta function
$L(s;\chi) := \sum_{n=1}^{\infty} \dfrac{\chi(n)}{n^s}$	Dirichlet L-function
RH	Riemann Hypothesis:
	$\zeta(s)$ only has nontrivial zeros with real part $\dfrac{1}{2}$.
GRH	Generalized Riemann Hypothesis:
	if χ is primitive and $L(s;\chi) = 0$ with $0 < \operatorname{Re} s < 1$, then $\operatorname{Re} s = \dfrac{1}{2}$.
LHS	left hand side
RHS	right hand side

0 Introduction

In his paper "The Pair Correlation of Zeros of the Zeta function" [23], H. L. Montgomery studied the distribution of zeros of the Riemann zeta function on the critical line and assuming RH he conjectured an asymptotic formula for the pair correlation of zeros. In fact, Montgomery's Pair Correlation Conjecture is a special case of the conjecture that normalized spacings between nontrivial zeros of $\zeta(s)$ are distributed like eigenvalues of random matrices from the Gaussian Unitary Ensemble (GUE).

It was observed that there is a natural correspondence between pair correlation of zeros and primes in short intervals. Assuming RH, Goldston and Montgomery [11, p. 186, eq. (16)] showed an equivalence between a certain form of the Pair Correlation Conjecture and an asymptotic formula for the variance of primes in short intervals. In [2], Chan made this equivalence more precise and based on this result also formulated a more precise Pair Correlation Conjecture.

This work takes into consideration second moments for primes in short intervals of the form

$$I(x;h) := \int_0^x (\psi(y;h) - h)^2 dy,$$

where $\psi(y;h) := \sum_{y<n\leq y+h} \Lambda(n)$ and $I(x;h)$ denotes the variance of primes in short intervals, as well as the pair correlation function, defined by

$$F(x,T) := \sum_{0,\gamma,\gamma'\leq T} x^{i(\gamma-\gamma')} w(\gamma-\gamma'), \qquad \text{where } w(u) := \frac{4}{4+u^2}$$

and γ, γ' denote imaginary parts of nontrivial zeros of the Riemann zeta function. The aforementioned equivalence between $I(x;h)$ and $F(x,T)$ states that under the assumption of RH, the asymptotic formulas

$$I(x;h) \sim hx \log\left(\frac{x}{h}\right) \qquad \text{uniformly in } 1 \leq h \leq x^{1-\varepsilon}, \varepsilon > 0, \quad x \to \infty, \tag{0.1}$$

and

$$F(x,T) \sim \frac{T}{2\pi} \log T \qquad \text{uniformly in } T \leq x \leq T^A, A \geq 1, \quad T \to \infty, \tag{0.2}$$

are equivalent to each other. Here (0.2) is known as the strong form of Montgomery's Pair Correlation Conjecture. Moreover, Goldston and Yıldırım [12, Thm. 1], proved (0.1) assuming RH as well as a strong quantitative form of the Twin Prime Conjecture, which states that

$$\sum_{n\le y}\Lambda(n)\Lambda(n+k)=\mathfrak{S}(k)y+O(y^{1/2+\varepsilon}) \tag{0.3}$$

for $0<|k|\le y$. Setting $h=x^\alpha$, we can express (0.1) as

$$I(x;h)\sim(1-\alpha)hx\log x,\qquad \alpha\in[0,1-\varepsilon].$$

Goldston, see [7, p. 366, eq. (1.5)] and [6, p, 154, eq. (1.5)], proved supporting lower bounds, namely

$$I(x;h)\ge\big(\frac{1}{2}-2\alpha-\varepsilon\big)hx\log x,\quad \alpha\in[0,1/4], \tag{0.4}$$

assuming GRH and

$$I(x;h)\ge\big(\frac{1}{2}-\varepsilon\big)hx\log x,\qquad 1\le h\le(\log x)^A,\quad A>0 \tag{0.5}$$

unconditionally for x large enough.

More generally, in [13], Goldston and Yıldırım considered second moments over arithmetic progressions, defined by

$$I(x;h,r,a):=\int_x^{2x}\big(\psi(y+h;r,a)-\psi(y;r,a)-\frac{h}{\varphi(r)}\big)^2,$$

where $(a,r)=1$ and $1\le r\le h\le x$ and showed that under GRH, one has

$$I(x;h,r,a)\ge\frac{1}{2}\frac{xh}{\varphi(r)}\log\big(\frac{xr}{h^3}\big)-O\big(\frac{xh}{\varphi(r)}(\log\log(3r))^2\big),\qquad r\le h\le(xr)^{1/3-\varepsilon}. \tag{0.6}$$

Employing this with $r=1$ improves (0.4) to

$$I(x;h)\ge\big(\frac{1}{2}-\frac{3}{2}\alpha-\varepsilon\big)hx\log x,\qquad \alpha\in[0,1/3]. \tag{0.7}$$

The proof of (0.4)–(0.7) essentially relies on the simple inequality

$$\int_0^x(\psi(y;h)-\psi_R(y;h))^2dy\ge 0, \tag{0.8}$$

where $\psi_R(x):=\sum_{n\le x}\lambda_R(n)$, $\psi_R(y;h):=\psi_R(y+h)-\psi_R(y)$ and

$$\lambda_R(n):=\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|(r,n)}d\mu(d),$$

see [7, p. 367, eq. (1.9)], and a lower bound for $I(x;h)$ can be derived by examining both $\int_0^x \psi(y;h)\psi_R(y;h)dy$ and $\int_0^x \psi_R^2(y;h)dy$. The function λ_R is a truncated form of Λ motivated by the identity

$$\frac{\varphi(n)}{n}\Lambda(n) = \sum_{r=1}^{\infty} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|(r,n)} d\mu(d), \qquad n > 1,$$

cp. [25, p. 373, eq. (1.7)], and copies the distribution of Λ over arithmetic progressions. It was first used in the paper [16] by Heath-Brown on the Goldbach problem and Goldston observed that it is the best approximation to Λ among sums of the form $\sum_{\substack{r\le R\\ r|n}} a(r,R)$, $a(1,R) = 1$, $a(r,R) \in \mathbb{R}$ in an L^2 sense, as mentioned in [14, p. 2].

The equivalence of (0.1) and (0.2) arises the question whether lower bounds for $F(x,T)$ can be obtained similarly by applying λ_R. In [10], Goldston et al. obtained such a result, namely

$$\Big(\frac{T}{2\pi}\log T\Big)^{-1} F(T^\alpha, T) \ge \frac{3}{2} - |\alpha| - \varepsilon \tag{0.9}$$

for any $\varepsilon > 0$, uniformly in $1 \le |\alpha| \le 3/2 - 2\varepsilon$ and all $T \ge T_0(\varepsilon)$ assuming GRH, cp. [10, p. 34, eq. (3.1)].

This work is organized as follows. In Chapter 1, using a modified approach based upon Goldston's method and using ideas of Hooley employed in [19], we examine which preferably weak additional assumptions beyond GRH lead to improved lower bounds compared to (0.7).

In Chapter 2 we consider the unconditional case and employing the Basic Mean Value Theorem, we improve the range for h for which (0.5) holds from a power of $\log x$ to a subexponential factor of x, which corresponds to the Vinogradov-Korobov zero free region for $\zeta(s)$, cp. [24, p. 194, eq. (6.26)]. Since it is the widest unconditional zero free region known so far, it also provides the widest region for h in which a nontrivial unconditional lower bound for $I(x;h)$ can be derived. We also give an application to the comparable case of lower bounds for the variance of primes in an arithmetic progression and improve former results obtained by Friedlander and Goldston [5, Thm. 3] and Hooley [19, pp. 53–54, eq. (6)–(7)], see p. 38 for a more detailed discussion of this.

In Chapter 3 we apply the methods of Chapter 1 to the pair correlation function, using techniques from [10]. Analogously to Chapter 1, our goal is to find a weak condition beyond GRH, which improves (0.9).

Introducing a suitably modified function $\widetilde{\lambda_R}(n)$ in Chapter 4, we examine $\psi(n;h)$ and its mean value h. For any $\vartheta > 0$ it is conjectured that

$$\psi(N;h) \sim h \qquad \text{uniformly in } N^{\vartheta} \leq h \leq N \tag{0.10}$$

and Huxley showed that (0.10) holds for any fixed $\vartheta > \dfrac{7}{12}$, cp. [17, p. 22]. Subject to RH, Cramer [24, p. 421, l. 24] established that there exists some $C > 0$ with

$$\psi(N+h) - \psi(N-h) > \frac{h}{2} \qquad \text{for } h = C\sqrt{N}\log N, \tag{0.11}$$

$N \geq 2$, from which it follows that the interval $(N, N + C\sqrt{N}\log N)$ contains at least $\sqrt{N}$ prime numbers, cp. [24, p. 421, Thm. 13.3]. We show unconditionally that for any $\varepsilon > 0$ and $\delta \in (0,1)$, there exists some $n_0 \in [N, 2N]$ with

$$\psi(n_0;h) > (1-\delta)h \qquad \text{for } 2\delta^{-1} \leq h \ll N^{1/6-\varepsilon} \tag{0.12}$$

if N is large enough in terms of ε and δ. Results like (0.11) and (0.12) also suggest that the second Hardy-Littlewood Conjecture, cp. [18, p. 375, Conj. (A)], which asserts that

$$\pi(N+h) - \pi(N) \leq \pi(h) \qquad \text{for } N, h \geq 2, \tag{0.13}$$

is false, although our method does not suffice to disprove it. In [18], Hensley and Richards showed that (0.13) and the Twin Prime Conjecture are incompatible.

Conjecture (0.10) can be specified to arithmetic progressions:

$$\psi(x+h;q,a) - \psi(x;q,a) = \frac{h}{\varphi(q)} + O_{\varepsilon}\left(x^{\varepsilon}\left(\frac{h}{q}\right)^{1/2}\right) \text{ uniformly in } 1 \leq q \leq h \leq x, \tag{0.14}$$

where $(a,q) = 1$, cp. [24, p. 422, Conj. 13.9] for the analogue conjecture in the case $h = x$ and with methods developed in Chapter 1–2, we show in Chapter 5 that the error term of (0.14) is sharp, more precisely, we prove that for ε, $\kappa > 0$ with $\kappa \leq 1/2 - \varepsilon$, infinitely many x and every $h \in [1, x^{\kappa/6-\varepsilon}] \cup [x^{\kappa}, x^{1/2-\varepsilon}]$ there exists some $q_0 \leq h$ and a constant $C(\varepsilon)$ with

$$\left|\psi(x+h;q_0,a) - \psi(x;q_0,a) - \frac{h}{\varphi(q)}\right| \geq C(\varepsilon)\left(\frac{h}{\varphi(q)}\log x\right)^{1/2}.$$

1 The conditional case

As this section will show, the strength of the error term occurring in the prime number theorem for arithmetic progressions mainly predicts the range in which nontrivial results concerning lower bounds for $I(x;h)$ can be established. Consequently, it seems unlikely that a nontrivial lower bound for $I(x;h)$ can be proved in a wider range for h using this approach under GRH. Additionally, another limiting factor comes from the error we make by evaluating $\int_0^x \psi_R^2(y;h)dy$, which is mainly of order h^2R^2, cp. [7, p. 369, eq. (2.8)] and the proof of the main Theorem stated in [7].

Applying a method of Hooley (cp. [19]) to examine the variance of primes in arithmetic progressions, we show that this error term estimates can be improved to terms of order $h^{3/2}R^2$, which gives an asymptotic formula for $\int_0^x \psi_R^2(y;h)dy$ valid for $1 \le h \le x^{2/5-\varepsilon}$, see Lemma 1.11 below. Hooley's method uses Vaaler's approximation to the sawtooth curve and the large sieve inequality.
However, to obtain lower bounds for $I(x;h)$ in larger ranges for h and to take advantage of our result concerning $\int_0^x \psi_R^2(y;h)dy$, we have to go further than GRH. For instance, we give such a stronger result by the following weakened and modified version of Montgomery's Conjecture concerning the error term of the Prime Number Theorem in arithmetic progressions.

Hypothesis M. *Let $m \in \mathbb{N}$ with $m \le \sqrt{x}$, $\varepsilon > 0$ and χ be a Dirichlet character mod m. Then we have the estimate*

$$\left|\psi(x;\chi) - \delta_\chi x\right| \ll \frac{x^{\frac{1}{2}+\varepsilon}}{m^{1/4}}.$$

For comparison, the original Montgomery-Conjecture, cp. [21, p. 34], asserts that for any a, m with $m \le x$ that

$$\psi(x;m,a) = E_{m,a}\frac{x}{\varphi(m)} + O\big(x^\varepsilon\big(\frac{x}{m}\big)^{1/2}\big).$$

The aim of this section is to prove the following theorem.

Theorem 1. *Assume Hypothesis M. Then for any $\varepsilon > 0$ and $1 \le h \le x^{2/5-\varepsilon}$ we have*

$$I(x;h) \ge hx\log\big(\frac{x^{1/2}}{h^{5/4}}\big) + o(hx\log x),$$

uniformly in h.

As in [6, Cor. 2], we can state the following Corollary, which follows immediately from Theorem 2.

Corollary 1. *Assuming Hypothesis M we have for $\delta > 0$ and $1 \le h \le x^{2/5-\delta}$:*

$$\max_{y\in[0,x]} |\psi(y;h) - h| \gg_\delta \sqrt{h \log x}.$$

Proof. We fix $\delta > 0$. An application of Theorem 1 yields

$$I(x;h) \ge \big(\frac{5}{4}\delta - \varepsilon\big) \log x \text{ for } 1 \le h \le x^{2/5-\delta},\ \varepsilon > 0,\ x > x_0(\varepsilon). \tag{1.1}$$

If for $1 \le h \le x^{2/5-\delta}$ we had

$$\max_{y\in[0,x]} |\psi(y;h) - h| < \sqrt{(\delta - \varepsilon) h \log x}$$

for infinitely many x, say, it follows that

$$I(x;h) \le (\delta - \varepsilon) hx \log x \text{ for } 1 \le h \le x^{2/5-\delta} \text{ and inf. many } x,$$

which contradicts (1.1). ■

1.1 Auxiliary Results

We provide some general auxiliary results, which will be needed during this Chapter and later.

1.1.1 General Lemmas

Lemma 1.1 (Large Sieve Inequality, [25, p. 157, eq. (8)]). *Let (a_n) be a sequence of complex numbers and M, N natural numbers. Then we have*

$$\sum_{m\le M} \sideset{}{^*}\sum_{a(m)} \Big| \sum_{n\le N} a_n e\big(n\frac{a}{m}\big)\Big|^2 \ll (N + M^2) \sum_{n\le N} |a_n|^2. \tag{1.2}$$

Lemma 1.2. *Under the assumption of Hypothesis M, we have for $m \le \sqrt{x}$ that*

$$\sideset{}{^*}\sum_{a(m)} E^2(x;m,a) \ll \frac{x^{1+\varepsilon}}{m^{1/2}}.$$

Proof. The proof goes along the lines of the classical GRH estimate

$$\sum_{a(m)}^{*} E^2(x;m,a) \ll x^{1+\varepsilon},$$

cp. [22, p. 145, Thm. 17.1]: By orthogonality, we have

$$\sum_{a(m)}^{*} E^2(x;m,a) = \frac{1}{\varphi(m)} \sum_{\chi(m)} |\psi(x;\chi) - \delta_\chi x|^2$$

and the Lemma follows, since $|\psi(x;\chi) - \delta_\chi x|^2 \ll \dfrac{x^{1+\varepsilon}}{m^{1/2}}$ by Hypothesis M. ■

Lemma 1.3. *For every $h \le x$ and $\varepsilon > 0$ we have*

$$\int_0^x \psi(y;h)dy = hx + h(\psi(x+h) - (x+h)) + O(h^2 x^\varepsilon) \tag{1.3}$$

and we obtain the unconditional result

$$\int_0^x \psi(y;h)dy = hx + O(hxe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) + O(h^2 x^\varepsilon) \tag{1.4}$$

with a suitable constant $c > 0$ as well as

$$\int_0^x \psi(y;h)dy = hx + O(hx^{1/2+\varepsilon}) + O(h^2 x^\varepsilon) \tag{1.5}$$

assuming RH.

Proof. First we can write

$$\begin{aligned} \int_0^x \psi(y;h)dy &= \int_0^x (\psi(y+h) - \psi(y))dy = \int_h^{x+h} \psi(y)dy - \int_0^x \psi(y)dy \\ &= \int_x^{x+h} \psi(y)dy - \int_0^h \psi(y)dy \end{aligned} \tag{1.6}$$

On using summation by parts we get

$$\begin{aligned} \int_x^{x+h} \psi(y)dy &= h\psi(x+h) - \sum_{x<m\le x+h} \Lambda(m)(m-x) \\ &= h(x+h+\psi(x+h) - (x+h)) - \sum_{x<m\le x+h} \Lambda(m)(m-x) \\ &= hx + h(\psi(x+h) - (x+h)) + O(h^2 x^\varepsilon), \end{aligned} \tag{1.7}$$

whereas $\int_0^h \psi(y)dy \ll h^2$ by Chebyshev's estimate $\psi(y) = O(y)$. Plugging this together with (1.7) into (1.6) gives (1.3).

Equation (1.4) now follows from (1.3) because of the Vinogradov-Korobov Prime Number Theorem, cp. [25, p. 194, eq. (6.28)], which can be formulated as

$$\psi(x) = x + O(xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) \qquad \text{for some } c > 0,$$

and noting that

$$\begin{aligned}\psi(x+h) - (x+h) &\ll (x+h)e^{-c(\log(x+h))^{3/5}(\log\log(x+h))^{-1/5}} \\ &\ll xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} + hx^{\varepsilon}.\end{aligned}$$

Analogously, we obtain (1.5) on using $\psi(x) - x = O(x^{1/2+\varepsilon})$ under RH and the fact that

$$\psi(x+h) - (x+h) \ll \sqrt{x+h}(\log(x+h))^2 \ll x^{1/2+\varepsilon}$$

for $h \le x$. ■

Lemma 1.4 (cp. [7, p. 374]). *Let (a_n), (b_n) be complex sequences and let $\|c\| := \max|c_n|$ for a complex sequence (c_n). Then we have*

$$\begin{aligned}\int_0^x \Big(\sum_{y<n\le y+h} a_n\Big)\Big(\sum_{y<m\le y+h} b_m\Big)dy =& h\sum_{n\le x} a_n b_n + \sum_{k\le h}(h-k)\Big(\sum_{n\le x} a_n b_{n+k} + \sum_{n\le x} b_n a_{n+k}\Big) \\ &+ O(\|a\|\|b\|h^3).\end{aligned}$$

Lemma 1.5 ([13, Lemma 3]). *For each integer $m \ge 1$ and real $R \ge 1$ we have*

$$\sum_{\substack{r\le R\\(r,m)=1}} \frac{\mu^2(r)}{\varphi(r)} = \frac{\varphi(m)}{m}(\log R + c + f(m)) + O\Big(\frac{g(m)}{\sqrt{R}}\Big),$$

where

$$c := \gamma + \sum_p \frac{\log p}{p(p-1)}, \quad f(m) := \sum_{p|m} \frac{\log p}{p}, \qquad g(m) := \sum_{d|m} \frac{\mu^2(d)}{\sqrt{d}}$$

and γ denotes Euler's constant.

Lemma 1.6 ([4, Lemma 2.1]). *For $m \geq 1$ and $y \geq m$ we have*

$$\sum_{\substack{d \leq y \\ (d,2m)=1}} \frac{\mu^2(d)}{\varphi_2(d)} = \frac{\varphi(2m)}{2m} \prod_{p \nmid 2m} \left(1 + \frac{1}{p(p-2)}\right) \log y + O(1), \tag{1.8}$$

where the implicit constant is absolute.

Lemma 1.7 ([8, Lemma 2]). *For real numbers c, d, we have for $y \geq 1$ real*

$$\sum_{r \leq y} \frac{\mu^2(r) r^c}{\varphi(r)^d} = \begin{cases} \frac{g(c-d+1;c,d)}{c-d+1} y^{c-d+1} + o_{c,d}(y^{c-d+1}), & \text{if } c-d > -1, \\ g(0; c-1, d) \log y + O_{c,d}(1), & \text{if } c-d = -1, \\ \zeta(c-d) g(0;c,d) + \frac{g(c-d+1;c,d)}{c-d+1} y^{c-d+1} + o_{c,d}(y^{c-d+1}), & \text{if } c-d < -1, \end{cases}$$

where

$$g(s;c,d) := \prod_p \left(1 - \frac{1 - p^{s-c+d}\left(1 - \left(1 - \frac{1}{p}\right)^d\right)}{(p-1)^d p^{2(s-c)+d}}\right).$$

In particular, if $c - d < -1$, the series $\sum_r \frac{\mu^2(r) r^c}{\varphi(r)^d}$ converges to $\zeta(c-d) g(0;c,d)$ and we have the estimate $\sum_{r > y} \frac{\mu^2(r) r^c}{\varphi(r)^d} = O_{c,d}(y^{c-d+1})$.

Lemma 1.8. *For real $R \geq 1$ and integers $d > 1$ we have the estimates*

(a) $\frac{d}{\varphi(d)} \ll \log d$, (b) $\sum_{r \leq R} \frac{\mu^2(r)\sigma(r)}{\varphi(r)} \ll R$, (c) $\sum_{r \leq R} \frac{2^{\omega(r)}}{\varphi(r)} \ll (\log R)^2$,

(d) $\sum_{r \leq R} \omega(r) \ll R \log\log R$

Proof. (a) follows e.g. from $d/\varphi(d) = \prod_{p|d}(1 - 1/p)^{-1}$ and Mertens' Theorem, (b) is stated in [13, Lemma 9, eq. (3.16)], (c) follows from $\sum_{r \leq R} 2^{\omega(r)} \ll R \log R$, cp. [24, p. 42], (a) and summing by parts and (d) follows from [24, p. 58, eq. (2.22)]. ∎

1.1.2 Lemmas involving sums of $\lambda_R(n)$ and $\Lambda(n)$

Following the approach in [7] resp. [19], we now present some Lemmas allowing us to derive asymptotic formulas of sums involving $\Lambda(n)$ and $\lambda_R(n)$, which are a crucial part of the proof of Theorem 2. Lemmas 1.9–1.11 are independent of Hypothesis M. In contrast to [7], we give a more elementary proof of Lemmas 1.11 and 1.12 by evaluating occurring terms directly instead of using the singular series.

The first Lemma is Lemma 1 of [19], which we prove in a different way here.

Lemma 1.9 (cp. [19, Lemma 1]). *For real $R, x \geq 1$ and integers a, m let*

$$\psi_R(x; m, a) := \sum_{\substack{n \leq x \\ n \equiv a \bmod m}} \Lambda_R(n).$$

Then for $0 \leq a < m \leq R$ we have

$$\psi_R(x; m, a) = E_{m,a} \frac{x}{\varphi(m)} + O(R) \tag{1.9}$$

and the error term can be expressed explicitly as

$$\sum_{\delta | a} \sum_{r \leq R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d | r \\ (d,m)=\delta}} d\mu(d) \big(\varrho\big(\frac{x}{\delta d' m'} - \frac{a}{\delta} \frac{\overline{d'}}{m'} \big) - \varrho\big(- \frac{a}{\delta} \frac{\overline{d'}}{m'} \big) \big), \tag{1.10}$$

where d', m' and $\overline{d'}$ are defined by

$$d = \delta d', \quad m = \delta m', \quad \text{and} \quad d' \overline{d'} \equiv 1 (m'). \tag{1.11}$$

Proof. First we have

$$\psi_R(x; m, a) = \sum_{r \leq R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d | r} d\mu(d) \# \{ n \leq x; d \mid n, n \equiv a(m) \} \tag{1.12}$$

and the simultaneous congruences

$$n \equiv 0(d), \qquad n \equiv a(m) \tag{1.13}$$

are only solvable if $\delta = (d, m) | a$, in which case they define a unique residue class mod $\dfrac{dm}{(d, m)}$. Writing $n = \delta n'$ and using (1.11), we can express the simultaneous congruences (1.13) for n as

$$n' \equiv 0(d'), \qquad n' \equiv \frac{a}{\delta} (m'),$$

the solutions n' mod $d'm'$ being given by

$$n' \equiv \frac{a}{\delta} d'\overline{d'}(d'm')$$

by the Chinese Remainder Theorem. Hence the inner sum in (1.12) equals

$$\begin{aligned}
&\sum_{\delta|a} \sum_{\substack{d|r\\(d,m)=\delta}} d\mu(d)\big([\frac{x}{\delta d'm'} - \frac{a}{\delta}\frac{d'\overline{d'}}{d'm'}] - [-\frac{a}{\delta}\frac{d'\overline{d'}}{d'm'}]\big)\\
=&\sum_{\delta|a} \sum_{\substack{d|r\\(d,m)=\delta}} d\mu(d)\frac{x}{\delta d'm'}\\
&-\sum_{\delta|a} \sum_{\substack{d|r\\(d,m)=\delta}} d\mu(d)\big(\varrho(\frac{x}{\delta d'm'} - \frac{a}{\delta}\frac{\overline{d'}}{m'}) - \varrho(-\frac{a}{\delta}\frac{\overline{d'}}{m'})\big),
\end{aligned}$$

so that we get

$$\begin{aligned}
\psi_R(x;m,a) =& \sum_{\delta|a} \frac{x}{\delta m'} \sum_{r\le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r\\(d,m)=\delta}} \frac{\mu(d)d}{d'}\\
&-\sum_{\delta|a}\sum_{r\le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r\\(d,m)=\delta}} d\mu(d)\big(\varrho(\frac{x}{\delta d'm'} - \frac{a}{\delta}\frac{\overline{d'}}{m'}) - \varrho(-\frac{a}{\delta}\frac{\overline{d'}}{m'})\big).
\end{aligned} \tag{1.14}$$

A crude estimate gives

$$\begin{aligned}
\sum_{\delta|a}\sum_{r\le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r\\(d,m)=\delta}} d\mu(d)\big(\varrho(\frac{x}{\delta d'm'} - \frac{a}{\delta}\frac{\overline{d'}}{m'}) - \varrho(-\frac{a}{\delta}\frac{\overline{d'}}{m'})\big) &\ll \sum_{r\le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r\\(d,m)|a}} d\\
&\ll \sum_{r\le R} \frac{\mu^2(r)\sigma(r)}{\varphi(r)} \ll R,
\end{aligned}$$

by Lemma 1.8 (b). Using (1.11), the main term on the RHS of (1.14) is seen to equal

$$\frac{x}{m} \sum_{r\le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|s} \mu(d)(d,m),$$

where

$$s := \prod_{\substack{p|r\\(p,m)|a}} p, \tag{1.15}$$

and since $\mu(d)(d,m)$ is a multiplicative function in d, we have for $\mu(r) \neq 0$

$$\sum_{d|s} \mu(d)(d,m) = \prod_{p|s}(1-(p,m)) = \begin{cases} \mu(r,a)\varphi(r,a), & r|m, \\ 0, & r \nmid m. \end{cases} \tag{1.16}$$

Therefore the main term on the RHS of (1.14) equals

$$\frac{x}{m} \sum_{\substack{r \le R \\ r|m}} \frac{\mu^2(r)\mu(r,a)\varphi(r,a)}{\varphi(r)} = \frac{x}{m} \sum_{r|m} \frac{\mu^2(r)\mu(r,a)\varphi(r,a)}{\varphi(r)}$$
$$= \frac{x}{m} \prod_{p|m} \left(1 + \frac{\mu(p,a)\varphi(p,a)}{(p-1)}\right) = E_{a,m} \frac{x}{\varphi(m)} \tag{1.17}$$

using $m \le R$ in the first equation and that $\mu^2(r)\mu(r,a)\varphi(r,a)/\varphi(r)$ is a multiplicative function of r in the second equation. ■

Lemma 1.10 ([7, Lemma 1, eq. (2.5), Lemma 2, eq. (2.7)]). *(A) For $1 \le R \ll x/\log x$, we have*

$$\sum_{n \le x} \lambda_R(n)\Lambda(n) = x \log R + O(x). \tag{1.18}$$

(B) *If $1 \le R \ll \sqrt{x}$, then*

$$\sum_{n \le x} \lambda_R^2(n) = x \log R + O(x). \tag{1.19}$$

Proof. We first prove (1.18) by noting that the definition of the function λ_R gives

$$\sum_{n \le x} \lambda_R(n)\Lambda(n) = \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r} d\mu(d) \sum_{\substack{n \le x \\ d|n}} \Lambda(n)$$
$$= \psi(x) \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} + \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r \\ d \ge 2}} d\mu(d) \sum_{\substack{n \le x \\ d|n}} \Lambda(n)$$
$$= \psi(x) \log R + O(\psi(x)) + \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r \\ d \ge 2}} d\mu(d) \sum_{\substack{n \le x \\ d|n}} \Lambda(n). \tag{1.20}$$

Now the innermost sum of the triple sum on the RHS of (1.20) is only non-zero, if $d = p$ is a prime number and since

$$\sum_{\substack{n \le x \\ p|n}} \Lambda(n) \le \log p \left[\frac{\log x}{\log p}\right] \ll \log x,$$

the contribution of this triple sum is bounded by

$$\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\varphi(r)}\log x \ll R\log x$$

using Lemma 1.8 (b), which is $O(x)$ provided $R \ll x/\log x$. Now inserting the Prime Number Theorem in the form

$$\psi(x) = x + O\big(\frac{x}{(\log x)^A}\big) \qquad \text{with } A \ge 1$$

into (1.20) shows (1.18).

Finally (1.19) can be proved as follows. First we have

$$\sum_{n\le x}\lambda_R^2(n) = \sum_{r,r'\le R}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{\substack{d|r\\ e|r'}} de\mu(d)\mu(e)\sum_{\substack{n\le x\\ d,e|n}} 1 \tag{1.21}$$

and we see that the simultaneous congruences

$$n \equiv 0(d), \qquad n \equiv 0(e)$$

for n occurring in the innermost sum of (1.21) are always solvable and define a unique residue class mod $\dfrac{de}{(d,e)}$, so that the expression in (1.21) equals

$$x\sum_{r,r'\le R}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{\substack{d|r\\ e|r'}}\mu(d)(d,e) + O\bigg(\sum_{r,r'\le R}\frac{\mu^2(r)\sigma(r)\mu^2(r')\sigma(r')}{\varphi(r)\varphi(r')}\bigg), \tag{1.22}$$

where the error is bounded by R^2 by Lemma 1.8 (b), which is $O(x)$ provided $R \ll \sqrt{x}$. Next, for any $d \mid r$, we have

$$\sum_{e|r'}\mu(d)(d,e) = \prod_{p|r'}(1-(p,d)) = \begin{cases}\prod_{p|r'}(1-p) = \mu(r')\varphi(r'), & r' \mid d,\\ 0, & r' \nmid d,\end{cases}$$

hence the double sum in (1.22) becomes

$$\begin{aligned} x\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\mu(d)\sum_{\substack{r'\le R\\ r'|d}}\mu(r') &= x\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\mu(d)\sum_{r'|d}\mu(r')\\ &= x\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\mu(d)\delta_{d,1}\\ &= x\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)} = x\log R + O(x) \end{aligned}$$

and this shows (1.19). ∎

Now the following Lemma is the main step in the proof of Theorem 1.

Lemma 1.11. *For $x > 1$ real and $1 \le h \le R \le x$ we have*

$$\sum_{k\le h}(h-k)\sum_{n\le x}\lambda_R(n)\lambda_R(n+k) = \frac{h^2x}{2} - \frac{hx}{2}\log h + O(hx) + O(h^{3/2}R^2x^{\varepsilon}) + O(h^3x^{\varepsilon}). \tag{1.23}$$

Here every error term is bounded by $O(hx)$ if we assume

$$R \ll \frac{x^{1/2-\varepsilon}}{h^{1/4}} \tag{1.24}$$

and $h \le x^{1/2-\varepsilon}$. Moreover for $1 \le y, R \le x$ we have

$$\sum_{k\le y}\sum_{n\le x}\lambda_R(n)\lambda_R(n+k) = x\sum_{k\le y}\mathfrak{S}_R(k) + O(y^{1/2}R^2x^{\varepsilon}) + O(y^2x^{\varepsilon}). \tag{1.25}$$

Proof. Using the definition of $\lambda_R(n)$ we have

$$\begin{aligned}
&\sum_{k\le h}(h-k)\sum_{n\le x}\lambda_R(n)\lambda_R(n+k)\\
=&\sum_{k\le h}(h-k)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}d\mu(d)(\psi_R(x+k;d,k)-\psi_R(k;d,k))\\
=&\sum_{k\le h}(h-k)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}d\mu(d)\psi_R(x;d,k)\\
&+O\Big(\sum_{k\le h}(h-k)k\sum_{r\le R}\frac{\mu^2(r)2^{\omega(r)}}{\varphi(r)}\Big),
\end{aligned} \tag{1.26}$$

where we employed the estimates

$$\psi_R(x+k;d,k) = \psi_R(x;d,k) + O\big(x^{\varepsilon}\frac{k}{d}\big), \quad \psi_R(k;d,k) = O\big(x^{\varepsilon}\frac{k}{d}\big),$$

which follow from $\lambda_R(n) \ll n^{\varepsilon}$, cp. [7, p. 374, eq. (3.4)], and then used the identity $\sum_{d|r}\mu^2(d) = 2^{\omega(r)}$ for r squarefree. Thus using Lemma 1.8 (c) and (1.9) of Lemma 1.9 together with the explicit description (1.10) of the error term, we can write the RHS of (1.26) as

$$\Sigma_1 + \Sigma_2 + O(h^3x^{\varepsilon})$$

with

$$\Sigma_1 := x\sum_{k\le h}(h-k)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\(d,k)=1}}\frac{d\mu(d)}{\varphi(d)}$$

and

$$\Sigma_2 := \sum_{k\le h}(h-k)\sum_{\delta|k}\sum_{r,r'\le R}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{\substack{d|r\\ d'|r'\\ (d,d')=\delta}}\mu(d)\mu(d')dd'\varrho(x;d,d',k,\delta),$$

where

$$d=\delta d_1,\quad d'=\delta d_2,\quad (d_1,d_2)=1,\quad d_2\overline{d_2}\equiv 1(d_1)$$

and

$$\varrho(x;d,d',k,\delta) := \varrho\big(\frac{x}{\delta d_1 d_2}-\frac{k}{\delta}\frac{\overline{d_2}}{d_1}\big)-\varrho\big(-\frac{k}{\delta}\frac{\overline{d_2}}{d_1}\big). \tag{1.27}$$

1. **Evaluation of Σ_1.**

Since for d squarefree, we have

$$\sum_{\substack{d|r\\ (d,k)=1}}\frac{d\mu(d)}{\varphi(d)} = \prod_{\substack{p|r\\ p\nmid k}}\big(1-\frac{p}{p-1}\big) = \prod_{\substack{p|r\\ p\nmid k}}\frac{(-1)}{p-1} = \prod_{p|r}\frac{(-1)}{p-1}\prod_{p|r,k}(1-p)$$
$$= \frac{\mu(r)}{\varphi(r)}\mu(r,k)\varphi(r,k) \tag{1.28}$$

by multiplicativity, we obtain

$$\Sigma_1 = x\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{k\le h}(h-k)\mu(r,k)\varphi(r,k). \tag{1.29}$$

Using the identity

$$\sum_{\ell|(r,k)}\mu(\ell)\ell = \mu(r,k)\varphi(r,k)$$

we conclude that

$$\sum_{k\le h}(h-k)\mu(r,k)\varphi(r,k) = \sum_{k\le h}(h-k)\sum_{\ell|(r,k)}\mu(\ell)\ell = \sum_{\substack{\ell\le h\\ \ell|r}}\mu(\ell)\ell\sum_{\substack{k\le h\\ \ell|k}}(h-k)$$
$$= \sum_{\substack{\ell\le h\\ \ell|r}}\mu(\ell)\ell^2\sum_{m\le\frac{h}{\ell}}\big(\frac{h}{\ell}-m\big)$$
$$= \frac{h^2}{2}\sum_{\substack{\ell\le h\\ \ell|r}}\mu(\ell)-\frac{h}{2}\sum_{\substack{\ell\le h\\ \ell|r}}\mu(\ell)\ell+O\Big(\sum_{\substack{\ell\le h\\ \ell|r}}\ell^2\Big),$$

where we used

$$\sum_{m\le y}(y-m)=\frac{y^2}{2}-\frac{y}{2}+O(1)$$

in the second last equation, and so we obtain

$$\Sigma_1=\mathcal{S}_1-\mathcal{S}_2+\mathcal{E}$$

with

$$\mathcal{S}_1:=\frac{h^2x}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le h\\ \ell|r}}\mu(\ell),$$

$$\mathcal{S}_2:=\frac{hx}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le h\\ \ell|r}}\mu(\ell)\ell,$$

and

$$\mathcal{E}\ll x\sum_{r\le R}\frac{\mu^2(r)}{\varphi^2(r)}\sum_{\substack{\ell\le h\\ \ell|r}}\ell^2.$$

1.1 Estimation of $\mathcal{E}$.

We have

$$\begin{aligned}\mathcal{E}\ll x\sum_{r\le R}\frac{\mu^2(r)}{\varphi^2(r)}\sum_{\substack{\ell\le h\\ \ell|r}}\ell^2=x\sum_{\ell\le h}\ell^2\sum_{\substack{r\le R\\ \ell|r}}\frac{\mu^2(r)}{\varphi^2(r)}&=x\sum_{\ell\le h}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)}\sum_{\substack{r'\le\frac{R}{\ell}\\ (r',\ell)=1}}\frac{\mu^2(r')}{\varphi^2(r')}\\ &\ll x\sum_{\ell\le h}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)}\sum_{r'=1}^{\infty}\frac{\mu^2(r')}{\varphi^2(r')}\\ &\ll x\sum_{\ell\le h}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)}\end{aligned}$$

and therefore

$$\mathcal{E}\ll hx \tag{1.30}$$

by Lemma 1.7.

1.2 Evaluation of $\mathcal{S}_1$.

We have

$$\mathcal{S}_1 = \frac{h^2x}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\ell|r}\mu(\ell) - \frac{h^2x}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{h<\ell\le R\\ \ell|r}}\mu(\ell), \tag{1.31}$$

where the first double sum equals

$$\frac{h^2x}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi(r)}\sum_{\ell|r}\mu(\ell) = \frac{h^2x}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi(r)}\delta_{r,1} = \frac{h^2x}{2}$$

and the second double sum can be evaluated as

$$\begin{aligned}
\frac{h^2x}{2}\sum_{h<\ell\le R}\mu(\ell)\sum_{\substack{r\le R\\ \ell|r}}\frac{\mu(r)}{\varphi^2(r)} &= \frac{h^2x}{2}\sum_{h<\ell\le R}\frac{\mu^2(\ell)}{\varphi^2(\ell)}\sum_{\substack{r'\le \frac{R}{\ell}\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\\
&\ll h^2x\sum_{h<\ell\le R}\frac{\mu^2(\ell)}{\varphi^2(\ell)}\sum_{r=1}^{\infty}\frac{\mu^2(r)}{\varphi^2(r)}\\
&\ll h^2x\sum_{\ell>h}\frac{\mu^2(\ell)}{\varphi^2(\ell)}\\
&\ll hx,
\end{aligned}$$

the last inequality following from the estimate $\sum_{\ell>h}\frac{\mu^2(\ell)}{\varphi^2(\ell)} \ll 1/h$, which we can deduce from Lemma 1.7 with $c=0$ and $d=2$. Thus in view of (1.31) we obtain

$$\mathcal{S}_1 = \frac{h^2x}{2} + O(hx). \tag{1.32}$$

1.3 Evaluation of $\mathcal{S}_2$.

We have

$$\mathcal{S}_2 = \frac{hx}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\ell|r}\mu(\ell)\ell - \frac{hx}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{h<\ell\le R\\ \ell|r}}\mu(\ell)\ell \tag{1.33}$$

and because of the identity

$$\sum_{\ell|r}\mu(\ell)\ell = \prod_{p|r}(1-p) = \mu(r)\varphi(r),$$

the first double sum of (1.33) is seen to equal

$$\frac{hx}{2}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)} = \frac{hx}{2}(\log R + O(1)) = \frac{hx}{2}\log R + O(hx), \tag{1.34}$$

while the second double sum of (1.33) equals

$$\begin{aligned}\frac{hx}{2}\sum_{h<\ell\le R}\mu(\ell)\ell\sum_{\substack{r\le R\\ \ell|r}}\frac{\mu(r)}{\varphi^2(r)} &= \frac{hx}{2}\sum_{h<\ell\le R}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\sum_{\substack{r'\le \frac{R}{\ell}\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\\ &= \mathcal{M}_1 + \mathcal{E}'\end{aligned}$$

with

$$\mathcal{M}_1 := \frac{hx}{2}\sum_{h<\ell\le R}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\sum_{\substack{r'=1\\ (r',\ell)=1}}^{\infty}\frac{\mu(r')}{\varphi^2(r')}$$

and

$$\mathcal{E}' \ll hx\sum_{h<\ell\le R}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\sum_{\substack{r'>\frac{R}{\ell}\\ (r',\ell)=1}}\frac{\mu^2(r')}{\varphi^2(r')}.$$

We can estimate the error $\mathcal{E}'$ as

$$\mathcal{E}' \ll hx\sum_{h<\ell\le R}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\sum_{r'>\frac{R}{\ell}}\frac{\mu^2(r')}{\varphi^2(r')} \ll \frac{hx}{R}\sum_{h<\ell\le R}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)} \ll hx,$$

the last estimate being a consequence of Lemma 1.7 with $c = d = 2$.

Next we evaluate $\mathcal{M}_1$. Since we have

$$\sum_{\substack{r'=1\\ (r',\ell)=1}}^{\infty}\frac{\mu(r')}{\varphi^2(r')} = \prod_{p\nmid\ell}\big(1-\frac{1}{(p-1)^2}\big) = \begin{cases}\frac{\mathfrak{S}}{2}\prod_{\substack{p|\ell\\ p>2}}\big(1-\frac{1}{(p-1)^2}\big)^{-1}, & \text{if } 2\mid\ell,\\ 0, & \text{if } 2\nmid\ell\end{cases} \tag{1.35}$$

by multiplicativity, we obtain

$$\begin{aligned}\mathcal{M}_1 &= \frac{h\mathfrak{S}x}{4}\sum_{\substack{h<\ell\le R\\ 2|\ell}}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\prod_{\substack{p|\ell\\ p>2}}\big(1-\frac{1}{(p-1)^2}\big)^{-1}\\ &= \frac{h\mathfrak{S}x}{2}\sum_{\substack{\frac{h}{2}<\ell'\le\frac{R}{2}\\ 2\nmid\ell'}}\frac{\mu^2(\ell')\ell'}{\varphi^2(\ell')}\prod_{p|\ell'}\big(1-\frac{1}{(p-1)^2}\big)^{-1}\end{aligned}$$

$$= \frac{h\mathfrak{S}x}{2} \sum_{\substack{\frac{h}{2} < \ell' \le \frac{R}{2} \\ 2 \nmid \ell'}} \frac{\mu^2(\ell')}{\varphi_2(\ell')},$$

since

$$\begin{aligned}
&\frac{\mu^2(\ell')\ell'}{\varphi^2(\ell')} \prod_{p|\ell'} \big(1 - \frac{1}{(p-1)^2}\big)^{-1} = \prod_{p|\ell'} \frac{p}{(p-1)^2} \prod_{p|\ell'} \big(1 - \frac{1}{(p-1)^2}\big)^{-1} \\
&= \prod_{p|\ell'} \frac{p}{(p-1)^2} \frac{(p-1)^2}{(p-1)^2 - 1} = \prod_{p|\ell'} \frac{p}{p^2 - 2p} \\
&= \prod_{p|\ell'} \frac{1}{p-2} = \frac{\mu^2(\ell')}{\varphi_2(\ell')}. \qquad (1.36)
\end{aligned}$$

Now we make use of (1.8) of Lemma 1.6 with $m = 1$ obtaining

$$\begin{aligned}
\mathcal{M}_1 = \frac{h\mathfrak{S}x}{2} \sum_{\substack{\frac{h}{2} < \ell' \le \frac{R}{2} \\ 2 \nmid \ell'}} \frac{\mu^2(\ell')}{\varphi_2(\ell')} &= \frac{hx}{2} \frac{\mathfrak{S}}{2} \prod_{p>2} \big(1 + \frac{1}{p(p-2)}\big) \log\big(\frac{R}{h}\big) + O(hx) \\
&= \frac{hx}{2} \frac{\mathfrak{S}}{2} \big(\frac{\mathfrak{S}}{2}\big)^{-1} \log\big(\frac{R}{h}\big) + O(hx) \\
&= \frac{hx}{2} \log\big(\frac{R}{h}\big) + O(hx),
\end{aligned}$$

hence the second double sum of (1.33) equals

$$\mathcal{M}_1 + \mathcal{E}' = \frac{hx}{2} \log\big(\frac{R}{h}\big) + O(hx),$$

so that in view of (1.34) we obtain

$$\mathcal{S}_2 = \frac{hx}{2} \log h + O(hx). \qquad (1.37)$$

From (1.30), (1.32) and (1.37) we infer that Σ_1 can be written as

$$\Sigma_1 = \frac{h^2 x}{2} - \frac{hx}{2} \log h + O(hx). \qquad (1.38)$$

2 Estimation of Σ_2.

We write the innermost sum of Σ_2 as

$$\mu^2(\delta)\delta^2 \sum_{\substack{d_1 | \frac{r}{\delta} \\ d_2 | \frac{r'}{\delta} \\ (d_1, d_2) = 1}} \mu(d_1)\mu(d_2) d_1 d_2 \varrho(x; \delta d_1, \delta d_2, k, \delta),$$

so that Σ_2 becomes

$$\sum_{k\le h}(h-k)\sum_{\delta|k}\mu^2(\delta)\delta^2\sum_{r,r'\le R}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{\substack{d_1|\frac{r}{\delta}\\ d_2|\frac{r'}{\delta}\\ (d_1,d_2)=1}}\mu(d_1)\mu(d_2)d_1d_2\varrho(x;\delta d_1,\delta d_2,k,\delta)$$

$$=\sum_{\delta\le h}\mu^2(\delta)\delta^2\sum_{\substack{d_1,d_2\le R/\delta\\ (d_1,d_2)=1}}\mu(d_1)\mu(d_2)d_1d_2\sum_{\substack{r,r'\le R\\ \delta d_1|r\\ \delta d_2|r'}}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{\substack{k\le h\\ \delta|k}}(h-k)\varrho(x;\delta d_1,\delta d_2;k,\delta)$$

$$=\sum_{\delta\le h}\mu^2(\delta)\delta^3\sum_{\substack{d_1,d_2\le R/\delta\\ (d_1,d_2)=1}}\mu(d_1)\mu(d_2)d_1d_2\sum_{\substack{r,r'\le R\\ \delta d_1|r\\ \delta d_2|r'}}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot\,\varrho(x;\delta d_1,\delta d_2,m\delta,\delta)$$

$$=\sum_{\delta\le h}\mu^2(\delta)\delta^3\sum_{\substack{d_1,d_2\le R/\delta\\ (d_1,d_2)=1}}\frac{\mu(d_1)\mu(d_2)d_1d_2}{\varphi(\delta d_1)\varphi(\delta d_2)}\sum_{\substack{r_1\le R/(\delta d_1)\\ r_2\le R/(\delta d_2)\\ (r_1,\delta d_1)=(r_2,\delta r_2)=1}}\frac{\mu^2(r_1)\mu^2(r_2)}{\varphi(r_1)\varphi(r_2)}\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot\,\varrho(x;\delta d_1,\delta d_2,m\delta,\delta)$$

$$\ll\sum_{\delta\le h}\mu^2(\delta)\delta\sum_{\substack{d_1,d_2\le R/\delta\\ (d_1,d_2)=1}}\mu^2(d_1)\mu^2(d_2)\log\big(\frac{R}{\delta d_1}\big)\log\big(\frac{R}{\delta d_2}\big)\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot\,|\varrho(x;\delta d_1,\delta d_2,m\delta,\delta)|, \tag{1.39}$$

where we used a consequence of Lemma 1.5, namely

$$\sum_{\substack{r_i\le R/(\delta d_i)\\ (r_i,\delta d_i)=1}}\frac{\mu^2(r_i)}{\varphi(r_i)}\ll\frac{\varphi(\delta d_i)}{\delta d_i}\log\big(\frac{R}{\delta d_i}\big)\qquad\text{for } i=1,2$$

in the last inequality.

Next, on using the triangle inequality and by writing out $\varrho(x;\delta d_1,\delta d_2;k,\delta)$ again, we see that the RHS of (1.39) is

$$\le\sum_{\delta\le h}\mu^2(\delta)\delta\sum_{\substack{d_1,d_2\le R/\delta\\ (d_1,d_2)=1}}\mu^2(d_1d_2)\log\big(\frac{R}{\delta d_1}\big)\log\big(\frac{R}{\delta d_2}\big)\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot\,\big(\big|\varrho\big(\frac{x}{\delta d_1d_2}-m\frac{\overline{d_2}}{d_1}\big)\big|+\big|\varrho\big(-m\frac{\overline{d_2}}{d_1}\big)\big|\big). \tag{1.40}$$

2.1 Application of Vaaler's Theorem

Vaaler's Theorem (cp. [1, p. 299, Thm. 6.1]) states that for every $L \geq 1$ real and any real w, there exist complex coefficients c_ℓ satisfying

$$c_\ell \ll \ell^{-1}, \tag{1.41}$$

such that

$$|\varrho(w)| \leq L^{-1} + \sum_{0<|\ell|\leq L} c_\ell e(\ell w). \tag{1.42}$$

Applying this, we see that the RHS of (1.40) is $\leq \mathcal{U}_1 + \mathcal{U}_2$ with

$$\mathcal{U}_1 := 2L^{-1} \sum_{\delta\leq h} \mu^2(\delta)\delta \sum_{\substack{d_1,d_2\leq R/\delta \\ (d_1,d_2)=1}} \mu^2(d_1)\mu^2(d_2) \log\big(\frac{R}{\delta d_1}\big) \log\big(\frac{R}{\delta d_2}\big) \sum_{m\leq h/\delta} \big(\frac{h}{\delta} - m\big)$$

and

$$\mathcal{U}_2 := \sum_{0<|\ell|\leq L} c_\ell \sum_{\delta\leq h} \mu^2(\delta)\delta \sum_{\substack{d_1,d_2\leq R/\delta \\ (d_1,d_2)=1}} \mu^2(d_1)\mu^2(d_2) \log\big(\frac{R}{\delta d_1}\big) \log\big(\frac{R}{\delta d_2}\big) \sum_{m\leq h/\delta} \big(\frac{h}{\delta} - m\big)$$
$$\cdot \big(e\big(\frac{\ell x}{\delta d_1 d_2} - m\frac{\ell\overline{d_2}}{d_1}\big) + e\big(-m\frac{\ell\overline{d_2}}{d_1}\big)\big).$$

Estimating trivially, we can conclude that

$$\begin{aligned}
\mathcal{U}_1 &\ll L^{-1} \sum_{\delta\leq h} \mu^2(\delta)\delta \sum_{\substack{d_1,d_2\leq R/\delta \\ (d_1,d_2)=1}} \mu^2(d_1)\mu^2(d_2) \log\big(\frac{R}{\delta d_1}\big) \log\big(\frac{R}{\delta d_2}\big)\big(\frac{h}{\delta}\big)^2 \\
&\ll L^{-1} \sum_{\delta\leq h} \mu^2(\delta)\delta\big(\frac{R}{\delta}\big)^2(\log R)^2\big(\frac{h}{\delta}\big)^2 \\
&= L^{-1}h^2R^2(\log R)^2 \sum_{\delta\leq h} \frac{\mu^2(\delta)}{\delta^3} \\
&\ll L^{-1}h^2R^2(\log R)^2.
\end{aligned} \tag{1.43}$$

It remains to deal with the second expression $\mathcal{U}_2$ involving exponential terms, where we only treat the part involving $e\big(\frac{\ell x}{\delta d_1 d_2} - m\frac{\ell\overline{d_2}}{d_1}\big)$, since the other part can be treated similarly. To this end we write for each ℓ the corresponding summand in $\mathcal{U}_2$ as

$$c_\ell \sum_{\delta\leq h} \mu^2(\delta)\delta \sum_{\substack{d_1,d_2\leq R/\delta \\ (d_1,d_2)=1}} \mu^2(d_1)\mu^2(d_2) \log\big(\frac{R}{\delta d_1}\big) \log\big(\frac{R}{\delta d_2}\big) \sum_{m\leq h/\delta} \big(\frac{h}{\delta} - m\big)$$

$$\cdot e\Big(\frac{\ell x}{\delta d_1 d_2} - m\frac{\ell\overline{d_2}}{d_1}\Big)$$

$$= c_\ell \sum_{\eta|\ell} \sum_{\delta\le h} \mu^2(\delta)\delta \sum_{\substack{d_1,d_2\le R/\delta\\(d_1,d_2)=1\\(d_1,\ell)=\eta}} \mu^2(d_1d_2)\log\big(\frac{R}{\delta d_1}\big)\log\big(\frac{R}{\delta d_2}\big)\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot e\Big(\frac{\ell x}{\delta d_1 d_2} - m\frac{\ell\overline{d_2}}{d_1}\Big)$$

$$= c_\ell \sum_{\delta\le h} \mu^2(\delta)\delta \sum_{\substack{\eta|\ell\\ \eta\le R/\delta}} \mu^2(\eta) \sum_{\substack{d_1'\le R/(\eta\delta)\\ d_2\le R/\delta\\ (d_1',d_2)=1\\ (d_1',\ell/\eta)=1}} \mu^2(d_1'd_2)\log\big(\frac{R}{\eta\delta d_1'}\big)\log\big(\frac{R}{\delta d_2}\big)\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot e\Big(\frac{(\ell/\eta)x}{d_1' d_2} - m\frac{(\ell/\eta)\overline{d_2}}{d_1'}\Big), \tag{1.44}$$

on substituting $d_1 = \eta d_1'$ in the last equation, where we just need to redefine $\overline{d_2}$ mod d_1' by $d_2\overline{d_2} \equiv 1(d_1')$. Since the factor $e\big(\frac{(\ell/\eta)x}{d_1'd_2}\big)$ occurring in the innermost sum of (1.44) does not depend on m, an application of the triangle inequality gives

$$\Bigg| c_\ell \sum_{\delta\le h} \mu^2(\delta)\delta \sum_{\substack{\eta|\ell\\ \eta\le R/\delta}} \mu^2(\eta) \sum_{\substack{d_1'\le R/(\eta\delta)\\ d_2\le R/\delta\\ (d_1',d_2)=1\\ (d_1',\ell/\eta)=1}} \mu^2(d_1'd_2)\log\big(\frac{R}{\eta\delta d_1'}\big)\log\big(\frac{R}{\delta d_2}\big)\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot e\Big(\frac{(\ell/\eta)x}{d_1' d_2} - m\frac{(\ell/\eta)\overline{d_2}}{d_1'}\Big)\Bigg|$$

$$\le \frac{1}{\ell} \sum_{\delta\le h} \mu^2(\delta)\delta \sum_{\substack{\eta|\ell\\ \eta\le R/\delta}} \mu^2(\eta) \sum_{\substack{d_1'\le R/(\eta\delta)\\ d_2\le R/\delta\\ (d_1',d_2)=1\\ (d_1',\ell/\eta)=1}} \mu^2(d_1'd_2)\log\big(\frac{R}{\eta\delta d_1'}\big)\log\big(\frac{R}{\delta d_2}\big)\Bigg|\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot e\Big(\frac{(\ell/\eta)x}{d_1' d_2} - m\frac{(\ell/\eta)\overline{d_2}}{d_1'}\Big)\Bigg|$$

$$\le \frac{1}{\ell} \sum_{\delta\le h} \delta \sum_{\eta|\ell} \mu^2(\eta) \sum_{\substack{d_1'\le R/(\eta\delta)\\ (d_1',\ell/\eta)=1}} \log\big(\frac{R}{\eta\delta d_1'}\big) \sum_{\substack{d_2\le R/\delta\\ (d_2,d_1')=1}} \log\big(\frac{R}{\delta d_2}\big)\Bigg|\sum_{m\le h/\delta}\big(\frac{h}{\delta}-m\big)$$

$$\cdot e\Big(-m\frac{(\ell/\eta)\overline{d_2}}{d_1'}\Big)\Bigg|$$

$$:= \mathcal{S}_{h,R,\ell},$$

on using (1.41) in the second estimate and dropping the restrictions that the integers δ, d_1' and d_2 are squarefree and $\eta \leq R/\delta$ in the last inequality.

2.2 Employment of the Large Sieve Inequality

We remark that the number $(\ell/\eta)\overline{d_2}$ runs through at most $R/(\delta d_1')$ many complete sets of reduced residues mod d_1' when d_2 does, and thus

$$\mathcal{S}_{h,R,\ell} \leq \frac{R(\log R)^2}{\ell} \sum_{\delta \leq h} \sum_{\eta | \ell} \mu^2(\eta) \sum_{d_1' \leq R/(\eta\delta)} \frac{1}{d_1'} \sum_{(a,d_1')}^{*} \Big| \sum_{m \leq h/\delta} \big(\frac{h}{\delta} - m\big) e\big(m\frac{a}{d_1'}\big) \Big|, \tag{1.45}$$

where we can evaluate the RHS in (1.45) using the Large Sieve Inequality, cp. Lemma 1.1 with $M = R/(\eta\delta)$, $N = h/\delta$, and employing Cauchy's inequality twice to obtain

$$\begin{aligned}
\mathcal{S}_{h,R,\ell} \ll & \frac{R(\log R)^2}{\ell} \sum_{\delta \leq h} \sum_{\eta | \ell} \mu^2(\eta) \Big(\sum_{d_1' \leq R/(\eta\delta)} \frac{1}{d_1'} \Big)^{1/2} \\
& \cdot \Big(\sum_{d_1' \leq R/\eta} \frac{1}{d_1'} \Big(\sum_{(a,d_1')}^{*} \Big| \sum_{m \leq h/\delta} \big(\frac{h}{\delta} - m\big) e\big(m\frac{a}{d_1'}\big) \Big| \Big)^2 \Big)^{1/2} \\
\ll & \frac{R(\log R)^{5/2}}{\ell} \sum_{\delta \leq h} \sum_{\eta | \ell} \mu^2(\eta) \Big(\sum_{d_1' \leq R/(\eta\delta)} \sum_{(a,d_1')}^{*} \Big| \sum_{m \leq h/\delta} \big(\frac{h}{\delta} - m\big) e\big(m\frac{a}{d_1'}\big) \Big|^2 \Big)^{1/2} \\
\ll & \frac{R(\log R)^{5/2}}{\ell} \sum_{\delta \leq h} \sum_{\eta | \ell} \mu^2(\eta) \big(\frac{h}{\delta} + \big(\frac{R}{\eta\delta}\big)^2 \big)^{1/2} \big(\frac{h}{\delta}\big)^{3/2} \\
\ll & \frac{h^2 R(\log R)^{5/2}}{\ell} \sum_{\delta \leq h} \frac{1}{\delta^2} \sum_{\eta | \ell} \mu^2(\eta) + \frac{h^{3/2} R^2 (\log R)^{5/2}}{\ell} \sum_{\delta \leq h} \frac{1}{\delta^{5/2}} \sum_{\eta | \ell} \frac{\mu^2(\eta)}{\eta} \\
\ll & \frac{\log \ell}{\ell} (\log R)^{5/2} (h^2 R + h^{3/2} R^2)
\end{aligned}$$

and hence we obtain the estimate

$$\mathcal{U}_2 \leq \sum_{0 < |\ell| \leq L} \mathcal{S}_{R,h,\ell} \ll (\log R)^{5/2} (\log L)^2 (h^2 R + h^{3/2} R^2),$$

which together with (1.43) implies

$$\Sigma_2 \ll \mathcal{U}_1 + \mathcal{U}_2 \ll L^{-1} h^2 R^2 (\log R)^2 + (\log R)^{5/2} (\log L)^2 (h^2 R + h^{3/2} R^2).$$

If we pick $L = \sqrt{h}$ and consider that $1 \le h \le R \le x$ by assumption, we are led to

$$\Sigma_2 \ll h^{3/2}R^2(\log R)^2 + (\log R)^{9/2}(h^2R + h^{3/2}R^2) \ll h^{3/2}R^2x^{\varepsilon},$$

which together with (1.38) gives (1.23).

• We next prove (1.25). Analogously to the proof of (1.23), we can infer

$$\begin{aligned}\sum_{k\le y}\sum_{n\le x}\lambda_R(n)\lambda_R(n+k) &= \sum_{k\le y}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\(d,k)=1}}\frac{d\mu(d)}{\varphi(d)} + \widetilde{\Sigma} + O(y^2x^{\varepsilon})\\ &= \sum_{k\le y}\mathfrak{S}_R(k) + \widetilde{\Sigma} + O(y^2x^{\varepsilon}) \quad \text{by (1.28) and definition of } \mathfrak{S}_R,\end{aligned}$$

where

$$\widetilde{\Sigma} = \sum_{k\le y}\sum_{\delta|k}\sum_{r,r'\le R}\frac{\mu^2(r)\mu^2(r')}{\varphi(r)\varphi(r')}\sum_{\substack{d|r\\d'|r'\\(d,d')=\delta}}\mu(d)\mu(d')dd'\varrho(x;d,d',k,\delta).$$

Estimating $\widetilde{\Sigma}$ like Σ_2 of p. 14 yields (1.25). ■

Lemma 1.12. *Let $1 \le h \le R \le x$. Then assuming Hypothesis M, we have*

$$\begin{aligned}\sum_{k\le h}(h-k)\sum_{n\le x}\lambda_R(n)\Lambda(n+k) =& \frac{h^2x}{2} - \frac{hx}{2}\log h + O(h^2R^{1/4}x^{1/2+\varepsilon})\\ &+ O(h^{3/2}R^{3/4}x^{1/2+\varepsilon}) + O(h^2Rx^{\varepsilon}), \qquad (1.46)\end{aligned}$$

the same asymptotic formula applying to

$$\sum_{k\le h}(h-k)\sum_{n\le x}\Lambda(n)\lambda_R(n+k).$$

Here every error term is bounded by $O(hx)$ if we assume that

$$R \ll \min\Big(\frac{x^{2-\varepsilon}}{h^4}, \frac{x^{2/3-\varepsilon}}{h^{2/3}}, \frac{x^{1-\varepsilon}}{h}\Big). \qquad (1.47)$$

Proof. Inserting the definition of $\lambda_R(n)$ and $E(y;d,k)$ we first have

$$\sum_{n\le x}\lambda_R(n)\Lambda(n+k) = \sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}d\mu(d)\sum_{\substack{n\le x\\d|n}}\Lambda(n+k)$$

$$= \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r} d\mu(d)(\psi(x+k;d,k) - \psi(k;d,k))$$

and since

$$\psi(x+k;d,k) = \psi(x;d,k) + O\big(\frac{k \log x}{d}\big), \quad \psi(k;d,k) = O\big(\frac{k \log x}{d}\big),$$

this gives

$$\sum_{n \le x} \lambda_R(n)\Lambda(n+k) = \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r} d\mu(d)\psi(x;d,k) + O\bigg(k \log x \sum_{r \le R} \frac{\mu^2(r) 2^{\omega(r)}}{\varphi(r)}\bigg),$$

whence

$$\begin{aligned}\sum_{k \le h}(h-k) \sum_{n \le x} \lambda_R(n)\Lambda(n+k) = &\sum_{k \le h}(h-k) \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r} d\mu(d)\psi(x;d,k) \\ &+ O\bigg(x^{\varepsilon} \sum_{k \le h}(h-k)k\bigg)\end{aligned}$$

by Lemma 1.8 (c), which by inserting

$$\psi(x;d,k) = E_{d,k}\frac{x}{\varphi(d)} + E(x;d,k)$$

equals

$$\begin{aligned} x\sum_{k \le h}(h-k) \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r \\ (d,k)=1}} \frac{d\mu(d)}{\varphi(d)} + \sum_{k \le h}(h-k) \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r} d\mu(d)E(x;d,k) \\ + O(h^3 x^{\varepsilon}). \qquad (1.48)\end{aligned}$$

In order to estimate the second triple sum in (1.48), we note that interchanging the order of summation of r and d yields

$$\begin{aligned}\sum_{k \le h}(h-k) \sum_{r \le R} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r} d\mu(d)E(x;d,k) &= \sum_{k \le h}(h-k) \sum_{d \le R} d\mu(d)E(x;d,k) \sum_{\substack{r \le R \\ d|r}} \frac{\mu^2(r)}{\varphi(r)} \\ &= \sum_{k \le h}(h-k) \sum_{d \le R} \frac{d\mu(d)}{\varphi(d)} E(x;d,k) \sum_{\substack{r' \le R/d \\ (r',d)=1}} \frac{\mu^2(r')}{\varphi(r')},\end{aligned}$$

where

$$\sum_{\substack{r' \le R/d \\ (r',d)=1}} \frac{\mu^2(r')}{\varphi(r')} \ll \frac{\varphi(d)}{d} \log\big(\frac{R}{d}\big)$$

by Lemma 1.5, so that

$$\sum_{k\le h}(h-k)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}d\mu(d)E(x;d,k)\ll\sum_{d\le R}\log\big(\frac{R}{d}\big)\sum_{k\le h}(h-k)|E(x;d,k)|. \tag{1.49}$$

To estimate the double sum in (1.49), we insert the condition $(d,k)=1$ introducing an error, which is bounded by

$$h\log R\sum_{k\le h}\sum_{\substack{d\le R\\(d,k)>1}}|E(x;d,k)|=h\log R\sum_{k\le h}\sum_{\substack{d\le R\\(d,k)>1}}\psi(x;d,k),$$

so that this additional error contributes at most

$$\begin{aligned}hR\log R\sum_{k\le h}\sum_{\substack{n\le x\\(n,k)>1}}\Lambda(n)=hR\log R\sum_{k\le h}\sum_{p|k}\sum_{\substack{m\le x\\p^m\le x}}\log p&\le hR\log R\sum_{k\le h}\sum_{p|k}\log p[\frac{\log x}{\log p}]\\&\le hR(\log x)^2\sum_{k\le h}\omega(k)\\&\ll h^2Rx^{\varepsilon}\quad\text{by Lemma 1.8 (d).}\end{aligned}$$

We note that for $d\le R$ with $(d,k)=1$, the numbers k run through at most $[\frac{h}{d}]+1$ many complete sets of reduced residues mod d. We deduce that by Cauchy's inequality,

$$\begin{aligned}&\sum_{d\le R}\log\big(\frac{R}{d}\big)\sum_{\substack{k\le h\\(d,k)=1}}(h-k)|E(x;d,k)|\\
\le&\sum_{d\le R}\log\big(\frac{R}{d}\big)\Big(\sum_{k\le h}(h-k)^2\Big)^{1/2}\Big(\sum_{k\le h}E^2(x;d,k)\Big)^{1/2}\\
\le&\sum_{d\le R}\log\big(\frac{R}{d}\big)\Big(\sum_{k\le h}(h-k)^2\Big)^{1/2}\big(\frac{h}{d}+1\big)^{1/2}\Big({\sum_{a(d)}}^{*}E^2(x,d,a)\Big)^{1/2}\\
\le&h^{3/2}\log R\sum_{d\le R}\Big(\big(\frac{h}{d}\big)^{1/2}+1\Big)\Big({\sum_{a(d)}}^{*}E^2(x,d,a)\Big)^{1/2},\end{aligned}$$

where this sum can be bounded using Hypothesis M (Lemma 1.2), which gives

$$\begin{aligned}&h^{3/2}\log R\sum_{d\le R}\Big(\big(\frac{h}{d}\big)^{1/2}+1\Big)\Big({\sum_{a(d)}}^{*}E^2(x,d,a)\Big)^{1/2}\\
\ll&h^{3/2}x^{1/2+\varepsilon}\sum_{d\le R}\Big(\big(\frac{h}{d}\big)^{1/2}+1\Big)\frac{1}{d^{1/4}}\end{aligned}$$

$$\ll h^2 R^{1/4} x^{1/2+\varepsilon} + h^{3/2} R^{3/4} x^{1/2+\varepsilon}.$$

Finally, by the estimates $\Lambda(m), \lambda_R(m) \ll m^\varepsilon$, we have

$$\sum_{k\le h}(h-k)\sum_{n\le x}\Lambda(n)\lambda_R(n+k) = \sum_{k\le h}(h-k)\sum_{k<n\le x}\Lambda(n-k)\lambda_R(n) + O(h^3x^\varepsilon)$$

and since $\psi(x;k,d) = \psi(x;-k,d)$, the result for this expression follows from (1.46). ∎

1.2 Proof of Theorem 1

We are now ready to prove Theorem 1.

Throughout the proof we assume that $1 \le h \le R$ as well as $h \le x^{1/2-\varepsilon}$. From the inequality

$$\int_0^x (\psi(y;h) - \psi_R(y;h))^2 dy \ge 0$$

we infer

$$\int_0^x \psi^2(y;h)dy \ge 2\int_0^x \psi(y;h)\psi_R(y;h)dy - \int_0^x \psi_R^2(y;h)dy \tag{1.50}$$

and the estimate $\lambda_R(n) \ll x^\varepsilon$ for $n \le x$ combined with Lemma 1.4 gives

$$\begin{aligned}\int_0^x \psi(y;h)\psi_R(y;h)dy =& h\sum_{n\le x}\lambda_R(n)\Lambda(n) + \sum_{k\le h}(h-k)\sum_{n\le x}\lambda_R(n)\Lambda(n+k)\\ &+\sum_{k\le h}(h-k)\Lambda(n)\lambda_R(n+k) + O(h^3x^\varepsilon)\end{aligned} \tag{1.51}$$

as well as

$$\begin{aligned}\int_0^x \psi_R^2(y;h)dy =& h\sum_{n\le x}\lambda_R^2(n) + 2\sum_{k\le h}(h-k)\sum_{n\le x}\lambda_R(n)\lambda_R(n+k)\\ &+O(h^3x^\varepsilon).\end{aligned} \tag{1.52}$$

Now we choose R subject to (1.24) and (1.47).

Then by applying Lemma 1.10 (A) on p. 12 and Lemma 1.12 on p. 24 to (1.51) we obtain

$$\int_0^x \psi(y;h)\psi_R(y;h)dy = hx\log\left(\frac{R}{h}\right) + h^2x + O(hx). \tag{1.53}$$

Moreover, inserting Lemma 1.10 (B) on p. 12 and (1.23) of Lemma 1.11 on p. 14 into (1.52) yields

$$\int_0^x \psi_R^2(y;h)dy = hx\log\big(\frac{R}{h}\big) + h^2x + O(hx) \tag{1.54}$$

and thus resubstituting (1.53) and (1.54) into (1.50) leads us to

$$\int_0^x \psi^2(y;h)dy \geq hx\log\big(\frac{R}{h}\big) + h^2x + O(hx). \tag{1.55}$$

The conditions on R from (1.24) and (1.47) now read

$$R \ll \min\big(\frac{x^{2-\varepsilon}}{h^4}, \frac{x^{1-\varepsilon}}{h}, \frac{x^{2/3-\varepsilon}}{h^{2/3}}, \frac{x^{1/2-\varepsilon}}{h^{1/4}}\big) = \frac{x^{1/2-\varepsilon}}{h^{1/4}}, \quad \text{if } 1 \leq h \leq x^{2/5-\varepsilon},$$

and substituting the choice of $R = \dfrac{x^{1/2-\varepsilon}}{h^{1/4}}$ into (1.55) gives

$$\int_0^x \psi^2(y;h)dy \geq hx\log\big(\frac{x^{1/2}}{h^{5/4}}\big) + h^2x + O(hx) \tag{1.56}$$

uniformly in $1 \leq h \leq x^{2/5-\varepsilon}$. It remains to establish a similar bound for $I(x;h)$, which equals

$$\int_0^x (\psi(y;h)-h)^2dy = \int_0^x \psi^2(y;h)dy - 2h\int_0^x \psi(y;h)dy + h^2x.$$

For this purpose we employ (1.5) of Lemma 1.3 together with the lower bound (1.56) for $\int_0^x \psi^2(y;h)dy$ to obtain

$$I(x;h) \geq hx\log\big(\frac{x^{1/2}}{h^{5/4}}\big) + o(hx\log x)$$

uniformly in $1 \leq h \leq x^{2/5-\varepsilon}$, which shows Theorem 1. ■

2 The unconditional case

In [6, Thm. 1], Goldston proved a lower bound for the second moment of prime numbers in the unconditional case, namely

$$I(x;h) \geq \big(\frac{1}{2} - \varepsilon\big) hx \log x \qquad \text{uniformly in } 1 \leq h \leq (\log x)^A, \tag{2.1}$$

a main ingredient of his proof being the Bombieri–Vinogradov Theorem, cp. [25, p. 161, eq. (1)], which states that

$$\sum_{q \leq Q} \max_{(a,q)=1} \sup_{y \leq x} |E(y;q,a)| \ll_A \sqrt{x} Q (\log x)^5,$$

for fixed $A > 0$, provided that $\frac{\sqrt{x}}{(\log x)^A} \leq Q \leq \sqrt{x}$, to estimate the error occurring in Lemma 1.12 of Chapter 1. The gained factor $(\log x)^{-A}$ then determines the possible range for h.

Applying the Basic Mean Value Theorem instead, see Theorem 5 below, which itself plays a central role in the proof of the Bombieri–Vinogradov Theorem, we show how to gain a subexponential factor in the proof of Lemma 1.12, which in turn widens the possible range for h.

The main result of this section is

Theorem 2. *Let* $1 \leq h \leq e^{c(\log x)^{3/5}(\log\log x)^{-1/5}}$ *with a certain constant* $c > 0$. *Then for* $\varepsilon > 0$ *and* $x \geq X(\varepsilon, c)$ *we have*

$$I(x;h) \geq \big(\frac{1}{2} - \varepsilon\big) hx \log x \quad \textit{uniformly in } h.$$

As in [6, Cor. 1], we can state the following Corollary, which follows immediately from the Theorem.

Corollary 2. *Let* c *be as in Theorem 2. Then for arbitrary* $\varepsilon > 0$ *and* $1 \leq h \leq e^{c(\log x)^{3/5}(\log\log x)^{-1/5}}$ *we have*

$$\max_{y \in [0,x]} |\psi(y;h) - h| \gg_\varepsilon \sqrt{h \log x}.$$

2.1 Ingredients of the proof

Theorem 3 (Pólya–Vinogradov Inequality; [25, p. 135, eq. (2)])**.** *Every primitive Dirichlet character χ mod m with $m > 1$ satisfies the inequality*

$$\left|\sum_{n\le N}\chi(n)\right| < \sqrt{m}\log m.$$

Corollary 3. *If a Dirichlet character χ mod m is induced by a primitive character χ^* mod m^* with $m^* > 1$, then*

$$\left|\sum_{n\le N}\chi(n)\right| < 2^{\omega(m)}\sqrt{m^*}\log m^*.$$

Proof. We have

$$\begin{aligned}\sum_{n\le N}\chi(n) = \sum_{\substack{n\le N\\(n,m)=1}}\chi^*(m) &= \sum_{n\le N}\chi^*(n)\sum_{s|n,m}\mu(s) = \sum_{\substack{s\le N\\s|m}}\mu(s)\sum_{\substack{n\le N\\s|n}}\chi^*(n)\\ &= \sum_{\substack{s\le N\\s|m}}\mu(s)\chi^*(s)\sum_{n'\le N/s}\chi^*(n')\end{aligned}$$

and hence

$$\left|\sum_{n\le N}\chi(n)\right| \le \sum_{s|m}\mu^2(s)\left|\sum_{n'\le N/s}\chi^*(n')\right| < \sum_{s|m}\mu^2(s)\sqrt{m^*}\log m^* = 2^{\omega(m)}\sqrt{m^*}\log m^*.$$

■

In order to control the contribution of moduli m which are small compared to x, we apply to the following sharpened version of the Siegel–Walfisz Theorem.

Theorem 4 ([20, Theorem 1.1])**.** *Let $A > 0$. Then for $(a,m) = 1$ and $m \le (\log x)^A$ we have that*

$$\psi(x;m,a) = \frac{x}{\varphi(m)} + O\big(xe^{-c_A(\log x)^{3/5}(\log\log x)^{-1/5}}\big)$$

for some constant c_A, which cannot be computed effectively.

Moreover, we shall need the

Theorem 5 (Basic Mean Value Theorem; [25, p. 162, eq. (2)])**.** *For $M \ge 1$ and $x \ge 2$ let*

$$T(x,M) := \sum_{m\le M}\frac{m}{\varphi(m)}\sum_{\chi(m)}{}^{*}\sup_{y\le x}|\psi(x,\chi)|. \tag{2.2}$$

Then we have the estimate

$$T(x,M) \ll (x + x^{5/6}M + \sqrt{x}M^2)(\log(xM))^3. \tag{2.3}$$

Lemma 2.1. *For a Dirichlet character χ' mod d we have*

$$\psi(x;\chi') - \delta_{\chi'} x = \sum_{a(d)}^{*} \chi'(a) E(x,d;a).$$

Proof. We have

$$\begin{aligned}
\sum_{a(d^*)}^{*} \chi'(a) E(x,d^*;a) &= \sum_{a(d^*)}^{*} \frac{\chi'(a)}{\varphi(d^*)} \sum_{\chi(d^*)} \overline{\chi}(a)(\psi(x;\chi) - \delta_{\chi} x) \\
&= \sum_{\chi(d^*)} \frac{1}{\varphi(d^*)} (\psi(x;\chi) - \delta_{\chi} x) \sum_{a(d^*)}^{*} \chi'(a) \overline{\chi}(a) \\
&= \psi(x;\chi') - \delta_{\chi'} x
\end{aligned}$$

by orthogonality. ■

From Theorems 3–5 we can deduce the following Proposition, which is the main part in the proof of Theorem 2.

Proposition 1. *Let $\varepsilon > 0$, $x \geq 2$ and $g : \mathbb{N} \to \mathbb{R}$ be a function supported on squarefree integers d with $|g(d)| \leq (\log x)^B$, $B \geq 1$, for each d. Moreover let $(\log x)^{B+6+\varepsilon} \leq h \leq x^{\varepsilon}$. Then there exists some $c > 0$, such that*

$$\begin{aligned}
\sum_{k \leq h} (h-k) \sum_{\substack{d \leq R \\ (d,k)=1}} g(d) E(x;d,k) \ll & h^2 x e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} + h^2 R \sqrt{x} (\log x)^{B+4} \\
& + hx.
\end{aligned}$$

Proof. For each pair d, k with $(d,k) = 1$ we have by definition of $E(x;d,k)$ and orthogonality of Dirichlet characters that

$$E(x;d,k) = \frac{1}{\varphi(d)} \sum_{\chi(d)} \overline{\chi}(k)(\psi(x;\chi) - \delta_{\chi} x),$$

which gives

$$\begin{aligned}
&\sum_{k \leq h} (h-k) \sum_{\substack{d \leq R \\ (d,k)=1}} g(d) E(x;d,k) \\
=& \sum_{k \leq h} (h-k) \sum_{\substack{d \leq R \\ (d,k)=1}} \frac{g(d)}{\varphi(d)} \sum_{\chi(d)} \overline{\chi}(k)(\psi(x;\chi) - \delta_{\chi} x) \\
=& \sum_{d \leq R} \frac{g(d)}{\varphi(d)} \sum_{\chi(d)} (\psi(x;\chi) - \delta_{\chi} x) \sum_{\substack{k \leq h \\ (d,k)=1}} (h-k) \overline{\chi}(k)
\end{aligned}$$

$$= \sum_{d \le R} \frac{g(d)}{\varphi(d)} \sum_{\chi(d)} (\psi(x;\chi) - \delta_\chi x) \sum_{k \le h} (h-k)\overline{\chi}(k), \tag{2.4}$$

where the last equation holds since $\chi(k) = 0$ for $(d,k) > 1$. Letting χ^* mod d^* be the uniquely determined primitive character which induces the character χ mod d, the triple sum in (2.4) equals

$$\mathcal{M} := \sum_{d \le R} \frac{g(d)}{\varphi(d)} \sum_{d^*|d} \sideset{}{^*}\sum_{\chi^*(d^*)} \sum_{\substack{\chi(d) \\ \chi^* \text{ induces } \chi}} (\psi(x;\chi) - \delta_\chi x) \sum_{k \le h} (h-k)\overline{\chi}(k),$$

where the second last sum of $\mathcal{M}$ actually contains only one character χ mod d. Now we can utilize $\psi(x;\chi) - \delta_\chi x = \psi(x;\chi^*) - \delta_{\chi^*} x + O((\log x)^2)$, cp. [3, p. 163, l. 6], to infer

$$\begin{aligned}\mathcal{M} =& \sum_{d \le R} \frac{g(d)}{\varphi(d)} \sum_{d^*|d} \sideset{}{^*}\sum_{\chi^*(d^*)} (\psi(x;\chi^*) - \delta_{\chi^*} x) \sum_{\substack{\chi(d) \\ \chi^* \text{ induces } \chi}} \sum_{k \le h} (h-k)\overline{\chi}(k) \\ &+ O\Big(h^2 (\log x)^2 \sum_{d \le R} \frac{|g(d)|}{\varphi(d)} \sum_{d^*|d} \varphi(d^*) \Big)\end{aligned}$$

with

$$h^2 \sum_{d \le R} \frac{|g(d)|}{\varphi(d)} \sum_{d^*|d} \varphi(d^*) \le h^2 (\log x)^B \sum_{d \le R} \frac{d\mu^2(d)}{\varphi(d)} \ll h^2 R (\log x)^B$$

noting that $\sum_{d^*|d} \varphi(d^*) = d$, the last estimate following from Lemma 1.7.

In a next step we decompose $\mathcal{M}$ into

$$\mathcal{M} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + O\big(h^2 R (\log x)^{B+2}\big) \tag{2.5}$$

with

$$\mathcal{T}_1 := \sum_{d \le R} \frac{g(d)}{\varphi(d)} \sum_{\substack{d^*|d \\ d^* \le (\log x)^A}} \sideset{}{^*}\sum_{\chi^*(d^*)} (\psi(x;\chi^*) - \delta_{\chi^*} x) \sum_{\substack{\chi(d) \\ \chi^* \text{ induces } \chi}} \sum_{k \le h} (h-k)\overline{\chi}(k),$$

$$\mathcal{T}_2 := \sum_{d \le R} \frac{g(d)}{\varphi(d)} \sum_{\substack{d^*|d \\ (\log x)^A < d^* \le h^2}} \sideset{}{^*}\sum_{\chi^*(d^*)} (\psi(x;\chi^*) - \delta_{\chi^*} x) \sum_{\substack{\chi(d) \\ \chi^* \text{ induces } \chi}} \sum_{k \le h} (h-k)\overline{\chi}(k)$$

and

$$\mathcal{T}_3 := \sum_{d \le R} \frac{g(d)}{\varphi(d)} \sum_{\substack{d^*|d \\ h^2 < d^* \le R}} \sideset{}{^*}\sum_{\chi^*(d^*)} (\psi(x;\chi^*) - \delta_{\chi^*} x) \sum_{\substack{\chi(d) \\ \chi^* \text{ induces } \chi}} \sum_{k \le h} (h-k)\overline{\chi}(k)$$

and continue investigating each expression $\mathcal{T}_i$.

• Estimation of $\mathcal{T}_1$

Changing the order of summation of d and d^* we obtain

$$\begin{aligned}
|\mathcal{T}_1| &\le h^2(\log x)^B \sum_{d^*\le(\log x)^A} \sum_{\chi^*(d^*)}{}^{*} |\psi(x;\chi^*)-\delta_{\chi^*}x| \sum_{\substack{d\le R\\ d^*|d}} \frac{\mu^2(d)}{\varphi(d)} \\
&= h^2(\log x)^B \sum_{d^*\le(\log x)^A} \frac{1}{\varphi(d^*)} \sum_{\chi^*(d^*)}{}^{*} |\psi(x;\chi^*)-\delta_{\chi^*}x| \sum_{\substack{d'\le R/d^*\\ (d',d^*)=1}} \frac{\mu^2(d')}{\varphi(d')} \\
&\ll h^2(\log x)^{B+1} \sum_{d^*\le(\log x)^A} \frac{1}{\varphi(d^*)} \sum_{\chi^*(d^*)}{}^{*} |\psi(x;\chi^*)-\delta_{\chi^*}x| \\
&\ll h^2(\log x)^{B+1} \sum_{d^*\le(\log x)^A} \varphi(d^*) \max_{(a,d^*)=1} |E(x;d^*,a)| \quad \text{by Lemma 2.1} \qquad (2.6) \\
&\ll h^2(\log x)^{2A+B+1} x e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} \quad \text{for some } c>0, \text{ by Theorem 4} \\
&\ll h^2 x e^{-c'(\log x)^{3/5}(\log\log x)^{-1/5}} \quad \text{for some } c' \text{ with } 0<c'<c. \qquad (2.7)
\end{aligned}$$

• Estimation of $\mathcal{T}_2$

In the estimation of $\mathcal{T}_2$ it is advantageous to refer to Corollary 3, from which it follows that

$$\left|\sum_{k\le h}(h-k)\overline{\chi}(k)\right| = \left|\int_1^h \left(\sum_{m\le t}\overline{\chi}(m)\right)dt\right| \le \int_1^h \left|\sum_{m\le t}\overline{\chi}(m)\right|dt \le h2^{\omega(d)}\sqrt{d^*}\log d^*$$

on summing by parts. Noting that $\delta_{\chi^*}=0$ for a primitive character with conductor $d^*>1$, we can bound $\mathcal{T}_2$ as

$$\begin{aligned}
|\mathcal{T}_2| &\le h(\log x)^{B+1} \sum_{(\log x)^A<d^*\le h^2} \sqrt{d^*} \sum_{\chi^*(d^*)}{}^{*} \sup_{y\le x} |\psi(y;\chi^*)| \sum_{\substack{d\le R\\ d^*|d}} \frac{2^{\omega(d)}\mu^2(d)}{\varphi(d)} \\
&= h(\log x)^{B+1} \sum_{(\log x)^A<d^*\le h^2} \frac{\sqrt{d^*}2^{\omega(d^*)}\mu^2(d^*)}{\varphi(d^*)} \sum_{\chi^*(d^*)}{}^{*} \sup_{y\le x} |\psi(y;\chi^*)| \sum_{\substack{d'\le R/d^*\\ (d',d^*)=1}} \frac{2^{\omega(d')}\mu^2(d')}{\varphi(d')} \\
&\ll h(\log x)^{B+3} \sum_{(\log x)^A<d^*\le h^2} \frac{(d^*)^{1/2+\varepsilon}}{\varphi(d^*)} \sum_{\chi^*(d^*)}{}^{*} \sup_{y\le x} |\psi(y;\chi^*)|
\end{aligned}$$

using $2^{\omega(d^*)} \ll (d^*)^\varepsilon$, which follows from and Lemma 1.8 (c) in the last estimate. On using summation by parts we can write

$$\sum_{(\log x)^A < d^* \le h^2} \frac{(d^*)^{1/2+\varepsilon}}{\varphi(d^*)} \sideset{}{^*}\sum_{\chi^*(d^*)} \sup_{y \le x} |\psi(y;\chi^*)| = \frac{T(x,h^2)}{h^{1-\varepsilon}} - \frac{T(x,(\log x)^A)}{(\log x)^{A(1/2-\varepsilon)}} + \Big(\frac{1}{2}-\varepsilon\Big)\int_{(\log x)^A}^{h^2} T(x,t)\frac{dt}{t^{3/2-\varepsilon}}$$

with $T(x,t)$ as in (2.2), where by Theorem 5 the RHS is majorized by

$$\begin{aligned}
&(x + h^2x^{5/6} + h^4\sqrt{x})h^{\varepsilon-1}(\log x)^3 + \big(x + (\log x)^A x^{5/6} + (\log x)^{2A}\sqrt{x}\big)(\log x)^{3-A(1/2-\varepsilon)} \\
&+ \int_{(\log x)^A}^{h^2} (x + tx^{5/6} + t^2\sqrt{x})(\log x)^3 \frac{dt}{t^{3/2-\varepsilon}} \\
\ll& \Big(\frac{x}{h^{1-\varepsilon}} + h^{1+\varepsilon}x^{5/6} + h^{3+\varepsilon}\sqrt{x}\Big)(\log x)^3 + x(\log x)^{3-A(1/2-\varepsilon)} \\
\ll& \frac{x}{h^{1-\varepsilon}}(\log x)^3 + x(\log x)^{3-A(1/2-\varepsilon)} \quad \text{since } h \le x^\varepsilon \text{ by assumption.}
\end{aligned}$$

Therefore relying on our assumption that $(\log x)^{B+6+\varepsilon} \le h$ and choosing $A = \dfrac{B+6}{1/2-\varepsilon}$ we have

$$\mathcal{T}_2 \ll h^\varepsilon x(\log x)^{B+6} + hx(\log x)^{B-A(1/2-\varepsilon)+6} \ll hx. \tag{2.8}$$

- Estimation of $\mathcal{T}_3$

Lastly we can bound $\mathcal{T}_3$ as

$$|\mathcal{T}_3| \le h^2(\log x)^{B+1} \sum_{h^2 < d^* \le R} \frac{1}{\varphi(d^*)} \sideset{}{^*}\sum_{\chi^*(d^*)} \sup_{y \le x} |\psi(y;\chi^*)|, \tag{2.9}$$

where summation by parts yields

$$\sum_{h^2 < d^* \le R} \frac{1}{\varphi(d^*)} \sideset{}{^*}\sum_{\chi^*(d^*)} \sup_{y \le x} |\psi(y;\chi^*)| = \frac{T(x,R)}{R} - \frac{T(x,h^2)}{h^2} + \int_{h^2}^{R} T(x,t)\frac{dt}{t^2}$$

and the RHS is bounded by

$$\begin{aligned}
&(x + Rx^{5/6} + R^2\sqrt{x})R^{-1}(\log x)^3 + (x + h^2x^{5/6} + h^4\sqrt{x})h^{-2}(\log x)^3 \\
&+ \int_{h^2}^{R} (x + tx^{5/6} + t^2\sqrt{x})(\log x)^3 \frac{dt}{t^2} \\
\ll& \Big(\frac{x}{h^2} + x^{5/6} + R\sqrt{x}\Big)(\log x)^3
\end{aligned}$$

according to Theorem 5. Applying this to (2.9) we obtain

$$\mathcal{T}_3 \ll x(\log x)^{B+4} + h^2R\sqrt{x}(\log x)^{B+4} \ll hx + h^2R\sqrt{x}(\log x)^{B+4}, \tag{2.10}$$

since $(\log x)^{B+6+\varepsilon} \le h$ by assumption. The assertion follows by plugging (2.6), (2.8) and (2.10) into (2.5). ■

Lemma 2.2. *Let* $\varepsilon > 0$, $h \le R \le x$ *and suppose that* $(\log x)^{8+\varepsilon} \le h \le x^\varepsilon$. *Then there exists some* $c > 0$, *such that*

$$\begin{aligned}\sum_{k\le h}(h-k)\lambda_R(n)\Lambda(n+k) =& \frac{h^2x}{2} - hx\log h + O(h^2xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) \\ &+ O(h^2R\sqrt{x}(\log x)^6) + O(hx),\end{aligned}$$

the same asymptotic formula applying to

$$\sum_{k\le h}(h-k)\sum_{n\le x}\Lambda(n)\lambda_R(n+k).$$

Proof. Proceeding as in the proof of Lemma 1.12 in Chapter 1, we are led to

$$\sum_{k\le h}(h-k)\lambda_R(n)\Lambda(n+k) = \Sigma_1 + \mathcal{E} + O(h^3x^\varepsilon)$$

with

$$\Sigma_1 := x\sum_{k\le h}(h-k)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d\mid r\\(d,k)=1}}\frac{d\mu(d)}{\varphi(d)}$$

and

$$\mathcal{E} := \sum_{k\le h}(h-k)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d\mid r\\(d,k)=1}} d\mu(d)E(x;d,k),$$

where

$$\Sigma_1 = \frac{h^2x}{2} - \frac{hx}{2}\log h + O(hx), \tag{2.11}$$

cp. (1.38) on p. 19. As for $\mathcal{E}$, we can change the order of summation of d and r to obtain

$$\begin{aligned}\mathcal{E} &= \sum_{k\le h}(h-k)\sum_{\substack{d\le R\\(d,k)=1}} d\mu(d)E(x;d,k)\sum_{\substack{r\le R\\d\mid r}}\frac{\mu^2(r)}{\varphi(r)} \\ &= \sum_{k\le h}(h-k)\sum_{\substack{d\le R\\(d,k)=1}}\frac{d\mu(d)}{\varphi(d)}E(x;d,k)\sum_{\substack{r'\le R/d\\(r',d)=1}}\frac{\mu^2(r')}{\varphi(r')}.\end{aligned} \tag{2.12}$$

By Lemma 1.8 (a), the function $\widetilde{g}_R(d) := \dfrac{d\mu(d)}{\varphi(d)} \sum_{\substack{r' \le R/d \\ (r',d)=1}} \dfrac{\mu^2(r')}{\varphi(r')}$ is bounded by $(\log x)^2$. Setting $g_R(d) := C\widetilde{g}_R(d)$ with a suitable constant $C > 0$ and noting that $h \ge (\log x)^{8+\varepsilon}$ by assumption, we can apply Proposition 1 to infer

$$\mathcal{E} \ll h^2 x e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} + h^2 R\sqrt{x}(\log x)^6 + hx$$

for some $c > 0$, and the Lemma follows. ∎

2.2 Proof of Theorem 2

Proof of Theorem 2. We let $\varepsilon > 0$ and suppose that $(\log x)^{8+\varepsilon} \le h \le x^{\varepsilon}$ as well as

$$R \ll \frac{x^{1/2-\varepsilon}}{h^{1/4}} \tag{2.13}$$

and $h \le R$. We start from the inequality

$$\begin{aligned}\int_0^x \psi^2(y;h)dy \ge& h\sum_{n\le x} \lambda_R(n)(2\Lambda(n) - \lambda_R(n)) \\ &+ 2\sum_{k\le h}(h-k)\sum_{n\le x}(\lambda_R(n)\Lambda(n+k) + \Lambda(n)\lambda_R(n+k)) \\ &- 2\sum_{k\le h}(h-k)\lambda_R(n)\lambda_R(n+k) + O(h^3x^{\varepsilon}),\end{aligned} \tag{2.14}$$

which follows by inserting (1.51) and (1.52) into (1.50). Then, by Lemma 1.8,

$$h\sum_{n\le x}\lambda_R(n)(2\Lambda(n) - \lambda_R(n)) = hx\log R + O(hx) \tag{2.15}$$

and Lemma 2.2 yields

$$\begin{aligned}&\sum_{k\le h}(h-k)\sum_{n\le x}(\lambda_R(n)\Lambda(n+k) + \Lambda(n)\lambda_R(n+k)) \\ =&h^2x - hx\log h + O(h^2Rx^{1/2+\varepsilon}) + O(h^2xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) + O(hx)\end{aligned} \tag{2.16}$$

for some $c > 0$. Lastly, by Lemma 1.11 and referring to (2.13), we have

$$\sum_{k\le h}(h-k)\lambda_R(n)\lambda_R(n+k) = \frac{h^2x}{2} - \frac{hx}{2}\log h + O(hx). \tag{2.17}$$

Employing (2.15)–(2.17) to (2.14) yields

$$\int_0^x \psi^2(y;h)dy \geq hx\log\big(\frac{R}{h}\big) + h^2x + O(h^2Rx^{1/2+\varepsilon}) + O(h^2xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) + O(hx) \tag{2.18}$$

for some $c>0$. Now we want R to be as large as possible to keep $h^2Rx^{1/2+\varepsilon} \ll hx$, which leads to the choice $R = \dfrac{x^{1/2-\varepsilon}}{h}$ and by applying this to (2.18) we obtain

$$\int_0^x \psi^2(y;h)dy \geq hx\log\big(\frac{x^{1/2-\varepsilon}}{h^2}\big) + h^2x + O(h^2xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) + O(hx). \tag{2.19}$$

Next, by squaring out,

$$I(x;h) = \int_0^x \psi^2(y;h)dy - 2h\int_0^x \psi(y;h)dy + h^2x$$

and since (1.4) of Lemma 1.3 tells us that

$$\int_0^x \psi(y;h)dy = hx + O(hxe^{-c'(\log x)^{3/5}(\log\log x)^{-1/5}}) + O(h^2x^{\varepsilon})$$

for some $c'>0$, we can deduce from (2.19) that

$$I(x;h) \geq hx\log\big(\frac{x^{1/2-\varepsilon}}{h^2}\big) + O(h^2xe^{-\min(c,c')(\log x)^{3/5}(\log\log x)^{-1/5}}) + O(h^3x^{\varepsilon}) + O(hx).$$

Thus, noting that $h \leq x^{\varepsilon}$ by assumption, we obtain

$$I(x;h) \geq \big(\frac{1}{2} - \varepsilon\big)hx\log x \qquad \text{for } (\log x)^{8+\varepsilon} \leq h \leq e^{\min(c,c')(\log x)^{3/5}(\log\log x)^{-1/5}}$$

and x large enough. Theorem 2 now follows, since (2.1) gives the assertion in the range $1 \leq h \leq (\log x)^{8+\varepsilon}$. ■

2.3 Application to the variance of primes

In this section we employ the methods used for the proof of Theorem 2 to investigate the mean square sum

$$G(x;q) := \sum_{a(q)}{}^{*} E^2(x;q,a).$$

Assuming RH as well as the usual strong form of the Twin Prime Conjecture, see (0.3) on p. 2, Friedlander and Goldston (cp. [5, Theorem 3]) established the asymptotic formula

$$G(x;q) \sim x\log q \qquad \text{uniformly in } x^{1/2+\varepsilon} \le q \le x \tag{2.20}$$

and proved an unconditional lower bound, namely

$$G(x;q) \ge \Big(\frac{1}{2}-\varepsilon\Big)x\log q \qquad \text{for } \frac{x}{(\log x)^A} \le q \le x, \quad A>0, \tag{2.21}$$

for x large enough. Hooley (see for example [19, pp. 53–54, eq. (6)–(7)]), improved on (2.21) by showing unconditionally that

$$G(x;q) \ge \Big(\frac{1}{2}-\varepsilon\Big)x\log x \quad \text{for} \quad \frac{x}{e^{C\sqrt{\log x}}} \le q \le x \text{ and some } C>0 \tag{2.22}$$

for x large enough. Analogously to the case of $I(x;h)$, the proof of (2.21) and (2.22) is essentially based on the inequality

$$\sum_{a(q)}{}^{*} (\psi(x;q,a)-\psi_R(x;q,a))^2 \ge 0.$$

The aim of this subsection is to improve on (2.22) by establishing

Theorem 6. *Let $\varepsilon>0$ and x be sufficiently large. Then there exists some $c>0$, such that*

$$G(x;q) \ge \Big(\frac{1}{2}-\varepsilon\Big)x\log x \quad \textit{for} \quad \frac{x}{e^{c(\log x)^{3/5}(\log\log x)^{-1/5}}} \le q \le x.$$

Throughout this section we need the estimate $\dfrac{q}{\varphi(q)} \ll \log\log q$, $q\ge 3$, cp. [24, p. 55, Thm. 2.9].
We continue with some preparatory results.

Proposition 2. *Let $\varepsilon > 0$, $x \geq 2$, $1 \leq y \leq x$ and $g : \mathbb{N} \to \mathbb{R}$ be a function supported on squarefree integers d with $|g(d)| \leq (\log x)^B$, $B \geq 1$, for each d and let*

$$x^{1-\varepsilon} \leq q \leq \frac{x}{(\log x)^{B+6+\varepsilon}}. \tag{2.23}$$

Then there exists some $c > 0$, such that

$$\sum_{j \leq x/q} \sum_{\substack{d \leq R \\ (d,jq)=1}} g(d)E(y;d,jq) \ll \frac{x^2}{q} e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} + \frac{x^{3/2}}{q} R(\log x)^{B+4} + x. \tag{2.24}$$

Proof. First by arguing as in the proof of Proposition 1, we see that the LHS of (2.24) equals

$$\sum_{d \leq R} \frac{g(d)}{\varphi(d)} \sum_{d^* | d} \sum_{\chi^*(d^*)}{}^{*} (\psi(y;\chi^*) - \delta_{\chi^*} x) \sum_{\substack{\chi(d) \\ \chi^* \text{ induces } \chi}} \overline{\chi}(q) \sum_{j \leq x/q} \overline{\chi}(j) + O\big(\frac{x}{q} R(\log x)^B\big),$$

which we write as

$$\mathcal{T}_{1,q} + \mathcal{T}_{2,q} + \mathcal{T}_{3,q} + O\big(\frac{x}{q} R(\log x)^B\big),$$

according to whether $d^* \leq (\log x)^A$, $(\log x)^A < d^* \leq (x/q)^2$ or $(x/q)^2 < d^* \leq R$. Since $(\log x)^{B+6+\varepsilon} \leq \frac{x}{q} \leq x^\varepsilon$ by (2.23), we can repeat the estimations of $\mathcal{T}_i$ in the proof of Proposition 1 with x/q in place of h and with y instead of x to obtain

$$\mathcal{T}_{1,q} + \mathcal{T}_{2,q} + \mathcal{T}_{3,q} \ll \frac{x^2}{q} e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} + \frac{x^{3/2}}{q} R(\log x)^{B+4} + x$$

for some $c > 0$, which shows Proposition 2. ■

The next two Lemmas are q–analogs of Lemmas 2.2 and 1.11.

Lemma 2.3. *Let $\varepsilon > 0$, $x^{1/2+\varepsilon} \leq q \leq x$ and $\frac{x}{q^{1-\varepsilon}} \leq R \leq \frac{q}{(\log x)^2}$. Then there exists some $c > 0$, such that*

$$\begin{aligned}\sum_{j \leq x/q} \sum_{n \leq x - jq} \lambda_R(n)\Lambda(n+jq) =& \frac{x^2}{2\varphi(q)} - \frac{x}{2} \log\big(\frac{x}{q}\big) + O\big(\frac{x^2}{q} e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}\big) \\ &+ O\big(\frac{x^{3/2}}{q} R(\log x)^6\big) + O(x \log\log q),\end{aligned}$$

the same asymptotic formula holding for

$$\sum_{j \leq x/q} \sum_{n \leq x - jq} \Lambda(n)\lambda_R(n+jq).$$

Proof. By definition of $\lambda_R(n)$, we have

$$\begin{aligned}\sum_{j\le x/q}\sum_{n\le x-jq}\lambda_R(n)\Lambda(n+jq) &= \sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r} d\mu(d)\sum_{\substack{n\le x-jq\\ d|n}}\Lambda(n+jq)\\ &= \sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r} d\mu(d)(\psi(x;d,jq)-\psi(jq;d,jq))\\ &= \Sigma_q+\mathcal{E}_q \qquad (2.25)\end{aligned}$$

with

$$\Sigma_q := \sum_{j\le x/q}(x-jq)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\ (d,jq)=1}}\frac{d\mu(d)}{\varphi(d)} \qquad (2.26)$$

and

$$\mathcal{E}_q := \sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r} d\mu(d)(E(x;d,jq)-E(jq;d,jq)). \qquad (2.27)$$

• Estimation of $\mathcal{E}_q$

We can introduce the condition $(d,jq)=1$ to $\mathcal{E}_q$ causing an error, which is bounded by

$$\log x\sum_{j\le x/q}\omega(jq)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r} d\mu^2(d) \ll \frac{x}{q}(\log x)^2\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\varphi(r)} \ll \frac{Rx}{q}(\log x)^2 \ll x,$$

using the estimate $\omega(n)\ll\log n$, cp. [24, p. 55, Thm. 2.10], Lemma 1.8 (b) and the assumptions on R. Noting that $jq\le x$ we can clearly estimate $\mathcal{E}_q$ using Proposition 2 (with the same function g_R we used in the proof of Lemma 2.2 on p. 36) to obtain

$$\mathcal{E}_q \ll \frac{x^2}{q}e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}+\frac{x^{3/2}}{q}R(\log x)^6+x \quad \text{for some } c>0. \qquad (2.28)$$

• Evaluation of Σ_q

Since

$$\sum_{\substack{d|r\\ (d,jq)=1}}\frac{d\mu(d)}{\varphi(d)} = \frac{\mu(r)}{\varphi(r)}\mu(r,jq)\varphi(r,jq) = \frac{\mu(r)}{\varphi(r)}\frac{\mu(r,q)\varphi(r,q)\mu(r,j)\varphi(r,j)}{\mu(r,j,q)\varphi(r,j,q)},$$

cp. (1.28) on p. 15 for the first equation, we have

$$\Sigma_q = q\sum_{r\le R}\frac{\mu(r)\mu(r,q)\varphi(r,q)}{\varphi^2(r)}\sum_{j\le x/q}\big(\frac{x}{q}-j\big)\frac{\mu(r,j)\varphi(r,j)}{\mu(r,j,q)\varphi(r,j,q)}$$

and using the identity

$$\sum_{\substack{\ell|(r,j)\\(\ell,q)=1}}\mu(\ell)\ell = \prod_{p|r,j}(1-p)\prod_{p|r,j,q}\frac{1}{1-p} = \frac{\mu(r,j)\varphi(r,j)}{\mu(r,j,q)\varphi(r,j,q)}$$

we obtain

$$\Sigma_q = q\sum_{r\le R}\frac{\mu(r)\mu(r,q)\varphi(r,q)}{\varphi^2(r)}\sum_{j\le x/q}\big(\frac{x}{q}-j\big)\sum_{\substack{\ell|r,j\\(\ell,q)=1}}\mu(\ell)\ell,$$

where

$$\begin{aligned}\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le\frac{x}{q}}}\mu(\ell)\ell\sum_{\substack{j\le\frac{x}{q}\\ \ell|j}}\big(\frac{x}{q}-j\big) &= \sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le\frac{x}{q}}}\mu(\ell)\ell^2\sum_{m\le\frac{x}{q\ell}}\big(\frac{x}{q\ell}-m\big)\\ &= \frac{x^2}{2q^2}\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le\frac{x}{q}}}\mu(\ell)-\frac{x}{2q}\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le\frac{x}{q}}}\mu(\ell)\ell+O\Bigg(\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le\frac{x}{q}}}\mu^2(\ell)\ell^2\Bigg).\end{aligned}$$

Thus we can write

$$\Sigma_q = \mathcal{S}_{1,q}-\mathcal{S}_{2,q}+\mathcal{E}'_q$$

with

$$\mathcal{S}_{1,q} := \frac{x^2}{2q}\sum_{r\le R}\frac{\mu(r)\mu(r,q)\varphi(r,q)}{\varphi^2(r)}\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le x/q}}\mu(\ell),$$

$$\mathcal{S}_{2,q} := \frac{x}{2}\sum_{r\le R}\frac{\mu(r)\mu(r,q)\varphi(r,q)}{\varphi^2(r)}\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le x/q}}\mu(\ell)\ell,$$

and

$$\mathcal{E}'_q \ll q\sum_{r\le R}\frac{\mu^2(r)\mu^2(r,q)\varphi(r,q)}{\varphi^2(r)}\sum_{\substack{\ell|r\\(\ell,q)=1\\ \ell\le x/q}}\mu^2(\ell)\ell^2,$$

where

$$\mathcal{E}'_q \ll q\sum_{\ell\le x/q}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)}\sum_{\substack{r'\le x/(q\ell)\\(r',\ell)=1}}\frac{\mu^2(r')\varphi(r',q)}{\varphi^2(r')}$$

$$
\begin{aligned}
&\ll q \sum_{\ell \le x/q} \frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)} \sum_{r' \le x/(q\ell)} \frac{\mu^2(r')\varphi(r',q)}{\varphi^2(r')} \\
&= q \sum_{\delta | q} \frac{\mu^2(\delta)}{\varphi(\delta)} \sum_{\ell \le x/q} \frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)} \sum_{\substack{r'' \le x/(q\ell\delta) \\ (r'',\delta)=1}} \frac{\mu^2(r'')}{\varphi^2(r'')} \\
&\ll x \sum_{\delta | q} \frac{\mu^2(\delta)}{\varphi(\delta)} \qquad \text{by Lemma 1.7} \\
&\ll x \log\log q \qquad (2.29)
\end{aligned}
$$

using $\sum_{\delta|q} \frac{\mu^2(\delta)}{\varphi(\delta)} = \prod_{p|q} \big(1 + \frac{1}{p-1}\big) = \frac{q}{\varphi(q)} \ll \log\log q$ in the last step.

• Moreover we have

$$
\mathcal{S}_{1,q} = \frac{x^2}{2q} \sum_{\substack{\delta | q \\ \delta \le R}} \frac{\mu^2(\delta)}{\varphi(\delta)} \sum_{\substack{r' \le R/\delta \\ (r',q)=1}} \frac{\mu(r')}{\varphi^2(r')} \sum_{\substack{\ell | r' \\ \ell \le x/q}} \mu(\ell)
$$

and

$$
\sum_{\substack{r' \le R/\delta \\ (r',q)=1}} \frac{\mu(r')}{\varphi^2(r')} \sum_{\substack{\ell | r' \\ \ell \le x/q}} \mu(\ell) = 1 + O\big(\frac{q}{x}\big),
$$

which can be deduced like in the estimation of $\mathcal{S}_1$ with x/q instead of h on p. 17, the only difference being the additional $(r',q) = 1$ to the outer sum. Therefore, noting that

$$
\sum_{\substack{\delta | q \\ \delta \le R}} \frac{\mu^2(\delta)}{\varphi(\delta)} = \sum_{\delta | q} \frac{\mu^2(\delta)}{\varphi(\delta)} + O\big(\frac{\tau(q)\log q}{R}\big) = \frac{q}{\varphi(q)} + O\big(\frac{q^{\varepsilon}}{R}\big) \qquad (2.30)
$$

we obtain

$$
\mathcal{S}_{1,q} = \frac{x^2}{2q} \sum_{\substack{\delta | q \\ \delta \le R}} \frac{\mu^2(\delta)}{\varphi(\delta)} = \frac{x^2}{2\varphi(q)} + O\big(\frac{x^2}{Rq^{1-\varepsilon}}\big). \qquad (2.31)
$$

• Finally, using $(\ell, q) = 1$, we obtain

$$
\begin{aligned}
\mathcal{S}_{2,q} &= -\frac{x}{2} \sum_{\substack{\ell \le x/q \\ (\ell,q)=1}} \frac{\mu^2(\ell)\ell}{\varphi^2(\ell)} \sum_{\substack{r' \le R/\ell \\ (r',\ell)=1}} \frac{\mu(r')\mu(r',q)\varphi(r',q)}{\varphi^2(r')} \\
&= -\frac{x}{2} \sum_{\substack{\ell \le x/q \\ (\ell,q)=1}} \frac{\mu^2(\ell)\ell}{\varphi^2(\ell)} \sum_{\substack{r'=1 \\ (r',\ell)=1}}^{\infty} \frac{\mu(r')\mu(r',q)\varphi(r',q)}{\varphi^2(r')}
\end{aligned}
$$

$$+O\Big(x\sum_{\substack{\ell\le x/q\\(\ell,q)=1}}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\sum_{r'>R/\ell}\frac{\mu^2(r')\varphi(r',q)}{\varphi^2(r')}\Big),$$

where the error term is bounded by

$$x\sum_{\substack{\ell\le x/q\\(\ell,q)=1}}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\sum_{\delta|q}\frac{\mu^2(\delta)}{\varphi(\delta)}\sum_{r''>R/(\ell\delta)}\frac{\mu^2(r'')}{\varphi^2(r'')}\ll\frac{x}{R}\sum_{\delta|q}\frac{\mu^2(\delta)\delta}{\varphi(\delta)}\sum_{\ell\le x/q}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)}$$
$$\ll\frac{x^2}{Rq}\tau(q)\log q\ll\frac{x^2}{Rq^{1-\varepsilon}}$$

using Lemma 1.7 for the first estimate. Now since

$$\sum_{\substack{r'=1\\(r',\ell)=1}}^{\infty}\frac{\mu(r')\mu(r',q)\varphi(r',q)}{\varphi^2(r')}=\sum_{\delta|q}\frac{\mu^2(\delta)}{\varphi(\delta)}\sum_{\substack{r''=1\\(r'',\ell q)=1}}^{\infty}\frac{\mu(r'')}{\varphi^2(r'')}$$
$$=\begin{cases}\dfrac{q}{\varphi(q)}\dfrac{\mathfrak{S}}{2}\displaystyle\prod_{\substack{p|\ell q\\p>2}}\big(1-\frac{1}{(p-1)^2}\big)^{-1}, & \text{if } 2|\ell q,\\ 0, & \text{if } 2\nmid \ell q,\end{cases}$$

cp. (1.35) on p. 18, the main term of $\mathcal{S}_{2,q}$ equals

$$-\frac{\mathfrak{S}x}{4}\frac{q}{\varphi(q)}\prod_{\substack{p|q\\p>2}}\big(1+\frac{1}{p(p-2)}\big)\sum_{\substack{\ell\le x/q\\(\ell,q)=1\\2|\ell q}}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\prod_{\substack{p|\ell\\p>2}}\big(1+\frac{1}{p(p-2)}\big)$$

$$=\begin{cases}-\dfrac{\mathfrak{S}x}{4}\dfrac{q}{\varphi(q)}\displaystyle\prod_{\substack{p|q\\p>2}}\big(1+\frac{1}{p(p-2)}\big)\sum_{\substack{\ell\le x/q\\(\ell,q)=1}}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\prod_{\substack{p|\ell\\p>2}}\big(1+\frac{1}{p(p-2)}\big), & \text{if } 2\mid q,\\ -\dfrac{\mathfrak{S}x}{4}\dfrac{q}{\varphi(q)}\displaystyle\prod_{\substack{p|q\\p>2}}\big(1+\frac{1}{p(p-2)}\big)\sum_{\substack{\ell\le x/q\\(\ell,q)=1\\2|\ell}}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\prod_{\substack{p|\ell\\p>2}}\big(1+\frac{1}{p(p-2)}\big), & \text{if } 2\nmid q,\end{cases}$$

which becomes

$$-\frac{\mathfrak{S}x}{2}\frac{q}{\varphi(q)}\prod_{\substack{p|q\\p>2}}\big(1+\frac{1}{p(p-2)}\big)\sum_{\substack{\ell'\le\frac{x(2,q)}{2q}\\(\ell',2q)=1}}\frac{\mu^2(\ell')}{\varphi_2(\ell')}=-\frac{x}{2}\log\big(\frac{x}{q}\big)+O(x\log\log q),$$

where the first equation follows from (1.36) on p. 19, and using Lemma 1.6 on p. 8 for the last equation. Thus we obtained

$$\mathcal{S}_{2,q}=-\frac{x}{2}\log\big(\frac{x}{q}\big)+O\big(\frac{x^2}{Rq^{1-\varepsilon}}\big)+O(x\log\log q) \tag{2.32}$$

and on combining (2.29)–(2.32) we can conclude

$$\begin{aligned}\Sigma_q &= \frac{x^2}{2\varphi(q)} - \frac{x}{2}\log\big(\frac{x}{q}\big) + O(\frac{x}{R}q^{\varepsilon}) + O\big(\frac{x^2}{Rq^{1-\varepsilon}}\big) + O(x\log\log q)\\ &= \frac{x^2}{2\varphi(q)} - \frac{x}{2}\log\big(\frac{x}{q}\big) + O(x\log\log q) \text{ since } \frac{x}{q^{1-\varepsilon}} \le R \text{ by assumption,} \qquad (2.33)\end{aligned}$$

which together with (2.28) shows the Lemma. ■

Lemma 2.4. *Let* $\varepsilon > 0$, $3 \le q \le x$ *and* $\dfrac{x}{q^{1-\varepsilon}} \le R$. *Then we have*

$$\sum_{j\le x/q}\sum_{n\le x-jq} \lambda_R(n)\lambda_R(n+jq) = \frac{x^2}{2\varphi(q)} - \frac{x}{2}\log\big(\frac{x}{q}\big) + O\big(\frac{R^2x}{q}\big) + O(x\log\log q)$$

Since we shall only consider the case $x^{1-\varepsilon} \le q \le x$ in the proof of Theorem 6, we do not need to refer to Lemma 1.11 in full force, which leads to the error term $O\big(R^2\frac{x^{1/2+\varepsilon}}{q^{1/2}}\big)$ instead of $O\big(\frac{R^2x}{q}\big)$.

Proof. We have

$$\begin{aligned}\sum_{j\le x/q}\sum_{n\le x-jq} \lambda_R(n)\lambda_R(n+jq) &= \sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r} d\mu(d)(\psi_R(x;d,jq) - \psi_R(jq;d,jq))\\ &= \sum_{j\le x/q}(x-jq)\sum_{\substack{d|r\\(d,jq)=1}}\frac{d\mu(d)}{\varphi(d)}\\ &\quad + O\bigg(\frac{Rx}{q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r} d\mu^2(d)\bigg) \qquad (2.34)\end{aligned}$$

on using

$$\psi_R(y;m,a) = E_{m,a}\frac{y}{\varphi(m)} + O(R),$$

cp. Lemma 1.9 on p. 10, for the second equation. Noting that $\sum_{d|r} d\mu^2(d) = \sigma(r)$ for r squarefree, the error term in (2.34) is bounded by

$$\frac{Rx}{q}\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\varphi(r)} \ll \frac{R^2x}{q} \qquad \text{by Lemma 1.8 (b),}$$

whereas the main term equals Σ_q defined by (2.26) on p. 40. The Lemma now follows by (2.33) on p. 44. ■

Lemma 2.5. *There exists some $c' > 0$, such that*

$$G(x;q) = \sum_{a=1}^{q} \psi^2(x;q,a) - \frac{x^2}{\varphi(q)} + O\big(\frac{x^2}{q} e^{-c'(\log x)^{3/5}(\log\log x)^{-1/5}}\big).$$

Proof. By [5, p. 323, eq. (3.1)], we have

$$\begin{aligned} G(x;q) = & \sum_{a=1}^{q} \psi^2(x;q,a) - \frac{x^2}{\varphi(q)} - \frac{2x}{\varphi(q)}(\psi(x) - x) + O\big(\frac{x(\log x)^2}{\varphi(q)}\big) \\ & + O((\log(qx))^3) \end{aligned} \tag{2.35}$$

and using the Prime Number Theorem in the form

$$\psi(x) = x + O(xe^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}) \qquad \text{for some } c > 0$$

and Lemma 1.8 (a) we can infer

$$\begin{aligned} \frac{x}{\varphi(q)}(\psi(x) - x) + \frac{x(\log x)^2}{\varphi(q)} + (\log(qx))^3 &\ll \frac{x^2}{q}(\log q) e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}} \\ &\ll \frac{x^2}{q} e^{-c'(\log x)^{3/5}(\log\log x)^{-1/5}} \end{aligned}$$

for some c' with $0 < c' < c$. ■

Proof of Theorem 6. We fix $\varepsilon > 0$ and suppose that $x^{1-\varepsilon} \le q \le \frac{x}{(\log x)^{8+\varepsilon}}$ as well as $\frac{x}{q^{1-\varepsilon}} \le R \ll \sqrt{x}$. First from the inequality

$$\sum_{a=1}^{q} (\psi(x;q,a) - \psi_R(x;q,a))^2 \ge 0$$

we can deduce that

$$\begin{aligned} \sum_{a=1}^{q} \psi^2(x;q,a) \ge & 2\sum_{a=1}^{q} \psi(x;q,a)\psi_R(x;q,a) - \sum_{a=1}^{q} \psi_R^2(x;q,a) \\ = & 2\sum_{a=1}^{q} \sum_{\substack{n,m \le x \\ n \equiv a \bmod q \\ m \equiv a \bmod q}} \Lambda(n)\lambda_R(m) - \sum_{a=1}^{q} \sum_{\substack{n,m \le x \\ n \equiv a \bmod q \\ m \equiv a \bmod q}} \lambda_R(n)\lambda_R(m) \\ = & 2\sum_{n \le x} \Lambda(n)\lambda_R(n) + 2\Bigg(\sum_{\substack{n,m \le x \\ n < m \\ n \equiv m (q)}} + \sum_{\substack{n,m \le x \\ n > m \\ n \equiv m (q)}} \Bigg) \Lambda(n)\lambda_R(m) \end{aligned}$$

$$
\begin{aligned}
&- \sum_{n\le x} \lambda_R^2(n) - \Big(\sum_{\substack{n,m\le x\\ n<m\\ n\equiv m(q)}} + \sum_{\substack{n,m\le x\\ n>m\\ n\equiv m(q)}} \Big) \lambda_R(n)\lambda_R(m)\\
=&2\sum_{n\le x} \Lambda(n)\lambda_R(n) - \sum_{n\le x} \lambda_R^2(n)\\
&+ 2\sum_{j\le x/q}\sum_{n\le x-jq} \big(\lambda_R(n)\Lambda(n+jq) + \Lambda(n)\lambda_R(n+jq)\big)\\
&- 2\sum_{j\le x/q}\sum_{n\le x-jq} \lambda_R(n)\lambda_R(n+jq). \qquad (2.36)
\end{aligned}
$$

Now by Lemma 1.10 (A) and (B) on p. 12, we have

$$
2\sum_{n\le x} \Lambda(n)\lambda_R(n) - \sum_{n\le x} \lambda_R^2(n) = x\log R + O(x) \qquad (2.37)
$$

and from Lemma 2.3 on p. 39 comes

$$
\begin{aligned}
&\sum_{j\le x/q}\sum_{n\le x-jq} \big(\lambda_R(n)\Lambda(n+jq) + \Lambda(n)\lambda_R(n+jq)\big)\\
=&\frac{x^2}{\varphi(q)} - x\log\big(\frac{x}{q}\big) + O\big(\frac{Rx^{3/2+\varepsilon}}{q}\big) + O\big(\frac{x^2}{q} e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}\big)\\
&+ O(x\log\log q) \qquad (2.38)
\end{aligned}
$$

for some $c > 0$. Finally, by Lemma 2.4 on p. 44, we have

$$
\sum_{j\le x/q}\sum_{n\le x-jq} \lambda_R(n)\lambda_R(n+jq) = \frac{x^2}{2\varphi(q)} - \frac{x}{2}\log\big(\frac{x}{q}\big) + O\big(\frac{R^2x}{q}\big). \qquad (2.39)
$$

Substituting (2.37)–(2.39) into (2.36) yields

$$
\begin{aligned}
\sum_{a=1}^{q} \psi^2(x;q,a) \ge& x\log\big(\frac{Rq}{x}\big) + \frac{x^2}{\varphi(q)} + O\big(\frac{x^2}{q} e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}\big)\\
&+ O\big(\frac{Rx^{3/2+\varepsilon}}{q}\big) + O(x\log\log q), \qquad (2.40)
\end{aligned}
$$

where we used our assumption that $R \ll \sqrt{x}$. Now we choose R as large as possible to keep $\frac{Rx^{3/2+\varepsilon}}{q} \ll x$, which gives $R = \frac{q}{x^{1/2+\varepsilon}}$. Inserting this into (2.40) yields

$$
\sum_{a=1}^{q} \psi^2(x;q,a) \ge x\log\big(\frac{q^2}{x^{3/2+\varepsilon}}\big) + \frac{x^2}{\varphi(q)} + O\big(\frac{x^2}{q} e^{-c(\log x)^{3/5}(\log\log x)^{-1/5}}\big)
$$

$$+ O(x \log\log q) \tag{2.41}$$

and since

$$G(x;q) = \sum_{a=1}^{q} \psi^2(x;q,a) - \frac{x^2}{\varphi(q)} + O\big(\frac{x^2}{q} e^{-c'(\log x)^{3/5}(\log\log x)^{-1/5}}\big)$$

by Lemma 2.5, we can deduce from (2.41) that

$$G(x;q) \geq x \log\big(\frac{q^2}{x^{3/2+\varepsilon}}\big) + O\big(\frac{x^2}{q} e^{-\min(c,c')(\log x)^{3/5}(\log\log x)^{-1/5}}\big) + O(x \log\log q).$$

Thus, noting that $x^{1-\varepsilon} \geq q$ by assumption, we obtain

$$G(x;q) \geq \big(\frac{1}{2} - \varepsilon\big) x \log x$$

for

$$\frac{x}{e^{\min(c,c')(\log x)^{3/5}(\log\log x)^{-1/5}}} \leq q \leq \frac{x}{(\log x)^{8+\varepsilon}}$$

and x large enough, which shows Theorem 6, since the range $\frac{x}{(\log x)^{8+\varepsilon}} \leq q \leq x$ is covered by (2.21). ■

In contrast to the case of $G(x;q)$ alone, more is known about the sum

$$S(x;Q) := \sum_{q \leq Q} G(x;q).$$

The Barban–Davenport–Halberstam Theorem, in a version sharpened by Gallagher, gives the upper bound

$$S(x;Q) \ll Qx \log x \qquad \text{for } \frac{x}{(\log x)^A} \leq Q \leq x,\ A > 0,$$

cp. [3, p. 169, eq. (1)]. Assuming GRH, Friedlander and Goldston ([5, Thm. 4]) showed the asymptotic formula

$$S(x;Q) = Qx \log Q - cQx + O(\min(Q^{3/2}x^{1/2}(\log x)^{1/2}, Qx)) + O(x^{3/2}(\log x)^6)$$

for $x^{1/2} \leq Q \leq x$ with a suitable constant $c > 0$.

As far as lower bounds for $S(x;Q)$ are concerned, Hooley (cp. [19, p. 54]) showed that

$$S(x;Q) > (1-\varepsilon)Qx \log Q \qquad \text{for } \frac{x}{e^{(\log x)^{3/5+\varepsilon}}} \leq Q \leq x,\ \varepsilon > 0,\ x > x_0(\varepsilon),$$

and as mentioned in [15, p. 2], Perelli showed a lower bound for $S(x;Q)$ in the range $x^{1-\delta} \leq Q \leq x$, where $\delta > 0$ is small enough. Recently, in [15], Harper and Soundararajan improved on this by showing the following

Theorem 7 ([15, Thm. 1]). *Let $\varepsilon > 0$, N be large enough and $\sqrt{N} \leq Q \leq N$. Then there exists an absolute constant C such that*

$$\sum_{q \leq Q} \sideset{}{^*}\sum_{a(q)} \big(\psi(N;q,a) - \frac{\psi_q(N)}{\varphi(q)}\big)^2 \geq (1-\varepsilon) QN \Big(\log\big(\frac{Q^2}{N}\big) - \log\log N \Big),$$

where

$$\psi_q(N) := \sum_{\substack{n \leq N \\ (n,q)=1}} \Lambda(n). \tag{2.42}$$

Their proof uses exponential sums and gives a lower bound by the minor arc contribution in the circle method.

Employing the Basic Mean Value Theorem, we can show

Theorem 8. *Let $\varepsilon > 0$ and x large enough. Then for $x^{2/3+\varepsilon} \ll Q \leq x$ we have*

$$S(x;Q) - S(x;Q/2) \geq \big(\frac{1}{2} - \varepsilon\big) Qx \log\big(\frac{Q^{3/2}}{x}\big).$$

Although Theorem 8 is weaker than Theorem 7, its proof illustrates how the Basic Mean Value Theorem can be used to obtain a lower bound for $S(x;Q)$ in nontrivial ranges for Q.
We continue with some preparations.

Lemma 2.6. *For s squarefree and $M \geq 1$ we have*

$$\sum_{\substack{m \leq M \\ (m,s)=1}} 1 = \frac{\varphi(s)}{s} M + O(2^{\omega(s)}).$$

Proof. We have

$$\begin{aligned}
\sum_{\substack{m \leq M \\ (m,s)=1}} 1 = \sum_{m \leq M} \sum_{\ell | s,m} \mu(\ell) = \sum_{\ell | s} \mu(\ell) \sum_{\substack{m \leq M \\ \ell | m}} 1 &= M \sum_{\substack{\ell | s \\ \ell \leq M}} \frac{\mu(\ell)}{\ell} + O\Big(\sum_{\ell | s} \mu^2(\ell) \Big) \\
&= M \sum_{\ell | s} \frac{\mu(\ell)}{\ell} + O\Big(M M^{-1} \sum_{\ell | r} \mu^2(\ell) \Big) + O(2^{\omega(s)}) \\
&= \frac{\varphi(s)}{s} M + O(2^{\omega(s)}).
\end{aligned}$$

■

Proposition 3. *Let $\varepsilon > 0$ and Q_1, Q, R be real numbers with $x^{1/2+\varepsilon} \le Q_1 < Q \le x$ and let*

$$\frac{x}{Q}(\log x)^2 \le R \le \frac{Q}{(\log x)^4}.$$

Then we have

$$\begin{aligned}\sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\(d,jq)=1}} d\mu(d)E(x;d,jq) =&(\psi(x)-x)x\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)}\\ &+O\big(\frac{x^{1+\varepsilon}}{Q_1}(x+x^{5/6}\sqrt{R}+\sqrt{x}R^{3/2})\big)\\ &+O(Qx). \qquad (2.43)\end{aligned}$$

Proof. By orthogonality, we first we write the LHS $\mathcal{M}$ of (2.43) as

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

with

$$\mathcal{M}_1 := \sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\(d,jq)=1}}\frac{d\mu(d)}{\varphi(d)}(\psi_d(x)-x),$$

where ψ_d is defined in (2.42), and

$$\mathcal{M}_2 := \sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\(d,jq)=1}}\frac{d\mu(d)}{\varphi(d)}\sum_{\substack{\chi(d)\\ \chi\ne\chi_0}}\psi(x;\chi).$$

• Estimation of $\mathcal{M}_1$

Since $\psi(x)-\psi_d(x) = \sum_{\substack{n\le x\\(n,d)>1}}\Lambda(n) = \sum_{p|d}\sum_{\substack{m\le x\\p^m\le x}}\log p \ll \omega(d)\log x \ll x^{\varepsilon}$ for $d \le x$,

we have

$$\begin{aligned}\mathcal{M}_1 =&(\psi(x)-x)\sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r\\(d,jq)=1}}\frac{d\mu(d)}{\varphi(d)}\\ &+O\Big(x^{1+\varepsilon}\sum_{q\le Q}\frac{1}{q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{d\mu^2(d)}{\varphi(d)}\Big),\end{aligned}$$

where on using $\sum_{d|r}\frac{d\mu^2(d)}{\varphi(d)} = \prod_{p|r}\frac{2p-1}{p-1} \le \frac{2^{\omega(r)}r}{\varphi(r)}$ for r squarefree as well as Lemma 1.8 (a) and (c) the error is bounded by $O(x^{1+\varepsilon})$. The main term of $\mathcal{M}_1$

equals

$$
\begin{aligned}
&(\psi(x)-x)\sum_{Q_1<q\le Q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d\mid r\\(d,q)=1}}\frac{d\mu(d)}{\varphi(d)}\sum_{\substack{j\le x/q\\(j,d)=1}}1\\
=&(\psi(x)-x)x\sum_{Q_1<q\le Q}\frac{1}{q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d\mid r\\(d,q)=1}}\mu(d)\\
&+O\bigg(Q|\psi(x)-x|\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d\mid r}\frac{\mu^2(d)d2^{\omega(d)}}{\varphi(d)}\bigg)\qquad\text{by Lemma 2.6,}
\end{aligned}
$$

where

$$
\begin{aligned}
Q|\psi(x)-x|\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d\mid r}\frac{\mu^2(d)d2^{\omega(d)}}{\varphi(d)}&=Q|\psi(x)-x|\sum_{d\le R}\frac{\mu^2(d)d2^{\omega(d)}}{\varphi^2(d)}\sum_{\substack{r'\le R/d\\(r',d)=1}}\frac{\mu^2(r')}{\varphi(r')}\\
&\ll Q(\log x)^2|\psi(x)-x|\sum_{d\le R}\frac{\mu^2(d)2^{\omega(d)}}{\varphi(d)}\\
&\ll Q(\log x)^4|\psi(x)-x|\quad\text{by Lemma 1.8 (c)}\\
&\ll Qx\quad\text{by the Prime Number Theorem.}
\end{aligned}
$$

Therefore, because of the identity

$$
\sum_{\substack{d\mid r\\(d,q)=1}}\mu(d)=\prod_{\substack{p\mid r\\p\nmid q}}(1+\mu(p))=\begin{cases}1, & \text{if } r\mid q,\\ 0, & \text{if } r\nmid q,\end{cases}
$$

we obtain

$$
\begin{aligned}
\mathcal{M}_1&=(\psi(x)-x)x\sum_{Q_1<q\le Q}\frac{1}{q}\sum_{\substack{r\le R\\r\mid q}}\frac{\mu^2(r)}{\varphi(r)}+O(Qx)\\
&=(\psi(x)-x)x\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)}+O\Big(\frac{x^2}{R}\sum_{q\le Q}\frac{\tau(q)\log q}{q}\Big)+O(Qx)\\
&=(\psi(x)-x)x\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)}+O\Big(\frac{x^2}{R}(\log x)^2\Big)+O(Qx)\\
&=(\psi(x)-x)x\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)}+O(Qx)\qquad\text{since }\frac{x}{Q}(\log x)^2\le R
\end{aligned}
\tag{2.44}
$$

on using (2.30) on p. 42 for the second equation and $\sum_{q\le Q}\frac{\tau(q)}{q}\ll\log Q$, which follows from [24, p. 38, Thm. 2.3] and summation by parts, for the last equation in (2.44).

• Estimation of $\mathcal{M}_2$

Proceeding as in the proof of Proposition 1, we see that $\mathcal{M}_2$ equals

$$\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{d\mu(d)}{\varphi(d)}\sum_{d^*|d}\sideset{}{^*}\sum_{\chi^*(d^*)}(\psi(x;\chi^*)-\delta_{\chi^*}x)\sum_{\substack{\chi(d)\\ \chi\neq\chi_0\\ \chi^* \text{ induces } \chi}}\sum_{Q_1<q\le Q}\overline{\chi}(q)\sum_{j\le x/q}\overline{\chi}(j)$$

$$+O\left(x(\log x)^2\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{d\mu^2(d)}{\varphi(d)}\sum_{d^*|d}\varphi(d^*)\sum_{q\le Q}\frac{1}{q}\right),$$

where the error term is bounded by

$$\begin{aligned}x(\log x)^3\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{d^2\mu^2(d)}{\varphi(d)} &= x(\log x)^3\sum_{d\le R}\frac{\mu^2(d)d^2}{\varphi^2(d)}\sum_{\substack{r'\le R/d\\ (r',d)=1}}\frac{\mu^2(r')}{\varphi(r')}\\ &\ll x(\log x)^4\sum_{d\le R}\frac{\mu^2(d)d^2}{\varphi^2(d)}\\ &\ll Rx(\log x)^4 \quad \text{by Lemma 1.7}\\ &\ll Qx \quad \text{since } R\le\frac{Q}{(\log x)^4}.\end{aligned}$$

If a primitive character χ^* mod d^* induces a non principal character χ, we have $d^*>1$ and on noting that $\delta_{\chi^*}=0$ for a primitive character with conductor $d^*>1$, it follows that the main term $\mathcal{M}_2'$ of $\mathcal{M}_2$ equals

$$\left|\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{d\mu(d)}{\varphi(d)}\sum_{d^*|d}\sideset{}{^*}\sum_{\chi^*(d^*)}\psi(x;\chi^*)\sum_{\substack{\chi(d)\\ \chi\neq\chi_0\\ \chi^* \text{ induces } \chi}}\sum_{j\le x/Q_1}\overline{\chi}(j)\sum_{Q_1<q\le\min(x/j,Q)}\overline{\chi}(q)\right|$$

$$\le\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{d\mu^2(d)}{\varphi(d)}\sum_{d^*|d}\sideset{}{^*}\sum_{\chi^*(d^*)}|\psi(x;\chi^*)|\sum_{\substack{\chi(d)\\ \chi\neq\chi_0\\ \chi^* \text{ induces } \chi}}\sum_{j\le x/Q_1}\left|\sum_{Q_1<q\le\min(x/j,Q)}\overline{\chi}(q)\right|$$

$$\ll\frac{x}{Q_1}\log x\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}\frac{2^{\omega(d)}d\mu^2(d)}{\varphi(d)}\sum_{d^*|d}\sqrt{d^*}\sideset{}{^*}\sum_{\chi^*(d^*)}|\psi(x;\chi^*)| \tag{2.45}$$

employing the Pólya–Vinogradov Inequality in the last step. Interchanging the order of summation and substituting $r'=r/d$ in the RHS of (2.45) and applying Lemma 1.8, we can infer

$$\mathcal{M}_2'\ll\frac{x}{Q_1}\log x\sum_{d^*\le R}\sqrt{d^*}\mu^2(d^*)\sideset{}{^*}\sum_{\chi^*(d^*)}|\psi(x;\chi^*)|\sum_{\substack{d\le R\\ d^*|d}}\frac{2^{\omega(d)}d\mu^2(d)}{\varphi^2(d)}\sum_{\substack{r'\le R/d\\ (r',d)=1}}\frac{2^{\omega(r')}r'\mu^2(r')}{\varphi^2(r')}$$

$$\ll \frac{x^{1+\varepsilon}}{Q_1} \sum_{d^* \le R} \sqrt{d^*}\mu^2(d^*) \sideset{}{^*}\sum_{\chi^*(d^*)} |\psi(x;\chi^*)| \sum_{\substack{d\le R\\ d^*|d}} \frac{2^{\omega(d)} d\mu^2(d)}{\varphi^2(d)}$$

$$\ll \frac{x^{1+\varepsilon}}{Q_1} \sum_{d^* \le R} \frac{2^{\omega(d^*)}(d^*)^{3/2}}{\varphi^2(d^*)} \sideset{}{^*}\sum_{\chi^*(d^*)} |\psi(x;\chi^*)| \sum_{\substack{d'\le R/d^*\\ (d',d^*)=1}} \frac{2^{\omega(d')} d'\mu^2(d')}{\varphi^2(d')}$$

$$\ll \frac{x^{1+\varepsilon}}{Q_1} \sum_{d^* \le R} \frac{\sqrt{d^*}}{\varphi(d^*)} \sideset{}{^*}\sum_{\chi^*(d^*)} \sup_{y\le x} |\psi(y;\chi^*)|.$$

Proposition 3 now follows, since on summing by parts and by the Basic Mean Value Theorem (Theorem 5, p. 30) we obtain

$$\begin{aligned}
&\sum_{d^* \le R} \frac{\sqrt{d^*}}{\varphi(d^*)} \sideset{}{^*}\sum_{\chi^*(d^*)} \sup_{y\le x} |\psi(y;\chi^*)| \\
=&\frac{T(x,R)}{\sqrt{R}} + \frac{1}{2}\int_1^R T(x,t)\frac{dt}{t^{3/2}} \\
\ll& R^{-1/2}(\log x)^3(x + Rx^{5/6} + R^2\sqrt{x}) + (\log x)^3 \int_1^R (x + tx^{5/6} + t^2\sqrt{x})\frac{dt}{t^{3/2}} \\
\ll&(\log x)^3(x + x^{5/6}\sqrt{R} + \sqrt{x}R^{3/2}).
\end{aligned}$$

■

Lemma 2.7. *Let $\varepsilon > 0$ and Q, Q_1, R be real numbers with $x^{1/2+\varepsilon} \ll Q_1 < Q \le x$ and let*

$$\max\Big(\frac{x}{Q}(\log x)^2, \frac{x}{Q_1^{1-\varepsilon}}\Big) \le R \le \frac{Q_1}{(\log x)^4}.$$

Then

$$\begin{aligned}
\sum_{Q_1<q\le Q} \sum_{j\le x/q} \sum_{n\le x-jq} \Lambda(n)\lambda_R(n+jq) =& \frac{x^2}{2} \sum_{Q_1<q\le Q} \frac{1}{\varphi(q)} - \frac{Qx}{2}\log\Big(\frac{x}{Q}\Big) + \frac{Q_1 x}{2}\log\Big(\frac{x}{Q_1}\Big) \\
&+ (\psi(x) - x)x \sum_{Q_1<q\le Q} \frac{1}{\varphi(q)} \\
&+ O\Big(\frac{x^{1+\varepsilon}}{Q_1}(x + x^{5/6}\sqrt{R} + \sqrt{x}R^{3/2})\Big) \\
&+ O(Qx\log\log Q).
\end{aligned}$$

Proof. Since $\frac{x}{q^{1-\varepsilon}} \le R \le \frac{q}{(\log x)^4}$ for every $q \in [Q_1, Q]$ by assumption, we can apply (2.25)–(2.27) of Lemma 2.3 on pp. 40–40 and then (2.33) on p. 44, and then sum over q to obtain

$$\begin{aligned}\sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{n\le x-jq}\Lambda(n)\lambda_R(n+jq) &= \frac{x^2}{2}\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)} - \frac{x}{2}\sum_{Q_1<q\le Q}\log\left(\frac{x}{q}\right)\\ &+ \sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}d\mu(d)(E(x;d,jq)-E(jq;d,jq))\\ &+O\bigg(x\sum_{q\le Q}\log\log q\bigg), \qquad (2.46)\end{aligned}$$

where

$$\begin{aligned}\sum_{Q_1<q\le Q}\log\left(\frac{x}{q}\right) &= Q\log\left(\frac{x}{Q}\right) - Q_1\log\left(\frac{x}{Q_1}\right) + O\bigg(\int_1^Q [t]\frac{dt}{t}\bigg)\\ &= Q\log\left(\frac{x}{Q}\right) - Q_1\log\left(\frac{x}{Q_1}\right) + O(Q) \qquad (2.47)\end{aligned}$$

by summation by parts. We can introduce the condition $(d,jq)=1$ to the triple sum on the RHS of (2.46) causing an error term, which is bounded by

$$\begin{aligned}\log x\sum_{q\le Q}\sum_{j\le x/q}\omega(jq)\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)}\sum_{d|r}d\mu^2(d) &\ll x(\log x)^2\sum_{q\le Q}\frac{1}{q}\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\varphi(r)}\\ &\ll Rx(\log x)^3 \ll Qx.\end{aligned}$$

Applying this as well as Proposition 3 to (2.46) yields the Lemma. ■

Lemma 2.8. *Let $\varepsilon>0$ and Q, Q_1, R be real numbers with $3\le Q_1<Q\le x$ and $\frac{x}{Q_1^{1-\varepsilon}}\le R\le x$. Then we have*

$$\begin{aligned}\sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{n\le x-jq}\lambda_R(n)\lambda_R(n+jq) =& \frac{x^2}{2}\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)} - \frac{Qx}{2}\log\left(\frac{x}{Q}\right)\\ &+\frac{Q_1x}{2}\log\left(\frac{x}{Q_1}\right) + O(R^2x^{1+\varepsilon})\\ &+O(Qx\log\log Q).\end{aligned}$$

Proof. Since $\frac{x}{q^{1-\varepsilon}}\le R\le x$ for every $q\in[Q_1,Q]$ by assumption, we can use Lemma 2.4 on p. 44 to infer

$$\sum_{Q_1<q\le Q}\sum_{j\le x/q}\sum_{n\le x-jq}\lambda_R(n)\lambda_R(n+jq) = \frac{x^2}{2}\sum_{Q_1<q\le Q}\frac{1}{\varphi(q)} - \frac{x}{2}\sum_{Q_1<q\le Q}\log\left(\frac{x}{q}\right)$$

$$+ O\Big(R^2 x \sum_{q\le Q} \frac{1}{q}\Big) + O\Big(x \sum_{3\le q\le Q} \log\log q\Big)$$

and the Lemma follows now by (2.47). ■

• Proof of Theorem 8

We let $\varepsilon > 0$ and for $Q \le x$ assume that $\frac{x}{Q^{1-\varepsilon}} \ll R \le \sqrt{x}$, (which implies $Q \gg x^{1/2+\varepsilon}$), so that we can utilize Lemma 1.10 (A) and (B) as well as Lemmas 2.7 and 2.8 with $Q_1 = Q/2$. By (2.36) on p. 45 we have

$$\begin{aligned}\sum_{q\sim Q/2} \sum_{a=1}^{q} \psi^2(x;q,a) \ge & \sum_{q\sim Q/2} \Big(2\sum_{n\le x} \Lambda(n)\lambda_R(n) - \sum_{n\le x} \lambda_R^2(n)\Big) \\ & + 2 \sum_{q\sim Q/2} \sum_{j\le x/q} \sum_{n\le x-jq} (\Lambda(n)\lambda_R(n+jq) + \lambda_R(n)\Lambda(n+jq)) \\ & - 2 \sum_{q\sim Q/2} \sum_{j\le x/q} \sum_{n\le x-jq} \lambda_R(n)\lambda_R(n+jq), \end{aligned} \tag{2.48}$$

where by Lemma 1.10 (A) and (B), we have

$$\sum_{q\sim Q/2} \Big(2\sum_{n\le x} \Lambda(n)\lambda_R(n) - \sum_{n\le x} \lambda_R^2(n)\Big) = \frac{Qx}{2} \log R + O(Qx), \tag{2.49}$$

while by Lemma 2.7,

$$\begin{aligned} & 2 \sum_{q\sim Q/2} \sum_{j\le x/q} \sum_{n\le x-jq} (\Lambda(n)\lambda_R(n+jq) + \lambda_R(n)\Lambda(n+jq)) \\ = & 2x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} - Qx \log\big(\frac{x}{Q}\big) + 2(\psi(x) - x)x \sum_{q\sim Q/2} \frac{1}{\varphi(q)} \\ & + O\big(\frac{x^{1+\varepsilon}}{Q}(x + x^{5/6}\sqrt{R} + \sqrt{x}R^{3/2})\big) + O(Qx\log\log Q) \end{aligned} \tag{2.50}$$

and Lemma 2.8 yields

$$\begin{aligned} 2 \sum_{q\sim Q/2} \sum_{j\le x/q} \sum_{n\le x-jq} \lambda_R(n)\lambda_R(n+jq) = & x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} - \frac{Qx}{2} \log\big(\frac{x}{Q}\big) \\ & + O(R^2 x^{1+\varepsilon}) + O(Qx\log\log Q). \end{aligned} \tag{2.51}$$

Plugging (2.49)–(2.51) into (2.48) yields

$$\sum_{q\sim Q/2} \sum_{a=1}^{q} \psi^2(x;q,a) \ge \frac{Qx}{2} \log R + 2x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} - Qx \log\big(\frac{x}{Q}\big)$$

$$
\begin{aligned}
&+ 2(\psi(x)-x)x \sum_{q\sim Q/2} \frac{1}{\varphi(q)} - x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} \\
&+ \frac{Qx}{2} \log\big(\frac{x}{Q}\big) + \mathcal{F} \\
&= \frac{Qx}{2} \log\big(\frac{RQ}{x}\big) + x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} + \mathcal{F},
\end{aligned} \tag{2.52}
$$

where $\mathcal{F}$ can be evaluated as

$$
\mathcal{F} \ll \frac{x^{1+\varepsilon}}{Q}(x + x^{5/6}\sqrt{R} + \sqrt{x}R^{3/2}) + R^2 x^{1+\varepsilon} + Qx \log\log Q. \tag{2.53}
$$

In the next step we choose R such that every summand of (2.53) involving R is bounded by $O(Qx)$ and hence such that $\mathcal{F} \ll Qx \log\log Q$. This leads to

$$
\frac{x}{Q^{1-\varepsilon}} \ll R \ll \min(Q^4 x^{-5/3-\varepsilon}, Q^{4/3}x^{-1/3-\varepsilon}, Q^{1/2}x^{-\varepsilon}\big) = Q^{1/2}x^{-\varepsilon} \text{ for } Q \gg x^{1/2+\varepsilon}
$$

and the condition $\frac{x}{Q} \ll Q^{1/2}x^{-\varepsilon}$ implies $x^{2/3+\varepsilon} \ll Q$.
Inserting this choice of R into (2.52) we obtain for $x^{2/3+\varepsilon} \ll Q$ and x large enough that

$$
\begin{aligned}
\sum_{q\sim Q/2} \sum_{a=1}^{q} \psi^2(x;q,a) \geq &\big(\frac{1}{2} - \varepsilon\big)Qx \log\big(\frac{Q^{3/2}}{x}\big) + x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} \\
&+ 2(\psi(x)-x)x \sum_{q\sim Q/2} \frac{1}{\varphi(q)}.
\end{aligned} \tag{2.54}
$$

Lastly by (2.35) on p. 45 we have

$$
\begin{aligned}
S(x;Q) - S(x;Q/2) = &\sum_{q\sim Q/2} \sum_{a=1}^{q} \psi^2(x;q,a) - x^2 \sum_{q\sim Q/2} \frac{1}{\varphi(q)} \\
&- 2(\psi(x)-x)x \sum_{q\sim Q/2} \frac{1}{\varphi(q)} + O\Big(x(\log x)^2 \sum_{q\leq Q} \frac{1}{\varphi(q)}\Big) \\
&+ O(Q(\log x)^3)
\end{aligned} \tag{2.55}
$$

and Theorem 8 follows by inserting (2.54) into (2.55). ■

3 An Application to the Pair Correlation Function

In this chapter we investigate the question whether a similar result to Theorem 1 in Chapter 1 also holds for the Pair Correlation function $F(x,T)$. In [10], Goldston et al. obtained

$$\Big(\frac{T}{2\pi}\log T\Big)^{-1}F(T^{\alpha},T)\geq\frac{3}{2}-|\alpha|-\varepsilon$$

uniformly in $1\leq|\alpha|\leq 3/2-2\varepsilon$ and all $T\geq T_0(\varepsilon)$ assuming GRH, cp. [10, eq. (3.1)]. The proof also makes use of the auxiliary function λ_R and it is natural that a similar result to Theorem 1 of Chapter 1 also holds for F. The aim of this chapter is to proof the following

Theorem 9. *Assume Hypothesis M and let $\varepsilon>0$. Then there exists some $T_0(\varepsilon)$, such that*

$$\Big(\frac{T}{2\pi}\log T\Big)^{-1}F(T^{\alpha},T)\geq\frac{5}{4}-\frac{3}{4}|\alpha|-\varepsilon$$

uniformly in $1\leq|\alpha|\leq 5/3-\varepsilon$ provided $T\geq T_0(\varepsilon)$.

The method of proof in [10] is to connect F to Dirichlet Polynomials involving $\Lambda(n)$ and then to find a lower bound for them: More precisely, the proof is based on the inequalities

$$\int_{-\infty}^{\infty}\Psi_U\Big(\frac{t}{T}\Big)\Big|\sum_{n\leq x}(\Lambda(n)-\lambda_R(n))n^{1/2-it}\Big|^2dt\geq 0$$

and

$$\int_{-\infty}^{\infty}\Psi_U\Big(\frac{t}{T}\Big)\Big|\sum_{n>x}(\Lambda(n)-\lambda_R(n))n^{-3/2-it}\Big|^2dt\geq 0,$$

where Ψ_U is a smooth function specified in Section 3.1. Similar to the case of $I(x;h)$, Theorem 9 does not follow from Hypothesis M alone, but also from improved estimations of error terms occurring in the evaluation of

$$\int_{-\infty}^{\infty}\Psi_U\Big(\frac{t}{T}\Big)\Big|\sum_{n\leq x}\lambda_R(n)n^{1/2-it}\Big|^2dt\quad\text{and}\quad\int_{-\infty}^{\infty}\Psi_U\Big(\frac{t}{T}\Big)\Big|\sum_{n>x}\lambda_R(n)n^{-3/2-it}\Big|^2dt$$

by means of Lemma 1.12 of Chapter 1. Continuing the more elementary approach of Chapter 1, we shall investigate sums of $\mathfrak{S}_R(\cdot)$ instead of $\mathfrak{S}(\cdot)$ for the proof of Theorem 9.

3.1 Preliminaries

To prove Theorem 9, we first provide some necessary preliminaries. Like in [11], we now introduce a smooth weight function Ψ_U with $U \geq 1$, which has the properties that supp $\Psi_U \subseteq [0,1]$, $0 \leq \Psi_U(t) \leq 1$, $\Psi_U(t) = 1$ for $1/U \leq t \leq 1-1/U$ and $\Psi^{(j)}(t) \ll U^j$ for $j \in \mathbb{N}$, cp. [11, p. 37]. Denoting the Fourier transform of Ψ_U with $\widehat{\Psi}_U$ as usual, we have

$$\begin{aligned} &(a) \quad \widehat{\Psi}_U(0) = 1 + O(1/U), \qquad (b) \quad \widehat{\Psi}_U(v) = \widehat{\Psi}_U(0) + O(v), \\ &(c) \quad \widehat{\Psi}_U(v) \ll \min\left(1, \frac{U}{v}\right) \end{aligned} \tag{3.1}$$

cp. [10, p. 45, eq. (7.3)] and [9, p. 184, l. 16]. The function Ψ_U is used to control some of the occurring error terms, whose estimation would be quite difficult without it.

The next Lemma connects the function F to integrals over certain Dirichlet polynomials. By this connection, lower bounds for these integrals imply a lower bound for F.

Lemma 3.1 ([10, Lemma 1]). *Assuming RH we have for $T \leq x$ and $U = (\log T)^B$ with $B > 1$ that*

$$F(x,T) = \frac{1}{2\pi x^2} I_1(x,T) + \frac{x^2}{2\pi} I_2(x,T) + O\Big(\frac{T(\log T)^2}{U}\Big) + O\Big(\frac{x^{1+\varepsilon}}{T}\Big),$$

where

$$I_1(x,T) = \int_{-\infty}^{\infty} \Psi_U\Big(\frac{t}{T}\Big) \Big| \mathcal{A}\Big(-\frac{1}{2} + it\Big) - \int_1^x u^{1/2-it} du \Big|^2 dt$$

and

$$I_2(x,T) = \int_{-\infty}^{\infty} \Psi_U\Big(\frac{t}{T}\Big) \Big| \mathcal{A}^*\Big(\frac{3}{2} + it\Big) - \int_x^{\infty} u^{-3/2-it} du \Big|^2 dt.$$

Here, for a complex number s, $\mathcal{A}(s) := \sum_{n \leq x} \frac{\Lambda(n)}{n^s}$ and $\mathcal{A}^*(s) := \sum_{n > x} \frac{\Lambda(n)}{n^s}$.

The next Lemma is a generalization of Lemma 1.7 of Chapter 1.

Lemma 3.2. *Letting*

$$G_m(s;c,d) := \sum_{\substack{r \leq s \\ (r,m)=1}} \frac{\mu^2(r) r^c}{\varphi(r)^d}$$

for real numbers c, d, we have

$$G_m(s;c,d) = \begin{cases} \frac{g_m(c-d+1;c,d)}{c-d+1} s^{c-d+1} + o_{c,d}(s^{c-d+1}), & \text{if } c-d > -1, \\ g_m(0;c-1,d)\log s + O_{c,d}(1), & \text{if } c-d = -1, \\ \zeta(c-d) g_m(0;c,d) + \frac{g_m(c-d+1;c,d)}{c-d+1} s^{c-d+1} + o_{c,d}(s^{c-d+1}), & \text{if } c-d < -1, \end{cases}$$

where

$$g_m(z;c,d) := \prod_{p|m} \big(1 + \frac{1}{(p-1)^d p^{z-c}}\big)^{-1} \prod_p \big(1 - \frac{1 - p^{z-c+d}(1 - \big(1-\frac{1}{p}\big)^d)}{(p-1)^d p^{2(z-c)+d}}\big).$$

Proof. This can be proven like [8, Lemma 2], the difference being considering the more general function

$$\Phi_m(z;c,d) = \sum_{\substack{r=1 \\ (r,m)=1}}^{\infty} \frac{\mu^2(r)}{\varphi(r)^d r^{z-c}} = \zeta(z-c+d) g_m(z;c,d)$$

instead of $\Phi_1(z;c,d)$ and proceeding in the same way as it is described there. ■

Lemma 3.3. *Letting*

$$H(s;c) := \sum_{\substack{r \le s \\ 2 \nmid r}} \frac{\mu^2(r) r^c}{\varphi_2(r)}$$

for a real and positive number c, we have

$$H(s;c) = \big(\frac{\mathfrak{S}}{2}\big)^{-1} \frac{s^c}{2} + o(s^c).$$

Proof. By substituting the identity

$$\frac{\varphi(r)}{\varphi_2(r)} = \sum_{d|r} \frac{\mu^2(d)}{\varphi_2(d)}$$

for r odd, cp. [4, eq. (2.8)], into $H(s;c)$ we infer

$$\begin{aligned} H(s;c) &= \sum_{\substack{r \le s \\ 2 \nmid r}} \frac{\mu^2(r) r^c}{\varphi(r)} \sum_{d|r} \frac{\mu^2(d)}{\varphi_2(d)} = \sum_{\substack{d\delta \le s \\ 2 \nmid d\delta \\ (d,\delta)=1}} \frac{\mu^2(d) d^c}{\varphi_2(d)\varphi(d)} \frac{\mu^2(\delta)\delta^c}{\varphi(\delta)} \\ &= \sum_{\substack{d \le s \\ 2 \nmid d}} \frac{\mu^2(d) d^c}{\varphi_2(d)\varphi(d)} \sum_{\substack{\delta \le s/d \\ (\delta,2d)=1}} \frac{\mu^2(\delta)\delta^c}{\varphi(\delta)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{d\le s\\ 2\nmid d}} \frac{\mu^2(d)d^c}{\varphi_2(d)\varphi(d)}\Big(g_{2d}(c;c,1)\big(\frac{s}{d}\big)^c + o\big(\frac{s}{d}\big)^c\Big)\\
&= s^c \sum_{\substack{d\le s\\ 2\nmid d}} \frac{\mu^2(d)g_{2d}(c;c,1)}{\varphi_2(d)\varphi(d)} + o\Big(s^c\sum_{\substack{d\le s\\ 2\nmid d}}\frac{\mu^2(d)}{\varphi_2(d)\varphi(d)}\Big)\\
&= s^c \sum_{\substack{d\le s\\ 2\nmid d}} \frac{\mu^2(d)g_{2d}(c;c,1)}{\varphi_2(d)\varphi(d)} + o(s^c)\\
&= \frac{s^c}{2}\sum_{\substack{d\le s\\ 2\nmid d}} \frac{\mu^2(d)}{d\varphi_2(d)} + o(s^c)\\
&= \frac{s^c}{2}\Big(\sum_{\substack{d=1\\ 2\nmid d}}^{\infty} \frac{\mu^2(d)}{d\varphi_2(d)} - \sum_{\substack{d>s\\ 2\nmid d}}\frac{\mu^2(d)}{d\varphi_2(d)}\Big) + o(s^c)\\
&= \frac{s^c}{2}\prod_{p>2}\big(1+\frac{1}{p(p-2)}\big) + O(s^{c-1}) + o(s^c)\\
&= \big(\frac{\mathfrak{S}}{2}\big)^{-1}\frac{s^c}{2} + o(s^c),
\end{aligned}
$$

where we employed the first case of Lemma 3.2 in the fourth equation and used $g_{2d}(c;c,1) = \prod_{p|2d}\big(1+\frac{1}{p-1}\big)^{-1} = \frac{\varphi(2d)}{2d} = \frac{\varphi(d)}{2d}$ in the seventh equation. ■

For the following we let

$$
f_R(y) := \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le y\\ \ell|r}}\mu(\ell), \quad g_R(y) := \frac{1}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le y\\ \ell|r}}\mu(\ell)\ell,
$$

$$
h_R(y) := \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le y\\ \ell|r}}\mu(\ell)\ell\varrho\big(\frac{y}{\ell}\big).
$$

We note that for $y \ge R$, we obtain

$$
g_R(y) = \frac{1}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\ell|r}\mu(\ell)\ell = \frac{1}{2}\sum_{r\le R}\frac{\mu^2(r)}{\varphi(r)} = \frac{1}{2}\log R + O(1). \tag{3.2}
$$

Lemma 3.4. *We have*

$$
\sum_{m\le y}\mathfrak{S}_R(m) = yf_R(y) - g_R(y) - h_R(y) \tag{3.3}
$$

and

$$h_R(y) \ll \log^{2/3} y \qquad \textit{for } y \le R. \tag{3.4}$$

Proof. By definition of $\mathfrak{S}_R(\cdot)$,

$$\begin{aligned}
\sum_{m\le y}\mathfrak{S}_R(m) &= \sum_{m\le y}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\mu(r,m)\varphi(r,m) = \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{m\le y}\mu(r,m)\varphi(r,m)\\
&= \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{m\le y}\sum_{\ell|r,m}\mu(\ell)\ell\\
&= \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le y\\ \ell|r}}\mu(\ell)\ell\sum_{\substack{m\le y\\ \ell|m}}1 = \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le y\\ \ell|r}}\mu(\ell)\ell\big[\frac{y}{\ell}\big]\\
&= \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le y\\ \ell|r}}\mu(\ell)\ell\big(\frac{y}{\ell}-\frac{1}{2}-\varrho\big(\frac{y}{\ell}\big)\big)\\
&= yf_R(y)-g_R(y)-h_R(y),
\end{aligned}$$

which is (3.3).

It remains to prove (3.4). We have

$$\begin{aligned}
h_R(y) &= \sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell|r\\ \ell\le y}}\mu(\ell)\ell\varrho\big(\frac{y}{\ell}\big) = \sum_{\ell\le y}\mu(\ell)\ell\varrho\big(\frac{y}{\ell}\big)\sum_{\substack{r\le R\\ \ell|r}}\frac{\mu(r)}{\varphi^2(r)}\\
&= \sum_{\ell\le y}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\varrho\big(\frac{y}{\ell}\big)\sum_{\substack{r'\le R/\ell\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\\
&= \sum_{\ell\le y}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\varrho\big(\frac{y}{\ell}\big)\Bigg(\sum_{\substack{r'=1\\ (r',\ell)=1}}^{\infty}\frac{\mu(r')}{\varphi^2(r')}-\sum_{\substack{r'>R/\ell\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\Bigg)\\
&= \sum_{\ell\le y}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\varrho\big(\frac{y}{\ell}\big)\prod_{p\nmid\ell}\big(1-\frac{1}{(p-1)^2}\big)+O\Bigg(R^{-1}\sum_{\ell\le y}\frac{\mu^2(\ell)\ell^2}{\varphi^2(\ell)}\Bigg)\\
&= \sum_{\ell\le y}\frac{\mu^2(\ell)\ell}{\varphi^2(\ell)}\varrho\big(\frac{y}{\ell}\big)\prod_{p\nmid\ell}\big(1-\frac{1}{(p-1)^2}\big)+O\big(\frac{y}{R}\big) \quad \text{by Lemma 3.2.} \qquad (3.5)
\end{aligned}$$

For ℓ even and squarefree we have

$$\frac{\ell}{\varphi^2(\ell)}\prod_{p\nmid\ell}\big(1-\frac{1}{(p-1)^2}\big) = \frac{\mathfrak{S}}{2}\prod_{p|\ell}\frac{p}{(p-1)^2}\prod_{\substack{p|\ell\\ p>2}}\big(1-\frac{1}{(p-1)^2}\big)^{-1}$$

$$= \mathfrak{S} \prod_{\substack{p|\ell \\ p>2}} \frac{1}{p-2} = \frac{\mathfrak{S}}{\varphi_2(\ell/2)}, \tag{3.6}$$

while $\prod_{p\nmid\ell} \big(1 - \frac{1}{(p-1)^2}\big) = 0$ for ℓ odd. By inserting this into (3.5) and noting that $y \leq R$ by assumption we get

$$h_R(y) = \mathfrak{S} \sum_{\substack{\ell' \leq y/2 \\ 2\nmid\ell'}} \frac{\mu^2(\ell')}{\varphi_2(\ell')} \varrho\big(\frac{y/2}{\ell'}\big) + O(1).$$

Thus employing the result

$$\sum_{\substack{m\leq M \\ (m,2m')=1}} \frac{\mu^2(m)}{\varphi_2(m)} \varrho\big(\frac{M}{m}\big) \ll \frac{m'}{\varphi(m')} \log^{2/3} M,$$

cp. [4, Lemma 2.2], with $m' = 1$ shows (3.4). ■

Lemma 3.5. *Let*

$$S_{2,R}(y) := \sum_{m\leq y} \mathfrak{S}_R(m)m^2 - \frac{y^3}{3} \qquad \textit{as well as} \qquad T_{2,R}(y) := \sum_{m>y} \frac{\mathfrak{S}_R(m)}{m^2} - \frac{1}{y}.$$

We have

$$\sum_{m\leq y} \mathfrak{S}_R(m)m^2 = \frac{y^3}{3} f_R(y) + \frac{1}{6}(\min(y,R))^2 - y^2 h_R(y) + \int_1^y 2t h_R(t)dt$$
$$+ o((\min(y,R))^2) + O\big(\frac{(\min(y,R))^3}{R}\big), \tag{3.7}$$

$$S_{2,R}(y) \ll (\min(y,R))^2, \tag{3.8}$$

and

$$T_{2,R}(y) \ll y^{-2}. \tag{3.9}$$

Proof. • We can prove (3.7) by noting that

$$\sum_{m\leq y} \mathfrak{S}_R(m)m^2 = y^2(y f_R(y) - g_R(y) - h_R(y))$$
$$- \int_1^y 2t(t f_R(t) - g_R(t) - h_R(t))dt \tag{3.10}$$

by partial summation using (3.3) of Lemma 3.4. Another summation by parts over ℓ gives

$$\frac{2}{3}y^3 f_R(y) - \int_1^y 2t^2 f_R(t)dt = \frac{2}{3}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le\min(y,R)\\ \ell|r}}\mu(\ell)\ell^3$$

and

$$y^2 g_R(y) - \int_1^y 2tg_R(t)dt = \frac{1}{2}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le\min(y,R)\\ \ell|r}}\mu(\ell)\ell^3,$$

so that

$$\begin{aligned}
&\left(\frac{2}{3}y^3 f_R(y) - \int_1^y 2t^2 f_R(t)dt\right) - \left(y^2 g_R(y) - \int_1^y 2tg_R(t)dt\right)\\
=&\frac{1}{6}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell\le\min(y,R)\\ \ell|r}}\mu(\ell)\ell^3\\
=&\frac{1}{6}\sum_{\ell\le\min(y,R)}\mu(\ell)\ell^3\sum_{\substack{r\le R\\ \ell|r}}\frac{\mu(r)}{\varphi^2(r)}\\
=&\frac{1}{6}\sum_{\ell\le\min(y,R)}\frac{\mu^2(\ell)\ell^3}{\varphi^2(\ell)}\sum_{\substack{r'\le R/\ell\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\\
=&\frac{1}{6}\sum_{\ell\le\min(y,R)}\frac{\mu^2(\ell)\ell^3}{\varphi^2(\ell)}\left(\sum_{\substack{r'=1\\ (r',\ell)=1}}^{\infty}\frac{\mu(r')}{\varphi^2(r')} - \sum_{\substack{r'>R/\ell\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\right)\\
=&\frac{1}{6}\sum_{\ell\le\min(y,R)}\frac{\mu^2(\ell)\ell^3}{\varphi^2(\ell)}\sum_{\substack{r'=1\\ (r',\ell)=1}}^{\infty}\frac{\mu(r')}{\varphi^2(r')} + O\left(\frac{1}{R}\sum_{\ell\le\min(y,R)}\frac{\mu^2(\ell)\ell^4}{\varphi^2(\ell)}\right)\\
=&\frac{1}{6}\sum_{\ell\le\min(y,R)}\frac{\mu^2(\ell)\ell^3}{\varphi^2(\ell)}\prod_{p\nmid\ell}\big(1-\frac{1}{(p-1)^2}\big) + O\big(\frac{1}{R}(\min(y,R))^3\big).
\end{aligned} \tag{3.11}$$

It follows from (3.6) that

$$\frac{\ell^3}{\varphi^2(\ell)}\prod_{p\nmid\ell}\big(1-\frac{1}{(p-1)^2}\big) = 4\mathfrak{S}\frac{(\ell/2)^2}{\varphi_2(\ell/2)}$$

and by inserting this into (3.11) and then applying Lemma 3.3 we obtain

$$\frac{2}{3}y^3 f_R(y) - \int_1^y 2t^2 f_R(t)dt - y^2 g_R(y) + \int_1^y 2tg_R(t)dt$$

$$
\begin{aligned}
=&\frac{2}{3}\mathfrak{S}\sum_{\substack{\ell'\le\min(y/2,R/2)\\ 2\nmid\ell'}}\frac{\mu^2(\ell')\ell'^2}{\varphi_2(\ell')}+O\big(\frac{1}{R}(\min(y,R))^3\big)\\
=&\frac{1}{6}(\min(y,R))^2+o((\min(y,R))^2)+O\big(\frac{1}{R}(\min(y,R))^3\big). \qquad (3.12)
\end{aligned}
$$

Then (3.7) follows by plugging (3.12) into (3.10).

• Now we prove (3.8). By (3.7) we have

$$
\sum_{m\le y}\mathfrak{S}_R(m)m^2=\frac{y^3}{3}f_R(y)-y^2h_R(y)+\int_1^y 2th_R(t)dt+O((\min(y,R)^2) \qquad (3.13)
$$

and analogously to the derivation of (3.12) we find on using $|\varrho(w)|\le 1$ that

$$
\begin{aligned}
y^2h_R(y)-\int_1^y 2th_R(t)dt =&4\mathfrak{S}\sum_{\substack{\ell'\le\min(y/2,R/2)\\ 2\nmid\ell'}}\frac{\mu^2(\ell')\ell'^2}{\varphi_2(\ell')}\varrho\big(\frac{y/2}{\ell'}\big)+o((\min(y,R))^2)\\
&+O\big(\frac{(\min(y,R))^3}{R}\big)\\
\ll&(\min(y,R))^2. \qquad (3.14)
\end{aligned}
$$

Moreover, for $y\le R$,

$$
\begin{aligned}
\frac{y^3}{3}(f_R(y)-1)=&\frac{y^3}{3}\left(f_R(y)-\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\ell|r}\mu(\ell)\right)=-\frac{y^3}{3}\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell|r\\ \ell>y}}\mu(\ell)\\
=&-\frac{y^3}{3}\sum_{\ell>y}\mu(\ell)\sum_{\substack{r\le R\\ \ell|r}}\frac{\mu(r)}{\varphi^2(r)}=-\frac{y^3}{3}\sum_{\ell>y}\frac{\mu^2(\ell)}{\varphi^2(\ell)}\sum_{\substack{r'\le R/\ell\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\ll y^2,
\end{aligned}
\qquad (3.15)
$$

while $f_R(y)=1$ for $y>R$. Thus (3.8) follows from inserting this and (3.14) into (3.13).

• We next prove (3.9). Summation by parts using (3.3) of Lemma 3.4 gives

$$
\begin{aligned}
\sum_{y<m\le M}\frac{\mathfrak{S}_R(m)}{m^2}=&M^{-2}\sum_{y<m\le M}\mathfrak{S}_R(m)+2\int_y^M\Big(\sum_{y<m\le t}\mathfrak{S}_R(m)\Big)\frac{dt}{t^3}\\
=&M^{-2}(Mf_R(M)-g_R(M)-h_R(M)-yf_R(y)+g_R(y)+h_R(y))\\
&+2\int_y^M(tf_R(t)-g_R(t)-h_R(t)-yf_R(y)+g_R(y)+h_R(y))\frac{dt}{t^3},
\end{aligned}
$$

from which it follows that

$$\begin{aligned}\sum_{m>y}\frac{\mathfrak{S}_R(m)}{m^2} =&2\int_y^\infty (tf_R(t)-g_R(t)-h_R(t)-yf_R(y)+g_R(y)+h_R(y))\frac{dt}{t^3}\\ =&-\frac{f_R(y)}{y}+2\int_y^\infty f_R(t)\frac{dt}{t^2}+\frac{g_R(y)}{y^2}-2\int_y^\infty g_R(t)\frac{dt}{t^3}+\frac{h_R(y)}{y^2}\\ &-2\int_y^\infty h_R(t)\frac{dt}{t^3}.\end{aligned}$$

Summing by parts over ℓ gives

$$\begin{aligned}&-2\frac{f_R(y)}{y}+2\int_y^\infty f_R(t)\frac{dt}{t^2}+\frac{g_R(y)}{y^2}-2\int_y^\infty g_R(t)\frac{dt}{t^3}\\ =&\sum_{r\le R}\frac{\mu(r)}{\varphi^2(r)}\sum_{\substack{\ell>y\\ \ell|r}}\frac{\mu(\ell)}{\ell}=\sum_{\ell>y}\frac{\mu^2(\ell)}{\ell\varphi^2(\ell)}\sum_{\substack{r'\le R/\ell\\ (r',\ell)=1}}\frac{\mu(r')}{\varphi^2(r')}\\ \ll& y^{-2}\end{aligned}$$

and analogously using $|\varrho(w)|\le 1$ we can show that

$$\frac{h_R(y)}{y^2}-2\int_y^\infty h_R(t)\frac{dt}{t^3}\ll y^{-2}.$$

Thus we can write

$$\sum_{m>y}\frac{\mathfrak{S}_R(m)}{m^2}=\frac{f_R(y)}{y}+O\Big(\frac{1}{y^2}\Big) \tag{3.16}$$

and (3.9) follows from

$$-\frac{f_R(y)}{y}+\frac{1}{y}=O\Big(\frac{1}{y^2}\Big),$$

which follows from (3.15). ■

3.2 Lower Bounds for $I_1(x,T)$ and $I_2(x,T)$

We recall the definition of $I_1(x,T)$ and $I_2(x,T)$ from Lemma 3.1:

$$I_1(x,T)=\int_{-\infty}^\infty \Psi_U\Big(\frac{t}{T}\Big)\Big|\mathcal{A}\Big(-\frac{1}{2}+it\Big)-\int_1^x u^{1/2-it}du\Big|^2 dt$$

and

$$I_2(x,T) = \int_{-\infty}^{\infty} \Psi_U\big(\frac{t}{T}\big)\Big|\mathcal{A}^*\big(\frac{3}{2}+it\big) - \int_x^{\infty} u^{-3/2-it}du\Big|^2 dt.$$

Here we choose $U = (\log T)^B$ with $B > 1$.
From now on we assume $R \geq \frac{x}{T}$.

3.2.1 Treatment of $I_1(x,T)$

By [10, p. 46, eq. (7.4)], which holds with no change if we replace $\mathfrak{S}(\cdot)$ by $\mathfrak{S}_R(\cdot)$, we have for the range $T \leq x \ll T^{2-\varepsilon}$, that

$$\begin{aligned} I_1(x,T) \geq & \frac{\widehat{\Psi}_U(0)}{2} T x^2 \log R + I_{11}(x,T) + 2R_1(x,T) - R_1'(x,T) \\ & + O(Tx^2) \end{aligned} \tag{3.17}$$

with

$$I_{11}(x,T) = 4\pi\big(\frac{T}{2\pi}\big)^3 \int_{T/2\pi x}^{\infty} \Big(\sum_{m \leq 2\pi xv/T} \mathfrak{S}_R(m)m^2 - \int_0^{2\pi xv/T} u^2 du\Big) \operatorname{Re}\widehat{\Psi}_U(v)\frac{dv}{v^3},$$

where $R_1(x,T)$ and $R_1'(x,T)$ are certain error terms, which we will consider in Section 3.2. (We note that in [10, eq. (7.4)], $R_1(x,T)$ and $R_1'(x,T)$ are incorporated in the single error term $O(Tx^2)$.) We continue to show

$$I_1(x,T) \geq \frac{Tx^2}{2} \log\big(\frac{RT}{x}\big)(1+o(1)) + 2R_1(x,T) - R_1'(x,T). \tag{3.18}$$

On substituting $y = 2\pi xv/T$ we can write

$$I_{11}(x,T) = 2Tx^2(J_1(x,T) + J_2(x,T)) \tag{3.19}$$

with

$$J_1(x,T) = \int_1^{x/T} S_{2,R}(y) \operatorname{Re}\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)\frac{dy}{y^3}$$

and

$$J_2(x,T) = \int_{x/T}^{\infty} S_{2,R}(y) \operatorname{Re}\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)\frac{dy}{y^3},$$

where $S_{2,R}(y)$ is as in Lemma 3.5. The estimation of $J_2(x,T)$ is quick; from (3.8) of Lemma 3.5 and (3.1) (c) comes

$$\begin{aligned} J_2(x,T) &= \Big(\int_{x/T}^{xU/T} + \int_{xU/T}^{\infty}\Big) S_{2,R}(y)\,\mathrm{Re}\,\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)\frac{dy}{y^3} \\ &\ll \int_{x/T}^{xU/T}\frac{dy}{y} + \frac{Ux}{T}\int_{xU/T}^{\infty}\frac{dy}{y^2} \\ &= \log U + O(1) \\ &\ll \log\log T. \end{aligned} \tag{3.20}$$

Next since

$$\mathrm{Re}\,\widehat{\Psi}_U(v) = \mathrm{Re}\,\widehat{\Psi}_U(0) + O(v) = \widehat{\Psi}_U(0) + O(v) \tag{3.21}$$

because of (3.1) (b), we have

$$J_1(x,T) = \widehat{\Psi}_U(0)\int_1^{x/T} S_{2,R}(y)\frac{dy}{y^3} + O\Big(\frac{T}{x}\int_1^{x/T}|S_{2,R}(y)|\frac{dy}{y^2}\Big),$$

where

$$\frac{T}{x}\int_1^{x/T}|S_{2,R}(y)|\frac{dy}{y^2} \ll 1,$$

which follows from (3.8) of Lemma 3.5.

Now for $\dfrac{x}{T} \le R$,

$$\begin{aligned} &\int_1^{x/T}\Big(\sum_{m\le y}\mathfrak{S}_R(m)m^2\Big)\frac{dy}{y^3} = \sum_{m\le x/T}\mathfrak{S}(m)m^2\int_m^{x/T}\frac{dy}{y^3} \\ =&\frac{1}{2}\sum_{m\le x/T}\mathfrak{S}_R(m) - \frac{1}{2}\big(\frac{T}{x}\big)^2\sum_{m\le x/T}\mathfrak{S}_R(m)m^2 \\ =&\frac{1}{2}\Big(\frac{x}{T}f_R\big(\frac{x}{T}\big) - g_R\big(\frac{x}{T}\big) - h_R\big(\frac{x}{T}\big)\Big) \\ &-\frac{1}{2}\big(\frac{T}{x}\big)^2\Big(\frac{1}{3}\big(\frac{x}{T}\big)^3 f_R\big(\frac{x}{T}\big) + \frac{1}{6}\big(\frac{x}{T}\big)^2 + o\Big(\big(\frac{x}{T}\big)^2\Big) + O\Big(\frac{(x/T)^3}{R}\Big) \\ &-\big(\frac{x}{T}\big)^2 h_R\big(\frac{x}{T}\big) + \int_1^{x/T}2th_R(t)dt\Big) \qquad \text{by (3.3) and (3.7)} \\ =&\frac{x}{3T} - \frac{1}{4}\log\big(\frac{x}{T}\big) + O\big(\log^{2/3}\big(\frac{x}{T}\big)\big) \qquad \text{by (3.2) and (3.4),} \end{aligned} \tag{3.22}$$

so that

$$J_1(x,T) = -\frac{1}{4}\widehat{\Psi}_U(0)\log\big(\frac{x}{T}\big) + o\big(\log\big(\frac{x}{T}\big)\big).$$

Inserting this together with (3.20) into (3.19) yields

$$I_{11}(x,T) = -\frac{Tx^2}{2}\widehat{\Psi}_U(0)\log\big(\frac{x}{T}\big) + o\big(Tx^2\log\big(\frac{x}{T}\big)\big)$$

and (3.18) follows by substituting this into (3.17) and then using (3.1) (a).

3.2.2 Treatment of $I_2(x,T)$

By [10, p. 47, eq. (7.10)] with $\mathfrak{S}_R(\cdot)$ instead of $\mathfrak{S}(\cdot)$,

$$I_2(x,T) \geq \widehat{\Psi}_U(0)T\frac{\log R}{2x^2} + I_{21}(x,T) + 2R_2(x,T) - R_2'(x,T) + O\big(\frac{T}{x^2}\big), \quad (3.23)$$

with

$$I_{21}(x,T) = \frac{2T}{x^2}\int_1^{H^*}\Bigg(\sum_{y<m\leq H^*}\frac{\mathfrak{S}_R(m)}{m^2} - \int_y^{H^*}\frac{du}{u^2}\Bigg)\operatorname{Re}\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)y\,dy,$$

where

$$H^* = \frac{x^{2/(1-\varepsilon)}}{T^{2-2\varepsilon}} \quad (3.24)$$

and $R_2(x,T)$, $R_2'(x,T)$ are certain error terms, which are incorporated in the single error term $O\big(\frac{T}{x^2}\big)$ in [10, eq. (7.10)]. We will consider them in Section 3.2 and continue showing

$$I_2(x,T) \geq \frac{T}{2x^2}\log\big(\frac{RT}{x}\big)(1+o(1)) + 2R_2(x,T) - R_2'(x,T). \quad (3.25)$$

In order to do this, we write $I_{21}(x,T)$ as

$$I_{21}(x,T) = \frac{2T}{x^2}\Bigg(J_3(x,T) + J_4(x,T) - \int_1^{H^*}T_{2,R}(H^*)\operatorname{Re}\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)y\,dy\Bigg) \quad (3.26)$$

with

$$J_3(x,T) = \int_1^{x/T}T_{2,R}(y)\operatorname{Re}\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)y\,dy$$

and

$$J_4(x,T) = \int_{x/T}^{H^*}T_{2,R}(y)\operatorname{Re}\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big)y\,dy.$$

The integral on the right in (3.26) can be estimated as $O(1)$ by using $T_{2,R}(H^*) \ll (H^*)^{-2}$, cp. (3.9) of Lemma 4.1, and noting that $\widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big) \ll 1$.
Employing (3.9) again we obtain

$$\begin{aligned} J_4(x,T) &= \Big(\int_{x/T}^{xU/T} + \int_{xU/T}^{H^*}\Big) T_{2,R}(y) \operatorname{Re} \widehat{\Psi}_U\big(\frac{yT}{2\pi x}\big) y dy \\ &\ll \int_{x/T}^{xU/T} \frac{dy}{y} + \frac{Ux}{T} \int_{xU/T}^{H^*} \frac{dy}{y^2} \\ &\ll \log U. \end{aligned}$$

Finally we estimate J_3. Referring to (3.21) we can write

$$J_3(x,T) = \widehat{\Psi}_U(0) \int_1^{x/T} T_{2,R}(y) y dy + O\Big(\frac{T}{x} \int_1^{x/T} |T_{2,R}(y)| y^2 dy\Big),$$

where

$$\frac{T}{x} \int_1^{x/T} |T_{2,R}(y)| y^2 dy \ll 1$$

by the estimate $T_{2,R}(y) \ll y^{-2}$. Changing the order of integration and summation yields

$$\begin{aligned} \int_1^{x/T} \Big(\sum_{m>y} \frac{\mathfrak{S}_R(m)}{m^2}\Big) y dy =& \sum_{m=1}^{\infty} \frac{\mathfrak{S}_R(m)}{m^2} \int_1^{\min(m,x/T)} y dy \\ =& \frac{1}{2} \sum_{m \le x/T} \mathfrak{S}_R(m) + \frac{1}{2}\big(\frac{x}{T}\big)^2 \sum_{m>x/T} \frac{\mathfrak{S}_R(m)}{m^2} \\ =& \frac{1}{2}\Big(\frac{x}{T} f_R\big(\frac{x}{T}\big) + g_R\big(\frac{x}{T}\big) - h_R\big(\frac{x}{T}\big)\Big) \qquad \text{by (3.3)} \\ & - \frac{1}{2}\frac{x}{T} f_R\big(\frac{x}{T}\big) + O(1) \qquad \text{by (3.16)} \\ =& -\frac{1}{4} \log\big(\frac{x}{T}\big) + O\big(\log^{2/3}\big(\frac{x}{T}\big)\big) \qquad \text{by (3.2) and (3.4),} \end{aligned}$$

so that

$$J_3(x,T) = -\frac{1}{4} \widehat{\Psi}_U(0) \log\big(\frac{x}{T}\big) + o\big(\log\big(\frac{x}{T}\big)\big).$$

Now (3.25) follows from plugging this into (3.23) and using (3.1) (a).

3.3 Treatment of the error terms and proof of Theorem 9

3.3.1 The error terms $R_i(x,T)$, $R_i'(x,T)$

Throughout this section we assume $T \le x \ll T^{2-\varepsilon}$.

• We first examine $R_1(x,T)$. Let $R = x^\nu$ with $0 \le \nu \le 1/2$. In [10, p. 42, eq. (6.8)], the error $R_1'(x,T)$ is directly estimated by $x^{3-\nu}+x^{2+2\nu}$. We will estimate $R_1'(x,T)$ later more precisely. Thus it follows from [10, eq. (6.8)] that we can estimate $R_1(x,T)$ as

$$R_1(x,T) \ll \left(\frac{x^3}{T} + T\sum_{k\le H}\int_{k\tau}^{x-k} K(u,k)d_uG_R(u,k)\right)x^\varepsilon, \tag{3.27}$$

where $d_uG_R(u,k)$ denotes the Lebesgue-Stieltjes measure associated with $G_R(u,k)$,

$$H = \frac{x}{\tau+1}, \quad \tau = T^{1-\varepsilon}, \quad G_R(v,k) = \sum_{r\le R}|E(v;r,k)|,$$

where $K(u,k)$ is a smooth function with

$$K(u,k) \ll u \qquad \text{and} \qquad \frac{\partial}{\partial u}K(u,k) \ll T^\varepsilon \quad \text{for } k \le u/\tau, \tag{3.28}$$

cp. [10, p. 43, eq. (6.10)]. Assuming Hypothesis M we have the estimate

$$\sum_{a(r)}^{*}|E(v;r,a)|^2 \ll \frac{v^{1+\varepsilon}}{r^{1/2}}, \tag{3.29}$$

cp. Lemma 1.1 of Chapter 1, and employing this instead of Hooley's GRH–result

$$\sum_{a(r)}^{*}\max_{v\le u}|E(v;r,a)|^2 \ll u(\log(2u))^4$$

into the deduction of [10, p. 43, eq. (6.11)] we obtain

$$\begin{aligned}\sum_{k\le y}G_R(v,k) &= \sum_{r\le R}\sum_{k\le y}|E(v;r,k)| \\ &= \sum_{r\le R}\sum_{\substack{k\le y\\(r,k)=1}}|E(v;r,k)| + O\left(\sum_{r\le R}\log r\right)\end{aligned}$$

$$\ll \sum_{r\le R} y^{1/2}\Big(\sum_{\substack{k\le y\\(r,k)=1}} |E(v;r,k)|^2\Big)^{1/2} + R\log R$$

$$\le y^{1/2}\sum_{r\le R}\big(1+\frac{y}{r}\big)^{1/2}\Big(\sum_{a(r)}{}^{*}|E(v;r,a)|^2\Big)^{1/2} + R\log R$$

$$\ll y^{1/2}\sum_{r\le R}\big(1+\frac{y}{r}\big)^{1/2}\frac{v^{1/2+\varepsilon}}{r^{1/4}} + R\log R$$

$$\ll (y^{1/2}R^{3/4}+yR^{1/4})v^{1/2+\varepsilon} + R\log R$$

and therefore

$$\begin{aligned} T\sum_{k\le H}\int_{k\tau}^{x-k} K(u,k)d_uG_R(u,k) \ll & xT\Big(\sum_{k\le H} G_R(x-k,k)\Big) \\ & + T^{1+\varepsilon}\Big(\int_{k\tau}^{x-k}\sum_{k\le H} G_R(u,k)du\Big) \\ \ll & x^{3/2+\varepsilon}T^{1+\varepsilon}(H^{1/2}R^{3/4}+HR^{1/4}) + RxT^{1+\varepsilon} \\ \ll & R^{3/4}x^2T^{1/2+\varepsilon} + R^{1/4}x^{5/2+\varepsilon} + RxT^{1+\varepsilon} \qquad (3.30) \end{aligned}$$

using summation by parts and (3.28) to derive the first inequality, cp. [10, p. 43, eq. (6.12)]. By inserting (3.30) into (3.27) we have

$$R_1(x,T) \ll \frac{x^3}{T} + R^{3/4}x^2T^{1/2+\varepsilon} + R^{1/4}x^{5/2+\varepsilon} + RxT^{1+\varepsilon} \qquad (3.31)$$

and we see that $R_1(x,T) \ll Tx^2$, provided this holds for every summand in (3.31), which in turn is the case, if

$$R \ll \min\big(T^{2/3-\varepsilon}, \frac{T^4}{x^{2+\varepsilon}}, \frac{x}{T^\varepsilon}\big) = T^{2/3-\varepsilon} \qquad \text{for } x \le T^{5/3-\varepsilon}. \qquad (3.32)$$

• Next we turn to $R_1'(x,T)$, which can be treated similarly. Defining $G_R'(v,k)$ by

$$G_R'(v,k) := \sum_{n\le v}\lambda_R(n)\lambda_R(n+k) - \mathfrak{S}_R(k)v, \qquad (3.33)$$

we have

$$R_1'(x,T) \ll \frac{x^3}{T} + T\sum_{k\le H}\Big(\int_{k\tau}^{x-k} K(u,k)d_uG_R'(u,k)\Big)x^\varepsilon, \qquad (3.34)$$

which can be derived in the same way as (3.27), where τ and H are as in (3.27). Now since

$$\sum_{k\le y} G_R'(v,k) \ll y^{1/2}R^2v^\varepsilon + y^2v^\varepsilon, \qquad (3.35)$$

cp. (1.25) of Lemma 1.12, we have

$$\begin{aligned}T\sum_{k\le H}\int_{k\tau}^{x-k}K(u,k)d_uG_R'(u,k) \ll & xT\Big(\sum_{k\le H}G_R'(x-k,k)\Big)\\ & + T^{1+\varepsilon}\Big(\int_{k\tau}^{x-k}\sum_{k\le H}G_R'(u,k)du\Big)\\ \ll & R^2xH^{1/2}T^{1+\varepsilon}+\frac{x^{3+\varepsilon}}{T}\\ \ll & R^2x^{3/2}T^{1/2+\varepsilon}+\frac{x^{3+\varepsilon}}{T}\end{aligned}$$

so that

$$R_1'(x,T)\ll\frac{x^{3+\varepsilon}}{T}+R^2x^{3/2}T^{1/2+\varepsilon}$$

by (3.34), and it follows that this is bounded by $O(Tx^2)$ if $R\ll(Tx)^{1/4-\varepsilon}$. Thus in view of (3.32) we obtain

$$2R_1(x,T)-R_1'(x,T)\ll Tx^2$$

provided

$$R\ll\min(T^{2/3-\varepsilon},(Tx)^{1/4-\varepsilon})=(Tx)^{1/4-\varepsilon}\qquad\text{for } x\le T^{5/3-\varepsilon}. \tag{3.36}$$

Now by [10, eq. (6.16)], we can estimate $R_2(x,T)$ as

$$R_2(x,T)\ll\Big(\frac{1}{Tx}+T\sum_{k\le H^*}\int_{\max(x,k\tau)}^{\infty}J(u,k)d_uG_R(u,k)\Big)x^{\varepsilon} \tag{3.37}$$

where $G_R(v,k)$ is as before, $\tau=T^{1-\varepsilon}$, H^* is given by (3.24) and $J(u,k)$ is a smooth function satisfying

$$J(u,k)\ll u^{-3}\qquad\text{and}\qquad\frac{\partial}{\partial u}J(u,k)\ll u^{-4}T^{\varepsilon}\quad\text{for } k\le u/\tau, \tag{3.38}$$

cp. [10, p. 44, eq. (6.17)]. (We note that in [10, eq. (6.16)], $R_2'(x,T)$ is directly estimated by $x^{-1-\nu}+x^{2\nu-2}$). Now on using summation by parts as well as employing (3.29) in the deduction of [10, eq. (6.20)], one has

$$\begin{aligned}T\sum_{k\le H^*}\int_{\max(x,k\tau)}^{\infty}J(u,k)d_uG_R(u,k) &\ll \frac{T^{1+\varepsilon}}{x^3}\big(\big(\frac{x}{T}\big)^{1/2}R^{3/4}+\frac{x}{T}R^{1/4}\big)x^{1/2+\varepsilon}\\ &\ll\frac{T^{1/2+\varepsilon}}{x^2}R^{3/4}+\frac{T^{\varepsilon}}{x^{3/2}}R^{1/4}\end{aligned}$$

and thus

$$R_2(x,T) \ll \frac{1}{Tx} + \frac{T^{1/2+\varepsilon}}{x^{5/2}} R^{3/4} + \frac{T^{\varepsilon}}{x^{3/2}} R^{1/4}$$

from (3.37), and similar to the case of R_1, we see that $R_2(x,T) \ll Tx^2$, if R satisfies condition (3.32).

• Similarly for $R_2'(x,T)$ we have

$$R_2'(x,T) \ll \Big(\frac{1}{Tx} + T \sum_{k \le H^*} \int_{\max(x,k\tau)}^{\infty} J(u,k) d_u G_R'(u,k)\Big) x^{\varepsilon} \ll \frac{1}{Tx^{1-\varepsilon}} + \frac{T^{1/2+\varepsilon}}{x^{5/2}} R^2$$

by employing (3.35) and we see that

$$R_2'(x,T) \ll T/x^2, \quad \text{if} \quad R \ll (Tx)^{1/4-\varepsilon}.$$

After all we have

$$2R_2(x,T) - R_2'(x,T) \ll \frac{T}{x^2}, \quad \text{if } R \ll (Tx)^{1/4-\varepsilon} \tag{3.39}$$

in the range $x \ll T^{5/3-\varepsilon}$, see also eq. (3.36).

3.3.2 Proof of Theorem 9

By plugging (3.36) with the choice $R = (Tx)^{1/4-\varepsilon}$ into (3.18) we obtain

$$I_1(x,T) \ge (1+o(1)) \frac{Tx^2}{2} \log\big(\frac{T^{5/4-\varepsilon}}{x^{3/4-\varepsilon}}\big) \qquad \text{for } T \le x \ll T^{5/3-\varepsilon}$$

and by using (3.39) with $R = (Tx)^{1/4-\varepsilon}$ in (3.25) we obtain

$$I_2(x,T) \ge (1+o(1)) \frac{T}{2x^2} \log\big(\frac{T^{5/4-\varepsilon}}{x^{3/4-\varepsilon}}\big) \qquad \text{for } T \le x \ll T^{5/3-\varepsilon}.$$

Inserting these inequalities into Lemma 3.1 gives the lower bound

$$F(x,T) \ge \frac{T}{2\pi}(1+o(1)) \log\big(\frac{T^{5/4-\varepsilon}}{x^{3/4-\varepsilon}}\big) + O\big(\frac{T(\log T)^2}{U}\big) + O\big(\frac{x^{1+\varepsilon}}{T}\big) \tag{3.40}$$

in the range $T \le x \ll T^{5/3-\varepsilon}$. Now since $x^{1+\varepsilon}/T \ll T$ for $x \ll T^{2-\varepsilon}$ and $T(\log T)^2/U = T(\log T)^{2-B} \ll T$ for $B \ge 2$, we can rewrite (3.40) as

$$F(x,T) \ge \frac{T}{2\pi}(1+o(1)) \log\big(\frac{T^{5/4-\varepsilon}}{x^{3/4-\varepsilon}}\big)$$

or

$$\big(\frac{T}{2\pi} \log T\big)^{-1} F(T^{\alpha},T) \ge \frac{5}{4} - \frac{3}{4}\alpha - \varepsilon,$$

respectively. The assertion of the Theorem now follows from this and the observation that $F(T^{\alpha},T) = F(T^{-\alpha},T)$. ■

4 A further inequality for $\psi(n;h)$

In [14], Goldston and Yıldırım asymptotically evaluated moments of the form

$$\sum_{n\sim N} \psi_R^2(n;h)\psi(n;h), \quad \sum_{n\sim N} \psi_R^3(n;h),$$

which enabled them to prove inequalities for $\psi(n;h)-h$ without absolute value, more precisely they proved

Theorem ([14, Theorem 3])**.** *Assume GRH. Then for any arbitrary small but fixed $\eta > 0$ and for sufficiently large N with $(\log N)^{14} \ll h \ll N^{1/7-\varepsilon}$ and writing $h = N^\alpha$, there exist $n_1, n_2 \in [N+1, 2N]$ such that*

$$\begin{aligned}\psi(n_1;h) - h &> \big(\frac{1}{2}\sqrt{1-5\alpha} - \eta\big)(h\log N)^{1/2}\\ \psi(n_2;h) - h &< \big(-\frac{1}{2}\sqrt{1-5\alpha} - \eta\big)(h\log N)^{1/2}.\end{aligned}$$

In this Chapter we prove the following

Theorem 10. *Let $\varepsilon > 0$ and $\delta \in (0,1)$. Then there exists some natural number $N_0(\varepsilon,\delta)$ such that for $N \geq N_0$ and $2\delta^{-1} \leq h \ll N^{1/6-\varepsilon}$ there exists an integer $n_0 \in [N+1, 2N]$ with*

$$\psi(n_0;h) > (1-\delta)h.$$

For the following we set

$$\mathcal{L}_z(R) := \sum_{\substack{r\leq R\\ p(r)>z}} \frac{\mu^2(r)}{\varphi(r)},$$

where $z \geq 1$ and $p(r)$ denotes the smallest prime factor of r. We define

$$\lambda_R'(n) := \sum_{\substack{r\leq R\\ p(r)>z}} \frac{\mu^2(r)}{\varphi(r)} \sum_{d|r,n} d\mu(d), \quad \widetilde{\lambda_R}(n) := \lambda_R'(n) - \mathcal{L}_z(R) \qquad (\star)$$

and note that by definition of $\lambda_R'(n)$ and $\mathcal{L}_z(R)$ we can write

$$\widetilde{\lambda_R}(n) = \sum_{\substack{r\leq R\\ p(r)>z}} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r,n\\ d>1}} d\mu(d).$$

Finally for $\delta > 0$ we let

$$\mathcal{M} = \mathcal{M}(\delta, N, h, \Lambda, \widetilde{\lambda_R}) := \sum_{n\sim N}(\psi(n;h) - (1-\delta)h)\Bigg(\sum_{n<m\leq n+h} \widetilde{\lambda_R}^2(m)\Bigg).$$

The distinction between $\widetilde{\lambda_R}(n)$ and $\lambda'_R(n)$, i.e. the case when $d = 1$ or not, turns out to be a crucial, see the proof of Lemma 4.3 below.

To derive Theorem 10, we need to show that $\mathcal{M}$ is positive for N sufficiently large, thus the proof of Theorem 10 relies upon deriving a lower bound for the expression $\mathcal{M}$.

We remark that we cannot show that $\mathcal{M}$ is positive if we replace $\psi(n;h)$ by $\pi(n;h)$ and h with $\pi(h)$: One can show that by the Prime Number Theorem, the part of $\mathcal{M}$ involving $\pi(h)$ contributes terms of order $\dfrac{h^2N}{\log h}$, whereas the part of $\mathcal{M}$ involving $\pi(n;h)$ is of order $\dfrac{h^2N}{\log N}$.
We continue with some preparations needed for the proof.

4.1 Preparatory Lemmas

The first Lemma is essentially shown in [14, p. 219–220]; we include the proof for completeness.

Lemma 4.1. *For positive and squarefree integers b_1, b_2 we have*

$$\sum_{a_1|b_1, a_2|b_2} \frac{a_1\mu(a_1)a_2\mu(a_2)}{\varphi([a_1,a_2])} = \frac{\mu(b_1)}{\varphi(b_1)}\frac{\mu(b_2)}{\varphi(b_2)} \prod_{p|b_1,b_2} (p^2-p-1)(p-1). \tag{4.1}$$

Proof. We set $\delta := (a_1, a_2)$ and $a'_1 := a_1/\delta$, $a'_2 := a_2/\delta$. Then we can write

$$\sum_{a_1|b_1, a_2|b_2} \frac{a_1\mu(a_1)a_2\mu(a_2)}{\varphi([a_1,a_2])} = \sum_{\delta|b_1,b_2} \frac{\delta^2}{\varphi(\delta)} \sum_{a'_1|b_1/\delta} \frac{a'_1\mu(a'_1)}{\varphi(a'_1)} \sum_{\substack{a'_2|b_2/\delta \\ (a'_2,a'_1)=1}} \frac{a'_2\mu(a'_2)}{\varphi(a'_2)}, \tag{4.2}$$

where

$$\begin{aligned}\sum_{\substack{a'_2|b_2/\delta \\ (a'_2,a'_1)=1}} \frac{a'_2\mu(a'_2)}{\varphi(a'_2)} &= \prod_{\substack{p|b_2/\delta \\ p\nmid a'_1}} (1-\frac{p}{p-1}) = \prod_{\substack{p|b_2/\delta \\ p\nmid a'_1}} \frac{(-1)}{p-1} = \prod_{p|b_2/\delta} \frac{(-1)}{p-1} \prod_{p|a'_1, b_2/\delta} (1-p) \\ &= \frac{\mu(b_2/\delta)}{\varphi(b_2/\delta)} \mu(a'_1, b_2/\delta)\varphi(a'_1, b_2/\delta)\end{aligned}$$

$$= \frac{\mu(b_2)}{\varphi(b_2)}\mu(\delta)\varphi(\delta)\mu(a_1', b_2/\delta)\varphi(a_1', b_2/\delta), \tag{4.3}$$

the last equality holding since $(b_2/\delta, \delta) = 1$, which follows from the fact that b_2 is squarefree. We plug (4.3) into (4.2) obtaining

$$\sum_{a_1|b_1, a_2|b_2} \frac{a_1\mu(a_1)a_2\mu(a_2)}{\varphi([a_1, a_2])} = \frac{\mu(b_2)}{\varphi(b_2)} \sum_{\delta|b_1,b_2} \delta^2\mu(\delta) \sum_{a_1'|b_1/\delta} \frac{a_1'\mu(a_1')}{\varphi(a_1')}\mu(a_1', b_2/\delta)\varphi(a_1', b_2/\delta), \tag{4.4}$$

where

$$\sum_{a_1'|b_1/\delta} \frac{a_1'\mu(a_1')}{\varphi(a_1')}\mu(a_1', b_2/\delta)\varphi(a_1', b_2/\delta) = \prod_{p|b_1/\delta} \big(1 - \frac{p}{p-1}\mu(p, b_2/\delta)\varphi(p, b_2/\delta)\big)$$

$$= \prod_{p|b_1/\delta, b_2/\delta} (1+p) \prod_{\substack{p|b_1/\delta \\ p\nmid b_2/\delta}} \big(1 - \frac{p}{p-1}\big) = \prod_{p|b_1/\delta, b_2/\delta} (1+p) \prod_{p|b_1/\delta} \frac{(-1)}{p-1} \prod_{p|b_1/\delta, b_2/\delta} (1-p)$$

$$= \frac{\mu(b_1/\delta)}{\varphi(b_1/\delta)}\mu\varphi\sigma(b_1/\delta, b_2/\delta) = \frac{\mu(b_1)\mu\varphi\sigma(b_1, b_2)}{\varphi(b_1)\sigma(\delta)},$$

the last equality following since b_1 is squarefree, and so (4.4) reads

$$\sum_{a_1|b_1, a_2|b_2} \frac{a_1\mu(a_1)a_2\mu(a_2)}{\varphi([a_1, a_2])} = \frac{\mu(b_1)}{\varphi(b_1)} \frac{\mu(b_2)}{\varphi(b_2)} \mu\varphi\sigma(b_1, b_2) \sum_{\delta|b_1,b_2} \frac{\mu(\delta)\delta^2}{\sigma(\delta)}$$

with

$$\mu\varphi\sigma(b_1, b_2) \sum_{\delta|b_1,b_2} \frac{\mu(\delta)\delta^2}{\sigma(\delta)} = \prod_{p|b_1,b_2} (1-p)(1+p)\big(1 - \frac{p^2}{p+1}\big) = \prod_{p|b_1,b_2} (p^2-p-1)(p-1),$$

which shows (4.1). ∎

Lemma 4.2. *For the functions defined in* $(\star)$ *on p. 73 the following holds: For* $R \leq N$*, we have*

$$\sum_{n\sim N} \lambda_R'(n) = N + O(R) \tag{4.5}$$

and for $1 \leq R \ll \sqrt{N}$ *we have*

$$\sum_{n\sim N} (\lambda_R'(n))^2 = N\mathcal{L}_z(R) + O(R^2) \tag{4.6}$$

as well as

$$\sum_{n\sim N} \widetilde{\lambda_R}^2(n) = N(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + O(R^2). \tag{4.7}$$

Proof. First (4.5) and (4.6) can be proven like (1.19) of Lemma 1.4 in Chapter 1, the only difference being the additional assumption $p(r) > z$ on the outer sum over r, while the inner sum over d remains unchanged.
(4.7) follows immediately from (4.6) and (4.5) by using $\widetilde{\lambda_R}(n) = \lambda'_R(n) - \mathcal{L}_z(R)$ and multiplying out. ∎

Lemma 4.3. *For $1 \leq h, R \leq N$ and $1 \leq z \leq R^2$ we have*

$$\sum_{k \leq h}(h-k)\sum_{n \sim N} \widetilde{\lambda_R}^2(n)\Lambda(n+k) \geq \frac{h^2N}{2}(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) - \frac{hN}{2}(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + O(h^3R^2N^{\varepsilon}) + O\big(h^2\big(\frac{N}{z} + N^{5/6} + R^2\sqrt{N}\big)\big)N^{\varepsilon}\big) \tag{4.8}$$

and the same lower bound applies for

$$\sum_{k \leq h}(h-k)\sum_{n \sim N}\Lambda(n)\widetilde{\lambda_R}^2(n+k).$$

The error terms are smaller than the main terms provided that $hR^2 \ll N^{1-\varepsilon}$.

Proof. We may restrict ourselves to showing (4.8), see also the proof of Lemma 1.12 in Chapter 1. By definition of $\widetilde{\lambda_R}(n)$ we have

$$\begin{aligned}\sum_{n \sim N}\widetilde{\lambda_R}^2(n)\Lambda(n+k) &= \sum_{\substack{r,r' \leq R \\ p(r),p(r')>z}} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r,e|r' \\ d,e>1}} d\mu(d)e\mu(e) \sum_{\substack{n \sim N \\ d,e|n}} \Lambda(n+k) \\ &= \sum_{\substack{r,r' \leq R \\ p(r),p(r')>z}} \frac{\mu^2(r)}{\varphi(r)} \sum_{\substack{d|r,e|r' \\ d,e>1}} d\mu(d)e\mu(e) \sum_{\substack{m=N+k+1 \\ m \equiv k \bmod d \\ m \equiv k \bmod e}}^{2N+k} \Lambda(m),\end{aligned} \tag{4.9}$$

where the simultaneous congruences occurring in the innermost sum of (4.9) are always compatible and determine a unique residue class $j \bmod([d,e])$, where j depends on k, d, e. Thus on writing

$$\begin{aligned}\sum_{\substack{m=N+k+1 \\ m \equiv j \bmod [d,e]}}^{2N+k} \Lambda(m) &= \frac{N}{\varphi([d,e])} + E(2N+k;[d,e],j) - E(N+k;[d,e],j) \\ &= \frac{N}{\varphi([d,e])} + E(2N;[d,e],j) - E(N;[d,e],j) + O(|k|\log N)\end{aligned}$$

we see that

$$\sum_{n\sim N}\widetilde{\lambda_R}^2(n)\Lambda(n+k)=\mathcal{S}+\mathcal{E}+O\Bigg(|k|\log N\Bigg(\sum_{r\le R}\frac{\mu^2(r)\sigma(r)}{\varphi(r)}\Bigg)^2\Bigg)$$
$$=\mathcal{S}+\mathcal{E}+O(|k|R^2\log N)\quad\text{by Lemma 1.8 (b)}\tag{4.10}$$

with

$$\mathcal{S}:=N\sum_{\substack{r,r'\le R\\ p(r),p(r')>z}}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r,e|r'\\ d,e>1}}\frac{d\mu(d)e\mu(e)}{\varphi([d,e])}$$

and

$$\mathcal{E}:=\sum_{\substack{r,r'\le R\\ p(r),p(r')>z}}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r,e|r'\\ d,e>1}}d\mu(d)e\mu(e)(E(2N;[d,e],j)-E(N;[d,e],j))$$

and we continue investigating $\mathcal{S}$ and $\mathcal{E}$.

1.1 Estimation of $\mathcal{E}$.

We have

$$\begin{aligned}\mathcal{E}&\ll\sum_{\substack{d,e\le R\\ p(d),p(e)>z}}\frac{de}{\varphi(d)\varphi(e)}|E(2N;[d,e],j)-E(N;[d,e],j)|\sum_{\substack{r_1\le R/d\\ (r_1,d)=1\\ p(r_1)>z}}\frac{\mu^2(r_1)}{\varphi(r_1)}\sum_{\substack{r_2\le R/e\\ (r_1,e)=1\\ p(r_2)>z}}\frac{\mu^2(r_2)}{\varphi(r_2)}\\
&\ll R^\varepsilon\sum_{\substack{d,e\le R\\ p(d),p(e)>z}}(|E(2N;[d,e],j)|+|E(N;[d,e],j)|)\\
&\ll R^\varepsilon\sum_{\substack{1<D\le R^2\\ p(D)>z}}(\max_{a \bmod D}|E(2N;D,a)|+\max_{a \bmod D}|E(N;D,a)|)\sum_{\substack{d,e\le R\\ [d,e]=D}}1\\
&\ll R^\varepsilon\sum_{z<D\le R^2}\tau^2(D)(\max_{a \bmod D}|E(2N;D,a)|+\max_{a \bmod D}|E(N;D,a)|)\\
&\ll N^\varepsilon\sum_{z<D\le R^2}(\max_{a \bmod D}|E(2N;D,a)|+\max_{a \bmod D}|E(N;D,a)|)\end{aligned}$$

the second last estimate following from the fact that the number of ways of writing D as lcm of two integers d,e is bounded by $\tau^2(D)$, since then d and e must be divisors of D and the last estimate following from $\tau(D)\ll D^\varepsilon\ll R^{2\varepsilon}\ll N^\varepsilon$.

Next we utilize the estimate

$$\sum_{z<D\le R^2}\max_{a \bmod D}|E(N;D,a)|\ll\big(\frac{N}{z}+N^{5/6}+R^2\sqrt{N}\big)(\log N)^4,$$

which follows from the Basic Mean Value Theorem and summation by parts, leading to

$$\mathcal{E} \ll \Big(\frac{N}{z} + N^{5/6} + R^2\sqrt{N}\Big)N^{\varepsilon}. \tag{4.11}$$

1.2 Evaluation of $\mathcal{S}$.

We can decompose $\mathcal{S}$ into

$$\mathcal{S} = \mathcal{S}_1 - 2\mathcal{S}_2 + \mathcal{S}_3$$

with

$$\mathcal{S}_1 = N \sum_{\substack{r,r'\leq R\\ p(r),p(r')>z}} \frac{\mu^2(r)}{\varphi(r)}\frac{\mu^2(r')}{\varphi(r')}\sum_{\substack{d|r\\ e|r'}}\frac{d\mu(d)e\mu(e)}{\varphi([d,e])},$$

$$\mathcal{S}_2 = N \sum_{\substack{r,r'\leq R\\ p(r),p(r')>z}} \frac{\mu^2(r)}{\varphi(r)}\frac{\mu^2(r')}{\varphi(r')}\sum_{\substack{d|r\\ e|r'\\ d=1}}\frac{d\mu(d)e\mu(e)}{\varphi([d,e])}$$

and $\mathcal{S}_3 = N\mathcal{L}_z^2(R)$, the latter sum corresponding to the case when $d = e = 1$.

Now we have

$$\begin{aligned}\mathcal{S}_2 = N\mathcal{L}_z(R)\sum_{\substack{r'\leq R\\ p(r')>z}}\sum_{e|r'}\frac{e\mu(e)}{\varphi(e)} &= N\mathcal{L}_z(R)\sum_{\substack{r'\leq R\\ p(r')>z}}\frac{\mu(r')}{\varphi^2(r')}\\ &= N\mathcal{L}_z(R) + O\Bigg(N\mathcal{L}_z(R)\sum_{\substack{1<r'\leq R\\ p(r')>z}}\frac{\mu^2(r')}{\varphi^2(r')}\Bigg)\end{aligned}$$

and therefore

$$\mathcal{S}_2 = N\mathcal{L}_z(R) + O\Big(\frac{N\mathcal{L}_z(R)}{z}\Big). \tag{4.13}$$

We turn to the evaluation of $\mathcal{S}_1$. By (4.1) of Lemma 4.1, we have

$$\mathcal{S}_1 = N \sum_{\substack{r,r'\leq R\\ p(r),p(r')>z}} \frac{\mu(r)}{\varphi^2(r)}\frac{\mu(r')}{\varphi^2(r')}\prod_{p|r,r'}(p^2-p-1)(p-1)$$

$$= N \sum_{\substack{\eta \le R \\ p(\eta) > z}} \frac{\mu^2(\eta)}{\varphi^3(\eta)} \prod_{p|\eta} (p^2 - p - 1) \sum_{\substack{s_1, s_2 \le R/\eta \\ (s_1,s_2)=(s_1,\eta)=(s_2,\eta)=1 \\ p(s_1), p(s_2) > z}} \frac{\mu(s_1)}{\varphi^2(s_1)} \frac{\mu(s_2)}{\varphi^2(s_2)}$$

and with the substitution $s = s_1 s_2$ the inner double sum can be combined to a single sum

$$\sum_{\substack{s \le (R/\eta)^2 \\ (s,\eta)=1 \\ p(s) > z}} \frac{\mu(s)}{\varphi^2(s)} = 1 + \sum_{\substack{1 < s \le (R/\eta)^2 \\ (s,\eta)=1 \\ p(s) > z}} \frac{\mu(s)}{\varphi^2(s)} = 1 + O\Big(\sum_{s > z} \frac{\mu^2(s)}{\varphi^2(s)} \Big) = 1 + O\big(\frac{1}{z}\big)$$

showing

$$\begin{aligned} \mathcal{S}_1 =& N \sum_{\substack{\eta \le R \\ p(\eta) > z}} \frac{\mu^2(\eta)}{\varphi^3(\eta)} \prod_{p|\eta} (p^2 - p - 1) + O\Big(\frac{N}{z} \sum_{\eta \le R} \frac{\mu^2(\eta)\eta^2}{\varphi^3(\eta)} \Big) \\ =& N \sum_{\substack{\eta \le R \\ p(\eta) > z}} \frac{\mu^2(\eta)}{\varphi^3(\eta)} \prod_{p|\eta} (p^2 - p - 1) + O\big(\frac{N \log R}{z} \big). \end{aligned} \tag{4.14}$$

(4.13) and (4.14) then give

$$\begin{aligned} \mathcal{S} &= N \sum_{\substack{r \le R \\ p(r) > z}} \frac{\mu^2(r)}{\varphi^3(r)} \prod_{p|r} (p^2 - p - 1) - 2N\mathcal{L}_z(R) + N\mathcal{L}_z^2(R) + O\big(\frac{N \log R}{z} \big) \\ &\ge N \sum_{\substack{r \le R \\ p(r) > z}} \frac{\mu^2(r)}{\varphi^3(r)} \prod_{p|r} (p-1)^2 - 2N\mathcal{L}_z(R) + N\mathcal{L}_z^2(R) + O\big(\frac{N \log R}{z} \big) \\ &= N\mathcal{L}_z^2(R) - N\mathcal{L}_z(R) + O\big(\frac{N \log R}{z} \big), \end{aligned} \tag{4.15}$$

using $\prod_{p|r} (p-1)^2 = \varphi^2(r)$ for r squarefree in the last step.

On inserting (4.15) and (4.11) into (4.10) we obtain

$$\sum_{n \sim N} \widetilde{\lambda_R}^2(n) \Lambda(n+k) \ge N\mathcal{L}_z^2(R) - N\mathcal{L}_z(R) + \big(\frac{N}{z} + N^{5/6} + R^2\sqrt{N} \big) N^{\varepsilon} + O(|k| R^2 N^{\varepsilon}),$$

from which it follows that

$$\begin{aligned} \sum_{k \le h} (h-k) \sum_{n \sim N} \widetilde{\lambda_R}^2(n) \Lambda(n+k) \ge& \big(\frac{h^2 N}{2} - \frac{hN}{2} \big) (\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + O(h^3 R^2 N^{\varepsilon}) \\ &+ O\big(h^2 \big(\frac{N}{z} + N^{5/6} + R^2\sqrt{N} \big) N^{\varepsilon} \big). \end{aligned}$$

This proves Lemma 4.3. ∎

4.2 Proof of Theorem 10

Throughout the proof we assume $1 \le hR^2 \ll N^{1-\varepsilon}$ and $1 \le z \le R$, so that we can make use of Lemmas 4.2 and 4.3. The assumption $z \le R$ also ensures that $\mathcal{L}_z(R) > 1$.

By Lemma 1.4 of Chapter 1, we have

$$\begin{aligned}\sum_{n\sim N}\psi(n;h)\Big(\sum_{n<m\le n+h}\widetilde{\lambda_R}^2(m)\Big) =& h\sum_{n\sim N}\Lambda(n)\widetilde{\lambda_R}^2(n)+\sum_{k\le h}(h-k)\sum_{n\sim N}\widetilde{\lambda_R}^2(n)\Lambda(n+k)\\ &+\sum_{k\le h}(h-k)\sum_{n\sim N}\Lambda(n)\widetilde{\lambda_R}^2(n+k)+O(h^3N^\varepsilon),\end{aligned} \tag{4.16}$$

where

$$\sum_{n\sim N}\Lambda(n)\widetilde{\lambda_R}^2(n)=\sum_{p\sim N}\log p\widetilde{\lambda_R}^2(p)=0, \tag{4.17}$$

since $\widetilde{\lambda_R}(p)=\sum_{\substack{r\le R\\ p(r)>z}}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d|r,p\\ d>1}}d\mu(d)=0$ for $p>R$. Now it follows from Lemma 4.3 that

$$\begin{aligned}&\sum_{k\le h}(h-k)\sum_{n\sim N}(\widetilde{\lambda_R}^2(n)\Lambda(n+k)+\Lambda(n)\widetilde{\lambda_R}^2(n+k))\\ \ge& h^2N(\mathcal{L}_z^2(R)-\mathcal{L}_z(R))-hN(\mathcal{L}_z^2(R)-\mathcal{L}_z(R)))+O(h^3R^2N^\varepsilon)\\ &+O\big(h^2\big(\frac{N}{z}+N^{5/6}+R^2\sqrt{N}\big)N^\varepsilon\big)\end{aligned} \tag{4.18}$$

and by applying (4.17) and (4.18) to (4.16) we obtain

$$\begin{aligned}\sum_{n\sim N}\psi(n;h)\Big(\sum_{n<m\le n+h}\widetilde{\lambda_R}^2(m)\Big)\ge& h^2N(\mathcal{L}_z^2(R)-\mathcal{L}_z(R))-hN(\mathcal{L}_z^2(R)-\mathcal{L}_z(R))\\ &+O(h^3R^2N^\varepsilon)\\ &+O\big(h^2\big(\frac{N}{z}+N^{5/6}+R^2\sqrt{N}\big)N^\varepsilon\big).\end{aligned} \tag{4.19}$$

On the other hand we have

$$\begin{aligned}\sum_{n\sim N}\Big(\sum_{n<m\le n+h}\widetilde{\lambda_R}^2(m)\Big) &= \sum_{N<m\le 2N+h}\widetilde{\lambda_R}^2(m)\sum_{m-h\le n<m}1\\ &= h\sum_{m\sim N}\widetilde{\lambda_R}^2(m)+O(h^2N^\varepsilon)\end{aligned}$$

$$= hN(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + O(h^2(R^2 + N^\varepsilon)) \qquad (4.20)$$

applying (4.7) of Lemma 4.2 in the last step. Then (4.19) and (4.20) lead to

$$\begin{aligned}
\mathcal{M} =& \sum_{n\sim N} (\psi(n;h) - (1-\delta)h) \Big(\sum_{n<m\le n+h} \widetilde{\lambda_R}^2(m) \Big) \\
\ge& (\delta h^2 N - hN)(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + O(h^3 R^2 N^\varepsilon) \\
&+ O\big(h^2\big(\frac{N}{z} + N^{5/6} + R^2\sqrt{N}\big)N^\varepsilon\big) \\
&\ge hN(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + O(h^3 R^2 N^\varepsilon) \\
&+ O\big(h^2\big(\frac{N}{z} + N^{5/6} + R^2\sqrt{N}\big)N^\varepsilon\big) \qquad \text{if } h \ge \frac{2}{\delta},
\end{aligned}$$

where every occurring error term is bounded by $o(hN)$, if the conditions

$$h \ll N^{1/6-\varepsilon}, \qquad R \ll \min\big(\frac{N^{1/2-\varepsilon}}{h}, \frac{N^{1/4-\varepsilon}}{h^{1/2}}\big) = \frac{N^{1/4-\varepsilon}}{h^{1/2}} \quad \text{for } h \ll N^{1/2-\varepsilon}$$

and $hN^\varepsilon \ll z$ $(\le R)$ are satisfied, which are all compatible in the range $2/\delta \le h \ll N^{1/6-\varepsilon}$. Hence we established

$$\mathcal{M} \ge hN(\mathcal{L}_z^2(R) - \mathcal{L}_z(R)) + o(hN)$$

and since $\mathcal{L}_z(R) > 1$ we can conclude that there exists some $N_0(\varepsilon, \delta)$ such that $\mathcal{M}$ is positive for $N \ge N_0$. ■

5 An alternate result for the second moment of primes over an arithmetic progression

We let
$$E'(y;h,r) := \max_{(a,r)=1} |E(y+h;r,a) - E(y;r,a)|,$$
and for $(a,r)=1$ and $1 \le r \le h \le x$ we let
$$I(x;h,r,a) := \int_0^x \big(\psi(y+h;r,a) - \psi(y;r,a) - \frac{h}{\varphi(r)}\big)^2 dy.$$
Here Özlük (see [13, Theorem B]) proved unconditionally that
$$I(x;h,r,a) > \big(\frac{1}{2} - \varepsilon\big)\frac{hx}{\varphi(r)} \log x$$
for any $\varepsilon > 0$ and for $1 \le r \le (\log x)^{1-\delta}$ and $h \le (\log x)^c$ uniformly in $r \le h$, where δ, $c > 0$. In this Chapter we shall prove the following

Theorem 11. *Let κ, $\varepsilon > 0$ with $\kappa \le 1/2 - \varepsilon$. Then for infinitely many x, the following holds true: For every $h \in [1, x^{\kappa/6}] \cup [x^{\kappa}, x^{1/2-\varepsilon}]$ there exists some $r_0 = r_0(h,\kappa,\varepsilon) \le h$ and a constant $C = C(\varepsilon) > 0$, such that*
$$I(x;h,r_0,a) \ge C(\varepsilon)\frac{hx}{\varphi(r_0)} \log x.$$

Moreover, we can state the following Corollary, which follows immediately from the Theorem.

Corollary 4. *Let κ, ε be as in the Theorem. Then for infinitely many x we have the following: For every $h \in [1, x^{\kappa/6}] \cup [x^{\kappa}, x^{1/2-\varepsilon}]$ there exists some $r_0 = r_0(h,\kappa,\varepsilon) \le h$ and a constant $C = C(\varepsilon) > 0$, such that*
$$\max_{y\le x} E'(y;h,r_0) \ge C(\varepsilon)\big(\frac{h}{\varphi(r_0)} \log x\big)^{1/2}.$$

In connection with this result it is surmised that, for $\varepsilon > 0$,

$$E'(x,h;q) \ll_\varepsilon x^\varepsilon \left(\frac{h}{q}\right)^{1/2}$$

for all $1 \leq q \leq h \leq x$. The analogue conjecture in the case $h = x$ is precisely Montgomery's Conjecture stated in Chapter 1.

5.1 Auxiliary Results

To prove the above Theorem, we shall utilize the following auxiliary results. The first Lemma is a special case of Lemma 1.4 in Chapter 1, whose proof we include for completeness.

Lemma 5.1. *For every $h \leq x$ we have*

$$\int_0^x \Big(\sum_{y<n\leq y+h} a_n\Big)dy = (h+O(1))\sum_{n\leq x} a_n + O(\|a\|h^2). \tag{5.1}$$

Proof. By Lemma 1.4 of Chapter 1 we have

$$\begin{aligned}(h+O(1))\int_0^x \Big(\sum_{y<n\leq y+h} a_n\Big)dy &= \int_0^x \Big(\sum_{y<n\leq y+h} a_n\Big)\Big(\sum_{y<m\leq y+h} 1\Big)dy \\ &= h\sum_{n\leq x} a_n + \sum_{0<|k|\leq h}(h-|k|)\sum_{n\leq x} a_n + O(\|a\|h^3) \\ &= h\sum_{n\leq x} a_n + (h^2-h+O(1))\sum_{n\leq x} a_n + O(\|a\|h^3)\end{aligned}$$

and we obtain (5.1) on dividing both sides by $h + O(1)$. ■

Lemma 5.2. *Let h, R be real numbers with $1 \leq h \leq R \leq x$. Then for any real number R_0 with $1 \leq R_0 \leq R$ we have*

$$\begin{aligned}\int_0^x \psi(y;h)\psi_R(y;h)dy =& hx\log\left(\frac{R}{h}\right) + h^2x \\ &+ O\Big(h^2\log x\sum_{r\leq R_0}\varphi(r)\max_{(a,r)=1}|E(x;r,a)|\Big) \\ &+ O\Big(\big(\frac{h^2x}{R_0} + h^2x^{5/6} + h^2R\sqrt{x}\big)x^\varepsilon\Big) + O(h^3x^\varepsilon).\end{aligned}$$

Proof. By (1.51) on p. 27, we have

$$\int_0^x \psi(y;h)\psi_R(y;h)dy$$
$$=h\sum_{n\le x}\Lambda(n)\lambda_R(n)+\sum_{k\le h}(h-k)\sum_{n\le x}(\Lambda(n)\lambda_R(n+k)+\lambda_R(n)\Lambda(n+k))+O(h^3x^{\varepsilon}),$$

where

$$\sum_{n\le x}\Lambda(n)\lambda_R(n)=x\log R+O(x)$$

by Lemma 1.10 (A) on p. 12. Moreover, we have

$$\sum_{k\le h}(h-k)\sum_{n\le x}\lambda_R(n)\Lambda(n+k)=\Sigma_1+\mathcal{E}$$

with

$$\Sigma_1=\frac{h^2x}{2}-hx\log h+O(hx)+O(h^3x^{\varepsilon})$$

and

$$\mathcal{E}=\sum_{k\le h}(h-k)\sum_{\substack{d\le R\\(d,k)=1}}d\mu(d)E(x;d,k)\sum_{\substack{r\le R\\d|r}}\frac{\mu^2(r)}{\varphi(r)}$$

cp. for example (2.11) and (2.12) on p. 35. Now by arguing as in the proof of Proposition 1 on p. 31, we can write $\mathcal{E}$ as

$$\mathcal{E}=\mathcal{T}_1'+\mathcal{T}_2' \tag{5.2}$$

with

$$\mathcal{T}_1':=\sum_{k\le h}(h-k)\sum_{d\le R}\frac{d\mu(d)}{\varphi(d)}\sum_{\substack{r\le R\\d|r}}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d^*|d\\d^*\le R_0}}\sideset{}{^*}\sum_{\chi(d^*)}\big(\psi(x;\chi^*)-\delta_{\chi^*}x\big)$$

and

$$\mathcal{T}_2':=\sum_{k\le h}(h-k)\sum_{d\le R}\frac{d\mu(d)}{\varphi(d)}\sum_{\substack{r\le R\\d|r}}\frac{\mu^2(r)}{\varphi(r)}\sum_{\substack{d^*|d\\R_0<d^*\le R}}\sideset{}{^*}\sum_{\chi(d^*)}\psi(x;\chi^*).$$

By (2.6) on p. 33 with $B=1$, we have

$$\mathcal{T}_1'\ll h^2(\log x)^2\sum_{d^*\le R_0}\varphi(d^*)\max_{(a,d^*)=1}|E(x;d^*,a)| \tag{5.3}$$

and $\mathcal{T}_2'$ can be estimated like $\mathcal{T}_3$ in the proof of Proposition 1 on p. 31 with R_0 instead of h^2 as

$$\mathcal{T}_2'\ll\big(\frac{h^2x}{R_0}+h^2x^{5/6}+h^2R\sqrt{x}\big)x^{\varepsilon}. \tag{5.4}$$

Since the same applies to the double sum $\sum_{k\le h}(h-k)\sum_{n\le x}\Lambda(n)\lambda_R(n+k)$, this shows the Lemma.

∎

5.2 Proof of Theorem 11

Given any $h \le x^{1/2-\varepsilon}$, we let $r_0 = r_0(h) \in \mathbb{N}$ with

$$I(x;h,r_0,a) = \max_{r \le h} I(x;h,r,a)$$

and distinguish two cases.

• Case 1

For every $h \in [x^{\kappa}, x^{1/2-\varepsilon}]$ we have

$$I(x;h,r_0,a) \gg_{\varepsilon} \frac{hx}{\varphi(r_0)} \log x, \qquad x > x_0(\varepsilon).$$

Then the assertion of the Theorem holds true.

• Case 2

There exists some $h_0 \in [x^{\kappa}, x^{1/2-\varepsilon}]$, such that for every $C = C(\varepsilon) > 0$ we have

$$I(x;h,r_0,a) \le C(\varepsilon) \frac{h_0 x}{\varphi(r_0)} \log x \qquad \text{for infinitely many } x. \tag{5.5}$$

Starting from (5.5), we now show in 5.1.2–5.1.6 that this implies $I(x;h) \gg_{\varepsilon} hx \log x$ for every $1 \le h \ll x^{\kappa/6-\varepsilon}$ for infinitely many x, which shows the Theorem with $r_0 = 1$.

We have $\varphi(r_0) I(x;h_0,r_0,a) \le C(\varepsilon) hx \log x$ and therefore, since $I(x;h,r,a) \le I(x;h,r_0,a)$ by definition of r_0, it follows that

$$I(x;h_0,r,a) \le C(\varepsilon) hx \log x \quad \text{uniformly in } r \le h_0, \text{ for infinitely many } x. \tag{5.6}$$

5.2.1 Estimation of $E(x;r,a)$

First we can infer from (5.1) of Lemma 5.1 with the choice

$$a_m = \begin{cases} \Lambda(m), & \text{if } m \equiv a \bmod r, \\ 0, & \text{otherwise,} \end{cases}$$

that

$$\int_0^x \Bigg(\sum_{\substack{y < n \le y+h_0 \\ n \equiv a \bmod r}} \Lambda(n) \Bigg) dy = (h_0 + O(1)) \psi(x;r,a) + O(h_0^2 \log x),$$

implying

$$\int_0^x \Big(\sum_{\substack{y<n\le y+h_0\\ n\equiv a \bmod r}} \Lambda(n) - \frac{h_0}{\varphi(r)}\Big)dy = (h_0+O(1))E(x;r,a) + O(h_0^2\log x),$$

which on using the triangle inequality as well as Cauchy's inequality gives

$$\begin{aligned}(h_0+O(1))|E(x;r,a)| &\ll \int_0^x \Big|\sum_{\substack{y<n\le y+h_0\\ n\equiv a \bmod r}} \Lambda(n) - \frac{h_0}{\varphi(r)}\Big|dy + h_0^2\log x\\ &\le x^{1/2}I(x;h_0,r,a)^{1/2} + h_0^2\log x\\ &\ll_\varepsilon x^{1/2}\big(\frac{h_0 x}{\varphi(r_0)}\log x\big)^{1/2} + h_0^2\log x \qquad \text{by (5.5)}\\ &\le x^{1/2}(h_0\log x)^{1/2} + h_0^2\log x. \end{aligned} \tag{5.7}$$

Thus we see that

$$\begin{aligned}\max_{(a,r)=1}|E(x;r,a)| &\ll_\varepsilon x^{1-\alpha_0/2}\log^{1/2}x + x^{\alpha_0}\log x\\ &\ll x^{1-\alpha_0/2}\log^{1/2}x\end{aligned} \tag{5.8}$$

on dividing both sides (5.7) by $h_0+O(1)$ and writing $h_0 = x^{\alpha_0}$.

5.2.2 A lower bound for $\int_0^x \psi^2(y;h)dy$

We first recall

$$\int_0^x \psi^2(y;h)dy \ge 2\int_0^x \psi(y;h)\psi_R(y;h)dy - \int_0^x \psi_R^2(y;h)dy \tag{5.9}$$

and since

$$\begin{aligned}\int_0^x \psi(y;h)\psi_R(y;h)dy =& hx\log\big(\frac{R}{h}\big) + h^2x + O\Big(h^2\log x\sum_{r\le R_0}\varphi(r)\max_{(a,r)=1}|E(x;r,a)|\Big)\\ &+O\Big(\big(\frac{h^2x}{R_0} + h^2x^{5/6} + h^2R\sqrt{x}\big)x^\varepsilon\Big) + O(h^3x^\varepsilon)\end{aligned} \tag{5.10}$$

for any $1\le R_0 < R$ by Lemma 5.2 and

$$\begin{aligned}&\int_0^x \psi_R^2(y;h)dy\\ =&h\sum_{n\le x}\lambda_R^2(n) + 2\sum_{k\le h}(h-k)\lambda_R(n)\lambda_R(n+k) + O(h^3x^\varepsilon) \qquad \text{by Lemma 1.4}\end{aligned}$$

$$=hx\log\big(\frac{R}{h}\big)+h^2x+O(h^{3/2}R^2x^{\varepsilon})+O(h^3x^{\varepsilon}), \tag{5.11}$$

using Lemma 1.10 (B) on p. 12 and Lemma 1.11 on p. 14 in the last step, we obtain the lower bound

$$\int_0^x \psi^2(y;h)dy \geq hx\log\big(\frac{R}{h}\big)+h^2x+O\Big(h^2\log x\sum_{r\leq R_0}\varphi(r)\max_{(a,r)=1}|E(x;r,a)|\Big)$$
$$+O\Big(\big(\frac{h^2x}{R_0}+h^2x^{5/6}+h^2R\sqrt{x}\big)x^{\varepsilon}+h^{3/2}R^2x^{\varepsilon}+h^3x^{\varepsilon}\Big) \tag{5.12}$$

for some $1\leq R_0<R$ by substituting (5.10) as well as (5.11) into (5.9).

5.2.3 The appropriate choice of R and R_0

As for the first error term of (5.12), we may use (5.8) to obtain the estimate

$$h^2\log x\sum_{r\leq R_0}\varphi(r)\max_{(a,r)=1}|E(x;r,a)|\ll h^2R_0^2x^{1-\alpha_0/2}\log^{3/2}x,$$

which by (5.12) leads us to the inequality

$$\int_0^x \psi^2(y;h)dy\geq hx\log\big(\frac{R}{h}\big)+h^2x+\mathcal{F}, \tag{5.13}$$

where $\mathcal{F}$ can be estimated as

$$\mathcal{F}\ll h^2R_0^2x^{1-\alpha_0/2}\log^2x+\big(\frac{h^2x}{R_0}+h^2x^{5/6}+h^2R\sqrt{x}\big)x^{\varepsilon}+h^{3/2}R^2x^{\varepsilon}+h^3x^{\varepsilon}. \tag{5.14}$$

In the next step we want to choose R resp. R_0 in an appropriate range to ensure $\mathcal{F}=O(hx)$, which especially holds true if every single term on the RHS of (5.14) is bounded by $O(hx)$.
Here the conditions $h^2R_0^2x^{1-\alpha_0/2}(\log x)^2\ll hx$ and $\frac{h^2x^{1+\varepsilon}}{R_0}\ll hx$ imply

$$hx^{\varepsilon}\ll R_0\ll\frac{x^{\alpha_0/4}}{h^{1/2}\log x}, \tag{5.15}$$

whereas the conditions $h^2Rx^{1/2+\varepsilon}\ll hx$ and $h^{3/2}R^2x^{\varepsilon}\ll hx$ require that

$$R\ll\min\Big(\frac{x^{1/2-\varepsilon}}{h},\big(\frac{x^{1/2-\varepsilon}}{h^{1/2}}\big)\Big)=\frac{x^{1/2-\varepsilon}}{h}. \tag{5.16}$$

5.2.4 A possible range for h

First of all (5.15) implies $h \ll x^{\alpha_0/6-\varepsilon}$, and moreover the condition $h^2 x^{5/6+\varepsilon} \ll hx$ to keep $\mathcal{F} = O(hx)$, cp. (5.14), implies $h \ll x^{1/6-\varepsilon}$, so that we obtain the range

$$1 \leq h \ll x^{\alpha_0/6-\varepsilon}. \tag{5.17}$$

Altogether, by putting (5.16) into (5.13) and choosing a suitable R_0 satisfying (5.15), we can establish the lower bound

$$\int_0^x \psi^2(y;h)dy \geq hx \log\big(\frac{x^{1/2-\varepsilon}}{h^2}\big) + h^2 x + O(hx), \tag{5.18}$$

provided that h satisfies (5.17).

5.2.5 End of proof

By squaring out,

$$I(x;h) = \int_0^x \psi^2(y;h)dy - 2h\int_0^x \psi(y;h)dy + h^2 x \tag{5.19}$$

and since

$$\int_0^x \psi(y;h)dy = h\psi(x) + O(x+h^2\log x) = hx + O(x+h|\psi(x)-x|+h^2\log x), \tag{5.20}$$

which follows from (5.1) of Lemma 5.1 and the estimate $\psi(x) \ll x$, it remains to control the error term $|\psi(x)-x|$, which also determines a possible range for h. In order to do this, we first observe that (5.1) of Lemma 5.1 with $a_n = \Lambda(n)$ together with the estimate $\psi(x) \ll x$ imply

$$\begin{aligned}(h_0+O(1))(\psi(x)-x) &= \sum_{n\leq x}(\psi(n;h_0)-h_0) + O(x+h_0^2\log x)\\ &= \sum_{n\leq x}(\psi(n;h_0)-h_0) + O(x).\end{aligned}$$

Next we have

$$\sum_{n\leq x}(\psi(n;h_0)-h_0)^2 \leq C(\varepsilon)h_0 x\log x \tag{5.21}$$

from our assumption (5.6) and hence

$$(h_0+O(1))|\psi(x)-x| \leq \sum_{n\leq x}|\psi(n;h_0)-h_0| + O(x)$$

$$
\begin{aligned}
&\le \sqrt{x}\Big(\sum_{n\le x}(\psi(n;h_0)-h_0)^2\Big)^{1/2} + O(x) \\
&\ll x\sqrt{h_0\log x} \qquad (5.22)
\end{aligned}
$$

using the triangle inequality and Cauchy's inequality, so we obtain

$$|\psi(x)-x| \ll \frac{x}{\sqrt{h_0}}\sqrt{\log x}$$

on dividing both sides of (5.22) by $h_0 + O(1)$. Thus we see

$$\int_0^x \psi(y;h)dy = h\psi(x) + O(x + h^2\log x) = hx + O(x)$$

by inserting (5.22) into (5.20). Using this as well as (5.18) in (5.19), we obtain

$$I(x;h) \ge hx\log\big(\frac{x^{1/2-\varepsilon}}{h^2}\big) + O(hx),$$

which establishes a nontrivial lower bound for $I(x;h)$ if

$$1 \le h \ll \min(x^{\alpha_0/6-\varepsilon}, x^{1/4-\varepsilon}) = x^{\alpha_0/6-\varepsilon}$$

by (5.17). In particular we have

$$I(x;h) \ge C(\varepsilon)hx\log x \qquad \text{for } 1\le h \ll x^{\kappa/6-\varepsilon}$$

for a constant $C(\varepsilon) > 0$, which shows the Theorem. ■

Bibliography

[1] O. Bordellès: Arithmetic Tales. London, Heidelberg, New York, Dordrecht: Springer-Verlag, 2012.

[2] T. H. Chan: More precise pair correlation of zeros and primes in short intervals. *J. London Math. Soc.* **68** (2003), pp. 579–598.

[3] H. Davenport: Multiplicative Number Theory. Berlin, Heidelberg, New York: Springer-Verlag, 1967.

[4] J. B. Friedlander and D. A. Goldston: Some singular series averages and the distribution of Goldbach numbers in short intervals. *Illinois J. of Math.* **39** (1) (1995), pp. 158–180.

[5] J. B. Friedlander and D. A. Goldston: Variance of distribution of primes in residue classes. *Quart. J. Math.* **47** (2) (1996), pp. 313–336.

[6] D. A. Goldston: The second moment for prime numbers. *Quart. J. Math.* **35** (2) (1984), pp. 153–163.

[7] D. A. Goldston: A lower bound for the second moments of primes in short intervals. *Expo. Math.* **13** (1995), pp. 366–376.

[8] D. A. Goldston: On Hardy and Littlewood's Contribution to the Goldbach Conjecture. In: *Proceedings of the Symposium on Analytic Number Theory, Amalfi, 1989*, pp. 115–155.

[9] D. A. Goldston and S. M. Gonek: Mean value theorems for long Dirichlet polynomials and tails of Dirichlet series. *Acta Arithmetica* **84** (2) (1998), pp. 155–192.

[10] D. A. Goldston, S. M. Gonek, A. E. Özlük and C. Snyder: On the pair correlation of the zeroes of the Riemann zeta-function. *Proc. London Math. Soc.* **80** (3) (2000), pp. 31–49.

[11] D. A. Goldston and H. L. Montgomery: Pair correlation of zeros and primes in short intervals. In: *Analytic Number Theory and Diophantine Problems.* Boston, Massachusetts: Birkhäuser, 1987, 83–203.

[12] D. A. Goldston and C. Yıldırım: Primes in short segments of arithmetic progressions. *Can. J. of Math.* **50** (3) (1998), pp. 563–580.

[13] D. A. Goldston and C. Yıldırım: On the second moment for primes in an arithmetic progression. *Acta Arithmetica* **100** (1) (2001), pp. 85–104.

[14] D. A. Goldston and C. Yıldırım: Higher correlation of short divisor sums related to primes II: Variations of the error term in the Prime Number Theorem. *Proc. London Math. Soc.* **95** (3) (2007), pp. 199–247.

[15] A. Harper and K. Soundararajan: Lower bounds for the variance of sequences in arithmetic progressions: primes and divisor functions. To appear in *Quart. J. Math.*

[16] D. R. Heath-Brown: The ternary Goldbach problem. *Rev. Mat. Iberoamericana* **1** (1) (1985), pp. 45–59.

[17] D. R. Heath-Brown: Primes in a short interval. *J. reine angew. Math.* **389** (1988), pp. 22–63.

[18] D. Hensley and I. Richards: Primes in intervals. *Acta Arithmetica* **25** (1973), pp. 375–391.

[19] C. Hooley: On the Barban Davenport Halberstam Theorem XIII. *Acta Arithmetica* **94** (1) (2000), pp. 53–86.

[20] D. Koukoulopoulos: Pretentious multiplicative functions and the prime number theorem for arithmetic progressions. *Compos. Math.* **149** (7) (2013), pp. 1129–1149.

[21] H. L. Montgomery: Primes in Arithmetic Progressions. *Mich. Math. J.* **17** (1970), pp. 33–39.

[22] H. L. Montgomery: Topics in Multiplicative Number Theory. Berlin, Heidelberg, New York: Springer–Verlag, 1971.

[23] H. L. Montgomery: The pair correlation of zeros of the zeta function. *Proc. Sym. Pure Math.* **24** (1973), pp. 181–193.

[24] H. L. Montgomery and R. C. Vaughan: Multiplicative Number Theory I. Classical Theory. New York: Cambridge University Press, 2006.

[25] R. C. Vaughan: Moments for primes in arithmetic progressions, I. *Duke Math. J.* **120** (2), pp. 371–383.