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Descent properties of topological chiral homology

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Abstract. We study descent properties of Jacob Lurie's topological chiral homology. We prove that this homology theory satisfies descent for a factorizing cover, as defined by Kevin Costello and Owen Gwilliam. We also obtain a generalization of Lurie's approach to this homology theory, which leads to a product formula for the infinity 1-category of factorization algebras, and its twisted generalization.

1. Introduction

1.1. **Factorization algebra and the topological chiral homology.** Lurie has introduced and studied the *topological chiral homology* [16, Chap. 5]. The construction has several other names; in particular, 'factorization homology' (without necessarily requiring 'topological' invariance of the 'coefficients') in [3, 9], or 'higher order Hochschild homology' (at least for coefficients in a commutative algebra) in [19], in which the work of Anderson [1] is mentioned for an earlier appearance of the notion.

Developing on Lurie's work, we study interesting counterparts on manifolds of *factorization algebras* defined by Beilinson and Drinfeld [5] on algebraic curves (which were shown by them to be equivalent to *chiral algebras*, introduced in the same work). Following some of the pioneers of the research of these objects on manifolds, we call them *factorization algebras*.

One motivation for studying factorization algebras on manifolds comes from the central role which they play in quantum field theory, generalizing the role of chiral algebras for conformal field theory. Namely, observables of a quantum (or a classical) field theory having locality form a factorization algebra, and this is the structure in terms of which one can rigorously understand quantization of a physical theory (in perturbative sense) [9], analogously to the deformation quantization of the classical mechanics [13].

Factorization algebras are closely related to *field theories* as functors on a cobordism category, as introduced by Atiyah [2] and Segal [20]. We study *locally constant* factorization algebras, which correspond to *topological* field theories.

A locally constant factorization algebra on the manifold $\mathbb{R}^n$ is equivalent to what is known as an E_n-algebra, first introduced in iterated loop space theory [6]. An E_1-algebra is an associative algebra, and an E_n*-algebra* can be inductively defined as an E_{n-1}-algebra with an additional structure of an associative algebra commuting with the E_{n-1}-structure. A locally constant factorization algebra can be considered as a global version of an E_n-algebra in a way which is analogous to the way in which a chiral algebra is a global version of a vertex operator algebra. In particular, from any locally constant factorization algebra on an n-dimensional manifold, one obtains an E_n-algebra around any point by restricting the algebra to an open ball around the point. This E_n-algebra is canonical up to a change of framing at the point, and can be thought of as a local form of the factorization algebra.

There is an issue that the notion of an E_n-algebra degenerates (unless $n \leq 1$) to that of a commutative algebra in a category whose higher homotopical structure is degenerate. Moreover, some further developments such as the theory of the Koszul duality for factorization algebras [18] require a nice higher homotopical structure in order to lead to fruitful results, even on the manifold $\mathbb{R}^1$. These issues force us to work in a homotopical setting. In order to work in such a setting, we use the convenient language of higher category theory. (For the main body, note our conventions stated in Section 2, which do not apply in this introduction.) We just remark here that associativity of an algebra in such a setting means a data for homotopy coherent associativity (which in particular is a structure rather than a property).

In this work, we study various problems from the point of view that a factorization algebra is a generalization of a sheaf on a manifold (the term 'locally constant' comes from this point of view). A factorization algebra takes values in a symmetric monoidal infinity 1-category. A *prealgebra* on a manifold M is a covariant functor A on the poset of open subsets of M, for which we have $A(U \sqcup V) \simeq A(U) \otimes A(V)$ for disjoint open subsets $U, V \subset M$, in a coherent way. (Covariance is chosen for consistency of the terminology with the intuition.) A is a *factorization algebra* if it satisfies a suitable gluing condition generalizing that for a sheaf. Indeed, a locally constant cosheaf is a locally constant factorization algebra with respect to the monoidal structure given by the coproduct.

The gluing condition of the factorization algebra of observables of a physical theory reflects locality of the theory. In the Atiyah–Segal framework, the same property corresponds to the possibility of extending the functor on cobordisms to higher codimensional manifolds. A theory is *fully extended* if it is extended to highest codimensional manifolds, namely, to points. The cobordism hypothesis of Baez and Dolan [4], proved in a much strengthened form by Hopkins and Lurie (unpublished) and Lurie [15], states that a fully extended topological field theory (on framed manifolds) is completely determined by its value for a point. Analogously, but in a simpler way, a factorization algebra which is systematically defined on all (framed) manifolds is determined by the E_n-algebra which appear as its local form [3].

A sheaf is defined by its sections. One is often more interested in the *derived* sections or the cohomology. Since we work in a homotopical setting for factorization algebras, the sections we consider for an algebra are *always* the 'derived' ones. Thus, study of factorization algebras can be considered as study of a kind of homology theory. This homology theory, for locally constant algebras, was defined by Lurie [16] and was called *topological chiral homology.* Following Francis and Costello, we also call it *factorization homology*, although this term is also used for not necessarily locally constant factorization algebras.

Let us give some ideas for our main results in Sections 1.2 and 1.4.

In the following, we assume that the target category $\mathcal{A}$ of prealgebras is a symmetric monoidal infinity 1-category which is closed under sifted homotopy colimits, and that the monoidal multiplication functors preserve sifted homotopy colimits variable-wise.

1.2. **Descent properties of factorization algebras.** We have developed descent properties of locally constant factorization algebras for covers and for bases of the topology. Our first result (Theorem 3.15, note the conventions stated in Section 2) proves (as a particular case, see Example 3.14) that the topological chiral homology satisfies descent for a *factorizing cover* in the sense of Costello and Gwilliam [9]. Therefore, this connects two approaches to factorization homology, namely, the 'Čech' approach of Costello and Gwilliam and Lurie's approach, which is analogous to the *singular* approach to the local coefficient (co)homology. (Costello and Gwilliam in fact considered not necessarily locally constant algebras.) This, combined with ideas of Francis, leads to a proof of a version of Ayala and Francis' theorem [3]. This will be contained in the sequel [18] of this paper. This theorem can be considered as giving an *Eilenberg–Steenrod* approach to factorization homology. One concludes from these theorems that all three approaches are equivalent.

Moreover, we generalize Lurie's approach to factorization homology in the following way. His definition of topological chiral homology uses the basis $\mathrm{Disk}(M)$ for the topology of a manifold M, consisting of open subdisks. He also uses disjoint unions of disks, which give another basis $\mathrm{Disj}(M)$ of M. This latter basis has a nice property in the spirit of Costello and Gwilliam, which we might call here *factorizingness.* Lurie's definition is stated in terms of the pair $\mathrm{Disk}(M) \to \mathrm{Disj}(M)$.

In Theorem 3.34, we have given a sufficient condition for a pair $\mathcal{E}_1 \to \mathcal{E}$ of bases to define the same notion of a locally constant factorization algebra, when it replaces the pair $\mathrm{Disk}(M) \to \mathrm{Disj}(M)$ in Lurie's definition. Even though the theorem is slightly technical, the sufficient condition we have found is easy to check in practice. For example, it is quite easy to check whether we can find a suitable $\mathcal{E}_1$ if $\mathcal{E}$ is a factorizing basis of M, closed under disjoint union in M, and consists of open submanifolds homeomorphic to disjoint unions of disks.

Thus, Theorem 3.34 is useful and it in particular leads to the following theorem, as well as to applications we discuss in Section 1.4. Let us denote by $\mathrm{Alg}_M(\mathcal{A})$ the infinity 1-category of locally constant factorization algebras on a manifold M.

Theorem 1.3 (Theorem 3.47)**.** *The association $M \mapsto \mathrm{Alg}_M(\mathcal{A})$ (which is contravariantly functorial in open embeddings) is a sheaf of infinity 1-categories.*

It follows that there is a reasonable notion of locally constant factorization algebras on an orbifold.

1.4. **Twisted product formula.** As an application of our investigation of the descent properties of factorization algebras, we have obtained the following basic theorem. In the special case where the manifolds are the Euclidean spaces, we recover a classical theorem of Dunn [10]. See Remark 1.7 below for the precise relation to his theorem.

The theorem relies on the relatively simple fact that the infinity 1-category $\mathrm{Alg}_F(\mathcal{A})$ of locally constant factorization algebras on a manifold F taking values in $\mathcal{A}$ (see the last paragraph of Section 1.1) is symmetric monoidal by the value-wise multiplication in $\mathcal{A}$. In particular, we can consider, on any manifold, locally constant factorization algebras taking values in $\mathrm{Alg}_F(\mathcal{A})$.

Theorem 1.5 (Theorem 4.17)**.** *Let B, F be manifolds. Then, the restriction functor*

$$\mathrm{Alg}_{F\times B}(\mathcal{A}) \to \mathrm{Alg}_B(\mathrm{Alg}_F(\mathcal{A}))$$

is an equivalence of symmetric monoidal infinity 1-categories.

Remark 1.6. If one swaps the factors of $B \times F$, then on the side of algebras, one recovers the canonical equivalence $\mathrm{Alg}_B(\mathrm{Alg}_F) \simeq \mathrm{Alg}_F(\mathrm{Alg}_B)$.

Remark 1.7. Dunn [10] in fact obtains an equivalence at the level of operads. In particular, in his case, the equivalence of algebras holds without any assumption on the target category. Even though our theorem applies to any manifold, the equivalence in this generality is proved only at the level of the category of algebras in this paper, since our proof depends on the property of the target category for the algebras.

Another slight difference between Theorem 1.5 and Dunn's result is that he considers Boardman and Vogt's little cubes operad [6] instead of factorization algebras on a Euclidean space. We can use Theorem 3.34 once again to show that the difference is not essential; see Remark 4.18 for the details.

Remark 1.8. A different proof of Theorem 1.5 is obtained by Ginot by relying on Dunn's theorem [11]. A version of Theorem 1.5 for general (i.e., not assumed locally constant) factorization algebras is described by Calaque in [8] with a (sketch of) proof by a strategy similar to ours (see Section 1.11). We remark that the theorem for locally constant algebras may not be a corollary of this since comparison of the 'locally constant' objects through Calaque's equivalence would perhaps not be straight-forward.

We have also obtained a natural generalization of Theorem 1.5, where the product is replaced by a fiber bundle (i.e., a ‘twisted’ product). In this case, the algebras on the target of the restriction functor need to be twisted. Namely, they should take values in an *algebra* of categories on B. Once we allow this twisting, it is natural to consider further twisting for algebras. Namely, we consider algebras on the total space E of a fiber bundle taking values in a locally constant factorization algebra $\mathcal{A}$ of categories on E. For such $\mathcal{A}$, we have defined an algebra $\mathrm{Alg}_{E/B}(\mathcal{A})$ of categories on the base manifold B, which is a twisted version of Alg_F in Theorem 1.5.

The next theorem is a generalization following from (the infinity 2-categorical generalizations of) Theorem 1.5 and the descent results.

Theorem 1.9 (Theorem 4.22)**.** *Let B be a manifold, and let $E \to B$ be a smooth fiber bundle over B. For a locally constant factorization algebra $\mathcal{A}$ on E of infinity 1-categories, there is a natural equivalence*

$$\mathrm{Alg}_E(\mathcal{A}) \xrightarrow{\sim} \mathrm{Alg}_B(\mathrm{Alg}_{E/B}(\mathcal{A}))$$

of infinity 1-categories, given by a suitable ‘restriction’ functor.

Remark 1.10. For this theorem, no assumptions on sifted colimits are needed for $\mathcal{A}$. If $\mathcal{A}$ is instead a single fixed symmetric monoidal category, there is actually a slight difference between an algebra in $\mathcal{A}$ (for which Theorem 1.5 may fail without assumption on sifted colimits) and an algebra taking values in the ‘constant’ algebra at $\mathcal{A}$ (to which Theorem 1.9 *always* applies). The assumption on sifted colimits simply ensures equivalence of these two notions of an algebra.

1.11. **Notes on related works.** The descent property of the topological chiral homology for a factorizing cover (as follows from Theorem 3.15, see Section 1.2) was proved earlier by Ginot, Tradler and Zeinalian [12]. Their proof uses a theorem on the descent of the infinity 1-category of factorization algebras, similar to our Theorem 1.3 but in non-locally constant setting. The theorem is due to Costello and Gwilliam [9]. Note that we prove Theorem 1.3 using Theorems 3.34 *and* 3.15. We do not know how to deduce Theorem 1.3 directly from the theorem of Costello and Gwilliam. The question is whether local constancy of a factorization algebra is a ‘local’ property in some useful manner, to which Theorem 3.34 gives one answer.

We learned about Calaque’s work [8] after our work was completed. He considers the notion of ‘factorizing basis’ based on a similar idea to our Definition 3.28. Using this, he considers a theorem [8, Thm. 2.1.9] which is similar in spirit to our Theorem 3.34, for not necessarily locally constant factorization algebras. Theorem 3.34 is more involved than this theorem, since it additionally answers a question on the localness of local constancy as mentioned. Calaque’s proof of the product formula mentioned in Remark 1.8 above uses [8, Thm. 2.1.9], similarly to our use of Theorem 3.34 for Theorem 1.5.

1.12. **Outline.** In Section 2, we introduce conventions which are used throughout the article. In Section 3, we review Lurie's definitions and results, and discuss descent properties of factorization algebras. In Section 4, we discuss further results including the twisted product formula.

2. Terminology and notations

By a 1-*category*, we always mean an *infinity* 1-category. We often call a 1-category (namely an infinity 1-category) simply a *category*. A category with discrete sets of morphisms (namely, a 'category' in the more traditional sense) will be called a *discrete* category.

In fact, all categorical and algebraic terms will be used in *infinity* (1-)categorical sense without further notice. Namely, categorical terms are used in the sense enriched in the *infinity* 1-category of spaces, or equivalently, of infinity groupoids, and algebraic terms are used freely in the sense generalized in accordance with the enriched categorical structures.

For example, for an integer $n \geq 1$, by an *n-category*, we mean an *infinity* n-category. We also consider multicategories. By default, multimaps in our multicategories will form a *space* with all higher homotopies allowed. Namely, our *multicategories* are 'infinity operads' in the terminology of Lurie's book [16].

Remark 2.1. We usually treat a space relatively to the structure of the standard (infinity) 1-category of spaces. Namely, a *space* for us is usually no more than an object of this category. Without loss of information, we shall freely identify a space in this sense with its fundamental infinity groupoid, and call it also a *groupoid*. Exceptions in which the term 'space' means not necessarily this include the Euclidean space, the total space of a fiber bundle, etc., in accordance with the common customs.

If $\mathcal{C}$ is a category and x is an object of $\mathcal{C}$, then we denote by $\mathcal{C}_{/x}$ the *over-category* of objects of $\mathcal{C}$ lying over x, i.e., equipped with a map to x. We denote by $\mathcal{C}_{x/}$ the *under-category* for x, in other words $((\mathcal{C}^{\mathrm{op}})_{/x})^{\mathrm{op}}$.

More generally, if a category $\mathcal{D}$ is equipped with a functor to $\mathcal{C}$, then we define $\mathcal{D}_{/x} := \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{/x}$, and similarly for $\mathcal{D}_{x/}$. Note that we have used here the functor $\mathcal{C}_{/x} \to \mathcal{C}$ which forgets the structure map to x. Note that the notation is abusive in that the name of the functor $\mathcal{D} \to \mathcal{C}$ is dropped from it. In order to avoid any confusion, we shall use this notation only when the functor $\mathcal{D} \to \mathcal{C}$ we consider is clear from the context.

By the *lax colimit* of a diagram in the category Cat of categories (of a limited size), indexed by a category $\mathcal{C}$, we mean the Grothendieck construction. We choose the variance of the laxness so the lax colimit projects to $\mathcal{C}$, to make it an op-fibration over $\mathcal{C}$, rather than a fibration over $\mathcal{C}^{\mathrm{op}}$. (In particular, if $\mathcal{C} = \mathcal{D}^{\mathrm{op}}$, so the functor is contravariant on $\mathcal{D}$, then the familiar fibered category over $\mathcal{D}$ is the op-lax colimit over $\mathcal{C}$ for us.) Of course, we can choose the variance for lax *limits* compatibly with this, so our lax colimit generalizes to that in any 2-category.

3. Descent properties of factorization algebras

In this section, we introduce the notion of a locally constant factorization algebra following Lurie (although he did not use this particular term), and then investigate its descent properties. This will be a study of the descent properties of Lurie's 'topological chiral homology'.

Many notions and notations we introduce in this section are from Lurie's book [16], which has an index and an index for notations.

3.1. **Locally constant factorization algebra.** Given a manifold M, let us denote by $\mathrm{Open}(M)$ the poset of open submanifolds of M. It (considered as a category where a map is an inclusion) has a partially defined symmetric monoidal structure given by the *disjoint* union in M,

$$\coprod_S : \mathrm{Open}(M)^{(S)} \to \mathrm{Open}(M),$$

where the domain here is the full subposet of $\mathrm{Open}(M)^S$ consisting of *pairwise disjoint* families of open submanifolds of M indexed by the finite set S.

Definition 3.2. Let $\mathcal{A}$ be a symmetric monoidal category. Then a *prefactorization algebra* (or just a *prealgebra*) on M (valued) in $\mathcal{A}$ is a symmetric monoidal functor $\mathrm{Open}(M) \to \mathcal{A}$.

We say that a prealgebra A is *locally constant* if it takes every inclusion $D \hookrightarrow D'$ between disks in M (namely, open submanifolds which are homeomorphic to an open disk) to an equivalence $A(D) \xrightarrow{\sim} A(D')$.

The category of *locally constant* prealgebras on M in $\mathcal{A}$ will be denoted by $\mathrm{PreAlg}_M(\mathcal{A})$.

Let M be a manifold. Let n denote its dimension. Then, following Lurie, we denote by $\mathrm{Disk}(M)$ the poset consisting of open submanifolds $U \subset M$ homeomorphic to an open disk of dimension n (by an unspecified homeomorphism). This poset has a structure of a symmetric multicategory where a multimap is a disjoint inclusion in M, so for every fixed source and target, the space of multimaps is either empty or contractible.

Given symmetric multicategories $\mathcal{A}$, $\mathcal{B}$, recall that an *algebra* on $\mathcal{B}$ in $\mathcal{A}$ is a morphism $\mathcal{B} \to \mathcal{A}$ of symmetric multicategories.

The following is a notion equivalent to an algebra over Lurie's multicategory $\mathbb{E}_M$; see [16, Thm. 5.2.4.9], also restated here as Theorem 3.17. Another equivalent notion has a natural name, and we use that name. All notions and the equivalence between them will be reviewed below.

Definition 3.3. Let $\mathcal{A}$ be a symmetric monoidal category. Then a *locally constant factorization algebra* (or just a '(locally constant) *algebra*', often in this work) on M valued in $\mathcal{A}$ is an algebra on $\mathrm{Disk}(M)$ in $\mathcal{A}$ whose underlying functor (of 'colors') inverts any map in $\mathrm{Disk}(M)$ (which is an inclusion of a single disk into another). The category of locally constant algebras on M in $\mathcal{A}$ will be denoted by $\mathrm{Alg}_M(\mathcal{A})$.

Remark 3.4. This definition makes sense for $\mathcal{A}$ just a symmetric multicategory, but for comparison with other notions, it is convenient to have $\mathcal{A}$ to be symmetric monoidal.

Following Lurie, let us denote by $\mathrm{Disj}(M)$ the poset of open submanifolds $U \subset M$ homeomorphic (by an unspecified homeomorphism) to the disjoint union of a finite number of disks. It has a partially defined monoidal structure given by the disjoint union in M. There is a functor $\mathrm{Disk}(M) \to \mathrm{Disj}(M)$ of multicategories, so a symmetric monoidal functor $A\colon \mathrm{Disj}(M) \to \mathcal{A}$ to a symmetric monoidal category $\mathcal{A}$ restricts to a morphism $\mathrm{Disk}(M) \to \mathcal{A}$ of symmetric multicategories. Moreover, any morphism $\mathrm{Disk}(M) \to \mathcal{A}$ extends uniquely to a symmetric monoidal functor $\mathrm{Disj}(M) \to \mathcal{A}$. Namely, an algebra on M can also be described as a symmetric monoidal functor $\mathrm{Disj}(M) \to \mathcal{A}$.

Remark 3.5. Again, this is still true if the monoidal structure of $\mathcal{A}$ is only partially defined, but this is not an important point for us.

Note that there is a (necessarily symmetric) monoidal full embedding $\mathrm{Disj}(M) \hookrightarrow \mathrm{Open}(M)$. Given a functor $\mathrm{Disj}(M) \to \mathcal{A}$, one has its left Kan extension $\mathrm{Open}(M) \to \mathcal{A}$ at least if $\mathcal{A}$ has colimits.

If the monoidal multiplication in $\mathcal{A}$ distributes over colimits, then the Kan extension $\mathrm{Open}(M) \to \mathcal{A}$ of a symmetric monoidal functor $\mathrm{Disj}(M) \to \mathcal{A}$ becomes symmetric monoidal in a unique way, so its restriction to $\mathrm{Disj}(M)$ becomes the original symmetric monoidal functor. In fact, Lurie proves that relevant colimits here can be described as sifted colimits (see Section 3.16 below). Therefore, it suffices to consider just sifted colimits.

To summarize, if the target category $\mathcal{A}$ has sifted colimits, and the monoidal multiplication in $\mathcal{A}$ distributes over sifted colimits (equivalently, sifted colimits are preserved by the monoidal multiplication), then we have a functor $\mathrm{Alg}_M(\mathcal{A}) \to \mathrm{PreAlg}_M(\mathcal{A})$ given by left Kan extension. This functor is clearly fully faithful, and it is left adjoint to the functor given by restriction through the functor $\mathrm{Disk}(M) \to \mathrm{Open}(M)$ of symmetric multicategories. In this way, $\mathrm{Alg}_M(\mathcal{A})$ is a right localization of the category of locally constant prealgebras.

Within the category of locally constant prealgebras, an algebra can be characterized as a prealgebra which, as a functor, is the left Kan extension of its restriction to $\mathrm{Disj}(M)$. We often identify Alg_M with this right localized full subcategory of PreAlg_M.

The following example is basic; see [3].

Example 3.6. Let $\mathcal{A}$ be a category closed under small colimits, and let us consider it as a symmetric monoidal category under the Cartesian coproduct. This symmetric monoidal multiplication $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ takes colimits in $\mathcal{A} \times \mathcal{A}$ to colimits in the target, so sifted colimits are preserved variable-wise, so the arguments above applies to this symmetric monoidal structure.

In this case, any functor $\mathrm{Disj}(M) \to \mathcal{A}$ has a unique lax symmetric monoidal structure, and this structure is strong monoidal if and only if the functor is the left Kan extension (in the canonical way) from its restriction to $\mathrm{Disk}(M)$.

It follows that a locally constant algebra in $\mathcal{A}$ with respect to the Cartesian coproduct is the same thing as a locally constant cosheaf in $\mathcal{A}$.

Dually, if $\mathcal{A}$ is closed under limits, then a locally constant algebra in $\mathcal{A}^{\mathrm{op}}$ with respect to the Cartesian product of $\mathcal{A}$ is the same thing as a locally constant *sheaf* valued in $\mathcal{A}$.

3.7. **Assumption on the target category.** From now on, in this paper, we assume that the target category $\mathcal{A}$ of prealgebras has sifted colimits, and the monoidal multiplication functor on $\mathcal{A}$ preserves sifted colimits variable-wise. Equivalently, the monoidal multiplication should preserve sifted colimits for all the variables at the same time.

3.8. **Descent for factorizing covers.** For a prealgebra on M, being the Kan extension of its restriction to disjoint union of disks is a kind of descent property. We shall observe a more general descent satisfied by a locally constant algebra.

Definition 3.9. Let $\mathcal{C}$ be a category and let $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$ be a functor. For $i \in \mathcal{C}$, denote $\chi(i)$ also by U_i within this definition. We shall call this datum a *factorizing cover* of M which is *nice in Lurie's sense*, or briefly, a *factorizing l-nice cover*, if for any nonempty finite subset $x \subset M$, the full subcategory $\mathcal{C}_x := \{i \in \mathcal{C} \mid x \subset U_i\}$ of $\mathcal{C}$ has contractible classifying space.

Remark 3.10. The definition is inspired by the definition of a factorizing cover by Costello and Gwilliam [9] and a condition introduced by Lurie for his *generalized Seifert–van Kampen theorem* [16, Appendix]. 'Nice' is Lurie's description of a cover satisfying his conditions, where he does not intend this to be a part of his terminology. However, we borrow the word 'nice' and make it our term for the notion above.

Example 3.11. If M is empty, then any cover of M, including the one indexed by the empty category, is factorizing l-nice.

Example 3.12. The inclusion $\mathrm{Disj}(M) \to \mathrm{Open}(M)$ determines a factorizing l-nice cover.

Example 3.13. Consider a cover of M by a filtered (or 'directed') inductive system of open submanifolds of M. Then this cover is factorizing l-nice.

Example 3.14. Suppose $\mathcal{U} = \{U_s\}_{s \in S}$ is an open cover of M indexed by a set S. For simplicity, assume that this cover is closed under taking finite disjoint union. If this is not satisfied, replace S by the set of finite subsets T of S for which U_t are pairwise disjoint for $t \in T$. (For example, if $M = \varnothing$, then the assumption excludes the empty cover indexed by $S = \varnothing$.)

Denote by $\mathbf{\Delta}_{/S}$ the category of combinatorial simplices whose vertices are labeled by elements of S. Namely, its objects are finite nonempty ordinal I

equipped with a set map $s\colon I \to S$. Then the cover determines a functor $\chi\colon (\mathbf{\Delta}_{/S})^{\mathrm{op}} \to \mathrm{Open}(M)$ by

$$(I,\, s\colon I \to S) \mapsto U_s := \bigcap_{i \in I} U_{s(i)}.$$

In Costello–Gwilliam's terminology, the cover $\mathcal{U}$ is *factorizing* if for this χ, the category $(\mathbf{\Delta}_{/S})^{\mathrm{op}}_x$ is nonempty for every finite subset $x \subset M$ (equivalently if there is $i \in S$ for which $x \subset U_i$).

It is immediate to see that χ determines a factorizing l-nice cover if (and only if) the cover is factorizing in Costello–Gwilliam's sense.

Given a prealgebra A on M, the descent complex for $\mathcal{U}$ of Costello and Gwilliam is equivalent to $\mathrm{colim}_{(\mathbf{\Delta}_{/S})^{\mathrm{op}}} A$.

The following generalizes the Kan extension property from the values for the disjoint unions of disks.

Theorem 3.15. *Let A be a locally constant algebra on M (in a symmetric monoidal category $\mathcal{A}$ satisfying our conditions stated in Section 3.7). Then for any factorizing l-nice cover determined by $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$, the canonical map $A(M) \leftarrow \mathrm{colim}_{\mathcal{C}} A\chi$ is an equivalence.*

For the proof, we need another description of locally constant algebras, due to Lurie. The proof begins right after Corollary 3.20.

3.16. **Isotopy invariance.** Let M be a manifold and n its dimension. Let $\mathbb{E}_M$ be the multicategory (i.e., an 'infinity operad') introduced by Lurie. Its objects are the open submanifolds of M homeomorphic to a disk of dimension n. The space of multimaps $\{U_i\}_{i \in S} \to V$ is that formed by an embedding

$$f\colon \coprod_i U_i \hookrightarrow V$$

together with an isotopy on each U_i from the defining inclusion $U_i \hookrightarrow M$ to $f\colon U_i \hookrightarrow M$.

It is immediate from this description that the underlying category (the category of 'colors') of $\mathbb{E}_M$ is a groupoid equivalent to (the fundamental infinity groupoid of) the space naturally formed by its objects.

Consider the obvious morphism $\mathrm{Disk}(M) \to \mathbb{E}_M$ of multicategories.

Theorem 3.17 (Lurie [16, Thm. 5.2.4.9])**.** *Restriction through the morphism $\mathrm{Disk}(M) \to \mathbb{E}_M$ induces a fully faithful functor between the categories of algebras on these multicategories. The essential image of the functor consists precisely of the locally constant algebras on M.*

In particular, a locally constant algebra on M extends uniquely (up to a contractible space) to an algebra on $\mathbb{E}_M$.

The property of an algebra on disks that the algebra extends to $\mathbb{E}_M$ can be understood as isotopy invariance (where the way to be invariant can be specified functorially) of the functor. By the above theorem, this property is equivalent to being locally constant.

Let $\mathrm{D}(M)$ be as defined by Lurie [16, Def. 5.3.2.11]. Its objects are open submanifolds of M which are homeomorphic to a finite disjoint union of disks. The space of maps $U \to V$ is the space formed by embeddings $f\colon U \hookrightarrow V$ together with an isotopy from the defining inclusion $U \hookrightarrow M$ to $f\colon U \hookrightarrow M$.

Disjoint union in M cannot be made into a partial monoidal structure on $\mathrm{D}(M)$ since the isotopies we used in defining a morphism in $\mathrm{D}(M)$ were required to be isotopies on the whole U, not just on each of its components. However, $\mathrm{D}(M)$ can be extended to a symmetric partial monoidal category which has the same objects but where the mentioned restriction on the maps is discarded. Let us denote this partial monoidal category by $\overline{\mathbb{E}}_M$. The composite $\mathbb{E}_M \to \mathrm{D}(M) \to \overline{\mathbb{E}}_M$ then has a canonical structure as a map of multicategories, and we can try to extend A to a symmetric monoidal functor on $\overline{\mathbb{E}}_M$.

To see that this is possible, let us further try discarding the restriction on the *objects*. Namely, an object of $\overline{\mathbb{E}}_M$ is an object of $\mathrm{D}(M)$, which can be considered as a disjoint family of disks in M, but we can instead include *any* family of disks (and define morphisms in the same way as in $\overline{\mathbb{E}}_M$). The result is the symmetric monoidal category freely generated from $\mathbb{E}_M$. Therefore, an algebra A on $\mathbb{E}_M$ can be extended to a symmetric monoidal functor on the free symmetric monoidal category, and then be restricted to $\overline{\mathbb{E}}_M$ through the symmetric monoidal inclusion. This symmetric monoidal functor on $\overline{\mathbb{E}}_M$, as an algebra on a multicategory, extends the algebra A on $\mathbb{E}_M$.

Moreover, there is a commutative square

$$\begin{array}{ccc} \mathrm{Disk}(M) & \longrightarrow & \mathbb{E}_M \\ \downarrow & & \downarrow \\ \mathrm{Disj}(M) & \longrightarrow & \mathrm{D}(M) \end{array}$$

which, with the functor $\mathrm{D}(M) \to \overline{\mathbb{E}}_M$, factorizes a square

$$\begin{array}{ccc} \mathrm{Disk}(M) & \longrightarrow & \mathbb{E}_M \\ \downarrow & & \downarrow \\ \mathrm{Disj}(M) & \longrightarrow & \overline{\mathbb{E}}_M, \end{array}$$

where the bottom functor underlies a symmetric monoidal functor. It follows that, by restricting to $\mathrm{D}(M)$ (the underlying functor of) the described symmetric monoidal functor on $\overline{\mathbb{E}}_M$ extending A, one gets a functor on $\mathrm{D}(M)$ which extends both

- (the underlying functor of) A on $\mathbb{E}_M$, and
- (the underlying functor of) the symmetric monoidal functor on $\mathrm{Disj}(M)$ uniquely extended from the algebra $A|_{\mathrm{Disk}(M)}$ on $\mathrm{Disk}(M)$.

Proposition 3.18 (Lurie [16, Prop. 5.3.2.13 (1)])**.** *The functor* $\mathrm{Disj}(M) \to \mathrm{D}(M)$ *is cofinal.*

That is, for a functor defined on $\mathrm{D}(M)$, its colimit over $\mathrm{D}(M)$ gives the colimit of the restriction of the same functor to $\mathrm{Disj}(M)$.

Proposition 3.19 (Lurie [16, Prop. 5.3.2.15])**.** *The category* $\mathrm{D}(M)$ *is sifted.*

Corollary 3.20. *Let A be a locally constant algebra on M. Consider it as an algebra on $\mathbb{E}_M$, and then extend its underlying functor to* $\mathrm{D}(M)$ *in the explained way. Denote the resulting functor on* $\mathrm{D}(M)$ *still by A. Then the canonical map*

$$A(M) \leftarrow \operatorname*{colim}_{\mathrm{D}(M)} A$$

is an equivalence.

We can now prove Theorem 3.15. Recall that a functor $\mathcal{C} \to \mathcal{D}$ is *cofinal* if for every functor f with domain $\mathcal{D}$, $\operatorname{colim} f$ (if it exists) is a colimit of f over $\mathcal{C}$ (in the canonical way); see [14, Def. 4.1.1.1 and Prop. 4.1.1.8].

Definition 3.21. Let $\mathcal{U}$ be a cover of a manifold M, given by a functor $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$, $i \mapsto U_i$. Then $\mathcal{U}$ is said to be *effectively factorizing l-nice* if the canonical functor $\operatorname{colim}_i \mathrm{D}(U_i) \to \mathrm{D}(M)$ is cofinal.

Remark 3.22. By Proposition 3.18, the condition of being an effectively factorizing l-nice cover is equivalent to that the functor $\operatorname{colim}_i \mathrm{Disj}(U_i) \to \mathrm{D}(M)$ is cofinal.

Theorem 3.17 immediately implies the following.

Lemma 3.23. *Let A be a locally constant algebra on M. Then for any effectively factorizing l-nice cover determined by* $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$*, the canonical map* $A(M) \leftarrow \operatorname{colim}_{\mathcal{C}} A\chi$ *is an equivalence.*

Theorem 3.15 is an immediate consequence of this and the following, 'factorizing' version of Lurie's higher homotopical generalization of the Seifert–van Kampen theorem. The factorizing version is actually a consequence of the original theorem. Our proof will be similar to the proof of [7, Thm. 5.1] by Boavida de Brito and Weiss, and will also use some arguments similar to those from the proofs of the theorems above of Lurie.

Proposition 3.24. *Let M be a manifold. Then every factorizing l-nice cover of M is effectively factorizing l-nice.*

In the proof, we shall use the following standard fact from basic homotopy theory. Its proof is included for completeness.

Lemma 3.25. *Let $\mathcal{G}$ be a groupoid. Then a functor $\mathcal{C} \to \mathcal{G}$ from a 1-category is cofinal if (and only if) the induced map $B\mathcal{C} \to \mathcal{G}$ is an equivalence.*

Proof. Assuming that $\mathcal{G} = B\mathcal{C}$, we want to prove that the colimit of any functor L defined over $\mathcal{G}$ is a colimit of L over $\mathcal{C}$. (The 'only if' part is trivial since $B\mathcal{C}$ is a colimit of the final diagram over $\mathcal{C}$ in the 1-category of groupoids.)

Note that it suffices to consider the case where L takes values in the opposite of the category of spaces, since whether an object is a colimit is tested by homming into another object. Let us conveniently change the variance of $\mathcal{C}$ and $\mathcal{G}$, and consider the limits of a *covariant* functor L defined on $\mathcal{G}$. Thus, we want to prove that for $\mathcal{G} = B\mathcal{C} = \operatorname{colim}_{\mathcal{C}} *$, where the colimit is taken in the category of groupoids, the induced map $\lim_{\mathcal{G}} L \to \lim_{\mathcal{C}} L$ is an equivalence.

The crucial fact here is that for any object i of $\mathcal{G}$, $L(i)$ is the homotopy fiber of the projection $\operatorname{colim}_{\mathcal{G}} L \to \mathcal{G}$. Namely, $L(i)$ is the space of sections of this map over the point i.

It follows that $\lim_{\mathcal{C}} L$ is the space of global sections if $\mathcal{G} = \operatorname{colim}_{\mathcal{C}} *$. Thus, we have proved that $\lim_{\mathcal{C}} L$ is functorially equivalent to a space which is independent of $\mathcal{C}$ as long as the map $B\mathcal{C} \to \mathcal{G}$ is an equivalence. (In particular, this independent space is identified with $\lim_{\mathcal{G}} L$ through the equivalence obtained in the case where the functor $\mathcal{C} \to \mathcal{G}$ is an equivalence.) This completes the proof of Lemma 3.25.

Alternatively, one can apply Joyal's generalization of Quillen's Theorem A [14], although, as we have shown, this is not necessary. Again, assuming $\mathcal{G} = B\mathcal{C}$, we want to show that, for any object x of $\mathcal{G}$, the under-category $\mathcal{C}_{x/}$ has contractible classifying space.

The point is that, since $\mathcal{G}$ is a groupoid, $\mathcal{C}_{x/}$ coincides with the fiber of the functor $\mathcal{C} \to \mathcal{G}$ over x. The result follows since the classifying space functor preserves fiber products over a groupoid (see Lemma 3.26 below). □

Lemma 3.26. *Let* Cat *denote the category of categories (with a fixed limit for the size), and* Gpd *its full subcategory consisting of groupoids. Then the classifying space functor* $B\colon \mathrm{Cat} \to \mathrm{Gpd}$ *preserves fiber products over a groupoid.*

Proof. The claim is that for $\mathcal{G} \in \mathrm{Gpd} \subset \mathrm{Cat}$, the functor

$$B\colon \mathrm{Cat}_{/\mathcal{G}} \to \mathrm{Gpd}_{/B\mathcal{G}} = \mathrm{Gpd}_{/\mathcal{G}}$$

preserves direct products. However, the functor

$$B_*\colon \mathrm{Fun}(\mathcal{G}, \mathrm{Cat}) \to \mathrm{Fun}(\mathcal{G}, \mathrm{Gpd})$$

can be identified with this functor since $B\colon \mathrm{Cat} \to \mathrm{Gpd}$ preserves colimits, and this concludes the proof. Indeed, B_* preserves direct products since $B\colon \mathrm{Cat} \to \mathrm{Gpd}$ does.

Alternatively, one may note that B is left adjoint to the inclusion $\mathrm{Gpd}_{/\mathcal{G}} \hookrightarrow \mathrm{Cat}_{/\mathcal{G}}$, and for $\mathcal{C}, \mathcal{D} \in \mathrm{Cat}_{/\mathcal{G}}$ and $\mathcal{H} \in \mathrm{Gpd}_{/\mathcal{G}}$, may equate

$$\mathrm{Map}_{\mathrm{Cat}_{/\mathcal{G}}}(\mathcal{C} \times_{\mathcal{G}} \mathcal{D}, \mathcal{H}) = \mathrm{Map}_{\mathrm{Cat}_{/\mathcal{G}}}(\mathcal{C}, \mathrm{Fun}_{\mathcal{G}}(\mathcal{D}, \mathcal{H}))$$

naturally with

$$\mathrm{Map}_{\mathrm{Gpd}_{/\mathcal{G}}}(B\mathcal{C}, \mathrm{Map}_{\mathcal{G}}(B\mathcal{D}, \mathcal{H})) = \mathrm{Map}_{\mathrm{Cat}_{/\mathcal{G}}}(\mathcal{C}, \mathrm{Map}_{\mathcal{G}}(B\mathcal{D}, \mathcal{H})),$$

where $\mathrm{Fun}_{\mathcal{G}}$ and $\mathrm{Map}_{\mathcal{G}}$ denote the internal hom functors in the Cartesian closed category $\mathrm{Cat}_{/\mathcal{G}} \simeq \mathrm{Fun}(\mathcal{G}, \mathrm{Cat})$ and in $\mathrm{Gpd}_{/\mathcal{G}}$ respectively. Indeed, the map $\mathrm{Map}_{\mathcal{G}}(B\mathcal{D}, \mathcal{H}) \to \mathrm{Fun}_{\mathcal{G}}(\mathcal{D}, \mathcal{H})$ induced from the canonical map $\mathcal{D} \to B\mathcal{D}$ can be seen to be an equivalence using the fact that $B\colon \mathrm{Cat} \to \mathrm{Gpd}$ preserves direct products. □

Proof of Proposition 3.24. Suppose that a factorizing l-nice cover $\mathcal{U}$ of M is given by a functor $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$, $i \mapsto U_i$. We want to show that the functor

$$\operatorname*{colim}_i \mathrm{D}(U_i) \to \mathrm{D}(M)$$

is cofinal.

Recall that for open $U \subset M$, the category $\mathrm{D}(U)$ was a comma category in the category Man of manifolds, in which the space of morphisms is the space of open embeddings. Namely, let D be the full subcategory of Man whose objects are equivalent to the disjoint union of a finite number of disks of dimension n, where $n = \dim M$. Then $\mathrm{D}(U)$ was the comma category whose object was a morphism from an object of D to U.

In other words, $\mathrm{D}(U) = \operatorname{lax\,colim}_{D\in \mathrm{D}} \mathrm{Emb}(D, U)$, where $\mathrm{Emb}(D, U)$, the infinity groupoid of embeddings, is the space of morphisms in Man, and the lax colimit is taken in the 2-category of categories.

It follows that it suffices to prove that, for every $D \in \mathrm{D}$, the map

$$\operatorname{lax}\operatorname*{colim}_{i\in\mathcal{C}} \mathrm{Emb}(D, U_i) \to \mathrm{Emb}(D, M)$$

is cofinal for every $D \in \mathrm{D}$.

In view of Lemma 3.25, it suffices to prove that the map

$$\operatorname*{colim}_{i\in\mathcal{C}} \mathrm{Emb}(D, U_i) \to \mathrm{Emb}(D, M)$$

is an equivalence.

Choose a homeomorphism $D \simeq S \times \mathbb{R}^n$ for a finite set S. In particular, we have picked a point in each component of D, corresponding to the origin in $\mathbb{R}^n$, together with a germ of chart at the chosen points. Then, given an embedding $D \hookrightarrow U$, restriction of it to the germs of charts at the chosen points gives us an injection $S \hookrightarrow U$ together with germs of charts in U at the image of S. This defines a homotopy equivalence of $\mathrm{Emb}(D, U)$ with the space of germs of charts around distinct points in U, labeled by S.

Furthermore, for any U, this space is fibered over the configuration space $\mathrm{Conf}(S, U) := \mathrm{Emb}(S, U)/\operatorname{Aut}(S)$, with fibers equivalent to $\mathrm{Germ}_0(\mathbb{R}^n)^S \times \operatorname{Aut}(S)$, where $\mathrm{Germ}_0(\mathbb{R}^n)$ is from [16, Not. 5.2.1.9], or to the underlying space of the wreath product $\mathrm{Germ}_0(\mathbb{R}^n) \wr \operatorname{Aut}(S)$ of groups, in a more explanatory expression.

Thus it suffices to show that the map

$$\operatorname*{colim}_{i\in\mathcal{C}} \mathrm{Conf}(S, U_i) \to \mathrm{Conf}(S, M)$$

is an equivalence of spaces.

In order to prove this, Lurie's generalized Seifert–van Kampen theorem implies that it suffices to prove that for every $x \in \mathrm{Conf}(S, M)$, the category $\{i \in \mathcal{C} \mid x \in \mathrm{Conf}(S, U_i)\}$ has contractible classifying space. However, $x \in \mathrm{Conf}(S, U_i)$ is equivalent to $\operatorname{supp} x \subset U_i$, where $\operatorname{supp} x$ is the subset of M corresponding to the configuration x, so the required condition is exactly our assumption that the cover is factorizing l-nice. $\square$

3.27. **Basic descent.** We continue with the assumptions introduced in Section 3.7. Namely, we assume that the target category $\mathcal{A}$ of prealgebras has sifted colimits, and the monoidal multiplication functors on $\mathcal{A}$ preserve sifted colimits (variable-wise).

Definition 3.28. Let M be a manifold, and let $\mathcal{U}$ be an effectively factorizing l-nice cover of M, given by a functor $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$, $i \mapsto U_i$. We say that $\mathcal{U}$ is an *(effectively) factorizing l-nice basis* for the topology of M, if for every open $V \subset M$, the functor $\chi\colon \mathcal{C}_{/V} \to \mathrm{Open}(M)_{/V} = \mathrm{Open}(V)$ determines an (effectively) factorizing l-nice cover of V.

Remark 3.29. There is an obvious non-factorizing version of these notions.

It is immediate to see that a factorizing l-nice basis is effectively so as well.

Example 3.30. Disjoint open disks of M form a factorizing l-nice basis of M.

The following proposition is a corollary of Lemma 3.23, in view of the definition of an effectively factorizing l-nice basis.

Proposition 3.31. *Let M be a manifold with an effectively factorizing l-nice basis $\mathcal{U}$. Then any factorization algebra A, as a functor, is a left Kan extension of its restriction to $\mathcal{U}$, namely, if the basis is given by a functor $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$, then A is a Kan extension along χ of $A\chi$.*

In fact, the converse to this is true in the following sense.

Proposition 3.32. *Let M be a manifold with an effectively factorizing l-nice basis $\mathcal{U}$. Suppose A is a prealgebra on M, then it is a locally constant factorization algebra if (and only if) the following conditions hold:*

- (i) *For any basic (in the basis) open U, the conditions*
  - (a) *A is locally constant when restricted to U,*
  - (b) *the map $\mathrm{colim}_{\mathrm{Disj}(U)} A \to A(U)$ is an equivalence*

  are satisfied.
- (ii) *The underlying functor of A is a left Kan extension of its restriction to the basis.*

Theorem 3.33. *The association $M \mapsto \mathrm{Alg}_M(\mathcal{A})$ (which is contravariantly functorial in open embeddings of codimension 0) satisfies descent for any effectively factorizing l-nice basis.*

Proof assuming Proposition 3.32. If A is a locally constant factorization algebra on a manifold U, then conditions (a) and (b) of the proposition are satisfied. $\square$

Let us seek for a proof of Proposition 3.32. Having Proposition 3.31, the only nontrivial point is to show that A is locally constant. Although Proposition 3.32 can be proved in a direct manner, we shall deduce it from a similar theorem in a more specific situation, with weaker looking local constancy assumption. The weaker assumption is more flexible, and the theorem will turn out to be useful.

The theorem is as follows. (We shall use its Corollary 3.45 for our proof of Proposition 3.32.)

Theorem 3.34. *Let M be a manifold, and $\mathcal{V}$ an effectively factorizing l-nice basis of M, given by a (necessarily symmetric) monoidal functor*

$$\psi\colon \mathcal{E} \to \mathrm{Open}(M), \quad i \mapsto V_i,$$

from a symmetric partial monoidal category $\mathcal{E}$, landing in fact in $\mathrm{Disj}(M)$. *Let $\mathcal{E}_1$ be a category mapping to (the underlying category of) $\mathcal{E}$, for which Hypothesis 3.38 below is satisfied. Then a prealgebra A in $\mathcal{A}$ on M is a locally constant factorization algebra on M if and only if it satisfies the following:*

- (i) *$A\psi$ sends every morphism in $\mathcal{E}_1$ to an equivalence.*
- (ii) *The underlying functor of A is a left Kan extension of its restriction $A\psi$ to the factorizing basis.*

In other words, any pair $\mathcal{E}_1 \to \mathcal{E}$ satisfying the hypotheses can replace the pair $\mathrm{Disk}(M) \to \mathrm{Disj}(M)$ in the definition of a locally constant factorization algebra.

Remark 3.35. For every $U \subset M$, the section $\mathcal{E}_{/U} \to \operatorname{lax\,colim}_{i\in\mathcal{E}_{/U}} \mathrm{Disj}(V_i)$ to the canonical functor $\operatorname{lax\,colim}_{i\in\mathcal{E}_{/U}} \mathrm{Disj}(V_i) \to \operatorname{lax\,colim}_{i\in\mathcal{E}_{/U}} * = \mathcal{E}_{/U}$, sending i to the image of the (existing!) terminal object of $\mathrm{Disj}(V_i)$ in the colimit, is cofinal.

In particular, the assumption that the basis is effectively factorizing l-nice is equivalent to that the composite

$$\mathcal{E}_{/U} \xrightarrow{\psi} \mathrm{Disj}(U) \to \mathrm{D}(U)$$

is cofinal for every U, since this can be written as the composite

$$\mathcal{E}_{/U} \to \operatorname*{colim}_{i\in\mathcal{E}_{/U}} \mathrm{Disj}(V_i) \to \mathrm{D}(U);$$

see Remark 3.22.

We need to introduce some notation to state the hypotheses. Note that a map $f\colon D \to E$ in $\mathrm{D}(M)$ is an equivalence if and only if the embedding $D \hookrightarrow E$ (call it g) contained as a part of data determining f is the disjoint union of embeddings of a single disk into another. That is, if and only if there is a one-to-one correspondence between the connected components of D and those of E, such that g embeds each component of D into the corresponding component of E.

Given a finite set S, we denote by $\mathrm{D}'_S(M)$ the groupoid whose objects are families $D = (D_s)_{s\in S}$ of disks labeled by elements of S, and pairwise disjointly embedded in M. A morphism $D \to E = (E_s)_{s\in S}$ is a map $\coprod_S D \to \coprod_S E$ in $\mathrm{D}(M)$ which preserves the labels, and is necessarily an equivalence. (Note the difference of this map from just a map $D_s \to E_s$ for every $s \in S$, in the case where S has more than one point. We denote our category by $\mathrm{D}'_S(M)$ instead of $\mathrm{D}_S(M)$ to emphasize this difference.)

Analogously, let $\mathrm{Disj}_S(M)$ denote the poset whose objects are families $D = (D_s)_{s\in S}$ of disks labeled by elements of S, and pairwise disjointly embedded in M. A morphism $D \to E = (E_s)_{s\in S}$ is an inclusion in M such that $D_s \subset E_s$ for every $s \in S$. (This is the same as a family of inclusions labeled by $s \in S$.) For example, if S consists of one element, then $\mathrm{Disj}_S = \mathrm{Disk}$.

Lemma 3.36. *The functor* $\mathrm{Disj}_S(M) \to \mathrm{D}'_S(M)$ *is cofinal.*

Proof. By Lemma 3.25, it suffices to prove that this functor identifies the groupoid $\mathrm{D}'_S(M)$ with the classifying space of $\mathrm{Disj}_S(M)$. The argument for this will be similar to the proof of Proposition 3.24.

Namely, let $D = (D_s)_{s\in S}$ be a family of free (i.e., not embedded) disks indexed by the elements of S. Then we deduce as before that it suffices to prove that the map

$$\operatorname*{colim}_{U\in \mathrm{Disj}_S(M)} \prod_{s\in S} \mathrm{Emb}(D_s, U_s) \to \mathrm{Emb}(\textstyle\coprod_S D, M)$$

is an equivalence.

It further follows in a similar manner as before, that it suffices to prove that the map

$$\operatorname*{colim}_{U\in \mathrm{Disj}_S(M)} \prod_S U \to \mathrm{Conf}(S, M)$$

is an equivalence.

The equivalence follows from applying the generalized Seifert–van Kampen theorem to the following open cover of $\mathrm{Conf}(S, M)$. The cover is indexed by the category $\mathrm{Disj}_S(M)$, and is given by the functor which associates to $U \in \mathrm{Disj}_S(M)$ the open subset $\prod_S U$ of $\mathrm{Conf}(S, M)$. It is immediate to see that this cover satisfies the assumption for the generalized Seifert–van Kampen theorem. □

Remark 3.37. Note that the last step of the proof of Lemma 3.36 implies that the classifying space of $\mathrm{Disj}_S(M)$ is equivalent to the labeled configuration space of M. Namely, the groupoid $\mathrm{D}'_S(M)$ models this space.

The hypotheses on the factorizing basis are the following. For a finite set S, denote by $\mathcal{E}_S$ the category of S-labeled families of objects of $\mathcal{E}_1$ for which the tensor product over S is defined in $\mathcal{E}$.

Hypothesis 3.38. We claim the following:

(i) $\psi_1 := \psi|_{\mathcal{E}_1}$ lands in $\mathrm{Disk}(M)$.

(ii) ψ_1 defines a (non-factorizing) effectively l-nice basis. (This is equivalent here to that $\psi_1 \colon (\mathcal{E}_1)_{/U} \to \mathrm{Disk}(U)$ is an equivalence on the classifying spaces for every open $U \subset M$; see Remark 3.35. $B\mathrm{Disk}(U)$ is equivalent to U.)

(iii) If a finite set S consists of one element, then $\mathcal{E}_S$ is the whole of $\mathcal{E}_1$.

(iv) For every finite set S, the square

$$\begin{array}{ccc} \mathcal{E}_S & \longrightarrow & \mathrm{Disj}_S(M) \\ {\scriptstyle \otimes_S}\big\downarrow & & \big\downarrow{\scriptstyle \sqcup_S} \\ \mathcal{E} & \xrightarrow{\psi} & \mathrm{Disj}(M) \end{array}$$

is Cartesian.

Remark 3.39. Considering the case where the finite set S consists of one element, we have a Cartesian square

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\psi_1} & \mathrm{Disk}(M) \\ \big\downarrow & & \big\downarrow \\ \mathcal{E} & \xrightarrow{\psi} & \mathrm{Disj}(M). \end{array}$$

In particular, the functor $\mathcal{E}_1 \to \mathcal{E}$ is a full embedding.

Other $\mathcal{E}_S$ are (non-full) subcategories of $\mathcal{E}$.

Remark 3.40. The consequence of the last condition of Hypothesis 3.38, which will be actually used in the proof, will be that for any object $D \in \mathrm{D}'_S(M)$, the square

(1)
$$\begin{array}{ccc} (\mathcal{E}_S)_{D/} & \longrightarrow & \mathrm{Disj}_S(M)_{D/} \\ {\scriptstyle \sqcup_S}\big\downarrow & & \big\downarrow{\scriptstyle \sqcup_S} \\ \mathcal{E}_{\bigsqcup_S D/} & \xrightarrow{\psi} & \mathrm{Disj}(M)_{\bigsqcup_S D/} \end{array}$$

is Cartesian. This follows from the assumption since the assumption implies that the square

$$\begin{array}{ccc} (\mathcal{E}_S)_{E/} & \longrightarrow & \mathrm{Disj}_S(M)_{E/} \\ {\scriptstyle \sqcup_S}\big\downarrow & & \big\downarrow{\scriptstyle \sqcup_S} \\ \mathcal{E}_{E/} & \xrightarrow{\psi} & \mathrm{Disj}(M)_{E/} \end{array}$$

is Cartesian for every $E \in \mathrm{D}(M)$, while the square

$$\begin{array}{ccc} (\mathcal{E}_S)_{D/} & \longrightarrow & \mathrm{Disj}_S(M)_{D/} \\ {\scriptstyle \sqcup_S}\big\downarrow & & \big\downarrow{\scriptstyle \sqcup_S} \\ (\mathcal{E}_S)_{\bigsqcup_S D/} & \xrightarrow{\psi} & \mathrm{Disj}_S(M)_{\bigsqcup_S D/} \end{array}$$

is always Cartesian for every $D \in \mathrm{D}'_S(M)$.

In order to have that the square (1) is Cartesian for every $D \in \mathrm{D}'_S(M)$, we do need the full force of the assumption, since if we have that the map $(\mathcal{E}_S)_{D/} \to [\mathcal{E} \times_{\mathrm{Disj}(M)} \mathrm{Disj}_S(M)]_{D/}$ is an equivalence for every $D \in \mathrm{D}'_S(M)$, then the colimit of this over all D will be the original assumption.

The following is a situation where the hypotheses are satisfied.

Example 3.41. Suppose we have a (non-factorizing) effectively l-nice basis given by a functor $\psi_1 \colon \mathcal{E}_1 \to \mathrm{Open}(M)$, $i \mapsto V_i$. Then we can freely generate a symmetric, partially monoidal category from $\mathcal{E}_1$ by using the partial monoidal structure of $\mathrm{Open}(M)$. Namely, we consider a category $\mathcal{E}$ whose objects are pairs consisting of a finite set S and a family $(i_s)_{s\in S}$ of objects of $\mathcal{E}_1$ for which the open submanifolds $V_{i_s} \subset M$ are pairwise disjoint. The symmetric partial monoidal structure on $\mathcal{E}$ is defined in the obvious way, and ψ_1 extends to a symmetric monoidal functor $\mathcal{E} \to \mathrm{Open}(M)$, which we shall denote by ψ.

In this case, the underlying functor of ψ defines an effectively factorizing l-nice basis of M at least if ψ_1 (and so ψ as well) is the inclusion of a full subposet.

If ψ_1 lands in $\mathrm{Disk}(M)$, then ψ lands in $\mathrm{Disj}(M)$, and the square given in Hypothesis 3.38 (iv) is Cartesian by our construction of the partial monoidal category $\mathcal{E}$.

Example 3.42. As a special case of Example 3.41 we can take $\mathcal{E}_1$ to be the full subposet of $\mathrm{Open}(M)$ consisting of open submanifolds *diffeomorphic* (rather than homeomorphic) to a disk. In this case, $\mathcal{E}$ is the full subposet of $\mathrm{Open}(M)$ consisting of open submanifolds diffeomorphic to the disjoint union of a finite number of disks.

Remark 3.43. $\mathcal{E}_1$ has a structure of a multicategory where for a finite set S, the space of multimaps $i \to j$ for $i = (i_s)_{s\in S}$, $i_s, j \in \mathcal{E}_1$, is nonempty only if $i \in \mathcal{E}_S$, and, in the case $i \in \mathcal{E}_S$, we have

$$\mathrm{Multimap}_{\mathcal{E}_1}(i,j) = \mathrm{Map}_{\mathcal{E}}(\textstyle\bigotimes_S i, j).$$

A symmetric monoidal functor on $\mathcal{E}$ restricts to an algebra on $\mathcal{E}_1$, and this gives an equivalence of categories. We may say that an algebra A on $\mathcal{E}_1$ (or equivalently, on $\mathcal{E}$) is *locally constant* if A inverts all unary maps of $\mathcal{E}_1$. We may denote the category of locally constant algebras by $\mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}_1}(\mathcal{A}) = \mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}}(\mathcal{A})$.

Our assumptions give a functor $\mathcal{E}_1 \to \mathrm{Disk}_1(M)$ of multicategories, and Theorem 3.34 may be stated as that the induced functor

$$\mathrm{Alg}_M(\mathcal{A}) \to \mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}_1}(\mathcal{A})$$

is an equivalence.

Proof of Theorem 3.34. Necessity follows from the definition of local constancy and Proposition 3.31.

For sufficiency, it suffices to prove that the given conditions on A imply that the underlying functor of the restriction of A to $\mathrm{Disj}(M)$ extends to $\mathrm{D}(M)$. Indeed, once we have this, then Proposition 3.18 and the effective l-niceness of the basis imply that, for every open $U \subset M$, the map $\mathrm{colim}_{\mathcal{E}_{/U}} A\psi \to \mathrm{colim}_{\mathrm{Disj}(U)} A$ is an equivalence, so A, which is assumed to be a left Kan extension from $\mathcal{E}$, will in fact be a Kan extension from $\mathrm{Disj}(M)$.

In order to extend the underlying functor of $A|_{\mathrm{Disj}(M)}$ to $\mathrm{D}(M)$, let us show that the right Kan extension $\overline{A}$ of $A|_{\mathrm{Disj}(M)}$ to $\mathrm{D}(M)$ coincides with A on

$\mathrm{Disj}(M)$. (For the solution for an issue here, see the remark after the proof.) It actually suffices to show that the map $\overline{A}(jV) \to A(V)$ is an equivalence for every V in the factorizing basis, where $j\colon \mathrm{Disj}(M) \to \mathrm{D}(M)$ is the functor through which we compare the two categories. Indeed, if D is an arbitrary object of $\mathrm{Disj}(M)$, and if we have equivalences

$$\operatorname*{colim}_{\mathrm{D}(D)} \overline{A} \xleftarrow{\sim} \operatorname*{colim}_{\mathcal{E}_{/D}} \overline{A} j\psi \xrightarrow{\sim} \operatorname*{colim}_{\mathcal{E}_{/D}} A\psi,$$

then by the Kan extension assumption on A, we have that the map $\overline{A}(D) \to A(D)$ is an equivalence.

In order to prove that the map

$$\overline{A}(jV) = \lim_{\mathrm{Disj}(M)_{jV/}} A \to A(V)$$

is an equivalence, we shall first replace the shape of the diagram over which this limit is taken, by a coinitial one. Decompose V into a disjoint union $\coprod_{s\in S} \psi(i_s)$, S a finite set, where $i_s \in \mathcal{E}_1$, so $U_s := \psi(i_s) = \psi_1(i_s)$ is a disk. Then we shall prove that the following functors are coinitial:

$$(\mathcal{E}_S)_{jU/} \xrightarrow{\psi} \mathrm{Disj}_S(M)_{jU/} \hookrightarrow \mathrm{Disj}(M)_{jV/},$$

where $U = (U_s)_{s\in S} \in \mathrm{Disj}_S(M)$ (so $jV = \coprod_S jU$). The inclusion

$$\mathrm{Disj}_S(M)_{jU/} \hookrightarrow \mathrm{Disj}(M)_{jV/}$$

is coinitial since it is obviously a left adjoint.

In order to prove that the functor

$$\psi\colon (\mathcal{E}_S)_{jU/} \to \mathrm{Disj}_S(M)_{jU/}$$

is coinitial, let us consider an object of $\mathrm{Disj}_S(M)_{jU/}$ which, as an object of $\mathrm{Disj}(M)_{jV/}$, is given by the pair consisting of an object D of $\mathrm{Disj}(M)$ and a map $f\colon jV \to jD$ in $\mathrm{D}(M)$. We would like to prove that the over-category $[(\mathcal{E}_S)_{jU/}]_{/(D,f)}$ has contractible classifying space, where (D,f) is considered as an object of $\mathrm{Disj}_S(M)_{jU/}$. Then, since f is required to be a map in $\mathrm{D}'_S(M)$, D can be written as a disjoint union $\coprod_{s\in S} D_s$ of disks, where the embedding part $g\colon V \hookrightarrow D$ of the data determining f embeds U_s into D_s. With this notation, it follows from definitions that the over-category $[(\mathcal{E}_S)_{jU/}]_{/(D,f)}$ is equivalent to $\prod_{s\in S} (\mathcal{E}_1)_{/D_s, gU_s/}$, where we consider D_s as an object of $\mathrm{Disk}(M) = \mathrm{Disj}_1(M)$, and gU_s as an object of $\mathrm{D}'_1(D_s)$, the full subcategory of $\mathrm{D}(D_s)$ consisting of disks.

However, the functor $j\psi_1\colon (\mathcal{E}_1)_{/D_s} \to \mathrm{D}'_1(D_s)$ is cofinal by the assumption of effective l-niceness, so we conclude that $(\mathcal{E}_1)_{/D_s, gU_s/}$ has contractible classifying space, which implies that $[(\mathcal{E}_S)_{jU/}]_{/(D,f)} \simeq \prod_{s\in S} (\mathcal{E}_1)_{/D_s, gU_s/}$ also has contractible classifying space. This proves coinitiality of the functor

$$\psi\colon (\mathcal{E}_S)_{jU/} \to \mathrm{Disj}_S(M)_{jU/}.$$

It follows that the map $\overline{A}(jV) \to \lim_{(\mathcal{E}_S)_{jU/}} A\psi$ is an equivalence, so in order to conclude the proof, it suffices to show that the map from this limit to $A(V)$ is an equivalence.

To analyze this limit, all the maps which appear in the diagram for this limit are equivalences since they are induced from (a finite family of) maps of $\mathcal{E}_1$, which $A\psi$ is assumed to invert.

It therefore suffices to show that the indexing category $(\mathcal{E}_S)_{jU/}$ of the limit has contractible classifying space. However, this follows from Lemma 3.36 since we have proved above that the functor $\psi\colon (\mathcal{E}_S)_{jU/} \to \mathrm{Disj}_S(M)_{jU/}$ is coinitial. □

Remark 3.44. In the above proof, we have used the right Kan extension of a functor taking values in $\mathcal{A}$. However, we do not need to assume existence of limits in $\mathcal{A}$ for the validity of Theorem 3.34. Indeed, our purpose for taking the Kan extension was to show that the prealgebra A was locally constant. In order to prove this in the described method, $\mathcal{A}$ could be fully embedded into a category which has all small limits (e.g., by the Yoneda embedding), and the right Kan extension could be taken in this larger category. Note that the monoidal structure of $\mathcal{A}$ was not used in this step of the proof.

Corollary 3.45. *Let M be a manifold and let $\mathcal{V}$ be an effectively factorizing l-nice basis of M considered in Theorem 3.34, equipped with all the data, and satisfying all the assumptions. Let $\mathcal{U}$ be another effectively factorizing l-nice basis of M, given by a functor $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$, $i \mapsto U_i$. Let $\psi = \chi\iota$ be a factorization, where $\iota\colon \mathcal{E} \to \mathcal{C}$. Then a prealgebra A (not assumed to be locally constant) on M is a locally constant factorization algebra if (and only if) the following are satisfied.*

(i) *$A\psi$ inverts all morphisms of $\mathcal{E}_1$.*
(ii) *The functor $A\chi$ on $\mathcal{C}$ is a left Kan extension of its restriction $A\psi$ to $\mathcal{E}$ through ι.*
(iii) *The underlying functor of A is a left Kan extension of its restriction $A\chi$ to the basis $\mathcal{U}$.*

Proof. A is a left Kan extension of its restriction to the basis $\mathcal{V}$, so Theorem 3.34 applies. □

Example 3.46. Consider the following *discrete* category Man_c^δ. An object is a compact smooth manifold with boundary. A map $\overline{U} \to \overline{V}$ is a smooth immersion of codimension 0 which restricts to an embedding $U \hookrightarrow V$, where U and V are the interior of $\overline{U}$ and $\overline{V}$ respectively. (In the notation, the superscript δ is to remind that the spaces of morphisms are discrete, and the subscript c is to remind that the objects are compact manifolds with boundary.) Man_c^δ is a symmetric monoidal category under disjoint union.

Let $\overline{M}$ be an object of this category, and let M denote its interior. Then, in the corollary, we can take $\mathcal{U}$ to be given by the map $\chi\colon (\mathrm{Man}_c^\delta)_{/\overline{M}} \to \mathrm{Open}(M)$ of partial monoidal posets sending $\overline{U} \to \overline{M}$ to its restriction $U \hookrightarrow M$ while taking $\mathcal{E}_1$ to be the full subposet of $(\mathrm{Man}_c^\delta)_{/\overline{M}}$ consisting of objects whose source has the diffeomorphism type of the closed disk, and $\mathcal{E}$ to be the symmetric partial monoidal category freely generated by $\mathcal{E}_1$. Here, we consider $(\mathrm{Man}_c^\delta)_{/\overline{M}}$ as a partial monoidal category under unions which is disjoint in interiors, and we induce a structure of symmetric multicategory on $\mathcal{E}_1$ from this. $\mathcal{E}$ is the full subposet of $(\mathrm{Man}_c^\delta)_{/\overline{M}}$ generated from $\mathcal{E}_1$ by the partial monoidal product.

In other words, a locally constant factorization algebra on this M could be defined as a symmetric monoidal functor on $(\mathrm{Man}_c^\delta)_{/\overline{M}}$ whose underlying functor satisfies the first two conditions of Corollary 3.45. The original notion is recovered by taking the left Kan extension of the underlying functor to $\mathrm{Open}(M)$, which acquires a canonical symmetric monoidal structure.

Proof of Proposition 3.32. Define

$$\mathcal{E} := \operatorname{lax} \operatorname*{colim}_{i \in \mathcal{C}} \mathrm{Disj}(U_i).$$

Let $\iota\colon \mathcal{E} \to \mathcal{C}$ be the canonical projection, $\psi := \chi\iota$, and

$$\mathcal{E}_1 := \operatorname{lax} \operatorname*{colim}_{i \in \mathcal{C}} \mathrm{Disk}(U_i) \subset \mathcal{E}.$$

It suffices to check that Corollary 3.45 applies. Firstly, $\psi\colon \mathcal{E} \to \mathrm{Open}(M)$ defines an effectively factorizing l-nice basis of M since for every open $U \subset M$, the functors $\mathrm{Disj}(U_i) \to \mathcal{E}_{/i}$ for $i \in \mathcal{C}_{/U}$, the functor $\operatorname{colim}_{i \in \mathcal{C}_{/U}} \mathcal{E}_{/i} \to \mathcal{E}_{/U}$, and so the composite

$$\operatorname*{colim}_{i \in \mathcal{C}_{/U}} \mathrm{Disj}(U_i) \to \operatorname*{colim}_{i \in \mathcal{C}_{/U}} \mathcal{E}_{/i} \to \mathcal{E}_{/U},$$

as well as the composite

$$\operatorname*{colim}_{i \in \mathcal{C}_{/U}} \mathrm{Disj}(U_i) \to \mathcal{E}_{/U} \to \mathrm{D}(U)$$

are cofinal. Similarly, ψ_1 defines an effectively l-nice basis. Moreover, for a finite set S, we have $\mathcal{E}_S = \operatorname{lax} \operatorname{colim}_{i \in \mathcal{C}_{/U}} \mathrm{Disj}_S(U_i)$, and the rest of Hypothesis 3.38 is satisfied. □

Finally, we prove the following result using Theorem 3.34. Let Man^δ denote the *discrete* category of manifolds and open embeddings.

Theorem 3.47. *The presheaf $M \mapsto \mathrm{Alg}_M(\mathcal{A})$ on Man^δ of categories is a sheaf.*

Proof. Let a cover of a manifold M be given by $\mathcal{U} = (U_s)_{s \in S}$ where S is an indexing set. Let $\mathcal{C} := (\mathbf{\Delta}_{/S})^{\mathrm{op}}$ be as in Example 3.14, and define $\chi\colon \mathcal{C} \to \mathrm{Open}(M)$ in the way described there. We would like to prove that the restriction functor

$$\mathrm{Alg}_M(\mathcal{A}) \to \lim_{i \in \mathcal{C}} \mathrm{Alg}_{\chi(i)}(\mathcal{A}) \tag{2}$$

is an equivalence. We shall construct an inverse.

For an open disk $D \in \mathrm{Disk}(M)$, define

$$\mathcal{C}_D := \{ i \in \mathcal{C} \mid D \subset \chi(i) \}.$$

Then this is either empty or has contractible classifying space. Indeed, we have $\mathcal{C}_D = (\mathbf{\Delta}_{/S_D})^{\mathrm{op}}$, where $S_D := \{ s \in S \mid D \in U_s \}$.

We plan to apply Theorem 3.34 to the following pair of bases. Namely, define $\mathcal{E}_1$ to be the full subposet of $\mathrm{Disk}(M)$ consisting of disks D such that

$\mathcal{C}_D$ is nonempty. This gives an l-nice basis of M. Then define a factorizing l-nice basis $\mathcal{E}$ as in Example 3.41. The full inclusion $\psi\colon \mathcal{E} \hookrightarrow \mathrm{Disj}(M)$ is a map of (symmetric) partial monoidal posets, and the pair $\mathcal{E}_1 \hookrightarrow \mathcal{E}$ of bases for the topology of M satisfies Hypothesis 3.38.

Let $(A_i)_{i\in\mathcal{C}} \in \lim_{i\in\mathcal{C}} \mathrm{Alg}_{\chi(i)}$ be given. Then define $B\colon \mathcal{E}_1 \to \mathcal{A}$ by

$$D \mapsto \lim_{i\in\mathcal{C}_D} A_i(D).$$

So $B(D)$ is canonically equivalent to $A_i(D)$ for any $i \in \mathcal{C}_D$. Extend this uniquely to a symmetric monoidal functor $B\colon \mathcal{E} \to \mathcal{A}$. Then the left Kan extension of the underlying functor $\mathcal{E} \to \mathcal{A}$ through $\psi\colon \mathcal{E} \to \mathrm{Open}(M)$ of B has a symmetric monoidal structure which makes it a locally constant factorization algebra by Theorem 3.34.

It is immediate that this construction of a locally constant factorization algebra gives an inverse to the restriction functor (2). □

Let us drop the name '$\mathcal{A}$' of the target symmetric monoidal category from the notation, and denote by Alg the sheaf $M \mapsto \mathrm{Alg}_M$ of Theorem 3.47. Then by general (and undoubtedly standard) constructions and arguments, Alg extends uniquely to a sheaf on the category Orb of orbifolds and local diffeomorphisms between them. Indeed, one finds equivalence of the categories of sheaves (of categories) on Man^δ, and on Orb, through restriction of sheaves.

The construction of the functor $f^*\colon \mathrm{Alg}_N \to \mathrm{Alg}_M$ for a local diffeomorphism $f\colon M \to N$ of manifolds for instance is as described (for not necessarily locally constant factorization algebras) in [9]. Namely, we obtain a cover $\mathcal{U}_f$ of M by open submanifolds U of M on which f restricts to an open embedding $U \hookrightarrow N$, and f^* is defined as the restriction

$$\mathrm{Alg}_N \to \lim_{i\in\mathcal{C}} \mathrm{Alg}_{\chi(i)} = \mathrm{Alg}_M,$$

where $\mathcal{C}$ and χ are as in the proof of Theorem 3.47 for the cover $\mathcal{U} = \mathcal{U}_f$. Note that this description of f^* is in fact forced since the construction is required to commute with composition of local diffeomorphisms.

The sheaf on Orb to extend Alg is (as it in fact needs to be) obtained by right Kan extension from the category of manifolds and local diffeomorphisms. For an orbifold X, it would perhaps make sense to refer to the value associated to X by this sheaf as the category 'of locally constant factorization algebras on X'. This defines a notion of the locally constant factorization algebra on X.

4. GENERALIZATIONS AND APPLICATIONS

4.1. **Push-forward.** We continue with the assumption stated in Section 3.7. Push-forward for factorization algebras has been extensively studied in, e.g., [3, 8, 9, 11]. We shall show in this section that Lurie's results and the analysis in the previous section allow for easy constructions and proofs for the push-forward for locally constant factorization algebras.

Let us first show that we can obtain a locally constant factorization algebra by pushing forward a locally constant factorization algebra along a 'locally constant' map.

Given any map $p\colon X \to M$ of manifolds, the map

$$p^{-1}\colon \mathrm{Open}(M) \to \mathrm{Open}(X)$$

is symmetric monoidal. It follows that any prealgebra on X can be precomposed with p^{-1} to give a prealgebra p_*A on M. Namely, we define

$$p_*A := A \circ p^{-1}.$$

We may ask when p_*A is locally constant, whenever A is a locally constant factorization algebra. It follows from Theorem 3.17 that a sufficient condition is that p is *locally trivial* in the sense that over every component of M, it is the projection of a fiber bundle. (Note that in this case, p can be regarded as giving a locally constant family of manifolds parametrized by points of M.)

Proposition 4.2. *If $p\colon X \to M$ is locally trivial, then for every locally constant factorization algebra on A, the prealgebra p_*A is a locally constant factorization algebra.*

Proof. We have seen that the prealgebra p_*A is locally constant, so it suffices to verify the gluing property. Given any open submanifold U of M, p^{-1} maps the factorizing l-nice cover $\mathrm{Disj}(U)$ of U to a factorizing l-nice cover of $p^{-1}U$. Therefore, the result follows from Theorem 3.15 applied to $A|_{p^{-1}U}$. □

Let us next provide the push-forward with functoriality on the groupoid of locally trivial maps. By definition, this groupoid is modeled by a Kan complex $K_\bullet$ whose k-simplex is a locally constant family over the standard k-simplex of locally trivial maps. In other words, a k-simplex is a map $p\colon X \times \Delta^k \to M \times \Delta^k$ over Δ^k which is locally trivial.

It follows from Theorem 3.17 and Corollary 3.20 that a locally constant algebra A on X is functorial on the groupoid of open submanifolds of X, which can be modeled by a Kan complex whose k-simplex is a locally constant family of open submanifolds parametrized over the standard k-simplex.

Now let p be a k-simplex of $K_\bullet$. Then for every open submanifold U of M, the projection $p^{-1}(U \times \Delta^k) \to \Delta^k$ gives a k-simplex of the space of open submanifolds of X. We obtain the desired functoriality of the push-forward immediately.

4.3. **Case of a higher target category.** A natural notion of a *twisted* factorization algebra would be the notion of an algebra taking values in a factorization *algebra* of categories (rather than in a symmetric monoidal category). A twisted algebra in this sense will turn out to be just a map between certain algebras taking values in the Cartesian symmetric monoidal category Cat of categories (of some limited size). In particular, the space of twisted algebras is a part of the structure of a category of $\mathrm{Alg}_M(\mathrm{Cat})$. However, in order to capture the structure of a *category* (rather than just a space) of twisted algebras, we need to take into account the structure of a 2-*category* of $\mathrm{Alg}_M(\mathrm{Cat})$, coming from the 2-category structure of Cat. We can consider algebras in a symmetric monoidal 2-category in general, and it is in fact natural to consider a symmetric monoidal n-category for any integer $n \geq 2$.

Definition 4.4. Let $n \geq 2$ be an integer, and $\mathcal{A}$ be a symmetric monoidal n-category. Let M be a manifold. Then a *locally constant factorization algebra* on M in $\mathcal{A}$ is an algebra in $\mathcal{A}$ over $\mathbb{E}_M$.

If $\mathcal{A}$ is an n-category, then algebras in $\mathcal{A}$ form an n-category.

The first thing to note is that the underlying 1-category of the n-category of factorization algebras in $\mathcal{A}$ is just the category of algebras in the underlying 1-category of $\mathcal{A}$.

In order to understand the structure of the n-category of factorization algebras, we would like to see that Theorem 3.17 holds in this context, for the n-categories of algebras.

Theorem 4.5. *Let $n \geq 1$ be an integer. Then restriction through the morphism* $\mathrm{Disk}(M) \to \mathbb{E}_M$ *induces a fully faithful functor between the n-categories of algebras on these multicategories, valued in a symmetric monoidal n-category $\mathcal{A}$. The essential image of the functor consists precisely of the locally constant algebras on* $\mathrm{Disk}(M)$*.*

In order to explain the proof of this theorem, let us first review the proof of Theorem 3.17. It follows from Theorem 4.6 (undefined terms will be explained below) and Lemma 4.7.

Theorem 4.6 (A special case of [16, Thm. 2.3.3.23])**.** *Let $\mathcal{C}$ and $\mathcal{O}$ be multicategories, and assume that the category of colors of $\mathcal{O}$ is a groupoid. Let $f\colon \mathcal{C} \to \mathcal{O}$ be a morphism, and assume that it is a* weak approximation *and induces a homotopy equivalence on the classifying spaces of the categories of colors. Then, for every multicategory $\mathcal{A}$, the functor*

$$f^*\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{A}) \to \mathrm{Alg}_{\mathcal{C}}(\mathcal{A})$$

induces an equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{A}) \xrightarrow{\sim} \mathrm{Alg}^{\mathrm{loc}}_{\mathcal{C}}(\mathcal{A})$*, where* $\mathrm{Alg}^{\mathrm{loc}}$ *denotes the category of* locally constant *algebras, and* Alg *the category of not necessarily locally constant algebras.*

Local constancy here means that the underlying functor of the algebra inverts all (unary) morphisms between colors. We do not need to explain the term *weak approximation*, since we just quote the following.

Lemma 4.7 ([16, Lem. 5.2.4.10, Lem. 5.2.4.11])**.** *The assumptions on f of Theorem 4.6 are satisfied by the map* $\mathrm{Disk}(M) \to \mathbb{E}_M$*.*

Thus, Theorem 3.17 extends to Theorem 4.5 once we prove the following.

Proposition 4.8. *Let $\mathcal{C}$ and $\mathcal{O}$ be multicategories, and let $f\colon \mathcal{C} \to \mathcal{O}$ be a morphism. Assume that f satisfies the* conclusion *of Theorem 4.6 (for example, by satisfying its assumptions). Then for every integer $n \geq 1$, the same conclusion is true also for any multicategory $\mathcal{A}$ enriched in n-categories, instead of just* 1*-dimensional $\mathcal{A}$ (so an equivalence* of $(n+1)$-categories *is the claimed conclusion).*

Proof. The proof will be by induction on n. Since we know that the conclusion is true at the level of the underlying 1-categories, it suffices to prove that the functor f^* is fully faithful.

Thus, suppose $n \geq 1$, and let $A, B \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{A})$. We need to recall the *Day convolution*. Namely, we construct a multicategory enriched in n-categories, which we shall denote by $\mathrm{Map}(A, B)$, equipped with a morphism to $\mathcal{O}$, so that the n-dimensional category $\mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{A})}(A, B)$ is by definition the fiber over the universal $\mathcal{O}$-algebra $\mathrm{id}\colon \mathcal{O} \to \mathcal{O}$ of the induced functor $\mathrm{Alg}_{\mathcal{O}}(\mathrm{Map}(A, B)) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{O})$. (This is actually a slight modification of Day's original construction, which captures lax, rather than genuine, morphisms of algebras.)

An object of $\mathrm{Map}(A, B)$ is a pair (x, ϕ), where x is an object (or a 'color') in $\mathcal{O}$ ($x \in \mathcal{O}$), and $\phi\colon A(x) \to B(x)$ in $\mathcal{A}$. Given a family $(x, \phi) = ((x_s, \phi_s))_{s \in S}$ of objects indexed by a finite set S, and an object (y, ψ), we define the $(n-1)$-category of multimaps by the equalizer diagram

$$\mathrm{Map}((x, \phi), (y, \psi)) \to \mathrm{Map}_{\mathcal{O}}(x, y) \rightrightarrows \mathrm{Map}_{\mathcal{A}}(A(x), B(y)),$$

where the two maps equalized are the composites

$$\mathrm{Map}_{\mathcal{O}}(x, y) \xrightarrow{B} \mathrm{Map}_{\mathcal{A}}(B(x), B(y)) \xrightarrow{\phi^*} \mathrm{Map}_{\mathcal{A}}(A(x), B(y))$$

and

$$\mathrm{Map}_{\mathcal{O}}(x, y) \xrightarrow{A} \mathrm{Map}_{\mathcal{A}}(A(x), A(y)) \xrightarrow{\psi_*} \mathrm{Map}_{\mathcal{A}}(A(x), B(y)).$$

For example, a multimap $(x, \phi) \to (y, \psi)$ is a pair (θ, α), where $\theta\colon x \to y$ in $\mathcal{O}$ and $\alpha\colon B(\theta)\phi \xrightarrow{\sim} \psi A(\theta)$ in $\mathrm{Map}(A(x), B(y))$, filling the square

$$\begin{array}{ccc} A(x) & \xrightarrow{\phi} & B(x) \\ {\scriptstyle A(\theta)}\big\downarrow & & \big\downarrow{\scriptstyle B(\theta)} \\ A(y) & \xrightarrow{\psi} & B(y). \end{array}$$

Note that $\mathrm{Map}((x, \phi), (y, \psi))$ is indeed an $(n-1)$-category since every fiber of the functor $\mathrm{Map}((x, \phi), (y, \psi)) \to \mathrm{Map}(x, y)$ is $(n-1)$-dimensional, where the base is 0-dimensional.

The functor $\mathrm{Map}(A, B) \to \mathcal{O}$ is given on objects by $(x, \phi) \mapsto x$, and on multimaps by the projection $\mathrm{Map}((x, \phi), (y, \psi)) \to \mathrm{Map}(x, y)$.

We shall denote $\mathrm{Map}(f^*A, f^*B)$ by $\mathrm{Map}_{\mathcal{C}}(A, B)$. The next lemma is immediate from the definitions.

Lemma 4.9. *The following canonical square of multicategories is Cartesian:*

$$\begin{array}{ccc} \mathrm{Map}_{\mathcal{C}}(A,B) & \longrightarrow & \mathcal{C} \\ \big\downarrow & & \big\downarrow f \\ \mathrm{Map}(A,B) & \longrightarrow & \mathcal{O}. \end{array}$$

We shall continue with the proof of Proposition 4.8. We have already seen that it suffices to prove that the functor

$$f^*\colon \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{A})}(A,B) \to \mathrm{Map}_{\mathrm{Alg}_{\mathcal{C}}(\mathcal{A})}(A,B)$$

is an equivalence. Lemma 4.9 implies that the following square is Cartesian:

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{C}}(\mathrm{Map}_{\mathcal{C}}(A,B)) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}(\mathcal{C}) \\ \big\downarrow & & \big\downarrow f_* \\ \mathrm{Alg}_{\mathcal{C}}(\mathrm{Map}(A,B)) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}(\mathcal{O}). \end{array}$$

From this and the definition of $\mathrm{Map}_{\mathrm{Alg}_{\mathcal{C}}(\mathcal{A})}(A,B)$, we obtain a Cartesian square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Alg}_{\mathcal{C}}(\mathcal{A})}(A,B) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}(\mathrm{Map}(A,B)) \times_{\mathrm{Alg}_{\mathcal{C}}(\mathcal{O})} \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}) \\ \big\downarrow & & \big\downarrow \\ * & \xrightarrow{\text{at } f} & \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}). \end{array}$$

From the inductive hypothesis or the hypothesis of Proposition 4.8 in the case $n = 1$, we also obtain a Cartesian square

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{A})}(A,B) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathrm{Map}(A,B)) \\ \big\downarrow & & \big\downarrow \\ * & \xrightarrow{\text{at } f} & \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}). \end{array}$$

Therefore, the following square is Cartesian:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{Alg}_{\mathcal{O}}(\mathcal{A})}(A,B) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathrm{Map}(A,B)) \\ f^* \big\downarrow & & \big\downarrow \\ \mathrm{Map}_{\mathrm{Alg}_{\mathcal{C}}(\mathcal{A})}(A,B) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}}(\mathrm{Map}(A,B)) \times_{\mathrm{Alg}_{\mathcal{C}}(\mathcal{O})} \mathrm{Alg}_{\mathcal{C}}^{\mathrm{loc}}(\mathcal{O}). \end{array}$$

Since in this square, the vertical map on the right is an inclusion between full subcategories of $\mathrm{Alg}_{\mathcal{C}}(\mathrm{Map}(A,B))$, it follows that the vertical map on the left identifies its source with the full subcategory of its target consisting of those maps of algebras which, as an algebra in $\mathrm{Map}(A,B)$, are locally constant.

The desired result now follows since the definition of a map of algebras implies that every map of locally constant $\mathcal{C}$-algebras is indeed locally constant in this sense. □

Definition 4.10. Let $n \geq 2$ be an integer, and $\mathcal{A}$ a symmetric monoidal n-category. A *prealgebra* on a manifold M in $\mathcal{A}$ is an algebra over $\mathrm{Open}(M)$ in $\mathcal{A}$. We say that a prealgebra A is *locally constant* if the restriction of A to a functor on $\mathrm{Disk}(M)$ is locally constant.

Our descent results in the case the target category was a 1-category described a locally constant factorization algebra as a prealgebra satisfying various local constancy and descent properties relative to a factorizing cover or basis satisfying certain hypotheses. Recall that these results depended on cofinality of functors to $\mathrm{D}(M)$. Now we would like to see if same proofs work in the case where the target category is now a symmetric monoidal n-category for $n \geq 2$. For example, we have proved that Theorem 3.17 holds in this context.

However, only this is a nontrivial result actually, and all of our other proofs work without any change. Namely, all of our descent results hold if our target is a symmetric monoidal n-category which (or equivalently, whose underlying symmetric monoidal 1-category) satisfies the assumptions of Section 3.7.

Finally, let us generalize Theorem 3.34 to twisted algebras. Thus, let M be a manifold, and let a basis for the topology of M be given as in Theorem 3.34, by a symmetric monoidal functor $\psi\colon \mathcal{E} \to \mathrm{Open}(M)$, $i \mapsto V_i$, equipped with all the data and satisfying all the assumptions. In particular, $V_i \in \mathrm{Disj}(M)$ for every $i \in \mathcal{E}$.

Lemma 4.11. *For $i \in \mathcal{E}$, if the composite*

$$\mathcal{E}_{/i} \to \mathcal{E}_{/V_i} \xrightarrow{\psi} \mathrm{Disj}(V_i) \to \mathrm{D}(V_i) \tag{3}$$

is cofinal, then the functor $\mathcal{E}_{/i} \to \mathrm{D}(V_i)$ is universal among the functors from $\mathcal{E}_{/i}$ which invert maps which are inverted in $\mathrm{D}(V_i)$. Namely, for any category $\mathcal{C}$, the restriction through (3)*,*

$$\mathrm{Fun}(\mathrm{D}(V_i), \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}_{/i}, \mathcal{C}),$$

is fully faithful with image consisting of functors $\mathcal{E}_{/i} \to \mathcal{C}$ which invert maps in $\mathcal{E}_{/i}$ inverted in $\mathrm{D}(V_i)$.

Remark 4.12. From Remark 3.35, the assumption of the cofinality follows if the first map $\mathcal{E}_{/i} \to \mathcal{E}_{/V_i}$ of the composition (3) is cofinal, e.g., by being an equivalence.

Proof of Lemma 4.11. In order to show that the restriction functor is fully faithful, we may first embed $\mathcal{C}$ by a fully faithful functor (e.g. the Yoneda embedding) into a category which has all small limits in it, and show that the restriction functor is fully faithful for this larger target category, in place of $\mathcal{C}$. Therefore, we do not lose generality by assuming that $\mathcal{C}$ has all small limits in it, as we shall do.

In this case, an argument similar to the proof of Theorem 3.34 implies that the restriction functor is the inclusion of a right localization of $\mathrm{Fun}(\mathcal{E}_{/i}, \mathcal{C})$. Namely, if $U \in \mathrm{D}(V_i)$ is of the form $\coprod_{s\in S} D_s$ for a family $D = (D_s)_{s\in S}$ of disjoint disks indexed by a finite set S, so $D \in \mathrm{Disj}_S(V_i)$, then we have

$\psi_S\colon (\mathcal{E}_S)_{/i} \to \mathrm{Disj}_S(V_i)$, and the resulting functor $((\mathcal{E}_S)_{/i})_{D/} \to (\mathcal{E}_{/i})_{U/}$ is coinitial since it has a right adjoint. It follows that the right Kan extension of a functor $F \in \mathrm{Fun}(\mathcal{E}_{/i}, \mathcal{C})$ to $\mathrm{D}(V_i)$ associates to U the limit $\lim_{((\mathcal{E}_S)_{/i})_{D/}} F$. The claim follows immediately from this, so we have proved the fully faithfulness of the restriction functor.

The identification of the image of the embedding is then also immediate. □

Let M be a manifold, and let Disj_M denote Disj considered as an algebra of categories on $\mathrm{Disk}(M)$. Then in the 2-category $\mathrm{Alg}_{\mathrm{Disk}(M)}(\mathrm{Cat})$ of (not necessarily locally constant) algebras of categories on $\mathrm{Disk}(M)$, Disj_M corepresents the functor $\mathcal{A} \mapsto \mathrm{Alg}_{\mathrm{Disk}(M)}(\mathcal{A})$.

Similarly, let D_M denote D as a (locally constant) algebra on $\mathrm{Disk}(M)$. The obvious functor $\mathrm{Disj} \to \mathrm{D}$ is a map of algebras. We obtain the following corollary by applying Lemma 4.11 to the basis $\mathrm{Disj}(M)$ for the topology of M.

Corollary 4.13. *Let M be a manifold, and $\mathcal{A}$ an algebra of categories on* $\mathrm{Disk}(M)$*. Then the restriction functor*

$$\mathrm{Map}_{\mathrm{Alg}_{\mathrm{Disk}(M)}}(\mathrm{D}_M, \mathcal{A}) \to \mathrm{Map}_{\mathrm{Alg}_{\mathrm{Disk}(M)}}(\mathrm{Disj}_M, \mathcal{A}) = \mathrm{Alg}_{\mathrm{Disk}(M)}(\mathcal{A})$$

through the map $\mathrm{Disj} \to \mathrm{D}$ *is fully faithful, and the image consists precisely of the locally constant algebras in $\mathcal{A}$.*

More generally, in our current situation as in Theorem 3.34, let $\mathrm{D}_{\mathcal{E}_1}$ denote the restriction of D_M through the functor $\psi\colon \mathcal{E}_1 \to \mathrm{Disk}(M)$ of multicategories; see Remark 3.43. Then Lemma 4.11 implies that if the functor (3) is cofinal for every $i \in \mathcal{E}_1$, then $\mathrm{D}_{\mathcal{E}_1}$ corepresents the functor $\mathcal{A} \mapsto \mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}_1}(\mathcal{A})$ on $\mathrm{Alg}_{\mathcal{E}_1}(\mathrm{Cat})$. As a consequence, we obtain the following twisted version of Theorem 3.34, from the 2-categorical generalization of Theorem 3.34 (in the case of the target 2-category Cat).

Theorem 4.14. *Let M be a manifold, and $\mathcal{A}$ a locally constant factorization algebra of categories on M. Then for a basis for the topology of M as in Theorem 3.34, if the functor* (3) *is cofinal for every $i \in \mathcal{E}_1$, then the following restriction functor is an equivalence:*

$$\mathrm{Alg}_M(\mathcal{A}) \to \mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}_1}(\mathcal{A}).$$

Remark 4.15. See Remark 4.12 for a sufficient condition for the assumption here to be satisfied.

4.16. **(Twisted) algebras on a (twisted) product.** We shall illustrate applications of Theorem 3.34 and its generalization Theorem 4.14.

Fix a target symmetric monoidal category $\mathcal{A}$ satisfying the assumptions of Section 3.7, and drop the name $\mathcal{A}$ from the notation. Note that, in this case, the category $\mathrm{Alg}_M = \mathrm{Alg}_M(\mathcal{A})$ for a manifold M is symmetric monoidal by the value-wise multiplication in $\mathcal{A}$.

Theorem 4.17. *Let B, F be manifolds. Then, the restriction functor*
$$\mathrm{Alg}_{F\times B} \to \mathrm{Alg}_B(\mathrm{Alg}_F)$$
is an equivalence of symmetric monoidal categories.

Proof. Note that the category Alg_F has sifted colimits, and they are preserved by the tensor product (since these are the same colimits and tensor product on the underlying objects).

The restriction functor is symmetric monoidal since the symmetric monoidal structures on the categories of algebras are value-wise, so it suffices to prove that it is an equivalence of categories. For this, we would like to use Theorem 3.34 on $M := F \times B$. For this purpose, we consider the following basis for the topology of M.

The basis will be indexed by the symmetric partially monoidal category $\mathcal{E}$ to be defined as follows. The objects of the underlying category of $\mathcal{E}$ are any objects D of $\mathrm{Disj}(M)$ for which there exist objects D' of $\mathrm{Disj}(B)$ and D'' of $\mathrm{Disj}(F)$, such that any component of D is a component of $D' \times D'' \subset M$.

Morphisms in $\mathcal{E}$ shall be just inclusions, so it is a full subposet of $\mathrm{Disj}(M)$. We denote the inclusion by $\psi\colon \mathcal{E} \hookrightarrow \mathrm{Disj}(M)$. Note that this determines a factorizing l-nice (and hence effectively factorizing l-nice by Proposition 3.24) basis of M.

The partial monoidal structure on $\mathcal{E}$ will be defined as follows. For any finite set S, let $\mathrm{Disj}(M)^{(S)}$ denote the full subposet of the Cartesian product $\mathrm{Disj}(M)^S$ on which the disjoint union operation to $\mathrm{Disj}(M)$ is defined. Then we define the poset $\mathcal{E}^{(S)}$ by the Cartesian square

$$\begin{array}{ccc} \mathcal{E}^{(S)} & \hookrightarrow & \mathrm{Disj}(M)^{(S)} \\ \downarrow & & \downarrow{\scriptstyle \sqcup_S} \\ \mathcal{E} & \xhookrightarrow{\ \psi\ } & \mathrm{Disj}(M). \end{array} \tag{4}$$

It is canonically a full subposet of $\mathcal{E}^S$, and we let it be the domain of definition of the S-fold monoidal operation of $\mathcal{E}$, where the operation is defined to be the left vertical map on the square (4). Since $\mathcal{E}$ is a poset, this determines a partial monoidal structure on $\mathcal{E}$.

We define the full subposet $\mathcal{E}_1$ of $\mathcal{E}$ to be the intersection $\mathcal{E} \cap \mathrm{Disk}(M)$ taken in $\mathrm{Disj}(M)$. (As a full subposet of $\mathrm{Disk}(M)$, $\mathcal{E}_1$ is $\mathrm{Disk}(F) \times \mathrm{Disk}(B)$.)

For this factorizing l-nice basis of M, equipped with auxiliary data required for Theorem 3.34, we would like to verify that Hypothesis 3.38 is satisfied. All but the hypothesis that $\psi_1 := \psi|_{\mathcal{E}_1} : \mathcal{E}_1 \to \mathrm{Open}(M)$ determines an effectively l-nice basis are easily verified from the construction. This remaining hypothesis follows from Lurie's generalized Seifert–van Kampen theorem, since it is immediate to see that ψ_1 determines an l-nice basis.

Now Theorem 3.34 implies that the restriction functor $\mathrm{Alg}_M \to \mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}}$ is an equivalence, where the target is the category of algebras on $\mathcal{E}$ *which is locally constant* with respect to $\mathcal{E}_1$ in the sense that the maps in $\mathcal{E}_1$ are all inverted.

However, the restriction functor $\mathrm{Alg}^{\mathrm{loc}}_{\mathcal{E}} \to \mathrm{Alg}_B(\mathrm{Alg}_F)$ is nearly tautologically (namely, up to introduction and elimination of the unit objects and the unit operations as necessary) an equivalence. This completes the proof. □

For example, a locally constant factorization algebra on $\mathbb{R}^2$ is the same as an associative algebra in the category of associative algebras since a locally constant factorization algebra on $\mathbb{R}^1$ can be directly seen to be the same as an associative algebra.

Inductively, a locally constant factorization algebra on $\mathbb{R}^n$ is an iterated associative algebra object.

Remark 4.18. A product manifold $M = B \times F$ has another interesting factorizing basis. Namely, there is a factoring basis of M consisting of the disjoint unions of disks in M of the form $D' \times D''$ for disks D' in B and D'' in F. As observed in Example 3.41, Theorem 3.34 applies to the factorizing basis freely generated by this basis. The result we obtain is another description of the category Alg_M, namely as the category of 'locally constant' algebras on this factorizing basis.

Iterating this, one finds a description of the category of locally constant algebras on $\mathbb{R}^n$ which identifies it essentially with the category of algebras over Boardman and Vogt's 'little cubes' [6]. Therefore, Theorem 4.17 can be considered as a generalization of a theorem of Dunn [10].

Remark 4.19. Dunn's theorem actually identifies the E_n-operad with the n-fold tensor product of the E_1-operad. In particular, unlike our theorem in the case of $\mathbb{R}^n$, the target category of the algebras need not satisfy our assumptions on sifted colimits.

Theorem 4.17 identified the algebras on a product manifold. A product of manifolds has a twisted version, namely, a fiber bundle. Accordingly there is a generalization of Theorem 4.17 which holds for a fiber bundle. Let us formulate and prove it.

Let $p\colon E \to B$ be a smooth fiber bundle over a smooth base manifold (i.e., a map with 'locally constant' fibers). Then we construct a locally constant algebra $\mathrm{Alg}_{E/B}$ of categories on B as follows. Given an open disk $D \subset B$, let $\mathrm{Alg}_{E/B}(D)$ be the category Alg_{E_x} for the unique (up to a contractible space of choices) point $x \in D$. Note that the manifold E_x is unambiguously specified by D in the infinity groupoid of manifolds where the spaces of morphisms are the spaces of diffeomorphisms.

An inclusion $D \hookrightarrow D'$ of disks in B induces an equivalence $\mathrm{Alg}_{E/B}(D) \to \mathrm{Alg}_{E/B}(D')$ of symmetric monoidal categories (specified uniquely up to a contractible space of choices). This association becomes an algebra on $\mathrm{Disk}(B)$ since given a disjoint inclusion $\coprod_{s\in S} D_s \hookrightarrow D'$ of disks in B, we have a functor

$$\prod_{s\in S} \mathrm{Alg}_{E/B}(D_s) \to \mathrm{Alg}_{E/B}(D') \tag{5}$$

defined as (the underlying functor of) the unique symmetric monoidal functor extending the symmetric monoidal functors $\mathrm{Alg}_{E/B}(D_s) \to \mathrm{Alg}_{E/B}(D')$. This defines $\mathrm{Alg}_{E/B}$ as a locally constant algebra of categories on B.

Alternatively, given a disk $D \subset B$, consider a trivialization of p over D. If F is the typical fiber in the trivialization, then we define $\mathrm{Alg}_{E/B}(D)$ to be Alg_F. A different trivialization with typical fiber F' specifies a diffeomorphism $F \xrightarrow{\sim} F'$ uniquely up to a contractible space of choices (we will have a family of diffeomorphisms parametrized by D). Moreover, the specified (family of) diffeomorphisms satisfy the cocycle condition. This eliminates the ambiguity of $\mathrm{Alg}_{E/B}(D)$.

With a trivialization as above fixed, we shall call F the *fiber over* D of p.

In this approach, the algebra structure of $\mathrm{Alg}_{E/B}$ is given by the symmetric monoidal structure of Alg_F. Namely, if a disjoint inclusion $\coprod_{s\in S} D_s \hookrightarrow D'$ of disks in B is given, then a trivialization of p over D' restricts to a trivialization over each D_s, and then all $\mathrm{Alg}_{E/B}(D_s)$ get canonically identified with $\mathrm{Alg}_F = \mathrm{Alg}_{E/B}(D')$, where F is the fiber over D' of p with respect to the chosen trivialization. So the monoidal operation $\bigotimes_S\colon \mathrm{Alg}_F^S \to \mathrm{Alg}_F$ becomes the desired operation (5).

This is compatible with the structure of a symmetric multicategory on $\mathrm{Disk}(B)$ since restriction of trivializations clearly is.

The relation of this approach to the previous approach is that a trivialization of p over a disk D in B gives an identification of E_x, $x \in D$, with the fiber of p over D.

Next, we shall construct the 'restriction' functor

$$\mathrm{Alg}_E \to \mathrm{Alg}_B(\mathrm{Alg}_{E/B}).$$

Given an algebra A on E, we shall associate to it an object of $\mathrm{Alg}_B(\mathrm{Alg}_{E/B})$ denoted by $A_{E/B}$ as follows.

Given an open disk $D \subset B$, we pick a trivialization of p over D, and denote by q the projection $p^{-1}D \to F$ with respect to the trivialization, where F is the fiber of p over D (with respect to the trivialization). Then we define

$$A_{E/B}(D) := q_* i^* A \in \mathrm{Alg}_F = \mathrm{Alg}_{E/B}(D),$$

where $i\colon p^{-1}D \hookrightarrow E$ is the inclusion.

We need to check the well-definedness of this construction. Recall that we identified different models of the fiber of p over D by comparing the family $F \times D$ over D, for any one model F, with the family $p^{-1}D$, by the trivialization making F be a model for the fiber over D.

Taking this into account, it is easy to see that, in order to eliminate the ambiguity of the construction, it suffices to give a path between the maps

$$q \times D\colon p^{-1}D \times D \to F \times D \quad \text{and} \quad p^{-1}D \times D \xrightarrow{\mathrm{pr}} p^{-1}D \simeq F \times D,$$

through locally trivial maps (maps with locally constant fibers) $p^{-1}D \times D \to F \times D$. Using the trivialization $p^{-1}D \simeq F \times D$ again, this is equivalent to giving a path between the two projections $F \times D \times D \to F \times D$ through locally trivial maps.

We may instead choose a path between the two projections $D^2 \to D$, through locally trivial maps. We pick an embedding of D into a vector space as an open *convex* subdisk, which does not add more information than a choice of a point from a contractible space. Then we have a path of locally trivial maps $D^2 \to D$:

$$(x, y) \mapsto x + t(y - x), \quad 0 \leq t \leq 1.$$

This clearly comes as a family over the said contractible space.

Let us now equip this association $D \mapsto q_* i^* A$ with a structure of an algebra over $\mathrm{Disk}(B)$. The construction is similar to the construction of the algebra structure of $\mathrm{Alg}_{E/B}$, which we have made before. Namely, if we are given an inclusion $D \hookrightarrow D'$ in B, where D is a disjoint union of disks, then a trivialization of p over D' restricts to a trivialization of p over D, and thus we can try to construct the desired map $A_{E/B}(D) \to A_{E/B}(D')$ as

$$A\big[(q|_{p^{-1}D})^{-1}(U) = q^{-1}(U) \cap p^{-1}D \hookrightarrow q^{-1}(U)\big]$$

for disks $U \subset F$, F the fiber over D'.

It remains to check that this construction is compatible with the construction we have made to eliminate the ambiguity for the association $D \mapsto A_{E/B}(D)$. Again, assuming that D' is an open convex subdisk of a vector space, it does no harm to restrict D to disjoint unions of open *convex* subdisks of D' (convexity in the same vector space).

Then the path of locally constant maps $(D')^2 \to D'$ given above restricts to a similar path on each component of D. This verifies the compatibility of the constructions.

Proposition 4.20. *Let $p\colon E \to B$ be a smooth fiber bundle as above. Then the following restriction functor is an equivalence:*

$$\mathrm{Alg}_E \to \mathrm{Alg}_B(\mathrm{Alg}_{E/B}).$$

Proof. The functor can be written as

$$\lim_{D \in \mathrm{Disj}(B)} \mathrm{Alg}_{p^{-1}D} \to \lim_{D \in \mathrm{Disj}(B)} \mathrm{Alg}_D(\mathrm{Alg}_{p^{-1}D/D}).$$

Indeed, we can apply Theorem 3.33 to the source, and the target is this limit essentially by definition.

The given functor is the limit of the restriction functors on $D \in \mathrm{Disj}(B)$.

However, on each D, the restriction functor can be identified with that in Theorem 4.17 by using the decomposition $p^{-1}D = F \times D$, where F is the fiber of p over D. Therefore it is an equivalence by the assertion of the theorem. It follows that the twisted version of the restriction functor is also an equivalence. ☐

Remark 4.21. From the discussions of Section 4.3, Proposition 4.20 holds for a higher target category by the same proof. If the target is an n-category, then the proposition states that we have an equivalence of n-categories of algebras. In the following, we shall use the 2-category case.

There is a natural further generalization of this. Namely, the algebra $\mathrm{Alg}_{E/B}$ can be constructed when the algebra on E is twisted. That is, let $\mathcal{A}$ be a locally constant (pre-)algebra on E of categories. Then, for a disk $D \subset B$, define the category

$$\mathrm{Alg}_{E/B}(\mathcal{A})(D) := \mathrm{Alg}_F(\mathcal{A}_{E/B}(D)),$$

where $\mathcal{A}_{E/B} \in \mathrm{Alg}_B(\mathrm{Alg}_{E/B}(\mathrm{Cat}))$ is the restriction of $\mathcal{A}$ as in Proposition 4.20 (Cat denoting the 2-category of categories in which $\mathcal{A}$ takes values), and F is the fiber of p over D, so

$$\mathcal{A}_{E/B}(D) \in \mathrm{Alg}_{E/B}(\mathrm{Cat})(D) = \mathrm{Alg}_F(\mathrm{Cat}).$$

Moreover, a restriction functor

$$\mathrm{Alg}_E(\mathcal{A}) \to \mathrm{Alg}_B(\mathrm{Alg}_{E/B}(\mathcal{A})) \tag{6}$$

can be defined by $A \mapsto A_{E/B}$, where $A_{E/B} \in \mathrm{Alg}_B(\mathrm{Alg}_{E/B}(\mathcal{A}))$ associates to a disk $D \subset B$ the object

$$q_* i^* A \in \mathrm{Alg}_F(q_* i^* \mathcal{A}) = \mathrm{Alg}_{E/B}(\mathcal{A})(D).$$

The algebra structure is exactly as before.

Theorem 4.22. *For a locally constant (pre-)algebra $\mathcal{A}$ on E of categories, the restriction functor* (6) *is an equivalence.*

Let us first establish this in the case where the fiber bundle is trivial. A global choice of a trivialization leads to simplification of the constructions as well.

Lemma 4.23. *Let B, F be manifolds, and $\mathcal{A}$ an object of $\mathrm{Alg}_B(\mathrm{Alg}_F(\mathrm{Cat}))$, or equivalently, a locally constant algebra of categories on $F \times B$, by Theorem 4.17. Then, the following restriction functor is an equivalence:*

$$\mathrm{Alg}_{F\times B}(\mathcal{A}) \to \mathrm{Alg}_B(\mathrm{Alg}_F(\mathcal{A})),$$

where $\mathrm{Alg}_F(\mathcal{A})$ is a locally constant algebra of categories on B defined by $\mathrm{Alg}_F(\mathcal{A})(D) := \mathrm{Alg}_F(\mathcal{A}(D))$.

Proof. Similar to the proof of Theorem 4.17. One simply notes that Theorem 4.14 applies here instead of Theorem 3.34; see Remark 4.15. □

Proof of Theorem 4.22. The 2-categorical generalization of Theorem 3.33 implies that the restriction functor $\mathrm{Alg}_E(\mathrm{Cat}) \to \lim_{D\in\mathrm{Disj}(B)} \mathrm{Alg}_{p^{-1}D}(\mathrm{Cat})$ is an equivalence of 2-categories. From this, one obtains that the restriction functor

$$\mathrm{Alg}_E(\mathcal{A}) \to \lim_{D\in\mathrm{Disj}(B)} \mathrm{Alg}_{p^{-1}D}(\mathcal{A})$$

is an equivalence.

Similarly, one would like to show that the restriction functor

$$\mathrm{Alg}_B(\mathrm{Alg}_{E/B}(\mathcal{A})) \to \lim_{D\in\mathrm{Disj}(B)} \mathrm{Alg}_D(\mathrm{Alg}_{p^{-1}D/D}(\mathcal{A}))$$

is an equivalence. However, since it is easy to verify from the definitions that the restriction of $\mathrm{Alg}_{E/B}(\mathcal{A})$ to $D \subset B$ is $\mathrm{Alg}_{p^{-1}D/D}(\mathcal{A})$, the equivalence also follows from Theorem 3.33.

By the naturality of the restriction functor, we have reduced the statement to the case where the base is a disjoint union of disks. In this case the fiber bundle is trivial on each component, and the statement follows from Lemma 4.23. □

Remark 4.24. The results of this section depend only on our descent results from Section 3. Therefore, by what we have seen in the previous section, all the results of this section have a version in which the target category may be of any dimension, and we get an equivalence of *higher* categories of algebras.

Note. This paper, together with [17] and [18], is based on the author's Ph.D. thesis (accepted in April 2014). The present article is logically independent of both [17] and [18].

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References

[1] D. W. Anderson, Chain functors and homology theories, in *Symposium on algebraic topology (Seattle, 1971)*, 1–12. Lecture Notes in Math., 249, Springer, Berlin, 1971. MR0339132

[2] M. Atiyah, Topological quantum field theories, Inst. Hautes Études Sci. Publ. Math. No. 68 (1988), 175–186 (1989). MR1001453

[3] D. Ayala and J. Francis, Factorization homology of topological manifolds, J. Topol. **8** (2015), no. 4, 1045–1084. MR3431668

[4] J. C. Baez and J. Dolan, Higher-dimensional algebra and topological quantum field theory, J. Math. Phys. **36** (1995), no. 11, 6073–6105. MR1355899

[5] A. Beilinson and V. Drinfeld, *Chiral algebras*, Amer. Math. Soc. Colloq. Publ., 51, Amer. Math. Soc., Providence, RI, 2004. MR2058353

[6] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Math., 347, Springer, Berlin, 1973. MR0420609

[7] P. Boavida de Brito and M. S. Weiss, Manifold calculus and homotopy sheaves, arXiv:1202.1305v2 (2013).

[8] D. Calaque, *Around Hochschild (co)homology*, Habilitation thesis, Université Claude Bernard Lyon 1, 2013.

[9] K. Costello and O. Gwilliam, *Factorization algebras in quantum field theory*, draft, www.math.northwestern.edu/~costello/renormalization.
[10] G. Dunn, Tensor product of operads and iterated loop spaces, J. Pure Appl. Algebra **50** (1988), no. 3, 237–258. MR0938617
[11] G. Ginot, Notes on factorization algebras, factorization homology and applications, in *Mathematical aspects of quantum field theories*, 429–552, Math. Phys. Stud, Springer, Cham, 2015. MR3330249
[12] G. Ginot, T. Tradler and M. Zeinalian, Higher Hochschild homology, topological chiral homology and factorization algebras, Comm. Math. Phys. **326** (2014), no. 3, 635–686. MR3173402
[13] M. Kontsevich, Operads and motives in deformation quantization, Lett. Math. Phys. **48** (1999), no. 1, 35–72. MR1718044
[14] J. Lurie, *Higher topos theory*, Ann. of Math. Stud., 170, Princeton Univ. Press, Princeton, NJ, 2009. MR2522659
[15] J. Lurie, On the classification of topological field theories, in *Current developments in mathematics, 2008*, 129–280, Int. Press, Somerville, MA, 2009. MR2555928
[16] J. Lurie, *Higher algebra*, 2012, www.math.harvard.edu/~lurie/outdated.html
[17] T. Matsuoka, Koszul duality between E_n-algebras and coalgebras in a filtered category, arXiv:1409.6943 (2014).
[18] T. Matsuoka, Koszul duality for locally constant factorization algebras, Serdica Math. J. **41** (2015), no. 4, 369–414. MR3467104
[19] T. Pirashvili, Hodge decomposition for higher order Hochschild homology, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 2, 151–179. MR1755114
[20] G. Segal, The definition of conformal field theory, in *Topology, geometry and quantum field theory*, 421–577, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004. MR2079383

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