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C^*-algebras associated to topological Ore semigroups

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Abstract. Let G be a locally compact group and let $P \subset G$ be a closed Ore semigroup containing the identity element. Let $V : P \to B(\mathcal{H})$ be an anti-homomorphism such that for every $a \in P$, V_a is an isometry and the final projections of $\{V_a \mid a \in P\}$ commute. We study the C^*-algebra generated by $\{\int f(a)V_a\,da \mid f \in L^1(P)\}$. We show that there exists a groupoid C^*-algebra which is universal for isometric representations with commuting range projections.

1. Introduction

It is fair to say that C^*-algebras of groups and their crossed products are the most studied C^*-algebras in the theory of operator algebras. Several authors have tried to study C^*-algebras associated to semigroups. For example, the Toeplitz algebra is the C^*-algebra associated to the additive semigroup $\mathbb{N}$. Recently, the theory of semigroup C^*-algebras has received renewed attention; see for example [2, 5, 6] and the references therein. The notion of crossed product by semigroups has also been studied by several authors most notably by Murphy in [8, 9, 11] and by Exel in [3]. However much of the literature focusses on discrete semigroups. In the topological direction, up to the author's knowledge, the only example studied is the Wiener–Hopf C^*-algebra. This was studied from the groupoid point of view first in [7] and then successively by Nica in [12, 13] and Hilgert and Neeb in [4].

Let G be a second countable locally compact group and $P \subset G$ a closed semigroup containing the identity element. We assume that $\mathrm{Int}(P)$ is dense in P and $PP^{-1} = G$. Let $V : P \to B(\mathcal{H})$ be an isometric representation on a Hilbert space $\mathcal{H}$, i.e., for $a \in P$, V_a is an isometry and $V_aV_b = V_{ba}$. For $f \in L^1(P)$, let

$$W_f := \int_{a\in P} f(a)V_a\,da.$$

The semigroup C^*-algebra or the Wiener–Hopf algebra, denoted $\mathcal{W}_V(P,G)$, associated to the representation V is the C^*-algebra generated by the operators $\{W_f \mid f \in L^1(P)\}$. If we consider the compression of the right regular representation of G on $L^2(G)$ onto $L^2(P)$, then one obtains the usual Wiener–Hopf algebra studied in [7]. In general, it is very difficult to understand the structure of $\mathcal{W}_V(P,G)$. However if we assume that the final projections $\{E_a := V_aV_a^* \mid a \in P\}$ form a commuting family of projections, then one can do better. Without this commutative assumption, the situation becomes complicated even for the simplest case of $P := \mathbb{N} \times \mathbb{N}$ as is illustrated by Murphy in [10]. The results obtained and the organization of the paper are described below.

From now on, we assume that the range projections commute. For $g = ab^{-1} \in G$, let $W_g := V_b^*V_a$ and let E_g be the final space of W_g. It is shown in Section 3, that W_g is well-defined and $\{E_g \mid g \in G\}$ forms a commuting family of projections. For $f \in L^1(G)$, let $W_f := \int f(g)W_g\,dg$. It is not difficult to show that $\mathcal{W}_V(P,G)$ is generated by $\{W_f \mid f \in L^1(G)\}$. Let Ω be the spectrum of the commutative C^*-algebra generated by $\{\int f(g)E_g\,dg \mid f \in L^1(G)\}$. The map $C(\Omega) \rtimes P \ni (T,a) \to V_a^*TV_a \in C(\Omega)$ provides an action of P on Ω. In Sections 4 and 5, we show that this action is injective. Let

$$\mathcal{G} := \Omega \rtimes P := \{(x, ab^{-1}, y) \in \Omega \times G \times \Omega \mid xa = yb\}$$

be the Deaconu–Renault groupoid where the groupoid operations are given by

$$\begin{aligned}(x,g,y)(y,h,z) &= (x,gh,z),\\ (x,g,y)^{-1} &= (y,g^{-1},x).\end{aligned}$$

For $f \in C_c(G)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x,g,y) = f(g)$. We apply the results of [14] to show that $\Omega \rtimes P$ has a Haar system. We also show that there exists a surjective representation $\lambda : C^*(\mathcal{G}) \to \mathcal{W}_V(P,G)$ such that for $f \in C_c(G)$,

$$\lambda(\tilde{f}) = \int f(g)\Delta(g)^{-\frac{1}{2}}W_{g^{-1}}\,dg.$$

Here Δ denotes the modular function of the group. This is achieved in Sections 4–6. For the Wiener–Hopf representation, the groupoid $\Omega \rtimes P$ is the groupoid considered in [7].

We show in Section 7, that there exists a universal space Ω_u on which P acts such that if $V : P \to B(\mathcal{H})$ is an isometric representation with commuting range projections, then there exists a representation $\lambda : C^*(\Omega_u \rtimes P) \to B(\mathcal{H})$ such that for $f \in C_c(G)$,

$$\lambda(\tilde{f}) = \int f(g)\Delta(g)^{-\frac{1}{2}}W_{g^{-1}}\,dg.$$

We end the paper by proving a version of Coburn's theorem for the semigroup $[0,\infty)^n$.

2. Preliminaries

For the convenience of the reader, we recall the essential facts from [14] that we need in this paper. The proofs can be found in [14]. Throughout this paper, G stands for a second countable, locally compact topological group and $P \subset G$ for a closed subsemigroup containing the identity element e. We also assume the following:

(C1) $G = PP^{-1}$.

(C2) The interior of P in G, denoted $\mathrm{Int}(P)$, is dense in P.

Semigroups for which (C1) is satisfied are called Ore semigroups. In this paper, we consider only semigroups with identity for which (C1) and (C2) are satisfied.

Let X be a compact Hausdorff space. A right action of P on X is a continuous map $X \times P \ni (x, a) \to xa \in X$ such that $xe = x$ and $(xa)b = x(ab)$ for $x \in X$ and $a, b \in P$. Moreover, we assume that the action is injective, i.e., for every $a \in P$, the map $X \ni x \to xa \in X$ is injective. Let X be a compact Hausdorff space on which P acts on the right injectively. Then the semi-direct product groupoid $X \rtimes P$ is defined as follows:

$$X \rtimes P := \{(x, g, y) \in X \times G \times X \mid \text{there exist } a, b \in P \text{ such that } \\ g = ab^{-1} \text{ and } xa = yb\}.$$

The groupoid multiplication and the inversion are given by

$$(x, g, y)(y, h, z) = (x, gh, z),$$
$$(x, g, y)^{-1} = (y, g^{-1}, x).$$

The map $X \rtimes P \ni (x, g, y) \to (x, g) \in X \times G$ is injective. Thus $X \rtimes P$ can be considered as a subset of $X \times G$ which we do from now. Moreover, $X \rtimes P$ is a closed subset of $X \times G$ and when $X \rtimes P$ is given the subspace topology, the groupoid $X \rtimes P$ becomes a topological groupoid. We denote the range and source maps by r and s, respectively.

For $x \in X$, let $Q_x := \{g \in G \mid (x, g) \in \mathcal{G}\}$. Then $r^{-1}(x) = \{x\} \times Q_x$. Note that for $x \in X$, we have $Q_x.P \subset Q_x$ and Q_x is closed. By [14, Lem. 4.1], for $x \in X$, $\mathrm{Int}(Q_x)$ is dense in Q_x and the boundary of Q_x has measure zero.

For $x \in X$, let λ^x be the measure on $\mathcal{G}$ defined as follows: For $f \in C_c(\mathcal{G})$,

$$\int f \, d\lambda^x = \int f(x, g) 1_{Q_x}(g) \, dg.$$

Here dg denotes the left Haar measure on G. In [14], it is shown that the groupoid $\mathcal{G} := X \rtimes P$ admits a Haar system if and only if the map $X \times \mathrm{Int}(P) \ni (x, a) \to xa \in X$ is open. In this case, the measures $(\lambda^x)_{x \in X}$ form a Haar system. We will use only this Haar system if $X \rtimes P$ admits one.

Suppose that $\mathcal{G} := X \rtimes P$ admits a Haar system. Then the action of P on X can be dilated to an action of G. That is there exists a locally compact Hausdorff space Y on which G acts on the right and a continuous P-equivariant injection $i : X \to Y$ such that

(i) the set $X_0 := i(X)\,\mathrm{Int}(P)$ is open in Y, and
(ii) $Y = \bigcup_{a\in P} i(X)a^{-1} = \bigcup_{a\in \mathrm{Int}(P)} X_0 a^{-1}$.

Moreover, the space Y is unique up to a G-equivariant homeomorphism. We will identify X as a subspace of Y via the injection i and will suppress the notation i. Also the groupoid $\mathcal{G}$ is isomorphic to the reduction $(Y \rtimes G)|_X$. With this notation, note that for $x \in X$, $1_{Q_x}(g) = 1_X(xg)$. Also we leave it to the reader to check that $1_{\mathrm{Int}(Q_x)}(g) = 1_{X_0}(xg)$.

For $f \in C_c(G)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x,g) = f(g)$. We also need the following proposition. The proof is a line by line imitation of that of [7, Prop. 3.5]. Hence we omit the proof. See also [14] for some remarks concerning the proof.

Let $\mathcal{G} := X \rtimes P$ and assume that it has a Haar system.

Proposition 2.1. *For $f \in C_c(G)$, let $\hat{f} \in C(X)$ be defined by the equation $\hat{f}(x) = \int f(g)1_X(xg)\,dg$. Suppose that the family $\{\hat{f} \mid f \in C_c(G)\}$ separates points of X. Then the $*$-algebra generated by $\{\tilde{f} \mid f \in C_c(G)\}$ is dense in $C_c(\mathcal{G})$ where $C_c(\mathcal{G})$ is given the inductive limit topology. As a consequence, $C^*(\mathcal{G})$ is generated by $\{\tilde{f} \mid f \in C_c(G)\}$.*

3. Isometric representations with commuting range projections

Definition 3.1. A map $V : P \to B(\mathcal{H})$ is called an isometric representation of P on the Hilbert space $\mathcal{H}$ if

(i) the maps $P \ni a \to V_a$ and $P \ni a \to V_a^*$ are strongly continuous,
(ii) for $a \in P$, V_a is an isometry, and
(iii) for $a, b \in P$, $V_aV_b = V_{ba}$.

For $a \in P$, let $E_a := V_aV_a^*$. If $\{E_a \mid a \in P\}$ is a commuting family of projections, we say that V has commuting range projections.

In the next example, we recall the Wiener–Hopf representation or the regular representation. The C^*-algebra associated to the Wiener–Hopf representation has been studied by several authors. See the papers [7, 4] and the references therein.

Example 3.2. Consider the Hilbert space $L^2(G)$ and consider $L^2(P)$ as a closed subspace of $L^2(G)$. For $\xi \in L^2(P)$ and $a \in P$, let $V_a(\xi)$ be defined as follows:

$$V_a(\xi)(x) := \begin{cases} \xi(xa^{-1})\Delta(a)^{\frac{-1}{2}} & \text{if } x \in Pa, \\ 0 & \text{if } x \notin Pa. \end{cases}$$

Here Δ denotes the modular function of the group G. Then the map $a \in P \to V_a \in B(L^2(P))$ is an isometric representation with commuting range projections. For $a \in P$, the range of V_a is $L^2(Pa)$. The C^*-algebra generated by $\{\int f(a)V_a\,da \mid f \in L^1(P)\}$ is called the Wiener–Hopf algebra associated to (P, G) and is denoted $\mathcal{W}(P, G)$.

The above example can be generalized as follows.

Example 3.3. Let A be a closed subset of G such that $e \in A$ and $A.P \subset A$. Following [15], such subsets of G are called P-modules. Note that by [14, Lem. 4.1], it follows that $\mathrm{Int}(A)$ is dense in A and the boundary of A has measure zero. Consider the Hilbert space $L^2(A)$ as a closed subspace of $L^2(G)$. For $\xi \in L^2(A)$ and $a \in P$, let $V_a(\xi)$ be defined as

$$V_a(\xi)(x) := \begin{cases} \xi(xa^{-1})\Delta(a)^{\frac{-1}{2}} & \text{if } x \in Aa, \\ 0 & \text{if } x \notin Aa. \end{cases}$$

Then the map $P \ni a \to V_a \in B(L^2(A))$ is an isometric representation with commuting range projections. We call the representation $P \ni a \to V_a \in B(L^2(A))$ the Wiener–Hopf representation associated to the P-module A, and the C^*-algebra generated by $\{\int f(a)V_a\, da \mid f \in L^1(P)\}$, denoted $\mathcal{W}_A(P,G)$, the Wiener–Hopf algebra associated to the P-module A.

The C^*-algebras $\mathcal{W}_A(P,G)$ when $G = \mathbb{Z}^k$ and $P = \mathbb{N}^k$ were analyzed by Salas in great detail in [15].

Throughout the paper, we fix an isometric representation $V : P \to B(\mathcal{H})$ with commuting range projections. For $g = ab^{-1}$, let $W_g := V_b^* V_a$ and let $E_g := W_g W_g^*$. First we show that W_g is well-defined and is a partial isometry.

Proposition 3.4. *Let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections.*

(i) *For $g \in G$, W_g is well-defined and is a partial isometry.*
(ii) *The family $\{E_g \mid g \in G\}$ forms a commuting family of projections.*
(iii) *If $g_1 g_2^{-1} \in P$, then $E_{g_1} \leq E_{g_2}$.*
(iv) *The map $G \in g \to W_g \in B(\mathcal{H})$ is strongly continuous.*
(v) *For $g, h \in G$, $W_g W_h = E_g W_{hg}$.*

Proof. Suppose $g = a_1 b_1^{-1} = a_2 b_2^{-1}$. Then $a_1^{-1} a_2 = b_1^{-1} b_2$. Since $PP^{-1} = G$, there exist $\alpha_1, \alpha_2 \in P$ such that $a_1^{-1} a_2 = \alpha_1 \alpha_2^{-1} = b_1^{-1} b_2$. Then $a_1 \alpha_1 = a_2 \alpha_2$ and $b_1 \alpha_1 = b_2 \alpha_2$.

Now observe that

$$\begin{aligned} V_{b_1}^* V_{a_1} &= V_{b_1}^* V_{\alpha_1}^* V_{\alpha_1} V_{a_1} \\ &= V_{b_1\alpha_1}^* V_{a_1\alpha_1} \\ &= V_{b_2\alpha_2}^* V_{a_2\alpha_2} \\ &= V_{b_2}^* V_{\alpha_2}^* V_{\alpha_2} V_{a_2} \\ &= V_{b_2}^* V_{a_2}. \end{aligned}$$

This proves that W_g is well-defined. Let $E_g := W_g W_g^*$. If $g = ab^{-1}$, then $E_g = V_b^* E_a V_b$ which is selfadjoint.

Now note that

$$\begin{aligned}
E_g^2 &= V_b^*E_aV_bV_b^*E_aV_b \\
&= V_b^*E_aE_bE_aV_b \\
&= V_b^*E_bE_a^2V_b \quad \text{(since } E_a \text{ and } E_b \text{ commute)} \\
&= V_b^*E_aV_b \\
&= E_g.
\end{aligned}$$

Thus E_g is a projection. This proves (i).

Let $g_1, g_2 \in G$ be given. Write $g_1 = a_1b_1^{-1}$ and $g_2 = a_2b_2^{-1}$ with $a_i, b_i \in P$. Choose $\alpha_1, \alpha_2 \in P$ such that $b_1\alpha_1 = b_2\alpha_2$. Let $a_i' = a_i\alpha_i$ and $b_i' = b_i\alpha_i$ for $i = 1, 2$. Then $g_i = a_i'(b_i')^{-1}$ for $i = 1, 2$. But now $b_1' = b_2'$. Thus

$$\begin{aligned}
E_{g_1}E_{g_2} &= V_{b_1'}^*E_{a_1'}V_{b_1'}V_{b_1'}^*E_{a_2'}V_{b_1'} \\
&= V_{b_1'}^*E_{a_2'}V_{b_1'}V_{b_1'}^*E_{a_1'}V_{b_1'} \\
&= E_{g_2}E_{g_1}.
\end{aligned}$$

This proves (ii).

Suppose $g_1g_2^{-1} = a$ for some $a \in P$. Write $g_2 = bc^{-1}$. Then $g_1 = (ab)c^{-1}$. Then

$$\begin{aligned}
E_{g_1} &= V_c^*V_{ab}V_{ab}^*V_c \\
&= V_c^*V_b(V_aV_a^*)V_b^*V_c \\
&\leq V_c^*V_bV_b^*V_c \\
&\leq E_{g_2}.
\end{aligned}$$

This proves (iii).

Note that the map $\mathrm{Int}(P) \times \mathrm{Int}(P) \ni (a, b) \to ab^{-1} \in G$ is surjective and open. Thus G is the quotient of $\mathrm{Int}(P) \times \mathrm{Int}(P)$. Since multiplication is strongly continuous on the unit ball of $B(\mathcal{H})$, it follows that the map $\mathrm{Int}(P) \times \mathrm{Int}(P) \ni (a, b) \to V_b^*V_a \in B(\mathcal{H})$ is strongly continuous. Consequently, the map $G \ni g \to W_g \in B(\mathcal{H})$ is strongly continuous. This proves (iv).

Let $g, h \in G$ be given. Write $g = ab^{-1}$ and $h = cd^{-1}$ with $a, b \in P$. Choose $\alpha, \beta \in P$ such that $d\beta = a\alpha$. Now note that $g = (a\alpha)(b\alpha)^{-1}$ and $h = (c\beta)(d\beta)^{-1}$. Thus we can write g and h as $g = a_1b_1^{-1}$ and $h = c_1a_1^{-1}$. Now calculate as follows:

$$\begin{aligned}
W_gW_h &= V_{b_1}^*V_{a_1}V_{a_1}^*V_{c_1} \\
&= V_{b_1}^*E_{a_1}V_{b_1}V_{b_1}^*V_{c_1} \quad \text{(since } E_{a_1} \text{ and } E_{b_1} \text{ commute)} \\
&= E_gW_{hg}.
\end{aligned}$$ □

For $f \in L^1(G)$, the Wiener–Hopf operator with symbol f is defined as

$$W_f := \int f(g)W_g\,dg.$$

We want to describe the C^*-algebra, denoted $\mathcal{W}_V(P,G)$, generated by the operators $\{W_f \mid f \in L^1(G)\}$.

Remark 3.5. One can show that the C^*-algebra $\mathcal{W}_V(P,G)$ is generated by $\{\int f(a)V_a\,da \mid f \in L^1(P)\}$. The proof is similar to that of [14, Prop. 2.2]. Hence we omit the proof.

First we consider a related commutative C^*-algebra. Note that by definition, for $g \in G$, $W_{g^{-1}} = W_g^*$. Moreover, the map $G \ni g \to E_g = W_g W_g^*$ is strongly continuous. For $f \in L^1(G)$, let

$$E_f := \int f(g)E_g\,dg.$$

Let

$$\mathcal{A} := C^*\{E_f \mid f \in L^1(G)\}.$$

Since $\{E_g \mid g \in G\}$ forms a commuting family of projections, it follows that $\mathcal{A}$ is a commutative C^*-subalgebra of $B(\mathcal{H})$. Note that $E_g = 1$ if $g \in P^{-1}$. If $f \in L^1(P^{-1})$, then $E_f = \int f(g)\,dg$. Thus, it follows that $\mathcal{A}$ is a commutative unital C^*-subalgebra of $B(\mathcal{H})$. Denote the spectrum of $\mathcal{A}$ by Ω.

Let

$$G_n := \underbrace{G \times G \times \cdots \times G}_{n \text{ times}}.$$

For $f \in C_c(G_n)$, let

$$E_f := \int f(g_1,g_2,\dots,g_n)E_{g_1}E_{g_2}\cdots E_{g_n}\,dg_1\,dg_2\cdots dg_n.$$

Let $\widetilde{\mathcal{A}} := \bigcup_{n=1}^{\infty}\{E_f \mid f \in C_c(G_n)\}$. Then $\widetilde{\mathcal{A}}$ forms a dense unital $*$-subalgebra of $\mathcal{A}$. Also note that for every n, the map $C_c(G_n) \ni f \to E_f \in \mathcal{A}$ is continuous when $C_c(G_n)$ is given the inductive limit topology and $\mathcal{A}$ is given the norm topology.

For $T \in B(\mathcal{H})$ and $a \in P$, let $\alpha_a(T) = V_a^* T V_a$. Clearly $\alpha_e = \mathrm{id}$ and $\alpha_a\alpha_b = \alpha_{ab}$.

Observe that $\alpha_a(V_b^*E_cV_b) = V_a^*V_b^*E_cV_bV_a = V_{ab}^*E_cV_{ab}$. Thus $\alpha_a(E_g) = E_{ga^{-1}}$ for $g \in G$. Since the final projection $V_aV_a^*$ commutes with E_g for every $g \in G$, it follows that $\alpha_a(E_{g_1}E_{g_2}) = \alpha_a(E_{g_1})\alpha_a(E_{g_2})$.

Proposition 3.6. *For $a \in P$, α_a leaves $\mathcal{A}$ invariant and the map $\alpha_a : \mathcal{A} \to \mathcal{A}$ is a unital $*$-homomorphism. For $T \in \mathcal{A}$, the map $P \ni a \to \alpha_a(T) \in \mathcal{A}$ is norm continuous.*

Proof. For $a \in P$ and $f \in C_c(G_n)$, let $\tilde{f}_a \in C_c(G_n)$ be defined by

$$\tilde{f}_a(g_1,g_2,\dots,g_n) = \Delta(a)^n f(g_1a,g_2a,\dots,g_na).$$

Then for $f \in C_c(G_n)$, the map $P \ni a \to \tilde{f}_a \in C_c(G_n)$ is continuous if $C_c(G_n)$ is given the inductive limit topology.

Let $a \in P$ and $f \in C_c(G_n)$ be given. Then

$$\begin{aligned}\alpha_a(E_f) &= \int f(g_1, g_2, \ldots, g_n)\alpha_a(E_{g_1}E_{g_2}\cdots E_{g_n})\, dg_1\, dg_2\cdots dg_n \\ &= \int f(g_1, g_2, \ldots, g_n)E_{g_1a^{-1}}E_{g_2a^{-1}}\ldots E_{g_na^{-1}}\, dg_1\, dg_2\cdots dg_n \\ &= \int f(g_1a, g_2a, \ldots, g_na)\Delta(a)^n E_{g_1}E_{g_2}\cdots E_{g_n}\, dg_1\, dg_2\cdots dg_n \\ &= E_{\tilde{f}_a}.\end{aligned}$$

Thus α_a leaves $\widetilde{\mathcal{A}}$ invariant. Since $\widetilde{\mathcal{A}}$ is dense in $\mathcal{A}$ and α_a is bounded, it follows that α_a leaves $\mathcal{A}$ invariant.

Observe that if $E_a = V_aV_a^*$ commutes with $T, S \in B(\mathcal{H})$ then $\alpha_a(TS) = \alpha_a(T)\alpha_a(S)$. By Proposition 3.4, it follows that E_a commutes with E_f for $f \in C_c(G_n)$. Thus E_a commutes with every element of $\mathcal{A}$. Hence $\alpha_a : \mathcal{A} \to \mathcal{A}$ is multiplicative. Clearly α_a is unital and $*$-preserving.

For $a \in P$, $\alpha_a : \mathcal{A} \to \mathcal{A}$ is contractive. Thus it is enough to show that for $T \in \widetilde{\mathcal{A}}$, the map $P \ni a \to \alpha_a(T) \in \mathcal{A}$ is continuous. Let $T = E_f$ for some $f \in C_c(G_n)$. Then $\alpha_a(T) = E_{\tilde{f}_a}$. Hence the map $P \ni a \to \alpha_a(T) = E_{\tilde{f}_a}$ is continuous as it is the composite of the continuous maps $P \ni a \to \tilde{f}_a \in C_c(G_n)$ and $C_c(G_n) \ni h \to E_h$ where $C_c(G_n)$ is given the inductive limit topology. ☐

Since $\mathcal{A} = C(\Omega)$, it follows that for every $a \in P$, there exists $\phi_a : \Omega \to \Omega$ such that $F \circ \phi_a = \alpha_a(F)$ for $F \in C(\Omega)$. The condition $\alpha_a\alpha_b = \alpha_{ab}$ translates to $\phi_a\phi_b = \phi_{ba}$ for $a, b \in P$. Also $\phi_e = \mathrm{id}$. Thus the map $\Omega \times P \ni (x, a) \to \phi_a(x) \in \Omega$ defines a right action of P on Ω. We henceforth write $\phi_a(x)$ as xa for $x \in \Omega$ and $a \in P$.

We claim that the map $\Omega \times P \ni (x, a) \to xa \in \Omega$ is continuous. Suppose $(x_n) \to x$ and $(a_n) \to a$. Let $F \in C(\Omega)$. By Proposition 3.6, it follows that $\alpha_{a_n}(F)$ converges uniformly to $\alpha_a(F)$. Since the convergence is uniform, it follows that $\alpha_{a_n}(F)(x_n)$ converges to $\alpha_a(F)(x)$. In other words, for every $F \in C(\Omega)$, $F(x_na_n)$ converges to $F(xa)$. Hence x_na_n converges to xa.

The goal of this paper is to prove the following statements.

(i) The right action of P on Ω is injective.
(ii) The semi-direct product groupoid $\mathcal{G} := \Omega \rtimes P$ has a Haar system.
(iii) For $f \in C_c(G)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x, g) = f(g)$ for $(x, g) \in \mathcal{G}$. There exists a surjective $*$-homomorphism $\pi : C^*(\mathcal{G}) \to \mathcal{W}(P, G)$ such that $\pi(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g)W_{g^{-1}}\, dg$ for $f \in C_c(G)$.

To prove the above statements, we need a better description of Ω, which forms the content of the next section. We end this section with a lemma which is useful in showing that $\Omega \rtimes P$ has a Haar system.

Lemma 3.7. *Let $f \in C_c(G)$ be such that* $\mathrm{supp}(f) \subset \mathrm{Int}(P)$. *Then for $T \in \mathcal{A}$, the integral $\int_{a\in P} f(a)V_aTV_a^*\, da \in \mathcal{A}$.*

Proof. It is enough to prove the statement for $T \in \widetilde{\mathcal{A}}$. Let $T = E_\phi$ for some $\phi \in C_c(G_n)$. For $a \in P$ and $g \in G$, we have $V_a^*E_{ga}V_a = E_g$. Hence $V_aE_gV_a^* =$

$E_{ga}E_a$. Now calculate as follows to find that

$$\begin{aligned}
&\int_{a\in P} f(a)V_aE_\phi V_a^*\,da \\
&\quad= \int_{a\in P} f(a)\phi(g_1,g_2,\ldots,g_n)V_aE_{g_1}E_{g_2}\cdots E_{g_n}V_a^*\,da\,dg_1\,dg_2\cdots dg_n \\
&\quad= \int_{a\in P} f(a)\phi(g_1,g_2,\ldots,g_n)E_aE_{g_1a}E_{g_2a}\cdots E_{g_na}\,da\,dg_1\,dg_2\cdots dg_n \\
&\quad= \int_{a\in P} \Delta(a)^{-n}f(a)\phi(g_1a^{-1},g_2a^{-1},\ldots,g_na^{-1}) \\
&\qquad\qquad \times E_aE_{g_1}E_{g_2}\cdots E_{g_n}\,da\,dg_1\,dg_2\cdots dg_n \\
&\quad= E_\psi \in \widetilde{\mathcal{A}},
\end{aligned}$$

where $\psi \in C_c(G_{n+1})$ is given by

$$\psi(g,g_1,g_2,\ldots,g_n) = \Delta(g)^{-n}f(g)\phi(g_1g^{-1},g_2g^{-1},\ldots,g_ng^{-1}). \qquad \square$$

4. What is Ω?

We first discuss the case when G is discrete. The discrete semigroup C^*-algebras are analyzed in great detail in the papers [5, 6]. Nevertheless, we discuss this case in the form that we need. This also motivates the topological case.

Let G be a discrete group and let $P \subset G$ be a semigroup such that $e \in P$ and $PP^{-1} = G$. In this case, the Wiener–Hopf C^*-algebra $\mathcal{W}_V(P,G)$ is simply the C^*-algebra generated by $\{W_g \mid g \in G\}$ and the commutative C^*-algebra $\mathcal{A}$ is the C^*-algebra generated by $\{E_g \mid g \in G\}$.

Let χ be a character of $\mathcal{A}$. Let us define the support of χ, denoted A_χ, as

$$A_\chi := \{g \in G \mid \chi(E_g) = 1\}.$$

Condition (iii) of Proposition 3.4 implies that $P^{-1}A_\chi \subset A_\chi$. Since $E_g = 1$ if $g \in P^{-1}$, it follows that $P^{-1} \subset A_\chi$.

Let $\mathcal{P}(G)$ be the power set of G. Identify $\mathcal{P}(G)$ with $\{0,1\}^G$, via the map $\mathcal{P}(G) \ni A \to 1_A \in \{0,1\}^G$, and endow it with the product topology. The group G acts on $\mathcal{P}(G)$. The right action is given, for $g \in G$ and $A \in \mathcal{P}(G)$, by $Ag := \{ag \mid a \in A\}$. Clearly the map $\Omega \ni \chi \to A_\chi \in \{0,1\}^G$ is continuous, injective and hence an embedding. We leave it to the reader to check that the above map is P-equivariant. From now, we view Ω as a subset of $\{0,1\}^G$.

Proposition 4.1. *We have the following.*

(i) *For $A \in \Omega$ and $a \in P$, $Aa^{-1} \in \Omega$ if and only if $a \in A$.*
(ii) *For $A \in \Omega$ and $g \in G$, $Ag \in \Omega$ if and only if $g^{-1} \in A$.*
(iii) *The action $\Omega \times P \to \Omega$ is open.*

Proof. Let $A \in \Omega$ and $a \in P$ be given. Suppose $B := Aa^{-1} \in \Omega$. Since $e \in B$, it follows that $a \in A$. Now suppose $a \in A$. Let χ be the character corresponding to A. Since $V_a^* E_{ga} V_a = E_g$, it follows that $V_a E_g V_a^* = E_a E_{ga}$. Thus the homomorphism $B(\mathcal{H}) \ni T \to V_a T V_a^* \in B(\mathcal{H})$ leaves $\mathcal{A}$ invariant.

Let $\tilde{\chi}$ be the character on $\mathcal{A}$ defined by $\tilde{\chi}(T) = \chi(V_a T V_a^*)$. Since $\chi(E_a) = 1$, it follows that $\tilde{\chi}$ is nonzero. Observe that for $T \in \mathcal{A}$,

$$\begin{aligned}(\tilde{\chi}a)(T) &= \tilde{\chi}(V_a^* T V_a)\\ &= \chi(V_a V_a^* T V_a V_a^*)\\ &= \chi(E_a)\chi(T)\chi(E_a)\\ &= 1_A(a)\chi(T)1_A(a)\\ &= \chi(T).\end{aligned}$$

Thus $\tilde{\chi}a = \chi$. Let B be the support of $\tilde{\chi}$. Then $A = Ba$. Thus $Aa^{-1} \in \Omega$. This proves (i).

Now let $A \in \Omega$ and $g = ab^{-1} \in G$. Suppose $g^{-1} = ba^{-1} \in A$. Then $b \in Aa \in \Omega$. By (i), it follows that $Ag = Aab^{-1} \in \Omega$. Now suppose $Ag \in \Omega$. Then $A = (Ag)g^{-1}$. Since $e \in Ag$, it follows that $g^{-1} \in A$. This proves (ii).

When G is discrete, $\mathrm{Int}(P) = P$ and $\Omega P = \Omega$. Thus, by [14, Thm. 4.3], to prove that the action $\Omega \times P \to \Omega$ is open, it is enough to show that Ωa is open in Ω for every $a \in P$. But note that by (i), for $a \in P$, $\Omega a = \{A \in \Omega \mid 1_A(a) = 1\}$ which is clearly open in Ω, as Ω has the subspace topology of $\{0,1\}^G$. ☐

A consequence of Proposition 4.1 is that the semi-direct product groupoid $\Omega \rtimes P$ has a Haar system. For $g \in G$, let $\delta_g \in C_c(\Omega \times P)$ be defined by $\delta_g(x,h) = 1$ if $h = g$ and $\delta_g(x,h) = 0$ if $h \neq g$. Then it is not difficult to show that there exists a representation $\pi : C_c(\Omega \times P) \to B(\mathcal{H})$ such that $\pi(\delta_g) = W_{g^{-1}}$ for every $g \in G$. We will prove this in the topological case.

Now let us turn our attention to the topological case. Let χ be a character of the commutative C^*-algebra $\mathcal{A}$. The *support* of χ, denoted A_χ, is defined as follows: For $g \in G$, $g \notin A_\chi$ if and only if there exists an open set U of G containing g such that $\chi(\int f(g)E_g\,dg) = 0$ for every $f \in C_c(U)$. Here $C_c(U) := \{f \in C_c(G) \mid \mathrm{supp}(f) \subset U\}$. Note that A_χ is closed.

Remark 4.2. Let χ be a character of $\mathcal{A}$ and let A be its support. Then for $g \in G$, $g \in A$ if and only if for every open set U containing g, there exists $f \in C_c(U)$ such that $f \geq 0$ and $\chi(\int f(g)E_g\,dg) > 0$.

Proposition 4.3. *Let χ be a character of $\mathcal{A}$ and let A be its support. We have the following.*

(i) $P^{-1} \subset A$ *and* $P^{-1}A \subset A$.
(ii) *The interior* $\mathrm{Int}(A)$ *is dense in* A.
(iii) *The boundary* $\partial(A)$ *has measure zero.*

Proof. Let $a \in \mathrm{Int}(P)$ and U an open set containing a^{-1}. Then $U \cap \mathrm{Int}(P)^{-1}$ is a nonempty open set containing a^{-1}. Choose $f \in C_c(G)$ such that $\mathrm{supp}(f) \subset U \cap \mathrm{Int}(P)^{-1}$, $f \geq 0$ and $\int f(g)\,dg = 1$. Since $E_g = 1$ for $g \in P^{-1}$, it follows that $\int f(g)E_g\,dg = \int f(g)\,dg = 1$. Thus $\chi(\int f(g)E_g\,dg) = 1$. This proves that

$a^{-1} \in A$. As a consequence, $\mathrm{Int}(P)^{-1} \subset A$. But $\mathrm{Int}(P)^{-1}$ is dense in P^{-1} and A is closed. Hence $P^{-1} \subset A$.

For $f \in C_c(G)$ and $g \in G$, let $L_g(f) \in C_c(G)$ be defined by $L_g(f)(x) = f(g^{-1}x)$.

Let $g \in A$ be given and $a \in P$. Let U be an open set containing $a^{-1}g$. Then aU is open and contains g. Thus there exists $f \in C_c(aU)$ such that $f \geq 0$ and $\chi(\int f(g)E_g\,dg) > 0$. Let $\tilde{f} = L_{a^{-1}}f$. Then $\tilde{f} \geq 0$ and $\mathrm{supp}(\tilde{f}) \subset U$. Now

$$\begin{aligned}\int \tilde{f}(g)E_g\,dg &= \int f(ag)E_g\,dg\\ &= \int f(g)E_{a^{-1}g}\,dg\\ &\geq \int f(g)E_g\,dg \quad \text{(by Proposition 3.4)}.\end{aligned}$$

Hence $\chi(\int \tilde{f}(g)E_g\,dg) > 0$. This implies that $a^{-1}g \in A$. Thus $P^{-1}A \subset A$. This proves (i). Statements (ii) and (iii) follow immediately from [14, Lem. 4.1]. ☐

Before proceeding further, let us review the Vietoris topology. Let X be a locally compact second countable Hausdorff space and let d be a metric on X inducing the topology. Let $\mathcal{C}(X)$ be the collection of closed subsets of X. Then $\mathcal{C}(X)$, endowed with the Vietoris topology, is compact and metrizable. We recall here the convergence of sequences of elements in $\mathcal{C}(X)$.

Let (A_n) be a sequence of closed subsets of X. Define

$$\begin{aligned}\liminf A_n &= \{x \in X \mid \limsup d(x, A_n) = 0\},\\ \limsup A_n &= \{x \in X \mid \liminf d(x, A_n) = 0\}.\end{aligned}$$

Then (A_n) converges in $\mathcal{C}(X)$ if and only if $\liminf A_n = \limsup A_n$. If $\liminf A_n = \limsup A_n = A$, then A_n converges to A. Observe that if $U \subset X$ is closed, then the subset $\{A \in \mathcal{C}(X) \mid A \cap U \neq \varnothing\}$ is open in $\mathcal{C}(X)$.

Consider $\mathcal{C}(G)$, the space of closed subsets of G, with the Vietoris topology. The group G acts on $\mathcal{C}(G)$ on the right. For $A \in \mathcal{C}(G)$ and $g \in G$, define $Ag = \{ag \mid a \in A\}$. Let

$$\Omega_u := \{A \in \mathcal{C}(G) \mid P^{-1} \subset A \text{ and } P^{-1}A \subset A\}.$$

We leave it to the reader to verify that Ω_u is a closed, and hence a compact, subset of $\mathcal{C}(G)$. Clearly Ω_u is P-invariant. The space Ω_u was first considered in [4].

Proposition 4.4. *The action $\Omega_u \times \mathrm{Int}(P) \to \Omega_u$ is open.*

Proof. Let $a \in \mathrm{Int}(P)$. It is enough to show that $\Omega_u \mathrm{Int}(P)a$ is open in Ω_u (see [14, Thm. 4.3]). We claim that

$$\Omega_u \mathrm{Int}(P)a = \{A \in \Omega_u \mid A \cap \mathrm{Int}(P)a \neq \varnothing\}$$

which will imply that $\Omega_u \mathrm{Int}(P)a$ is open.

Let $A \in \Omega_u \mathrm{Int}(P)a$. Then $A = Bba$ for some $B \in \Omega_u$ and $b \in \mathrm{Int}(P)$. Since $e \in B$, it follows that $ba \in A$. Hence $A \cap \mathrm{Int}(P)a$ is nonempty. Now suppose

$A \in \Omega_u$ and $A \cap \mathrm{Int}(P)a$ is nonempty. Choose $b \in \mathrm{Int}(P)$ such that $ba \in A$. Since $P^{-1}A \subset A$, it follows that $P^{-1}ba \subset A$, equivalently $P^{-1} \subset Aa^{-1}b^{-1}$, and $P^{-1}Aa^{-1}b^{-1} \subset Aa^{-1}b^{-1}$. This proves that $B = Aa^{-1}b^{-1} \in \Omega_u$. Then $A = Bba \in \Omega_u \mathrm{Int}(P)a$. □

We summarize a few facts regarding the space Ω_u in the following remark.

Remark 4.5. Note the following.

(i) $\Omega_u \mathrm{Int}(P)a = \{A \in \Omega_u \mid a \in \mathrm{Int}(A)\}$. If $A \cap \mathrm{Int}(P)a \neq \varnothing$, then $a \in \mathrm{Int}(P)^{-1}A$ which is open and contained in A. Thus $a \in \mathrm{Int}(A)$. Now suppose $a \in \mathrm{Int}(A)$; then $\mathrm{Int}(A) \cap Pa$ is nonempty. Since $\mathrm{Int}(P)a$ is dense in Pa, it follows that $\mathrm{Int}(A) \cap \mathrm{Int}(P)a$ is nonempty and hence $A \cap \mathrm{Int}(P)a$ is nonempty.

(ii) If $A \in \Omega_u$, then $\mathrm{Int}(A)$ is dense in A and the boundary $\partial(A)$ has measure zero. This follows from [14, Lem. 4.1].

(iii) Let $A \in \Omega_u$ and $g \in G$. Then $(A, g) \in \Omega_u \rtimes P$ if and only if $Ag \in \Omega_u$ if and only if $g^{-1} \in A$. We leave this verification to the reader.

(iv) The map $\Omega_u \ni A \to 1_A \in L^\infty(G)$ is continuous and injective and hence an embedding. Here $L^\infty(G)$ is given the weak* topology. Let $\mathcal{G}_u := \Omega_u \rtimes P$. Then Proposition 4.4 implies that $\mathcal{G}_u$ has a Haar system. Moreover, a Haar system on $\mathcal{G}_u$ is given by $(1_{Q_A}(g)\,dg)_{A\in\Omega_u}$. For $A \in \Omega_u$, observe that $Q_A := \{g \in G \mid (A, g) \in \mathcal{G}_u\}$ is A^{-1}.

By the definition of a Haar system, it follows that for $f \in C_c(\mathcal{G}_u)$, $\Omega_u \ni A \to \int f(x, g)1_A(g^{-1})\,dg$ is continuous. In particular, for $f \in C_c(G)$, the function $\Omega_u \ni A \to \int f(g)1_A(g^{-1})\,dg$ is continuous. As a consequence, the map $\Omega_u \ni A \to 1_A \in L^\infty(G)$ is continuous.

Suppose $A, B \in \Omega_u$ such that $1_A = 1_B$ in $L^\infty(G)$. Then $A \setminus B$ and $B \setminus A$ has measure zero. If $A \setminus B$ is nonemtpy then $\mathrm{Int}(A) \setminus B$ is nonempty since $\mathrm{Int}(A)$ is dense in A. But $\mathrm{Int}(A) \setminus B$ is open and hence cannot have measure zero. Thus $A \setminus B = \varnothing$. Similarly $B \setminus A = \varnothing$. Hence $A = B$. This proves that the map $\Omega_u \ni A \to 1_A \in L^\infty(G)$ is injective. Thus we can consider Ω_u as a compact subset of $L^\infty(G)$.

Let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Denote the commutative C^*-algebra generated by $\{\int f(g)E_g\,dg \mid f \in L^1(G)\}$ by $\mathcal{A}$ and let Ω be the spectrum of $\mathcal{A}$. For $f \in L^1(G)$, let $E_f := \int f(g)E_g\,dg$.

Proposition 4.6. *Let χ be a character of $\mathcal{A}$ and let A be its support. For $f \in C_c(G)$ the following equalities hold.*

(i) $\chi(\int f(g)1_{A^c}(g)E_g\,dg) = 0$.

(ii) *If* $\mathrm{supp}(f) \subset \mathrm{Int}(A)$*, then* $\chi(\int f(g)E_g\,dg) = \int f(g)\,dg$.

(iii) $\chi(\int f(g)E_g\,dg) = \int f(g)1_A(g)\,dg$.

Proof. First observe that if $\mathrm{supp}(f) \subset A^c$, where A^c denotes the complement of A, then $\chi(E_f) = 0$. This follows from the definition of A and by a partition of unity argument.

Now write $A^c = \bigcup_{n=1}^{\infty} K_n$ with K_n compact and K_n increasing. This is possible as A^c is open. Choose $\phi_n \in C_c(G)$ such that $0 \leq \phi_n \leq 1$, $\phi_n = 1$ on K_n and $\mathrm{supp}(\phi_n) \subset A^c$. Note that $\phi_n \to 1_{A^c}$ pointwise. Hence $f\phi_n$ converges to $f1_{A^c}$ in $L^1(G)$. This implies that $E_{f\phi_n}$ converges to $E_{f1_{A^c}}$. Since $\chi(E_{f\phi_n}) = 0$, it follows that $\chi(E_{f1_{A^c}}) = 0$. This proves (i).

Let $f \in C_c(G)$ be such that $\mathrm{supp}(f) \subset \mathrm{Int}(A)$. Let $g \in \mathrm{supp}(f)$. Then $\mathrm{Int}(A) \cap Pg$ is nonempty. Since $\mathrm{Int}(P)$ is dense in P, it follows that $\mathrm{Int}(A) \cap \mathrm{Int}(P)g$ is nonempty. Let $s \in \mathrm{Int}(P)$ be such that $sg \in \mathrm{Int}(A)$. Then $(sg)g^{-1} \in \mathrm{Int}(P)$. Since $\mathrm{Int}(P)$ is open, we can choose open sets U and V contained in $\mathrm{Int}(A)$, with compact closures, such that $(g, sg) \in U \times V \subset \mathrm{Int}(A) \times \mathrm{Int}(A)$ and $VU^{-1} \subset \mathrm{Int}(P)$. Then by Proposition 3.4, it follows that for $g_1 \in V$ and $g_2 \in U$, $E_{g_1}E_{g_2} = E_{g_1}$.

Since $\mathrm{supp}(f)$ is compact, it follows that there exist finitely many nonempty open sets $(U_i)_{i=1}^n$ and $(V_i)_{i=1}^n$ with compact closures, contained in $\mathrm{Int}(A)$, such that $\mathrm{supp}(f) \subset \bigcup_{i=1}^n U_i$ and $V_iU_i^{-1} \subset \mathrm{Int}(P)$. A partition of unity argument allows us to write f as $f = \sum_{i=1}^n f_i$ with $\mathrm{supp}(f_i) \subset U_i$. Thus to prove (ii), it is enough to show $\chi(E_{f_i}) = \int f_i(g)\,dg$.

Since V_i is a nonempty open set contained in A, by Remark 4.2, it follows that there exists $\phi_i \in C_c(G)$ such that $\mathrm{supp}(\phi_i) \subset V_i$ and $\chi(E_{\phi_i}) \neq 0$. Observe the following:

$$\begin{aligned}
E_{\phi_i}E_{f_i} &= \int_{V_i \times U_i} \phi_i(g_1)f_i(g_2)E_{g_1}E_{g_2}\,dg_1\,dg_2 \\
&= \int_{V_i \times U_i} \phi_i(g_1)f_i(g_2)E_{g_1}\,dg_1\,dg_2 \\
&= \Big(\int f_i(g_2)\,dg_2\Big)\int \phi_i(g_1)E_{g_1}\,dg_1 \\
&= \Big(\int f_i(g_2)\,dg_2\Big)E_{\phi_i}.
\end{aligned}$$

Since χ is multiplicative, it follows that

$$\chi(E_{\phi_i})\chi(E_{f_i}) = \Big(\int f_i(g)\,dg\Big)\chi(E_{\phi_i}).$$

Now $\chi(E_{\phi_i}) \neq 0$. Hence $\chi(E_{f_i}) = \int f_i(g)\,dg$. This proves (ii).

Now let $f \in C_c(G)$ be given. By (i), it follows that $\chi(E_f) = \chi(E_{f1_A})$. But since the boundary of A has measure zero, it follows that $1_{\mathrm{Int}(A)} = 1_A$ a.e. Thus $\chi(E_f) = \chi(E_{f1_{\mathrm{Int}(A)}})$. Write $\mathrm{Int}(A) = \bigcup_n K_n$ with K_n compact and K_n increasing. Choose $\phi_n \in C_c(G)$ such that $\phi_n = 1$ on K_n and $\mathrm{supp}(\phi_n) \subset \mathrm{Int}(A)$. Then $\phi_n \to 1_{\mathrm{Int}(A)}$ pointwise and hence $f\phi_n$ converges to $f1_{\mathrm{Int}(A)}$ in $L^1(G)$. Note that $\mathrm{supp}(f\phi_n) \subset \mathrm{Int}(A)$. Now calculate, as follows, to find that

$$\begin{aligned}
\chi(E_f) &= \chi(E_{f1_{\mathrm{Int}(A)}}) \\
&= \lim_n \chi(E_{f\phi_n})
\end{aligned}$$

$$
\begin{aligned}
&= \lim_n \int f(g)\phi_n(g)\,dg \quad \text{(by (ii))} \\
&= \int f(g)1_{\mathrm{Int}(A)}(g)\,dg \\
&= \int f(g)1_A(g)\,dg \quad (\text{since } 1_A = 1_{\mathrm{Int}(A)} \text{ in } L^\infty(G)).
\end{aligned}
$$

This proves (iii). □

Proposition 4.7. *For $\chi \in \Omega$, let A_χ be its support. Then the map $\Omega \ni \chi \to A_\chi \in \Omega_u$ is one-one, continuous and P-equivariant. Consequently, the action of P on Ω is injective.*

Proof. By Proposition 4.3, it follows that $A_\chi \in \Omega_u$ if $\chi \in \Omega$. For $f \in C_c(G)$, by Proposition 4.6, $\chi(E_f) = \int f(g)1_{A_\chi}(g)\,dg$. Hence the map $\Omega \ni \chi \to A_\chi \in L^\infty(G)$ is one-one and continuous where $L^\infty(G)$ is given the weak* topology. By part (iv) of Remark 4.5, it follows that $\Omega \ni \chi \to A_\chi \in \Omega_u$ is one-one and continuous.

Let $f \in C_c(G)$, $a \in P$ and $\chi \in \Omega$ and A be the support of χ. Observe that

$$
\begin{aligned}
(\chi.a)\Big(\int f(g)E_g\,dg\Big) &= \chi\Big(\int f(g)V_a^*E_gV_a\Big)\,dg \\
&= \chi\Big(\int f(g)E_{ga^{-1}}\,dg\Big) \\
&= \chi\Big(\int f(ga)\Delta(a)E_g\,dg\Big) \\
&= \int f(ga)\Delta(a)1_A(g)\,dg \\
&= \int f(g)1_A(ga^{-1})\,dg \\
&= \int f(g)1_{Aa}(g)\,dg.
\end{aligned}
$$

Hence the support of $\chi.a$ is Aa. Thus the map $\Omega \ni \chi \to A_\chi \in \Omega_u$ is a continuous P-equivariant embedding. □

Thus we can and will consider Ω as a subset of Ω_u with the subspace topology.

5. Haar system on $\Omega \rtimes P$

In this section, we show that the semi-direct product $\Omega \rtimes P$ admits a Haar system. We prove that the action $\Omega \times \mathrm{Int}(P) \to \Omega$ is open. To prove this, we need an analog of Proposition 4.1 in the topological setting.

Proposition 5.1. *Let $a \in \mathrm{Int}(P)$ and let $A \in \Omega$. Then $a \in A$ if and only if $Aa^{-1} \in \Omega$.*

Proof. Let $a \in \mathrm{Int}(P)$ and $A \in \Omega$ be given. Suppose $B := Aa^{-1} \in \Omega$. Since $e \in B$, it follows that $a \in A$. Now suppose $a \in A$. In addition, assume that $a \in \mathrm{Int}(A)$. Let χ be the character defining A. Then for $f \in C_c(G)$,

$$\chi\Big(\int f(g)E_g\,dg\Big) = \int f(g)1_A(g)\,dg.$$

Choose a decreasing sequence of open sets (U_n) in G such that

(i) the intersection $\bigcap_{n=1}^{\infty} U_n = \{a\}$,
(ii) if U is open in G and $a \in U$, then there exists N such that $U_n \subset U$ for $n \geq N$, and
(iii) for every n, we have $U_n \subset \mathrm{Int}(P) \cap \mathrm{Int}(A)$.

This is possible, for we can choose a metric and let (U_n) be the open balls containing a with $\mathrm{diam}(U_n) \to 0$. For every $n \in \mathbb{N}$, choose $f_n \in C_c(G)$ such that $f_n \geq 0$, $\int f_n(g)\,dg = 1$ and $\mathrm{supp}(f_n) \subset U_n$. Note that

$$\chi(E_{f_n}) = \int f_n(g)1_A(g)\,dg = 1$$

since $\mathrm{supp}(f_n) \subset \mathrm{Int}(A)$.

Let ϕ_n be the linear functional on the commutative C^*-algebra $\mathcal{A}$ defined by

$$\phi_n(T) = \chi\Big(\int_{b\in P} f_n(b)V_bTV_b^*\,db\Big).$$

Note that ϕ_n is well-defined by Lemma 3.7 and is clearly positive. Note that

$$\begin{aligned}\phi_n(1) &= \chi\Big(\int_{b\in\mathrm{supp}(f_n)} f_n(b)E_b\,db\Big)\\ &= \int_{b\in\mathrm{supp}(f_n)} f_n(b)1_A(b)\,db\\ &= \int_{b\in\mathrm{supp}(f_n)} f_n(b)\,db \quad (\text{since } \mathrm{supp}(f_n) \subset \mathrm{Int}(A))\\ &= 1.\end{aligned}$$

Thus ϕ_n is a state for every n. But the set of states on a unital C^*-algebra is weak* compact. By choosing a subsequence if necessary, we can assume without loss of generality that (ϕ_n) converges in the weak* topology and let ϕ be its limit.

Recall that for $a \in P$, $\alpha_a : \mathcal{A} \to \mathcal{A}$ is given by $\alpha_a(T) = V_a^*TV_a$. By Proposition 4.7, it follows that for every $a \in \mathcal{A}$, α_a is surjective.

Claim: $\phi \circ \alpha_a = \chi$.

It is enough to show that $(\phi \circ \alpha_a)(T) = \chi(T)$ for $T \in \widetilde{\mathcal{A}}$. Let $T = E_\psi$ for some $\phi \in C_c(G_m)$.

Observe that for $n \in \mathbb{N}$,

$$
\begin{aligned}
&\phi_n(\alpha_a(E_\psi)) \\
&\quad = \chi\Big(\int_{b\in P} f_n(b) V_b \alpha_a(E_\psi) V_b^*\, db\Big) \\
&\quad = \chi\Big(\int_{b\in P} f_n(b)\psi(g_1, g_2, \ldots, g_m) V_b V_a^* E_{g_1} \cdots E_{g_m} V_a V_b^*\, db\, dg_1 \cdots dg_m\Big) \\
&\quad = \chi\Big(\int_{b\in P} f_n(b)\psi(g_1, g_2, \ldots, g_m) E_b E_{g_1 a^{-1} b} \cdots E_{g_m a^{-1} b}\, db\, dg_1 \cdots dg_m\Big) \\
&\quad = \chi\Big(\int_{b\in P} \Delta(b^{-1}a)^m f_n(b)\psi(g_1 b^{-1}a, g_2 b^{-1}a, \ldots, g_m b^{-1}a) \\
&\qquad\qquad \times E_b E_{g_1} E_{g_2} \cdots E_{g_m}\, db\, dg_1\, dg_2 \cdots dg_m\Big).
\end{aligned}
$$

Let $\epsilon > 0$ be given. Since ψ is continuous and compactly supported, it follows that there exists an open set U such that $a \in U$ and for $b \in U$ and $g_1, g_2, \ldots, g_m \in G$,

$$
\begin{aligned}
\int \big|\Delta(b^{-1}a)^m \psi(g_1 b^{-1}a, g_2 b^{-1}a, \ldots, g_m b^{-1}a)& \\
- \psi(g_1, g_2, \ldots, g_m)\big|\, dg_1\, dg_2 \cdots dg_m \leq \epsilon&.
\end{aligned}
$$

Choose $N \geq 1$ such that for $n \geq N$, $U_n \subset U$ and thus $\operatorname{supp}(f_n) \subset U$. Note that for $n \geq N$,

$$
\begin{aligned}
|\phi_n(\alpha_a(E_\psi)) - \chi(E_\psi)| &= |\phi_n(\alpha_a(E_\psi)) - \chi(E_{f_n} E_\psi)| \\
&\leq \int_{b\in U_n} f_n(b)\Big(\int \big|\Delta(b^{-1}a)^m \psi(g_1 b^{-1}a, \ldots, g_m b^{-1}a) \\
&\qquad\qquad - \psi(g_1, g_2, \ldots, g_m)\big|\, dg_1 \cdots dg_m\Big)\, db \\
&\leq \epsilon \int_{b\in U_n} f_n(b) \\
&\leq \epsilon.
\end{aligned}
$$

Thus it follows that $\phi_n(\alpha_a(E_\psi)) \to \chi(E_\psi)$ and hence $\phi \circ \alpha_a = \chi$. This proves the claim.

Since α_a is surjective on $\mathcal{A}$, it follows that ϕ is a character of $\mathcal{A}$. Let $B \in \Omega$ be the support of ϕ. Then $\phi \circ \alpha_a = \chi$ translates to the equation $Ba = A$. Thus $Aa^{-1} \in \Omega$. Now suppose $a \in A$. Let (s_n) be a sequence in $\operatorname{Int}(P)$ converging to the identity element e. Then $s_n^{-1}a \in \operatorname{Int}(P)$ eventually, for $s_n^{-1}a \to a$ and $a \in \operatorname{Int}(P)$. But $\operatorname{Int}(P)^{-1}A \subset \operatorname{Int}(A)$. Hence $s_n^{-1}a \in \operatorname{Int}(A)$. By what we have proved, it follows that $Aa^{-1}s_n \in \Omega$ eventually. However Ω is a compact subset of Ω_u and $(Aa^{-1}s_n)$ converges to Aa^{-1}. From this we conclude that $Aa^{-1} \in \Omega$. □

Just like in the discrete case, we have the following theorem.

Proposition 5.2. *Let $A \in \Omega$ and $g \in G$. Then $Ag \in \Omega$ if and only if $g^{-1} \in A$. Also the semi-direct product groupoid $\Omega \rtimes P$ has a Haar system.*

Proof. Let $g \in G$ and $A \in \Omega$ be given. Suppose $Ag \in \Omega$. Since $e \in Ag$, it follows that $g^{-1} \in A$. Now suppose $g^{-1} \in A$. As $G = (\mathrm{Int}(P))(\mathrm{Int}(P))^{-1}$, write $g = ab^{-1}$ with $a, b \in \mathrm{Int}(P)$. Then $ba^{-1} \in A$ or $b \in Aa \in \Omega$. By Proposition 5.1, it follows that $Aab^{-1} \in \Omega$. Hence $Ag \in \Omega$.

To prove that $\Omega \rtimes P$ has a Haar system, it is enough to show that the action $\Omega \times P \to \Omega$ is open. By [14, Thm. 4.3], it is enough to show that $\Omega\,\mathrm{Int}(P)a$ is open in Ω for every $a \in P$. Let $a \in P$ be given.

Claim: $\Omega\,\mathrm{Int}(P)a = \{A \in \Omega \mid A \cap \mathrm{Int}(P)a \neq \varnothing\}$.

Suppose $A \cap \mathrm{Int}(P)a$ is nonempty. Then there exists $s \in \mathrm{Int}(P)$ such that $sa \in A$. By Proposition 5.1, $B := Aa^{-1}s^{-1} \in \Omega$. Thus $A = Bsa \in \Omega\,\mathrm{Int}(P)a$. Suppose $A \in \Omega\,\mathrm{Int}(P)a$. Then $A = Bsa$ for some $B \in \Omega$ and $s \in \mathrm{Int}(P)$. Since $e \in B$, it follows that $sa \in A$. Thus $A \cap \mathrm{Int}(P)a$ is nonempty. This proves the claim.

The set $\{A \in \mathcal{C}(G) \mid A \cap \mathrm{Int}(P)a \neq \varnothing\}$ is open in $\mathcal{C}(G)$, when $\mathcal{C}(G)$ is given the Vietoris topology. This implies that $\Omega\,\mathrm{Int}(P)a$ is open in Ω. □

Remark 5.3. Consider the groupoid $\Omega_u \rtimes P$. Then by Proposition 5.2 and statement (iii) of Remark 4.5, it follows that Ω is an invariant subset of Ω_u. Moreover, the groupoid $\Omega \rtimes P$ is just the restriction $\Omega_u \rtimes P|_\Omega$.

We end this section by describing Ω in the case of the Wiener–Hopf representation associated to a P-module A. Let A be a closed subset of G such that $e \in A$ and $A.P \subset A$. Recall that the Wiener–Hopf representation $V : P \to B(L^2(A))$ is given, for $a \in P$ and $\xi \in L^2(A)$, by

$$V_a(\xi)(x) := \begin{cases} \xi(xa^{-1})\Delta(a)^{\frac{-1}{2}} & \text{if } x \in Aa, \\ 0 & \text{if } x \notin Aa. \end{cases}$$

Here Δ denotes the modular function of the group G. Note that for $g \in G$ and $\xi \in L^2(A)$, W_g is given by

$$W_g(\xi)(x) := \begin{cases} \xi(xg^{-1})\Delta(g)^{\frac{-1}{2}} & \text{if } xg^{-1} \in A, \\ 0 & \text{if } xg^{-1} \notin A. \end{cases}$$

Let $M : L^\infty(A) \to B(L^2(A))$ be the multiplication representation. Observe that for $g \in G$, $E_g = W_g W_g^* = M(1_{A.g})$ where $A.g := \{xg \mid x \in A\} \cap A$. Denote the algebra of bounded continuous functions on A by $C_b(A)$. Since $\overline{\mathrm{Int}(A)} = A$, it follows that M is a faithful representation of $C_b(A)$. For

$f \in C_c(G)$, let $1_A * f \in C_b(A)$ be defined by

$$\begin{aligned} 1_A * f(t) &= \int 1_A(ts) f(s^{-1}) ds \\ &= \int 1_A(ts^{-1}) f(s) \Delta(s)^{-1} ds \\ &= \int 1_{A^{-1}t}(s) f(s) \Delta(s)^{-1} ds. \end{aligned}$$

Observe that given $a \in A$, there exists $f \in C_c(G)$ such that $(1_A * f)(a) = 1$. In order to see this, choose $f \in C_c(G)$ such that $\mathrm{supp}(f) \subset \mathrm{Int}(A)^{-1}a$ and $\int f(s)\Delta(s)^{-1} ds = 1$. For such an f, we have $(1_A * f)(a) = 1$.

Now let $f \in C_c(G)$ and $\xi \in L^2(A)$ be given. Calculate as below to find that

$$\begin{aligned} \Big\langle \Big(\int f(g) E_g \, dg \Big) \xi, \xi \Big\rangle &= \int f(g) \langle E_g \xi, \xi \rangle \, dg \\ &= \int f(g) \Big(\int_{x \in A} 1_{Ag}(x) |\xi(x)|^2 \, dx \Big) \, dg \\ &= \int_{x \in A} \Big(\int f(g) 1_{Ag}(x) \, dg \Big) |\xi(x)|^2 \, dx \\ &= \int_{x \in A} \Big(\int f(g) 1_A(xg^{-1}) \, dg \Big) |\xi(x)|^2 \, dx \\ &= \int_{x \in A} \Big(\int 1_A(xg) f(g^{-1}) \Delta(g)^{-1} \, dg \Big) |\xi(x)|^2 \, dx \\ &= \int_{x \in A} (1_A * \hat{f})(x) |\xi(x)|^2 \, dx \\ &= \langle M(1_A * \hat{f}) \xi, \xi \rangle, \end{aligned}$$

where $\hat{f}(g) = f(g)\Delta(g)$.

Thus the C^*-algebra $\mathcal{A}$ generated by $\{\int f(g) E_g \, dg \mid f \in C_c(G)\}$ is isomorphic to the C^*-subalgebra of $C_b(A)$ generated by $\{1_A * f \mid f \in C_c(G)\}$. Thus for $a \in A$, there exists a character χ_a of $\mathcal{A}$ such that

$$\begin{aligned} \chi_a \Big(\int f(g) E_g \, dg \Big) &= (1_P * \hat{f})(a) \\ &= \int 1_{A^{-1}a}(s) \hat{f}(s) \Delta(s)^{-1} ds \\ &= \int 1_{A^{-1}a}(s) f(s) ds. \end{aligned}$$

This implies that the support of χ_a is $A^{-1}a$. Also $\{\chi_a \mid a \in P\}$ separates the elements of $C_b(A)$ and hence those of $\mathcal{A}$. This implies that $\{A^{-1}a \mid a \in A\}$ is dense in Ω. As a consequence, it follows that Ω is the closure of $\{A^{-1}a \mid a \in A\}$ in the space of closed subsets of G with respect to the Vietoris topology.

Remark 5.4. For a P-module A, let us denote the closure $\{A^{-1}a \mid a \in A\}$ in the space of closed subsets by Ω_A. By Remark 4.5, observe that $B \in \Omega_A$ if and only if there exists a sequence $a_n \in A$ such that $1_{A^{-1}a_n} \to 1_B$ in $L^\infty(G)$ where $L^\infty(G)$ is endowed with the weak* topology. The "compactification" Ω_P of P is called the Wiener–Hopf compactification of P and is considered in [7, 14].

For explicit models of Ω_P, when P is a polyhedral cone of $\mathbb{R}^n$ or when P is the cone of positive elements in a Jordan algebra, we refer the reader to [7]. When P is the cone of positive matrices, Ω_P is described as a closed subset of unitary matrices in [16].

Remark 5.5. For a P-module A, the C^*-algebra $\mathcal{W}_A(P,G)$ is isomorphic to the reduced C^*-algebra of the groupoid $\Omega_A \rtimes P$. For $A = P$, this is proved in [7, Thm. 3.7]. Since the proof is exactly the same as that of [7, Thm. 3.7], we omit the proof.

It is beyond the scope of the present paper to give a detailed description of Ω_A for various P-modules A even when $P = [0,\infty) \times [0,\infty)$ and $G = \mathbb{R}^2$. We present a few simple examples below. Let $P = [0,\infty) \times [0,\infty)$ and $G = \mathbb{R}^2$ in the following examples.

Example 5.6. Let $A = [0,\infty) \times [0,\infty)$. For $(x,y) \in [0,\infty] \times [0,\infty]$, let

$$A_{(x,y)} := \{(a,b) \in \mathbb{R}^2 \mid a \leq x, b \leq y\}.$$

Then the map $[0,\infty] \times [0,\infty] \ni (x,y) \to A_{(x,y)} \in \Omega_A$ is a homeomorphism. Moreover, the groupoid $\Omega_A \rtimes P$ is isomorphic to the reduction of the transformation groupoid $(-\infty,\infty]^2 \rtimes \mathbb{R}^2$ onto the closed subset $[0,\infty] \times [0,\infty]$. Here $\mathbb{R}^2$ acts by translation with the usual convention that $\infty + a = \infty$ if $a \in \mathbb{R}$.

Example 5.7. Let $A = [0,\infty) \times \mathbb{R}$. Then $\Omega_A := \{A_{(x,\infty)} \mid x \in [0,\infty]\}$. The groupoid $\Omega_A \rtimes P$ is isomorphic to the reduction of the transformation groupoid $(-\infty,\infty]^2 \rtimes \mathbb{R}^2$ onto the closed subset $[0,\infty] \times \{\infty\}$. Clearly this groupoid is nonisomorphic to the one in Example 5.6. For the equivalence relation on the unit space partitions the unit space into two classes in this example whereas one obtains four equivalence classes for the groupoid in the previous example.

Example 5.8. Let $A = ([0,\infty) \times [0,\infty)) \cup ([1,\infty) \times [-1,\infty))$. For $(x,y) \in A$, let $A_{(x,y)} = -A + (x,y)$. For $y \in [-1,\infty)$, let $A_{(\infty,y)} = (-\infty,\infty) \times (-\infty, y+1]$. For $x \in [0,\infty)$, let $A_{(x,\infty)} = (-\infty,x] \times (-\infty,\infty)$ and let $A_{(\infty,\infty)} = \mathbb{R}^2$. We claim that

$$\begin{aligned}\Omega_A = \{A_{(x,y)} \mid (x,y) \in A\} &\cup \{A_{(x,\infty)} \mid x \in [0,\infty)\} \\ &\cup \{A_{(\infty,y)} \mid y \in [-1,\infty)\} \cup \{\mathbb{R}^2\}.\end{aligned}$$

Let $B \in \Omega_A$ and $(x_n,y_n) \in A$ such that $1_{-A+(x_n,y_n)} \to 1_B$ in $L^\infty(\mathbb{R}^2)$. This implies that for $f \in C_c(\mathbb{R})$ and $g \in C_c(\mathbb{R})$,

$$\int f(x)g(y)1_{-A+(x_n,y_n)}(x,y)\,dx\,dy \to \int f(x)g(y)1_B(x,y)\,dx\,dy$$

or in other words

$$(1) \qquad \int_{-\infty}^{x_n-1} f(x)\Big(\int_{-\infty}^{y_n+1} g(y)\,dy\Big)\,dx + \int_{x_n-1}^{x_n} f(x)\Big(\int_{-\infty}^{y_n} g(y)\,dy\Big)\,dx \to \int f(x)g(y)1_B(x,y)\,dx\,dy.$$

By passing to a subsequence, if necessary, we can assume that (x_n, y_n) is convergent in $(-\infty,\infty] \times (-\infty,\infty]$.

Case 1: Suppose $(x_n) \to x$ and $(y_n) \to y$ with $(x,y) \in A$. Then $B = -A + (x,y)$.

Case 2: Suppose $(x_n) \to \infty$ and $y_n \to y$ with $y \geq -1$. Then (1) implies that $B = A_{(\infty,y)}$.

Case 3: Suppose $(x_n) \to x$ and $y_n \to \infty$. Then again (1) implies that $B = (-\infty, x] \times \mathbb{R}$.

Case 4: Suppose $(x_n) \to \infty$ and $y_n \to \infty$. Then (1) implies that $B = \mathbb{R}^2$.

This proves the $\subset$ inclusion. We leave the inclusion $\supset$ to the reader. In fact, let $\overline{A}$ be the closure of A in $(-\infty,\infty] \times (-\infty,\infty]$. The map $\overline{A} \ni (x,y) \to A_{x,y} \in \Omega_A$ is a homeomorphism and the groupoid $\Omega_A \rtimes P$ is isomorphic to the reduction of $(-\infty,\infty]^2 \rtimes \mathbb{R}^2$ onto $\overline{A}$. It is however not clear to the author at present if this groupoid and the groupoid in Example 5.6 are isomorphic or not.

6. Covariant representations

In this section, let X be a compact Hausdorff space and assume that P acts on X on the right injectively. Let $X_0 := X\,\mathrm{Int}(P)$. We also assume that the semi-direct product $\mathcal{G} := X \rtimes P$ admits a Haar system. Let Y be a dilation of X, as explained in Section 1, on which the group G acts. For $y \in Y$, let $Q_y := \{g \in G \mid y.g \in X\}$. Recall that for $y \in Y$ and $g \in G$, $1_{Q_y}(g) = 1_X(yg)$ and $1_{\mathrm{Int}(Q_y)}(g) = 1_{X_0}(yg)$. Also note that for every $y \in Y$, the set Q_y is closed and $Q_yP \subset Q_y$. Thus by [14, Lem. 4.1], it follows that for every $y \in Y$, the boundary of Q_y has measure zero and $\overline{\mathrm{Int}(Q_y)} = Q_y$.

To state the next lemma, we need to fix some notations. Let $a \in P$ and let (U_n) be a decreasing sequence of open subsets of G such that $\bigcap_{n=1}^{\infty} U_n = \{a\}$, and if U is open and contains a, then $U_n \subset U$ eventually. Note that for every n, $U_n \cap Pa$ is nonempty. Hence $U_n \cap \mathrm{Int}(P)a$ is nonempty for every n. Choose $f_n \in C_c(G)$ such that $f_n \geq 0$, $\int f_n(g)\,dg = 1$ and $\mathrm{supp}(f_n) \subset U_n \cap \mathrm{Int}(P)a$. For $x \in X$, let

$$F_n(x) = \int f_n(g) 1_X(xg^{-1})\,dg.$$

Then $F_n \in C(X)$. The continuity of the function F_n follows from the fact that $(1_{Q_x}(g)\,dg)_{x \in X}$ is a Haar system on $X \rtimes P$. Observe that (F_n) is uniformly bounded.

Lemma 6.1. *The sequence F_n converges pointwise to $1_{X_0 a}$.*

Proof. Since for $x \in X$, $1_{Q_x} = 1_{\mathrm{Int}(Q_x)}$ a.e., it follows that F_n is given by the equation

$$F_n(x) = \int f_n(g) 1_{X_0}(xg^{-1})$$

for $x \in X$. Let $x \in X$. From $Q_x P \subset Q_x$, it is easily verifiable that $a^{-1} \in \mathrm{Int}(Q_x)$ if and only if $\mathrm{Int}(Q_x) \cap a^{-1}P^{-1}$ is nonempty. Suppose $a^{-1} \in \mathrm{Int}(Q_x)$, i.e., $xa^{-1} \in X_0$.

Let $U := \{g \in G \mid xg^{-1} \in X_0\}$. Then U is open and contains a. Thus there exists N such that $n \geq N$ implies $\mathrm{supp}(f_n) \subset U$ eventually. Then for $n \geq N$, we have $F_n(x) = \int f_n(g)\, dg = 1$. Now suppose $xa^{-1} \notin X_0$, i.e., $a^{-1} \notin \mathrm{Int}(Q_x)$. Then $\mathrm{Int}(Q_x)^{-1} \cap Pa$ is empty. Thus for $g \in Pa$, $xg^{-1} \notin X_0$. Since $\mathrm{supp}(f_n) \subset Pa$, it follows that $F_n(x) = 0$. This proves that (F_n) converges pointwise to $1_{X_0 a}$. □

Lemma 6.2. *There exists a sequence (s_n) in $\mathrm{Int}(P)$ such that $s_{n+1}^{-1} s_n \in \mathrm{Int}(P)$ and (s_n) converges to the identity element e.*

Proof. Let (U_n) be a countable base (of open sets) at e. We can assume that U_n is decreasing. Now $U_1 \cap P$ contains e and is nonempty. Since $\mathrm{Int}(P)$ is dense in P, it follows that $U_1 \cap \mathrm{Int}(P)$ is nonempty. Pick $s_1 \in U_1 \cap \mathrm{Int}(P)$. Now suppose that $s_1, s_2, \ldots, s_n$ are chosen such that $s_k \in \mathrm{Int}(P) \cap U_k$ for $1 \leq k \leq n$ and $s_{k+1}^{-1} s_k \in \mathrm{Int}(P)$ for $1 \leq k \leq n-1$. Since $e \in s_n(\mathrm{Int}(P))^{-1} \cap U_{n+1}$, it follows that $s_n \mathrm{Int}(P)^{-1} \cap U_{n+1} \cap P$ is nonempty. But $\overline{\mathrm{Int}(P)} = P$. Thus $s_n \mathrm{Int}(P)^{-1} \cap U_{n+1} \cap \mathrm{Int}(P)$ is nonempty; let s_{n+1} be one of its elements.

Then it is clear that the sequence (s_n) constructed as above converges to e and $s_{n+1}^{-1} s_n \in \mathrm{Int}(P)$ for every n. □

Consider a sequence (s_n) as in Lemma 6.2 converging to the identity e. Let $a \in \mathrm{Int}(P)$ and set $t_n := s_n^{-1} a$. Then observe that $t_{n+1} t_n^{-1} \in \mathrm{Int}(P)$. Since $a \in \mathrm{Int}(P)$, $\mathrm{Int}(P)$ is open and (t_n) converges to a, we can assume without loss of generality that $t_n \in \mathrm{Int}(P)$ for every n. With this notation, we have the following lemma.

Lemma 6.3. *The sequence $(1_{X_0 t_n})$ decreases pointwise to 1_{Xa}.*

Proof. Since $t_{n+1} t_n^{-1} \in \mathrm{Int}(P)$, it follows that $X_0 t_{n+1} \subset X_0 t_n$ for every n. Since $at_n^{-1} \in \mathrm{Int}(P)$, it follows that $Xa \subset X_0 t_n$ for every n. Thus $Xa \subset \bigcap_{n=1}^{\infty} X_0 t_n$. Now suppose $y \in X_0 t_n$ for every n. Then $yt_n^{-1} \subset X_0$ for every n. Note that $(yt_n^{-1}) \to ya^{-1}$. Since the closure of X_0 in Y is X, it follows that $ya^{-1} \in X$. Hence $y \in Xa$. This proves that $Xa = \bigcap_{n=1}^{\infty} X_0 t_n$. □

Let $B(X)$ be the space of bounded Borel measurable functions on X. For ϕ in $B(X)$ and $g \in G$, let $R_g(\phi)$ be defined by

$$R_g(\phi)(x) := \begin{cases} \phi(x.g) & \text{if } x.g \in X, \\ 0 & \text{if } x.g \notin X. \end{cases}$$

Then $R_g(\phi) \in B(X)$.

Definition 6.4. Let $\pi : C(X) \to B(\mathcal{H})$ be a unital $*$-representation and $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Denote the extension of π to $B(X)$, obtained via the Riesz representation theory, by π itself (see [1]). For $g = ab^{-1}$, let $W_g := V_b^* V_a$. The pair (π, V) is said to be a covariant representation of (X, P) if for $\phi \in B(X)$,

$$W_g \pi(\phi) W_g^* = \pi(R_{g^{-1}}(\phi)).$$

Remark 6.5. Since $G = (\mathrm{Int}(P))(\mathrm{Int}(P))^{-1}$, it follows that (π, V) is a covariant representation if and only if $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ and $V_a \pi(\phi) V_a^* = \pi(R_{a^{-1}}(\phi))$ for $\phi \in B(X)$ and $a \in \mathrm{Int}(P)$. We leave this verification to the reader.

We fix a few notations that will be useful for the rest of this section.

Notations. Let Y be the dilation of X, as explained in Section 1, on which G acts. Then $\mathcal{G} := X \rtimes P$ is a closed subset of $X \times G$ and also of $Y \rtimes G$. For $\phi \in C_c(Y)$, we let $\widehat{\phi} \in C(X)$ be the restriction. Define $\pi_Y : C_c(Y) \to B(\mathcal{H})$ by $\pi_Y(\phi) = \pi(\widehat{\phi})$. For $\phi \in C_c(Y)$ and $g \in G$, let $R_g(\phi) \in C_c(Y)$ be given by $R_g(\phi)(y) = \phi(y.g)$ for $y \in Y$.

For $\psi \in C_c(Y \rtimes G)$ and $g \in G$, let $\psi_g \in C_c(Y)$ be defined by $\psi_g(y) = \psi(y, g)$. For $\chi \in C_c(\mathcal{G})$ and $g \in G$, let $\chi_g \in B(X)$ be defined by $\chi_g(x) = \chi(x, g)$ if $(x, g) \in \mathcal{G}$ and $\chi_g(x) = 0$ if $(x, g) \notin \mathcal{G}$.

Let $\pi : C(X) \to B(\mathcal{H})$ be a unital $*$-representation. For $\xi \in \mathcal{H}$, let $d\mu_{\xi,\xi}$ be the probability measure on X such that

$$\int \phi(x)\, d\mu_{\xi,\xi}(x) = \langle \pi(\phi)\xi, \xi \rangle \quad \text{for } \phi \in C(X).$$

The same equality holds for $\phi \in B(X)$.

Proposition 6.6. *Let $\pi : C(X) \to B(\mathcal{H})$ be a unital $*$-representation and let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Then the following are equivalent.*

(i) *The pair (π, V) is a covariant representation.*
(ii) *For $a \in \mathrm{Int}(P)$ and $\phi \in C(X)$, $V_a^* \pi(\phi) V_a = \pi(R_a(\phi))$ and $\pi(1_{X_0 a}) = E_a$.*

Proof. For $a \in P$, let $\sigma_a : X \to X$ be the map sending x to xa.

Suppose (π, V) is a covariant representation. Then the covariance relation implies that for $g \in G$, $E_g := W_g W_g^* = \pi(1_{Xg \cap X})$.

Let $f \in C_c(G)$. Set $F(x) = \int f(g)1_X(xg^{-1})\,dg$. Then $F \in C(X)$ and

$$\begin{aligned}\langle \pi(F)\xi, \xi\rangle &= \int F(x)\,d\mu_{\xi,\xi}(x)\\ &= \int f(g)1_{Xg}(x)\,dg\,d\mu_{\xi,\xi}(x)\\ &= \int f(g)\Big(\int 1_{Xg}(x)\,d\mu_{\xi,\xi}(x)\Big)\,dg\\ &= \int f(g)\langle \pi(1_{Xg\cap X})\xi,\xi\rangle\,dg\\ &= \int f(g)\langle E_g\xi,\xi\rangle\,dg.\end{aligned}$$

Thus $\pi(F) = \int f(g)E_g$.

Let $a \in \mathrm{Int}(P)$ be given. Choose a sequence (f_n) as in Lemma 6.1 and let $F_n(x) := \int f_n(g)1_{Xg}(x)$ for $x \in X$. Note that F_n is uniformly bounded. By Lemma 6.1, it follows that F_n converges pointwise to 1_{X_0a}. On the other hand, we have $\pi(F_n) = \int f_n(g)E_g\,dg$. Since $g \to E_g$ is strongly continuous, it is easily verifiable that $\int f_n(g)E_g\,dg$ converges strongly to E_a. Hence $\pi(1_{X_0a}) = E_a$. Clearly by definition $V_a^*\pi(\phi)V_a = \pi(R_a(\phi))$. This proves that (i) implies (ii).

Now assume condition (ii). The equality $V_a^*\pi(\phi)V_a = \pi(R_a(\phi))$ for $a \in P$ and $\phi \in C(X)$ translates to the fact that for $a \in P$ and $\xi \in \mathcal{H}$, the push-forward measure $(\sigma_a)_*(\mu_{\xi,\xi})$ equals $\mu_{V_a\xi,V_a\xi}$. Hence $V_a^*\pi(\phi)V_a = \pi(R_a(\phi))$ for $a \in P$ and $\phi \in B(X)$. Now by Remark 6.5, it is enough to show that $V_a\pi(\phi)V_a^* = \pi(R_{a^{-1}}(\phi))$ for $a \in \mathrm{Int}(P)$ and $\phi \in B(X)$. Now let $a \in \mathrm{Int}(P)$ and $\phi \in B(X)$ be given. Then by assumption (ii), we have $V_a^*\pi(R_{a^{-1}}\phi)V_a = \pi(\phi)$. Hence $V_a\pi(\phi)V_a^* = E_a\pi(R_{a^{-1}}(\phi))E_a$. By the strong continuity of $g \to E_g$, by assumption (ii) and Lemma 6.3, it follows that $\pi(1_{Xa}) = E_a$. Hence

$$V_a\pi(\phi)V_a^* = \pi(1_{Xa}R_{a^{-1}}(\phi)) = \pi(R_{a^{-1}}(\phi)). \qquad \square$$

Theorem 6.7. *Let X be a compact Hausdorff space on which P acts injectively. Let $\mathcal{G} := X \rtimes P$. Assume that $\mathcal{G}$ has a Haar system. For $\phi \in C(X)$ and $f \in C_c(G)$, let $\phi \otimes f \in C_c(\mathcal{G})$ be defined by $(\phi \otimes f)(x,g) = \phi(x)f(g)$. We denote $1 \otimes f$ by $\tilde{f}$.*

Let (π, V) be a covariant representation of (X,P) on a Hilbert space $\mathcal{H}$. Then there exists a representation $\lambda : C^*(\mathcal{G}) \to B(\mathcal{H})$ satisfying the following.

(i) *For $f \in C_c(G)$, $\lambda(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}}f(g)W_{g^{-1}}\,dg$. Here Δ is the modular function of the group G.*
(ii) *For $\phi \in C(X)$ and $f \in C_c(G)$, $\lambda(\phi \otimes f) = \pi(\phi)\lambda(\tilde{f})$.*

Proof. Let $\phi \in C_c(\mathcal{G})$. We claim that $G \ni g \to \pi(\phi_g)W_{g^{-1}} \in B(\mathcal{H})$ is strongly continuous. Let $\Phi \in C_c(Y \rtimes G)$ be an extension of ϕ. Since (π, V) is covariant, it follows that $E_g = \pi(1_{Xg\cap X})$. Now observe that $\pi(\widehat{\Phi_g})W_{g^{-1}} = \pi(\phi_g)W_{g^{-1}}$. For we have

$$\pi(\widehat{\Phi_g})W_{g^{-1}} = \pi(\widehat{\Phi_g})E_{g^{-1}}W_{g^{-1}} = \pi(\widehat{\Phi_g}1_{Xg^{-1}})W_{g^{-1}} = \pi(\phi_g)W_{g^{-1}}.$$

But $g \to \widehat{\Phi_g} \in C(X)$ is continuous. Hence the map $G \ni g \to \pi(\widehat{\Phi_g}) \in B(\mathcal{H})$ is strongly continuous and consequently $G \ni g \to \pi(\phi_g)W_{g^{-1}} \in B(\mathcal{H})$ is strongly continuous.

For $\phi \in C_c(\mathcal{G})$, let $\lambda(\phi) \in B(\mathcal{H})$ be

$$\lambda(\phi) := \int \Delta(g)^{-\frac{1}{2}} \pi(\phi_g) W_{g^{-1}}\, dg.$$

Also we have shown that if $\Phi \in C_c(Y \rtimes G)$ is an extension of $\phi \in C_c(\mathcal{G})$, then

$$\lambda(\phi) = \int \Delta(g)^{-\frac{1}{2}} \pi_Y(\Phi_g) W_{g^{-1}}\, dg.$$

For $\phi \in C_c(\mathcal{G})$, calculate as follows to find that

$$\begin{aligned}
\lambda(\phi)^* &= \int W_g \pi(\overline{\phi_g}) \Delta(g)^{-\frac{1}{2}}\, dg \\
&= \int W_g \pi(\overline{\phi_g}) W_g^* W_g \Delta(g)^{-\frac{1}{2}}\, dg \\
&= \int \pi(R_{g^{-1}}(\overline{\phi_g})) W_g \Delta(g)^{-\frac{1}{2}}\, dg \\
&= \int \pi(R_g(\overline{\phi_{g^{-1}}})) W_{g^{-1}} \Delta(g)^{\frac{1}{2}} \Delta(g)^{-1}\, dg \\
&= \int \pi((\phi^*)_g) W_{g^{-1}} \Delta(g)^{-\frac{1}{2}}\, dg \\
&= \lambda(\phi^*).
\end{aligned}$$

Thus λ preserves the adjoint.

Now let $\phi, \psi \in C_c(\mathcal{G})$ be given and let $\Phi, \Psi \in C_c(Y \rtimes G)$ be extensions of ϕ and ψ, respectively. Consider the function on $Y \rtimes G$ defined by the equation

$$\Phi \circ \Psi(y, g) = \int \Phi(y, h)\Psi(y.h, h^{-1}g) 1_X(y.h)\, dh.$$

A simple application of the dominated convergence theorem together with the fact that $1_X(y.h) = 1_{X_0}(y.h)$ for almost all y implies that $\Phi \circ \Psi$ is continuous. Clearly $\Phi \circ \Psi$ is compactly supported and is an extension of $\phi * \psi$.

Let $g \in G$ and $\xi \in \mathcal{H}$ be given. Then

$$\begin{aligned}
\langle \pi_Y((\Phi \circ \Psi)_g)\xi, \xi\rangle &= \int \Phi \circ \Psi(x, g)\, d\mu_{\xi,\xi}(x) \\
&= \int \Big(\int \Phi(x, h)\Psi(x.h, h^{-1}g) 1_X(xh)\, dh \Big)\, d\mu_{\xi,\xi}(x) \\
&= \int \Big(\int \Phi(x, h)\Psi(x.h, h^{-1}g) 1_{Xh^{-1}}(x)\, d\mu_{\xi,\xi}(x) \Big)\, dh \\
&= \int \langle \pi_Y(\Phi_h)\pi_Y(R_h(\Psi_{h^{-1}g}))\pi_Y(1_{Xh^{-1}})\xi, \xi\rangle\, dh \\
&= \int \langle \pi_Y(\Phi_h)\pi_Y(R_h(\Psi_{h^{-1}g}))E_{h^{-1}}\xi, \xi\rangle\, dh.
\end{aligned}$$

Thus for $g \in G$,

$$\pi_Y((\Phi \circ \Psi)_g) = \int \pi_Y(\Phi_h)\pi_Y(R_h(\Psi_{h^{-1}g}))E_{h^{-1}}\,dh.$$

We calculate

$$\begin{aligned}
\lambda(\phi)\lambda(\psi) &= \int \Delta(gh)^{-\frac{1}{2}}\pi_Y(\Phi_g)W_{g^{-1}}\pi_Y(\Psi_h)W_{h^{-1}}\,dg\,dh \\
&= \int \Delta(gh)^{-\frac{1}{2}}\pi_Y(\Phi_g)W_g^*\pi_Y(\Psi_h)W_gW_{g^{-1}}W_{h^{-1}}\,dg\,dh \\
&= \int \Delta(gh)^{-\frac{1}{2}}\pi_Y(\Phi_g)\pi_Y(R_g(\Psi_h))W_{g^{-1}}W_{h^{-1}}\,dg\,dh \\
&= \int \Delta(gh)^{-\frac{1}{2}}\pi_Y(\Phi_g)\pi_Y(R_g(\Psi_h))E_{g^{-1}}W_{h^{-1}g^{-1}}\,dg\,dh \\
&\qquad\qquad \text{(by Proposition 3.4)} \\
&= \int \Big(\int \Delta(k)^{-\frac{1}{2}}\pi_Y(\Phi_g)\pi_Y(R_g(\Psi_{g^{-1}k}))E_{g^{-1}}W_{k^{-1}}\,dk\Big)\,dg \\
&= \int \Big(\int \pi_Y(\Phi_g)\pi_Y(R_g(\Psi_{g^{-1}k})E_{g^{-1}}\,dg\Big)\Delta(k)^{-\frac{1}{2}}W_{k^{-1}}\,dk \\
&= \int \Delta(k)^{-\frac{1}{2}}\pi_Y((\Phi\circ\Psi)_k)W_{k^{-1}} \\
&= \lambda(\phi * \psi).
\end{aligned}$$

Hence λ preserves the multiplication.

For $\phi \in B(X)$, one has $\|\pi(\phi)\| \leq \|\phi\|_\infty$ where $\|\cdot\|_\infty$ is the sup norm on $B(X)$. Let K be a compact subset of G. Then for $\phi \in C_c(\mathcal{G})$ with $\mathrm{supp}(\phi) \subset X \times K$, observe that

$$\begin{aligned}
\|\lambda(\phi)\| &\leq \int \Delta(g)^{-\frac{1}{2}}\|\pi(\phi_g)\|\,dg \\
&\leq \int_{g\in K} \Delta(g)^{-\frac{1}{2}}\|\phi_g\|\,dg \\
&\leq \big(\sup_{g\in K}\Delta(g)^{-\frac{1}{2}}\big)\|\phi\|_\infty \int 1_K(g)\,dg.
\end{aligned}$$

Thus it is clear that the map $\lambda : C_c(\mathcal{G}) \to B(\mathcal{H})$ is continuous when $C_c(\mathcal{G})$ is given the inductive limit topology and $B(\mathcal{H})$ is given the norm topology. By Renault's disintegration theorem, one obtains a bonafide representation $\lambda : C^*(\mathcal{G}) \to B(\mathcal{H})$ extending $\lambda : C_c(\mathcal{G}) \to B(\mathcal{H})$. Conditions (i) and (ii) follow just from definitions. □

7. The main theorem

Let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Let $\mathcal{A}$ and Ω be as in Sections 3–5. Denote the open set $\Omega\,\mathrm{Int}(P)$ by Ω_0. Let $\pi : C(\Omega) \to B(\mathcal{H})$ be the representation induced by the inclusion

$\mathcal{A} \subset B(\mathcal{H})$. Denote the extension to $B(\Omega)$ by π itself. First let us show that (π, V) is a covariant representation.

Lemma 7.1. *The pair (π, V) is a covariant representation.*

Proof. By definition, it follows that for $a \in P$, $V_a^*\pi(\phi)V_a = \pi(R_a(\phi))$ for $\phi \in C(X)$. Now fix $a \in P$. Choose a sequence $(f_n) \in C_c(G)$ as in Lemma 6.1. Set $F_n := \int f_n(g)E_g\, dg \in C(\Omega)$. Then by the strong continuity of $g \to E_g$, it is clear that F_n converges strongly to E_a. Now by definition, for $A \in \Omega$, we have $F_n(A) = \int f_n(g)1_A(g)\, dg$. By Proposition 5.2, it follows that $F_n(A) = \int f_n(g)1_\Omega(Ag^{-1})\, dg$ for $A \in \Omega$. By Lemma 6.1, it follows that F_n converges pointwise to $1_{\Omega_0 a}$. Hence $\pi(1_{\Omega_0 a}) = E_a$. The proof follows now from Proposition 6.6. □

Proposition 7.2. *Let $\mathcal{H}$ be a Hilbert space and let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Let $\Omega \subset \Omega_u$ be the spectrum of the commutative C^*-algebra $\{\int f(g)E_g\, dg \mid f \in C_c(G)\}$ described as in Sections 3–5. Then there exists a $*$-homomorphism $\lambda : C^*(\Omega \rtimes P) \to B(\mathcal{H})$ such that for $f \in C_c(G)$,*

$$\lambda(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g) W_{g^{-1}}\, dg.$$

Moreover, the range of λ is generated by $\{\int f(g)W_g\, dg \mid f \in C_c(G)\}$.

Proof. For $f \in C_c(G)$, let $\widehat{f} \in C(\Omega)$ be defined by

$$\widehat{f}(A) := \int f(g)1_\Omega(Ag)\, dg = \int f(g)1_A(g^{-1}).$$

By Remark 4.5 (iv) and the fact that $\Omega \subset \Omega_u$, it follows that $\{\widehat{f} \mid f \in C_c(G)\}$ separates points of Ω. Thus by Proposition 2.1, the $*$-algebra generated by $\{\tilde{f} \mid f \in C_c(G)\}$ is dense in $C^*(\Omega \rtimes P)$. Now the proof follows directly from Lemma 7.1 and Proposition 6.7. □

Remark 7.3. The unit space Ω of the groupoid $\Omega \rtimes P$ in Theorem 7.2 depends on the chosen isometric representation V. See the examples considered at the end of Section 5.

Theorem 7.4. *Let $\mathcal{H}$ be a Hilbert space and let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Let*

$$\Omega_u := \{A \in \mathcal{C}(G) \mid P^{-1} \subset A \text{ and } P^{-1}A \subset A\}$$

with the Vietoris topology. Consider the right action of P on Ω_u by right multiplication. Then there exists a $*$-homomorphism $\lambda : C^*(\Omega_u \rtimes P) \to B(\mathcal{H})$ such that for $f \in C_c(G)$,

$$\lambda(\tilde{f}) = \int \Delta(g)^{-\frac{1}{2}} f(g) W_{g^{-1}}\, dg.$$

Moreover, the range of λ is generated by $\{\int f(g)W_g\, dg \mid f \in C_c(G)\}$.

Proof. By Remark 5.3, it follows that $\Omega \rtimes P$ is isomorphic to the restriction $\Omega_u \rtimes P|_\Omega$ and Ω is an invariant subset of Ω_u. Consider the natural map $\mathrm{res}: C_c(\Omega_u \rtimes P) \to C_c(\Omega \rtimes P)$ which on $C_c(\Omega_u \rtimes P)$ is simply the restriction. Let $\tilde{\lambda} : C^*(\Omega \rtimes P) \to B(\mathcal{H})$ be the representation as in Proposition 7.2. Now one completes the proof by setting $\lambda := \tilde{\lambda} \circ \mathrm{res}$. □

Remark 7.5. Proposition 7.4 says that the C^*-algebra of the groupoid $\Omega_u \rtimes P$ can be interpreted as the "universal" C^*-algebra which encodes the isometric representations with commuting range projections. However, the space Ω_u is quite large to describe explicitly even for the simple example of the quarter plane $[0,\infty) \times [0,\infty) \subset \mathbb{R}^2$.

The following two results are a part of folklore in operator algebras and we indicate how our results can be applied to derive them.

Example 7.6. Let $P := \mathbb{N}$ and $G := \mathbb{Z}$ with the discrete topology. Consider the one-point compactification $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$. The semigroup $\mathbb{N}$ acts on $\mathbb{N}_\infty$ by translation with the convention that $\infty + n = \infty$ for $n \in \mathbb{N}$. It is easy to verify that the map $\mathbb{N}_\infty \ni n \to (-\infty, n] \in \Omega_u$ is an $\mathbb{N}$-equivariant homeomorphism. Here $(-\infty, \infty]$ is just $\mathbb{N}$. The groupoid $\mathbb{N}_\infty \rtimes \mathbb{N}$ is amenable and $C^*_{\mathrm{red}}(\mathbb{N}_\infty \rtimes \mathbb{N})$ is just the Toeplitz-algebra. Now Theorem 7.4 is just the well-known Coburn's theorem.

Example 7.7. Let $\mathbb{R}_+ = [0,\infty)$. Let $P := \mathbb{R}_+$ and $G := \mathbb{R}$ with the usual Euclidean topology and with addition as the group operation. Consider the one-point compactification $[0,\infty] := [0,\infty) \cup \{\infty\}$. The semigroup $[0,\infty)$ acts on $\mathbb{R} \cup \{\infty\}$ by translation with the convention that $\infty + x = \infty$ for $x \in [0,\infty)$. It is easily verifiable that the map $[0,\infty] \ni x \to (-\infty, x] \in \Omega_u$ is an $\mathbb{R}_+$-equivariant homeomorphism. The groupoid $[0,\infty] \rtimes \mathbb{R}_+$ is amenable and $C^*_{\mathrm{red}}([0,\infty] \rtimes [0,\infty))$ is the usual Wiener–Hopf algebra $\mathcal{W}([0,\infty), \mathbb{R})$; see [7].

Observe that if $V : \mathbb{R}_+ \to B(\mathcal{H})$ is an isometric representation, then the range projections $\{E_t := V_t V_t^* \mid t \geq 0\}$ commute. For if $t = r + s$, then $E_t E_r = V_r V_s V_s^* V_r^* V_r V_r^* = E_t$. Hence if $t > r$, then $E_t E_r = E_t$. Now the claim follows from the fact that $\mathbb{R}_+$ is totally ordered. Thus if $V : \mathbb{R}_+ \to B(\mathcal{H})$ is an isometric representation, then there exists a representation $\pi : \mathcal{W}([0,\infty), \mathbb{R}) \to B(\mathcal{H})$ such that

$$\pi(\tilde{f}) = \int_0^\infty f(t) V_t^* \, dt + \int_{-\infty}^0 f(t) V_{-t} \, dt$$

for $f \in C_c(\mathbb{R})$.

As an application of our results, we end this article by deriving a version of Coburn's theorem for the semigroup $[0,\infty)^n$ of $\mathbb{R}^n$. Let $n \geq 1$ be an integer. Let $G := \mathbb{R}^n$ and $P = [0,\infty)^n$ for the rest of the paper. For $t := (t_1, t_2, \ldots, t_n), s := (s_1, s_2, \ldots, s_n) \in \mathbb{R}^n$, let

$$t \vee s := (t_1 \vee s_1, t_2 \vee s_2, \ldots, t_n \vee s_n),$$

where $x \vee y$ denotes the maximum of real numbers x, y. For $i = 1, 2, \dots, n$, let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the ith coordinate projection. For $i = 1, 2, \dots, n$, let $\epsilon_i \in \mathbb{R}^n$ be the vector with 1 at the ith coordinate and zero elsewhere. For $t, s \in \mathbb{R}^n$, we write $t \leq s$ if $s - t \in P$.

For $x = (x_1, x_2, \dots, x_n) \in [0, \infty]^n$, let

$$A_x := \{y = (y_1, \dots, y_n) \in \mathbb{R}^n \mid y_i \leq x_i\}.$$

Let Ω_P be the closure of $\{-P + a \mid a \in P\}$ in the space of closed subsets of $\mathbb{R}^n$. We leave it to the reader to verify that $[0, \infty]^n \ni x \to A_x \in \Omega_P$ is a P-equivariant homeomorphism. Let $\mathcal{G} := [0, \infty]^n \rtimes P$ be the semi-direct product groupoid. The groupoid $\mathcal{G}$ is isomorphic to the reduction of the transformation groupoid $(-\infty, \infty]^n \rtimes \mathbb{R}^n$ onto the closed subset $[0, \infty]^n$. Here $\mathbb{R}^n$ acts on $(-\infty, \infty]^n$ by translation with the usual convention that $\infty + x = \infty$ for a real number. For $f \in C_c(\mathbb{R}^n)$, let $\tilde{f} \in C_c(\mathcal{G})$ be defined by $\tilde{f}(x, t) = f(t)$.

Theorem 7.8. *Let $V : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Let $E_t := V_t V_t^*$ for $t \in P$. Suppose that for $t, s \in P$, $E_t E_s = E_{t \vee s}$. Then there exists a unique $*$-homomorphism $\pi : C^*(\mathcal{G}) \to B(\mathcal{H})$ such that*

$$\pi(\tilde{f}) = \int_{\mathbb{R}^n} f(t) W_{-t} \, dt$$

for every $f \in C_c(\mathbb{R})$. Here $(W_t)_{t \in \mathbb{R}^n}$ stands for the partial isometries defined as in Section 3 corresponding to the representation $V : P \to B(\mathcal{H})$.

Proof. Let $\mathcal{A}$ be the commutative C^*-algebra generated by $\{\int f(t) E_t \, dt \mid f \in C_c(\mathbb{R}^n)\}$ and let $\Omega \subset \Omega_u$ be the spectrum of $\mathcal{A}$ where $E_t = W_t W_t^*$. First observe that $E_x E_y = E_{x \vee y \vee 0}$ for $x, y \in \mathbb{R}^n$. To see this, let $x, y \in \mathbb{R}^n$ and write $x = t_1 - s_1$ and $y = t_2 - s_2$ with $t_1, t_2, s_1, s_2 \in P$. We can assume that $s_1 = s_2$ and let $s := s_1$. We observe

$$\begin{aligned} E_x E_y &= V_s^* E_{t_1} V_s V_s^* E_{t_2} V_s \\ &= V_s^* E_{t_1 \vee s \vee t_2} V_s \\ &= E_{t_1 \vee t_2 \vee s - s} \\ &= E_{x \vee y \vee 0}. \end{aligned}$$

First we claim that $\Omega \subset \Omega_P$. Let $A \in \Omega$ be given and denote the character on $\mathcal{A}$ corresponding to A by χ_A. Then for $f \in C_c(\mathbb{R}^n)$,

$$\chi_A\Big(\int f(t) E_t \, dt\Big) = \int f(t) 1_A(t) \, dt.$$

For $i = 1, 2, \dots, n$, let $\alpha_i := \sup \pi_i(A)$. We use here the convention that if $\pi_i(A)$ is not bounded above, then $\sup \pi_i(A) = \infty$. Note that $\alpha_i \geq 0$ for $(0, 0, \dots, 0) \in A$.

We claim that $A = A_\alpha$ where $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n) \in [0, \infty]^n$. Clearly $A \subset A_\alpha$ by the choice of α. First assume that $\alpha_i > 0$ for every $i = 1, 2, \dots, n$. We show that the open "rectangle" $\prod_{i=1}^n (0, \alpha_i)$ is a subset of A. Let $x \in \prod_{i=1}^n (0, \alpha_i)$ be given and write $x := (x_1, x_2, \dots, x_n)$.

Claim: There exist $y_1, y_2, \dots, y_n \in \mathrm{Int}(A)$ (we write $y_i := (y_{i1}, y_{i2}, \dots, y_{in})$) such that $y_{ii} > x_i$ and $\pi_i(y_i) > \pi_i(y_j)$ for $j \neq i$.

Since $\pi_i(A) = \alpha_i$, it follows that for every i, there exists a sequence $(x_i^k) \in A$ such that $\pi_i(x_i^k) > 0$ and increases strictly to α_i. For $i = 1, 2, \dots, n$, write x_i^k as $x_i^k = (x_{i1}^k, x_{i2}^k, \dots, x_{in}^k)$. Let $z_i^k = x_i^k - \sum_{j \neq i} \max\{x_{ij}^k, 0\}\epsilon_j$. Since $A - P \subset A$, it follows that $z_i^k \in A$ for every i and k. Set $y_i^k := z_i^k - (\frac{1}{k}, \frac{1}{k}, \dots, \frac{1}{k})$. Since $A - \mathrm{Int}(P) \subset \mathrm{Int}(A)$, it follows that $y_i^k \in \mathrm{Int}(A)$ for every i and k. Observe that $y_{ji}^k < y_{ii}^k$ if $i \neq j$ and $y_{ii}^k \to \alpha_i$. Thus for k large, $y_{ii}^k > x_i$. This proves the claim.

Let $y_1, y_2, \dots, y_n \in \mathrm{Int}(A)$, with $y_i = (y_{i1}, y_{i2}, \dots, y_{in})$, be such that $y_{ii} > y_{ji}$ for $j \neq i$ and $y_{ii} > x_i$. To show that $x \in A$, it is enough to show that $y := y_1 \vee y_2 \vee \cdots \vee y_n = (y_{11}, y_{22}, \dots, y_{nn}) \in A$. For observe that $y \geq x$ and $A - P \subset A$.

Now we claim that $y \in A$. To obtain a contradiction, suppose that $y \notin A$. Then there exist open subsets U_i of $\mathbb{R}^n$ such that $y_i \in U_i$ and if $t_i \in U_i$ then $t_1 \vee t_2 \vee \cdots \vee t_n \notin A$. We can choose, and will choose, the open sets U_i such that $U_i \subset \mathrm{Int}(A)$, $\pi_i(t) > 0$ for $t \in U_i$ and $\pi_i(s) < \pi_i(t)$ for $(s,t) \in U_j \times U_i$ whenever $j \neq i$. Note then that if $t_i \in U_i$, then $t_1 \vee t_2 \vee \cdots \vee t_n = (\pi_1(t_1), \pi_2(t_2), \dots, \pi_n(t_n))$.

Let $f_i \in C_c(\mathbb{R}^n)$ be such that $\mathrm{supp}(f_i) \subset U_i$, $f_i \geq 0$ and $\int f_i(t)\, dt = 1$. Denote the operator $\int f_i(t) E_t\, dt \in \mathcal{A}$ by E_i. Since $U_i \subset \mathrm{Int}(A)$, it follows that $\chi_A(E_i) = \int f_i(t) 1_A(t)\, dt = 1$. Let $E := E_1 E_2 \cdots E_n$. Observe that

$$E := \int f_1(t_1) f_2(t_2) \cdots f_n(t_n) E_{(\pi_1(t_1), \pi_2(t_2), \dots, \pi_n(t_n))}\, dt_1\, dt_2 \cdots dt_n$$

or E can be written as

$$E := \int G(x_1, x_2, \dots, x_n) E_{(x_1, x_2, \dots, x_n)}\, dx_1\, dx_2 \cdots dx_n,$$

where G is given by

$$G(x_1, x_2, \dots, x_n) = \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \cdots \times \mathbb{R}^{n-1}} f_1^{x_1}(s_1) f_2^{x_2}(s_2) \cdots f_n^{x_n}(s_n) ds_1 ds_2 \cdots ds_n.$$

Here $f_i^{x_i}$ is given by

$$f_i^{x_i}(u_1, u_2, \dots, u_{n-1}) = f_i(u_1, u_2, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{n-1}).$$

Note that since $\mathrm{supp}(f_i) \subset U_i$, it follows that G vanishes on A. Hence $\chi_A(E) = 0$. But this is a contradiction to the fact that $\chi_A(E_i) = 1$ and $E = E_1 E_2 \cdots E_n$. This proves the claim that $y \in A$.

Consequently, for the open rectangle it follows that $\prod_{i=1}^n (0, \alpha_i) \subset A$. Since A is closed, it follows that $\prod_{i=1}^n [0, \alpha_i] \subset A$. Now the fact that $A - P \subset A$ implies that $A_\alpha = \prod_{i=1}^n [0, \alpha_i] - P \subset A - P \subset A$. Hence $A_\alpha = A$ when $\alpha_i > 0$ for every i.

Now let $A \in \Omega$ and let $\alpha_i = \sup \pi_i(A)$. Then $A' := A + (1,1,\ldots,1) \in \Omega$. Note that $\sup \pi_i(A') = \alpha_i + 1$. Thus by what we have proved, it follows that

$$A + (1,1,\ldots,1) = A_{(\alpha_1+1,\alpha_2+1,\ldots,\alpha_n+1)} = A_\alpha + (1,1,\ldots,1).$$

Hence $A = A_\alpha$. This completes the proof that $\Omega \subset \Omega_P$.

The groupoid $\Omega \rtimes P := \{(A,g) \mid A \in \Omega,\, g^{-1} \in A\}$ is the restriction of the groupoid $\Omega_P \rtimes P$ onto the closed invariant subset Ω. Let res : $C^*(\Omega_P \rtimes P) \to C^*(\Omega \rtimes P)$ be the "restriction" map which on $C_c(\Omega_P \rtimes P)$ is simply the restriction. Let $\lambda : C^*(\Omega \rtimes P) \to B(\mathcal{H})$ be the $*$-homomorphism as in Proposition 7.2. Then the composition $\lambda \circ \mathrm{res}$ is the desired map. □

Remark 7.9. Let $G := \mathbb{R}^n$ and $P := [0,\infty)^n$.

(i) When $V : P \to B(L^2(P))$ is the Wiener–Hopf representation, the $*$-homomorphism of Theorem 7.8 is in fact an isomorphism. This follows from [7, Thm. 3.7] and the fact that the groupoid $[0,\infty]^n \rtimes [0,\infty)^n$ is amenable.

(ii) Now Theorem 7.8 can be restated as follows: Let $V : P \to B(L^2(P))$ be the Wiener–Hopf representation and let $\widetilde{V} : P \to B(\mathcal{H})$ be an isometric representation such that $\widetilde{V}_t\widetilde{V}_t^*\widetilde{V}_s\widetilde{V}_s^* = \widetilde{V}_{t\vee s}\widetilde{V}_{t\vee s}^*$ for every $t,s \in P$. Then there exists a unique $*$-homomorphism $\pi : \mathcal{W}(P,G) \to \mathcal{W}_V(P,G)$ such that for $f \in L^1(P)$,

$$\pi\Big(\int f(t)V_t\,dt\Big) = \int f(t)\widetilde{V}_t\,dt.$$

Here $\mathcal{W}(P,G)$ denotes the Wiener–Hopf C^*-algebra and $\mathcal{W}_V(P,G)$ denotes the C^*-algebra generated by $\{\int f(t)\widetilde{V}_t\,dt \mid f \in L^1(P)\}$.

We end with the following question: Is it possible to formulate an analog of Coburn's theorem for an arbitrary topological Ore semigroup? More precisely, let $V : P \to B(L^2(P))$ be the Wiener–Hopf representation and let $\widetilde{V} : P \to B(\mathcal{H})$ be an isometric representation with commuting range projections. Denote the C^*-algebra generated by $\{\int f(a)V_a\,da \mid f \in L^1(P)\}$ by $\mathcal{W}_V(P,G)$. What conditions on $\widetilde{V}$ force the existence of a $*$-homomorphism $\pi : \mathcal{W}(P,G) \to \mathcal{W}_V(P,G)$ such that $\pi(\int f(a)V_a\,da) = \int f(a)\widetilde{V}_a\,da$ for $f \in L^1(P)$?

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