

# An improvement of de Jong-Oort's purity theorem

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**Abstract.** Consider an  $F$ -crystal over a noetherian scheme  $S$ . De Jong-Oort's purity theorem states that the associated Newton polygons over all points of  $S$  are constant if this is true outside a subset of codimension bigger than 1. In this paper we show an improvement of the theorem, which says that the Newton polygons over all points of  $S$  have a common break point if this is true outside a subset of codimension bigger than 1.

## 1. INTRODUCTION

De Jong-Oort's purity theorem [4, Thm. 4.1] states that for an  $F$ -crystal over a noetherian scheme  $S$  of characteristic  $p$  the associated Newton polygons over all points of  $S$  are constant if this is true outside a subset of codimension bigger than 1. This theorem has been strengthened and generalized by Vasiliu [14], who has shown that each stratum of the Newton polygon stratification defined by an  $F$ -crystal over any reduced, not necessarily noetherian  $\mathbb{F}_p$ -scheme  $S$  is an affine  $S$ -scheme. In the case of a family of  $p$ -divisible groups, alternative proofs of the purity have been given by Oort [11] and Zink [15]. In this paper we show an improvement, which implies that for an  $F$ -crystal over a noetherian scheme  $S$  the Newton polygons over all points have a common break point if this is true outside a subset of codimension bigger than 1. As to a stronger statement analogous to that in Vasiliu's paper, our method does not apply. The main result is the following theorem.

**Theorem 1.1.** *Let  $S$  be a locally noetherian scheme of characteristic  $p$  and  $\mathcal{E}$  be an  $F$ -crystal over  $S$ . Fix  $s \in S$ . If there exists an open neighborhood  $U$  of  $s$  in  $S$  such that the Newton polygons  $NP(\mathcal{E})_x$  over all points  $x \in U \setminus \{s\}^-$  have a common break point, then either  $\text{codim}(\{s\}^-, U) \leq 1$  or  $NP(\mathcal{E})_s$  has the same break point.*

The following example explains how Theorem 1.1 improves de Jong-Oort purity theorem. Look at Figure 1. Consider the spectrum of some local noetherian integral domain of dimension 2 and characteristic  $p$ . Then we ask:

does there exist an  $F$ -crystal such that the associated Newton polygon over the closed point is  $\xi$ , over a finite number of points of codimension 1 is  $\gamma$  and over each of all other points is  $\eta$ ? Theorem 1.1 tells us that the answer is negative, while it cannot be easily seen from [4, Thm. 4.1].

In our main theorem, the condition on “one of the break points” cannot be generalized to an arbitrary point of the Newton polygon which is not a break point. Consider a family of elliptic curves  $f : \mathcal{X} \rightarrow S$ , where  $S$  is a curve over a field  $k$  of positive characteristic. Look at Figure 2. Assume that all the fibers of  $f$  are ordinary except over a closed point  $0 \in S$ . Then Figure 2 shows all the Newton polygons associated to the family of abelian surfaces  $\mathcal{X} \times_k \mathcal{X} \rightarrow S \times_k S$ . Namely, over the special point  $(0, 0)$  the associated Newton polygon is  $\xi$ ; over each point in  $\{0\} \times S \cup S \times \{0\}$ , it is  $\gamma'$ ; over each of all other points, it is  $\eta'$ . We see that outside the one-point set  $\{(0, 0)\}$  of codimension 2, the Newton polygons have a common point  $P$ .

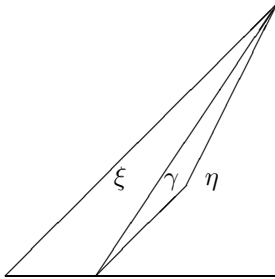


Figure 1

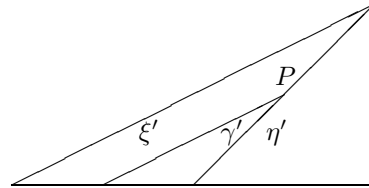


Figure 2

This paper is organized as follows. In Section 2, we review some facts about  $F$ -crystals before showing that if the Newton polygon of an  $F$ -crystal over a field has a break point  $(1, m)$ , then there exists a unique subcrystal of rank 1 and slope  $m$ . In Section 3, we describe the kernel of  $\text{Gal}(\overline{K}/K) \rightarrow \pi_1(X, \overline{\eta})$  as the normal subgroup generated by local kernels (Proposition 3.2) and particularly obtain another description when  $X$  is the spectrum of a discrete valuation ring (Corollary 3.9). In Section 4, we define the Galois representation associated to an  $F$ -crystal, and discuss the relationship between its ramification property and Newton slopes (see Proposition 4.6). Section 5 contains the proof of Theorem 1.1. The proof essentially follows the proof of [4, Thm. 4.1], yet is more accessible because the relationship between the ramification property of the representation and the Newton slopes has been clarified.

The author would like to thank her advisor, Aise Johan de Jong, without whose patient and effective instruction this paper would not have come into existence. Also the author owes a lot to the referees who have made many suggestions and particularly pointed out a significant improvement of the main theorem, which had originally treated the first few break points of the Newton polygon instead of any break point.

2. RESULTS ON  $F$ -CRYSTALS

**2.1. Conventions.** In this paper,  $k$  always denotes a field of characteristic  $p$ , where  $p$  is a prime number;  $\bar{k}$  denotes an algebraic closure of  $k$ ;  $S$  denotes a connected scheme of characteristic  $p$ . We use the term *crystal* to mean a crystal of finite locally free  $\mathcal{O}_{\text{cris}}$ -modules. See [1, p. 226]. Here  $\mathcal{O}_{\text{cris}}$  denotes the structure sheaf on the category  $\text{CRIS}(S/\text{Spec } \mathbb{Z}_p)$  (big crystalline site of  $S$ ). If  $T \rightarrow S$  is a morphism, then we use  $\mathcal{E}|_T$  to denote the pullback of  $\mathcal{E}$  to  $\text{CRIS}(T/\text{Spec } \mathbb{Z}_p)$ . For a crystal  $\mathcal{E}$ , we denote by  $\mathcal{E}^{(n)}$  the pullback of  $\mathcal{E}$  by the  $n$ th iterate of the Frobenius endomorphism of  $S$ . An  $F$ -crystal over  $S$  is a pair  $(\mathcal{E}, F)$ , where  $\mathcal{E}$  is a crystal over  $S$  and  $F: \mathcal{E}^{(1)} \rightarrow \mathcal{E}$  is a morphism of crystals. We usually denote an  $F$ -crystal by  $\mathcal{E}$ , with the map  $F$  being understood. Recall that  $\mathcal{E}$  is a *nondegenerate  $F$ -crystal* if the kernel and cokernel of  $F$  are annihilated by some power of  $p$ , see [12, 3.1.1]. All  $F$ -crystals in this paper will be nondegenerate.

A perfect scheme  $S$  in characteristic  $p$  is a scheme such that the Frobenius map  $(-)^p: \mathcal{O}_S \rightarrow \mathcal{O}_S$  is an isomorphism. A crystal over a perfect scheme  $S$  is simply given by a finite locally free sheaf of  $W(\mathcal{O}_S)$ -modules (see [6, p. 141]).

**2.2.** Suppose that  $S = \text{Spec } k$ . Choose a Cohen ring  $\Lambda$  for  $k$ , and let  $\sigma: \Lambda \rightarrow \Lambda$  be a lift of *Frobenius* on  $k$ . By [2, Prop. 1.3.3], we know that an  $F$ -crystal  $\mathcal{E}$  over  $k$  is given by a triple  $(M, \nabla, F)$  over  $\Lambda$ , where  $M$  is a finite free  $\Lambda$ -module of rank  $r$ ,  $\nabla$  is an integrable, topologically quasi-nilpotent connection, and  $F$  is a horizontal  $\sigma$ -linear self-map of  $M$ .

**2.3.** Let  $k^{pf}$  be the perfect closure of  $k$ . Note that under the identification  $k^{pf} = \varinjlim(k \rightarrow k \rightarrow \dots)$ , and by [13, Chapter II, Prop. 10], we obtain

$$W(k^{pf}) = p\text{-adic completion of } \varinjlim(\Lambda \xrightarrow{\sigma} \Lambda \xrightarrow{\sigma} \dots).$$

Furthermore  $\sigma$  can be extended to an endomorphism of  $W(k^{pf})$ , which is a lift of Frobenius on  $k^{pf}$ , still denoted by  $\sigma$ . Thus we get an injection  $\Lambda \rightarrow W(k^{pf})$  compatible with  $\sigma$ .

The pullback  $\mathcal{E}|_{\text{Spec}(k^{pf})}$  of  $\mathcal{E}$  corresponds to the pair  $(M \otimes_{\Lambda} W(k^{pf}), F \otimes \sigma)$ . According to [6, 1.3], we can describe Newton slopes associated to  $\mathcal{E}|_{\text{Spec}(k^{pf})}$  as follows. Choose an algebraic closure  $\bar{k}$  of  $k^{pf}$  and some positive integer  $N$  divisible by  $r!$ . Consider the valuation ring  $R = W(\bar{k})[X]/(X^N - p) = W(\bar{k})[p^{1/N}]$  and denote its fraction field by  $K$ . We extend  $\sigma$  to an automorphism of  $R$  by requiring that  $\sigma(X) = X$ . Then by Dieudonné (cp. [7]),  $M \otimes_{W(\bar{k})} K$  admits a  $K$ -basis  $e_1, \dots, e_r$  such that  $(F \otimes \sigma)(e_i) = p^{\lambda_i} e_i$  for  $\lambda_i \in \mathbb{Q}$  and  $0 \leq \lambda_1 \leq \dots \leq \lambda_r$ . These  $\lambda_i$ 's are defined to be the *Newton slopes* of  $(M, F)$  or  $\mathcal{E}$ .

For each  $\lambda$ , we define  $\text{mult}(\lambda)$  as the number of times  $\lambda$  occurs among  $\{\lambda_1, \dots, \lambda_r\}$ . By Dieudonné again, the product  $\lambda \cdot \text{mult}(\lambda) \in \mathbb{Z}_{\geq 0}$ . The Newton Polygon of  $(M, F)$  is a polygonal chain consisting of line segments  $S_1, \dots, S_r$ , where  $S_i$  connects the two points  $(i-1, \lambda_1 + \dots + \lambda_{i-1})$  and  $(i, \lambda_1 + \dots + \lambda_i)$ . The points at which the Newton polygon changes slope are called *break points*.

We now turn to an  $F$ -crystal  $\mathcal{E}$  over an arbitrary  $\mathbb{F}_p$ -scheme  $S$ . For every point  $s \in S$ , let  $s : \text{Spec } k(s) \rightarrow S$  be the natural map. We can assign to  $s$  the Newton polygon associated to  $\mathcal{E}|_{\text{Spec } k(s)}$ , denoted by  $NP(S, \mathcal{E})_s$  or  $NP(\mathcal{E})_s$ .

The following result about the existence of some special subcrystal will be significant in proving the theorem.

**Proposition 2.4.** *Let  $(\mathcal{E}, F)$  be a crystal over  $S = \text{Spec}(k)$ . If the first break point of  $NP(S, \mathcal{E})$  is  $(1, m)$  for  $m \in \mathbb{Z}_{\geq 0}$ , then it has a unique subcrystal  $\mathcal{E}_1 \subset \mathcal{E}$  of rank 1 and slope  $m$  such that any subcrystal of slope  $m$  is a subcrystal of  $\mathcal{E}_1$ .*

*Proof.* Let  $(M, \nabla, F)$  be the triple corresponding to the crystal  $\mathcal{E}$  by 2.2. Then the existence of the required subcrystal is equivalent to the existence of a unique  $\Lambda$ -submodule  $M_1$  of rank 1 and slope  $m$ , preserved by the action of  $\nabla$ . We will first find a submodule of rank 1 and slope  $m$ , then show it is preserved by  $\nabla$ . The uniqueness of such a submodule follows from the fact that the lowest slope is of multiplicity 1.

Choose  $\bar{k} \supset k^{pf} \supset k$ . From 2.3, we have a faithfully flat homomorphism  $\Lambda \xrightarrow{i} W(\bar{k})$ . Let  $\bar{M} = M \otimes_{\Lambda} W(\bar{k})$ . By [6, Thm. 2.6.1], there is an isogeny  $\psi : \bar{M} \rightarrow N$ , where  $\frac{1}{p^m}F_N : N \rightarrow N$  is a  $\sigma$ -linear self-map. By [6, Thm. 1.6.1],  $N$  has a unique free submodule  $N_1$  of rank 1 and slope  $m$  such that  $N/N_1$  is free as a  $W(\bar{k})$ -module. Let  $\bar{M}_1 = \psi^{-1}(N_1)$ . It is clear that  $\bar{M}_1$  is a module of rank 1 and slope  $m$ , and  $\bar{M}_2 = \bar{M}/\bar{M}_1$  is a free  $\Lambda$ -module of rank  $r - 1$  and slopes  $> m$ .

Since  $\psi$  is an isogeny, there exists some  $D \in \mathbb{Z}_{>0}$  such that  $p^D\psi^{-1}(N) \subset \bar{M}$ . As for every  $\nu > 0$ ,  $p^D\left(\frac{F}{p^m}\right)^\nu = p^D\psi^{-1}\left(\frac{F_N}{p^m}\right)^\nu\psi$ , thus  $p^D\left(\frac{F}{p^m}\right)^\nu : \bar{M} \rightarrow \bar{M}$ . Actually we can choose  $D_\nu \in [0, D]$  such that the matrix of  $f^\nu = p^{D_\nu}\left(\frac{F}{p^m}\right)^\nu \pmod p$  does not vanish. Let  $f^\nu = p^{D_\nu}\left(\frac{F}{p^m}\right)^\nu : M \rightarrow M$ , then  $f^\nu \pmod p$  does not vanish either. Since the Newton slopes of  $\bar{M}_2$  are greater than  $m$ , according to [6, 1.4.3], for each  $n > 0$  there exists  $c_n > 0$  such that  $f^\nu(\bar{M}_2) \subset p^n\bar{M}_2$  for all  $\nu \geq c_n$ . Let  $f_n^\nu : \bar{M}/p^n\bar{M} \rightarrow \bar{M}/p^n\bar{M}$ . Then  $\text{Im}(f_n^\nu) \subset \bar{M}_1/p^n\bar{M}_1$ . Let  $\bar{E}_n^\nu = \langle \text{Im}(f_n^\nu) \rangle$ . Note that  $\langle G \rangle$  is denoted as the smallest  $R/p^nR$ -submodule of  $M$  containing  $G$ , where  $R$  is a discrete valuation ring with  $p$  as its uniformizer,  $M$  is a finite free  $R/p^nR$ -module and  $G \subset M$  a subset.

Let  $f_n^\nu : M/p^nM \rightarrow M/p^nM$ ,  $E_n^\nu = \langle \text{Im}(f_n^\nu) \rangle$ , and  $E_n = \bigcap_{\nu \geq c_n} E_n^\nu$ . We get  $\bar{E}_n^\nu = E_n^\nu \otimes_{\Lambda} W(\bar{k})$ , and  $\bar{E}_n = E_n \otimes_{\Lambda} W(\bar{k}) = \bigcap_{\nu \geq c_n} \bar{E}_n^\nu$ . By the above argument, when  $\nu \geq c_n$ ,  $\bar{E}_n^\nu \simeq \bar{M}_1/p^n\bar{M}_1$ , a free  $W(\bar{k})/p^nW(\bar{k})$ -module of rank 1.

As  $\Lambda \xrightarrow{i} W(\bar{k})$  is faithfully flat and  $\bar{E}_n^\nu = E_n^\nu \otimes_{\Lambda} W(\bar{k})$  is a free  $W(\bar{k})/p^nW(\bar{k})$ -module of rank 1 for  $\nu \geq c_n$ , then  $E_n^\nu$  is a free  $\Lambda/p^n\Lambda$ -module of rank 1, hence so is  $E_n$ . Also the surjectivity of  $\bar{E}_{n+1} \rightarrow \bar{E}_n$  implies the surjectivity

of  $E_{n+1} \rightarrow E_n$ . Let  $M_1 = \varprojlim_{n \geq 0} E_n$ , it is easy to see that  $M_1$  is a free  $\Lambda$ -module. Since  $M_1 \otimes_{\Lambda} W(\bar{k}) = \overline{M}_1$  has slope  $m$ , so does  $M_1$ . It is clear that any subcrystal of slope  $m$  is contained in  $M_1$ .

Now we show  $\nabla(M_1) \subset M_1 \otimes_{\Lambda} \Omega_{\Lambda}$ . Here  $\Omega_{\Lambda} = \varprojlim_n \Omega_{(\Lambda/p^n \Lambda)/\mathbb{Z}}^1$  is the  $p$ -adic module of differentials. Let  $\{e_1, \dots, e_r\}$  be a basis of  $M$  and  $e_1 \in M_1$ . Suppose  $\nabla(e_1) = \sum_{i=1}^r e_i \otimes \eta^i$ . We need to show  $\eta^i = 0$  for  $i > 1$ . As  $F^{\nu}$  is a horizontal  $\sigma^{\nu}$ -linear self map for  $\nu > 0$ , it exchanges with  $\nabla$  in the following sense:  $\widetilde{F}^{\nu} \circ \nabla = \nabla \circ F^{\nu}$ , where  $\widetilde{F}^{\nu} = F^{\nu} \otimes \widetilde{\sigma}^{\nu}$  is the endomorphism of  $M \otimes_{\Lambda} \Omega_{\Lambda}$  and  $\widetilde{\sigma}^{\nu}$  is the map  $\Omega_{\Lambda} \rightarrow \Omega_{\Lambda}$  given by  $\alpha d\beta \mapsto \sigma^{\nu}(\alpha) d\sigma^{\nu}(\beta)$ . Then from  $\widetilde{F}^{\nu} \circ \nabla(e_1) = \nabla \circ F^{\nu}(e_1)$  we deduce that

$$p^{m\nu} \mu_{\nu} \sum_{i>1} e_i \otimes \eta^i = \sum_{i>1} F^{\nu}(e_i) \otimes \widetilde{\sigma}^{\nu}(\eta^i) \text{ mod } M_1 \otimes \Omega_{\Lambda},$$

where  $\mu_{\nu} \in \Lambda^*$ . By [6, 1.4.3],  $F^{\nu}(M/M_1) \subset p^{m\nu+1}(M/M_1)$  for  $\nu \gg 0$ . By comparing terms before  $e_i$  in the above equation, we get  $\eta^i \in p\Omega_{\Lambda}$ . Replace  $\eta^i$  by  $p\widetilde{\eta}^i$  on the right side, then we get  $\eta^i \in p^2\Omega_{\Lambda}$ . By repeating,  $\eta^i \in p^n\Omega_{\Lambda}$  for every  $n$ . By [2, Prop. 1.3.1],  $\Omega_{\Lambda}$  is a free  $\Lambda$ -module. Then  $\eta^i = 0$  for  $i > 1$ . Hence  $M_1$  is preserved by  $\nabla$ . □

**Remark 2.5.** The proposition can be generalized in the following way: Let  $(\mathcal{E}, F)$  be a crystal over  $S = \text{Spec}(k)$ . If the first break point of  $NP(S, \mathcal{E})$  is  $(\mu_1, \mu_1 \lambda_1)$ , where  $\lambda_1$  is the lowest Newton slope and  $\mu_1$  is its multiplicity, then there is a unique sub-crystal  $\mathcal{E}' \subset \mathcal{E}$  of rank  $\mu_1$  with its Newton slopes all equal to  $\lambda_1$ .

Applying the lemma to  $(\wedge^{\mu_1} \mathcal{E}, \wedge^{\mu_1} F)$ , we obtain a subcrystal  $\mathcal{E}_1$  of  $\wedge^{\mu_1} \mathcal{E}$ . To see that  $\mathcal{E}_1$  is of the form  $\wedge^{\mu_1} \mathcal{E}'$  for some subcrystal  $\mathcal{E}' \subset \mathcal{E}$ , we need to use the Plücker coordinate and check if  $\mathcal{E}_1$  satisfies the Plücker equations. By extending the scalars to the fraction field  $K$  of  $W(\bar{k})[X]/(X^N - p)$  for some proper  $N$ , we obtain that  $\mathcal{E} \otimes K$  admits a  $K$ -basis over which the matrix of  $F$  is diagonalized, hence the unique subcrystal  $\mathcal{E}_1 \otimes K$  of rank 1 and slope  $m$  satisfies the Plücker equations, and so does  $\mathcal{E}_1$ .

### 3. FACTS ABOUT FUNDAMENTAL GROUPS

3.1. Let  $X$  be a noetherian normal integral scheme with its generic point  $\eta$ . Let  $\bar{\eta}$  be a geometric point over  $\eta$ . By [5, Exposé V, Prop. 8.2], the canonical map  $\phi : \text{Gal}(\bar{K}/K) \rightarrow \pi_1(X, \bar{\eta})$  is surjective, and the kernel is  $\text{Gal}(\bar{K}/M)$ , where  $\bar{K}$  is some algebraic closure of the fraction field  $K$  of  $X$  and  $M$  is the union of all finite subextensions  $K \subset L \subset \bar{K}$  such that  $L$  is unramified over  $X$ , which means that the normalization of  $X$  in  $L$  is unramified over  $X$ .

This section focuses on describing the kernel of  $\phi$  in terms of local kernels. Assume that the completion  $\widehat{\mathcal{O}}_{X,x}$  of the local ring  $\mathcal{O}_{X,x}$  at every point  $x$  is an integral domain. Denote the fraction field and residue field of  $\widehat{\mathcal{O}}_{X,x}$  by  $K_x$  or  $k(x)$  respectively. Let  $\bar{K}_x$  be an algebraic closure of  $K_x$  and  $\bar{\eta}_x$  be the geometric

point defined by  $\text{Spec } \overline{K}_x \rightarrow \text{Spec } \widehat{\mathcal{O}}_{X,x}$ . Fix some injection  $\omega : \overline{K} \rightarrow \overline{K}_x$  such that we have the following commutative diagram:

$$\begin{array}{ccc} K & \longrightarrow & \overline{K} \\ \downarrow & & \downarrow \omega \\ K_x & \longrightarrow & \overline{K}_x \end{array}$$

Thus we have maps of Galois groups depending on  $\omega$ :

$$\psi_x : \text{Gal}(\overline{K}_x/K_x) \rightarrow \text{Gal}(\overline{K}/K), \quad \phi_x : \text{Gal}(\overline{K}_x/K_x) \rightarrow \pi_1(\text{Spec } \widehat{\mathcal{O}}_{X,x}, \overline{\eta}_x)$$

**Proposition 3.2.** *Let  $X$  be a noetherian normal integral scheme with  $K$  as its function field. Let  $\phi$ ,  $\phi_x$  and  $\psi_x$  be the same as above. If assuming that the completion  $\widehat{\mathcal{O}}_{X,x}$  of the local ring  $\mathcal{O}_{X,x}$  at each closed point  $x \in X$  is a normal domain and that the same condition holds for the normalization of  $X$  in every finite separable extension  $L/K$ , then  $\text{Ker } \phi = H$ , where  $H$  is the normal closed subgroup of  $\text{Gal}(\overline{K}/K)$  generated by  $\{\psi_x(\text{Ker } \phi_x) \mid x \text{ is a closed point of } X\}$ .*

Note that if moreover  $X$  is an excellent scheme, the conditions on the local ring  $\mathcal{O}_{X,x}$  at every closed point  $x \in X$  are satisfied. In the following, let  $x \in X$  be a closed point and  $L/K$  be a finite separable subextension in  $\overline{K}$  if no other description is given.

3.3. Let  $\widetilde{X}$  be the normalization of  $X$  and  $\widetilde{\mathcal{O}}_{X,x}$  be the integral closure of  $\mathcal{O}_{X,x}$  in  $L$ . By [13, Chapter I, Prop. 8],  $\widetilde{X} \rightarrow X$  is a finite morphism, and  $\widetilde{\mathcal{O}}_{X,x}$  is a finitely generated  $\mathcal{O}_{X,x}$ -module. Let  $\{x_i \in \widetilde{X}, i \in I\}$  be the set of points over  $x$ . Since  $\widetilde{\mathcal{O}}_{X,x}$  is a semilocal ring and a finite  $\mathcal{O}_{X,x}$ -module, then by [9, Chapter I, Thm. 4.2],  $\widetilde{\mathcal{O}}_{X,x} \otimes_{\mathcal{O}_{X,x}} \widehat{\mathcal{O}}_{X,x} = \prod_{i \in I} \widehat{\mathcal{O}}_{\widetilde{X},x_i}$ , where  $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$  is the completion of the local ring of  $x_i \in \widetilde{X}$ , and  $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$  is a finite  $\widehat{\mathcal{O}}_{X,x}$ -algebra. Thus we have the following cartesian diagram:

$$\begin{array}{ccccc} \text{Spec } \widehat{\mathcal{O}}_{X,x} & \longleftarrow & \text{Spec } \prod_{i \in I} \widehat{\mathcal{O}}_{\widetilde{X},x_i} & \longleftarrow & \text{Spec } \widehat{\mathcal{O}}_{\widetilde{X},x_i} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Spec } \mathcal{O}_{X,x} & \longleftarrow & \text{Spec } \widetilde{\mathcal{O}}_{X,x} & \longleftarrow & \text{Spec } \mathcal{O}_{\widetilde{X},x_i} \end{array}$$

3.4. By [9, Chapter I, Prop. 3.5],  $L$  is unramified over  $X$  if and only if  $\Omega^1_{\widetilde{X}/X} = 0$ . As the branch locus where  $\Omega^1_{\widetilde{X}/X} \neq 0$  is a closed subset, and  $\Omega^1$  behaves well with respect to base change, then  $L$  is unramified over  $X$  if and only if

$$\Omega^1_{\text{Spec } \mathcal{O}_{\widetilde{X},x_i} / \text{Spec } \mathcal{O}_{X,x}} = 0, \text{ for every closed point } x_i \in \widetilde{X} \text{ over } x \in X.$$

As  $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$  is a faithfully flat  $\mathcal{O}_{\widetilde{X},x}$ -module, this is also equivalent to

$$(1) \quad \Omega^1_{\text{Spec } \widehat{\mathcal{O}}_{\widetilde{X},x_i} / \text{Spec } \widehat{\mathcal{O}}_{X,x}} = 0 \text{ for every closed point } x_i \in \widetilde{X} \text{ over } x \in X.$$

3.5. Let  $L.K_x = \omega(L)K_x$ . It is clear that  $L.K_x$  is a separable extension of  $K_x$ . Actually  $L.K_x$  is the fraction field of  $\widehat{\mathcal{O}}_{\widetilde{X},x_i}$  for some  $i$ . Since by base change the diagram in 3.3 from  $\text{Spec } \widehat{\mathcal{O}}_{X,x} \rightarrow \text{Spec } \mathcal{O}_{X,x}$  to  $\text{Spec } K_x \rightarrow \text{Spec } K$ , we have  $L \otimes_K K_x \simeq \prod_{i \in I} \text{Frac}(\widehat{\mathcal{O}}_{\widetilde{X},x_i})$ ; by choosing  $\omega$ , one has to choose  $L \rightarrow \text{Frac}(\widehat{\mathcal{O}}_{\widetilde{X},x_i})$ . Hence  $L/K$  is fixed by  $H$  if and only if for every closed point  $x \in X$ , there exists some  $x_i$  over  $x$  such that  $\text{Frac}(\widehat{\mathcal{O}}_{\widetilde{X},x_i})$  is unramified over  $\text{Spec } \widehat{\mathcal{O}}_{X,x}$ ; by assumption this is equivalent to  $\text{Spec } \widehat{\mathcal{O}}_{\widetilde{X},x_i} \rightarrow \text{Spec } \widehat{\mathcal{O}}_{X,x}$  being unramified. Hence  $L/K$  is fixed by  $H$  if and only if

$$(2) \quad \Omega^1_{\text{Spec } \widehat{\mathcal{O}}_{\widetilde{X},x_i} / \text{Spec } \widehat{\mathcal{O}}_{X,x}} = 0$$

for every closed point  $x \in X$ , some  $x_i \in \widetilde{X}$  over  $x$ .

3.6. Assume that  $L/K$  is a finite Galois extension. Let  $U \subset X$  be an affine neighborhood of  $x$  and  $\widetilde{U}$  be its normalization in  $L$ . By [8, Chapter 2, 5.E], for two given points  $x_i, x_j \in \widetilde{U}$  over  $x \in U$ , there exists a  $U$ -automorphism of  $\widetilde{U}$  mapping  $x_i$  to  $x_j$ , hence  $\Omega^1_{\text{Spec } \mathcal{O}_{\widetilde{U},x_i} / \text{Spec } \mathcal{O}_{U,x}} \simeq \Omega^1_{\text{Spec } \mathcal{O}_{\widetilde{U},x_j} / \text{Spec } \mathcal{O}_{U,x}}$ . It follows that  $\widetilde{X} \rightarrow X$  is unramified at some point over  $x$  if and only if it is unramified at every point over  $x$ .

*Proof of Proposition 3.2:* Let  $N$  be the subfield of  $\overline{K}$  fixed by  $H$ . Since both  $H$  and  $\text{Ker } \psi$  are normal subgroups, then it suffices to show that  $N = M$ . Assume  $L/K$  is a finite Galois subextension. From discussions in 3.6, the conditions (1) and (2) are equivalent. Then we have  $L \subset M \Leftrightarrow L \subset N$ , hence  $M = N$ .  $\square$

We will also need the following facts about Galois groups.

**Claim 3.7.** Let  $(K, \nu)$  be a henselian field with a nonarchimedean valuation  $\nu$ . Let  $(K_\nu, \nu)$  be its completion. Denote by  $\overline{K}$  (resp.  $\overline{K}_\nu$ ) the algebraic closure of  $K$  (resp.  $K_\nu$ ). Then the homomorphism  $\text{Gal}(\overline{K}_\nu / K_\nu) \rightarrow \text{Gal}(\overline{K} / K)$  is surjective.

**Fact 3.8.** Let  $l/k$  be an algebraic extension. Let  $K = k((t))$ ,  $\widehat{L} = l((t))$ , and  $L = \bigcup m((t))$  where  $m$  runs over all finite subextensions of  $k$  in  $l$ . There is an obvious valuation  $\nu$  on  $L$  and  $\widehat{L}$  by sending  $t^n$  to  $n$ . It is clear that  $\widehat{L}$  is the completion of  $L$ . As the valuation ring of  $L$  is  $R = \bigcup m[[t]]$ , from Definition (6.1) in [10, Chapter II],  $L$  is a henselian field. Thus  $\text{Gal}_{\widehat{L}} \rightarrow \text{Gal}_L$  is surjective.

**Corollary 3.9.** Let  $R$  be a discrete valuation ring of characteristic  $p$  with fraction field  $K$  and residue field  $k$ . Let  $s \in \text{Spec } R$  be the closed point and  $\overline{\eta}$  be a geometric point over the generic point. Let  $R_s$  be the completion of  $R$ , which is of the form  $k[[t]]$ . Then the kernel of the canonical homomorphism  $\text{Gal}_K \xrightarrow{\phi} \pi_1(\text{Spec } R, \overline{\eta})$  is the normal subgroup of  $\text{Gal}_K$  generated by the image of the composition  $\text{Gal}_{\overline{k}((t))} \rightarrow \text{Gal}_{k((t))} \xrightarrow{\psi_s} \text{Gal}_K$ .

*Proof.* Since  $\text{Spec } R$  satisfies the assumption in Proposition 3.2,  $\text{Ker } \phi$  is generated by  $\psi_s(\text{Ker } \phi_s)$ . Apply Fact 3.8 to the case when  $l = k^{sep}$ . Note that  $L$  is the maximal unramified algebraic extension of  $X = \text{Spec } k[[t]]$  in the sense of 3.1, and hence  $\text{Ker } \phi_s$  in 3.1 is the normal subgroup generated by the image of  $\text{Gal}_{k^{sep}((t))} \rightarrow \text{Gal}_L \rightarrow \text{Gal}_{k((t))}$ . Apply Fact 3.8 to the field extension  $\bar{k}/k^{sep}$  to see that  $\text{Gal}_{\bar{k}((t))} \rightarrow \text{Gal}_{k^{sep}((t))}$  is surjective. In conclusion,  $\text{Ker } \phi_s$  is the image of  $\text{Gal}_{\bar{k}((t))} \rightarrow \text{Gal}_{k((t))}$ .  $\square$

#### 4. GALOIS REPRESENTATIONS ASSOCIATED TO $F$ -CRYSTALS OF RANK 1

4.1. Consider an  $F$ -crystal  $\mathcal{E}$  of rank 1 and slope  $m$  over  $k$ . Let  $(M, \nabla, F)$  over  $\Lambda$  be the triple defining the crystal  $\mathcal{E}$ . If  $\{e\}$  is chosen to be the basis of  $M$ , then  $F(e) = p^m \mu e$ , where  $\mu$  is a unit in  $\Lambda \subset W(k^{pf})$ . By 2.3, there exists some unit  $\alpha \in W(\bar{k})$  such that  $F(e \otimes \alpha) = p^m e \otimes \alpha$ , i.e.  $\sigma(\alpha)\mu = \alpha$ . As every  $g \in \text{Gal}(\bar{k}/k) = \text{Gal}(\bar{k}/k^{pf})$  can be uniquely lifted as a  $W(k^{pf})$ -automorphism of  $W(\bar{k})$ , it is easy to show that  $g(\alpha)\alpha^{-1} \in \mathbb{Z}_p^*$ . Thus we get a continuous homomorphism  $\rho : \text{Gal}(\bar{k}/k) \rightarrow \mathbb{Z}_p^*$  by sending  $g$  to  $g(\alpha)\alpha^{-1}$ .

**Definition 4.2.** Let  $\mathcal{E}$  be an  $F$ -crystal over a noetherian integral scheme  $X$  of characteristic  $p$ . Let  $K$  be the fraction field of  $X$  and  $\eta$  be the generic point. Assume that the first break point of  $NP(X, \mathcal{E})_\eta$  is  $(1, m)$  for  $m \in \mathbb{Z}_{\geq 0}$  and let  $\mathcal{E}_1 \subset \mathcal{E}_\eta$  be the unique subcrystal of rank 1 and slope  $m$  same as in Proposition 2.4. By the above discussion we obtain from the crystal  $\mathcal{E}_1$  a continuous homomorphism  $\rho : \text{Gal}(\bar{K}/K) \rightarrow \mathbb{Z}_p^*$ . We call it *the Galois representation associated to  $\mathcal{E}$* , or *the associated representation of  $\mathcal{E}$* .

4.3. Let  $X$  and  $Y$  be noetherian integral schemes. Let  $f : X \rightarrow Y$  be a morphism mapping the generic point of  $X$  to the generic point of  $Y$ . Assume that  $\mathcal{E}$  is a crystal over  $Y$  satisfying the assumption of the definition. Then the representation associated to  $\mathcal{E}|_X$  is the composition  $\text{Gal}_{K(X)} \rightarrow \text{Gal}_{K(Y)} \rightarrow \mathbb{Z}_p^*$ .

**Lemma 4.4.** *Let  $(\mathcal{E}, F_1)$  and  $(\mathcal{E}', F_2)$  be two  $F$ -crystals over a noetherian integral scheme  $X$  satisfying the assumptions in Definition 4.2. If there exists an isogeny  $\psi : \mathcal{E} \rightarrow \mathcal{E}'$ , then their associated representations are identical.*

*Proof.* Let  $\mathcal{E}_1$  (resp.  $\mathcal{E}'_1$ ) be the subcrystal of  $(\mathcal{E})_\eta$  (resp.  $(\mathcal{E}')_\eta$ ) obtained in Proposition 2.4. As  $\psi \circ F_1 = F_2 \circ \psi$ , then  $\psi(\mathcal{E}_1) \subset \mathcal{E}'_1$ . Actually we can choose a basis  $e_i$  of  $\mathcal{E}_1$  (resp.  $\mathcal{E}'_1$ ) so that  $\psi(e_1) = p^n e_2$  for some  $n \in \mathbb{N}$ , and  $F_i e_i = p^m \mu e_i$  for some unit  $\mu \in \Lambda$ . Then it is obvious that the Galois representations associated to  $\mathcal{E}$  and  $\mathcal{E}'$  are identical.  $\square$

**Lemma 4.5.** *Let  $\mathcal{E}$  be an  $F$ -crystal of rank 1 and slope  $m \in \mathbb{N}$  over  $S = \text{Spec } k$ , where  $k$  is a field of characteristic  $p$ . If the associated representation is trivial, then  $\mathcal{E}$  is a trivial crystal, i.e. there exists some basis  $\{e\}$  of  $\mathcal{E}$  such that  $F(e) = p^m e$ , and  $\nabla(e) = 0$ .*

*Proof.* Let  $e$  be a basis of  $\mathcal{E}$ , and  $F(e) = p^m \mu e$ . From 4.1, there is some unit  $\alpha \in W(\bar{k})$  such that  $\sigma(\alpha)\mu = \alpha$ ; the associated representation is trivial if and



only if the unit  $\alpha \in W(k^{pf})$ . It suffices to show that  $\alpha \in \Lambda$ , and  $\nabla(e) = 0$  follows automatically.

Let  $U^n(k) = 1 + p^n\Lambda$  and  $U^n(k^{pf}) = 1 + p^nW(k^{pf})$ . First choose  $\mu \in U^1(k)$  and  $\alpha \in U^1(k^{pf})$ . Considering  $\sigma(\alpha)\mu = \alpha \pmod{p}$ , we have  $\overline{\alpha^p\mu} = \overline{\alpha}$ , where  $\overline{\alpha} = (\alpha \pmod{p}) \in k^{pf}$ . Moreover, the equation implies that  $\overline{\alpha}$  is separable over  $k$ , and hence  $\overline{\alpha} \in k$ . Choose  $\gamma_0 \in \Lambda$  such that  $\gamma_0 \pmod{p} = \overline{\alpha}$ . Replace the basis  $e$  by  $\gamma_0 e$ , then replace  $\mu$  by  $\sigma(\gamma_0)\mu\gamma_0^{-1}$  and  $\alpha$  by  $\alpha \cdot \gamma_0^{-1}$ . Then  $\sigma(\alpha)\mu = \alpha$  still holds, and  $\mu \in U^1(k)$ ,  $\alpha \in U^1(k^{pf})$ .

The induction step: Assume  $\mu_{n-1} \in U^n(k)$ ,  $\alpha_{n-1} \in U^n(k^{pf})$ , and  $\alpha_{n-1} = \sigma(\alpha_{n-1})\mu_{n-1}$ . It suffices to show that there exists some  $\gamma_n \in U^n(k)$  such that  $\gamma_n = \alpha_{n-1} \pmod{p^{n+1}}$ . Write  $\mu_{n-1} = 1 + p^n\nu_n$ ,  $\alpha_{n-1} = 1 + p^n\delta_n$  for some  $\nu_n \in \Lambda$ ,  $\delta_n \in W(k^{pf})$ . By assumption we have

$$\sigma(\delta_n) + \nu_n = \delta_n \pmod{p} \text{ or } \overline{\delta_n^p} + \overline{\nu_n} = \overline{\delta_n}.$$

As  $\overline{\delta_n} \in k^{pf}$  and the above equation implies that it is separable over  $k$ , then  $\overline{\delta_n} \in k$ . Hence we can choose  $\gamma_n \in U^n(k)$  such that  $\gamma_n = \alpha_{n-1} \pmod{p^{n+1}}$ .

Let  $\mu_n = \sigma(\gamma_n)\mu_{n-1}\gamma_n^{-1}$  and  $\alpha_n = \alpha_{n-1} \cdot \gamma_n^{-1}$ . We can easily see that they satisfy the induction assumptions. Thus we can get a sequence  $\{\gamma_n \in U^n(k) \mid n \geq 1\}$ . As  $\Lambda$  is complete,  $\prod_n \gamma_n$  converges to some  $\beta \in \Lambda$ . It is not hard to see that  $\alpha \cdot \beta^{-1} = 1$ , and thus  $\alpha \in \Lambda$ . □

**Proposition 4.6.** *Let  $R$  be a discrete valuation ring of characteristic  $p$  with fraction field  $K$  and residue field  $k$ . Let  $\mathcal{E}$  be an  $F$ -crystal over  $\text{Spec } R$ . Let  $\eta$  and  $s$  be the generic and closed point of  $\text{Spec } R$ . Assume that the first break point of  $NP(\mathcal{E})_\eta$  is  $(1, m)$ . Then the following two conditions are equivalent:*

- (a) *The Galois representation associated to  $\mathcal{E}$  is unramified, i.e., it factors through  $\phi : \text{Gal}_K \rightarrow \pi_1(\text{Spec } R)$ .*
- (b) *The first break point of  $NP(\mathcal{E})_s$  is  $(1, m)$ .*

*Proof.* First consider  $\text{Spec } \overline{k}[[t]] \rightarrow \text{Spec } R$ . By Corollary 3.9 and 4.3, Condition (a) is equivalent to the triviality of the associated representation of  $\mathcal{E}|_{\text{Spec } \overline{k}[[t]]}$ . Moreover, as the Newton polygons of  $E$  are preserved after pulled back to  $\text{Spec } \overline{k}[[t]]$ , Condition (b) holds if and only if the first break point of  $NP(\mathcal{E}|_{\text{Spec } \overline{k}[[t]]})_s$  is  $(1, m)$ . Hence it suffices to prove the proposition for  $R = k[[t]]$  with  $k$  algebraically closed. Note that in this case (a) is equivalent to the following: (a)' The Galois representation associated to  $\mathcal{E}$  is trivial.

Condition (b) $\Rightarrow$ (a)': By [6, Cor. 2.6.2],  $\mathcal{E}$  is isogenous to an  $F$ -crystal  $\mathcal{E}'$  which is divisible by  $p^m$ , which contains a subcrystal  $\mathcal{E}'_1$  of rank 1 and slope  $m$ . By Lemma 4.4, the Galois representation in question is the same as the one associated to  $\mathcal{E}'_1|_{\text{Spec } K}$ . By [6, Thm. 2.7.4],  $\mathcal{E}'_1$  becomes isogenous to a constant  $F$ -crystal over  $k((t))^{pf}$ , and therefore the associated representation is trivial.

(a)' $\Rightarrow$ (b): By Lemma 4.5,  $\mathcal{E}_{\text{Spec } K}$  has a trivial subcrystal of rank 1 and slope  $m$ . Then we get an injection  $\Phi : \mathcal{L}_{\text{Spec } K} \rightarrow \mathcal{E}_{\text{Spec } K}$ , where  $\mathcal{L}$  is a trivial  $F$ -crystal of rank 1 and slope  $m$  over  $\text{Spec } R$ . Apply [3, Main Theorem] to  $\mathcal{E}, \mathcal{L}$  and  $\Phi$ . We obtain a nontrivial map  $\mathcal{L} \rightarrow \mathcal{E}$ . Restricting to  $s$ , we see

that  $\mathcal{E}_s$  contains a subcrystal of rank 1 and slope  $m$ . On the other hand, by Grothendieck's specialization theorem [6, 2.3.1],  $NP(\mathcal{E})_s$  lies on or above  $NP(\mathcal{E})_\eta$ . Hence  $(1, m)$  is the first break point of  $NP(\mathcal{E})_s$ .  $\square$

## 5. THE PROOF

Let  $P = (i, m)$  be the common break point. Assume that  $\text{codim}(U, \{s\}^-) > 1$ , now we need to show that  $P$  is also a break point of  $NP(\mathcal{E})_s$ .

Step 1: Reduce to the special case when the common break point  $P$  is of the form  $(1, m)$ .

In the general case, let  $\mathcal{E}' = \wedge^i \mathcal{E}$ . By assumption,  $(1, m)$  is the first break point of  $NP(\mathcal{E}')_x$  for all  $x \in U \setminus \{s\}^-$ . Applying the result for the special case, we obtain that  $(1, m)$  is a break point of  $NP(\mathcal{E}')_s$ , and hence  $P$  is a break point of  $NP(\mathcal{E})_s$ .

Step 2: First as  $S$  is locally noetherian, we may shrink  $S$  to an open affine neighborhood  $\text{Spec } A$  of  $s$  such that  $(\text{Spec } A \setminus \{s\}^-) \subset U$ . Then we follow the same reduction steps as in the proof of [4, Thm. 4.1]. We obtain that there exists a noetherian complete local normal domain  $A$  of dimension 2 with algebraically closed residue field  $k$  and a morphism  $\phi : \text{Spec } A \rightarrow S$  that maps closed point to  $s$  and other points into  $U$ . Hence it suffices to prove the statement when  $S$  is the spectrum of a noetherian complete local normal domain  $A$  of dimension 2 with algebraically closed residue field  $k$ ,  $s$  is the closed point and  $U = S \setminus \{s\}$ .

Up to now, we have shown that it suffices to prove the following simplified statement: *Let  $A$  be a noetherian complete local normal integral domain of dimension 2 with algebraically closed residue field  $k$ . Let  $s \in S = \text{Spec } A$  be the closed point and  $U = S \setminus \{s\}$ . If  $(1, m)$  is the first break point of  $NP(\mathcal{E})_x$  for every  $x \in U$ , then  $(1, m)$  is the first break point of  $NP(\mathcal{E})_s$ .*

Let  $K$  be the fraction field of  $A$ . Consider the Galois representation  $\rho : \text{Gal}(\overline{K}/K) \rightarrow \mathbb{Z}_p^*$  defined in 4.2. Let  $H$  be the kernel of the composition of  $\rho$  and  $\mathbb{Z}_p^* \xrightarrow{\text{mod } p} \mathbb{F}_p^*$ . Let  $L$  be the subfield of  $\overline{K}$  fixed by  $H$ . As  $\text{Gal}(\overline{K}/K)/H$  is a finite set,  $L$  is a finite Galois extension of  $K$ . Let  $\tilde{A}$  be the integral closure of  $A$  in  $L$ . By a standard argument, we see that  $\tilde{A}$  is a noetherian complete local normal domain of dimension 2 with residue field  $k$ . Consider the finite morphism  $\text{Spec } \tilde{A} \rightarrow \text{Spec } A$ . It is not hard to see that we only need to prove the statement for  $\tilde{A}$ . Replacing  $A$  by  $\tilde{A}$ , we may assume that  $\text{Im}(\rho) \subset 1 + p\mathbb{Z}_p$  if  $p > 2$  or  $\text{Im}(\rho) \subset 1 + p^2\mathbb{Z}_p$  if  $p = 2$ , such that the homomorphism  $\text{Gal}(\overline{K}/K) \xrightarrow{\log \circ \rho} \mathbb{Z}_p$  is valid.

Let  $x \in U$ . Assume that  $\phi$ ,  $\phi_x$  and  $\psi_x$  are the same as in 3.2. Since  $\mathcal{E}|_{\text{Spec } \hat{\mathcal{O}}_{U,x}}$  satisfies the assumptions and Condition (b) in Proposition 4.6, the associated representation of  $\mathcal{E}|_{\text{Spec } \hat{\mathcal{O}}_{U,x}}$ , which is the composition of  $\psi_x$  and  $\rho$ , factors through  $\phi_x$ . It follows that  $\psi_x(\text{Ker } \phi_x) \subset \text{Ker } \rho$ . By Proposition 4.6, we obtain that  $\rho$  factors through  $\phi$ . Thus we obtain a map  $\iota : \pi_1(U, \bar{\eta}) \rightarrow \mathbb{Z}_p^* \rightarrow \mathbb{Z}_p$ .

Take a resolution of singularities  $\tilde{S} \rightarrow S$ ; if  $A$  happens to be regular, let  $\tilde{S}$  be the blowup of the special point of  $\text{Spec } A$ . Then the main result of [4, Sec. 3] implies that  $\iota$  can be extended to  $\tilde{\iota} : \pi_1(\tilde{S}, \bar{\eta}) \rightarrow \mathbb{Z}_p$ . Let  $\xi$  denote the generic point of a component of the exceptional fiber of  $\tilde{S} \rightarrow S$ . Now we have the following diagrams:

$$\begin{array}{ccc}
 \text{Spec } K & \longrightarrow & U \\
 \downarrow & & \downarrow \\
 \text{Spec } \mathcal{O}_{\tilde{S}, \xi} & \longrightarrow & \tilde{S}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \text{Gal}(\bar{K}/K) & \longrightarrow & \pi_1(U, \bar{\eta}) \\
 \downarrow & \searrow \text{dotted} & \downarrow \\
 \pi_1(\text{Spec } \mathcal{O}_{\tilde{S}, \xi}, \bar{\eta}) & \longrightarrow & \pi_1(\tilde{S}, \bar{\eta}) \longrightarrow \mathbb{Z}_p^*
 \end{array}$$

*Proof.* By definition the representation associated to  $\mathcal{E}_{\text{Spec } \mathcal{O}_{\tilde{S}, \xi}}$  is the dotted arrow, and it is unramified by the above commutative diagram. By Proposition 4.6 again,  $(1, m)$  is the first break point of  $NP(\tilde{S}, \mathcal{E})_\xi$ . Since  $\xi$  is mapped to  $s$ ,  $(1, m)$  is thus the first break point of  $NP(\mathcal{E})_s$ .  $\square$

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