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Classifying spaces for proper actions of mapping class groups

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Abstract. We describe a construction of cocompact models for the classifying spaces $\underline{E}\Gamma^s_{g,r}$, where $\Gamma^s_{g,r}$ stands for the mapping class group of an oriented surface of genus g with r boundary components and s punctures. Our construction uses a cocompact model for $\underline{E}\Gamma^0_{g,0}$ as an input, a case which has been dealt with in [3]. We then proceed by induction on r and s.

1. Introduction

The mapping class group Γ_g of a closed, connected and oriented surface S_g of genus g is defined as the group of connected components of the group of orientation preserving homeomorphisms of S_g. This group has been the object of many recent studies. Of particular interest are its finite subgroups; these are for $g > 1$ precisely the finite groups which occur as groups of symmetries of the surface S_g equipped with a complex structure (a Riemann surface). The interplay of algebra, topology and analysis in the study of Γ_g make it one of the most fascinating groups. For a discrete group Γ we denote by $\underline{E}\Gamma$ its *classifying space for proper actions*; $\underline{E}\Gamma$ is a proper Γ-CW-complex characterized by the property that for every finite subgroup $F < \Gamma$, the fixed point subcomplex $\underline{E}\Gamma^F$ is contractible. The goal of this note is to provide a uniform construction of cocompact models for the classifying spaces for proper actions for all the mapping class groups $\Gamma^s_{g,r}$ of oriented surfaces of genus g, with r boundary components and s punctures. The proof uses an induction on r and s and relies on [3] where the case of $\Gamma_g := \Gamma^0_{g,0}$ is presented. For the case of $\Gamma^s_{g,0}$ with $s \geq 0$, cocompact models have also been constructed in a recent note by Ji and Wolpert [13].

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2. The Definition of the Mapping Class Group

2.1. The undecorated case Let S_g denote a closed, connected and oriented topological surface of genus g. It is well-known that S_g admits a unique smooth structure; we shall also write S_g for the corresponding smooth oriented manifold. There are four basic ways of viewing the mapping class group Γ_g of the surface S_g, one being purely topological, the second more geometric, the third homotopical and the fourth algebraic, involving the fundamental group of the surface only. The definitions we have in mind have the following form:

(I) $$\Gamma_g = \mathrm{Homeo}_+(S_g)/\,\mathrm{Homeo}^0(S_g)$$

(II) $$\Gamma_g = \mathrm{Diffeo}_+(S_g)/\,\mathrm{Diffeo}^0(S_g)$$

(III) $$\Gamma_g = \mathrm{Hoequ}_+(S_g)$$

(IV) $$\Gamma_g = \mathrm{Out}_+(\pi_1(S_g, s_0))$$

We shall first give some background information and comments concerning these equivalent definitions. Let $s_0 \in S_g$ denote a base point. The fundamental group $\pi_1(S_g, s_0)$ of S_g has a presentation

$$\langle a_1, b_1, \ldots, a_g, b_g \mid \prod [a_i, b_i] \rangle =: \Pi_g$$

and thus $\pi_1(S_g, s_0)_{ab} \cong H_1(S_g; \mathbb{Z}) \cong \mathbb{Z}^{2g}$. Since S_g is assumed to be oriented one has $H_2(S_g; \mathbb{Z}) \cong \mathbb{Z}$. A homotopy equivalence $f : S_g \to S_g$ is said to be orientation preserving, if the induced map

$$H_2(f) : H_2(S_g; \mathbb{Z}) \to H_2(S_g; \mathbb{Z})$$

is the identity map. It is useful to notice that in case $g > 0$ this is equivalent to the requirement that the induced isomorphism

$$H_1(f) : H_1(S_g; \mathbb{Z}) \to H_1(S_g; \mathbb{Z})$$

preserves the intersection pairing. We shall denote by $\mathrm{Hoequ}(S_g)$ the group of homotopy classes of homotopy equivalences of S_g and by $\mathrm{Hoequ}_+(S_g)$ its subgroup consisting of those classes, which are orientation preserving.

Let $\mathrm{Homeo}(S_g)$ denote the topological group of homeomorphisms of S_g, with the compact-open topology. We shall write $\mathrm{Homeo}_+(S_g)$ for the subgroup of orientation preserving homeomorphisms, and $\mathrm{Homeo}^0(S_g)$ for the connected component of the identity. The mapping class group Γ_g of the surface S_g is then defined as the discrete group of connected components

(I) $$\Gamma_g = \mathrm{Homeo}_+(S_g)/\,\mathrm{Homeo}^0(S_g) = \pi_0(\mathrm{Homeo}_+(S_g)).$$

We will consider (I) as our basic definition for Γ_g and want to compare it with (II), (III) and (IV). Consider now S_g as a smooth oriented manifold. In accordance to the notation used above, we write $\mathrm{Diffeo}_+(S_g)$ for the group of orientation preserving diffeomorphisms of S_g with the C^∞-topology, and $\mathrm{Diffeo}^0(S_g)$ for the connected component of the identity. It was proved by Dehn [4] that $\mathrm{Homeo}_+(S_g)/\,\mathrm{Homeo}^0(S_g)$ is generated by "*Dehn twists*", which are diffeomorphisms obtained by splitting S_g along a simple closed smooth curve, rotating one part by 2π, and gluing the surface back together. It follows that the natural map

$$\mathrm{Diffeo}_+(S_g) \to \mathrm{Homeo}_+(S_g)/\,\mathrm{Homeo}^0(S_g)$$

is surjective. The kernel is precisely $\mathrm{Diffeo}^0(S_g)$; namely, if $f : S_g \to S_g$ is a diffeomorphism isotopic to the identity (i.e. $f \in \mathrm{Homeo}^0(S_g)$), then f is a fortiori homotopic to the identity, and therefore, according to Earle and Eells [5], the map f can be connected by a path in $\mathrm{Diffeo}_+(S_g)$ to the identity map. We have thus established that

$$\text{(II)} \qquad \Gamma_g = \mathrm{Diffeo}_+(S_g)/\,\mathrm{Diffeo}^0(S_g) = \pi_0(\mathrm{Diffeo}_+(S_g)).$$

In case $g = 0$, that is $S_0 = S^2$ the 2-sphere, $\mathrm{Diffeo}_+(S^2)$ is connected; the inclusion of $SO(3)$ in $\mathrm{Diffeo}_+(S^2)$ is actually a homotopy equivalence by Smale's result [20]. Thus $\Gamma_0 = \{e\}$. For $g > 0$ however, the mapping class groups Γ_g turn out to be nontrivial. The group Γ_1 can be most easily understood using the definitions (III) and (IV) respectively, which we shall discuss now.

By a result due to Nielsen [19], the natural map $\mathrm{Homeo}_+(S_g) \to \mathrm{Hoequ}_+(S_g)$ is surjective, and Baer proved [1] that any homeomorphism which is homotopic to the identity, is actually isotopic to the identity, showing that the kernel is precisely $\mathrm{Homeo}^0(S_g)$, (compare also Mangler [17]). Therefore, we conclude

$$\text{(III)} \qquad \Gamma_g = \mathrm{Hoequ}_+(S_g).$$

Because for $g > 0$ the surface S_g is an Eilenberg-Mac Lane space $K(\Pi_g, 1)$, where $\Pi_g = \pi_1(S_g, s_0)$, the pointed homotopy set $[(S_g, s_0), (S_g, s_0)]_\bullet$ maps via the induced map on the fundamental group bijectively onto $\mathrm{Hom}(\Pi_g, \Pi_g)$. The set of free homotopy classes $[S_g, S_g]$ may be identified with the set of orbits of the usual Π_g-action on the pointed homotopy set $[(S_g, s_0), (S_g, s_0)]_\bullet$ and this action is well-known to correspond to the conjugation action on the fundamental group Π_g. Passing to orbit spaces with respect to the Π_g-action, we obtain therefore a natural bijection $[S_g, S_g] \cong \mathrm{Rep}(\Pi_g, \Pi_g)$, where $\mathrm{Rep}(\Pi_g, \Pi_g)$ stands for the set of conjugacy class of homomorphisms $\Pi_g \to \Pi_g$. Homotopy equivalences correspond under this identification to automorphisms modulo inner automorphisms of Π_g. If we denote by $\mathrm{Out}(\Pi_g)$ the group of outer automorphisms of Π_g, we can view this group as a subset of $\mathrm{Rep}(\Pi_g, \Pi_g)$, and the map defined above yields a bijective homomorphism $\mathrm{Hoequ}(S_g) \to \mathrm{Out}(\Pi_g) \subset \mathrm{Rep}(\Pi_g, \Pi_g)$, where as earlier, $\mathrm{Hoequ}(S_g)$ stands for the group of homotopy classes of homotopy equivalences of S_g. If we write Out_+ for the "orientation preserving" outer automorphisms, that is, the subgroup of $\mathrm{Out}(\Pi_g)$ consisting of those elements which when acting on $H_1(S_g; \mathbb{Z})$ preserve the intersection pairing, we infer that $\mathrm{Hoequ}_+(S_g) \cong \mathrm{Out}_+(\Pi_g)$. Note that the formula is also correct in case $g = 0$. From (III) we conclude then that

$$\text{(IV)} \qquad \Gamma_g = \mathrm{Out}_+(\pi_1(S_g, s_0)).$$

Thus $\Gamma_1 = \mathrm{Out}_+(\mathbb{Z} \oplus \mathbb{Z}) \cong \mathrm{Sp}(2, \mathbb{Z}) = \mathrm{SL}(2, \mathbb{Z})$ and $\Gamma_0 = \{e\}$. Since the action of Γ_g on $H_1(S_g; \mathbb{Z})$ preserves the symplectic intersection pairing, one can define a natural map $\Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z})$, which is known to be surjective, with torsion-free kernel (the Torelli group). Therefore, by choosing a torsion-free subgroup of finite index in $\mathrm{Sp}(2g, \mathbb{Z})$, the pre-image in Γ_g is a torsion-free subgroup of finite index: Γ_g is virtually torsion-free; as a matter of fact, Γ_g is a virtual duality group, cp. [9].

2.2. Mapping Class Groups with decorations If $S^s_{g,r}$ denotes an oriented surface of genus g with r boundary components and s punctures, the associated mapping class group $\Gamma^s_{g,r}$ is $\pi_0(\mathrm{Diffeo}_+(S^s_{g,r}; \mathrm{rel}))$, where the diffeomorphisms are supposed to be the identity on the boundary of $S^s_{g,r}$ and fixing the punctures (we treat here the punctures as distinguished points on the surface). Choosing a base point in $S^s_{g,r}$ different from the distinguished points, one has an evaluation map $\mathrm{Diffeo}_+(S^s_{g,r}; \mathrm{rel}) \to S^s_{g,r}$, which is a fibration with fiber $\mathrm{Diffeo}_+(S^{s+1}_{g,r}; \mathrm{rel})$. The associated long exact homotopy sequence has the form

$$\cdots \to \pi_1(\mathrm{Diffeo}^0(S^s_{g,r}; \mathrm{rel})) \xrightarrow{\alpha} \pi_1(S^s_{g,r}) \to \Gamma^{s+1}_{g,r} \to \Gamma^s_{g,r} \to 1.$$

The image of α is known to be a central subgroup. For $2g + r + s > 2$, the fundamental group $\pi_1(S^s_{g,r})$ is a nonabelian surface group or a nonabelian free group and is therefore centerless. As a result, we have for $2g + r + s > 2$ the following *Birman short exact sequence*

$$\text{(B)} \qquad 1 \to \pi_1(S^s_{g,r}) \to \Gamma^{s+1}_{g,r} \to \Gamma^s_{g,r} \to 1.$$

Moreover, by replacing a boundary component by a punctured disc, on has for $s > 0$ and $2g + 2r + s > 2$ a central extension (*Dehn twist sequences*)

$$\text{(D)} \qquad 1 \to \mathbb{Z} \to \Gamma^{s-1}_{g,r+1} \to \Gamma^s_{g,r} \to 1$$

with central subgroup $\mathbb{Z}$ generated by a Dehn twist. (Relevant information on these exact sequences can be found in [2, 6, 8, 10, 18]). The group $\Gamma^s_{0,1}$ can be identified with the pure braid group on s strands; for $r > 0$, the groups $\Gamma^s_{g,r}$ are torsion-free.

We will see that once one has constructed a cocompact $\underline{E}\Gamma_g$, one can use these exact sequences to construct a cocompact model for the more general groups $\Gamma^s_{g,r}$. For the groups $\Gamma^s_{g,0}$ with $s > 0$ and $g > 1$, there is a construction of a spine of dimension $4g-4+s$ in Teichmüller space, admitting a proper and cocompact action of $\Gamma^s_{g,0}$, cp. Harer [9]. Harer also shows that for $2g+s+r > 2$, the groups $\Gamma^s_{g,r}$ are virtual duality groups and he shows that for $g > 1$ the *vcd* of Γ_g is $4g-5$, and for $g > 0$ and $r+s > 0$, the *vcd* of $\Gamma^s_{g,r}$ equals $4g+2r+s-4$ [9, Thm. 4.1]. This shows that Harer's spine is of optimal dimension, but Harer does not prove that the spine is a $\underline{E}\Gamma^s_{g,0}$; what is missing is a proof that finite subgroups of $\Gamma^s_{g,0}$ have contractible fixed point sets.

Remark 2.3. In a recent preprint, Ji and Wolpert [13] gave a proof that, in our notation, $\Gamma^s_{g,0}$ admits for all $s \geq 0$ a cocompact classifying space for proper actions. Their classifying space is defined intrinsically, in terms of the geometry of Riemann surfaces, whereas ours, obtained via Theorem 3.5, depends on choosing an equivariant triangulation of Teichmüller space.

3. The construction of a cocompact $\underline{E}\Gamma_g$

A G-subspace Y of a G-space X is called a strong G-deformation retract, if there is a homotopy $H : X \times I \to X$ such that for $(g, x, t) \in G \times X \times I$, $H(gx, t) = gH(x, t)$, and for $(y, t) \in Y \times I$, $H(y, t) = y$, and $H(x, 0) = x$, $H(x, 1) \in Y$. Thus, if $X = \underline{E}G$ and $Y \subset X$ is a strong G-deformation retract, then Y is G-homotopy equivalent to X and therefore Y is a model for $\underline{E}G$ too.

We first show that Teichmüller space $\mathcal{T}_g$ is a model for $\underline{E}\Gamma_g$, and then we use a theorem due to Broughton [3] to get a Γ_g-subspace of $\mathcal{T}_g$ which is a strong Γ_g-deformation retract and which is a cocompact Γ_g-space.

The following are some basic facts on Teichmüller spaces. Let $S_g^{\mathbb{C}}$ denote a Riemann surface of genus $g \geq 2$. The universal cover of $S_g^{\mathbb{C}}$ can be identified with the upper half plane U, with holomorphic covering projection $U \to S_g^{\mathbb{C}}$. The group $\mathrm{Aut}(U)$ of holomorphic automorphisms of U can be identified with $\mathrm{PSL}(2, \mathbb{R})$, acting by linear fractional transformations on U. A discrete subgroup Π of $\mathrm{PSL}(2, \mathbb{R})$ with compact quotient U/Π is called a *cocompact* Fuchsian group. It has a presentation of the form

$$\Pi = \langle \alpha_1, \beta_1, \cdots, \alpha_g, \beta_g; \gamma_1, \cdots, \gamma_t \mid \textstyle\prod_{i=1}^{g} [\alpha_i, \beta_i] \prod_{j=1}^{t} \gamma_j; \gamma_1^{n_1}; \cdots; \gamma_t^{n_t} \rangle$$

with $n_1, \ldots, n_t > 0$; $\sigma(\Pi) := (g; n_1, \ldots, n_t)$ is called the *signature* of Π. In case that $t = 0$, we write Π_g for Π so that Π_g is isomorphic to $\pi_1(S_g)$. One then considers the space of representations

$$R(\Pi) = \{\rho : \Pi \overset{mono}{\longrightarrow} \mathrm{PSL}(2, \mathbb{R}) \mid \rho(\Pi) \text{ discrete in } \mathrm{PSL}(2, \mathbb{R})\},$$

which we equip with the subspace topology of $\mathrm{PSL}(2, \mathbb{R})^{2g+t}$ (for the discussion of such representation spaces in a more general context, see Weil [21]). The group $\mathrm{PSL}(2, \mathbb{R})$ acts by conjugation on $R(\Pi)$ and the orbit space $R(\Pi)/\mathrm{PSL}(2, \mathbb{R})$ is known to have two homeomorphic components (*extended Teichmüller space*). Pick one and call it *Teichmüller space* $\mathcal{T}(\Pi)$; in case $\Pi = \Pi_g$ is the fundamental group of S_g, we write $\mathcal{T}_g$ for $\mathcal{T}(\Pi_g)$. Note that a point $x \in \mathcal{T}_g$ can be represented by $\rho_x \in R(\Pi_g)$, where $\rho_x : \Pi_g \to \mathrm{PSL}(2, \mathbb{R})$ is unique up to conjugation by an element of $\mathrm{PSL}(2, \mathbb{R})$. Thus $x \in \mathcal{T}_g$ corresponds to a Riemann surface $S_g^{\mathbb{C}}(x)$ of the form $U/\rho_x(\Pi_g)$, and the Riemann surface corresponding to x is unique up to conformal equivalence. Conversely, given a Riemann surface $S_g^{\mathbb{C}}$, $g \geq 2$, by passing to its universal cover U, one obtains an injective homomorphism $\rho : \Pi_g \to \mathrm{PSL}(2, \mathbb{R})$, which is unique up to an orientation preserving inner automorphism of Π_g, thus defining a unique $\mathrm{Out}_+(\Pi_g) = \Gamma_g$ orbit in $\mathcal{T}_g$. The orbit space $\mathcal{T}_g/\Gamma_g = \mathcal{M}_g$ is called the *moduli space* of Riemann surfaces of genus g; it follows that its points are in bijective correspondence with holomorphic isomorphism classes of Riemann surfaces of genus g.

We now return to the more general Teichmüller space $\mathcal{T}(\Pi)$ of a cocompact Fuchsian group Π with presentation as above. According to Greenberg [7] the space $\mathcal{T}(\Pi)$ is a real-analytic manifold and the following holds.

Proposition 3.1. *Let* Π *be a cocompact Fuchsian group, with signature* $\sigma(\Pi) = (r; n_1, \ldots, n_t)$. *Then the analytic manifold* $\mathcal{T}(\Pi)$ *is diffeomorphic to* $\mathbb{R}^{6(r-1)+2t}$. *If* Λ *is another cocompact Fuchsian group with signature* $\sigma(\Lambda) = (s; m_1, \ldots, m_u)$ *and* $\iota : \Pi \to \Lambda$ *is injective, then the induced map* $\iota^* : \mathcal{T}(\Lambda) \to \mathcal{T}(\Pi)$ *is a real-analytic diffeomorphism onto its image* $I \subset \mathcal{T}(\Pi)$, *and* I *is a closed subset diffeomorphic to* $\mathbb{R}^{6(s-1)+2u}$.

The group of outer automorphism $\mathrm{Out}(\Pi)$ of Π acts on $R(\Pi)/\operatorname{PSL}(2,\mathbb{R})$ in an obvious way: $\gamma[\rho] = [\rho \circ \tilde{\gamma}^{-1}]$ for $\tilde{\gamma} \in \mathrm{Aut}(\Pi)$ representing γ and $\rho \in R(\Pi)$ representing $[\rho]$. Assume now that $S_g^{\mathbb{C}}$ is a Riemann surface of genus $g > 1$. The *Uniformization Theorem* asserts that $S_g^{\mathbb{C}}$ is the quotient of U by a discrete group of isometries $\rho(\Pi_g) < \operatorname{PSL}(2,\mathbb{R})$, with ρ corresponding to a point $[\rho] \in \mathcal{T}_g$. The group $\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}})$ of holomorphic automorphisms of $S_g^{\mathbb{C}}$ gives rise to a group of lifts to $\operatorname{PSL}(2,\mathbb{R})$, which by covering space theory is equal to the normalizer $N_{\operatorname{PSL}(2,\mathbb{R})}(\rho(\Pi_g))$ of $\rho(\Pi_g)$ in $\operatorname{PSL}(2,\mathbb{R})$, so that

$$\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}}) \cong N_{\operatorname{PSL}(2,\mathbb{R})}(\rho(\Pi_g))/\rho(\Pi_g).$$

The natural composite map $\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}}) \to \mathrm{Diffeo}_+(S_g) \to \Gamma_g$ is injective, because $\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}})$ is known to act faithfully on the space of holomorphic differentials of $S_g^{\mathbb{C}}$ (the details for this argument are easy, but not relevant for what follows). Note that $\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}})$ is classically known to be a finite group, of order bounded by $84(g-1)$, the Hurwitz bound. The action of $\mathrm{Out}(\Pi_g)$ on the extended Teichmüller space restricts to an action of $\Gamma_g = \mathrm{Out}_+(\Pi_g)$ on $\mathcal{T}_g$. We can smoothly triangulate $\mathcal{T}_g$ so that this action is simplicial, cp. [12]. The stabilizer F_x of a point $x \in \mathcal{T}_g$ can by our discussion above be identified with the group of complex automorphisms $\mathrm{Aut}_{\mathbb{C}}(U/\rho(\Pi_g))$, where the representation ρ corresponds to $x \in \mathcal{T}_g$. It follows that the stabilizers F_x are finite groups. To prove that $\mathcal{T}_g$ is actually an $\underline{E}\Gamma_g$, it remains to show that for $F < \Gamma_g$ the fixed point space $\mathcal{T}_g^F$ is contractible. That it is not empty follows from Kerckhoff's solution of the Nielsen Realization Problem [15]:

Theorem 3.2. *Let* $g > 1$ *and* $F < \Gamma_g$ *a finite subgroup. Then there exists a Riemann surface* $S_g^{\mathbb{C}} = U/\rho(\Pi_g)$ *and a subgroup* $F^{\mathbb{C}}$ *of the group of holomorphic automorphisms* $\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}})$ *such that the natural map* $\mathrm{Aut}_{\mathbb{C}}(S_g^{\mathbb{C}}) \to \Gamma_g$ *maps* $F^{\mathbb{C}}$ *isomorphically onto* F.

In the situation above, the point $x = [\rho]$ of $\mathcal{T}_g$ corresponding to ρ then lies in $\mathcal{T}_g^F$. Moreover, the group of lifts $\Lambda := \{\phi : U \to U\} < \operatorname{PSL}(2,\mathbb{R})$ of the maps $S_g^{\mathbb{C}}(x) \to S_g^{\mathbb{C}}(x)$ in $F^{\mathbb{C}}$, can be identified with a cocompact Fuchsian group, contained in the normalizer $N_{\operatorname{PSL}(2,\mathbb{R})}(\rho(\Pi_g))$ in $\operatorname{PSL}(2,\mathbb{R})$. Thus there is an exact sequence

$$1 \to \Pi_g \to \Lambda \to F \to 1,$$

giving rise to a real-analytic restriction map $\mathcal{T}(\Lambda) \to \mathcal{T}(\Pi_g) = \mathcal{T}_g$. The following is proved in Harvey's paper [11, Cor. 3]; Harvey's proof is worked out under the assumption that $\mathcal{T}_g^F$ is nonempty (Kerckhoff's Theorem [15]), which was not known at the time.

Proposition 3.3. *Let $g \geq 2$ and F a finite subgroup of Γ_g. Then there is a cocompact Fuchsian group $\Lambda < \mathrm{PSL}(2,\mathbb{R})$ containing Π_g as a normal subgroup of finite index, with $\Lambda/\Pi_g \cong F$ such that the natural inclusion $\mathcal{T}(\Lambda) \to \mathcal{T}_g$ has image $\mathcal{T}_g^F$. In particular, $\mathcal{T}_g^F$ is contractible and thus $\mathcal{T}_g$ is a model $\underline{E}\Gamma_g$ for Γ_g.*

Remark 3.4. There are other ways to show that $\mathcal{T}_g^F$ is contractible. In [13, Prop. 2.3] this is deduced using properties of the Weil-Petersson metric on $\mathcal{T}_g$ (it is geodesically convex and nonpositively curved so that $\mathcal{T}_g$ is a CAT(0) space [22]). The result can also be proved using "earthquake paths", in conjunction with Kerckhoff's Theorem [15], see [13, Rem. 2.4] for more details.

If the genus g equals one, $\mathcal{T}_1$ can be identified with the upper half plane, on which $\Gamma_1 = \mathrm{SL}(2,\mathbb{Z})$ acts by linear fractional transformations. It is well-known that $\mathcal{T}_1$ contains a tree T as a strong Γ_1-deformation retract, on which $\mathrm{SL}(2,\mathbb{Z})$ acts cocompactly (the orbit space $T/\mathrm{SL}(2,\mathbb{Z})$ is an interval, corresponding to the decomposition of $\mathrm{SL}(2,\mathbb{Z})$ as $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$). This tree T is a cocompact model for Γ_1.

More generally, the following theorem has been proved by Broughton [3, Thm 2.7]:

Theorem 3.5. *For any genus $g \geq 1$, Teichmüller space $\mathcal{T}_g$ contains a (simplicial) Γ_g-subspace which is a strong Γ_g-deformation retract, and which is a cocompact $\underline{E}\Gamma_g$.*

4. Cocompact models for $\underline{E}\Gamma_{g,r}^s$

The following result of Lück [16, Thm 3.2] is very useful for the construction of cocompact models $\underline{E}G$ for a group G given by an extension.

Proposition 4.1. *Let $1 \to H \to G \to Q \to 1$ be an exact sequence of groups and assume that*

1. *for every finite subgroup $F < Q$ and every extension $1 \to H \to \Gamma \to F \to 1$ there exists a cocompact model $\underline{E}\Gamma$,*
2. *Q admits a cocompact model $\underline{E}Q$.*

Then G admits a cocompact model $\underline{E}G$ too.

We want to apply this Proposition to prove our main theorem:

Theorem 4.2. *For all $g, r, s \geq 0$, the mapping class group $\Gamma_{g,r}^s$ possesses a cocompact model $\underline{E}\Gamma_{g,r}^s$.*

For its proof, we will need the following two Lemmas.

Lemma 4.3. *Let $1 \to \mathbb{Z}^n \to \Gamma \to F \to 1$ be an exact sequence of groups with F finite. Then Γ admits an n-dimensional cocompact $\underline{E}\Gamma$ homeomorphic to $\mathbb{R}^n$, with Γ acting by affine maps.*

Proof. The F-action on $\mathbb{Z}^n$ extends, via the standard inclusion $\iota : \mathbb{Z}^n \to \mathbb{R}^n$, to a representation

$$\phi : F \to \mathrm{Aut}(\mathbb{Z}) \to \mathrm{GL}(n, \mathbb{R}).$$

Since F is finite, $H^2(F; \underline{\mathbb{R}}^n) = 0$ so that the induced map

$$\iota_* : H^2(F; \underline{\mathbb{Z}}^n) \to H^2(F; \underline{\mathbb{R}}^n)$$

is trivial. We therefore obtain a commutative diagram of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma & \longrightarrow & F & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle =} & & \\ 1 & \longrightarrow & \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \rtimes F & \longrightarrow & F & \longrightarrow & 1\,, \end{array}$$

with ρ an injective homomorphism. The representation $\phi : F \to \mathrm{GL}(n, \mathbb{R})$ induces a homomorphism $\Phi : \mathbb{R}^n \rtimes F \to \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ with finite kernel. It follows that $\Phi \circ \rho : \Gamma \to \mathbb{R}^n \rtimes \mathrm{GL}(n, \mathbb{R})$ defines an affine, proper and cocompact action on $\mathbb{R}^n$, and it follows that for $G < \Gamma$, the fixed point set of this action is empty, if G is infinite, and a nonempty affine subspace, if G is finite, proving our assertion. □

Lemma 4.4. *Let* $1 \to H \to \Gamma \to F \to 1$ *be an exact sequence of groups with* F *finite.*

(1) *If* H *is finitely generated free, then* Γ *admits a one-dimensional cocompact* $\underline{E}\Gamma$.
(2) *If* $H = \pi_1(S_g)$ *with* $g > 0$*, then* Γ *admits a two-dimensional cocompact* $\underline{E}\Gamma$ *(for* $g > 1$*, one can choose* $\underline{E}\Gamma = U$ *the upper half plane, with* Γ *acting by hyperbolic isometries).*

Proof. If H is finitely generated and free, then Γ is the fundamental group of a finite graph of groups, with finite vertex stabilizers, cp. [14]. Thus Γ admits a one-dimensional cocompact $\underline{E}\Gamma$. In case $H = \pi_1(S_g)$ with $g > 0$, H is either $\mathbb{Z}^2$ and Γ then admits a two-dimensional cocompact $\underline{E}\Gamma$ by the previous lemma, or H is a nonabelian surface group. Since in that second case the center of H is trivial, the extension group Γ is uniquely determined up to isomorphism by the action $F \to \mathrm{Out}(H)$. In particular, the extension will be split over the kernel K of the action map $F \to \mathrm{Out}(H)$. Therefore, Γ contains a finite normal subgroup $\tilde{K}$ isomorphic to K so that the extension

$$H \to \Gamma/\tilde{K} \to F/K$$

has a faithful action $F/K \to \mathrm{Out}\, H$. By Kerckhoff's Theorem [15, Thm. 1], F/K acts faithfully and isometrically on S_g with respect to some Riemannian metric with curvature -1 (not necessarily preserving the orientation of S_g). Thus, the group Λ of lifts to the universal cover U is isomorphic to $\Gamma/\tilde{K}$ and acts properly and cocompactly on U (not necessarily preserving the orientation: it is a discrete subgroup of the group of isometries of U, which contains $\mathrm{PSL}(2, \mathbb{R})$ as a subgroup of index two). Since the action is by hyperbolic isometries, it follows that U is an $\underline{E}\Gamma$. □

Proof of Theorem 4.2. We will proceed by induction, using Proposition 4.1 and Lemma 4.4, in conjunction with the exact sequences (B) and (D) of Section 2.2. To verify (1) of Proposition 4.1 for our situation, we just need to check that extensions of the form $1 \to H \to \Gamma \to F \to 1$ with F finite and H either a finitely generated free group or $H = \pi_1(S_g)$ admit a cocompact $\underline{E}\Gamma$. This has been done in Lemma 4.4. For $g \geq 2$ we start our induction with $\Gamma^0_{g,0} = \Gamma_g$, for which the theorem has been proved in Theorem 3.5. The exact sequence (B) together with Lemma 4.4 then yields the result for $\Gamma^s_{g,0}$, for all s. Using the exact sequences (D) and (B), together with Lemma 4.4, permits us then to pass to $\Gamma^s_{g,r}$ for all (r,s). It remains to deal with the cases of $g < 2$. The case of $g = 0$: it is well-known that $\Gamma^s_{0,0} = \{e\}$ for $s < 4$. We can therefore use (B) to pass to $\Gamma^s_{0,0}$, $s \geq 4$. Then we can use (D) to pass from $\Gamma^s_{0,0}$ with $s \geq 3$ to $\Gamma^{s-1}_{0,1}$; the two groups missed are $\Gamma^1_{0,1}$ and $\Gamma^0_{0,1}$, which are known to be trivial. From there on we can pass to all the remaining groups $\Gamma^s_{0,r}$. The genus 1 case: we know the result for $\Gamma^0_{1,0} = \Gamma_1$ and $\Gamma^1_{1,0}$ is known to be isomorphic to Γ_1. We can thus pass to $\Gamma^s_{1,0}$ with $s > 1$ using the exact sequence (B). Finally, we can apply (D) to get to all the groups $\Gamma^s_{1,r}$, finishing the proof of the theorem. □

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