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L^2-Invariants of Groups and Discrete Measured Groupoids

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Introduction

L^2-Betti numbers, Novikov-Shubin invariants, L^2-Torsion and L^2-signatures are often embraced by the term L^2-invariants. Their common characteristic is that they are numerically-valued topological invariants of spaces, which are are defined on the universal covering by functional-analytic methods.

In the present work only L^2-Betti numbers and Novikov-Shubin invariants will be considered. A priori, they take values in $[0, \infty]$ resp. $[0, \infty] \cup \{\infty^+\}$, where ∞^+ is a formal symbol. Both are invariants of the spectrum of the Laplacians acting on differential forms of the universal covering. They turn out to be homotopy invariants.

L^2-Betti numbers made their first appearance in 1976, when Atiyah defined them for universal coverings of compact manifolds in terms of the heat kernel [3]. Subsequently, simplicial [16] and homological definitions [43], [44] were developed. The homological definition is analogous to the definition of Betti numbers as the ranks of the singular homology modules. L^2-Betti numbers are the dimensions of modules over the group von Neumann algebra $\mathcal{N}(G)$. Typically, G is the fundamental group of a space.

Novikov-Shubin invariants were first defined in 1986 by S. P. Novikov and M. A. Shubin [48]. While L^2-Betti numbers are concerned with the spectrum **at zero**, Novikov-Shubin invariants codify information about the spectrum **near zero**. Novikov-Shubin invariants also admit an interpretation in terms of homology [46].

In this thesis we are led by the following fundamental questions.

- What are the invariance properties of L^2-invariants beyond homotopy invariance?
- What are the possible values of L^2-invariants? Are they integer- or rational-valued?

The first question is of particular interest, if one studies L^2-Betti numbers of groups, i.e. of classifying spaces of groups. Concerning geometric group theory, the first question that comes to mind is, whether L^2-Betti numbers constitute quasi-isometry invariants. Because L^2-Betti numbers behave multiplicatively with respect to finite coverings, the correct question has to be: Do L^2-Betti numbers of quasi-isometric groups coincide up to a constant factor? The answer is: No. In 3.42 on p. 64 we present a well-known counterexample.

However, P. Pansu [49] observed that the vanishing of the L^2-Betti numbers is a quasi-isometry invariant among groups admitting a finite classifying space.

An important theorem proven by D. Gaboriau [24] indicates that the L^2-Betti numbers are related to the measure-theoretic nature of the group rather than to its geometry. Before we formulate his result, we have to introduce a substitute for *Quasi-Isometry* in the measure-theoretic context.

DEFINITION. *Two groups are measure equivalent if they act (essentially) freely and measure preserving on some (standard Borel) measure space such that the actions commute and have finite measure fundamental domains.*

THEOREM (Gaboriau). *The L^2-Betti numbers of measure equivalent groups coincide up to a non-zero multiplicative factor.*

To give an idea of measure equivalence let me mention the following examples. Compare 2.31, 2.21 and 3.42.

- Lattices of finite covolume in the same locally compact, second countable, Hausdorff group are measure equivalent.
- All infinite countable amenable groups are measure equivalent.
- $F_3 * (F_3 \times F_3)$ and $F_4 * (F_3 \times F_3)$ are quasi-isometric but **not** measure equivalent. Here F_n denotes the free group of rank n.

A major part of this thesis is devoted to a new approach to Gaboriau's theorem.

Novikov-Shubin invariants of groups are **not** invariant under measure equivalence. In his fundamental essay [29, p. 241] Gromov gives some positive indications for the following conjecture.

CONJECTURE. *The Novikov-Shubin invariants of groups are invariant under quasi-isometry.*

We remark that the considerations in Gromov's essay refer to groups admitting a finite classifying space.

In 4.25 we prove that the Novikov-Shubin invariants of amenable quasi-isometric groups coincide provided a mild condition on the groups is satisfied, which is the case for groups admitting a finite classifying space. The proof is not purely geometric but also uses similar measure-theoretic tools as in the new proof of Gaboriau's theorem.

Now we turn to the second fundamental question. Atiyah asked in [3] whether L^2-Betti numbers of the universal covering of a compact manifold are always rational. Subsequently, the corresponding conjecture was named after Atiyah.

CONJECTURE (Atiyah Conjecture). *The L^2-Betti numbers of the universal covering of a compact manifold are always rational.*

Meanwhile, the Atiyah conjecture is proven in a lot of cases. For instance, it is true if the fundamental group is elementary amenable such that the orders of finite subgroups are bounded [38]. J. Lott and W. Lück formulated the analogous conjecture for the Novikov-Shubin invariants [41] on which we focus in the last chapter.

CONJECTURE. *The Novikov-Shubin invariants of the universal covering of a compact manifold are positive rational unless they are ∞ or ∞^+.*

J. Lott verified the conjecture for free abelian fundamental groups [40]. D. Voiculescu observed that his free probability theory can be used to give a proof for free fundamental groups. We use another method to prove it for free groups and relate it to another interesting, purely algebraic question about power series over the group ring.

We now describe the results of this thesis in more detail.

The first definition of L^2-Betti numbers for arbitrary countable groups is due to Cheeger and Gromov [8]. One of its drawbacks is that, in general, the L^2-Betti numbers cannot be interpreted as the dimension of a module – as one may expect having in mind the classical Betti numbers. This can be resolved using the following definition of W. Lück.

DEFINITION. *Let X be a topological space with an action of the group G. Then the n-th L^2-Betti number of X is defined as the $\mathcal{N}(G)$-dimension of the n-th homology of the chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} \mathrm{C}^{\mathrm{sing}}_\bullet(X)$, where $\mathrm{C}^{\mathrm{sing}}_\bullet(X)$ is the singular chain complex of X:*

$$b_n^{(2)}(X; \mathcal{N}(G)) = \dim_{\mathcal{N}(G)}\big(H_n(X; \mathcal{N}(G))\big).$$

The n-th L^2-Betti number of a group G is given by

$$b_n^{(2)}(G) = \dim_{\mathcal{N}(G)}\big(H_n(EG; \mathcal{N}(G))\big) \quad \Big(= \dim_{\mathcal{N}(G)}\big(\mathrm{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C})\big)\Big),$$

where EG is any universal free G-space.

Here $\dim_{\mathcal{N}(G)}$ is the dimension function for arbitrary modules over the group von Neumann algebra $\mathcal{N}(G)$, which was developed in [42], [43], [44]. This homological definition has many advantages. For instance, the apparatus of homological algebra, like spectral sequences, can be applied to compute $\mathrm{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C})$. Of course, Tor denotes the derived functor of the respective tensor product.

D. Gaboriau defines the notion of *L^2-Betti numbers of a countable standard measure preserving equivalence relation* using techniques which are motivated by the ones in the article [8] of Cheeger and Gromov. The orbit equivalence relation of a countable group acting freely and measure preserving on a probability space is an example of a countable standard measure preserving equivalence relation. Each countable group admits such an action, and Gaboriau shows that the L^2-Betti numbers of the group coincide with the L^2-Betti numbers of its associated orbit equivalence relation. This leads to a proof of the theorem mentioned above because measure equivalence can be expressed in terms of orbit equivalence relations.

Motivated by the advantages of the homological definition in the group case, we give the following very general definition in 3.32:

DEFINITION. *Let $\underline{G}$ be a discrete measured groupoid. Its n-th L^2-Betti number $b_n^{(2)}(\underline{G})$ is defined as*

$$b_n^{(2)}(\underline{G}) = \dim_{\mathcal{N}(\underline{G})}\Big(\operatorname{Tor}_n^{\mathbb{C}\underline{G}}\left(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0)\right)\Big).$$

A few explanations are in order. A *discrete measured groupoid* is a (small) groupoid together with a measurable structure and an invariant measure in the sense of [2]. See 1.11 for the precise definition. Countable groups, countable standard measure preserving equivalence relations and holonomy groupoids of foliations with an invariant measure (restricted to a transversal) provide examples. The groupoid ring $\mathbb{C}\underline{G}$ and the von Neumann algebra $\mathcal{N}(\underline{G})$ of a discrete measured groupoid $\underline{G}$ are explained in the sections 1.2 and 1.3. If $\underline{G}$ is a group, then the groupoid ring is the ordinary group ring. Furthermore, $L^\infty(\underline{G}^0)$ is defined as the algebra of essentially bounded measurable functions on the set of objects $\underline{G}^0$ of $\underline{G}$. Notice that for a group G we have $G^0 = pt$ and $L^\infty(pt) = \mathbb{C}$.

In 3.37 we prove the following statement, which will lead to a new proof of Gaboriau's theorem (see 3.38).

THEOREM. *The L^2-Betti numbers $b_n^{(2)}(\mathcal{R}(G \curvearrowright X))$ of the orbit equivalence relation of a free, measure preserving group action $G \curvearrowright X$ coincide with $b_n^{(2)}(G)$.*

In section 3.2 we develop some tools concerning the homological algebra of finite von Neumann algebras which are needed for the proof. These tools are built around handling Tor-groups of the form

$$\operatorname{Tor}_n^R(\mathcal{A}, M),$$

where $\mathcal{B} \subset \mathcal{A}$ are finite von Neumann algebras, M is an R-module, and R is an intermediate ring $\mathcal{B} \subset R \subset \mathcal{A}$, which has the following property, called *dimension-compatibility*, as an $\mathcal{B}$-$\mathcal{B}$-bimodule:

$$\dim_{\mathcal{B}}(N) = 0 \Rightarrow \dim_{\mathcal{B}}(R \otimes_{\mathcal{B}} N) = 0.$$

One of the questions we deal with is: Does the $\mathcal{A}$-dimension of the Tor-group stay the same if we replace M or R by $\dim_{\mathcal{B}}$-isomorphic modules resp. rings? Compare the theorems 3.29 and 3.31.

We already mentioned that the Novikov-Shubin invariants of groups are not invariant under measure equivalence. Although we could define the Novikov-Shubin invariants of the orbit equivalence relation of a free, measure-preserving group action (or more generally for a discrete measured groupoid) in the same way as its L^2-Betti numbers, it is **not** true that they coincide with the Novikov-Shubin invariants of the respective group.

Instead we can prove the following theorem. See 4.25.

THEOREM. *Let G and H be amenable, quasi-isometric groups. Then the Novikov-Shubin invariants of G and H coincide provided a mild technical condition is satisfied.*

For instance, this technical condition, called *induction-friendly*, is satisfied in each of the following cases (compare 4.19).

- Both groups have a finite classifying space for proper actions.

- Both groups are virtually nilpotent.
- Both groups are elementary amenable and do not contain infinite locally finite subgroups.

In 4.24 we prove that Novikov-Shubin invariants of groups are invariant under a special kind of measure equivalence, called *bounded measure equivalence*. Since we can show that a quasi-isometry between amenable groups induces a bounded measure equivalence (see 2.38), this will imply the theorem above.

Furthermore, in 4.26 another consequence of these techniques is obtained:

THEOREM. *Let G and H satisfy the same technical condition as before. Assume that G and H act isometrically on the same second countable, proper metric space X such that both actions are proper and cocompact. Then the Novikov-Shubin invariants of G and H coincide.*

In the last chapter of this thesis we deal with the problem of rationality and positivity of Novikov-Shubin invariants for free groups. In 5.30 the following result is proven.

THEOREM. *Let F be a virtually free group. The Novikov-Shubin invariants of a finite free F-CW complex are positive rational unless they are ∞^+.*

Denote by F an arbitrary virtually free group in the sequel. Novikov-Shubin invariants are concerned with the spectrum of certain operators near zero. These operators are matrices over the group ring and correspond to matrix representations of the differentials in the cellular chain complex. The information about the spectrum of a self-adjoint operator $a \in \mathcal{N}(F)$ can be extracted from the power series

$$\sum_{n=0}^{\infty} \operatorname{tr}_{\mathcal{N}(F)}(a^n) z^n \in \mathbb{C}[\![z]\!].$$

Here $\operatorname{tr}_{\mathcal{N}(F)}(a) \in \mathbb{C}$ is the von Neumann trace of a. If $a \in \mathbb{C}F$, then $\operatorname{tr}_{\mathcal{N}(F)}(a)$ is given by the coefficient of the unit element of a. We deduce the positivity and rationality from the fact that this power series is algebraic for an operator $a \in \mathbb{C}F$ over the group ring. More generally, we prove the following theorem. See 5.21.

THEOREM. *The map*

$$\operatorname{tr} : \mathbb{C}F[\![z]\!] \to \mathbb{C}[\![z]\!], \quad \sum_{n \geq 0} a_n z^n \mapsto \sum_{n \geq 0} \operatorname{tr}_{\mathcal{N}(F)}(a_n) z^n$$

maps rational series to algebraic ones.

Here a power series over $\mathbb{C}F$ is rational if it lies in the division closure of the ring inclusion $\mathbb{C}F[z] \subset \mathbb{C}F[\![z]\!]$, i.e. in the smallest division-closed subring of $\mathbb{C}F[\![z]\!]$ containing $\mathbb{C}F[z]$. A power series $P(z)$ in $\mathbb{C}[\![z]\!]$ is algebraic if there exists a polynomial $q(w, z) \in \mathbb{C}[w, z]$ of positive degree in w such that $q(P(z), z) = 0$. The theorem above leads to the following two problems, which the author finds interesting in themselves.

- Determine the image of the rational series under tr in $\mathbb{C}[\![z]\!]$ for other classes of groups.

- More generally, how does tr increase the complexity of power series for various groups?

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Notations and Conventions

All our rings are associative and have a unit. Ring homomorphisms are always unital. A 'generic' groupoid is denoted by an underlined symbol like $\underline{G}$. The source, target and inverse map in a groupoid are always denoted by the letters s, t and i, and the set of objects in $\underline{G}$ is denoted by $\underline{G}^0$ and considered as a subset of $\underline{G}$.

The space of continuous mappings between topological spaces X, Y is denoted by $C(X,Y)$. The continuous, complex-valued functions on X are denoted by $C(X)$. The symbol $L^\infty(X)$ always stands for the algebra of essentially bounded, complex-valued measurable functions on a measure space modulo almost null functions. We adopt the usual convention and speak of functions in $L^\infty(X)$ instead of equivalence classes of functions. The standard symbol for the characteristic function of a set A is χ_A.

The letters $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{N}$, $\mathcal{M}$ usually stand for finite von Neumann algebras. By a finite von Neumann algebra we understand a von Neumann algebra with a specified finite trace. This trace is usually normalized. Unital inclusions of finite von Neumann algebras are assumed to preserve the trace.

A statement involving an expression like $H_\bullet(C) = \mathrm{Tor}_\bullet^R(A,B)$ means $\forall n$: $H_n(C) = \mathrm{Tor}_n^R(A,B)$. A union $\bigcup_{i\in I} M_i$ is called directed if for every i and j in I there exists a $k \in I$ such that $M_i \cup M_j \subset M_k$. If G is a group acting on X and $F \subset G$, $X' \subset X$ are subsets, then FX' (or $F \cdot X'$) denotes the set $\{gx;\ g \in F,\ x \in X'\}$.

Chapter 1

Discrete Measured Groupoids

In this chapter our basic objects of study are introduced: Discrete measured groupoids, groupoid rings and the associated von Neumann algebra. In the first section we review and provide the necessary measure-theoretic tools. After that we define the notion of a discrete measured groupoid and mention the most important examples. The groupoid ring of a discrete measured groupoid will be explained in the second section, and some algebraic notions and facts for its study are provided. The construction of the von Neumann algebra of a discrete measured groupoid is well-known, and a detailed account for the case of countable standard equivalence relations can be found in [20], [21]. In the third section we provide the construction for discrete measured groupoids. The theory of locally compact groupoids with (quasi-)invariant measure is extensively dealt with in [2].

1.1 Definition and Examples

We review some fundamental definitions and facts from measure theory.

A set X together with a σ-algebra $\mathcal{A}$ will be a called a *measurable* space. We say that two measurable spaces $(X_i, \mathcal{A}_i)$, $i = 1, 2$ are isomorphic if there is a measurable map $f : X_1 \to X_2$ with a measurable inverse f^{-1}. A measurable space $(X, \mathcal{A})$ with a measure on $\mathcal{A}$ is called a *measure space*. A measurable map $f : (X, \mathcal{A}) \to (Y; \mathcal{B})$ from a measure space $(X, \mathcal{A})$ with measure μ to a measurable space $(Y, \mathcal{B})$ induces a measure on $(Y; \mathcal{B})$ denoted by $f_*\mu$. For a topological space X we denote the Borel algebra on X by $\mathcal{B}(X)$. A measure on the Borel algebra is called a *Borel measure*.

A *Polish space* is a separable topological space which is metrizable by a complete metric. The measurable space $(X, \mathcal{A})$ is called a *standard Borel space* if it is isomorphic to a Polish space $(Y, \mathcal{B}(Y))$ equipped with its Borel algebra. We stress that the Polish space is not part of the datum, only its existence is required. However, the elements in $\mathcal{A}$ of a standard Borel space $(X, \mathcal{A})$ are also called Borel sets, and measurable maps and isomorphisms between standard Borel spaces are called Borel maps resp. Borel isomorphisms. Further, a measure on $\mathcal{A}$ is also called a Borel measure.

Let μ be a Borel measure on a topological Hausdorff space X. Then μ is called *regular* if

(i) each compact subset $K \subset X$ satisfies $\mu(K) < \infty$,

(ii) each Borel set $A \subset X$ satisfies $\mu(A) = \inf\{\mu(U);\ A \subset U \text{ and } U \text{ is open}\}$,

(iii) each open $U \subset X$ satisfies $\mu(U) = \sup\{\mu(K);\ K \subset U \text{ and } K \text{ is compact}\}$.

Every finite Borel measure on a Polish space is regular ([10, proposition 8.1.10. on p. 258]). A locally compact topological group has always a non-zero regular measure that is invariant under left translations [10, theorem 9.2.1 on p. 305]. This measure is unique up to a scalar, and it is called the (left) *Haar measure*.

Now let X be a locally compact Hausdorff space, and μ be a regular Borel measure on X. Then the union of all the open subsets of X that have measure zero under μ is itself an open set that has measure zero [10, proposition 7.4.1 on p. 226]. The complement of this set is called the *support* of μ.

The category of standard Borel spaces is very convenient for doing measure theory, comparable to the category of compactly generated spaces in topology. The following results provide evidence for that.

THEOREM 1.1 (Facts about Polish Spaces and Standard Borel Spaces).

(i) Every locally compact second countable Hausdorff space is Polish ([35, theorem 5.3 on p. 29]).

(ii) Each closed and each open subset of a Polish space is Polish ([10, proposition 8.1.1. on p. 251]).

(iii) Finite and countable disjoint unions of Polish spaces are Polish ([10, proposition 8.1.2. on p. 253]).

(iv) Finite and countable products of Polish spaces are Polish ([10, proposition 8.1.3. on p. 253]).

(v) A measurable subset of a standard Borel space is a standard Borel space ([35, corollary 13.4 on p. 82]).

(vi) A bijective Borel map between standard Borel spaces is a Borel isomorphism ([35, theorem 14.12 on p. 88]).

THEOREM 1.2. *Let $f : X \to Y$ be a Borel map between standard Borel spaces. If $f_{|A}$ is injective for the Borel set $A \subset X$ then $f(A)$ is Borel, and $f_{|A} : A \to f(A)$ is a Borel isomorphism.*

A proof can be found in [35, corollary 15.2 on p. 89].

THEOREM 1.3. *Let $f : X \to Y$ be a Borel map between standard Borel spaces that is countable-to-1, i.e. the preimages $f^{-1}(\{y\})$, $y \in Y$, are countable. Then the image $f(X)$ is measurable in Y, and there is a countable partition of X into measurable subsets X_i, $i \in \mathbb{N}$, such that all $f_{|X_i}$ are injective, and $f_{|X_1} : X_1 \to f(X)$ is a Borel isomorphism. If f is even bounded finite-to-1, i.e. there is some $N \in \mathbb{N}$ such that $\#f^{-1}(\{y\}) \leq N$ for all $y \in Y$, then the partition can be chosen to have at most N sets.*

Proof. By [35, theorem 18.10 on p. 123] there is a sequence $(X_n)_{n\geq 0}$ of Borel sets with $\bigcup_{n\geq 0} X_n = X$ and injective $f_{|X_i}$ for $i \geq 0$. By the previous theorem $f(X) = \bigcup_{n\geq 0} f(X_n)$ is Borel. It is easy to change the X_n such that they are disjoint.

Now we will show that we can change the X_n further (without loosing the described properties) such that $f_{|X_1} : X_1 \to f(X)$ is a Borel isomorphism and, in the case $\#f^{-1}(\{y\}) \leq N$, it is $X_n = \emptyset$ for $n > N$. Put

$$X_1^{(1)} = X_1 \cup \bigcup_{n=2}^{\infty} \Big(X_n - \bigcup_{i=1}^{n-1} f^{-1}\big(f(X_i)\big) \Big) \tag{1.1}$$

$$X_n^{(1)} = X_n \cap \bigcup_{i=1}^{n-1} f^{-1}\big(f(X_i)\big) \quad \text{for } n \geq 2. \tag{1.2}$$

We obtain the properties:

(i) $f\left(X_1^{(1)}\right) = f(X)$,

(ii) $f\left(X_i^{(1)}\right) \subset f\left(X_1^{(1)}\right)$ for $i \geq 1$,

(iii) $\bigcup_{i\geq 1} X_i^{(1)} = \bigcup_{i\geq 1} X_i = X$ (disjoint unions),

(iv) $f_{|X_i^{(1)}}$ is injective for all $i \geq 1$.

Now apply the construction in (1.1), (1.2) again to the sequence $(X_n^{(1)})_{n\geq 2}$ to obtain a sequence $(X_n^{(2)})_{n\geq 2}$. Because of $X_2^{(2)} \subset \bigcup_{n\geq 2} X_n^{(1)}$ we get $X_2^{(2)} \cap X_1^{(1)} = \emptyset$. Now keep repeating this construction. For every $n \in \mathbb{N}$ we get a sequence of subsets $\left(X_{n+i}^{(n)}\right)_{i\geq 0}$ satisfying:

(i) $X_n^{(n)} \cap X_{n-1}^{(n-1)} = \emptyset$,

(ii) $f\left(X_n^{(n)}\right) = f\left(\bigcup_{j\geq 0} X_{n+j}^{(n-1)}\right)$,

(iii) $f\left(X_{n+i}^{(n)}\right) \subset f\left(X_n^{(n)}\right)$ for $i \geq 0$,

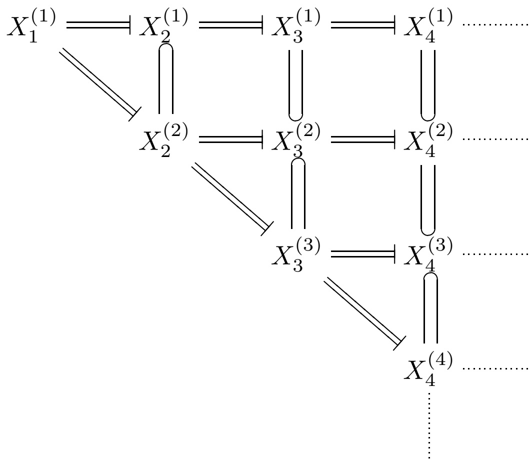
(iv) $\bigcup_{i\geq 0} X_{n+i}^{(n)} = \bigcup_{j\geq 0} X_{n+j}^{(n-1)}$ (disjoint unions),

(v) $f_{|X_{n+i}^{(n)}}$ is injective for all $i \geq 0$.

We define the following relation between subsets in X:

$$A \models B \overset{\text{def}}{\iff} f(A) \subset f(B) \text{ and } A \cap B = \emptyset.$$

Some of the relations between the sequences of subsets are illustrated in the next diagram.



We have the implication: $x \in X_j^{(i)} \Rightarrow x \in X_j^{(i+1)}$ or $x \in X_{i+1}^{(i+1)}$. Thereby every $x \in X$ must lie in some diagonal element $X_i^{(i)}$. So the diagonal sequence $(X_n^{(n)})_{n\in\mathbb{N}}$ is a Borel partition of X such that $f_{X_n^{(n)}}$ is injective for $n \geq 1$ and $f(X_1^{(1)}) = f(X)$. If $\# f^{-1}(\{y\}) \leq N$ for all $y \in Y$, then we must have $X_n^{(n)} = \emptyset$ for $n > N$. So we get a partition with at most N different sets. $\square$

DEFINITION 1.4 (Groupoid). A *groupoid* is a small category where all morphisms are invertible. We usually identify a groupoid $\underline{G}$ with the set of its morphisms. The set of objects $\underline{G}^0$ can be considered as a subset (via the identity morphisms). There are four canonical maps, namely

the *source map* $s : \underline{G} \to \underline{G}^0, (f : x \to y) \mapsto x$,

the *target map* $t : \underline{G} \to \underline{G}^0, (f : x \to y) \mapsto y$,

the *inverse map* $i : \underline{G} \to \underline{G}, f \to f^{-1}$ and

the *composition* $\circ : \underline{G}^{(2)} := \{(f, g) \in \underline{G} \times \underline{G};\ s(f) = t(g)\} \to \underline{G}, (f, g) \mapsto f \circ g$. The composition will also be denoted by $g_1 g_2$ instead of $g_1 \circ g_2$.

DEFINITION 1.5 (Discrete Measurable Groupoid).
A *discrete measurable groupoid* $\underline{G}$ is a groupoid $\underline{G}$ equipped with the structure of a standard Borel space such that the composition and the inverse map are measurable maps and $s^{-1}(\{x\})$ is countable for all $x \in \underline{G}^0$ (or, equivalently, $t^{-1}(\{x\})$ is countable).

LEMMA 1.6. *The source and target maps of a discrete measurable groupoid $\underline{G}$ are measurable, and $\underline{G}^0 \subset \underline{G}$ is a Borel subset.*

Proof. One can write the source map s as the composition of the measurable maps $\underline{G} \to \underline{G}^{(2)}, g \mapsto (i(g), g)$ and $\circ : \underline{G}^{(2)} \to \underline{G}$. For t it is analogous. Therefore they are measurable. Theorem 1.3 implies that the image $\underline{G}^0 = \operatorname{im} s \subset \underline{G}$ is measurable. $\square$

LEMMA 1.7 (Measures on Groupoids).
Let μ be a probability measure on the set of objects $\underline{G}^0$ of a discrete measurable groupoid $\underline{G}$. Then, for any measurable subset $A \subset \underline{G}$, the function

$$\underline{G}^0 \to \mathbb{C},\ x \mapsto \#\big(s^{-1}(x) \cap A\big)$$

is measurable, and the measure μ_s on $\underline{G}$ defined by

$$\mu_s(A) = \int_{\underline{G}^0} \#\big(s^{-1}(x) \cap A\big) d\mu(x)$$

is σ-finite. It will be called the left counting measure of μ. *The analogous statement holds if we replace s by t, and the corresponding measure μ_t is called the* right counting measure of μ.

Proof. By 1.3 we know that there is a Borel partition $\underline{G} = \bigcup_{n\in\mathbb{N}} X_n$ such that the maps $s_{|X_i}$ are injective, and s maps Borel sets to Borel sets. Therefore the function $x \mapsto \#\{s^{-1}(x) \cap A\}$ can be written as $\sum_{i\in\mathbb{N}} \chi_{s(X_i \cap A)}$, which is Borel. Then σ-finiteness follows from $\mu_s(X_i) \leq \mu(\underline{G}^0) < \infty$. The proof for t is analogous. ☐

REMARK 1.8. The integral $\int f d\mu_s$ of a μ_s-integrable function $f : \underline{G} \to \mathbb{C}$ is given by $\int_{\underline{G}^0} \sum_{g\in s^{-1}(x)} f(g) d\mu(x)$. This can be tested on characteristic functions.

LEMMA 1.9. *The following conditions on μ (as above) are equivalent.*

(i) $\mu_s = \mu_t$,

(ii) $i_*\mu_s = \mu_s$,

(iii) for every Borel subset $E \subset \underline{G}$ such that $s_{|E}$ and $t_{|E}$ are injective we have $\mu(s(E)) = \mu(t(E))$.

Proof. The equivalence of (i) and (ii) is clear. If $E \subset \underline{G}$ is a Borel subset such that the restrictions $s_{|E}$ and $t_{|E}$ are injective, then we have $\mu_s(E) = \mu(s(E))$ and $\mu_t(E) = \mu(t(E))$. Thus we obtain the implication (i)⇒(iii). Now suppose (iii) is true, and let $A \subset \underline{G}$ be a Borel subset. When we apply 1.3 to $s : \underline{G} \to \underline{G}^0$, we obtain a countable Borel partition X_i^s, $i \in I$ of $\underline{G}$ such that each restriction $s_{|X_i^s}$ is injective. Similarly, we get a countable Borel partition X_j^t, $j \in J$ such that each $t_{|X_j^t}$ is injective. Because we assume (iii), we know that for each $(i,j) \in I \times J$

$$\begin{aligned}\mu_s\Big(A \cap \big(X_i^s \cap X_j^t\big)\Big) = \mu\big(s\big(A \cap \big(X_i^s \cap X_j^t\big)\big)\big) &= \mu\Big(t\big(A \cap \big(X_i^s \cap X_j^t\big)\big)\Big) \\ &= \mu_t\Big(A \cap \big(X_i^s \cap X_j^t\big)\Big).\end{aligned}$$

This implies $\mu_s(A) = \mu_t(A)$. ☐

DEFINITION 1.10 (Invariant Measure).
A probability measure μ on $\underline{G}^0$ is called *invariant* if it satisfies one of the conditions (and hence all) in 1.9.

DEFINITION 1.11 (Discrete Measured Groupoid).
A discrete measurable groupoid $\underline{G}$ together with an invariant measure on $\underline{G}^0$ is called a *discrete measured groupoid*.

For the rest of this paper $\underline{G}$ will always denote a discrete measured groupoid.

DEFINITION 1.12 (Restriction of a Discrete Measured Groupoid).
Let $\underline{G}$ be a discrete measured groupoid with invariant measure μ and $A \subset \underline{G}^0$ be a Borel subset. Then $\underline{G}_{|A} = s^{-1}(A) \cap t^{-1}(A)$ with the normalized measure $\frac{1}{\mu(A)}\mu_{|A}$ is a discrete measured groupoid, called the *restriction* of $\underline{G}$ to A.

We call a functor between groupoids a *groupoid homomorphism*. A *groupoid isomorphism* is a groupoid homomorphism with an inverse. To stress it, a groupoid isomorphism is an isomorphism of categories, not only an equivalence. A measurable groupoid isomorphism between discrete measurable groupoids has a measurable inverse by 1.2. A measurable groupoid homomorphism $f : \underline{G} \to \underline{H}$ between discrete measured groupoids with invariant measures $\mu_{\underline{G}^0}$ and $\mu_{\underline{H}^0}$ on $\underline{G}^0$ resp. $\underline{H}^0$ is called measure-preserving if $f_*\mu_{\underline{G}^0} = \mu_{\underline{H}^0}$.

DEFINITION 1.13 (Isomorphism of Groupoids).
An *isomorphism of discrete measured groupoids* is a measure-preserving groupoid isomorphism. An isomorphism $f : \underline{G}_{|A} \to \underline{H}_{|B}$ between the restricted discrete measured groupoids $\underline{G}_{|A}$ and $\underline{H}_{|B}$, where $A \subset \underline{G}^0$ and $B \subset \underline{H}^0$ are Borel subsets such that $t(s^{-1}(A))$ $(= s(t^{-1}(A)))$ and $t(s^{-1}(B))$ have full measure in $\underline{G}^0$ resp. $\underline{H}^0$, is called a *weak isomorphism* between $\underline{G}$ and $\underline{H}$. A weak isomorphism $\underline{G}_{|A} \to \underline{H}_{|B}$ with $A \subset \underline{G}^0$ and $B \subset \underline{H}^0$ having full measure is called a *measure isomorphism* between $\underline{G}$ and $\underline{H}$

EXAMPLE 1.14 (Group). The simplest example of a discrete measured groupoid is given by a countable group G. Here $\underline{G}^0$ consists of a single point with measure 1.

EXAMPLE 1.15 (Orbit Equivalence Relation). Assume the countable group G acts on a standard Borel space X with probability measure μ by μ-preserving Borel automorphisms. The *orbit equivalence relation* given by

$$\mathcal{R}(G \curvearrowright X) = \{(x, gx); x \in X, g \in G\} \subset X \times X$$

is a discrete measured groupoid. First of all, every equivalence relation is a groupoid in an obvious sense. Further, $\mathcal{R}(G \curvearrowright X)$ is a standard Borel space because it is the image of a measurable countable-to-1 map between standard Borel spaces (see 1.3), namely of the multiplication map

$$G \times X \to X \times X, (g, x) \mapsto (x, gx).$$

Obviously, the inverse map given by $(x, y) \mapsto (y, x)$ and the composition given by $((x, y), (y, z)) \mapsto (x, z)$ are measurable.

Remains to show that μ is invariant in the sense of 1.9, i.e. $\mu_s(A) = \mu_t(A)$ for $A \subset \mathcal{R}(G \curvearrowright X)$. For $g \in G$ define the Borel set $X(g) = \{(x, gx);\ x \in X\} \subset \mathcal{R}(G \curvearrowright X)$. Every measurable $A \subset \mathcal{R}(G \curvearrowright X)$ can be written as the countable union $A = \bigcup_{g \in G}(A \cap X(g))$. Therefore it suffices to show $\mu_s(A) = \mu_t(A)$ for $A \subset X(g)$, which follows from $\mu_t(A) = \mu(t(A)) = \mu(g \cdot s(A)) = \mu(s(A)) = \mu_s(A)$.

Note that we did not assume that the group acts freely or at least essentially freely. However, this will be the most important case, so we call such group actions standard. Here is the precise definition:

DEFINITION 1.16 (Standard Action).
An action $G \curvearrowright X$ of a countable group G is called *standard* if X is a standard Borel space with probability measure μ, G acts by μ-preserving Borel-automorphism and the action is essentially free, i.e. the stabilizer of almost every $x \in X$ is trivial.

LEMMA 1.17. *Every countable group admits a standard action: If G is finite then the G-action on G itself equipped with the normalized counting measure is standard. If G is infinite then the shift action of G on the standard Borel space* $X = \{0,1\}^G$ *is standard. Here X is equipped with the product measure* μ *of the equiprobability on* $\{0,1\}$.

Proof. Note that $X = \{0,1\}^G$ is Polish due to 1.1, (iv), hence $(X, \mathcal{B}(X))$ is a standard Borel space. We show that the shift action on X is essentially free. Define $X^g = \{x \in X; g \cdot x = x\}$. It is enough to show $\mu(X^{g_0}) = 0$ for every $g_0 \neq 1$. For a subset $F \subset G$ put

$$X(F) = \{(e_g) \in X; \text{ either } e_g = 0 \text{ for all } g \in F \text{ or } e_g = 1 \text{ for all } g \in F\}.$$

We get $\mu(X(F)) = 2^{1-|F|}$, where we put $2^{-\infty} = 0$. It is $X^{g_0} \subset X(\langle g_0 \rangle)$. Thus, if the generated subgroup $\langle g_0 \rangle$ of $g_0 \in G$ is infinite, then $\mu(X^{g_0}) = 0$. If not, consider an infinite set $\{g_1, g_2, \ldots\}$ of representatives of $G/\langle g_0 \rangle$. Then

$$\mu\left(\bigcap_{i=1}^{n} X(\langle g_0 \rangle \cdot g_i)\right) = \left(2^{1-|\langle g_0 \rangle|}\right)^n \longrightarrow 0 \quad \text{for } g_0 \neq 1$$

holds, and $X^{g_0} \subset \bigcap_{i=1}^{\infty} X(\langle g_0 \rangle \cdot g_i)$ implies $\mu(X^{g_0}) = 0$. ☐

EXAMPLE 1.18 (Holonomy Groupoid).
We give a brief, informal outline of the holonomy groupoid of a foliation. In [51, p. 64-78] details and proofs of the statements below can be found.

Let M be a manifold with a foliation $\mathcal{F}$. Let $a, b \in M$ be on the same leaf L and γ be a path in L from a to b. Consider foliation charts U, V containing a, b with transverse sections A, B passing through a, b respectively. The idea of holonomy (along γ) is to obtain paths from points of A close to a to points of B close to b by sliding along the leaves following γ. The correspondence between the starting and ending points of the paths defines a local diffeomorphism T_γ from a neighborhood of a in A to a neighborhood of b in B. Its germ $[\gamma]$, called the *holonomy class* of γ, is independent of all the choices made in the construction and depends only on γ.

The set of triples

$$G(M, \mathcal{F}) = \{(a, [\gamma], b);\ a, b \in M, \gamma \text{ path from } x \text{ to } y \text{ in } L.\}$$

has a well-defined groupoid structure induced by multiplication of paths. The manifold M can be identified with $G(M, \mathcal{F})^0$. We call $G(M, \mathcal{F})$ the *holonomy groupoid* of $(M, \mathcal{F})$. It can be made into a topological groupoid, in particular it has a measurable structure. However, $G(M, \mathcal{F})$ is far from being discrete. A discrete measurable groupoid is obtained by restricting to an appropriate subset of M as follows: One can find a locally finite, countable cover $(U_n)_{n\in\mathbb{N}}$ of foliation charts for M and transverse sections T_n of the U_n's such that for each $n \in \mathbb{N}$, $\overline{T}_n \cap \overline{\bigcup_{i\neq n} T_i} = \emptyset$. Let $T = \bigcup_n T_n$. The subset T is called a *complete transversal for* $(M, \mathcal{F})$. Then the restriction $G(M,\mathcal{F})_{|T}$ is a discrete measurable groupoid. But an invariant measure for $G(M,\mathcal{F})_{|T}$ does not always exist. Without further explanation, we remark that any foliation of a compact manifold with some leaf of non-exponential growth admits an invariant measure [12, p. 72].

1.2 Groupoid Rings

Let $\underline{G}$ be a discrete measured groupoid with invariant measure μ. For a function $\phi : \underline{G} \to \mathbb{C}$ and $x \in \underline{G}^0$ we put

$$\begin{aligned} S(\phi)(x) &= \#\{g \in \underline{G};\ \phi(g) \neq 0, s(g) = x\} \in \mathbb{N} \cup \{\infty\}, \\ T(\phi)(x) &= \#\{g \in \underline{G};\ \phi(g) \neq 0, t(g) = x\} \in \mathbb{N} \cup \{\infty\}. \end{aligned} \tag{1.3}$$

We denote the measure on $\underline{G}$ induced by μ (see 1.7) by $\mu_{\underline{G}}$. As usual, the set of complex-valued, measurable, essentially bounded functions (module almost null functions) on $\underline{G}$ with respect to $\mu_{\underline{G}}$ is denoted by $L^\infty(\underline{G}, \mu_{\underline{G}})$. Recall that a function f is called *essentially bounded* if there is a constant C such that $|f(x)| \leq C$ for almost all x. Next we define the groupoid ring of $\underline{G}$, which is a subset of $L^\infty(\underline{G}, \mu_{\underline{G}})$.

DEFINITION 1.19 (Groupoid Ring).
The *groupoid ring* $\mathbb{C}\underline{G}$ of $\underline{G}$ is defined as

$$\mathbb{C}\underline{G} = \left\{\phi \in L^\infty(\underline{G}, \mu_{\underline{G}}); S(\phi) \text{ and } T(\phi) \text{ are essentially bounded on } \underline{G}^0\right\}.$$

LEMMA 1.20 (Ring Structure of the Groupoid Ring).
The set $\mathbb{C}\underline{G}$ is a ring with involution containing $L^\infty(\underline{G}^0)$ as a subring. The addition is the pointwise addition in $L^\infty(\underline{G}, \mu_{\underline{G}})$, the multiplication is given by the so-called convolution product

$$(\phi \cdot \eta)(g) = \sum_{\substack{g_1, g_2 \in \underline{G} \\ g_2 \circ g_1 = g}} \phi(g_1) \cdot \eta(g_2), \quad \phi, \psi \in \mathbb{C}\underline{G},\ g \in \underline{G},$$

and the involution is defined by $(\phi^*)(g) = \overline{\phi(i(g))}$.

Proof. Strictly speaking, the elements in $\mathbb{C}\underline{G} \subset L^\infty(\underline{G})$ are classes of measurable functions. The convolution product $\phi \cdot \eta$ is defined in terms of representatives ϕ, ψ with bounded $S(\phi), T(\phi), S(\psi), T(\psi)$. Notice that the sum in the definition of $\phi \cdot \psi$ is finite. We will show that $\phi \cdot \psi$ is measurable. That the convolution product is well-defined on $\mathbb{C}\underline{G}$ follows then from the easy observation that two measurable functions $\phi, \psi : \underline{G} \to \mathbb{C}$ coincide almost everywhere if and only if there is a measurable subset $N \subset \underline{G}^0$ of full measure such that $\phi(g) = \psi(g)$ holds if $s(g) \in N$ or $t(g) \in N$.

We will frequently use the theorems 1.2 and 1.3 in the following. Note that s and t are bounded finite-to-1 on $\phi^{-1}(\mathbb{C} - \{0\})$. Hence there is a finite Borel partition $X_i, i \in I$ of $\phi^{-1}(\mathbb{C} - \{0\})$ such that all $s_{|X_i}, t_{|X_i}$ are injective. Every $\phi \in \mathbb{C}\underline{G}$ can be written as a finite sum $\phi = \sum_{i \in I} \phi_i$ such that ϕ_i vanishes outside X_i. So for the measurability of $\phi \cdot \psi$ we can assume that ϕ and ψ vanish outside X_i, X_j respectively. The restrictions $s_{|X_i} : X_i \to s(X_i)$, $t_{|X_j} : X_j \to t(X_j)$ have Borel inverses s'_i, t'_j respectively. Then we get

$$(\phi \cdot \psi)(g) = \begin{cases} \phi\Big(s'_i\big(s(g)\big)\Big)\psi\Big(t'_j\big(t(g)\big)\Big) & \text{if } s(g) \in s(X_i), t(g) \in t(X_j) \\ & \text{and } t\big(s'_i\big(s(g)\big)\big) = s\big(t'_j\big(t(g)\big)\big), \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\phi \cdot \psi$ is measurable.

We omit the routine verification of associativity, distributivity and so on. By extending a function from $\underline{G}^0$ to $\underline{G}$ by zero, we can consider $L^\infty(\underline{G}^0)$ as a subset of $\mathbb{C}\underline{G}$, and it is a subring indeed. Furthermore, the involutions on $L^\infty(\underline{G}^0)$ and $\mathbb{C}\underline{G}$ are compatible. ☐

In the preceding proof we saw that every $\phi \in \mathbb{C}\underline{G}$ can be written as a finite sum $\phi = \sum_{i \in I} \phi_i$, where the support of ϕ_i lies in X_i, and X_i is such that $s_{|X_i}$, $t_{|X_i}$ are injective. Hence ϕ_i is of the form $f \cdot \chi_{X_i}$ (convolution product) with $f \in L^\infty(\underline{G}^0)$. We record this for later reference:

LEMMA 1.21. *Every $\phi \in \mathbb{C}\underline{G}$ can be written as a finite sum $\phi = \sum_{i=1}^n f_i \cdot \chi_{E_i}$, where $f_i \in L^\infty(\underline{G}^0)$ and each Borel subset $E_i \subset \underline{G}$ has the property that $s_{|E_i}$ and $t_{|E_i}$ are injective.*

Next we explain how $L^\infty(\underline{G}^0)$ becomes a (left) $\mathbb{C}\underline{G}$-module equipped with a $\mathbb{C}\underline{G}$-epimorphism from $\mathbb{C}G$ to $L^\infty(\underline{G}^0)$. In the language of [7], this means that $\mathbb{C}\underline{G}$ is an augmented ring with the augmentation module $L^\infty(\underline{G}^0)$. Define the so-called *augmentation homomorphism* $\epsilon : \mathbb{C}\underline{G} \to L^\infty(\underline{G}^0)$ by

$$\epsilon : \mathbb{C}\underline{G} \to L^\infty(\underline{G}^0),\ \epsilon(\phi)(x) = \sum_{g \in s^{-1}(x)} \phi(g) \text{ for } x \in \underline{G}^0. \tag{1.4}$$

It becomes a homomorphism of $\mathbb{C}\underline{G}$-modules when we equip $L^\infty(\underline{G}^0)$ with the $\mathbb{C}\underline{G}$-module structure defined below, but it is not a homomorphism of rings unless $\underline{G}$ is a group.

Due to the previous lemma, measurability of $\epsilon(\phi)$ has to be checked only for case $\phi = f \cdot \chi_E$ with injective $s_{|E}$, $t_{|E}$, for which it is clear. Further, note that ϵ restricted to $L^\infty(\underline{G}^0)$ is the identity.

LEMMA 1.22 (Module Structure).
The augmentation homomorphism ϵ induces a $\mathbb{C}\underline{G}$-module structure on $L^\infty(\underline{G}^0)$ via

$$\eta \bullet f = \epsilon(\eta \cdot f),\ \eta \in \mathbb{C}\underline{G},\ f \in L^\infty(\underline{G}^0),$$

where $\cdot$ denotes the convolution product in $\mathbb{C}\underline{G}$.

Proof. We only show the associativity of the scalar multiplication, i.e. $(\eta_1 \cdot \eta_2) \bullet f = \eta_1 \bullet (\eta_2 \bullet f)$. This follows immediately from the equality $\epsilon(\phi_1 \cdot \epsilon(\phi_2)) =$

$\epsilon(\phi_1 \cdot \phi_2)$, $\phi_i \in \mathbb{C}\underline{G}$, which is obtained by the following computation.

$$\begin{aligned}\epsilon\big(\phi_1 \cdot \epsilon(\phi_2)\big)(x) = \sum_{g \in s^{-1}(x)} \big(\phi_1 \cdot \epsilon(\phi_2)\big)(g) &= \sum_{g \in s^{-1}(x)} \phi_1(g)\epsilon(\phi_2)(t(g)) \\ &= \sum_{g \in s^{-1}(x)} \phi_1(g) \sum_{g' \in s^{-1}(t(g))} \phi_2(g') \\ &= \sum_{g'' \in s^{-1}(x)} (\phi_1 \cdot \phi_2)(g'') \\ &= \epsilon(\phi_1 \cdot \phi_2)(x).\end{aligned}$$

□

NOTE. The symbol • for the scalar multiplication in the preceding lemma was chosen for a clear distinction. In the following we will not distinguish the different types of multiplication in the notation, and always use the symbol · (or nothing). There should be no confusion.

REMARK 1.23 (Groupoid Ring of the Restriction).
The groupoid ring of the restriction $\mathbb{C}\underline{G}_{|A}$ of $\underline{G}$ to A, called the *restricted groupoid ring*, is canonically isomorphic to $\chi_A \mathbb{C}\underline{G}\chi_A$.

DEFINITION 1.24 (Crossed Product Ring).
Let R be a ring and G be a group. Given a homomorphism $c : G \to \operatorname{Aut}(R)$, $g \mapsto c_g$ we define the *crossed product* $R *_c G$ of R with G as the free R-module with basis G. It carries a ring structure, where the multiplication is given by

$$\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{g \in G} b_g g\right) = \sum_{g \in G} \sum_{\substack{g_1, g_2 \in G \\ g_1 g_2 = g}} \left(a_{g_1} c_{g_1}(b_{g_2})\right) g.$$

DEFINITION 1.25. Let $G \curvearrowright X$ be a standard action. Then we denote by $L^\infty(X) * G$ the crossed product $L^\infty(X) *_c G$ obtained by the homomorphism $c : G \to \operatorname{Aut}(L^\infty(X))$, $g \mapsto l_{g^{-1}}$. Here $l_g(x) = gx$ is left translation by $g \in G$.

LEMMA 1.26 (Crossed Product inside the Groupoid Ring).
Let $G \curvearrowright X$ be a standard action. Then the ring homomorphism

$$L^\infty(X) * G \to \mathbb{C}\mathcal{R}(G \curvearrowright X), \ \sum_{g \in G} f_g \cdot g \mapsto \big((gx, x) \mapsto f_g(gx)\big)$$

is injective and $\phi \in \mathbb{C}\mathcal{R}(G \curvearrowright X)$ is in the image if and only if there is a finite subset $F \subset G$ such that $g \notin F$ implies $\phi(gx, x) = 0$ for almost all $x \in X$.

Note that the map is well defined because the action is essentially free. It is an easy computation that is a ring homomorphism. The other claims are obvious.

In the sequel we regard $L^\infty(X) * G$ as a subring of $\mathbb{C}\mathcal{R}(G \curvearrowright X)$.

REMARK 1.27. The restriction of the $\mathbb{C}\mathcal{R}(G \curvearrowright X)$-module structure on $L^\infty(X)$ to $L^\infty(X) * G$ is isomorphic to the $L^\infty(X) * G$-module structure obtained by the isomorphism $L^\infty(X) \cong (L^\infty(X) * G) \otimes_{\mathbb{C}G} \mathbb{C}$.

In the rest of this section we review some algebraic definitions and facts that will be applied to groupoid rings and their von Neumann algebras in chapter 3.

DEFINITION 1.28 (Semihereditary Rings).
A ring R is called *left (resp. right) semihereditary* if every finitely generated submodule of a projective left (resp. right) R-module is projective. A ring is called *semihereditary* if it is both left and right semihereditary.

A large class of examples for semihereditary rings is given by the following theorem. Compare [45, theorem 6.7 on p. 239, p. 288].

THEOREM 1.29. *Every von Neumann algebra $\mathcal{N}$ is a semihereditary ring.*

For the definition of a von Neumann algebra see the next section.

DEFINITION 1.30 (Torsionless Modules).
Let R be a ring. An R-module M is called *torsionless*, if the canonical map $M \to M^{**}$ from M into the double dual module is injective, i.e. if for every $m \in M$, $m \neq 0$ there is an R-homomorphism $f : M \to R$ with $f(m) \neq 0$.

Clearly, the property *torsionless* is preserved by taking submodules and arbitrary sums and products of modules. If R is a domain, i.e. contains no nonzero zero-divisor, then every torsionless module is torsionfree. But the example of $\mathbb{Q}$ as a $\mathbb{Z}$-module (note that $\hom_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$) shows that a torsionfree module need not to be torsionless.

THEOREM 1.31 (Semihereditary Rings and Flatness).
Let R be a semihereditary ring. Then the following holds.

(i) All torsionless R-modules are flat.

(ii) Any direct product of flat R-modules is flat.

(iii) Submodules of flat R-modules are flat.

You find these results and their proofs in [36, theorem 4.67 on p. 146], [36, theorem 4.47 on p. 139], [36, theorems 4.66 and 4.67 on p. 145/146].
As a consequence we get the following theorem.

THEOREM 1.32. *$\mathbb{C}\underline{G}$ is flat over $L^\infty(\underline{G}^0)$.*

Proof. On p. 29-32 we will see that there is an inclusion of rings

$$L^\infty(\underline{G}^0) \subset \mathbb{C}\underline{G} \subset \mathcal{N}(\underline{G})$$

where $\mathcal{N}(\underline{G})$ is a finite von Neumann algebra whose trace extends that of $L^\infty(\underline{G}^0)$. By 1.48 $\mathcal{N}(\underline{G})$ is a flat module over $L^\infty(\underline{G}^0)$. Then the previous theorems 1.29, 1.31 imply that $\mathbb{C}\underline{G}$ is flat. □

For later reference, we state some definitions and easy facts about Morita equivalence of rings. For details and proofs we refer to [36, section 18].

DEFINITION 1.33 (Morita equivalence).
Two rings are called *Morita equivalent* if there exists a category equivalence, called *Morita equivalence*, between their module categories.

LEMMA 1.34. *Every Morita equivalence is exact and preserves projective modules.*

DEFINITION 1.35 (Full Idempotent).
An idempotent p in a ring R is called *full* if the additive subgroup in R generated by the elements rpr' with $r, r' \in R$, denoted by RpR, coincides with R.

LEMMA 1.36 (Morita Equivalence for Full Idempotents).
Let p be a full idempotent in R. Then R and pRp are Morita equivalent, and mutual inverse category equivalences are given by tensoring with the bimodules Rp, pR respectively.

1.3 The von Neumann Algebra of a Discrete Measured Groupoid

We briefly recall some of the fundamental notions and results in the theory of (finite) von Neumann algebras. The material can be found in any introductionary textbook on that topic. We consulted the books [33], [34] and [15].

Let $\mathcal{B}(\mathcal{H})$ be the bounded operators on a complex Hilbert space $\mathcal{H}$. A *von Neumann algebra (acting on $\mathcal{H}$)* is a $*$-closed subalgebra of $\mathcal{B}(\mathcal{H})$, which is closed under the weak operator topology and contains the identity. The *double commutant theorem* of von Neumann says that a $*$-subalgebra $A \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra if and only if A equals its double commutant A''. The commutant B' of a subset $B \in \mathcal{B}(\mathcal{H})$ is defined as

$$B' = \{t \in \mathcal{B}(\mathcal{H});\ tb = bt \text{ for all } b \in B\}.$$

Any separable abelian von Neumann algebra is $*$-isomorphic to the algebra $L^\infty(X)$ of essentially bounded measurable functions on X for some standard Borel space X with a Radon measure.

There is a *partial order on the set of self-adjoint elements* in a von Neumann algebra defined by

$$a \geq b \text{ if } a - b \text{ is positive.}$$

An operator $a \in \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ is defined to be *positive* if $\langle a(x), x\rangle_{\mathcal{H}} \geq 0$ for all $x \in \mathcal{H}$. Equivalently, a is positive if and only if a can be written as $a = b^*b$ with $b \in \mathcal{N}$. So, being positive can be expressed intrinsically, using only the algebra structure and not a specific representation on a Hilbert space.

In a von Neumann algebra every increasing net of self-adjoint operators bounded above has a *supremum*, i.e. a least upper bound and the net converges strongly to it.

A map between von Neumann algebras is called *normal* if it preserves suprema of increasing nets of self-adjoint operators. A $*$-isomorphism of von Neumann algebras is always normal. The image of a normal $*$-homomorphism between von Neumann algebras is again a von Neumann algebra.

A von Neumann algebra $\mathcal{N}$ is called *finite* if it possesses a *finite faithful normal trace* $\mathrm{tr} : \mathcal{N} \to \mathbb{C}$. This means that tr is a linear functional satisfying the trace property $\mathrm{tr}(ab) = \mathrm{tr}(ba)$ for all $a, b \in \mathcal{N}$ and is faithful, i.e. $\mathrm{tr}(a) = 0 \Leftrightarrow a = 0$ for positive $a \in \mathcal{N}$.

The example of the von Neumann algebra of essentially bounded measurable functions on a finite measure space with the integral as the trace shows that the trace is not unique, in general. But it is unique up to a scalar factor if the von Neumann algebra is a *factor*, i.e. if its center consists only of scalar multiplies of the identity.

We adopt the convention that the trace is part of the datum of a finite von Neumann algebra.

We say that a $*$-homomorphism $\phi : \mathcal{N} \to \mathcal{M}$ between finite von Neumann algebras is *trace preserving* if $\mathrm{tr}_{\mathcal{M}}(\phi(n)) = \mathrm{tr}_{\mathcal{N}}(n)$ for $n \in \mathcal{N}$ holds. In particular, ϕ is injective. A trace-preserving $*$-homomorphism is always normal.

The *spectral calculus* for C^*-algebras with respect to continuous functions has an extension to Borel functions for von Neumann algebras. The following theorem is taken from [33, theorem 5.29. on p.322].

THEOREM 1.37 (Spectral Calculus with Borel Functions).
Let A be a normal operator on a complex Hilbert space $\mathcal{H}$, and let $\mathcal{B}(sp(A))$ be the algebra of bounded Borel functions on the spectrum $sp(A)$ of A. Then there is a unique normal $*$-homomorphism ϕ from $\mathcal{B}(sp(A))$ into the abelian von Neumann algebra $\mathcal{A}$ generated by A such that $\phi(1) = id$ and $\phi(id) = A$.

Here *normal* has the obvious meaning with respect to the usual (pointwise defined) partial ordering of functions. If A and ϕ are as above we use the suggestive notation $f(A) = \phi(f)$. For a normal operator A the family of projections

$$E_\lambda^A = \chi_{(-\infty,\lambda]}(A), \ \lambda \in \mathbb{R}$$

is called the *spectral family of A*.

DEFINITION 1.38 (Spectral Density Function).
The *spectral density function* $F(A;\mathcal{N}) : [0,\infty) \to [0,\infty)$ of an operator A in a finite von Neumann algebra $\mathcal{N}$ is defined as

$$F(A;\mathcal{N})(\lambda) = \mathrm{tr}_{\mathcal{N}}\left(E_{\lambda^2}^{A^*A}\right).$$

The spectral calculus is functorial for trace-preserving $*$-homomorphism of finite von Neumann algebras. See [33, proposition 5.12.14] for the - more general - case of normal $*$-homomorphisms.

THEOREM 1.39 (Functoriality of Spectral Calculus).
If $\phi : \mathcal{A} \to \mathcal{B}$ is a trace-preserving $*$-homomorphism between finite von Neumann algebras then $sp_{\mathcal{A}}(A) = sp_{\mathcal{B}}(\phi(A))$ and $\phi(f(A)) = f(\phi(A))$ for each normal $A \in \mathcal{A}$ and each Borel function $f \in \mathcal{B}(sp(A))$.

For the following result we refer to [15, proposition 1 on p.17].

THEOREM 1.40 (Restricted von Neumann Algebra).
If $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ is a finite von Neumann algebra and $p \in \mathcal{N}$ a non-zero projection, then $p\mathcal{N}p \subset \mathcal{B}(p\mathcal{H})$ is a finite von Neumann algebra with the normalized trace

$$\mathrm{tr}_{p\mathcal{N}p}(T) = \frac{1}{\mathrm{tr}_{\mathcal{N}}(p)}\,\mathrm{tr}_{\mathcal{N}}(T)$$

and commutant $p\mathcal{N}'p = \mathcal{N}'p$. If $N \subset \mathcal{N}$ generates $\mathcal{N}$ (as a von Neumann algebra), then pNp generates $p\mathcal{N}p$.

Spectral calculus is compatible with restriction: Let $T \in p\mathcal{N}p$ be selfadjoint. For a Borel function f on $\mathbb{R}$ we can apply spectral calculus to T with respect to $p\mathcal{N}p$ and $\mathcal{N}$, the results being denoted by $f(T;p\mathcal{N}p)$, $f(T;\mathcal{N})$ respectively. The relation between them is $pf(T;\mathcal{N})p = f(T;p\mathcal{N}p)$. This is obvious if f is a polynomial. But this suffices because $f(T)$ (with respect to $\mathcal{N}$ or $p\mathcal{N}p$) is the strong operator limit of $p_i(T)$, $i \in I$, where each p_i is a polynomial. Let us record this result for later reference.

THEOREM 1.41 (Spectral Calculus and Restriction).
Let p be a projection in $\mathcal{N}$, $T \in p\mathcal{N}p$, and let f be a Borel function on $\mathbb{R}$. Then

$$pf(T;\mathcal{N})p = f(T;p\mathcal{N}p).$$

A finite von Neumann algebra $\mathcal{N}$ has a canonical action on the Hilbert space $l^2\mathcal{N}$. Here $l^2\mathcal{N}$ is defined as the Hilbert space completion of the complex vector space $\mathcal{N}$ with the inner product $\langle a, b\rangle = \mathrm{tr}_{\mathcal{N}}(a^*b)$. Left multiplication by $n \in \mathcal{N}$ is a linear map $\mathcal{N} \to \mathcal{N}$ which extends to bounded map $l^2\mathcal{N} \to l^2\mathcal{N}$. We obtain a left $\mathcal{N}$-module structure on $l^2\mathcal{N}$. Analogously, we get a right $\mathcal{N}$-module structure by right multiplication. The following theorem is fundamental for the study of finite von Neumann algebras [15, theorem 1 on p. 80, theorem 2 on p. 99].

THEOREM 1.42 (Regular Representation).
Let $\mathcal{N}$ be a finite von Neumann algebra. Then the map from $\mathcal{N}$ to the left $\mathcal{N}$-equivariant bounded operators on $l^2\mathcal{N}$

$$\mathcal{N} \longrightarrow \mathcal{B}(l^2\mathcal{N}, l^2\mathcal{N})^{\mathcal{N}}$$

induced by sending $n \in \mathcal{N}$ to right multiplication by n is an isometric $*$-antihomomorphism of $\mathbb{C}$-algebras.

There is a useful concept to produce examples of finite von Neumann algebras, namely that of a unital Hilbert-algebra.

DEFINITION 1.43 (Unital Hilbert Algebras).
Let A be a unital $\mathbb{C}$-algebra with an involution and a positive definite inner product $\langle\ ,\ \rangle : A \times A \to \mathbb{C}$. Denote the Hilbert space completion of $(A, \langle\ ,\ \rangle)$ by $\mathcal{H}_A$. We say that A is a *unital Hilbert algebra* if the following holds.

(i) $\langle y, x\rangle = \langle x^*, y^*\rangle$ for $x, y \in A$.

(ii) $\langle xy, z\rangle = \langle y, x^*z\rangle$ for $x, y, z \in A$.

(iii) For each $x \in A$ the mapping $y \mapsto xy$ is continuous with respect to the inner product.

The mapping $L_x : A \to A, y \mapsto xy$ extends by continuity to a bounded operator on $\mathcal{H}_A$. The axioms (i) and (iii) above ensure that $R_x : A \to A, y \mapsto yx = (x^*y^*)^*$ is also continuous and extends to a bounded operator on $\mathcal{H}_A$.

DEFINITION 1.44 (The von Neumann Algebra of a Hilbert Algebra).
The *left von Neumann algebra $\mathcal{L}(A)$ associated to A* (resp. *right von Neumann algebra $\mathcal{R}(A)$ associated to A*) is the weak closure of the set of operators L_a, $a \in A$ (resp. R_a, $a \in A$) in $\mathcal{B}(\mathcal{H}_A)$.

For the next statement see [15, p. 78,79] and [15, theorem 1 on p. 80].

THEOREM 1.45. *The maps $A \to \mathcal{L}(A)$, $a \mapsto L_a$ and $A \to \mathcal{R}(A)$, $a \mapsto R_a$ are injective $*$-homomorphisms resp. $*$-antihomomorphisms, and $\mathcal{L}(A)$ and $\mathcal{R}(A)$ are commutants of each other.*

The von Neumann algebra of a unital Hilbert algebra is always equipped with a canonical finite trace:

THEOREM 1.46 (Finite Traces).
The maps

$$\mathrm{tr}_{\mathcal{L}(A)} : \mathcal{L}(A) \to \mathbb{C},\ T \mapsto \langle T(1_A), 1_A \rangle_{\mathcal{H}_A}$$

resp.

$$\mathrm{tr}_{\mathcal{R}(A)} : \mathcal{R}(A) \to \mathbb{C},\ T \mapsto \langle T(1_A), 1_A \rangle_{\mathcal{H}_A}$$

define finite normal faithful traces on $\mathcal{L}(A)$ resp. $\mathcal{R}(A)$.

Proof. We give the proof for $\mathrm{tr}_{\mathcal{L}(A)}$. The trace property $\mathrm{tr}_{\mathcal{L}(A)}(st) = \mathrm{tr}_{\mathcal{L}(A)}(ts)$ needs only be checked on the dense subalgebra A. Using (i), (ii) in 1.43 we conclude $\mathrm{tr}_{\mathcal{L}(A)}(st) = \langle st, 1 \rangle = \langle t, s^* \rangle = \langle s, t^* \rangle = \langle ts, 1 \rangle = \mathrm{tr}_{\mathcal{L}(A)}(ts)$. Obviously, 1_A is a *cyclic* vector for the commutant $\mathcal{L}(A)' = \mathcal{R}(A)$, i.e. $\mathcal{R}(A)1_A$ is dense in $\mathcal{H}_A$. By [15, proposition 5 on p. 5] it follows that 1_A is *separating* for $\mathcal{L}(A)$, i.e. $T(1_A) = 0$ for $T \in \mathcal{L}(A)$ implies $T = 0$. Hence $\mathrm{tr}_{\mathcal{L}(A)}$ is faithful. It is clear that $\mathrm{tr}_{\mathcal{L}(A)}$ is normal. □

THEOREM 1.47 (Functoriality of Hilbert Algebras).
Let $\phi : A \to B$ be an isometric $*$-homomorphism of Hilbert algebras such that the operator norms of L_a and $L_{\phi(a)}$ for $a \in A$ satisfy the inequality $\left\| L_{\phi(a)} \right\| \le \|L_a\|$. Then ϕ extends to a trace-preserving $*$-homomorphism of the associated von Neumann algebras $\Phi : \mathcal{L}(A) \to \mathcal{L}(B)$. The same holds for the right handed version $\mathcal{R}(A)$.

Proof. The map ϕ extends to an isometric inclusion of the associated Hilbert spaces $\phi : \mathcal{H}_A \to \mathcal{H}_B$. The extension $\Phi : \mathcal{L}(A) \to \mathcal{L}(B)$ is defined as follows. Let $t \in \mathcal{L}(A)$. Then t, considered as an element of $\mathcal{H}_A$, is the limit of a sequence of elements $t_i \in A \subset \mathcal{H}_A$ such that $C := \sup_{i \in \mathbb{N}} \|L_{t_i}\| < \infty$ [15, proposition 4 on p.82]. Observe that

$$\|\phi(t) \cdot b\| = \lim \|\phi(t_i) \cdot b\| \le \lim \left\| L_{\phi(t_i)} \right\| \cdot \|b\| \le \lim \|L_{t_i}\| \cdot \|b\| \le C\|b\|.$$

The first equality follows from the continuity of $\phi : \mathcal{H}_A \to \mathcal{H}_B$ and R_b. So $b \mapsto \phi(t) \cdot b$ extends to a bounded operator $\Phi(t)$ on $\mathcal{H}_B$. For $t \in \mathcal{L}(A)$ the operator $\Phi(t)$ lies in the strong closure of B, thus $\Phi(t) \in \mathcal{L}(B)$. An easy continuity argument shows that Φ is a $*$-homomorphism. It is clear that Φ is trace-preserving. □

The next theorem is a generalization of [45, theorem 6.29 on p. 253] to arbitrary finite von Neumann algebras.

THEOREM 1.48 (Flatness for Inclusions of Finite von Neumann Algebras).
Any trace-preserving $*$-homomorphism between finite von Neumann algebras is a faithfully flat ring extension.

Proof. We can assume that the trace-preserving $*$-homomorphism is an inclusion $\mathcal{N} \subset \mathcal{M}$. We show that $\mathcal{M}$ is a torsionless right $\mathcal{N}$-module. This will imply flatness by 1.29 and 1.31.

Let $m \in \mathcal{M}$, $m \neq 0$. So, we have to show that there is a $\mathcal{N}$-homomorphism $f : \mathcal{M} \to \mathcal{N}$ such that $f(m) \neq 0$. Since m is the sum of four unitaries [33, theorem 4.1.7 on p. 242], there is a unitary $u \in \mathcal{M}$ such that $\mathrm{tr}(m^*u) \neq 0$. The map $\mathcal{N} \to \mathcal{M}$, $n \mapsto nu$ extends to an $\mathcal{N}$-equivariant isometric embedding $i : l^2\mathcal{N} \to l^2\mathcal{M}$. Taking the orthogonal projection onto the image of i yields an $\mathcal{N}$-equivariant bounded split $f : l^2\mathcal{M} \to l^2\mathcal{N}$ of i. Because of $\langle m, u\rangle_{l^2\mathcal{M}} = \mathrm{tr}(m^*u) \neq 0$ we get $f(m) \neq 0$. Remains to show that $f(\mathcal{M}) \subset \mathcal{N}$. Due to 1.42, we only need to know that $\mathcal{N} \to l^2\mathcal{N}$, $n \mapsto n \cdot f(m) = f(n \cdot m)$ extends to a bounded operator on $l^2\mathcal{N}$. This is obtained from the boundedness of both f and right multiplication with m.
Concerning faithfulness, note that the above reasoning shows (for $u = 1$) that $\mathcal{N}$ is a direct summand of $\mathcal{M}$. In particular, $\mathcal{N} \subset \mathcal{M}$ is faithful. □

REMARK 1.49. The previous proof yields some more information. Let $M \subset \mathcal{M}$ be a finitely generated $\mathcal{N}$-submodule. In 3.10 we will see that M splits as $M = \mathbf{P}M \oplus \mathbf{T}M$ where $\mathbf{P}M$ is projective and $\mathbf{T}M$ is the kernel of the canonical map $M \to M^{**}$. But M is torsionless because $\mathcal{M}$ is torsionless, so we obtain $\mathbf{T}M = 0$, and M is projective. Thus every finitely generated $\mathcal{N}$-submodule of $\mathcal{M}$ is projective.

We want to associate to each discrete measured groupoid $\underline{G}$ a finite von Neumann algebra $\mathcal{N}(\underline{G})$. In order to do that, we have to show that $\mathbb{C}\underline{G}$ is a Hilbert algebra.

THEOREM 1.50 (Groupoid Ring as a Hilbert Algebra).
The groupoid ring $\mathbb{C}\underline{G}$ as a $\mathbb{C}$-algebra with involution is a unital Hilbert algebra, where the inner product is given by

$$\langle \phi, \eta \rangle = \int_{\underline{G}} \phi(g) \cdot \overline{\eta(g)} d\mu_{\underline{G}}(g).$$

Proof. First we show that the mapping $\eta \mapsto \phi \cdot \eta$ is continuous with respect to the inner product. One computes

$$\|\phi \cdot \eta\|^2 = \int_{\underline{G}^0} \sum_{t(g)=x} |(\phi \cdot \eta)(g)|^2 \, d\mu(x) = \int_{\underline{G}^0} \sum_{t(g)=x} \left| \sum_{g_2 g_1 = g} \phi(g_1)\eta(g_2) \right|^2 d\mu(x).$$

Let $S(\phi)$, $T(\phi)$ be as in (1.3). There is a constant $N \in \mathbb{N}$ such that $S(\phi)(x) \leq N$ and $T(\phi)(x) \leq N$ almost everywhere. Let $C \in \mathbb{R}_{\geq 0}$ be the essential supremum of $|\phi|$. Recall the well-known inequality

$$\left(\sum_{i=1}^{n} a_i \right)^2 \leq n \cdot \sum_{i=1} a_i^2 \tag{1.5}$$

for real numbers a_i. Now the integrand can be estimated almost everywhere as follows.

$$\begin{aligned}
\sum_{t(g)=x}\left|\sum_{g_2g_1=g}\phi(g_1)\eta(g_2)\right|^2 &\leq \sum_{t(g)=x}\left(\sum_{g_2g_1=g}|\phi(g_1)||\eta(g_2)|\right)^2\\
&\leq \sum_{t(g)=x}C^2\left(\sum_{\substack{g_2g_1=g\\ g_1\in S(\phi)(s(g))}}|\eta(g_2)|\right)^2\\
&\leq N\cdot C^2\sum_{t(g)=x}\sum_{\substack{g_2g_1=g\\ g_1\in S(\phi)(s(g))}}|\eta(g_2)|^2 \quad \text{by (1.5)}\\
&= N\cdot C^2\cdot\sum_{t(g_2)=x}\sum_{\substack{t(g_1)=s(g_2)\\ \phi(g_1)\neq 0}}|\eta(g_2)|^2\\
&\leq N^2\cdot C^2\cdot\sum_{t(g)=x}|\eta(g)|^2 \quad (T(\phi)(s(g_2))\leq N)
\end{aligned}$$

So $\|\phi\cdot\eta\|\leq C\cdot N\cdot\|\eta\|$ follows. The equality $\langle\phi,\eta\rangle=\langle\eta^*,\phi^*\rangle$ is obtained from

$$\begin{aligned}
\langle\phi,\eta\rangle = \int_{\underline{G}^0}\sum_{s(g)=x}\phi(g)\overline{\eta(g)}d\mu(x) &= \int_{\underline{G}^0}\sum_{t(g)=x}\phi(g^{-1})\overline{\eta(g^{-1})}d\mu(x)\\
&= \int_{\underline{G}^0}\sum_{t(g)=x}\eta^*(g)\overline{\phi^*(g)}d\mu(x) = \langle\eta^*,\phi^*\rangle.
\end{aligned}$$

Finally, (ii) of 1.43 follows from

$$\begin{aligned}
\langle\phi\cdot\eta,\sigma\rangle &= \int_{\underline{G}^0}\sum_{t(g)=x}(\phi\cdot\eta)(g)\overline{\sigma(g)}d\mu(x)\\
&= \int_{\underline{G}^0}\sum_{\substack{t(g)=x\\ g_2g_1=g}}\phi(g_1)\eta(g_2)\overline{\sigma(g)}d\mu(x)\\
&= \int_{\underline{G}^0}\sum_{\substack{t(g_2)=x\\ t(g_1)=s(g_2)}}\eta(g_2)\phi(g_1)\overline{\sigma(g_2g_1)}d\mu(x)\\
&= \int_{\underline{G}^0}\sum_{t(g_2)=x}\eta(g_2)\left(\overline{\sum_{s(g_1)=s(g_2)}\overline{\phi(i(g_1))}\sigma(g_2i(g_1))}\right)d\mu(x) = \langle\eta,\phi^*\sigma\rangle.
\end{aligned}$$

□

DEFINITION 1.51 (von Neumann Algebra of a Discrete Measured Groupoid). Define the *von Neumann algebra* $\mathcal{N}(\underline{G})$ *of a discrete measured groupoid* $\underline{G}$ as the right von Neumann algebra $\mathcal{R}(\underline{G})$ associated to the Hilbert algebra $\mathbb{C}\underline{G}$ (see 1.44).

REMARK 1.52 (Groupoid Ring inside its von Neumann Algebra).
Let $C : \mathbb{C}\underline{G} \to \mathbb{C}\underline{G}$ be the ring homomorphism given by $C(\phi)(g) = \overline{\phi(g)}$ for $g \in \underline{G}$. The map $\mathbb{C}\underline{G} \to \mathcal{N}(\underline{G})$ which sends ϕ to $R_{C(\phi)^*}$, i.e. to right multiplication with the conjugate of $C(\phi)$ is an injective $*$-homomorphism of $\mathbb{C}$-algebras. See 1.45.
Using this map, we regard the groupoid ring as a subring of its von Neumann algebra.

THEOREM 1.53. *Let $G \curvearrowright X$ be a standard action, and let $\mathcal{R} = \mathcal{R}(G \curvearrowright X)$ be its orbit equivalence relation. The composition $\mathbb{C}G \hookrightarrow L^\infty(X) * G \hookrightarrow \mathbb{C}\mathcal{R}(G \curvearrowright X)$ (see 1.26) is an isometric $*$-homomorphism ϕ of Hilbert algebras such that the operator norms satisfy the inequality $\left\|R_{\phi(a)}\right\| \leq \|R_a\|$. The same holds also for the left handed version $L_{\phi(a)}$, L_a.*

Proof. Recall that a standard action is essentially free, by definition. We have the equality

$$\phi\left(\sum_{i=1}^n a_i g_i\right)(x,y) = \begin{cases} a_i & \text{if } x = g_i y,\ i = 1,\ldots,n \\ 0 & \text{else} \end{cases}$$

almost everywhere for an element $a = \sum_{i=1}^n a_i g_i$ in $\mathbb{C}G$ where the g_i are mutually disjoint. Put $r = \phi(a)$. The fact that ϕ is isometric follows from

$$\|r\| = \int_X \sum_{g\in G} |r(gx,x)|^2 d\mu(x) = \int_X \sum_{i=1}^n |a_i|^2 d\mu(x) = \mu(X) \cdot \sum_{i=1}^n |a_i|^2 = \|a\|.$$

Now we want to show that $\left\|R_{\phi(a)}\right\| \leq \|R_a\|$. The computation

$$\begin{aligned}\sum_{g\in G}\left|\sum_{i=1}^n a_i r(x, g_i g x)\right|^2 &= \left\|\left(\sum_{g\in G} r(x, g^{-1}x) g\right)\left(\sum_{i=1}^n a_i g_i\right)\right\|^2_{\mathbb{C}G} \\ &\leq \|R_a\|^2 \cdot \left(\sum_{g\in G}\left|r(x,g^{-1}x)\right|^2\right) = \|R_a\|^2 \cdot \left(\sum_{g\in G} |r(x,gx)|^2\right)\end{aligned}$$

implies

$$\begin{aligned}\|r\cdot\phi(a)\|^2 = \int_X \sum_{g\in G} |(r\phi(a))(x,gx)|^2 d\mu(x) &= \int_X \sum_{g\in G}\left|\sum_{i=1}^n a_i r(x, g_i g x)\right|^2 d\mu(x) \\ &\leq \|R_a\|^2 \int_X \sum_{g\in G} |r(x,gx)|^2 d\mu(x) = \|R_a\|^2 \|r\|^2\end{aligned}$$

and therefore we obtain $\left\|R_{\phi(a)}\right\| \leq \|R_a\|$. □

By theorem 1.47 we obtain the

COROLLARY 1.54. *The $*$-homomorphism $\phi : \mathbb{C}G \to \mathbb{C}\mathcal{R}$ extends to a trace-preserving $*$-homomorphism $\mathcal{N}(G) \to \mathcal{N}(\mathcal{R})$ of von Neumann algebras.*

REMARK 1.55. Note that for a Borel subset $A \subset \underline{G}^0$ the von Neumann algebras $\chi_A \mathcal{N}(\underline{G})\chi_A$ and $\mathcal{N}(\underline{G}_{|A})$, considered as subalgebras of $\mathcal{B}(\chi_A \mathcal{H}_{\underline{G}} \chi_A)$, contain the same dense set $\chi_A \underline{G} \chi_A = \underline{G}_{|A}$. If $\underline{G}_{|A}$ carries the normalized measure, we get an identification of $\chi_A \mathcal{N}(\underline{G})\chi_A$ and $\mathcal{N}(\underline{G}_{|A})$, which is compatible with the traces.

REMARK 1.56. A measure isomorphism $f : \underline{G}_2 \to \underline{G}_1$ of discrete measured groupoids induces the following commutative diagram of ring homomorphisms where the vertical maps are isomorphisms.

$$\begin{array}{ccccc} L^\infty(\underline{G}_1^0) & \hookrightarrow & \mathbb{C}\underline{G}_1 & \hookrightarrow & \mathcal{N}(\underline{G}_1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ L^\infty(\underline{G}_2^0) & \hookrightarrow & \mathbb{C}\underline{G}_2 & \hookrightarrow & \mathcal{N}(\underline{G}_2) \end{array} .$$

The right vertical map is trace-preserving. Moreover, the module structures on $L^\infty(\underline{G}_1)$ and $L^\infty(\underline{G}_2)$ with respect to the respective groupoid ring are also compatible with the isomorphism induced by f.

Proof. Obviously, the map $\mathbb{C}\underline{G}_1 \to \mathbb{C}\underline{G}_2$, $\phi \mapsto \phi \circ f$ is an isomorphism of $*$-algebras. The map is isometric with respect to the inner product defined in 1.50 because f is measure preserving. So it is an isometric isomorphism of unital Hilbert algebras. By 1.47 the map extends to a trace-preserving isomorphism of the associated von Neumann algebras. The module structure on $L^\infty(\underline{G})$ for a discrete measured groupoid $\underline{G}$ is defined using the augmentation map (see (1.4))

$$\epsilon : \mathbb{C}\underline{G} \to L^\infty(\underline{G}^0),\ \phi \mapsto \left(g \mapsto \sum_{g \in s^{-1}(x)} \phi(g) \right).$$

Clearly, ϵ is compatible with respect to isomorphisms of discrete measured groupoids, and so is the module structure. ☐

Chapter 2

Geometric and Measurable Group Theory

In this chapter we study (discrete) groups in a geometric and measure-theoretic context. To every finitely generated group a metric, the so-called *word-metric*, is assigned. In geometric group theory properties of this metric are studied, and groups are classified into geometric families. The notion of isometry is far too rigid in this context and is replaced by the notion of *quasi-isometry*. We give a reminder of the basic geometric notions of groups in the following section. In the section after that we consider groups acting on measure spaces. Quasi-isometry has a direct analog in this measure-theoretic context, called *measure equivalence*. In the last section we deal with the relation between quasi-isometry and measure equivalence.

2.1 Quasi-Isometry

DEFINITION 2.1 (Word Metric).
Let G be a finitely generated group with a finite generating set S. The *(word) length* $l_S(g)$ of an element $g \in G$ (with respect to S) is the smallest integer n for which there exists a sequence $s_1, s_2, \ldots, s_n$ of elements in $S \cup S^{-1}$ such that $g = s_1 s_2 \cdots s_n$. The *word metric* d_S on G is the metric defined by $d_S(g_1, g_2) = l_S(g_1^{-1} g_2)$.

DEFINITION 2.2 (Quasi-Isometry).
A (not necessarily continuous) mapping $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is called a *quasi-isometry* if there exist constants $\lambda \geq 1$, $C \geq 0$ such that

$$\frac{1}{\lambda} d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + C$$

for all $x, y \in X$ and if $f(X)$ is *C-dense* in Y, i.e. each point in Y is within distance C from some point in $f(X)$. Metric spaces (X, d_X), (Y, d_Y) are *quasi-isometric* if there exists a quasi-isometry $f : X \to Y$.

REMARK 2.3. For discrete metrics, i.e. if the distance of two different points is bounded below, the estimates in the definition of quasi-isometry can be replaced by Lipschitz estimates.

It is easy to see that for the same group equipped with two different word metrics the identity map is a quasi-isometry. We record:

LEMMA 2.4 (Independence of the generating set).
The quasi-isometry type of a finitely generated group does not depend on the choice of a generating set.

There are many results about the geometric characterization of finitely generated groups. We say a class of groups is *geometric* if it closed under quasi-isometries. The following classes are geometric: finitely presented, virtually free, virtually abelian, virtually nilpotent, amenable and word-hyperbolic groups. The following classes are not geometric, for example: virtually solvable groups and groups with Kazhdan property (T). A landmark in the subject is the theorem of Gromov [28] saying that a finitely generated group has polynomial growth if and only if it is virtually nilpotent, i.e. it has a nilpotent subgroup of finite index. Virtually nilpotent and solvable groups are examples of amenable groups. A group containing $\mathbb{Z} * \mathbb{Z}$ is not amenable.

DEFINITION 2.5 (Amenable Groups).
A group G is *amenable* if there exists a (left) G-invariant linear operator (called a *mean*) $\mu : l^\infty(G;\mathbb{R}) \to \mathbb{R}$ from the bounded functions on G to $\mathbb{R}$ such that $\mu(1) = 1$ and

$$\inf\{f(g);\ g \in G\} \le \mu(f) \le \sup\{f(g);\ g \in G\}$$

for all $f \in l^\infty(G;\mathbb{R})$.

This definition is not geometrical. However, amenable groups can be geometrically characterized by the Følner condition [50, proposition 4.9 on p. 131].

DEFINITION 2.6 (Følner Condition).
A group G is amenable if and only if it satisfies the *Følner condition*, i.e. for any finite set $S \subset G$ with $S^{-1} = S$ and $\epsilon > 0$ there exists a finite non-empty subset $A \subset G$ such that for its *S-boundary* $\partial_S A = \{a \in A; \exists s \in S : as \notin A\}$ we have

$$|\partial_S A| \le \epsilon |A|.$$

DEFINITION 2.7 (Proper Action).
A (topological) group action $G \curvearrowright X$ is called *proper* if for every compact subset $K \subset X$ there are only finitely many $g \in G$ such that $K \cap g \cdot K \neq \emptyset$.

REMARK 2.8. This definition of *proper* coincides with the Bourbaki definition if X is locally compact and Hausdorff (see [5, thm. 1, 4, ch. III]).

DEFINITION 2.9 (Proper and Geodesic Metric Spaces).
A metric space is called *proper* if the bounded, closed sets are compact. In particular, a proper metric space is locally compact. A metric space X is called *geodesic* if for every pair $x, y \in X$ of points there is an isometric embedding $f : [0, d(x,y)] \to X$ with $f(0) = x$ and $f(d(x,y)) = y$.

The following theorem is a key observation in geometric group theory. It implies that the fundamental group of a compact Riemannian manifold is quasi-isometric to its universal covering (with its induced metric). So fundamental groups of Riemannian manifolds with isometric universal coverings, like hyperbolic manifolds of the same dimension, are quasi-isometric. The theorem goes back to Efremovic, Milnor and Švarc. See [47], [58]. A proof can be found in [14, theorem 23 on p. 87].

THEOREM 2.10 (Isometric Group Actions and Quasi-Isometry).
Let (X, d) be a proper and geodesic metric space, and let G be a group acting by isometries on X. Assume the action is proper and cocompact. Then G is finitely generated and quasi-isometric to X. Moreover, for any $x_0 \in X$ the mapping $G \to X$, $g \mapsto g \cdot x_0$ is a quasi-isometry.

THEOREM 2.11 (Set-Theoretic Coupling).
Let Σ be a non-empty set on which the finitely generated groups G and H act from the left resp. from the right by commuting actions. Assume there is a subset $K \subset \Sigma$ satisfying the following properties.

(i) $G \cdot K = K \cdot H = \Sigma$.

(ii) The sets $\{h \in H;\ K \cdot h \cap K \neq \emptyset\}$ *and* $\{g \in G;\ g \cdot K \cap K \neq \emptyset\}$ *are finite.*

(iii) For each $g \in G$ there is a finite subset $F(g) \subset H$ such that $g \cdot K \subset K \cdot F(g)$, and for each $h \in H$ there is a finite subset $F(h) \subset G$ such that $K \cdot h \subset F(h) \cdot K$.

Then G and H are quasi-isometric.

Proof. Fix an element $x_0 \in K$, and let $\alpha : G \to H$ be a map such that $g \cdot x_0 \in K \cdot \alpha(g)$ holds for all $g \in G$. Let S be a finite, symmetric set of generators of G with respect to which we define the word metric on G. Choose also some word metric on H. We will show that α is a quasi-isometry. Recall that $l_H(h)$ denotes the length of an element $h \in H$ in the word metric, i.e. $l_H(h) = d_H(1, h)$. Put

$$F = \{h \in H; Kh \cap K \neq \emptyset\},\ k = \max\{l_H(h); h \in F\}.$$

Here the set F is finite because of (ii), so k is finite. Then $\alpha(G)$ is k-dense in H: For $h \in H$ there is always an $g \in G$ such that $g\cdot x_0 \cdot h^{-1} \in K$. So $K\cdot h \cap K \cdot \alpha(g) \neq \emptyset$ and therefore $K \cap K \cdot (\alpha(g) \cdot h^{-1}) \neq \emptyset$ holds. We obtain $d_H(h, \alpha(g)) \leq k$.

By assumption, there is a finite $T \subset H$ such that $S \cdot K \subset K \cdot T$. Define

$$t = \max\{l_H(h); h \in T\}.$$

Then we obtain

$$d_H(\alpha(g_1), \alpha(g_2)) \leq t \cdot d_G(g_1, g_2) + k \tag{2.1}$$

by the following reasoning. Write $g_2 g_1^{-1} = s_1 s_2 \cdots s_n$ with $s_i \in S$ and $n = d_G(g_1, g_2)$. Then we get

$$g_2 g_1^{-1} \cdot K = (s_1 \cdots s_n) \cdot K \subset (s_1 \cdots s_{n-1}) \cdot K \cdot T \subset \ldots \subset K \cdot T^n.$$

Because of $g_2 x_0 \alpha(g_1)^{-1} \in g_2 g_1^{-1} K \cap K \alpha(g_2)\alpha(g_1)^{-1} \neq \emptyset$ this yields

$$K \cdot T^n \cap K \cdot \alpha(g_2)\alpha(g_1)^{-1} \neq \emptyset,$$

and so $T^n\alpha(g_1)\alpha(g_2)^{-1} \subset F$. Thus we get $l_H(\alpha(g_2)\alpha(g_1)^{-1}) \leq nt + k$, implying (2.1).

The desired lower bound for $d_H(\alpha(g_1), \alpha(g_2))$ is obtained analogously, using the assumptions on G in (ii), (iii). □

REMARK 2.12. The set Σ above is called a *(set-theoretic) coupling*. The converse of the preceding theorem is also true and will follow from 2.14. There Σ is a locally compact space on which G and H act properly and cocompactly, and K can be chosen to be any compact set with $G{\cdot}K = K{\cdot}H = \Sigma$. Consequently, such a Σ is called a *topological coupling*. To stress it, if some (set-theoretic) coupling exists, then there exists a topological one. A natural question is: Can we enrich every topological coupling with an invariant measure? A positive answer would imply that every pair of quasi-isometric groups is measure equivalent (a notion defined in 2.16). This is false, in general. See 3.42. A topological coupling with an invariant measure gives rise to a so-called *bounded measure equivalence* (defined in 2.23).

Now we collect some technical statements on point set topology, which we will use frequently.

LEMMA 2.13 (Technicalities).

(i) **(Quotients of proper actions)**
Let Ω be a locally compact, second countable, Hausdorff space on which a group G acts properly. Then the quotient Ω/G is locally compact, second countable and Hausdorff.

(ii) *Let G be a group acting on a locally compact space Ω. Then for every compact set $K' \subset \Omega/G$ there is a compact set $K \subset \Omega$ such that $K' \subset p(K)$ for the projection map p.*

(iii) **(Function spaces)**
Let X be a locally compact, second countable space and Y be a locally compact, second countable, Hausdorff space. Then the space $C(X,Y)$ of continuous mappings from X to Y is metrizable and second countable (in the compact-open topology).

(iv) **(Arzela-Ascoli)**
Let X, Y be proper, metric spaces, and let $H \subset C(X,Y)$ be a subset. If there is $C > 0$ such that all $f \in H$ have Lipschitz constant at most C and $H(x) = \{f(x);\ f \in H\} \subset Y$ is relatively compact for some $x \in X$, then H is relatively compact in $C(X,Y)$. On the other hand, if $H \subset C(X,Y)$ is relatively compact, then $H(x)$ is relatively compact for all $x \in X$.

Proof. Concerning (i), being Hausdorff is the statement of [5, III, 4, prop. 3 on p. 253]. Notice that the quotient map $p : \Omega \to G\backslash\Omega$ is open. So, if U_i, $i \in I$ is a countable basis of the topology of X then $p(U_i)$ is a countable basis for $G\backslash X$. Being locally compact follows from [5, III, 4, prop. 9 on p. 257].

For (ii) choose for every $x \in \Omega$ a compact neighborhood U_x. Then the neighborhoods $p(U_x)$ cover Ω/G and by compactness one can choose a finite set $I \subset \Omega$ such that K' is covered by $\{p(U_x); x \in I\}$. Then $K := \bigcup_{x\in I} U_x$ does the job.
Part (iii) is a special case of [6, X, 3,cor. of thm. 1 on p. 299]. There Y is required

to be metrizable and uniform. But a locally compact, second countable, Hausdorff space is metrizable. Compare 1.1. Moreover, every locally compact space is uniformizable by [5, II, 4,cor. 2 of thm. 1 on p. 200].

Part (iv) follows from [6, X, 2, cor. 3 of thm. 2 on p. 292]. There it is required that H is only equicontinuous but $H(x)$ is relatively compact for every $x \in X$. So let us check in our case that $H(x_0)$ being relatively compact for some $x_0 \in X$ implies it for all $x \in X$. By the Lipschitz estimate it follows that $H(x)$ lies in the $C \cdot r$-neighborhood of $H(x_0)$ for $r = d(x, x_0)$. By properness $H(x)$ is relatively compact. □

Gromov observed [29, 0.2.C], [30, p. 94] that quasi-isometry between groups can also be characterized in terms of group actions. We provide a proof of Gromov's criterion because we could not find a complete proof in the literature.

THEOREM 2.14 (Gromov's Dynamic Criterion for Quasi-Isometry).
The finitely generated groups G and H are quasi-isometric if and only if there exists a non-empty locally compact space Ω such that G and H act properly and cocompactly on Ω and the two actions commute. Furthermore, if G and H are quasi-isometric, the non-empty space Ω can be chosen to be locally compact, second countable and Hausdorff.

Proof. Assume there exists such a space Ω as above. Let us say G acts from the left and H from the right. According to 2.13, (ii) there is a compact subset $K \subset \Omega$ such that the restrictions of the projections $p : \Omega \to G\backslash\Omega$ and $q : \Omega \to \Omega/H$ to K are surjective. Then the pair Ω, K satisfies the conditions (i)-(iii) in 2.11, where (ii), (iii) are direct consequences of the fact that the actions are proper and K is compact.

Next we show the "only-if"-part of the statement. If G and H are quasi-isometric, then there is a constant $\lambda \geq 1$ such that

$$\Omega = \left\{ \phi : G \to H;\ \frac{1}{\lambda} d_G(x,y) \leq d_H(\phi(x), \phi(y)) \leq \lambda d_G(x,y), \right.$$
$$\left. d_H(\phi(G), z) \leq \lambda \text{ for all } x, y \in X,\ z \in Y \right\}$$

is non-empty. Compare remark 2.3. The left G-action on Ω is given by $g \cdot \phi = (x \mapsto \phi(g^{-1} \cdot x))$ and the right H-action by $\phi \cdot h = (x \mapsto \phi(x) \cdot h)$. The space Ω is a closed subset of the topological space of continuous (i.e. all) maps from G to H equipped with the compact-open topology and carries the subspace topology. Concretely, a sequence in Ω converges if and only if it converges pointwise. By the technical theorem 2.13 Ω is second countable and metrizable, in particular Hausdorff.

The space Ω is locally compact: For $\phi_0 \in \Omega$ the open neighborhood $\{\phi \in \Omega;\ \phi(1) = \phi_0(1)\}$ of ϕ_0 is compact by Arzela-Ascoli.

Both actions are proper: Let $K \subset \Omega$ be a compact subset. By the Arzela-Ascoli theorem, $K(1) = \{\phi(1);\ \phi \in K\}$ is compact, i.e. finite. Suppose $g \cdot K \cap K \neq \emptyset$. Then there is a $\phi \in K$ such that $\phi(g^{-1}) \in K(1)$. Hence we can conclude

$$l_G(g) \leq \lambda \cdot d_H(\phi(g^{-1}), \phi(e)) \leq \lambda \cdot \max\{d_H(x,y);\ x, y \in K(1)\} < \infty.$$

So there can be only finitely many $g \in G$ with $g \cdot K \cap K \neq \emptyset$. Further, $K \cap K \cdot h \neq \emptyset$ for $h \in H$ implies that h lies in $\{x^{-1} \cdot y;\ x, y \in K(1)\}$, which is finite. Hence both actions are proper.

Both actions are cocompact: For every $\phi \in \Omega$ choose $h_\phi \in H$ with $\phi(1) \cdot h_\phi = 1$ and $g_\phi \in G$ with $d_H(\phi(g_\phi^{-1}), 1) \le \lambda$. Then the sets $K_H = \{\phi \cdot h_\phi;\ \phi \in \Omega\}$ and $K_G = \{g_\phi \cdot \phi;\ \phi \in \Omega\}$ are relatively compact by the Arzela-Ascoli theorem, and they surject onto $G\backslash\Omega$ resp. Ω/H under the respective quotient map. So cocompactness follows. □

2.2 Measure Equivalence

Gromov's dynamic criterion shows that cocompact discrete subgroups in the same locally compact group are quasi-isometric. In this situation there is a natural measure present, the Haar measure. Replacing *cocompact* by *finite covolume* leads to the measure-theoretic version of quasi-isometry, called *measure equivalence*, which goes back to Gromov [29, 0.5.E] and Zimmer. In recent years, the interest in measure equivalence was stimulated by the work of Alex Furman. His articles [22], [23] contain a lot of fundamental results about measure equivalence. For example, he proves [22, corollary 1.4] that Kazhdan's property (T) is an invariant under measure equivalence. As mentioned before, property (T) is not invariant under quasi-isometry.

DEFINITION 2.15 (Fundamental Domain).
Let Ω be a standard Borel space with a Borel measure μ, and let G be a group acting by Borel automorphisms on Ω. A *fundamental domain* for the G-action is a Borel subset $X \subset \Omega$ such that $g \cdot X \cap X = \emptyset$ for $g \in G - \{1\}$ and $G \cdot X = \Omega$ hold. A *measure fundamental domain* for the G-action is a Borel subset $X \subset \Omega$ such that $\mu(g \cdot X \cap X) = 0$ for $g \in G - \{1\}$ and $\mu(\Omega - G \cdot X) = 0$ hold.

DEFINITION 2.16 (Measure Equivalence).
The countable groups G and H are called *measure equivalent* if there is a standard Borel space Ω with a non-zero Borel measure on which G and H act by measure-preserving Borel automorphisms such that the actions commute and possess *measure fundamental domains* X resp. Y of finite measure.

The triple (Ω, X, Y) is called a *measure coupling* of G and H. The measure coupling is called *strict* if X and Y are fundamental domains. The *index of* (Ω, X, Y) is the quotient $\frac{\mu(X)}{\mu(Y)}$.

REMARK 2.17. Note that the actions on a measure coupling are essentially free, due to the existence of measure fundamental domains. We call Ω itself a measure coupling, when the measure fundamental domains are fixed, or the specific choice is not relevant in respective context. On a measure coupling of G and H we usually think of G acting from the left and H acting from the right.

DEFINITION 2.18 (Cocycles of a Measure Coupling).
Let (Ω, X, Y) be a measure coupling of G and H. The Borel mappings $\sigma_X : H \times X \to G$ and $\sigma_Y : G \times Y \to H$, determined up to null-sets by

$$xh \in \sigma_X(h, x)X \text{ for each } h \in H \text{ and almost all } x \in X,$$
$$gy \in Y\sigma_Y(g, y) \text{ for each } g \in G \text{ and almost all } y \in Y,$$

are called the *cocycles* of (Ω, X, Y). They are called *essentially bounded*, if the restrictions $(\sigma_X)_{|\{h\}\times X}$ and $(\sigma_Y)_{\{g\}\times Y}$ are essentially bounded for all $h \in H$,

$g \in G$ respectively[1].

REMARK 2.19. If (Ω, X, Y) were strict, then the previous definition makes also sense in an obvious strict sense.

REMARK 2.20. Measure equivalence defines an equivalence relation on countable groups [22, 2]. Transitivity can be seen as follows. Let (Ω, X, X') be a measure coupling of G and H and (Σ, Y, Y') be a measure coupling of H and K. Then $\Omega \times_H \Sigma$ gives rise to a measure coupling between G and K. The (standard) Borel structure and the measure on $\Omega \times_H \Sigma$ is defined by the identification (as a set)

$$\Omega \times_H \Sigma \cong \Omega \times Y \quad (\cong X' \times \Sigma).$$

Note that the canonical isomorphism $\Omega \times Y \cong X' \times \Sigma$ is measure preserving. The measure fundamental domain for G resp. K is $X \times Y$ resp. $X' \times Y'$.

The easiest example of measure equivalent groups are the commensurable ones. If H is a finite index subgroup of G then G with the counting measure is a measure coupling for G and H. More interesting is the example 2.31.

In [22, corollary 1.3] it is shown that the following astonishing theorem is a consequence of results in the paper [13], which itself builds on the work in [18], [19].

THEOREM 2.21 (Amenable Groups and Measure Equivalence).
The class of groups which are measure equivalent to $\mathbb{Z}$ consists precisely of all infinite countable amenable groups.

REMARK 2.22. As a consequence of the previous theorem, there are measure equivalent groups that are not quasi-isometric, for instance $\mathbb{Z}$ and $\mathbb{Z}^2$. There are also quasi-isometric groups that are not measure equivalent. An example is discussed in 3.42.

The notions in the following two definitions become important in chapter 4, where we show that Novikov-Shubin invariants of groups are invariant under bounded measure equivalence (4.24).

DEFINITION 2.23 (Bounded Measure Equivalence).
Two groups G and H are called *boundedly measure equivalent*, if they possess a measure coupling (Ω, X, Y) satisfying the following additional properties:

(i) There are finite subsets $F \subset G$, $L \subset H$ satisfying $X \subset YL$ and $Y \subset FX$ up to null-sets.

(ii) The cocycles of (Ω, X, Y) are essentially bounded.

Then (Ω, X, Y) is called a *bounded measure coupling*. We say that (Ω, X, Y) is a *strict bounded measure coupling*, if (Ω, X, Y) is a strict measure coupling, (i) is satisfied everywhere and the cocycles are bounded.

LEMMA 2.24. *If G and H have a (bounded) measure coupling (Ω, X, Y) with index C, then there exists a strict (bounded) measure coupling (Ω', X', Y') with index C such that the $G \times H$-action on Ω' is free.*

[1] Here a map to a group is called bounded if it has finite image.

Proof. Let (Ω, X, Y) be a measure coupling of G and H. Put

$$X_0 = \{x \in X; x \notin gX \text{ for } g \in G, g \neq 1\}.$$

We know that $X_0 \subset X$ has full measure. Now define

$$\Omega' = \bigcap_{h \in H} G X_0 h. \tag{2.2}$$

The Borel set Ω' is $G \times H$-invariant and has full measure in Ω. Then $X' = \Omega' \cap X_0$ is a fundamental domain for $G \curvearrowright \Omega'$, and $Y' = \Omega' \cap Y$ is still a measure fundamental domain for $\Omega' \curvearrowleft H$. Now we can repeat the same procedure with Y', whereby the fact that we have a fundamental domain for the G-action is preserved. Finally, we get a measure coupling (Ω', X', Y') with fundamental domains such that $\Omega' \subset \Omega$, $X' \subset X$ and $Y' \subset Y$ have full measure (hence with the same index). Furthermore, if (Ω, X, Y) was bounded, then (Ω', X', Y') is bounded.

Now let (Ω, X, Y) be a bounded measure coupling with (without loss of generality) fundamental domains X, Y. We show that it be can be replaced by one with bounded cocycles. By assumption, there is for every $h \in H$ a finite set $F(h) \subset G$ such that $Xh \subset F(h)X$ holds up to null-sets. Therefore

$$X_0 = \{x \in X;\ xh \in F(h)X \text{ for all } h \in H\} \subset X$$

has full measure. Now define Ω' as in (2.2). Then $X' = \Omega' \cap X_0$ is a fundamental domain for $G \curvearrowleft \Omega'$, and $Y' = \Omega' \cap Y$ is a fundamental domain for $\Omega' \curvearrowright H$. Because X is a fundamental domain for Ω, we have $X' = \Omega' \cap X$. Thereby we get that $xh \in F(h)X'$ for every $x \in X'$. So (Ω', X', Y') has a bounded cocycle σ_X. Applying the same procedure to Y' preserves σ_X being bounded, and we finally get a measure coupling (Ω', X', Y') with bounded cocycles and fundamental domains such that $\Omega' \subset \Omega$, $X' \subset X$ and $Y' \subset Y$ have full measure.

Now let (Ω, X, Y) be a bounded measure coupling with fundamental domains X, Y, bounded cocycles satisfying (i) in 2.23 (non-strictly). Then we can argue similarly as before using the set $X_0 = \{x \in X; x \in YL\} \subset X$ to get a strict measure coupling (Ω', X', Y') such that $\Omega' \subset \Omega$, $X' \subset X$ and $Y' \subset Y$ have full measure.

Next we show that a strict (bounded) measure coupling (Ω, X, Y) can be replaced by one with a free $G \times H$-action. Let F be a probability space with an essentially free $G \times H$- action as in 1.17. Replacing F by its subset of points with trivial isotropy, we can assume that the $G \times H$-action on F is free. The diagonal action of $G \times H$ on $\Omega' = \Omega \times F$ is free, and $(\Omega', X \times F, Y \times F)$ is a strict (bounded) measure coupling of G and H with the same index as (Ω, X, Y), where Ω' is equipped with the product measure. ☐

LEMMA 2.25 (Bounded ME implies QI).
Finitely generated boundedly measure equivalent groups are quasi-isometric.

Proof. Let (Ω, X, Y) be a strict bounded measure coupling of the finitely generated groups G and H. Strictness is no restriction by the preceding lemma. Put $K = X \cup Y$. It suffices to show that $K \subset \Omega$ satisfies the conditions (i)-(iii) in 2.11, where (i) is immediately clear. By assumption, X and Y are contained in finitely many translates of Y resp. X. Thus there are finite subsets $F \subset G$, $L \subset H$ with $K \subset FX$ and $K \subset YL$. This implies (ii) because X and Y are fundamental domains. Further, this yields (iii) because of the cocycles being bounded. $\square$

DEFINITION 2.26 (Weak Orbit Equivalence).
The standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are called *weakly orbit equivalent*, if there are Borel subsets $A \subset X$, $B \subset Y$ meeting almost every orbit and a Borel isomorphism $f : A \to B$, which preserves the normalized measures on A resp. B and satisfies

$$f(G \cdot x \cap A) = H \cdot f(x) \cap B \text{ for almost all } x \in A. \tag{2.3}$$

The map f is called a *weak orbit equivalence*. If A and B have full measure in X resp. Y, then $G \curvearrowright X$ and $H \curvearrowright Y$ are called *orbit equivalent*.

The *index* of a weak orbit equivalence $f : A \to B$ is the quotient $\frac{\mu(B)}{\mu(A)}$.

REMARK 2.27. We remark that the inverse of a weak orbit equivalence is a weak orbit equivalence. For the definition of weak orbit equivalence it is equivalent to require that (2.3) holds for all $x \in A$ by the following argument. Let $A' \subset A$ be a Borel subset of full measure such that (2.3) holds for all $x \in A'$. Then $f_{|A'}$ is a Borel isomorphism onto $f(A') = B'$, and B' is a Borel subset of B. See 1.2. It is easy to see that $f : A' \to B'$ is a weak isomorphism satisfying (2.3) everywhere.

REMARK 2.28 (Weak Orbit Equivalence in Terms of Groupoids).
Note that the standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are weakly orbit equivalent if and only if the associated orbit equivalence relations are weakly isomorphic as discrete measured groupoids. See 1.13. We remind the reader that the notion of a standard action and of an isomorphism of discrete measured groupoids have been defined in 1.16, 1.13 respectively.

DEFINITION 2.29 (Cocycles of a Weak Orbit Equivalence).
Let $f : A \to B$ be a weak orbit equivalence between the standard actions $G \curvearrowright X$ and $H \curvearrowright Y$. The Borel mapping σ from the subset $\{(g,a);\ g \in G, a \in A \cap g^{-1}A\} \subset G \times A$ to H, determined up to null-sets by the condition $f(ga) = \sigma(g,a)f(a)$ is called the *cocycle* of f. We say that σ is *essentially bounded*, if for each $g \in G$ the restriction $\sigma_{|\{g\} \times (A \cap g^{-1}A)}$ is essentially bounded.

DEFINITION 2.30 (Bounded Weak Orbit Equivalence).
The standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are called *boundedly weakly orbit equivalent* if there is a weak orbit equivalence $f : A \to B$ with the following additional properties:

(i) There are finite subsets $F \subset G$, $L \subset H$ satisfying $X = FA$, $Y = LB$ up to null-sets.

(ii) The cocycles of f and f^{-1} are essentially bounded.

EXAMPLE 2.31 (Lattices in Locally Compact Groups).
Discrete subgroups K, L with **finite covolume** (called *lattices*) in the same locally compact, second countable Hausdorff group G equipped with a (left invariant) Haar measure μ are **measure equivalent** with G as a measure coupling.

A (left invariant) Haar measure on G is a non-trivial, regular Borel measure which is invariant under left translations. *Finite covolume* means that the (left) actions of K and L on G have measure fundamental domains of finite measure. If G has a lattice then every left invariant Haar measure is right invariant and vice versa by the lemma below. Therefore G is a measure coupling with K acting from the left and L acting from the right by multiplication.

Every discrete **cocompact** subgroup $H \subset G$ has finite covolume, in particular: The action of such an H on G by left (or right) translations is proper. By 2.13 the quotient is locally compact, second countable and Hausdorff, so it is a standard Borel space by 1.1. There is a compact $K \subset G$ such that $p(K) = G/H$ for the quotient map $p : G \to G/H$. The map p is countable-to-1, and hence there is a Borel set $X \subset K$ with $p(X) = G/H$ and $p_{|X}$ being injective (see 1.3). So X is a relatively compact, fundamental domain, and regularity yields $\mu(X) \leq \mu(K) < \infty$.

Two discrete, cocompact subgroups in the same locally compact, second countable Hausdorff group G are **boundedly measure equivalent**, and G is the bounded measure coupling, of course. The conditions (i), (ii) are direct consequences of the fact that the fundamental domains are relatively compact, and the actions on G are proper.

LEMMA 2.32 (Unimodularity).
Let G be a locally compact Hausdorff group. If G has a discrete subgroup of finite covolume then G is unimodular, *i.e. every left invariant Haar measure is also right invariant.*

Proof. Let μ be a left invariant Haar measure of G, and let $\Gamma \subset G$ a lattice with finite covolume. Denote by $X \subset G$ a finite measure fundamental domain for $\Gamma \curvearrowright G$. By [10, p. 313] there is a Borel function $\Delta : G \to \mathbb{R}_{>0}$, called the *modular function*, such that

$$\mu(Y \cdot h) = \Delta(h) \cdot \mu(Y)$$

for every Borel subset $Y \subset G$ and each $h \in G$. We want to show $\Delta(h) = 1$ for every $h \in G$. Fix $h \in G$. Then $X \cdot h$ is also a measure fundamental domain for Γ. Therefore we obtain

$$\begin{aligned}\Delta(h) \cdot \mu(X) = \mu(X \cdot h) &= \sum_{g \in \Gamma} \mu(X \cdot h \cap g \cdot X) \\ &= \sum_{g \in \Gamma} \mu(g^{-1} \cdot X \cdot h \cap X) \\ &= \mu(X).\end{aligned}$$

Now $0 < \mu(X) < \infty$ implies $\Delta(h) = 1$. □

The following theorem is formulated and proven in [23, theorem 3.3] (without the bounded case), where credit is also given to Zimmer and Gromov.

THEOREM 2.33 (Equivalence of ME and WEO).
The countable groups G, H are measure equivalent with respect to a measure coupling of index C if and only if there exist standard actions of G and H that are weakly orbit equivalent by a weak orbit equivalence of index C. Further, if G and H are boundedly measure equivalent, then there are boundedly weakly orbit equivalent standard actions of G and H.

Proof. Let (Ω, X, Y) be a bounded measure coupling of G and H with measure μ. We show that there are boundedly weakly orbit equivalent standard actions of G and H. By 2.24 we can assume that the measure coupling is strict, and the $G \times H$-action on Ω is free. As always, we consider G as acting from the left and H from the right. Notice that we can convert left into right actions, so a weak orbit equivalence between a left and a right action makes perfectly sense.

The bounded cocycles of (Ω, X, Y) are denoted by σ_X, σ_Y as in 2.18. With the identifications (as sets) $X = G\backslash\Omega$ and $Y = \Omega/H$ one sees that there are natural free actions of H resp. G on X resp. Y. They are explicitly given by

$$\begin{aligned} x \cdot h &= \sigma_X(h, x)^{-1} x h \quad \text{for } x \in X,\ h \in H, \\ g \cdot y &= g y \sigma_Y(g, y)^{-1} \quad \text{for } y \in Y,\ g \in G. \end{aligned}$$

Both actions are μ-preserving because they are piecewise μ-preserving.

For $x \in \Omega$ let $\gamma(x) \in H$ be the unique element in H such that $x\gamma(x) \in Y$, and let $\delta(x) \in G$ be the unique element with $\delta(x)x \in X$. The mappings $\gamma : X \to H$, $\delta : Y \to G$ are bounded because (Ω, X, Y) is strictly bounded.

The Borel map $p : \Omega \to Y, x \mapsto x\gamma(x)$ is G-equivariant and countable-to-1. Moreover, p is measure preserving on Borel sets on which it is injective. There is a Borel set $A \subset X$ such that $f = p_{|A}$ is a Borel isomorphism onto the image $B = p(X)$, due to 1.3. The map f is measure preserving. One easily sees that, if we normalize the measures on X, Y, then f is a weak orbit equivalence with the same index as the measure equivalence, but we need to be more explicit in order to prove that it is bounded.

Because of the equivariance of p we have $f(a \cdot h) = p(\sigma_X(h, a)^{-1} a h) = \sigma_X(h, a)^{-1} \cdot p(a) = \sigma_X(h, a)^{-1} \cdot f(a)$, so the cocycle of f is bounded. For the cocycle of f^{-1} consider $x, w \in A$, $g \in G$ with $g \cdot f(x) = f(w)$. This implies $g x \gamma(x) \sigma_Y(g, f(x))^{-1} = w\gamma(w)$, hence $x \cdot (\gamma(x)\sigma_Y(g, f(x))^{-1}\gamma(w)^{-1}) = w$. Because γ and σ_Y are bounded, we get that there is a finite set $F(g) \subset H$, depending only on g, such that $w \in x \cdot F(g)$. Therefore we have $f^{-1}(g \cdot f(x)) = w \in f^{-1}(f(x)) \cdot F(g)$, and so the cocycle of f^{-1} is bounded.

Now let $y \in Y$. Then we have $y = p(\delta(y)^{-1}(\delta(y)y)) \in \delta(y)^{-1} \cdot B$. Because δ is bounded, we obtain that finitely many translates of B cover Y. For every $x \in X$ there is $a \in A$ with $x\gamma(x) = p(x) = p(a) = a\gamma(a)$, hence $x = a\gamma(a)\gamma(x)^{-1} = a \cdot (\gamma(a)\gamma(x)^{-1})$. Thus finitely many translates of A cover X because γ is bounded.

The proof for the non-bounded case is included in the reasoning above. For the converse of the statement, we refer to Furman's proof in [23, theorem 3.3]. □

REMARK 2.34. We remark that the converse of the preceding theorem is also true in the bounded case. Having no application for it, we omit its (slightly tedious) proof.

2.3 Quasi-Isometry versus Measure Equivalence

The following theorem is well known for Riemannian manifolds. The more general case is shown in the same way.

THEOREM 2.35 (Group of Isometries).
Let X be a second countable, proper metric space. Then the group of isometries $\mathrm{Isom}(X)$ *with its compact-open topology is locally compact, second countable and Hausdorff. Furthermore, if G is a group acting properly and cocompactly on X by isometries, then G is a discrete and cocompact subgroup of* $\mathrm{Isom}(X)$.

Proof. By lemma 2.13 we know that $\mathrm{Isom}(X)$ is second countable and Hausdorff, even metrizable. Let $\phi \in \mathrm{Isom}(X)$ and $x \in X$. Choose a compact neighborhood V of $y = \phi(x)$. Then $U_\phi = C(\{x\}, V) \cap \mathrm{Isom}(X)$ is a relatively compact neighborhood of ϕ by Arzela-Ascoli. So $\mathrm{Isom}(X)$ is locally compact.

Now we prove the second part of the statement. Let $g_0 \in G$ and $\epsilon > 0$. There are only finitely many $g \in G$ such that $g \cdot x \in B_\epsilon(g_0 \cdot x)$ (ball of radius ϵ) for fixed $x \in X$ and $\epsilon > 0$ because the G-action is proper. So $U := \{f \in \mathrm{Isom}(X);\ f(x) \in B_\epsilon(g_0 \cdot x)\}$ is an open neighborhood of g_0 such that $U \cap G$ is finite. This implies that G is a discrete subgroup because $\mathrm{Isom}(X)$ is Hausdorff. The argument for cocompactness is essentially the same as in the proof of Gromov's criterion. Fix $x_0 \in X$. Since $G \curvearrowright X$ is cocompact there is a compact $K \subset X$ with $G \cdot K = X$. Choose for every $\phi \in \mathrm{Isom}(X)$ an element $g_\phi \in G$ with $g_\phi \cdot \phi(x) \in K$. Then $K_G = \{g_\phi \cdot \phi;\ \phi \in \mathrm{Isom}(X)\} \subset \mathrm{Isom}(X)$ is relatively compact by the Arzela-Ascoli theorem 2.13, and it surjects onto the quotient of $\mathrm{Isom}(X)$ by G. □

The following theorem should be compared to 2.10.

THEOREM 2.36 (Isometric Group Actions and Measure Equivalence).
Let X be a second countable, proper metric space, and let G and H be groups acting properly and cocompactly on X by isometries. Then G and H are boundedly measure equivalent.

Proof. By the previous theorem 2.35 G and H are cocompact lattices in the locally compact, second countable, Hausdorff group $\mathrm{Isom}(X)$. As explained in 2.31, it follows that G and H are boundedly measure equivalent. □

The following fundamental theorem can be found in [10, proposition 7.2.8 on p. 209]

THEOREM 2.37 (Riesz Representation Theorem).
Let Ω be a locally compact Hausdorff space, and let I be a positive linear functional on the continuous functions $C_c(\Omega)$ with compact support. Then there is a unique regular Borel measure μ on Ω such that

$$I(f) = \int_\Omega f d\mu$$

holds for each $f \in C_c(\Omega)$.

THEOREM 2.38 (QI⇒BME for Amenable Groups).
If two finitely generated amenable groups are quasi-isometric, then they are boundedly measure equivalent.

Proof. Let G and H be finitely generated, amenable, quasi-isometric groups. We remind the reader that the notion of a standard action and of an isomorphism of discrete measured groupoids have been defined in 1.16, 1.13 respectively. There is a non-empty locally compact and second countable Hausdorff space Ω on which G and H act properly and cocompactly such that the two actions commute (see 2.14). First we equip Ω with a $G \times H$-invariant measure.

Fix a point $x_0 \in \Omega$. Let $f \in C_c(\Omega)$ be a continuous function on Ω with compact support. The function $\phi_f : G \to \mathbb{R}$, defined by $\phi_f(g) = \sum_{h \in H} f(gx_0h)$, is bounded: First of all, the sum is finite for every $g \in G$ because the H-action on Ω is proper and f has compact support. Let $K \subset \Omega$ be a compact subset such that $K \cdot H = \Omega$ (compare 2.13, (ii)). For the function $\Sigma(f) \in C(\Omega)$, given by $\Sigma(f)(x) = \sum_{h \in H} f(xh)$, we get $\operatorname{im}(\Sigma(f)) = \operatorname{im}\left(\Sigma(f)_{|K}\right)$. But $\Sigma(f)_{|K}$ is bounded, hence $\Sigma(f)$ and ϕ_f are bounded.

Now let $\mu_G : l^\infty(G, \mathbb{R}) \to \mathbb{R}$ be a mean for G as in 2.5. The positive linear functional $I : C_c(\Omega) \to \mathbb{R}$ is defined by $I(f) = \mu_G(\phi_f)$. Clearly, the functional I is invariant under the $G \times H$-action on $C_c(\Omega)$, induced by the $G \times H$-action on Ω. Due to the Riesz representation theorem (2.37), I induces a $G \times H$-invariant, regular measure μ on Ω. To see that μ is non-zero, choose a non-negative function $f \in C_c(\Omega)$ with $f_{|K} \equiv 1$ (which exists by Uryson's lemma). Then $\phi_f(g) \geq 1$ for all $g \in G$, and so $I(f) \geq 1$ due to properties of the mean μ_G.

Now we show that we can replace Ω by another space with the same properties as Ω, on which G and H act essentially freely. Due to 1.17 there is a compact, second countable Hausdorff space (hence a Polish space) S with a Borel probability measure on which G acts measure-preserving and essentially freely ($S = \{0,1\}^G$ for infinite G). As every finite Borel measure on a Polish space is regular [10, proposition 8.1.10 on p. 258], this measure is regular. The groups G and H act on $\Omega' = \Omega \times S$ measure-preserving by $g(x,s) = (gx, gs)$ resp. $(x,s)h = (xh, s)$, where we equip Ω' with the product measure. This product measure is a regular Borel measure [10, proposition 7.6.2 on p. 242]. The space Ω' is again locally compact, second countable and Hausdorff. Obviously, the actions on Ω' are commuting, proper, cocompact. Moreover, the G-action is essentially free. If the H-action on Ω is not essentially free, then we repeat the same procedure for $\Omega' \curvearrowleft H$. So we can assume that the G- and H-action on Ω are essentially free, without loss of generality.

We want to show that Ω is a bounded measure coupling, so first we have to ensure the existence of measure fundamental domains. The quotient space Ω/H is locally compact, second countable and Hausdorff because the H-action is proper. See 2.13, (i). Therefore Ω/H (with its Borel algebra) is a standard Borel space (see 1.1). The projection $p : \Omega \to \Omega/H$ is countable-to-1, and there is a compact subset $K \subset \Omega$ such that $p(K) = \Omega/H$ (2.13, (ii)). By 1.3 there is a Borel subset $Y \subset K$ such that $p(Y) = \Omega/H$ and $p_{|Y}$ is a Borel isomorphism. Thus we get $Y \cdot H = \Omega$ and $\mu(Yh \cap Y) = 0$ for $h \neq 1$ because the action is essentially free. Furthermore, we have $\mu(Y) \leq \mu(K) < \infty$ because μ is regular. So Y is a measure fundamental domain of finite measure for $\Omega \curvearrowleft H$. Analogously, the action $G \curvearrowright \Omega$ has a relatively compact measure fundamental domain X.

Now we can conclude that Ω is a bounded measure coupling. The conditions (i), (ii) in the definition 2.23 follow easily from X, Y being relatively compact and $G \curvearrowright \Omega$, $\Omega \curvearrowleft H$ being proper. For example, we know that $Y \subset G \cdot X$, hence there is a finite subset $F_X \subset G$ with $Y \subset F_X \cdot X$ because of $G \curvearrowright \Omega$ being proper. $\square$

Chapter 3

L^2-Betti Numbers

L^2-Betti numbers were invented by Atiyah in his study of elliptic operators on covering spaces [3]. The p-th L^2-Betti number $b_p^{(2)}(\tilde{M})$ of a compact Riemannian manifold is a priori a non-negative real number which measures the size of the smooth harmonic L^2-integrable p-forms on the universal covering $\tilde{M}$ of M. There are no such non-trivial forms if and only if $b_p^{(2)}(\tilde{M})$ vanishes. We can express $b_p^{(2)}(\tilde{M})$ in terms of the heat kernel.

$$b_p^{(2)}(\tilde{M}) = \lim_{t\to\infty} \int_{\mathcal{F}} \mathrm{tr}_{\mathbb{R}} \left(e^{-t\Delta_p}(x,x) \right) dvol_x$$

Here Δ_p is the Laplacian acting on the L^2-integrable p-forms, $e^{-t\Delta_p}(x,y)$ is the integral kernel (heat kernel) of the operator $e^{-t\Delta_p}$ understood in the sense of spectral calculus. In particular, $e^{-t\Delta_p}(x,x) : \mathrm{Alt}^p(T\tilde{M}) \to \mathrm{Alt}^p(T\tilde{M})$ is a linear operator for every $x \in \tilde{M}$, of which you can take the ordinary trace $\mathrm{tr}_{\mathbb{R}}(e^{-t\Delta_p}(x,x)) \in \mathbb{R}$. The letter $\mathcal{F}$ denotes a fundamental domain for the $\pi_1(M)$-action on $\tilde{M}$.

On a technical level, Atiyah considers the space of L^2-integrable harmonic p-forms as a Hilbert module over the von Neumann algebra of the fundamental group. Then $b_p^{(2)}(\tilde{M})$ is its dimension in the sense of von Neumann.

Nowadays, L^2-Betti numbers of spaces with a G-action can be defined by the twisted singular homology with coefficients in the group von Neumann algebra of G. The key ingredient for that is the dimension theory for arbitrary modules over finite von Neumann algebras developed by W. Lück in [42], [43], [44].

Throughout this chapter the letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{N}, \mathcal{M}$ always denote finite von Neumann algebras, which come equipped with normalized traces.

3.1 Dimension Theory

We begin by reviewing the dimension function for modules (in the algebraic sense) over a finite von Neumann algebra and explain the L^2-Betti numbers of topological spaces. For a complete, systematic account of dimension theory see Lück's book [45].

DEFINITION 3.1. Let P be a finitely generated projective $\mathcal{N}$-module. Choose an idempotent matrix $A = (A_{ij}) \in M_n(\mathcal{N})$ such that $P \cong \mathcal{N}^n \cdot A$. Then the dimension $\dim_{\mathcal{N}}(P)$ is defined as

$$\dim_{\mathcal{N}}(P) = \sum_{i=1}^{n} \mathrm{tr}_{\mathcal{N}}(A_{ii}) \in [0, \infty).$$

One can easily show that this definition is independent of the choice of A.

REMARK 3.2. It is not necessary but possible to require $A = A^*$ in the previous definition.

DEFINITION 3.3. For an arbitrary $\mathcal{N}$-module N define

$$\dim_{\mathcal{N}}(N) = \sup\{\dim_{\mathcal{N}}(P);\ P \subset N \text{ fin. gen. projective submodule}\} \in [0, \infty].$$

For finitely generated projective modules this definition is consistent with the previous one. Of course, the idea of extending the definition of $\dim_{\mathcal{N}}$ from projective to arbitrary modules as in 3.3 is the obvious one, but you cannot expect to get a notion with reasonable properties for general rings.

Now we can go on and define L^2-Betti numbers of spaces and groups as the dimension of the twisted singular homology with coefficients in the group von Neumann algebra.

DEFINITION 3.4. Let X be topological space with an action of the group G. Then the *n-th L^2-Betti number of X* is defined as the dimension of the n-th homology of the chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} \mathrm{C}^{\mathrm{sing}}_\bullet(X)$, where $\mathrm{C}^{\mathrm{sing}}_\bullet(X)$ is the singular chain complex of X:

$$b^{(2)}_n(X; \mathcal{N}(G)) = \dim_{\mathcal{N}(G)}\big(H_n(X; \mathcal{N}(G))\big).$$

The *n-th L^2-Betti number of a group G* is given by

$$b^{(2)}_n(G) = \dim_{\mathcal{N}(G)}\big(H_n(EG; \mathcal{N}(G)\big),$$

where EG is any universal free G-space.

REMARK 3.5. This definition coincides with the heat kernel definition in the case of universal coverings of compact Riemannian manifolds. For a proof, based on [16], see [45, 1.4.2]. Note that it is obvious from the definition above (but not from the heat kernel definition) that the $b^{(2)}_i(X)$ are G-homotopy invariants for G-spaces.

REMARK 3.6. The homology $H_\bullet(EG; \mathcal{N}(G))$ equals the group homology with coefficients in the $\mathbb{Z}G$-module $\mathcal{N}(G)$. Group homology can be expressed as the derived functor of the tensor product $- \otimes_{\mathbb{Z}G} \mathbb{Z}$:

$$H_\bullet(EG; \mathcal{N}) = \mathrm{Tor}^{\mathbb{Z}G}_\bullet(\mathcal{N}(G), \mathbb{Z}).$$

Thus the L^2-Betti number of a group can be expressed purely by homological algebra.

We remark that, in general, the values of the L^2-Betti numbers of the universal covering $\tilde{M}$ of M are not related to the values of the classical Betti numbers of M, although the alternating sum equals the Euler characteristic of M in both cases. See [45, theorem 1.35] for a proof.

THEOREM 3.7 (Euler-Poincare formula).
Let M be a finite CW-complex. Then

$$\sum_{i=0}^{\dim M} (-1)^i b_i^{(2)}(\tilde{M}) = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M.

We return to abstract dimension theory and list some fundamental properties of the dimension function. The following theorem is taken from [43, Theorem 0.6]. See also [45, Theorem 6.7 on p.239,].

THEOREM 3.8. *The dimension function $\dim_{\mathcal{N}}$ satisfies the following properties.*

(i) A projective $\mathcal{N}$-module P is trivial if and only if $\dim_{\mathcal{N}}(P) = 0$.

(ii) Additivity.
If $0 \to A \to B \to C \to 0$ is an exact sequence of $\mathcal{N}$-modules then

$$\dim_{\mathcal{N}}(B) = \dim_{\mathcal{N}}(A) + \dim_{\mathcal{N}}(C)$$

holds, where we put $\infty + r = r + \infty = \infty$ for $r \in [0, \infty]$.

(iii) Cofinality.
Let $M = \bigcup_{i \in I} M_i$ be a directed union of submodules $M_i \subset M$. Then

$$\dim_{\mathcal{N}}(M) = \sup_{i \in I}\{\dim_{\mathcal{N}}(M_i)\}.$$

DEFINITION 3.9 (Dimension Isomorphism).
An $\mathcal{N}$-homomorphism $f : M \to N$ between $\mathcal{N}$-modules M, N is called a $\dim_{\mathcal{N}}$-*isomorphism* if $\dim_{\mathcal{N}}(\ker f) = \dim_{\mathcal{N}}(\operatorname{coker} f) = 0$.

As a slogan, $\dim_{\mathcal{N}}$-isomorphisms should be dealt with like isomorphisms. Indeed, there is a suitable localization of the category of $\mathcal{N}$-modules in which $\dim_{\mathcal{N}}$-isomorphisms become isomorphisms. Let us recall the relevant notions.

Let $\mathcal{A}$ be an abelian category. A non-empty full subcategory $\mathcal{B}$ of $\mathcal{A}$ is called a *Serre subcategory* if for all short exact sequences in $\mathcal{A}$

$$0 \to M' \to M \to M'' \to 0$$

M belongs to $\mathcal{B}$ if and only if both M' and M'' do. Then there is a quotient category $\mathcal{A}/\mathcal{B}$ with the same objects as $\mathcal{A}$ and a functor $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{B}$. Moreover, $\mathcal{A}/\mathcal{B}$ is abelian, π is exact and $\pi(f)$ is an isomorphism if and only if $\ker(f)$ and $\operatorname{coker}(f)$ lie in $\mathcal{B}$ for a morphism f in $\mathcal{A}$. The category $\mathcal{A}/\mathcal{B}$ can be characterized by the following property. Let $\mathcal{C}$ another abelian category. Let $\phi : \mathcal{A} \to \mathcal{C}$ be an exact functor such that $\phi(M) = 0$ for $M \in \mathrm{ob}\,\mathcal{B}$. Then there is a functor $\psi : \mathcal{A}/\mathcal{B} \to \mathcal{C}$, unique up to natural isomorphism, such that the diagram

$$\begin{array}{ccc} \mathcal{A} & & \\ \downarrow{\scriptstyle \pi} & \searrow^{\phi} & \\ \mathcal{A}/\mathcal{B} & \xrightarrow{\psi} & \mathcal{C} \end{array}$$

commutes up to natural equivalence.

The properties in 3.8 imply that the subcategory

$$\mathcal{N}\mathrm{-Mod}_0 = \{\, \mathcal{N}\text{-modules } M \text{ with } \dim_{\mathcal{N}}(M) = 0\} \subset \mathcal{N}\mathrm{-Mod}$$

is Serre. This has useful consequences. For instance, there is a 5-lemma for $\dim_{\mathcal{N}}$-isomorphisms because there is one for general abelian categories.

The following theorem can be found in [42, Lemma 3.4]. See also [45, Lemma 6.28 on p. 252].

THEOREM 3.10. *Every finitely generated $\mathcal{N}$-module N splits as*

$$N = \mathbf{P}N \oplus \mathbf{T}N$$

where $\mathbf{P}N$ is finitely generated projective and $\mathbf{T}N$ is the kernel of the canonical homomorphism $N \to N^{**}$ into the double dual, mapping $x \in N$ to $N^* \to \mathcal{N}, f \mapsto f(x)$. Furthermore, $\dim_{\mathcal{N}}(\mathbf{T}N) = 0$ holds. The modules $\mathbf{P}N$ and $\mathbf{T}N$ are called the projective *resp.* torsion part of N.

The next theorem is taken from [42, lemma 3.4].

THEOREM 3.11. *If N is a finitely presented $\mathcal{N}$-module then the torsion part $\mathbf{T}N$ possesses an exact resolution of the form*

$$0 \to \mathcal{N}^n \xrightarrow{\cdot A} \mathcal{N}^n \to \mathbf{T}N \to 0,$$

where $A \in M_n(\mathcal{N})$ is positive.

The following theorem [42, theorem 1.2] is a consequence of $\mathcal{N}$ being semihereditary.

THEOREM 3.12. *The subcategory of finitely presented modules in the category of $\mathcal{N}$-modules is abelian.*

The next theorem is exercise 6.3. (with solution on p. 530!) in [45, p. 289]. It is formulated for group von Neumann algebras there, but the proof is exactly the same for arbitrary finite von Neumann algebras.

THEOREM 3.13. *Let M be a submodule of a finitely generated projective $\mathcal{N}$-module P. For every $\epsilon > 0$, there is submodule $P' \subset M$, which is a direct summand in P and satisfies $\dim_{\mathcal{N}}(M) \le \dim_{\mathcal{N}}(P') + \epsilon$.*

Now we prove some more results about dimension theory, needed in the next section.

LEMMA 3.14. *Let $(p_i)_{i\in\mathbb{N}}$ be a sequence of projections in $\mathcal{N}$ such that $\mathrm{tr}_{\mathcal{N}}(p_i) \to 1$ for $i \to \infty$. Then $(p_i)_{i\in\mathbb{N}}$ converges to the identity in the strong operator topology.*

Proof. The finite von Neumann algebra $\mathcal{N}$ is identified with the subalgebra of equivariant operators on $l^2\mathcal{N}$. The operator on $l^2\mathcal{N}$ represented by p_i is denoted by P_i. We have to show

$$\lim_{i\to\infty} P_i(x) = x \tag{3.1}$$

for $x \in l^2\mathcal{N}$. If suffices to show (3.1) for unitary $x \in \mathcal{N}$ because $\mathcal{N} \subset l^2\mathcal{N}$ is dense, and every element in $\mathcal{N}$ is the sum of four unitary elements [33, theorem 4.1.7 on p. 242]. But for unitary $u \in \mathcal{N}$ we have

$$\begin{aligned}\|u - P_i(u)\|_{l^2\mathcal{N}} = \operatorname{tr}_{\mathcal{N}}(u(1-p_i)(1-p_i)^* u^*) &= \operatorname{tr}_{\mathcal{N}}((1-p_i)(1-p_i)^*)\\ &= \operatorname{tr}_{\mathcal{N}}(1-p_i) \to 0.\end{aligned}$$

□

LEMMA 3.15. *Let $n \in \mathcal{N}$ be an element in a finite von Neumann algebra $\mathcal{N}$. Assume there exists a sequence of projections $p_i \in \mathcal{N}$ such that $\operatorname{tr}_{\mathcal{N}}(p_i) \to 1$ and $p_i \cdot n = 0$. Then n must be zero.*

Proof. It is an easy observation that the map $\mathcal{N} \to \mathcal{N}, m \mapsto m \cdot n$ is continuous with respect to the weak operator topology[1]. By the preceding lemma, the p_i converge strongly (hence weakly) to 1. Thus $0 = p_i n \to n$, and the claim follows. □

The following theorem is a local criterion for the vanishing of the dimension of modules.

THEOREM 3.16 (Local Criterion).
Let M be an $\mathcal{N}$-module. Its dimension $\dim_{\mathcal{N}}(M)$ vanishes if and only if for every element $m \in M$ there is a sequence $p_i \in \mathcal{N}$ of projections such that

$$\lim_{i\to\infty} \operatorname{tr}_{\mathcal{N}}(p_i) = 1 \ \textit{ and } \ p_i \cdot m = 0 \textit{ for all } i \in \mathbb{N}.$$

Furthermore, if $q \in \mathcal{N}$ is a given projection with $qm = 0$ for an element m in M with $\dim_{\mathcal{N}}(M) = 0$, then the sequence p_i can be chosen such that $q \le p_i$.

Proof. First assume $\dim_{\mathcal{N}}(M) = 0$. Consider an element $m \in M$, and let $q \in \mathcal{N}$ be a projection such that $qm = 0$. For a given $\epsilon > 0$, we want to construct a projection $p \in \mathcal{N}$ such that $\operatorname{tr}_{\mathcal{N}}(p) \ge 1 - \epsilon$, $p \cdot m = 0$ and $p \ge q$.
Let $\langle m \rangle \subset M$ be the submodule generated by m. We have the epimorphism

$$\phi : \mathcal{N}(1-q) \to \langle m \rangle,\ n(1-q) \mapsto nm$$

and $\dim_{\mathcal{N}}(\ker\phi) = \dim_{\mathcal{N}}(\mathcal{N}(1-q)) - \dim_{\mathcal{N}}(\langle m \rangle) = 1 - \operatorname{tr}_{\mathcal{N}}(q)$. By 3.13 there is a submodule $P \subset \ker\phi$ such that P is a direct summand in $\mathcal{N}(1-q)$ and $\dim_{\mathcal{N}}(\ker\phi) \le \dim_{\mathcal{N}}(P) + \epsilon$. Hence $\mathcal{N}q \oplus P \subset \mathcal{N}q \oplus \mathcal{N}(1-q) = \mathcal{N}$ is a direct summand in $\mathcal{N}$, i.e. it is has the form $\mathcal{N}p$ for a projection p. Its trace is $\operatorname{tr}_{\mathcal{N}}(p) = \operatorname{tr}_{\mathcal{N}}(q) + \dim_{\mathcal{N}}(P) \ge 1 - \epsilon$. Moreover, $\mathcal{N}q \subset \mathcal{N}p$ implies $qp = q$, i.e. $q \le p$, and $pm = 0$ is obvious.

Now we prove the converse. It suffices to prove that M has no non-trivial

[1]We remark that multiplication as a map from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$ is not weakly continuous.

finitely generated projective submodules (see 3.3).
Suppose $P \subset M$ is a non-trivial finitely generated projective submodule. Then there is a non-trivial $\mathcal{N}$-homomorphism $\phi : P \to \mathcal{N}$. Choose non-zero element $y = \phi(x) \neq 0$ in the image of ϕ. There is a sequence of projections $p_i \in \mathcal{N}$ such $\mathrm{tr}_{\mathcal{N}}(p_i) \to 1$ and $p_i \cdot x = 0$. In particular, $p_i \cdot y = \phi(p_i \cdot x) = 0$. But by the previous lemma 3.15 this yields $y = 0$. So, no such non-trivial P can exist. □

THEOREM 3.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite von Neumann algebras, and let F be an exact functor from the category of $\mathcal{A}$-modules to the category of $\mathcal{B}$-modules which preserves colimits. Assume there is a constant $C > 0$ such that*

$$\dim_{\mathcal{B}}(F(P)) = C \cdot \dim_{\mathcal{A}}(P) \tag{3.2}$$

holds for every finitely generated projective $\mathcal{A}$-module P. Then $\dim_{\mathcal{B}}(F(M)) = C \cdot \dim_{\mathcal{A}}(M)$ holds for every $\mathcal{A}$-module M.

Proof. We prove this for finitely presented, finitely generated and arbitrary modules, subsequently.
Step 1: Let M be a finitely presented $\mathcal{A}$-module.

Then M splits as $M = \mathbf{P}M \oplus \mathbf{T}M$ (3.10, 3.11), where $\mathbf{P}M$ is projective and $\mathbf{T}M$ admits an exact resolution

$$0 \to \mathcal{A}^n \to \mathcal{A}^n \to \mathbf{T}M \to 0.$$

Additivity yields $\dim_{\mathcal{A}}(\mathbf{T}M) = 0$. Applying the exact functor F to this resolution, additivity also implies $\dim_{\mathcal{B}}(F(\mathbf{T}M)) = 0$. We have $\dim_{\mathcal{A}}(\mathbf{P}M) = \dim_{\mathcal{B}}(F(\mathbf{P}M))$ by assumption. Hence, $\dim_{\mathcal{A}}(M) = \dim_{\mathcal{B}}(F(M))$.
Step 2: Let M be finitely generated.

Then there is a finitely generated free $\mathcal{A}$-module P with an epimorphism $P \to M$. Let K be its kernel. K can be written as the directed union of its finitely generated submodules $K = \bigcup_{i \in I} K_i$. By cofinality and additivity (3.8) we conclude

$$\begin{aligned}
\dim_{\mathcal{A}}(M) = \dim_{\mathcal{A}}(P) - \dim_{\mathcal{A}}(K) &= \dim_{\mathcal{A}}(P) - \sup_{i \in I}\{\dim_{\mathcal{A}}(K_i)\} \\
&= \inf_{i \in I}\{\dim_{\mathcal{A}}(P) - \dim_{\mathcal{A}}(K_i)\} \\
&= \inf_{i \in I}\{\dim_{\mathcal{B}}(P/K_i)\}
\end{aligned}$$

We have $F(K) = \mathrm{colim}_{i \in I} F(K_i)$ with injective structure maps because F is colimit-preserving and exact. Thus we can conclude similarly to obtain

$$\dim_{\mathcal{B}}(F(M)) = \inf_{i \in I}\{\dim_{\mathcal{B}}(F(P/K_i))\}.$$

Then the claim follows from the first step.
Step 3: Let M be an arbitrary module.

Every module is the directed union of its finitely generated submodules, which reduces the claim to the preceding step due to cofinality. □

THEOREM 3.18 (Dimension and Induction).
Let $\phi : \mathcal{N} \to \mathcal{M}$ be a trace-preserving $*$-homomorphism between finite von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$. Then for every $\mathcal{N}$-module N we have

$$\dim_{\mathcal{N}}(N) = \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} N).$$

Proof. The functor $\mathcal{M} \otimes_{\mathcal{N}} -$ is exact according to 1.48. Moreover, the tensor product preserves colimits. Due to the previous theorem, we only have to show that $\dim_{\mathcal{N}}(P) = \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} P)$ holds for a finitely generated projective $\mathcal{N}$-module P.

There is an idempotent matrix $A \in M_n(\mathcal{N})$ such that $P \cong \mathcal{N}^n \cdot A$. We have $\mathcal{M} \otimes_{\mathcal{N}} P \cong \mathcal{M}^n \cdot \phi(A)$, and so we get

$$\dim_{\mathcal{N}}(P) = \sum_{i=1}^{n} \mathrm{tr}_{\mathcal{N}}(A_{ii}) = \sum_{i=1}^{n} \mathrm{tr}_{\mathcal{M}}(\phi(A_{ii})) = \dim_{\mathcal{M}}(\mathcal{M} \otimes_{\mathcal{N}} P).$$

□

DEFINITION 3.19 (Dimension-Full Idempotent).
Let $\mathcal{A}$ be a finite von Neumann algebra, and let R be a ring containing $\mathcal{A}$ as a subring. An idempotent $p \in R$ is called $\dim_{\mathcal{A}}$*-full* if the inclusion of

$$RpR := \left\{ \sum_{i=1}^{n} r_i \cdot p \cdot s_i;\ n \in \mathbb{N}, r_i, s_i \in R \right\}$$

into R is a $\dim_{\mathcal{A}}$-isomorphism of $\mathcal{A}$-modules.

Recall that p is called full if $RpR = R$. See 1.35. We remind the reader that the restricted von Neumann algebra $p\mathcal{N}p$ is equipped with the normalized trace $\mathrm{tr}_{p\mathcal{N}p} = \frac{1}{\mathrm{tr}_{\mathcal{N}}(p)} \mathrm{tr}_{\mathcal{N}}$. Compare 1.40.

If $p \in \mathcal{N}$ is full, then $\mathcal{N}$ and $p\mathcal{N}p$ are Morita equivalent by 1.36. So the (mutually inverse) functors

$$F = \mathcal{N}p \otimes_{p\mathcal{N}p} - : p\mathcal{N}p\text{−Mod} \to \mathcal{N}\text{−Mod} \tag{3.3}$$

$$G = p\mathcal{N} \otimes_{\mathcal{N}} - : \mathcal{N}\text{−Mod} \to p\mathcal{N}p\text{−Mod} \tag{3.4}$$

are exact and preserve projectives. The question is: Is this still true, if p is $\dim_{\mathcal{N}}$-full?

It is not clear that G preserves projectives, but G is exact for every idempotent p because $G(M) = pM$ is a direct summand in M (as an abelian group). Obviously, the functor F preserves projectives for every idempotent p. Fortunately, F is also exact, which we prove next.

THEOREM 3.20. *Let p be a $\dim_{\mathcal{N}}$-full idempotent in $\mathcal{N}$. Then $\mathcal{N}p$ is a right flat $p\mathcal{N}p$-module.*

Proof. Consider the image $\bar{1}$ of 1 in the cokernel of the inclusion $\mathcal{N}p\mathcal{N} \subset \mathcal{N}$. By assumption, the cokernel has dimension zero. By the local criterion 3.16 there is a sequence $(p_i)_{i\in\mathbb{N}}$ of projections in $\mathcal{N}$ such that $p_i\bar{1} = 0$, i.e. $p_i \in \mathcal{N}p\mathcal{N}$, $\mathrm{tr}_{\mathcal{N}}(p_i) \to 1$ and $p_i \geq p$. From $p \leq p_i$ we get $p = p_i p p_i \in p_i \mathcal{N} p_i$. Furthermore, $p = p_i p p_i$ and $p_i \in \mathcal{N}p\mathcal{N}$ imply

$$p_i \in (p_i \mathcal{N} p_i)\, p\, (p_i \mathcal{N} p_i),$$

hence p is a full idempotent in $p_i\mathcal{N}p_i$. The rings $p_i\mathcal{N}p_i$ and $p\mathcal{N}p$ are Morita equivalent (1.36). Thus the right $p\mathcal{N}p$-module $p_i\mathcal{N}p$ is projective. The $p\mathcal{N}p$-homomorphism

$$\mathcal{N}p \longrightarrow \prod_{i\in\mathbb{N}} p_i\mathcal{N}p,\ n \mapsto (p_i n)_{i\in\mathbb{N}}$$

is injective by 3.15. Flatness is inherited to products and submodules over semihereditary rings (see 1.31). Recall that a von Neumann algebra is semihereditary (1.29). Therefore the product on the right is flat, and its submodule $\mathcal{N}p$ is also flat as a $p\mathcal{N}p$-module. □

Next we want to study how the functors F and G (3.3) behave with respect to the dimension. We start with the functor F, which is easier because F preserves projectives.

THEOREM 3.21 (Dimension and Restriction I).
Let p be a $\dim_{\mathcal{N}}$-full projection in $\mathcal{N}$. For every $p\mathcal{N}p$-module M we have

$$\dim_{\mathcal{N}}(\mathcal{N}p \otimes_{p\mathcal{N}p} M) = \operatorname{tr}_{\mathcal{N}}(p)\dim_{p\mathcal{N}p}(M).$$

Proof. We know that $\mathcal{N}p$ is flat over $p\mathcal{N}p$ due to the preceding theorem. The functor $\mathcal{N}p \otimes_{p\mathcal{N}p} -$ respects colimits. By 3.17 it suffices to check the claim for a finitely generated projective $p\mathcal{N}p$-module M. Let $A \in M_n(p\mathcal{N}p)$ be an idempotent matrix such that

$$M \cong (p\mathcal{N}p)^n \cdot A.$$

Then $\dim_{p\mathcal{N}p}(M) = \sum_{i=1}^n \operatorname{tr}_{p\mathcal{N}p}(A_{ii})$ by definition. The $\mathcal{N}$-module $\mathcal{N}p \otimes_{p\mathcal{N}p} M$ is finitely generated projective, and we have

$$\mathcal{N}p \otimes_{p\mathcal{N}p} M \cong (\mathcal{N}p)^n \cdot A = \mathcal{N}^n \cdot A.$$

For the right equal sign note that $(1-p)A = 0$. Hence we can conclude

$$\begin{aligned}\dim_{\mathcal{N}}(\mathcal{N}p \otimes_{p\mathcal{N}p} M) = \sum_{i=1}^n \operatorname{tr}_{\mathcal{N}}(A_{ii}) &= \operatorname{tr}_{\mathcal{N}}(p) \cdot \sum_{i=1}^n \operatorname{tr}_{p\mathcal{N}p}(A_{ii}) \\ &= \operatorname{tr}_{\mathcal{N}}(p) \cdot \dim_{p\mathcal{N}p}(M).\end{aligned}$$

□

THEOREM 3.22. *Let R be a ring containing $\mathcal{N}$ as a subring, and let p be a $\dim_{\mathcal{N}}$-full idempotent in R. Then*

$$\phi : Rp \otimes_{pRp} pM \to M,\ n \otimes m \mapsto nm \tag{3.5}$$

is a $\dim_{\mathcal{N}}$-isomorphism for every R-module M.

Proof. First we show that ϕ is $\dim_{\mathcal{N}}$-surjective. The local criterion 3.16 applied to the cokernel of the inclusion $RpR \subset R$ provides a sequence p_i of projections in $\mathcal{N}$, which lie in RpR and satisfy $\operatorname{tr}_{\mathcal{N}}(p_i) \to 1$. But every element in the cokernel of ϕ is annihilated by the p_i, so this yields $\dim_{\mathcal{N}}(\operatorname{coker}\phi) = 0$, again due to the local criterion.

Now we can prove that ϕ is a $\dim_{\mathcal{N}}$-isomorphism. Consider the exact sequence

$$0 \to \ker\phi \to Rp \otimes_{pRp} pM \to M \to \operatorname{coker}\phi \to 0.$$

Applying the exact functor $pR \otimes_R -$ produces an isomorphism in the middle because $pR \otimes_R (Rp \otimes_{pRp} pM) = pRp \otimes_{pRp} pM = pR \otimes_R M$. Hence $p\ker\phi = 0$. Because ϕ is already shown to be $\dim_{\mathcal{N}}$-surjective, we obtain $\dim_{\mathcal{N}}(\ker\phi) = 0$. Hence ϕ is a $\dim_{\mathcal{N}}$-isomorphism. □

THEOREM 3.23 (Dimension and Restriction II).
Let p be a $\dim_{\mathcal{N}}$-full projection in $\mathcal{N}$. Then for every $\mathcal{N}$-module M we have

$$\dim_{\mathcal{N}}(M) = \operatorname{tr}_{\mathcal{N}}(p) \cdot \dim_{p\mathcal{N}p}(pM).$$

Proof. By the preceding theorem we have $\dim_{\mathcal{N}}(\mathcal{N}p \otimes_{p\mathcal{N}p} pM) = \dim_{\mathcal{N}}(M)$. Now the claim is obtained by 3.21. □

3.2 Homological Algebra for Finite von Neumann Algebras

This section provides the technical tools for the new proof of Gaboriau's theorem (see 3.38). Besides that, we think that some theorems are interesting in their own right.

The object we try to understand is the Tor-term

$$\operatorname{Tor}_{\bullet}^{R}(\mathcal{A}, M)$$

resp. its $\mathcal{A}$-dimension, where $\mathcal{B} \subset R \subset \mathcal{A}$ are ring inclusions and M is an R-module. Here R satisfies the following property as an $\mathcal{B}$-$\mathcal{B}$-bimodule:

$$\dim_{\mathcal{B}}(N) = 0 \Rightarrow \dim_{\mathcal{B}}(R \otimes_{\mathcal{B}} N) = 0$$

We say that R is *dimension-compatible* as a bimodule. The questions we deal with are: What happens to the $\mathcal{A}$-dimension of the Tor-term,

- if we replace M by a $\dim_{\mathcal{B}}$-isomorphic R-module (3.28),
- if we replace R by a $\dim_{\mathcal{B}}$-isomorphic subring $\mathcal{B} \subset R' \subset R$ (3.29),
- if we take the restricted version $\operatorname{Tor}_{\bullet}^{pRp}(p\mathcal{A}p, pM)$, where p is a projection in $\mathcal{B}$ (3.31).

DEFINITION 3.24 (Dimension-Compatible Bimodules).
An $\mathcal{A}$-$\mathcal{B}$-bimodule M is called *dimension-compatible* if for every $\mathcal{B}$-module N the implication

$$\dim_{\mathcal{B}}(N) = 0 \Rightarrow \dim_{\mathcal{A}}(M \otimes_{\mathcal{B}} N) = 0$$

holds.

We record some easy facts about dimension-compatible bimodules.

LEMMA 3.25 (Properties of Dimension-Compatible Bimodules).

(i) If M is a dimension-compatible $\mathcal{A}$-$\mathcal{B}$-bimodule and N is a dimension-compatible $\mathcal{B}$-$\mathcal{C}$-bimodule, then $M \otimes_{\mathcal{B}} N$ is a dimension-compatible $\mathcal{A}$-$\mathcal{C}$-bimodule.

(ii) Quotients and direct summands of dimension-compatible bimodules are dimension-compatible.

(iii) Let $\mathcal{B} \subset \mathcal{A}$ be an inclusion of finite von Neumann algebras. Then $\mathcal{A}$ is a dimension-compatible $\mathcal{A}$-$\mathcal{B}$-bimodule.

Here the third assertion follows from 3.18. Next we show that groupoid rings provide examples of dimension-compatible bimodules. The definitions of a discrete measured groupoid $\underline{G}$ and its groupoid ring $\mathbb{C}\underline{G}$ are given in 1.11 on p. 19 resp. 1.19 on p. 21. Recall that such a groupoid $\underline{G}$ is equipped with a measure μ on its set of objects $\underline{G}^0$. The measure μ satisfies a certain invariance property (see 1.7).

LEMMA 3.26. *$\mathbb{C}\underline{G}$ is a dimension-compatible $L^\infty(\underline{G}^0)$-$L^\infty(\underline{G}^0)$-bimodule.*

Proof. Let M be an $L^\infty(\underline{G}^0)$-module with $\dim_{L^\infty(\underline{G}^0)}(M) = 0$. We have to show

$$\dim_{L^\infty(\underline{G}^0)} \big(\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M\big) = 0. \tag{3.6}$$

By the local criterion 3.16 equation (3.6) follows, if for a given $x \in \mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$ a sequence of annihilating projections $\chi_{A_i} \in L^\infty(\underline{G}^0)$ exists such that

$$\chi_{A_i} x = 0 \text{ and } \operatorname{tr}_{L^\infty(\underline{G}^0)}(\chi_{A_i}) = \mu(A_i) \to 1. \tag{3.7}$$

Suppose this is true for a set S of $L^\infty(\underline{G}^0)$-generators of $\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$. Then (3.7) holds for any element in $\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$ by the following observation.

If χ_{A_i} and χ_{B_i} are annihilating projections for the elements r resp. s as in 3.7, whose traces converge to 1, then the $\chi_{A_i \cap B_i}$ are annihilating projections for $f \cdot r + g \cdot s$, $f, g \in L^\infty(\underline{G}^0)$, whose traces also converge to 1.

A set S of $L^\infty(\underline{G}^0)$-generators of $\mathbb{C}\underline{G} \otimes_{L^\infty(\underline{G}^0)} M$ is given by elements of the form $\chi_E \otimes m$, where χ_E is the characteristic function of a Borel subset $E \subset \underline{G}$, such that $s_{|E}$ and $t_{|E}$ are injective, and m is an element in M. This follows from 1.21.

Before we prove (3.7) for the elements in S (and hence (3.6)), we show that for any Borel subset $A \subset \underline{G}^0$ there is a Borel subset $A' \subset \underline{G}^0$ such that

$$\mu(A') \geq \mu(A) \text{ and } \chi_{A'} \cdot \chi_E = \chi_E \cdot \chi_A. \tag{3.8}$$

We have the identities $\chi_{A'}\chi_E = \chi_{s^{-1}(A')\cap E}$ and $\chi_E\chi_A = \chi_{t^{-1}(A)\cap E}$. Put

$$A' = s\big(E \cap t^{-1}(A)\big) \cup \big(\underline{G}^0 - s(E)\big).$$

Because $s_{|E}$ is injective we get

$$s^{-1}(A') \cap E = s^{-1}\Big(s\big(E \cap t^{-1}(A)\big)\Big) \cap E = E \cap t^{-1}(A).$$

This yields $\chi_{A'} \cdot \chi_E = \chi_E \cdot \chi_A$. The invariance of μ yields

$$\begin{aligned}
\mu(A') &= \mu\Big(s\big(E \cap t^{-1}(A)\big)\Big) + \mu\big(\underline{G}^0 - s(E)\big) \\
&= \mu\Big(t\big(E \cap t^{-1}(A)\big)\Big) + \mu\big(\underline{G}^0 - t(E)\big) \\
&= \mu\big(t(E) \cap A\big) + \mu\big(\underline{G}^0 - t(E)\big) \\
&\geq \mu(A).
\end{aligned}$$

Compare 1.9. Equation (3.8) follows.

Now we can prove (3.7) for $\chi_E \otimes m \in S$ as follows. Because of $\dim_{L^\infty(\underline{G}^0)}(M) = 0$

there are $A_i \subset \underline{G}^0$ with $\chi_{A_i} m = 0$ and $\mu(A_i) \to 1$, due to the local criterion 3.16. By (3.8) there are $A'_i \subset \underline{G}^0$ with $\chi_{A'_i}\chi_E = \chi_E\chi_{A_i}$ and $\mathrm{tr}_{L^\infty(\underline{G}^0)}(\chi_{A'_i}) = \mu(A'_i) \to 1$. Now (3.7) is obtained from

$$\chi_{A'_i}(\chi_E \otimes m) = \chi_E\chi_{A_i} \otimes m = \chi_E \otimes \chi_{A_i} m = 0.$$

This finishes the proof. □

LEMMA 3.27.
Let $\mathcal{N}$ be a finite von Neumann algebra, R a ring and B_1, B_2 $\mathcal{N}$-R-bimodules. A bimodule map $B_1 \to B_2$, which is a $\dim_{\mathcal{N}}$-isomorphism, induces $\dim_{\mathcal{N}}$-isomorphisms

$$\mathrm{Tor}_\bullet^R(B_1, M) \to \mathrm{Tor}_\bullet^R(B_2, M)$$

for every R-module M.

Proof. Let B an $\mathcal{N}$-R-bimodule with $\dim_{\mathcal{N}}(B) = 0$. Let M be an arbitrary R-module and $P_\bullet$ a free R-resolution of M. Then $\dim_{\mathcal{N}}(B \otimes_R P_\bullet) = 0$ follows from the additivity and cofinality of $\dim_{\mathcal{N}}$ (see 3.8). Hence

$$\dim_{\mathcal{N}}\left(H_\bullet(B \otimes_R P_\bullet)\right) = \dim_{\mathcal{N}}\left(\mathrm{Tor}_\bullet^R(B, M)\right) = 0.$$

In the general case of a $\dim_{\mathcal{N}}$-isomorphism $\phi : B_1 \to B_2$, we consider the short exact sequences

$$0 \to \ker\phi \to B_1 \to \mathrm{im}\,\phi \to 0,$$
$$0 \to \mathrm{im}\,\phi \to B_2 \to \mathrm{coker}\,\phi \to 0.$$

$\ker\phi$ and $\mathrm{coker}\,\phi$ have vanishing dimension. We obtain long exact sequences for the Tor-terms:

$$\cdots \to \mathrm{Tor}_1^R(B_1, M) \to \mathrm{Tor}_1^R(\mathrm{im}\,\phi, M) \to \underbrace{\ker\phi \otimes M}_{=\mathrm{Tor}_0^R(\ker\phi, M)} \to B_1 \otimes M \to \mathrm{im}\,\phi \otimes M \to 0$$
$$\cdots \to \mathrm{Tor}_1^R(B_2, M) \to \mathrm{Tor}_1^R(\mathrm{coker}\,\phi, M) \to \mathrm{im}\,\phi \otimes M \to B_2 \otimes M \to \underbrace{\mathrm{coker}\,\phi \otimes M}_{=\mathrm{Tor}_0^R(\mathrm{coker}\,\phi, M)} \to 0.$$

We already know $\dim_{\mathcal{N}}\left(\mathrm{Tor}_\bullet^R(\ker\phi, M)\right) = 0$ and $\dim_{\mathcal{N}}\left(\mathrm{Tor}_\bullet^R(\mathrm{coker}\,\phi, M)\right) = 0$, hence

$$\mathrm{Tor}_\bullet^R(B_1, N) \to \mathrm{Tor}_\bullet^R(\mathrm{im}\,\phi, N),$$
$$\mathrm{Tor}_\bullet^R(\mathrm{im}\,\phi, N) \to \mathrm{Tor}_\bullet^R(B_2, N)$$

are $\dim_{\mathcal{N}}$-isomorphisms, and so their composition. □

LEMMA 3.28.
Let $\mathcal{B} \subset R \subset \mathcal{A}$ be an inclusion of rings where $\mathcal{A}$, $\mathcal{B}$ are finite von Neumann algebras. Let B be an $\mathcal{A}$-R-bimodule. We assume the following.

(i) R is dimension-compatible as a $\mathcal{B}$-$\mathcal{B}$-bimodule.

(ii) B is dimension-compatible as an $\mathcal{A}$-$\mathcal{B}$-bimodule.

(iii) B is flat as a right $\mathcal{B}$-module.

Then every R-homomorphism $M \to N$, which is a $\dim_{\mathcal{B}}$-isomorphism, induces a $\dim_{\mathcal{A}}$-isomorphism

$$\mathrm{Tor}_{\bullet}^{R}(B,M) \to \mathrm{Tor}_{\bullet}^{R}(B,N).$$

Proof. First we get the following flatness properties.

- The $\mathcal{A}$-R-bimodule $B \otimes_{\mathcal{B}} R$ is flat as a right R-module because B is flat as a right $\mathcal{B}$-module.
- R is a $\mathcal{B}$-submodule of the flat $\mathcal{B}$-module $\mathcal{A}$ (1.48). Hence R is flat as a right $\mathcal{B}$-module by 1.31 and 1.29.
- Therefore $B \otimes_{\mathcal{B}} R$ is also flat as a right $\mathcal{B}$-module.

Multiplication yields a surjective $\mathcal{A}$-R-bimodule homomorphism

$$m : B \otimes_{\mathcal{B}} R \to B.$$

$B \otimes_{\mathcal{B}} R$ is dimension-compatible (as an $\mathcal{A}$-$\mathcal{B}$-bimodule) because B and R are dimension-compatible. The map m splits as an $\mathcal{A}$-$\mathcal{B}$-homomorphism by the map $B \to B \otimes_{\mathcal{B}} R$, $b \mapsto b \otimes 1$. Hence, as an $\mathcal{A}$-$\mathcal{B}$-bimodule, $\ker m$ is a direct summand in $B \otimes_{\mathcal{B}} R$ and therefore also dimension-compatible. We record the properties of $\ker m$:

- $\ker m$ is an $\mathcal{A}$-R-bimodule.
- $\ker m$ is dimension-compatible as an $\mathcal{A}$-$\mathcal{B}$-bimodule.
- $\ker m$ is flat as a right $\mathcal{B}$-module because it is the submodule of a flat $\mathcal{B}$-module (1.31, 1.29).

Notice that $\ker m$ satisfies all properties imposed on B. Let M be an R-module with $\dim_{\mathcal{B}}(M) = 0$. So we have $\dim_{\mathcal{A}}(\ker m \otimes_{\mathcal{B}} M) = 0$ and hence for its quotient

$$\dim_{\mathcal{A}}(\ker m \otimes_R M) = 0.$$

The short exact sequence $0 \to \ker m \to B \otimes_{\mathcal{B}} R \to B \to 0$ induces a long exact sequence of Tor-terms

$$\ldots \to 0 \to \mathrm{Tor}_2^R(B,M) \to \mathrm{Tor}_1^R(\ker m, M) \to \underbrace{\mathrm{Tor}_1^R(B \otimes_{\mathcal{B}} R, M)}_{=0} \to$$
$$\to \mathrm{Tor}_1^R(B,M) \to \ker m \otimes_R M \to (B \otimes_{\mathcal{B}} R \otimes_R M) \to B \otimes_R M \to 0,$$

where the zero terms are due to $B \otimes_{\mathcal{B}} R$ being R-flat. We obtain

$$\begin{gathered}\dim_{\mathcal{A}}(B \otimes_R M) = \dim_{\mathcal{A}}\left(\mathrm{Tor}_1^R(B,M)\right) = 0,\\ \mathrm{Tor}_{i+1}^R(B,M) \cong \mathrm{Tor}_i^R(\ker m, M) \quad i \geq 1.\end{gathered} \tag{3.9}$$

Now we apply (3.9) to $\ker m$ instead of B and get

$$\dim_{\mathcal{A}}\left(\mathrm{Tor}_1^R(\ker m, M)\right) = 0,\ \dim_{\mathcal{A}}\left(\mathrm{Tor}_2^R(B,M)\right) = 0.$$

Repeating this ("dimension shifting") yields

$$\dim_{\mathcal{A}}\left(\mathrm{Tor}_i^R(B,M)\right) = 0 \text{ for } i \geq 0.$$

Deducing the general case of a $\dim_{\mathcal{B}}$-isomorphism $\phi : M \to N$ from the case $\dim_{\mathcal{B}}(M) = 0$ uses exactly the same method as in the proof of 3.27. □

THEOREM 3.29.
Let $\mathcal{B} \subset R_1 \subset R_2 \subset \mathcal{A}$ be an inclusion of rings where $\mathcal{A}$, $\mathcal{B}$ are finite von Neumann algebras. We assume the following.

(i) R_2 is dimension-compatible as a $\mathcal{B}$-$\mathcal{B}$-bimodule.

(ii) The inclusion $R_1 \subset R_2$ is a $\dim_{\mathcal{B}}$-isomorphism.

Then

$$\dim_{\mathcal{A}}\left(\operatorname{Tor}_{\bullet}^{R_1}(\mathcal{A}, M)\right) = \dim_{\mathcal{A}}\left(\operatorname{Tor}_{\bullet}^{R_2}(\mathcal{A}, M)\right)$$

holds for every R_2-module M.

Proof. By 3.27 the induced map

$$\operatorname{Tor}_i^{R_1}(R_2, M) \longleftarrow \operatorname{Tor}_i^{R_1}(R_1, M) = \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} \tag{3.10}$$

is a $\dim_{\mathcal{B}}$-isomorphism. Note that this is not an R_2-homomorphism from M to the left Tor-term in dimension zero: The map is given by the inclusion $M \to R_2 \otimes_{R_1} M, m \mapsto 1 \otimes m$. But its left inverse is the R_2-homomorphism $R_2 \otimes_{R_1} M \to M$, given by multiplication, which has to be a $\dim_{\mathcal{B}}$-isomorphism, too.

Let $P_{\bullet} \to M$ be a projective R_1-resolution of M. The Künneth spectral sequence, applied to $\mathcal{A}$ and the complex $R_2 \otimes_{R_1} P_{\bullet}$ (see [61, theorem 5.6.4 on p. 143]), has the E^2-term

$$E^2_{pq} = \operatorname{Tor}_p^{R_2}\Big(\mathcal{A}, H_q(R_2 \otimes_{R_1} P_{\bullet})\Big) = \operatorname{Tor}_p^{R_2}\Big(\mathcal{A}, \operatorname{Tor}_q^{R_1}(R_2, M)\Big)$$

and converges to

$$H_{p+q}\Big(\mathcal{A} \otimes_{R_2} (R_2 \otimes_{R_1} P_{\bullet})\Big) = \operatorname{Tor}_{p+q}^{R_1}(\mathcal{A}, M).$$

Now we can apply the preceding lemma 3.28 to the inclusion $\mathcal{B} \subset R_2 \subset \mathcal{A}$. Here recall that $\mathcal{A}$ is flat as a right $\mathcal{B}$-module (1.48) and dimension-compatible as an $\mathcal{A}$-$\mathcal{B}$-bimodule. So we obtain

$$\dim_{\mathcal{A}}\left(E^2_{pq}\right) = \begin{cases} \dim_{\mathcal{A}}\Big(\operatorname{Tor}_p^{R_2}(\mathcal{A}, M)\Big) & \text{if } q = 0 \\ 0 & \text{if } q > 0. \end{cases}$$

So the spectral sequence collapses up to dimension, and additivity (3.8) yields

$$\dim_{\mathcal{A}}\left(\operatorname{Tor}_p^{R_1}(\mathcal{A}, M)\right) = \dim_{\mathcal{A}}\left(E^{\infty}_{p0}\right) = \dim_{\mathcal{A}}\left(E^2_{p0}\right) = \dim_{\mathcal{A}}\left(\operatorname{Tor}_p^{R_2}(\mathcal{A}, M)\right).$$

□

REMARK 3.30. Actually, the theorem above provides more than just an equality of dimension. There is a natural zig-zag of $\dim_{\mathcal{A}}$-isomorphisms between $\operatorname{Tor}_{\bullet}^{R_1}(\mathcal{A}, M)$ and $\operatorname{Tor}_{\bullet}^{R_2}(\mathcal{A}, M)$.

THEOREM 3.31.
Let $\mathcal{B} \subset R \subset \mathcal{A}$ be an inclusion of rings, where $\mathcal{A}$, $\mathcal{B}$ are finite von Neumann algebras. Let $p \in \mathcal{B}$ be a projection. We assume the following.

(i) R is dimension-compatible as a $\mathcal{B}$-$\mathcal{B}$-bimodule.

(ii) p is $\dim_{\mathcal{B}}$-full in R.

Then the equality

$$\dim_{p\mathcal{A}p}\Big(p\operatorname{Tor}_\bullet^R(\mathcal{A}, M)\Big) = \dim_{p\mathcal{A}p}\Big(\operatorname{Tor}_\bullet^{pRp}(p\mathcal{A}p, pM)\Big)$$

holds for every R-module M.

Proof. Theorem 3.23 yields for $i > 0$

$$\begin{aligned}\dim_{\mathcal{B}} \operatorname{Tor}_i^{pRp}(Rp, pM) &= \operatorname{tr}_{\mathcal{B}}(p) \cdot \dim_{p\mathcal{B}p}\Big(p\operatorname{Tor}_i^{pRp}(Rp, pM)\Big) \\ &= \operatorname{tr}_{\mathcal{B}}(p) \cdot \dim_{p\mathcal{B}p}\Big(\operatorname{Tor}_i^{pRp}(pRp, pM)\Big) \\ &= 0.\end{aligned}$$

Furthermore, the multiplication map m

$$\operatorname{Tor}_0^{pRp}(Rp, pM) = Rp \otimes_{pRp} pM \longrightarrow M$$

is a $\dim_{\mathcal{B}}$-isomorphism by 3.22.

Now let $P_\bullet \to pM$ be a pRp-projective resolution of pM. The Künneth spectral sequence applied to $\mathcal{A}$ and the complex $Rp \otimes_{pRp} P_\bullet$ ([61, theorem 5.6.4 on p. 143]) has the E^2-term

$$E_{ij}^2 = \operatorname{Tor}_i^R\Big(\mathcal{A}, H_j(Rp \otimes_{pRp} P_\bullet)\Big) = \operatorname{Tor}_i^R\Big(\mathcal{A}, \operatorname{Tor}_j^{pRp}(Rp, pM)\Big)$$

and converges to

$$H_{i+j}\big(\underbrace{\mathcal{A} \otimes_R (Rp \otimes_{pRp} P_\bullet)}_{=\mathcal{A}p \otimes_{pRp} P_\bullet}\big) = \operatorname{Tor}_{i+j}^{pRp}\big(\mathcal{A}p, pM\big).$$

We know $\operatorname{Tor}_\bullet^{pRp}(Rp, pM)$ up to $\dim_{\mathcal{B}}$-isomorphism. An application of Lemma 3.28 to that module yields

$$\dim_{\mathcal{A}}\left(E_{ij}^2\right) = \begin{cases}\dim_{\mathcal{A}}\Big(\operatorname{Tor}_i^R(\mathcal{A}, M)\Big) & \text{if } j = 0 \\ 0 & \text{if } j > 0.\end{cases}$$

The spectral sequence collapses up to dimension. This implies

$$\dim_{\mathcal{A}}\Big(\operatorname{Tor}_i^{pRp}(\mathcal{A}p, pM)\Big) = \dim_{\mathcal{A}}\left(E_{i0}^\infty\right) = \dim_{\mathcal{A}}\left(E_{i0}^2\right) = \dim_{\mathcal{A}}\left(\operatorname{Tor}_i^R(\mathcal{A}, M)\right).$$

Due to 3.23 and the exactness of the functor $p\mathcal{N} \otimes_{\mathcal{N}} -$, the claim follows. □

3.3 L^2-Betti Numbers of Discrete Measured Groupoids

We introduce the notion of *L^2-Betti numbers of discrete measured groupoids*. This should be seen in analogy to the L^2-Betti numbers of groups (see 3.4 and 3.6). We identify the L^2-Betti numbers of a group G with the L^2-Betti numbers of a standard action of G (3.37). The main result is the (proportionality) invariance of the L^2-Betti numbers of groups under measure equivalence (3.38), first proven in [24], [25].

The notions of a *discrete measured groupoid*, its *groupoid ring* and the corresponding *von Neumann algebra* are defined in 1.11, 1.19, and 1.44.

DEFINITION 3.32 (L^2-Betti numbers of Discrete Measure Groupoids). Let $\underline{G}$ be a discrete measured groupoid. Its *n-th L^2-Betti number* $b_n^{(2)}(\underline{G})$ is defined as

$$b_n^{(2)}(\underline{G}) = \dim_{\mathcal{N}(\underline{G})}\left(\operatorname{Tor}_n^{\mathbb{C}\underline{G}}\left(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0)\right)\right).$$

REMARK 3.33. Notice that L^2-Betti numbers of measure isomorphic discrete measured groupoids coincide (see 1.56).

LEMMA 3.34. *Let $\underline{G}$ be a discrete measured groupoid, and let $A \subset \underline{G}^0$ be a Borel subset such that $t(s^{-1}(A))$ has full measure in $\underline{G}^0$. Then the characteristic function $\chi_A \in L^\infty(\underline{G}^0) \subset \mathbb{C}\underline{G}$ of A is a $\dim_{L^\infty(\underline{G}^0)}$-full idempotent in $\mathbb{C}\underline{G}$.*

Proof. We have to show that the inclusion $\mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G} \subset \mathbb{C}\underline{G}$ is a $\dim_{L^\infty(\underline{G}^0)}$-isomorphism. Recall: In a discrete measured groupoid $\underline{G}$ we denote the source, target and inverse maps by s, t and i and the measure on $\underline{G}^0$ by μ.

Let $s^{-1}(A) = \bigcup_{n\in\mathbb{N}} E_n$ be a partition into Borel sets E_n with the property that $s_{|E_n}$ is injective. The partition exists by theorem 1.3. We have

$$\chi_{i(E_n)} \cdot \chi_A \cdot \chi_{E_n} = \chi_{i(E_n)}\chi_{s^{-1}(A)\cap E_n} = \chi_{t(s^{-1}(A)\cap E_n)}.$$

The right equal sign is due to the injectivity of $s_{|E_n}$. Hence $\sum_{n=1}^N \chi_{t(s^{-1}(A)\cap E_n)} \in \mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G}$, where we understand $\mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G}$ as in 3.19 on p. 53. So we get

$$\chi_{t(s^{-1}(A)\cap\bigcup_{n=1}^N E_n)} = f \cdot \left(\sum_{n=1}^N \chi_{t(s^{-1}(A)\cap E_n)}\right) \in \mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G}$$

for a suitable $f \in L^\infty(\underline{G}^0)$. This implies that $\chi_{t(s^{-1}(A)\cap\bigcup_{n=1}^N E_n)} \cdot [\phi] = 0$ for every $[\phi] \in \mathbb{C}\underline{G}/\mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G}$. Because $t(s^{-1}(A))$ has full measure, we get

$$\lim_{N\to\infty} \mu\Big(t\big(s^{-1}(A) \cap \bigcup_{n=1}^N E_n\big)\Big) = \mu\big(\underline{G}^0\big) = 1.$$

So $\mathbb{C}\underline{G} \cdot \chi_A \cdot \mathbb{C}\underline{G} \subset \mathbb{C}\underline{G}$ is a $\dim_{L^\infty(\underline{G}^0)}$-isomorphism, due to the local criterion (3.16). ☐

THEOREM 3.35 (Restriction). *Let $\underline{G}$ be a discrete measured groupoid, and let $A \subset \underline{G}^0$ be a Borel subset such that $t(s^{-1}(A))$ has full measure in $\underline{G}^0$. Then*

$$b_\bullet^{(2)}(\underline{G}) = \mu(A) \cdot b_\bullet^{(2)}(\underline{G}_{|A}).$$

Proof. By the preceding lemma 3.34 the characteristic function χ_A is a $\dim_{L^\infty(\underline{G}^0)}$-full idempotent in $\mathbb{C}\underline{G}$. By 3.23 we have

$$\begin{aligned} b_\bullet^{(2)}(\underline{G}) &= \dim_{\mathcal{N}(\underline{G})}\Big(\operatorname{Tor}_\bullet^{\mathbb{C}\underline{G}}\left(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0)\right)\Big) \\ &= \mu(A)\cdot \dim_{\chi_A\mathcal{N}(\underline{G})\chi_A}\Big(\chi_A \operatorname{Tor}_\bullet^{\mathbb{C}\underline{G}}(\mathcal{N}(\underline{G}), L^\infty(\underline{G}^0))\Big). \end{aligned}$$

By 3.26 $\mathbb{C}\underline{G}$ is dimension-compatible as a $L^\infty(\underline{G}^0)$-$L^\infty(\underline{G}^0)$-bimodule. So by 3.31 we can continue the computation of $b_\bullet^{(2)}(\underline{G})$ as follows.

$$\begin{aligned} &= \mu(A)\cdot \dim_{\chi_A\mathcal{N}(\underline{G})\chi_A}\Big(\operatorname{Tor}_\bullet^{\chi_A\mathbb{C}\underline{G}\chi_A}\left(\chi_A\mathcal{N}(\underline{G})\chi_A, \chi_A L^\infty(\underline{G}^0)\right)\Big) \\ &= \mu(A)\cdot b_\bullet^{(2)}\left(\underline{G}_{|A}\right). \end{aligned}$$

For the last step note that we have

$$\mathcal{N}(\underline{G}_{|A}) = \chi_A\mathcal{N}(\underline{G})\chi_A,\ \mathbb{C}\underline{G}_{|A} = \chi_A\mathbb{C}\underline{G}\chi_A,\ \chi_A L^\infty(\underline{G}^0) = L^\infty(A).$$

Compare 1.23 and 1.55. □

For the next lemma we note that the crossed product $L^\infty(X)*G$ for a standard action $G\curvearrowright X$ is defined in 1.24. The crossed product can be considered as a subring of the groupoid ring $\mathbb{C}\mathcal{R}(G\curvearrowright X)$ of the orbit equivalence relation of $G\curvearrowright X$. See 1.26.

LEMMA 3.36. *The inclusion $L^\infty(X)*G \subset \mathbb{C}\mathcal{R}(G\curvearrowright X)$ is a $\dim_{L^\infty(\underline{G}^0)}$-isomorphism.*

Proof. Write $\mathcal{R} = \mathcal{R}(G\curvearrowright X)$ for short. We apply the local criterion (3.16) to show that the quotient $\mathbb{C}\mathcal{R}/L^\infty(X)*G$ has dimension zero.

Let $\phi \in \mathbb{C}\mathcal{R}$ and $[\phi]$ be its image in the quotient. Choose an enumeration $G = \{g_1, g_2, \ldots\}$. Define Borel subsets $X_n \subset X$ by

$$X_n = \{x\in X;\ \phi(g_i\cdot x, x) = 0 \text{ for all } i > n\}.$$

Let χ_n be the characteristic function of X_n. If $i > n$ then

$$(\chi_n\cdot\phi)(g_i\cdot x, x) = 0$$

holds for almost all $x\in X$. Therefore $\chi_n\cdot\phi \in L^\infty(X)*G$ (see 1.26). So $\chi_n\cdot[\phi] = 0 \in \mathbb{C}\mathcal{R}/L^\infty(X)*G$. But the X_n form an exhaustion of X, so $\mu(X_n)\to\mu(X) = 1$. Then the local criterion yields $\dim_{L^\infty(X)}\left(\mathbb{C}\mathcal{R}/L^\infty(X)*G\right) = 0$. □

THEOREM 3.37. *Let $G\curvearrowright X$ be a standard action of a countable group. Then the L^2-Betti numbers of G and its orbit equivalence relation $\mathcal{R}(G\curvearrowright X)$ coincide.*

$$b_\bullet^{(2)}(G) = b_\bullet^{(2)}\left(\mathcal{R}(G\curvearrowright X)\right)$$

Proof. Write $\mathcal{R} = \mathcal{R}(G\curvearrowright X)$ for short. The crossed product ring $L^\infty(X)*G$ is flat as a right $\mathbb{C}G$-module because of the equality

$$(L^\infty(X)*G)\otimes_{\mathbb{C}G} M = L^\infty(X)\otimes_{\mathbb{C}} M.$$

Hence we obtain

$$
\begin{aligned}
b_n^{(2)}(G) &= \dim_{\mathcal{N}(G)}\left(\operatorname{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G),\mathbb{C})\right)\\
&= \dim_{\mathcal{N}(\mathcal{R})}\left(\mathcal{N}(\mathcal{R})\otimes_{\mathcal{N}(G)}\operatorname{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G),\mathbb{C})\right) \qquad \text{by (3.18)}\\
&= \dim_{\mathcal{N}(\mathcal{R})}\left(\operatorname{Tor}_n^{\mathbb{C}G}(\mathcal{N}(\mathcal{R}),\mathbb{C})\right) \qquad \text{by (1.48)}\\
&= \dim_{\mathcal{N}(\mathcal{R})}\left(\operatorname{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}),(L^\infty(X)*G)\otimes_{\mathbb{C}G}\mathbb{C})\right)\\
&= \dim_{\mathcal{N}(\mathcal{R})}\left(\operatorname{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}),L^\infty(X))\right).
\end{aligned}
$$

Now we will apply theorem 3.29 to the ring inclusions

$$L^\infty(X)\subset L^\infty(X)*G\subset\mathbb{C}\mathcal{R}\subset\mathcal{N}(\mathcal{R}).$$

As an $L^\infty(X)$-$L^\infty(X)$-bimodule, $\mathbb{C}\mathcal{R}$ is dimension-compatible by 3.26. The inclusion $L^\infty(X)*G\subset\mathbb{C}\mathcal{R}$ is a $\dim_{L^\infty(\underline{G}^0)}$-isomorphism by 3.36. So all assumptions in theorem 3.29 are satisfied, and we can conclude

$$
\begin{aligned}
\dim_{\mathcal{N}(\mathcal{R})}\left(\operatorname{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}),L^\infty(X))\right) &= \dim_{\mathcal{N}(\mathcal{R})}\left(\operatorname{Tor}_n^{\mathbb{C}\mathcal{R}}(\mathcal{N}(\mathcal{R}),L^\infty(X))\right)\\
&= b_n^{(2)}(\mathcal{R}).
\end{aligned}
$$

This finishes the proof. □

Concerning the following theorem, we remind the reader that the notions *measure equivalence* and *weak isomorphism* are explained in 2.16 on p. 38 and 2.26 on p. 41.

THEOREM 3.38 (Invariance under Measure Equivalence).
Let G and H be countable groups. If G and H are measure equivalent with respect to a measure coupling of index C, then the L^2-Betti numbers of G and H are proportional by the factor C:

$$b_\bullet^{(2)}(G)=C\cdot b_\bullet^{(2)}(H)$$

Proof. By 2.33 there exist standard actions $G\curvearrowright X$ and $H\curvearrowright Y$, Borel subsets $A\subset X$, $B\subset Y$ meeting almost every orbit with $C=\frac{\mu(A)}{\mu(B)}$ and an isomorphism between the restricted orbit equivalence relations $\mathcal{R}(G\curvearrowright X)_{|A}$ and $\mathcal{R}(H\curvearrowright Y)_{|B}$. Now the theorem follows from 3.37 and 3.35. □

REMARK 3.39. Measure equivalent groups can have measure couplings with different indices. But then the L^2-Betti numbers must vanish by the preceding theorem.

As a consequence of the Euler-Poincare formula (3.7), we get:

COROLLARY 3.40. *The sign of the Euler characteristic* $\operatorname{sign}(\chi(G))\in\{-1,0,1\}$ *is an invariant of measure equivalence among groups admitting a finite classifying space.*

The following corollary is obtained using theorem 2.36.

COROLLARY 3.41. *Let G and H be groups, which admit a finite classifying space. If they act both on the same second countable, proper metric space properly and cocompactly by isometries, then*

$$\operatorname{sign}(\chi(G))=\operatorname{sign}(\chi(H)).$$

EXAMPLE 3.42. The following example is well-known. Compare [14, p.106] and [26, 2.3]. Consider the groups

$$G = F_3 * (F_3 \times F_3)$$
$$H = F_4 * (F_3 \times F_3)$$

where F_n denotes the free group of rank n. These groups are quasi-isometric [14, p. 106], but the computation

$$\chi(G) = \chi(F_3) + \chi(F_3)^2 - 1 = -2 + 4 - 1 = 1$$
$$\chi(H) = \chi(F_4) + \chi(F_3)^2 - 1 = -3 + 4 - 1 = 0$$

shows that G and H are not measure equivalent. Moreover, there is no second countable, proper metric space on which they act both properly and cocompactly by isometries.

Chapter 4

Novikov-Shubin Invariants

Like the L^2-Betti numbers, Novikov-Shubin invariants were originally defined in terms of the heat kernel of the universal covering of a compact manifold [48]. The i-th Novikov-Shubin invariant $\alpha_i^\Delta(\tilde{M})$ of M quantifies the speed of convergence of the limit

$$b_i^{(2)}(\tilde{M}) = \lim_{t\to\infty} \underbrace{\int_{\mathcal{F}} \operatorname{tr}_{\mathbb{R}}\left(e^{-t\Delta_i}(x,x)\right) dvol_x}_{\theta_i(t)} .$$

Compare p. 47. To give an impression, we have

$$\alpha_i^\Delta(\tilde{M}) = p, \text{ if } \left(\theta_i(t) - b_i^{(2)}(\tilde{M})\right) \sim t^{-p},$$

and $\alpha_i^\Delta(\tilde{M})$ is infinite, if the decay of $\theta_i(t) - b_i^{(2)}(\tilde{M})$ is exponential.

For extending the definition of L^2-Betti numbers from universal coverings of compact manifolds (or finite simplicial complexes) to arbitrary G-spaces, the notion of a dimension function for modules over finite von Neumann algebras is the key ingredient.

An analogous program has been carried out for Novikov-Shubin invariants in [46], based on the work in [42], [43], [44].

4.1 Introduction to Capacity Theory

In [46] the *capacity* of a module over a finite von Neumann algebra is defined (which is the essentially the inverse of the Novikov-Shubin invariant). We review this notion and some further definitions and theorems from [46].

DEFINITION 4.1 (Novikov-Shubin Invariant of an Operator).
The *Novikov-Shubin invariant* $\alpha(A;\mathcal{A}) \in [0,\infty] \cup \{\infty^+\}$ of an operator $A \in \mathcal{A}$ in a finite von Neumann algebra $\mathcal{A}$ is defined as

$$\alpha(A;\mathcal{A}) = \begin{cases} \liminf\limits_{\lambda\to 0^+} \dfrac{\ln(F(A)(\lambda) - F(A)(0))}{\ln(\lambda)} \in [0,\infty] & \text{if } F(A)(\lambda) > F(A)(0) \text{ for } \lambda > 0, \\ \infty^+ & \text{else.} \end{cases}$$

Here $F(A)(\lambda) = F(A;\mathcal{A})(\lambda) = \mathrm{tr}_{\mathcal{A}}\left(E^{A^*A}_{\lambda^2}\right)$ is the spectral density function introduced in 1.38. We understand ∞^+ as a formal symbol.

REMARK 4.2 (Finite Dimensional von Neumann Algebras).
The Novikov-Shubin invariant of an operator in a finite dimensional von Neumann algebra is always ∞^+. That's because such an operator A lies in some matrix algebra $M_n(\mathbb{C})$, and, if A is positive, E^A_λ is the projection onto the sum of eigenspaces of eigenvalues $\leq \lambda$. Thus $F(A)$ is a (right-continuous) step function.

REMARK 4.3. We remind the reader that the $n\times n$-matrices $M_n(\mathcal{A})$ over a finite von Neumann algebra $\mathcal{A}$ form also a finite von Neumann algebra with the normalized trace

$$\mathrm{tr}_{M_n(\mathcal{A})}((a_{ij})) = \frac{1}{n}\sum_{i=1}^n \mathrm{tr}_{\mathcal{A}}(a_{ii}).$$

In [42, definition 3.7] it is shown that the following definition is well defined.

DEFINITION 4.4. The *Novikov-Shubin invariant* $\alpha(M;\mathcal{A})$ of a finitely presented $\mathcal{A}$-module M with $\dim_{\mathcal{A}}(M) = 0$ is defined by choosing an exact sequence (compare 3.11)

$$0 \to \mathcal{A}^n \xrightarrow{\cdot A} \mathcal{A}^n \to M \to 0, \quad A \in M_n(\mathcal{A}) \text{ positive},$$

and putting

$$\alpha(M;\mathcal{A}) := \alpha(A; M_n(\mathcal{A})).$$

DEFINITION 4.5 (Capacity of a Finitely Presented Module).
Let M be a finitely presented $\mathcal{A}$-module with $\dim_{\mathcal{A}}(M) = 0$. Define its *capacity*

$$c(M;\mathcal{A}) := \frac{1}{\alpha(M;\mathcal{A})} \in [0,\infty] \cup \{0^-\},$$

where 0^- is a new formal symbol. Here we put $0^{-1} = \infty, \infty^{-1} = 0$ and $(\infty^+)^{-1} = 0^-$. We also write $c(M)$, if the von Neumann algebra meant is clear from the context.

DEFINITION 4.6 (Measurable and Cofinal-Measurable Modules).
An $\mathcal{A}$-module is called *measurable* if it is the quotient of a finitely presented $\mathcal{A}$-module M with $\dim_{\mathcal{A}}(M) = 0$. We call an $\mathcal{A}$-module *cofinal-measurable* if each finitely generated submodule is measurable. In particular, this implies $\dim_{\mathcal{A}}(M) = 0$.

DEFINITION 4.7 (Capacity of Arbitrary Modules).
The *capacity* $c(M;\mathcal{A})$ of a measurable $\mathcal{A}$-module M is defined as

$$c(M;\mathcal{A}) = \inf\{c(L);\ L \text{ finitely presented, } \dim_{\mathcal{A}}(L) = 0,\ M \text{ quotient of } L\}.$$

The *capacity* of an arbitrary $\mathcal{A}$-module can now be defined as

$$c(M;\mathcal{A}) = \sup\{c(L);\ L \text{ measurable, } L \subset M\}.$$

In [46] it is shown that all the definitions of $c(M;\mathcal{A})$ are consistent with each other.

DEFINITION 4.8 (Novikov-Shubin Invariant of a Module).
The *Novikov-Shubin invariant* $\alpha(M;\mathcal{A}) \in [0,\infty] \cup \{\infty^+\}$ of an arbitrary $\mathcal{A}$-module is defined as the inverse of $c(M;\mathcal{A})$.

REMARK 4.9. The reason for introducing $c(M)$ as the inverse of $\alpha(M)$ is that it measures the size of a zero-dimensional module M. For instance, a finitely presented zero-dimensional module is zero if and only if its capacity is 0^-. Moreover, some formulas are more suggestive for the capacity (like in 4.16). In the sequel we will use both notions.

DEFINITION 4.10 (Capacity of Spaces).
Let X be topological space with an action of the group G. Then the *n-th capacity of X* is defined as the capacity of the n-th homology of the chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} \mathrm{C}^{\mathrm{sing}}_{\bullet}(X)$ where $\mathrm{C}^{\mathrm{sing}}_{\bullet}(X)$ is the singular chain complex of X:

$$c_n(X;\mathcal{N}(G)) = c(H_n\left(X;\mathcal{N}(G)\right)).$$

The *n-th capacity of the group G* is given by

$$c_n(G) = c_n(H_n(EG;\mathcal{N}(G)) \quad (= \mathrm{Tor}^{\mathbb{Z}G}_{\bullet}(\mathcal{N}(G);\mathbb{Z})),$$

where EG is any universal free G-space.
Of course, we define $\alpha_n(X;\mathcal{N}(G))$ as the inverse of $c_n(X;\mathcal{N}(G))$ etc.

REMARK 4.11. We briefly relate the preceding definition to the presentation in [45]. There the p-th Novikov-Shubin invariant of a finite free G-CW complex X is defined in terms of the p-th differential c_p of its cellular chain complex, which consists of finitely generated free $\mathbb{Z}G$-modules. After a choice of basis, the differential is given by multiplication with a matrix over the integral group ring. It can be shown that $\alpha_{p-1}(X;\mathcal{N}(G))$ (in our sense) is given by

$$\frac{1}{2} \cdot \alpha\Big(c_p^* c_p; M_m(\mathcal{N}(G)\Big)$$

Compare [45, lemma 6.97]. We remark that our definition of $\alpha_p(X;\mathcal{N}(G))$ coincides with the one given in [45] up to a dimension shift of 1.

REMARK 4.12 (Easy Facts).
The following are easy consequences of the definitions above. See [46, p. 161]. For a finitely generated projective $\mathcal{A}$-module P we have $c(P) = 0^-$. A cofinal-measurable $\mathcal{A}$-module M satisfies $c(M) = 0^-$ if and only if M is trivial. For an arbitrary $\mathcal{A}$-module N the equation $c(N) = 0^-$ holds if and only if any $\mathcal{A}$-map $f: M \to N$ from a finitely presented $\mathcal{A}$-module M with $\dim_{\mathcal{A}}(M) = 0$ to N is trivial.

REMARK 4.13 (Homotopy Invariance).
Note that the capacities resp. Novikov-Shubin invariants are homotopy invariants according to the definition above. Before the algebraic approach was available, this was shown for the analytic definition in [31].

REMARK 4.14 (Relation to the Heat Kernel Definition).
The Novikov-Shubin invariants of the universal covering of a compact Riemannian manifold give back the analytic Novikov-Shubin invariants $\alpha^{\Delta}_{\bullet}(\tilde{M})$, which were mentioned in the introduction on p. 65. The precise relation is

$$\alpha^{\Delta}_n(\tilde{M}) = \frac{1}{2} \cdot \min\{\alpha_n(\tilde{M}), \alpha_{n-1}(\tilde{M})\}.$$

In [45, section 2.3] you find a detailed proof going back to Efremovic.

In general, computations of Novikov-Shubin invariants of spaces and groups are hard. We record a few computations.

THEOREM 4.15 (Computations).

- *As a consequence of a theorem of Varopoulos [59], the zeroth Novikov-Shubin invariant of a finitely generated group G can be computed as follows (see [46, proposition 3.2]).*

$$\alpha_0(G) = \begin{cases} \infty^+ & \text{if } G \text{ is finite or non-amenable} \\ n & \text{if } G \text{ has polynomial growth of degree } n \\ \infty & \text{if } G \text{ is infinite and amenable, but not virtually nilpotent.} \end{cases}$$

- *Free abelian groups are computed in [46, theorem 3.7]:*

$$\alpha_p(\mathbb{Z}^n) = \begin{cases} n & \text{if } 0 \le p \le n-1 \\ \infty^+ & \text{otherwise.} \end{cases}$$

$$\alpha_p(\mathbb{Z}^\infty) = \infty \text{ for all } p \ge 0.$$

- *Let M be a hyperbolic manifold of dimension n. Then [40, proposition 46 on p. 499]:*

$$\alpha_p(\tilde{M}) = \begin{cases} 1 & n \text{ odd, } 2p \in \{n-1, n-3\} \\ \infty^+ & \text{otherwise.} \end{cases}$$

- *The Novikov-Shubin-invariants of a closed Riemannian manifold M whose universal covering $\tilde{M}$ is the Heisenberg group H^{2n+1} are given by (see [52]):*

$$\alpha_p(\tilde{M}) = \begin{cases} n+1 & \text{if } p = n \\ 2(n+1) & \text{if } 0 \le p \le \dim M - 1,\ p \ne n \\ \infty^+ & \text{otherwise.} \end{cases}$$

Now we return to abstract capacity theory. The capacity of $\mathcal{A}$-modules has some pleasant properties, which we quote from [46, theorem 2.7]:

THEOREM 4.16.

(i) Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be an exact sequence of $\mathcal{A}$-modules. Then

$c(M_0) \le c(M_1)$

$c(M_2) \le c(M_1)$*, provided M_1 is cofinal-measurable.*

$c(M_1) \le c(M_0) + c(M_2)$ *if* $\dim_{\mathcal{A}}(M_1) = 0$.

(ii) Let $M = \bigcup_{i \in I} M_i$ be a directed union of submodules. Then

$$c(M) = \sup\{c(M_i);\ i \in I\}.$$

(iii) Let M be the colimit of a directed system of $\mathcal{A}$-modules A_i, $i \in I$. Then

$$c(M) \leq \liminf_{i \in I} c(M_i).$$

If every M_i is measurable and all structure maps of the colimit are surjective then

$$c(M) = \inf_{i \in I} c(M_i).$$

(iv) For every family M_i, $i \in I$ of $\mathcal{A}$-modules we have

$$c\left(\oplus_{i \in I} M_i\right) = \sup\{c(M_i);\ i \in I\}.$$

THEOREM 4.17.
Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of finite von Neumann algebras. Then the following holds.

(i) If M is a cofinal-measurable $\mathcal{A}$-module, then $c(M;\mathcal{A}) = c(\mathcal{B} \otimes_{\mathcal{A}} M;\mathcal{B})$.

(ii) If M is a finitely presented $\mathcal{A}$-module, then $c(M;\mathcal{A}) = c(\mathcal{B} \otimes_{\mathcal{A}} M;\mathcal{B})$.

Proof. In [46, lemma 2.12] (i) is proven for inclusions of group von Neumann algebras. The reasoning there for finitely presented modules has to be modified in the general case as follows.

Let M be a finitely presented $\mathcal{A}$-module with $\dim_{\mathcal{A}}(M) = 0$. Then there is a short exact sequence

$$0 \to \mathcal{A}^n \xrightarrow{\cdot A} \mathcal{A}^n \to M \to 0.$$

where $A \in M_n(\mathcal{A})$ is positive (3.11). The functor $\mathcal{B} \otimes_{\mathcal{A}} -$ is exact and preserves dimensions (see 1.48, 3.18). Hence

$$0 \to \mathcal{B}^n \xrightarrow{\cdot A} \mathcal{B}^n \to \mathcal{B} \otimes_{\mathcal{A}} M \to 0$$

is also exact. Theorem 1.39 implies equality $F(A;\mathcal{A}) = F(A;\mathcal{B})$ of the spectral density functions with respect to $\mathcal{A}$ and $\mathcal{B}$. Hence $c(M;\mathcal{A}) = c(\mathcal{B} \otimes_{\mathcal{A}} M;\mathcal{B})$ for finitely presented M.
Now the proof for measurable and cofinal-measurable modules can be copied literally from the paper cited above.
For (ii) note that a finitely presented $\mathcal{A}$-module M splits as

$$M = \mathbf{P}M \oplus \mathbf{T}M$$

into a projective part $\mathbf{P}M$ and into a finitely presented torsion part $\mathbf{T}M$ with $\dim_{\mathcal{A}}(\mathbf{T}M) = 0$ (3.10). The capacity is determined by the torsion part $c(M;\mathcal{A}) = c(\mathbf{T}M;\mathcal{A})$ (see 4.16 (iv) and 4.12). Similarly, we obtain that $c(\mathcal{B} \otimes_{\mathcal{A}} M;\mathcal{B}) = c(\mathcal{B} \otimes_{\mathcal{A}} \mathbf{T}M;\mathcal{B})$. The module $\mathbf{T}M$ has dimension zero and is finitely presented, hence cofinal-measurable. Now (ii) follows from (i). $\square$

DEFINITION 4.18 (Induction-Friendly Modules).
An $\mathcal{N}$-module N is *induction-friendly* if for every inclusion $\mathcal{N} \subset \mathcal{M}$ of finite von Neumann algebras the equality $c(\mathcal{M} \otimes_{\mathcal{N}} N;\mathcal{M}) = c(N;\mathcal{N})$ holds. A group G is *induction-friendly* resp. *cofinal-measurable* if the $\mathcal{N}(G)$-modules $\mathrm{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C})$, $n \geq 0$ are induction-friendly resp. cofinal-measurable.

Note that by the previous theorem a cofinal-measurable group is induction-friendly.

THEOREM 4.19. *Let G be a group. Then the following holds.*

(i) G is induction-friendly, if the trivial $\mathbb{C}G$-module $\mathbb{C}$ has a projective $\mathbb{C}G$-resolution of finite type.

(ii) G is cofinal-measurable, if it has a cofinal-measurable normal subgroup.

(iii) An infinite finitely generated virtually nilpotent group is cofinal-measurable.

(iv) An infinite elementary amenable group containing no infinite locally finite subgroup is cofinal-measurable.

Proof. The group homology $\operatorname{Tor}_{\bullet}^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C})$ is given by the homology of the complex $\mathcal{N}(G) \otimes_{\mathbb{C}G} P_{\bullet}$, where $P_{\bullet}$ is a projective $\mathbb{C}G$-resolution of $\mathbb{C}$. If each P_n is finitely generated, then each $\mathcal{N}(G) \otimes_{\mathbb{C}G} P_n$ is a finitely generated projective $\mathcal{N}(G)$-module, and hence the homology $H_{\bullet}(\mathcal{N}(G) \otimes_{\mathbb{C}G} P_{\bullet})$ is finitely presented because the category of finitely presented $\mathcal{N}(G)$-modules is abelian. See 3.12. Now (i) follows from 4.17.

The other assertions are taken from [46, theorems 3.7, 3.9]. □

The following theorem should be compared to its version for L^2-Betti numbers (3.21). For the capacity it is crucial to consider full idempotents in contrast to dim-full ones. Recall that a full idempotent p in a ring R induces a Morita equivalence between R and pRp given by the functors $Rp \otimes_{pRp} -$ and $pR \otimes_R -$. See 1.36.

THEOREM 4.20 (Capacity and Restriction).
Let $\mathcal{N}$ be a finite von Neumann algebra and $p \in \mathcal{N}$ a full projection. Then for every $p\mathcal{N}p$-module M we have

$$c(\mathcal{N}p \otimes_{p\mathcal{N}p} M; \mathcal{N}) = c(M; p\mathcal{N}p).$$

Equivalently, for every $\mathcal{N}$-module N we have

$$c(pN; p\mathcal{N}p) = c(N; \mathcal{N}).$$

Proof. The functor $\mathcal{N}p \otimes_{p\mathcal{N}p} -$ is exact, preserves finitely presented modules and colimits (1.34). Moreover, it sends zero dimensional modules to zero dimensional ones by 3.18.

First we consider the case that the $p\mathcal{N}p$-module M is finitely presented and has dimension zero. Then there exists a short exact sequence of the form

$$0 \to \big(p\mathcal{N}p\big)^n \xrightarrow{\cdot A} \big(p\mathcal{N}p\big)^n \to M \to 0.$$

where $A \in M_n(p\mathcal{N}p)$ is positive (3.11). Applying the functor $\mathcal{N}p \otimes_{p\mathcal{N}p} -$ and stabilizing, we obtain a short exact sequence

$$0 \to \big(\mathcal{N}p\big)^n \oplus \big(\mathcal{N}(1-p)\big)^n \xrightarrow{(\cdot A)\oplus id} \big(\mathcal{N}p\big)^n \oplus \big(\mathcal{N}\big(1-p\big)\big)^n \to \mathcal{N}p \otimes_{p\mathcal{N}p} M \to 0.$$

The map in the middle is an $\mathcal{N}$-equivariant endomorphism of $\mathcal{N}^n$. It is represented by right multiplication with the matrix $A + (1-p)\,\mathrm{id}_{\mathcal{N}^n} \in M_n(\mathcal{N})$.

For any two operators a, b in a finite von Neumann algebra $\mathcal{A}$ satisfying $a \cdot b = b \cdot a = 0$ and for any Borel function f with $f(0) = 0$ the results of spectral calculus with respect to f satisfy

$$f(a + b; \mathcal{A}) = f(a; \mathcal{A}) + f(b; \mathcal{A}) \tag{4.1}$$

by the following argument. The operators a, b generate an abelian von Neumann algebra $\langle a, b\rangle \cong L^\infty(X)$ in $\mathcal{A}$. So it suffices to check (4.1) in $L^\infty(X)$. There it is clear that (4.1) is true, if we keep in mind that spectral calculus in $L^\infty(X)$ with respect to f is given by composing $f(\phi) = f \circ \phi$, $\phi \in L^\infty(X)$.

Notice that we have

$$\chi_{(0,\lambda]}((1-p)\operatorname{id}_{\mathcal{N}^n}) = 0 \ \text{ for } \lambda < 1, \tag{4.2}$$

because $(1-p)\operatorname{id}_{\mathcal{N}^n}$ is a projection.

Because p is full there are $r_i, s_i \in \mathcal{N}$ ($i = 1, \ldots, n$) such that $\sum_{i=1}^n r_i \cdot p \cdot s_i = 1 \in \mathcal{N}$. This yields

$$\begin{aligned}
&F(A + (1-p)\operatorname{id}_{\mathcal{N}^n}; \mathcal{N})(\lambda) - F(A + (1-p)\operatorname{id}_{\mathcal{N}^n}; \mathcal{N})(0) = \\
&\quad = \operatorname{tr}_{\mathcal{N}}\big(\chi_{(0,\lambda]}(A + (1-p)\operatorname{id}_{\mathcal{N}^n}; \mathcal{N})\big) \\
&\quad = \operatorname{tr}_{\mathcal{N}}\big(\chi_{(0,\lambda]}(A; \mathcal{N})\big) \qquad \text{(by (4.2), (4.1))} \\
&\quad = \operatorname{tr}_{\mathcal{N}}\left(\sum_{i=1}^n r_i p s_i \chi_{(0,\lambda]}(A; \mathcal{N})\right) \\
&\quad = \operatorname{tr}_{\mathcal{N}}\left(\left(\sum_{i=1}^n r_i p s_i\right) p \chi_{(0,\lambda]}(A; \mathcal{N}) p\right) \qquad \text{(by the trace property)} \\
&\quad = \operatorname{tr}_{\mathcal{N}}\big(p \chi_{(0,\lambda]}(A; \mathcal{N}) p\big) \\
&\quad = \frac{1}{\operatorname{tr}_{\mathcal{N}}(p)} \cdot \operatorname{tr}_{p\mathcal{N}p}\big(\chi_{(0,\lambda]}(A; p\mathcal{N}p)\big) \qquad \text{(by 1.41)} \\
&\quad = \frac{1}{\operatorname{tr}_{\mathcal{N}}(p)} \cdot \big(F(A; p\mathcal{N}p)(\lambda) - F(A; p\mathcal{N}p)(0)\big).
\end{aligned}$$

Thus $\alpha(F(A; \mathcal{N})) = \alpha(F(A; p\mathcal{N}p)$ follows. ☐

4.2 Invariance under Quasi-Isometry

In this section we prove that capacities (equivalently Novikov-Shubin invariants) of amenable groups are invariant under quasi-isometry provided the groups are induction-friendly. See 4.25.

LEMMA 4.21. *Let $G \curvearrowright X$ be a standard action, and let $A \subset X$ be a Borel subset such that X is the union of finitely many translates of A up to null-sets. Then the characteristic function χ_A is a full idempotent in the crossed product $L^\infty(X) * G$.*

Proof. By assumption, there are $g_1, \ldots, g_m \in G$ such that $\bigcup_{i=1}^m g_i A = X$ (up to null-sets). So

$$\sum_{i=1}^m g_i \cdot \chi_A \cdot g_i^{-1} = \sum_{i=1}^m \chi_{g_i \cdot A} \geq 1$$

is invertible, hence $\big(L^\infty(X) * G\big)\chi_A\big(L^\infty(X) * G\big) = L^\infty(X) * G$ (where the left term is understood in the sense of 3.19). ☐

LEMMA 4.22. *Let $G \curvearrowright X$ be a standard action, and let $A \subset X$ be a Borel subset such that X is the union of finitely many translates of A up to null-sets. If G is induction-friendly then*

$$c_n(G) = c\left(\operatorname{Tor}_n^{\chi_A(L^\infty(X)*G)\chi_A}\left(\mathcal{N}\left(\mathcal{R}(G\curvearrowright X)_{|A}\right), L^\infty(A)\right); \mathcal{N}\left(\mathcal{R}(G\curvearrowright X)_{|A}\right)\right).$$

Proof. Put $\mathcal{R} = \mathcal{R}(G\curvearrowright X)$. For the following computation we use the exactness of $\mathcal{N}(G) \subset \mathcal{N}(\mathcal{R})$ (1.48) and the exactness of $L^\infty(X)*G$ as a right $\mathbb{C}G$-module, which follows from the equality $(L^\infty(X)*G) \otimes_{\mathbb{C}G} M = L^\infty(X) \otimes_{\mathbb{C}} M$.

$$\begin{aligned}
c_n(G) &= c\Big(\operatorname{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C}); \mathcal{N}(G)\Big) \\
&= c\Big(\mathcal{N}(\mathcal{R}) \otimes_{\mathcal{N}(G)} \operatorname{Tor}_n^{\mathbb{C}G}(\mathcal{N}(G), \mathbb{C}); \mathcal{N}(\mathcal{R})\Big) \\
&= c\Big(\operatorname{Tor}_n^{\mathbb{C}G}(\mathcal{N}(\mathcal{R}), \mathbb{C}); \mathcal{N}(\mathcal{R})\Big) \\
&= c\Big(\operatorname{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}), L^\infty(X)*G \otimes_{\mathbb{C}G} \mathbb{C}); \mathcal{N}(\mathcal{R})\Big) \\
&= c\Big(\operatorname{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}), L^\infty(X)); \mathcal{N}(\mathcal{R})\Big)
\end{aligned}$$

By the preceding lemma χ_A is a full idempotent in $L^\infty(X)*G$, hence the rings $L^\infty(X)*G$ and $\chi_A(L^\infty(X)*G)\chi_A$ are Morita equivalent. Compare with 1.36. Let $P_\bullet$ be a projective $L^\infty(X)*G$-resolution of $L^\infty(X)$. Then $\chi_A P_\bullet$ is a projective $\chi_A(L^\infty(X)*G)\chi_A$-resolution of $L^\infty(A)$ (1.34). The Morita equivalence implies

$$\big(L^\infty(X)*G\chi_A\big) \otimes_{\chi_A L^\infty(X)*G\chi_A} \big(\chi_A P_\bullet\big) \cong P_\bullet.$$

Therefore we obtain

$$\begin{aligned}
&\chi_A \operatorname{Tor}_n^{L^\infty(X)*G}(\mathcal{N}(\mathcal{R}), L^\infty(X)) \\
&\quad= \operatorname{Tor}_n^{L^\infty(X)*G}(\chi_A\mathcal{N}(\mathcal{R}), L^\infty(X)) \\
&\quad= H_n\big(\chi_A\mathcal{N}(\mathcal{R}) \otimes_{L^\infty(X)*G} P_\bullet\big) \\
&\quad= H_n\Big(\big(\chi_A\mathcal{N}(\mathcal{R})\big) \otimes_{L^\infty(X)*G} \big((L^\infty(X)*G)\chi_A\big) \otimes_{\chi_A(L^\infty(X)*G)\chi_A} \big(\chi_A P_\bullet\big)\Big) \\
&\quad= H_n\Big(\big(\chi_A\mathcal{N}(\mathcal{R})\chi_A\big) \otimes_{\chi_A(L^\infty(X)*G)\chi_A} \big(\chi_A P_\bullet\big)\Big) \\
&\quad= \operatorname{Tor}_n^{\chi_A(L^\infty(X)*G)\chi_A}\big(\chi_A\mathcal{N}(\mathcal{R})\chi_A, L^\infty(A)\big).
\end{aligned}$$

Recall that $\mathcal{N}\left(\mathcal{R}_{|A}\right) = \chi_A\mathcal{N}(\mathcal{R})\chi_A$. See 1.55. Theorem 4.20 finishes the proof. □

LEMMA 4.23. *Let $G \curvearrowright X$ and $H \curvearrowright Y$ be standard actions and let $f : A \to B$ be a bounded weak orbit equivalence, where $A \subset X$ and $B \subset Y$ are Borel subsets. Then the induced isomorphism between the (restricted) groupoid rings restricts to an isomorphism of the (restricted) crossed product rings, i.e. there is the following commuting diagram.*

$$\begin{array}{ccc}
\mathbb{C}\mathcal{R}(H\curvearrowright Y)_{|B} & \xrightarrow[\cong]{f^*} & \mathbb{C}\mathcal{R}(G\curvearrowright X)_{|A} \\
\uparrow & & \uparrow \\
\chi_B\big(L^\infty(Y)*H\big)\chi_B & \xrightarrow{\cong} & \chi_A\big(L^\infty(X)*G\big)\chi_A
\end{array}$$

Proof. Let $\phi \in \chi_B(L^\infty(Y)*H)\chi_B \subset \mathbb{C}\mathcal{R}(H\curvearrowright Y)_{|B}$. There is a finite subset $F \subset H$ such that for $h \notin F$ we have $\phi(hy, y) = 0$ almost everywhere. Compare 1.26. There is a finite subset $L \subset G$ such that $g \notin L$ implies $f(gx) \notin Ff(x)$ almost everywhere because the cocycle of f^{-1} is bounded. Therefore we have for $g \notin L$ that $f^*\phi(gx, x) = 0$ almost everywhere, i.e. $f^*\phi \in \chi_A(L^\infty(X)*G)\chi_A$. Because the inverse f^{-1} restricts to the restricted crossed product rings by the same argument, we obtain an isomorphism between them. $\square$

THEOREM 4.24 (Invariance under Bounded Measure Equivalence).
Let G and H be induction-friendly groups. If G and H are boundedly measure equivalent then

$$c_\bullet(G) = c_\bullet(H).$$

Proof. There is a bounded weak orbit equivalence $X \supset A \xrightarrow{f} B \subset Y$ between suitable standard actions $G\curvearrowright X$ and $H\curvearrowright Y$ by 2.33. By 1.56 and 4.23 f induces the following commutative diagram of ring homomorphisms

$$\begin{array}{ccccccc}
L^\infty(B) & \hookrightarrow & \chi_B\big(L^\infty(Y)*H\big)\chi_B & \hookrightarrow & \mathbb{C}\mathcal{R}(H\curvearrowright Y)_{|B} & \hookrightarrow & \mathcal{N}\left(\mathcal{R}(H\curvearrowright Y)_{|B}\right) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
L^\infty(A) & \hookrightarrow & \chi_A\big(L^\infty(X)*G\big)\chi_A & \hookrightarrow & \mathbb{C}\mathcal{R}(G\curvearrowright X)_{|A} & \hookrightarrow & \mathcal{N}\left(\mathcal{R}(G\curvearrowright X)_{|A}\right).
\end{array}$$

where the right vertical map is trace-preserving, and the left vertical map is compatible with respect to the module structures of the crossed product rings. So we obtain the equality

$$\begin{aligned}
&c\left(\operatorname{Tor}_n^{\chi_A L^\infty(X)*G\chi_A}\left(\mathcal{N}\left(\mathcal{R}(G\curvearrowright X)_{|A}\right), L^\infty(A)\right); \mathcal{N}\left(\mathcal{R}(G\curvearrowright X)_{|A}\right)\right) \\
&\quad = c\left(\operatorname{Tor}_n^{\chi_B L^\infty(Y)*H\chi_B}\left(\mathcal{N}\left(\mathcal{R}(H\curvearrowright Y)_{|B}\right), L^\infty(B)\right); \mathcal{N}\left(\mathcal{R}(H\curvearrowright Y)_{|B}\right)\right).
\end{aligned}$$

Now 4.22 finishes the proof. $\square$

Using the theorems 2.38 and 2.36 we obtain the following two corollaries.

COROLLARY 4.25 (Invariance under Quasi-Isometry).
Let G and H be finitely generated amenable, induction-friendly groups. If G and H are quasi-isometric then

$$c_\bullet(G) = c_\bullet(H).$$

COROLLARY 4.26 (Actions on a Metric Space).
Let G and H be induction-friendly groups. Assume that G and H act isometrically on the same second countable, proper metric space X such that both actions are proper and cocompact. Then

$$c_\bullet(G) = c_\bullet(H).$$

REMARK 4.27 (Not Invariant Under Measure Equivalence).
By 2.21 all infinite amenable groups are measure equivalent. But the values $c_i(\mathbb{Z}^n)$ differ for different $n \in \mathbb{N}$. Compare 4.15. Thus capacities of groups are not invariant under measure equivalence.

Chapter 5

The Rationality Conjecture for Novikov-Shubin Invariants

Due to the analytical definition of L^2-invariants, their values are a priori real. The natural question about their possible values has been studied since the very beginning of the theory. Conjecturally, L^2-Betti numbers and Novikov-Shubin invariants of finite free G-CW complexes, say, for instance, universal coverings of compact CW-complexes, are expected to be rational. The rationality conjecture for the L^2-Betti numbers is referred to as the *Atiyah conjecture*.

Meanwhile, there is a large number of results concerning the Atiyah conjecture. See [27], [37], [38], [39], [53], [54]. The conjecture about the rationality (and positivity) of Novikov-Shubin invariants was formulated by J. Lott and W. Lück [41, conjecture 7.1 on p. 56]. Surveys about both conjectures and their various formulations can be found in [45, p. 112-114, 370-376]. We state the precise formulation for the Novikov-Shubin invariants.

CONJECTURE. *The Novikov-Shubin invariants of a finite free G-CW complex are positive rational unless they are ∞ or ∞^+.*

Equivalent to that is the following formulation. See [45, p. 113, proof of 10.5 on p. 371].

CONJECTURE. *The Novikov-Shubin invariant of a matrix $A \in M_n(\mathbb{Z}G)$ over the group ring is positive rational unless it is ∞ or ∞^+.*

We prove the conjecture for virtually free groups in 5.29, 5.30. Moreover, the value ∞ does not occur in that case. The conjecture has also been verified for abelian groups by J. Lott [40, proposition 39 on p. 494] (see also [45, p.113]). Also in this case the value ∞ does not occur.

We stress that the positivity statement of the conjecture is at least as important as the rationality statement. For instance, $\alpha(A) > 0$ implies that there are constants $\alpha > 0, K > 0$ such that the spectral density function $F_A(\lambda)$ satisfies $F_A(\lambda) - F_A(0) \leq K\lambda^\alpha$ for small $\lambda > 0$, which is an important estimate in various circumstances.

For free groups it turns out that the conjecture follows from the fact that certain power series are algebraic. In the first section the necessary inputs from the theory of power series in noncommuting variables are developed. We believe that the main result there (see 5.21) is of independent interest.

In the second section we explain how to extract information about the spectrum of an operator from a certain power series. For illustration, we provide an explicit determination of the spectral measure of the Markov operator of the free group.

5.1 Power Series in Noncommuting Variables

Before we dive into the world of formal power series with noncommuting variables, we recall some relevant notions in the commutative case.

The ring of formal power series over a (not necessarily commutative) ring R in a set of variables X is denoted by $R[\![X]\!]$. The ring of polynomials is denoted by $R[X]$. Let us review some well-known notions and facts for the case that the coefficient ring k of $k[\![X]\!]$ is a commutative field.

An element in $k[\![X]\!]$ is invertible if and only if its constant term is invertible, i.e. non-zero. The quotient field of the integral domain $k[X]$ of polynomials is denoted by $k(X)$. The quotient field of the integral domain $k[\![X]\!]$, denoted by $k(\!(X)\!)$, contains both $k(X)$ and $k[\![X]\!]$. Let D be an integral domain that contains $k(X)$. Then $P \in D$ is said to be *algebraic over* $k(X)$, if there exist $p_0, \ldots, p_d \in k(X)$, not all 0, such that

$$p_d P^d + p_{d-1} P^{d-1} + \cdots + p_0 = 0.$$

By clearing denominators, the p_i above can be assumed to be polynomials, i.e. $p_i \in k[X]$. The algebraic elements in D form a $k(X)$-subalgebra of D. The power series in $k[\![X]\!] \subset k((X))$, which are algebraic over $k(X)$, are called *algebraic (formal) power series*. We denote the set of all algebraic formal power series by $k_{alg}[\![X]\!]$.

Before we develop analogous notions for the non-commutative setting, we have to recall some concepts from ring theory.

DEFINITION 5.1 (Division Closure and Rational Closure).
Let S be a ring and $R \subset S$ be a subring.

(i) R is *division closed* if for every element in R, which is invertible in S, the inverse already lies in R.

(ii) R is *rationally closed* if for every matrix over R, which is invertible in S, the entries of the inverse already lie in R.

(iii) The *division closure* of R in S denoted by $\mathcal{D}(R \subset S)$ is the smallest division closed subring containing R.

(iv) The *rational closure* of R in S denoted by $\mathcal{R}(R \subset S)$ is the smallest rationally closed subring containing R.

Note that the intersections of division closed resp. rationally closed subrings is division resp. rationally closed. The rational closure has the advantage that it can be explicitly described as follows (see [11, theorem 1.2 on p. 383]).

THEOREM 5.2 (Explicit Description of the Rational Closure).
Let $R \subset S$ be a ring extension. Then $s \in S$ is an element of $\mathcal{R}(R \subset S)$ if and only if there is a matrix $A \in M(R)$ over R, which is invertible over S, such that s is an entry of $A^{-1} \in M(S)$. Further, $s \in \mathcal{R}(R \subset S)$ holds if and only if there is a matrix $A \in M_n(R)$, which is invertible over S, and a column vector $b \in R^n$ such that s is a component of the solution u of the matrix equation $Au = b$.

Consider a homomorphism $\phi : S_1 \to S_2$ of ring extensions $R_1 \subset S_1$, $R_2 \subset S_2$, i.e. f restricts to $f_{|R_1} : R_1 \to R_2$. Then ϕ extends canonically to a homomorphism $M(\phi) : M(S_1) \to M(S_2)$, which restricts to the matrix rings of R_1, R_2 respectively. Since $M(\phi)$ maps invertible elements in $M(S_1)$ to invertible elements in $M(S_2)$, the preceding theorem implies the next corollary.

COROLLARY 5.3. *The rational closure is functorial with respect to homomorphisms of ring extensions, i.e. for a homomorphism $f : S_1 \to S_2$ restricting to the subrings $R_1 \subset S_1$, $R_2 \subset S_2$, we have $\phi(\mathcal{R}(R_1 \subset S_1)) \subset \mathcal{R}(R_2 \subset S_2)$.*

Next we explain the concept of formal power series in noncommuting variables. Let X be a finite set, called an *alphabet*, and let X^* be the free monoid generated by the elements of X. Thus X^* consists of all finite words (strings) $x_1 \ldots x_n$ of elements in X including the empty word $1 \in X^*$. The product in X^* is given by concatenation of words. The *length* of $w = x_1 \ldots x_n \in X^*$ is given by n, that is the number of letters in w. We write S^+ for $S - \{1\}$ where $S \subset X^*$, and we write X^+ for $X^* - \{1\}$.

A *(formal) power series* in the set of noncommuting variables X over a (not necessarily commutative) ring R is a function $P : X^* \to R$. We write $\langle P, w\rangle$ for $P(w)$, and then use the suggestive notation

$$P = \sum_{w \in X^*} \langle P, w\rangle w.$$

The power series P is called a *polynomial in X* if it has finite support, i.e. $\langle P, w\rangle \neq 0$ holds only for finitely many $w \in X^*$. The set of all formal power series and all polynomials over R in X is denoted by $R\langle\langle X\rangle\rangle$, $R\langle X\rangle$ respectively.

The set of formal power series over R in noncommuting variables has a ring structure (even an R-algebra structure for commutative R). The addition is componentwise and the product is given by

$$\left(\sum_{w \in X^*} a_w w\right) \cdot \left(\sum_{w \in X^*} b_w w\right) = \sum_{w \in X^*} \left(\sum_{uv = w} a_u b_v\right) w.$$

This is like the ordinary product of power series besides the fact that the variables do not commute with each other, whereas they do commute with the elements of R. We will sometimes omit the point $\cdot$ in the notation. The set of all polynomials $R\langle X\rangle$ is a subring of $R\langle\langle X\rangle\rangle$. A term of the form $a \cdot w$, $a \in R$, $w \in X^*$ is called a *monomial*, and the *degree* of $a \cdot w$ is defined as the length of w.

We say that a sequence $P_1, P_2, \ldots$ of formal power series *converges* to $P \in R\langle\langle X\rangle\rangle$, if for every $w \in X^*$ there are only finitely many $i \in \mathbb{N}$ such that $\langle P_i, w\rangle \neq \langle P, w\rangle$.

The *augmentation homomorphism* $\epsilon : R\langle\langle X\rangle\rangle \to R$ is the ring map given by $\epsilon(P) = \langle P, 1\rangle$. In the commutative case $\epsilon : R[\![X]\!] \to R$ is defined analogously.

There is a canonical epimorphism $\phi : R\langle\langle X\rangle\rangle \to R[\![X]\!]$ from the formal power series ring in noncommuting variables to the one in commuting variables.

LEMMA 5.4 (Invertible Formal Power Series).
A formal power series $P \in R\langle\langle X\rangle\rangle$ is invertible if and only if $\epsilon(P)$ is invertible in R. In this case the inverse is given by

$$P^{-1} = \sum_{k=0}^{\infty}(1-\epsilon(P)^{-1}\cdot P)^k \cdot \epsilon(P)^{-1}.$$

The analogous statement holds for $R[\![X]\!]$.

Proof. First of all, the sum converges to a formal power series because the minimal length of a monomial in $(1-\epsilon(P)^{-1}\cdot P)^k$ goes to infinity for $k\to\infty$. Put $T := \sum_{k=0}^{\infty}(1-\epsilon(P)^{-1}\cdot P)^k\cdot\epsilon(P)^{-1}$. Then T is a right inverse of P because of $(1-\epsilon(P)^{-1}P)\cdot T = T-\epsilon(P)^{-1}$, and so $\epsilon(P)^{-1}PT = \epsilon(P)^{-1}$, $PT = 1$ follow. Similarly, we see that $S := \epsilon(P)^{-1}\cdot\sum_{k=0}^{\infty}(1-P\cdot\epsilon(P)^{-1})^k$ is a left inverse. Thus P is invertible, and $S = T$ must be true. □

DEFINITION 5.5 (Rational Power Series).
The rational closure $\mathcal{R}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$ is called the *ring of rational power series over R in the noncommuting variables X*, and is denoted by $R_{rat}\langle\langle X\rangle\rangle$. We define $R_{rat}[\![X]\!]$ analogously.

THEOREM 5.6. *Let R be an arbitrary ring, and X be a finite set. Then the division closure $\mathcal{D}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$ of the polynomials $R\langle X\rangle$ in the ring of formal power series $R\langle\langle X\rangle\rangle$ coincides with the rational closure $\mathcal{R}(R\langle X\rangle \subset R\langle\langle X\rangle\rangle)$. The analogous statement for $R[\![X]\!]$ also holds.*

Proof. Let $P\in\mathcal{R}(R\langle X\rangle\subset R\langle\langle X\rangle\rangle)$. Then, by 5.2, there is a matrix $A\in M_n(R\langle X\rangle)$ $(= M_n(R)\langle X\rangle)$, invertible over $R\langle\langle X\rangle\rangle$ and a vector $b\in(R\langle X\rangle)^n$ such that P is a component of the solution u of the matrix equation $Au=b$. By 5.4 we know that the augmentation $\epsilon(A)\in M_n(R)$ must be invertible (over R). So, multiplying with $\epsilon(A)^{-1}$ from the left, we can assume that the matrix equation has the form $(\mathrm{id}+A)u=b$, where A has zero constant term. More generally, consider a system of equations

$$\begin{array}{ccccccccc}
(1+A_{11})u_1 & + & A_{22}u_2 & + & \dots & + & A_{1n}u_n & = & b_1\\
A_{21}u_1 & + & (1+A_{22})u_2 & + & \dots & + & A_{2n}u_n & = & b_2\\
\vdots & & \vdots & & & & \vdots & & \vdots\\
A_{n1}u_1 & + & A_{n2}u_2 & + & \dots & + & (1+A_{nn})u_n & = & b_n,
\end{array}$$

where all A_{ij} lie in $\mathcal{D}(R\langle X\rangle\subset R\langle\langle X\rangle\rangle)$ and have zero constant term. By induction over n, we show that then there is a unique solution $u=(u_1,\dots,u_n)$ in $\mathcal{D}(R\langle X\rangle\subset R\langle\langle X\rangle\rangle)$. For $n=1$ we get $u_1=(1+A_{11})^{-1}b_1\in\mathcal{D}(R\langle X\rangle\subset R\langle\langle X\rangle\rangle)$. Assume it is true for $n-1$. Then we can do *Gaussian elimination*. Multiply the first equation on the left with $-A_{j1}(1+A_{11})^{-1}\in\mathcal{D}(R\langle X\rangle\subset R\langle\langle X\rangle\rangle)$ and add to the jth equation for $2\le j\le n$. We obtain a system of $n-1$ equations with the same structure, which has, by induction hypothesis, a unique solution $u_2,\dots,u_n$ lying in the division closure. Solving the first equation, we get $u_1\in\mathcal{D}(R\langle X\rangle\subset R\langle\langle X\rangle\rangle)$. The proof for $R[\![X]\!]$ is analogous. □

REMARK 5.7. If k is a commutative field, then

$$\mathcal{D}(k[X] \subset k[\![X]\!]) = \mathcal{R}(k[X] \subset k[\![X]\!]) = k(X) \cap k[\![X]\!].$$

The intersection $k(X) \cap k[\![X]\!] \subset k((X))$ consists of the quotients P/Q of polynomials in $k(X)$ such that $Q(0) \neq 0$.

However, there is no calculus of fractions for $R_{rat}\langle\langle X\rangle\rangle$. In particular, not every rational series in $R_{rat}\langle\langle X\rangle\rangle$ is a quotient of two polynomials. One can easily see that there is no polynomial $S \in R\langle x, y\rangle$ such that

$$S \cdot \big((1-x)^{-1} + (1-y)^{-1}\big) \in R\langle x, y\rangle.$$

Moreover, there are no polynomials $S, T \in R\langle x, y, z\rangle$ with

$$S \cdot \big((1-x)^{-1} + (1-y)^{-1} + (1-z)^{-1}\big) \cdot T \in R\langle x, y, z\rangle.$$

The previous theorem should be regarded as good news because the division closure, which cannot be described explicitly in general, can be replaced by the rational closure in our situation, which is very explicit. We remark that several papers and books, we cite, use the division closure for the definition of rational series.

There is an important theorem characterizing the sequence of coefficients of a rational power series in noncommuting variables. It says that this sequence is *recognizable* – a term coming from the theory of formal languages. The theorem is due to Schützenberger [55]. See also [57, theorem 6.5.7 on p. 202].

THEOREM 5.8 (Recognizability of Rational Power Series).
A formal power series $P \in R\langle\langle X\rangle\rangle$ is rational if and only if there is a monoid homomorphism $\mu : X^* \to M_n(R)$ from X^* into the matrix ring $M_n(R)$ for some $n \in \mathbb{N}$ such that $\langle P, w\rangle = \mu(w)_{1n}$ for all $w \in X^*$.

EXAMPLE 5.9. From 5.3 we see that the canonical homomorphism $\phi : R\langle\langle X\rangle\rangle \to R[\![X]\!]$ restricts to a map $R_{rat}\langle\langle X\rangle\rangle \to R_{rat}[\![X]\!]$. On the other hand, it is not true that $\phi(P) \in R_{rat}[\![X]\!]$ implies $P \in R_{rat}\langle\langle X\rangle\rangle$. A counterexample (see [57, example 6.6.2 on p. 203]) is given by the formal power series

$$P(x, y) = \sum_{n=0}^{\infty} x^n y^n \in \mathbb{C}\langle\langle x, y\rangle\rangle.$$

$P = P(x, y)$ maps to the geometric series

$$\sum_{n=0}^{\infty} (xy)^n = \frac{1}{1 - xy}$$

under ϕ. But P is not rational, which can be seen by the Recognizability theorem. Suppose there is a monoid homomorphism $\mu : \{x, y\}^* \to M_n(\mathbb{C})$ such that $\mu(x^i y^j)_{1n} = \delta_{ij}$. Put $A = \mu(x)$, $B = \mu(y)$. So we have $(A^i B^j)_{1n} = \delta_{ij}$. By the Cayley Hamilton theorem in linear algebra every matrix is a zero of its characteristic polynomial, so there are scalars $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$ such that $A^n = a_{n-1}A^{n-1} + \cdots + a_0 A^0$. But this implies

$$1 = (A^n B^n)_{1n} = \sum_{i=0}^{n-1} a_i (A^i B^n)_{1n}$$

which is a contradiction. Hence P cannot be rational.

EXAMPLE 5.10 (Word Problem of $\mathbf{Z}/2 \times \mathbf{Z}/2$).
Let G be a finitely generated group and $S \subset G$ be a finite subset that generates G as a monoid. The *language of the word problem* $\mathcal{W}(G)$ (with respect to S) is defined as the set of words $w = x_1 x_2 \cdots x_n \in S^*$ that reduce to the identity in G. We associate the formal power series

$$P_G = \sum_{w \in \mathcal{W}(G)} w \in \mathbb{Z}\langle\langle S \rangle\rangle.$$

in the noncommuting variables S to $\mathcal{W}(G)$. For instance, we have

$$P_{\mathbb{Z}/2} = (1 - x^2)^{-1} = \sum_{n \geq 0} x^{2n},$$

where x represents the generator of $\mathbb{Z}/2$. Now we want to consider the non-trivial example

$$G = \mathbb{Z}/2 \times \mathbb{Z}/2 = \{x, y;\ x^2 = y^2 = 1, x \cdot y = y \cdot x\}$$

with the monoid generators $S = \{x, y\}$. For $z \in \{1, x, y, x \cdot y\} \subset G$ define $\mathcal{W}(z)$ as the set of words in S^* that reduce to z in G. Put $P_z = \sum_{w \in \mathcal{W}(z)} w$. The obvious fact that a word in $\mathcal{W}(z)$ is trivial or ends with x or y leads to the following system of equations for $P_1, P_x, P_y, P_{x \cdot y}$.

$$\begin{aligned} P_1 &= 1 + P_x x + P_y y \\ P_x &= P_1 x + P_{x \cdot y} y \\ P_y &= P_1 y + P_{x \cdot y} x \\ P_{x \cdot y} &= P_x y + P_y x. \end{aligned}$$

Now we could apply noncommutative Gaussian elimination and end up with an explicit solution of $P_G = P_1$.

DEFINITION 5.11 (Proper Algebraic System).
Let X be an alphabet, and let $Z = \{z_1, \ldots, z_n\}$ be an alphabet disjoint from X. A *proper algebraic system* is a set of equations $z_i = p_i$, $1 \leq i \leq n$ such that

(i) $p_i = p_i(X, Z) \in R\langle X \cup Z \rangle$ for all $1 \leq i \leq n$,

(ii) $\langle p_i, 1 \rangle = 0$ and $\langle p_i, z_j \rangle = 0$ for all $1 \leq i, j \leq n$, i.e. p_i has no constant term and no linear terms in the z_j.

A *solution* to the proper algebraic system is an n-tuple $(S_1, \ldots, S_n) \in R\langle\langle X \rangle\rangle^n$ of formal power series in X each having zero constant term and satisfying

$$S_i = p_i(X, Z)_{z_j = S_j} \text{ for } 1 \leq i \leq n.$$

Here $p_i(X, Z)_{z_j = S_j}$ means that we formally substitute each z_j by S_j in $p_i(X, Z)$. Each S_j is called a *component* of the solution. Sometimes we call the n-tuple $(p_1, \ldots, p_n)$ a proper algebraic system.

THEOREM 5.12 (Unique Solution).
Every proper algebraic system $(p_1, \ldots, p_n)$ has a unique solution $(S_1, \ldots, S_n) \in R\langle\langle X \rangle\rangle^n$.

Proof. The following proof is constructive, and the method is called *successive approximation* [57, proposition 6.6.3 on p. 203]. Put $S_i^{(0)} = 0$ for each $1 \leq i \leq n$, and define inductively

$$S_i^{(m+1)} = p_i(X, Z)_{z_j = S_j^{(m)}}.$$

Then $S_i^{(m+1)}$ and $S_i^{(m)}$ agree in all monomials of degree $\leq m$. This can be seen by induction on m. It is clear for $m = 0$ because of $p_i(0;0) = 0$. Assume it is true for $m \geq 0$. Then $S_i^{(m+1)}$ and $S_i^{(m+2)}$ agree in all monomials of degree $\leq m+1$ because $p_i(0;Z)$ has no non-zero linear terms, and so only monomials of degree $\leq m$ in $S_i^{(m)}$ resp. $S_i^{(m+1)}$ contribute to the monomials of degree $\leq m+1$ in $S_i^{(m+1)}$ resp. $S_i^{(m+2)}$. The limit

$$S_i := \lim_{m\to\infty} S_i^{(m)}, \quad 1 \leq i \leq n$$

is well defined, and $(S_1, \ldots, S_n)$ is a solution. The uniqueness is obtained by a similar inductive argument. ☐

DEFINITION 5.13 (Algebraic Power Series in Noncommuting Variables).
A formal power series $p \in R\langle\langle X\rangle\rangle$ is *algebraic* if $P - \langle P, 1\rangle$ is a component of the solution of some proper algebraic system. The set of all algebraic formal power series in $R\langle\langle X\rangle\rangle$ is denoted by $R_{alg}\langle\langle X\rangle\rangle$.

EXAMPLE 5.14. The power series $\sum_{n\geq 1} x^n y^n \in \mathbb{Z}\langle\langle x, y\rangle\rangle$ (compare example 5.9) is algebraic because it satisfies the equation $z = xy + xzy$. If we apply the method of successive approximation to this equation, the first stages of the computation would be:

$$\begin{aligned}
S^0 &= 0\\
S^1 &= xy\\
S^2 &= xy + x(xy)y = xy + x^2y^2\\
S^3 &= xy + x(xy + x^2y^2)y = xy + x^2y^2 + x^3y^3.
\end{aligned}$$

The following theorem (whose proof is rather easy) can be found in [11, theorem 9.17 on p. 135].

THEOREM 5.15. *$R_{alg}\langle\langle X\rangle\rangle \subset R\langle\langle X\rangle\rangle$ is a subring containing $R_{rat}\langle\langle X\rangle\rangle$.*

As we would expect from a reasonable notion of *algebraicity* in the noncommutative world, it is compatible with *algebraicity* in the commutative setting. See [57, theorem 6.6.10 on p. 207] and [57, theorem 6.1.12 on p. 168] for the proofs of the next two theorems.

THEOREM 5.16. *Let k be a commutative field. Then the algebraic formal power series (in noncommuting variables) are mapped to algebraic formal power series (in commuting variables) under the canonical homomorphism $\phi : k\langle\langle X\rangle\rangle \to k[\![X]\!]$.*

THEOREM 5.17. *Let k be a commutative field, and let $P \in k(x)((x_1, \ldots, x_n))$ be algebraic over $k(x)(x_1, \ldots, x_n)$. If $P(1, \ldots, 1)$ is a well-defined element in $k((x))$, then $P(1, \ldots, 1) \in k((x))$ is algebraic over $k(x)$.*

DEFINITION 5.18 (Hadamard Product).
Let $P, Q \in R\langle\langle X\rangle\rangle$. The *Hadamard product* $P \odot Q$ of P and Q is defined as

$$P \odot Q = \sum_{w \in X^*} \langle P, w\rangle \langle Q, w\rangle w.$$

The following theorem by Schützenberger [56] is central to the theory of formal power series in noncommuting variables, and it is the crucial ingredient in the proof of our main result 5.21. For a proof see also [57, proposition 6.6.12 on p. 208].

THEOREM 5.19 (Schützenberger's Theorem).
Let R be a commutative ring. The Hadamard product of two rational formal power series in $R\langle\langle X\rangle\rangle$ is again rational, and the Hadamard product of an algebraic with a rational formal power series in $R\langle\langle X\rangle\rangle$ is algebraic.

EXAMPLE 5.20 (Word Problem of Free Groups).
We come back to the situation of example 5.10. Consider the free group F^n in n letters $x_1, \ldots, x_n$. It is generated as a monoid by $S = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$. The language of the word problem $\mathcal{W}(F^n)$ is the set of those words in S^* that reduce to the identity under the relations

$$x_i x_i^{-1} = x_i^{-1} x_i = 1,\ 1 \leq i \leq n.$$

We will construct a proper algebraic system for $P_{F^n} = \sum_{w \in \mathcal{W}(F^n)} w \in \mathbb{Z}\langle\langle S\rangle\rangle$. The algebraicity of P_{F^n} was first observed by Chomsky and Schützenberger [9].

We say that an element in $\mathcal{W}(F^n)$ is *atomic* if it cannot be written as the product of two words in $\mathcal{W}(F^n)^+$. For $t \in S$ we define G_t as the subset of $\mathcal{W}(F^n)$ consisting of atomic words whose first letter is t, i.e.

$$G_t = \{w \in \mathcal{W}(F^n);\ w = tv, w \neq uu' \text{ for } u, u' \in \mathcal{W}(F^n)^+\}.$$

Define $P_t = \sum_{w \in G_t} w \in \mathbb{Z}\langle\langle S\rangle\rangle$ and put

$$\bar{t} = \begin{cases} x_i & \text{if } t = x_i^{-1} \\ x_i^{-1} & \text{if } t = x_i. \end{cases}$$

Claim 0: Each word in G_t must end in $\bar{t}$.

For $w \in G_t$ of length ≤ 2 that is certainly true. Assume it is true for words in G_t of length smaller than m where $m > 2$. Let $w \in G_t$ be of length m. Since w reduces to the identity, it contains a substring of the form $x\bar{x}$ with $x \in S$. This substring is not at the first or last position, otherwise w would not be atomic. So we get that $w = tw_1 x\bar{x} w_2 b$ with $w_1, w_2 \in S^*$, $b \in S$. The word tw_1w_2b is also atomic, and by the induction hypothesis it follows $b = \bar{t}$.

Now define the subset $B_t \subset S^*$ by requiring $G_t = tB_t\bar{t}$, and put $Q_t = \sum_{w \in B_t} w$.

Claim 1: Every $w \in \mathcal{W}(F^n)^+$ can be uniquely written as $w = uv$ with $u \in \mathcal{W}(F^n)$, $v \in G_t$ for some $t \in S$.

For a given $w \in \mathcal{W}(F^n)^+$ define the string v as the string of minimal length in $\mathcal{W}(F^n)^+$ such that there is a factorization $w = uv$. Then there must be a $t \in S$ with $v \in G_t$. The converse of the first claim is trivial:

Claim 2: Every product uv with $u \in \mathcal{W}(F^n)$, $v \in G_t$ lies in $\mathcal{W}(F^n)^+$.

Claim 3: Every $w \in B_t^+$ can be uniquely written as $w = uv$ with $u \in B_t$, $v \in G_q$, $q \neq \bar{t}$.

By the first claim there exist a unique u and $v \in G_q$ with $w = uv$. Because $tw\bar{t} = tuv\bar{t} \in G_t$ is atomic, the string v cannot end with t, hence $q \neq \bar{t}$. It is clear that $tu\bar{t}$ reduces to the identity. If $tu\bar{t}$ would not be atomic then $tw\bar{t} = tuv\bar{t}$ would not be either, and so we must have $u \in B_t$.

Claim 4: Every word $w = uv$ with $u \in B_t$, $v \in G_q$, $q \neq \bar{t}$ lies in B_t^+.

Suppose that $tw\bar{t}$ is not atomic. Then let $tw\bar{t} = u'v'$ be a factorization with $u', v' \in \mathcal{W}(F^n)^+$ and u' having minimal length. We have $u' \in G_t$. Since $tu\bar{t}$ is atomic we must have $u' = tur\bar{t}$. It is $r \in \mathcal{W}(F^n)^+$ because of $q \neq \bar{t}$. So we have $r, r' \in \mathcal{W}(F^n)^+$ with $v = rr'$. This contradicts v being atomic, hence $tw\bar{t} \in G_t$, i.e. $w \in B_t$.

Algebraically, the first and second claim can be expressed by the equation

$$P_{F^n} = 1 + P_{F^n} \sum_{t \in S} P_t, \tag{5.1}$$

and the third and fourth claim translate into

$$Q_t = 1 + Q_t \sum_{\substack{q \in S \\ q \neq \bar{t}}} P_q \quad \text{for all } t \in S. \tag{5.2}$$

The equations (5.1), (5.2) yield the following proper algebraic system with the solution $(P_{F^n}^+, Q_{x_1}^+, Q_{x_1^{-1}}^+, \ldots, Q_{x_n}^+, Q_{x_n^{-1}}^+)$. Here we use the abbreviation $P^+ = P - \langle P, 1 \rangle$ for a power series P.

$$\begin{aligned} P_{F^n}^+ &= (P_{F^n}^+ + 1) \sum_{q \in S} q(Q_q^+ + 1)\bar{q} \\ Q_t^+ &= (Q_t^+ + 1) \sum_{\substack{q \in S \\ q \neq \bar{t}}} q(Q_q^+ + 1)\bar{q}, \qquad t \in S \end{aligned}$$

Hence P_{F^n} is algebraic.

Now we want to consider formal power series in one variable over group rings. Let G be a group and R be a ring. The *von Neumann trace on the group ring RG* is the mapping

$$\mathrm{tr}_{RG} : RG \to R, \sum_{g \in G} a_g g \mapsto a_1.$$

For $R = \mathbb{C}$ this is the restriction of $\mathrm{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) \to \mathbb{C}$ to $\mathbb{C}G$. This map extends to a map of the associated power series rings (in one variable)

$$\mathrm{tr}_{RG} : RG[\![z]\!] \to R[\![z]\!], \sum_{n \geq 0} a_n z^n \mapsto \sum_{n \geq 0} \mathrm{tr}_{RG}(a_n) z^n.$$

THEOREM 5.21 (Power Series over the Group Ring of Free Groups).

(i) Let H be a subgroup of G with finite index $n < \infty$, and let k be a commutative ring with $\frac{1}{n} \in k$. Then

$$\operatorname{tr}_{kG}(kG_{rat}[\![z]\!]) \subset \operatorname{tr}_{kH}(kH_{rat}[\![z]\!]).$$

(ii) Let k be a commutative field and F be a virtually free group. Then

$$\operatorname{tr}_{kF}(kF_{rat}[\![z]\!]) \subset k_{alg}[\![z]\!].$$

Proof. (i) By choosing a system of representatives $\{g_1, \ldots, g_n\}$ for G/H we get an isomorphism of left kH-modules

$$kG \xrightarrow{\cong} \bigoplus_{i=1}^{n} kH,\; g \longmapsto (h_1, \ldots, h_n) \text{ with } h_i = \begin{cases} gx_i^{-1} & \text{if } gx_i^{-1} \in H \\ 0 & \text{else.} \end{cases}$$

This induces the injection ϕ of rings

$$\phi : kG = \hom_{kG}(kG, kG) \hookrightarrow \hom_{kH}(kG, kG) \cong M_n(kH).$$

Let $\Sigma : M_n(kH) \to kH$ be the map defined by taking the sum of the diagonal entries. The canonical extensions of ϕ and Σ to the respective power series rings are denoted by the same symbol. A little computation shows that the von Neumann traces tr_{kG} and tr_{kH} on kG and kH satisfy

$$\frac{1}{n} \operatorname{tr}_{kH} \circ \Sigma \circ \phi = \operatorname{tr}_{kG}.$$

By 5.3 the map ϕ restricts to a homomorphism $\phi : kG_{rat}[\![z]\!] \to M_n(kH)_{rat}[\![z]\!]$. We get the inclusion

$$\operatorname{tr}_{kG}(kG_{rat}[\![z]\!]) \subset (\operatorname{tr}_{kH} \circ \Sigma)(M_n(kH)_{rat}[\![z]\!]).$$

But from the explicit description of the rational closure (5.2) it is clear that the entries of $M_n(kH)_{rat}[\![z]\!]$ lie in $kH_{rat}[\![z]\!]$. Therefore $\Sigma(M_n(kH)_{rat}[\![z]\!]) \subset kH_{rat}[\![z]\!]$, and the claim follows.

(ii) For the second assertion we can restrict to free groups because of the first part. Furthermore, every free group F is the union of its finitely generated subgroups F_i, $i \in I$. As one knows, the F_i are also free. It is easy to see that $kF_{rat}[\![z]\!]$ is the union of the $(kF_i)_{rat}[\![z]\!]$. So it suffices to deal with finitely generated free groups.

Let F be the free group in n letters $x_1, x_2, \ldots, x_n$, and let S be the alphabet $S = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$. For greater clarity, we denote the empty string in S^* by e. In the sequel we shall frequently use the fact that a formal power series (with commuting or noncommuting variables) is invertible if and only if its augmentation is invertible in the coefficient ring (see 5.4).

By rearranging terms, we get the following ring inclusions.

$$k\langle S\rangle[\![z]\!] \subset k[\![z]\!]\langle\!\langle S\rangle\!\rangle \subset k(\!(z)\!)\langle\!\langle S\rangle\!\rangle \supset k(z)\langle\!\langle S\rangle\!\rangle \supset k(z)\langle S\rangle$$

Thus it makes formally sense to claim

$$(k\langle S\rangle)_{rat}[\![z]\!] \subset (k(z))_{rat}\langle\!\langle S\rangle\!\rangle. \tag{5.3}$$

Let us show this. An element in $(k\langle S\rangle)_{rat}[\![z]\!]$ is an entry in the inverse of some matrix $A \in M_n(k\langle S\rangle[z]) = M_n(k)\langle S\rangle[z]$ which is invertible over $k\langle S\rangle[\![z]\!]$. In particular, the coefficient of z^0 in A is invertible in $M_n(k)\langle S\rangle$. Hence the coefficient of ez^0 of A is invertible in $M_n(k)$. In particular, the coefficient of e, which lies in $M_n(k[z])$, is invertible in $M_n(k[\![z]\!])$, hence invertible in $M_n(k(z))$. Thus A is invertible in $M_n(k(z))\langle\langle S\rangle\rangle = M_n(k(z)\langle\langle S\rangle\rangle)$ implying (5.3).

The monoid homomorphism $\pi : S^* \to F$ is uniquely defined by $\pi(x_i) = x_i$ and $\pi(x_i^{-1}) = x_i^{-1}$ for $1 \leq i \leq n$. It extends to a homomorphism $\pi : k\langle S\rangle \to kF$ and then (coefficient-wise) to $\pi : k\langle S\rangle[\![z]\!] \to kF[\![z]\!]$.

Now consider $P \in (kF)_{rat}[\![z]\!]$. The power series P is a component of the solution u of some matrix equation $Au = b$, where $A \in M_n(kF[z]) = M_n(kF)[z]$ is a matrix which is invertible over $kF[\![z]\!]$, and b is a vector in $(kF[z])^n$. Without loss of generality, we can assume that the coefficient of z^0 in A is the identity matrix. Compare the proof of 5.6. Choose a lift $\bar{b}$ of b to $(k\langle S\rangle[z])^n$, i.e. $\pi(\bar{b}) = b$. Obviously, one can choose a lift $\bar{A} \in M_n(k\langle S\rangle[z]) = M_n(k\langle S\rangle)[z]$ of A such that the z^0-coefficient of $\bar{A}$ is the identity matrix. In particular, $\bar{A}$ is invertible in $M_n(k\langle S\rangle[\![z]\!])$. Therefore the respective entry of the solution $\bar{u}$ of the matrix equation $\bar{A}\bar{u} = \bar{b}$ maps to P under π. Thus we have

$$P \in \pi\left((k\langle S\rangle)_{rat}[\![z]\!]\right).$$

Let $\bar{P} \in (k\langle S\rangle)_{rat}[\![z]\!] \overset{(5.3)}{\subset} (k(z))_{rat}\langle\langle S\rangle\rangle$ be a preimage of P. Denote by $\phi : k(z)\langle\langle S\rangle\rangle \to k(z)[\![S]\!]$ the canonical homomorphism. Let $P_F \in \mathbb{Z}\langle\langle S\rangle\rangle$ be the power series associated to the word problem of F with respect to S. We have seen in 5.20 that P_F is algebraic. Therefore $\bar{P} \odot P_F$ is algebraic, i.e. $\bar{P} \odot P_F \in (k(z))_{alg}\langle\langle S\rangle\rangle$ by 5.19. So $\phi(\bar{P} \odot P_F) \in k(z)[\![S]\!]$ is algebraic by 5.16. Substituting every $s \in S$ by 1, we get a formally well defined power series $\phi(\bar{P} \odot P_F)(1, \ldots, 1) \in k[\![z]\!]$ with

$$\mathrm{tr}_{kF}(P) = \phi(\bar{P} \odot P_F)(1, \ldots, 1) \in k[\![z]\!].$$

Finally, from 5.17 the algebraicity of $\mathrm{tr}_{kF}(P)$ is obtained. □

EXAMPLE 5.22 (Markov Operator for Free Abelian Groups).
Consider the so-called *Markov operator*

$$M = x + x^{-1} + y + y^{-1} \in \mathbb{C}\mathbb{Z}^2$$

of the free abelian group $\mathbb{Z}^2 = \langle x, y; xy = yx\rangle$. We want to compute the trace $T(z) \in \mathbb{C}[\![z]\!]$ of the rational power series

$$(1 - Mz)^{-1} = \sum_{n=0}^{\infty} M^n z^n \in \mathbb{C}\mathbb{Z}^2[\![z]\!].$$

It is clear that the trace of M^{2n+1} is zero. The trace of M^{2n} can be combinatorially described as the number $N(n)$ of strings w of length $2n$ in the alphabet $\{x, x^{-1}, y, y^{-1}\}$ such that the number of x resp. y in w equals the number of x^{-1} resp. y^{-1} in w.

To calculate $N(n)$ first consider all strings in the alphabet $\{a, b\}$ of length $2n$ with n-many a's and n-many b's in it – there are $\binom{2n}{n}$-many. For each such string w produce all strings you get by marking exactly n letters in w with an apostrophe – again $\binom{2n}{n}$ possibilities. You end up with all strings (these are $\binom{2n}{n}^2$-many) of length $2n$ in the alphabet $\{a, a', b, b'\}$ such that the number of the b' resp. a' in it equals the number of the a resp. b in it. This implies $N(n) = \binom{2n}{n}^2$, hence

$$T(z) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 z^{2n} \in \mathbb{C}[\![z]\!].$$

But this power series can be shown to be not algebraic. Compare [57, 6.3. on p. 217]. We remark that $T(z)$ is *D-finite*, i.e. it satisfies a linear differential equation with polynomial coefficients.

Example 5.23 (Markov Operator for the Free Groups).
Consider the Markov operator of the free group F^k of rank k in the letters x_i, $1 \le i \le k$.

$$M = \sum_{i=1}^{k} x_i + x_i^{-1} \in \mathbb{C}F^k$$

Let $X = \{x_1, \ldots, x_k, x_1^{-1}, \ldots, x_k^{-1}\}$. We compute the trace $T = T(z) \in \mathbb{C}[\![z]\!]$ of the power series

$$(1 - Mz)^{-1} = \sum_{i=0}^{\infty} M^i z^i \in \mathbb{C}F^k[\![z]\!].$$

This problem was studied by a large number of people. One can also apply Voiculescu's machinery of free probability to solve it (see [60, p. 28]). The algebraicity of T is shown in [62]. The following argument uses 5.20. We begin with a general remark.

Let Σ an alphabet. For $w \in \Sigma^*$ denote by $|w|$ the length of w. An easy computation shows that the following map ψ is a ring homomorphism.

$$\psi : \mathbb{C}\langle\!\langle \Sigma \rangle\!\rangle \longrightarrow \mathbb{C}[\![z]\!]$$
$$\sum_{w \in \Sigma^*} a_w w \mapsto \sum_{n \ge 0} \left(\sum_{|w|=n} a_w \right) z^n$$

Note that in the notation of 5.20 we have $T = \psi(P_{F^k})$. By symmetry we have that $S := \psi(P_t)$ is the same for all $t \in X$. Because of $P_t = tQ_t\bar{t}$ we obtain $\psi(Q_t) = z^{-2}S$. The equations (5.1) and (5.2) yield the following system of equations after applying ψ.

$$T = 1 + 2kTS$$
$$z^{-2}S = 1 + z^{-2}(2k-1)S^2$$

The solution T of this system satisfies the algebraic equation

$$(1 - 4k^2z^2)T(z)^2 + (2k-2)T(z) - (2k-1) = 0.$$

5.2 Rationality and Positivity for Free Groups

In this section we study spectra of operators in a finite von Neumann algebra. More precisely, we compute the spectral density functions of self-adjoint operators. Applying the results of the last section, we will show that the Novikov-Shubin invariants of operators in $M_n(\mathbb{C}F) \subset M_n(\mathcal{N}(F))$, where F is a virtually free group, are positive and rational unless they are ∞^+.

The idea is to consider a power series built from the operator, which codifies all its spectral information. Then there is an explicit way (an application of the Riemann-Stieltjes inversion formula) to extract all the spectral information, we need, from this power series.

The Riemann-Stieltjes inversion formula is a well-known tool in this context, and also used in [60] to compute spectra of operators.

DEFINITION 5.24 (Cauchy Transform).
The *Cauchy transform* G_μ of a finite, compactly supported Borel measure μ on $\mathbb{R}$ is defined as the function on $\mathbb{C}^+ = \{z \in \mathbb{C}; \operatorname{Im} z > 0\}$ given by

$$G_\mu(z) := \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}.$$

In our context the measure will be the spectral measure of a self-adjoint operator in a finite von Neumann algebra $\mathcal{A}$. Let us recall its definition. Let $A \in \mathcal{A}$ be a self-adjoint operator. Associated to it is the family of spectral projections $E^A_\lambda \in \mathcal{A}$, $\lambda \in \mathbb{R}$ (compare p. 26). They induce a compactly supported probability Borel measure μ_A on $\mathbb{R}$ defined by

$$\mu_A((\lambda,\mu]) = \operatorname{tr}_{\mathcal{A}}\left(E^\mu_A\right) - \operatorname{tr}_{\mathcal{A}}\left(E^\lambda_A\right).$$

Recall that $\mu_A(\{\lambda\}) \neq 0$ if and only if λ is an eigenvalue of A. The Cauchy transform G_{μ_A} of μ_A can be expressed explicitly by A.

LEMMA 5.25 (Cauchy Transform of the Spectral Measure).
We have the following equality of holomorphic functions on $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$.

$$G_{\mu_A}(z) = \operatorname{tr}_{\mathcal{A}}((z-A)^{-1}).$$

REMARK 5.26. In the sequel $\int_a^b$ means $\int_{[a,b]}$, and $\int_a^{b^-}$ stands for $\int_{[a,b)}$ etc.

Proof. The support of μ lies in the spectrum of A, in particular in $[-\|A\|, \|A\|]$. For $z > \|A\|$ the operator $z - A$ is invertible. We get for $\|z\| > \|A\|$

$$\begin{aligned} G_{\mu_A}(z) = \int_{-\|A\|}^{\|A\|} \frac{d\mu_A(t)}{z-t} = \int_{-\|A\|}^{\|A\|} \left(\sum_{n=0}^{\infty} \frac{t^n}{z^{n+1}}\right) d\mu_A(t) &= \sum_{n=0}^{\infty} \left(\int_{-\|A\|}^{\|A\|} \frac{t^n}{z^{n+1}} d\mu_A(t) \right) \\ &= \sum_{n\geq 0} \operatorname{tr}_{\mathcal{A}}(A^n z^{-n-1}) \\ &= \operatorname{tr}_{\mathcal{A}}((z-A)^{-1}). \end{aligned}$$

Here recall that $\sum_{n\geq 0} A^n z^{-n-1}$ converges to $(z-A)^{-1}$ in the norm topology [33, lemma 3.1.5 on p. 175], and that $\operatorname{tr}_{\mathcal{A}}$ is continuous with respect to the ultraweak topology. □

THEOREM 5.27 (Riemann-Stieltjes Inversion Formula).
Let μ be a finite, compactly supported Borel measure on $\mathbb{R}$. Let $a, b \in \mathbb{R}$ such that $\mu(\{a\}) = \mu(\{b\}) = 0$. Then

$$\mu([a,b]) = \lim_{y\to 0^+}\left(-\frac{1}{\pi}\int_a^b \operatorname{Im} G_\mu(x+iy)dx\right).$$

Proof. In [32, p. 92-93] it is proven that μ is the weak limit of the measures $-\frac{1}{\pi}\operatorname{Im} G_\mu(x+iy)dx$. By [4, Satz 30.12 on p. 228] this yields the statement provided $\mu(\{a\}) = \mu(\{b\}) = 0$. □

LEMMA 5.28. *Let μ be a finite, compactly supported Borel measure on $\mathbb{R}$. If $G_\mu(z)$ has a holomorphic extension around $t_0 \in \mathbb{R}$, then $\mu(\{t_0\}) = 0$ holds.*

Proof. Write μ as $\mu = \alpha \cdot \delta_{t_0} + \mu_0$, $\alpha \in \mathbb{R}_{\geq 0}$, where δ_{t_0} is the Dirac measure concentrated at t_0, and the measure μ_0 satisfies $\mu_0(\{t_0\}) = 0$. Then we get

$$G_\mu(z) = \alpha \cdot \frac{1}{z-t_0} + \int_{\mathbb{R}} \frac{d\mu_0(t)}{z-t}.$$

Because $G_\mu(z)$ has an analytic extension around t_0, we have in particular

$$\lim_{y\to 0^+} iy \cdot G_\mu(t_0+iy) = 0.$$

Next we show that

$$\lim_{y\to 0^+}\int_{\mathbb{R}} \frac{iy}{(t_0+iy)-t}d\mu_0(t) = 0. \tag{5.4}$$

This would imply $\alpha = 0$ and finish the proof. The absolutes values of the summands on the right side in

$$\frac{iy}{(t_0+iy)-t} = \frac{y^2}{(t_0-t)^2+y^2} + i\frac{y(t_0-t)}{(t_0-t)^2+y^2}$$

are ≤ 1 for all $y \neq 0$ and $t \in \mathbb{R}$. Because of σ-additivity and the finiteness of μ_0 we have $\lim_{\epsilon\to 0^+}\mu_0([t_0-\epsilon, t_0+\epsilon]) = \mu_0(\{t_0\}) = 0$. For $n \in \mathbb{N}$ choose $\epsilon > 0$ such that $\mu_0([t_0-\epsilon, t_0+\epsilon]) < \frac{1}{2n}$ holds. Set $\mathbb{R}(\epsilon) = \mathbb{R} - [t_0-\epsilon, t_0+\epsilon]$. By the majorized convergence theorem we obtain

$$\lim_{y\to 0^+}\int_{\mathbb{R}(\epsilon)} \frac{iy}{(t_0+iy)-t}d\mu_0(t) =$$
$$\int_{\mathbb{R}(\epsilon)} \lim_{y\to 0^+}\frac{y^2}{(t_0-t)^2+y^2}d\mu_0(t) + \int_{\mathbb{R}(\epsilon)} \lim_{y\to 0^+}\frac{y(t_0-t)}{(t_0-t)^2+y^2}d\mu_0(t) = 0.$$

For every $y > 0$ we get the estimate

$$\left|\int_{t_0-\epsilon}^{t_0+\epsilon} \frac{iy}{(t_0+iy)-t}d\mu_0(t)\right| \leq$$
$$\int_{t_0-\epsilon}^{t_0+\epsilon}\left|\frac{y^2}{(t_0-t)^2+y^2}\right|d\mu_0(t) + \int_{t_0-\epsilon}^{t_0+\epsilon}\left|\frac{y(t_0-t)}{(t_0-t)^2+y^2}\right|d\mu_0(t) \leq \frac{1}{n}$$

Hence the limit in (5.4) is bounded above by $\frac{1}{n}$. Because $n \in \mathbb{N}$ was chosen arbitrarily, (5.4) follows. □

THEOREM 5.29 (Rationality and Positivity).
Let F be a virtually free group and $\mathbb{Q} \subset k \subset \mathbb{C}$ be a field. Let $A \in M_n(kF) \subset M_n(\mathcal{N}(F))$ be a self-adjoint operator in the finite von Neumann algebra $M_n(\mathcal{N}(F))$, which lives over the group ring. Then the following holds.

(i) The Novikov-Shubin invariant $\alpha(A)$ is positive rational unless it is ∞^+.

(ii) The operator A has a finite number of eigenvalues, and they lie in the algebraic closure of k.

(iii) The real function $\lambda \mapsto \operatorname{tr}_{M_n(\mathcal{N}(F))}\left(E_A^\lambda\right)$ is piecewise smooth.

Proof. The entries of $z(1 - Az)^{-1} \in M_n(kF)[\![z]\!]$ lie in the rational closure $(kF)_{rat}[\![z]\!]$. Due to theorem 5.21, the formal power series

$$q(z) = \operatorname{tr}_{M_n(kF)}\big(z(1 - Az)^{-1}\big) = \sum_{i=1}^{n} \operatorname{tr}_{k(F)}\big((z(1 - Az)^{-1})_{ii}\big)$$

is algebraic over $k(z)$. From a non-trivial algebraic equation of $q(z)$ it is obvious that $q(z^{-1})$ also satisfies a non-trivial algebraic equation over $k(z)$. In the domain $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$ the function $q(z^{-1})$ is convergent, and we have $G_{\mu_A}(z) = q(z^{-1})$, due to 5.25. Therefore there is a non-constant polynomial $P(w, z) = p_n(z)w^n + \cdot + p_0(z)w^0 \in k[w, z]$, $p_i(z) \in k[z]$, $p_n \neq 0$ such that

$$P(G_{\mu_A}(z), z) = 0$$

holds in $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$ – thus in every domain $G_{\mu_A}(z)$ can be analytically extended to. We can assume that $P(w, z)$ is irreducible (compare [1, p. 293]). Let $Z \subset \mathbb{C}$ be the finite set consisting of the zeroes of p_n and the zeroes of the discriminant of P. We remind the reader that the discriminant of P is a polynomial over k, and the zeroes of the discriminant are exactly the points z_0 such that $Q(w) = P(w, z_0)$ has multiple roots. In particular, Z lies in the algebraic closure of k. From the domain $\{z \in \mathbb{C}^+; \|z\| > \|A\|\}$ the function $G_{\mu_A}(z)$ can be analytically extended along any arc which does not pass through a point of Z [1, p. 294]. Equivalently, $G_{\mu_A}(z)$ can be analytically extended to every simply connected domain not containing Z.

For an eigenvalue λ of A we have $\mu_A(\{\lambda\}) > 0$, and 5.28 implies that the eigenvalues lie in Z. Thus they are contained in the algebraic closure of k.

Put $F(\lambda) = \operatorname{tr}_{M_n(\mathcal{N}(F))}(E_\lambda^A)$. Remember that $F(\lambda)$ is the spectral density function of A, as defined on p. 26 provided A is positive. Let $\lambda \in \mathbb{R} - Z$. Then there is an open ball U around λ such that G_{μ_A} can be analytically extended to U. In particular, for $\epsilon \in U \cap \mathbb{R}$ we have $\mu_A(\{\epsilon\}) = 0$ by 5.28, and the Riemann-Stieltjes inversion formula yields

$$F(\lambda) - F(\epsilon) = \lim_{y \to 0^+} \left(-\frac{1}{\pi} \cdot \int_{\epsilon}^{\lambda} \operatorname{Im} G_{\mu_A}(x + iy) dx \right).$$

Now the majorized convergence theorem implies

$$F(\lambda) - F(\epsilon) = -\frac{1}{\pi} \cdot \int_{\epsilon}^{\lambda} \operatorname{Im} G_{\mu_A}(x) dx.$$

Thus $F(\lambda)$ is smooth outside of Z, and the derivative there is

$$F'(\lambda) = -\frac{1}{\pi} \cdot \operatorname{Im} G_{\mu_A}(\lambda). \tag{5.5}$$

Next we show $\alpha(A) \in \mathbb{Q}_{>0} \cup \{\infty^+\}$. The Novikov-Shubin invariant is defined using the spectral density function of A^*A, so we can assume that A is positive. Because G_{μ_A} is algebraic there exists $k \in \mathbb{N}$ such that $G_{\mu_A}(z^k)$ can be analytically extended to a pointed neighborhood of 0 having 0 as a pole (see [1, theorem 4 on p. 297]). Therefore $G_{\mu_A}(z^k)$ has an expansion as a Laurent series with finitely many terms of negative exponent. Put $S(\lambda) = F(\lambda^k)$. From $S'(\lambda) = -\frac{k}{\pi} \cdot \operatorname{Im} G_{\mu_A}(\lambda^k)\lambda^{k-1}$ for small $\lambda > 0$ we see, by integrating, that $S(\lambda)$ has the form

$$S(\lambda) - S(\epsilon) = \sum_{i=N}^{\infty} c_i \lambda^i + c\ln(\lambda) - \sum_{i=N}^{\infty} c_i \epsilon^i - c\ln(\epsilon) \tag{5.6}$$

with $N \in \mathbb{Z}$, $c, c_i \in \mathbb{R}$ and $0 < \epsilon \leq \lambda$ small enough. For fixed λ and $\epsilon \to 0^+$ (5.6) stays bounded because the spectral density function is bounded. In particular, we get $\lim_{\epsilon \to 0^+} \sum_{i=N}^{\infty} c_i \epsilon^{i+1} = 0$ because of $\lim_{\epsilon \to 0^+} \epsilon \ln(\epsilon) = 0$. This implies $c_i = 0$ for $i < 0$, and so c must be zero for (5.6) to stay bounded. Using the fact that $S(\lambda) = F(\lambda^k)$ is right-continuous, we finally get

$$S(\lambda) - S(0) = \sum_{i=M}^{\infty} c_i \lambda^i$$

with $M > 0$. If all c_i are zero, then $F(\lambda) - F(0)$ is constant for small λ, and then $\alpha(A) = \infty^+$ follows. Now consider the case that not all c_i are zero. Without loss of generality, we assume that $c_M \neq 0$. By the l'Hospital rule we get

$$\lim_{\lambda \to 0^+} \frac{\ln(F(\lambda^k) - F(0))}{\ln(\lambda)} = \lim_{\lambda \to 0^+} \frac{\ln(S(\lambda) - S(0))}{\ln(\lambda)} = M.$$

Therefore we obtain

$$\alpha(A) = \lim_{\lambda \to 0^+} \frac{\ln(F(\lambda) - F(0))}{\ln(\lambda)} = \frac{M}{k} \in \mathbb{Q}_{>0}.$$

□

In 4.11 we mentioned that the Novikov-Shubin of finite G-CW complexes are essentially the Novikov-Shubin invariants of the differentials – viewed as matrices over the group ring. As a consequence, we get the following corollary. Compare [45, p. 113].

COROLLARY 5.30. *Let F be a virtually free group. The Novikov-Shubin invariants of a finite free F-CW complex are positive rational unless they are ∞^+.*

REMARK 5.31. In general, it is not clear that the limes inferior in the definition of $\alpha(A)$ can be replaced by a limit. The previous proof shows that it is possible for operators over the group ring of a virtually free group. We say that these operators have the *limit property*.

Part (ii) of 5.29, i.e. the algebraicity of the eigenvalues, is shown in [17] by a different method. There it is proven not only for virtually free but for all ordered groups satisfying the strong Atiyah conjecture over the complex group ring.

EXAMPLE 5.32 (Spectral Measure for the Markov Operator of the Free Group). Using the Riemann-Stieltjes inversion formula we are able to compute an explicit formula for the spectral measure μ_A of the Markov operator $A = x + x^{-1} + y + y^{-1}$ of the free group $\mathbb{Z} * \mathbb{Z} = \langle x, y \rangle$. Put $F(\lambda) = \operatorname{tr}_{\mathcal{N}(\mathbb{Z}*\mathbb{Z})}(E_A^\lambda)$. The power series $T(z) = \sum_{n \geq 0} \operatorname{tr}_{\mathcal{N}(\mathbb{Z}*\mathbb{Z})}(A^n) z^n$ satisfies the equation (see 5.23)

$$(1 - 16z^2)T(z)^2 + 2T(z) - 3 = 0.$$

The explicit solution to this equation is

$$T(z) = \frac{3}{1 \pm 2\sqrt{1 - 12z^2}}.$$

Because of $G_{\mu_A}(z) = z^{-1}T(z^{-1})$ we obtain

$$G_{\mu_A}(z) = \frac{3}{z \pm 2\sqrt{z^2 - 12}} = \frac{3(z \mp 2\sqrt{z^2 - 12})}{z^2 - 4(z^2 - 12)}.$$

There are the ("boundary") conditions $F'(\lambda) \geq 0$ and $F'(\lambda) = 0$ outside a compact set. With this in mind, equation (5.5) implies

$$F'(\lambda) = \begin{cases} \dfrac{6\sqrt{12 - \lambda^2}}{\pi(48 - 3\lambda^2)} & \text{if } |\lambda| < \sqrt{12} \\ 0 & \text{if } |\lambda| > \sqrt{12}. \end{cases}$$

Thus the support of μ_A is $[-\sqrt{12}, \sqrt{12}]$, and in that interval we have the equality of measures

$$\mu_A = \frac{6\sqrt{12 - \lambda^2}}{\pi(48 - 3\lambda^2)} d\lambda.$$

Integrating yields for $|\lambda| < \sqrt{12}$

$$F(\lambda) - F(0) = \frac{2}{\pi} \arcsin\left(\frac{\lambda}{2\sqrt{3}}\right) + \frac{1}{2\pi} \arctan\left(\frac{2(3 - \lambda)}{\sqrt{12 - \lambda^2}}\right) + \frac{1}{2\pi} \arctan\left(\frac{2(-3 - \lambda)}{\sqrt{12 - \lambda^2}}\right).$$

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