Gap distributions in circle packings

Zeev Rudnick and Xin Zhang

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Abstract. We determine the distribution of nearest neighbor spacings between the tangencies to a fixed circle in a class of circle packings generated by reflections. We use a combination of geometric tools and the theory of automorphic forms.

1. Introduction

1.1. **A class of circle packings.** In recent years, there has been increasing interest in the quantitative study of aspects of circle packings, in particular for Apollonian packings. For instance, the asymptotic count of the number of circles with bounded curvature has been determined [8, 10], as well as several questions concerning the arithmetic of Apollonian packings, see [4, 12].

In this paper, we investigate local statistics for the distribution of tangencies to a fixed circle in a circle packing, in a class of packings generated by reflections. The packings that we study are formed by starting with an initial finite configuration $\mathcal{K}$ of circles in the plane with disjoint interiors, and such that the gaps between circles are curvilinear triangles, as in Figure 1. We then form the group $\mathfrak{S}$ generated by reflections in the dual circles defined by the triangular gaps of the configuration, see Section 2. Applying the elements of $\mathfrak{S}$ to the initial configuration $\mathcal{K}$, we obtain an infinite circle packing $\mathcal{P}$, see Figure 1 for an example.

We create such initial configurations by starting with a finite triangulation G of the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. By the circle packing theorem of Koebe, Andreev and Thurston [13], there is a circle packing $\mathcal{K}$ of $\widehat{\mathbb{C}}$ having G as its tangency graph, that is, a collection of disks with disjoint interiors on $\widehat{\mathbb{C}}$
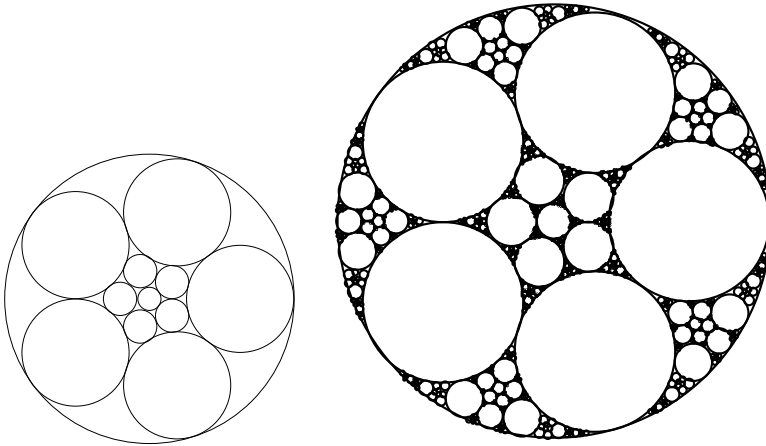
FIGURE 1. An Apollonian-9 packing found by Butler, Graham, Guettler and Mallows. On the left is the initial configuration, for which the tangency graph is the icosahedron.

corresponding to the vertices of G where two disks are tangent if and only if the corresponding vertices are connected in G. The gaps between the disks are the connected components of the complement $\widehat{\mathbb{C}} \setminus \bigcup_{C \in \mathcal{K}} C$ of the disks in $\widehat{\mathbb{C}}$. They are triangular because we assume that G is a triangulation, that is, each face of G is a triangle. After stereographic projection from the point ∞, we obtain a circle packing $\mathcal{K}$ in the finite plane $\mathbb{C}$.

1.2. **Tangencies.** We fix a base circle $C_0 \in \mathcal{P}$ and consider the subset $\mathcal{P}_0 \subset \mathcal{P}$ of circles tangent to C_0. Let $\mathcal{I} \subseteq C_0$ be an arc (if C_0 is a line, take $\mathcal{I}$ to be a bounded interval). Let $\mathcal{A}_{\mathcal{I}}$ be the set of tangencies in $\mathcal{I}$, and $\mathcal{A}_{T,\mathcal{I}}$ be the subset of $\mathcal{A}_{\mathcal{I}}$ whose corresponding circles in $\mathcal{P}_0$ have curvatures bounded by T, see Figure 2.

We will show that the cardinality of $\mathcal{A}_{T,\mathcal{I}}$ grows linearly with T:

(1) $$\#\mathcal{A}_{T,\mathcal{I}} \sim l(\mathcal{I}) c_{\mathcal{P},C_0} T, \quad T \to \infty,$$

where $l(\mathcal{I})$ is the standard arclength of $\mathcal{I}$, and $c_{\mathcal{P},C_0}$ is a constant depending only on $\mathcal{P}$ and C_0. In particular, the tangencies are uniformly distributed in C_0. To compare, the total number of circles $C \in \mathcal{P}$ with $\kappa(C) < T$ is $\sim \mathrm{const}(\mathcal{P}) T^{\delta}$ for some $\delta > 1$, see [8, 10].

The goal of our paper is to study the distribution of nearest neighbor spacings (gaps) in $\mathcal{A}_{\mathcal{I}}$. Let $\{x_{T,\mathcal{I}}^i\}$ be the sequence of tangencies in $\mathcal{A}_{T,\mathcal{I}}$ ordered by some orientation (say, the counter-clockwise orientation). The nearest-neighbor gaps, or spacings, between the tangencies are

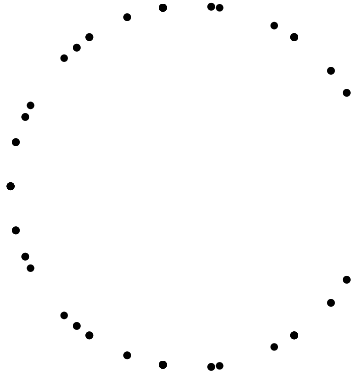$$d(x_{T,\mathcal{I}}^i, x_{T,\mathcal{I}}^{i+1}),$$

FIGURE 2. Tangencies $\mathcal{A}_{T,C_0}$ associated to the Apollonian-9 packing of Figure 1, with C_0 being the bounding circle.

where d denotes the arc-length distance. The mean spacing is

$$\langle d_T \rangle := \frac{l(\mathcal{I})}{\#\mathcal{A}_{T,\mathcal{I}}} \sim \frac{1}{c_{\mathcal{P},C_0} T}.$$

We define the gap distribution function to be

$$F_{T,\mathcal{I}}(s) = F_{T;\mathcal{P},C_0}(s) = \frac{1}{\#\mathcal{A}_{T,\mathcal{I}}} \# \left\{ x_T^i \mid \frac{d(x_T^i, x_T^{i+1})}{\langle d_T \rangle} \leq s \right\}.$$

Clearly the definition of $F_{T,\mathcal{I}}$ does not depend on the orientation that we choose. We will show there is a limiting gap distribution, which is conformally invariant and independent of $\mathcal{I}$:

Theorem 1.3. *Given $\mathcal{P}, C_0, \mathcal{I}$ as above, there exists a continuous piecewise smooth function $F(s) = F_{\mathcal{P},C_0}(s)$, such that*

$$\lim_{T \to \infty} F_{T,\mathcal{I}}(s) = F(s).$$

Moreover, the limit distribution $F_{\mathcal{P},C_0}(s)$ is conformally invariant: if $M \in \mathrm{PSL}(2,\mathbb{C})$, $M\mathcal{P} = \tilde{\mathcal{P}}$, $MC_0 = \tilde{C}_0$ then

$$F_{\tilde{\mathcal{P}},\tilde{C}_0}(s) = F_{\mathcal{P},C_0}(s).$$

Our method does not just give the existence, but actually explicitly calculates the limiting spacing distribution in terms of certain areas, showing that it is a continuous, piecewise smooth function. We also show that $F(s)$ is supported away from the origin, which is a very strong form of "level repulsion".

In Section 7 we explicitly compute the limiting distribution for some examples. As a warm-up, we start with classical Apollonian packings, where we start with three mutually tangent circles and then fill in each curvilinear triangle with the unique circle which is tangent to all three sides of that triangle, and then repeat this process with each newly created curvilinear triangle. In that case we recover a theorem of Hall [7] on the gap distribution of Farey
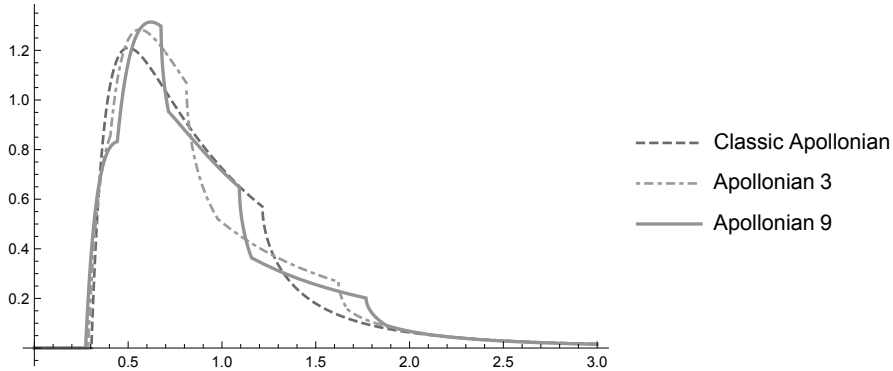
FIGURE 3. The density $F'(s)$ of the gap distribution for classical Apollonian packings (dashed curve), which is the same as that discovered by Hall for Farey sequence, for the Apollonian-3 packings of Guettler and Mallows (dotted curve), and for the Apollonian-9 packing of Butler, Graham, Guettler and Mallows (solid curve).

points. See [1, 3] for a history and further results in this direction, and see [9] for an approach using homogeneous dynamics.

We then compute the gap distribution for two other classes of packings: the Guettler and Mallows [6] packing in Figure 4, called also Apollonian-3, where the curvilinear triangle is filled by three new circles, each tangent to exactly two sides, and for which the associated tangency graph is the octahedron; and for the configuration (called Apollonian-9) found by Butler, Graham, Guettler and Mallows [2, Fig. 11], for which the tangency graph is the icosahedron, as in Figure 1. The density functions of these three cases are displayed in Figure 3.

1.4. **Our method.** We work on models where $\mathcal{K}$ consists of two horizontal lines, with $C_0 = \mathbb{R}$ and $C_1 = \mathbb{R} + \mathbf{i}$, and the remaining circles lie in the strip between them. We call such models generalized Ford configurations. Let Γ be the subgroup of $\mathfrak{S}$ consisting of orientation preserving elements of $\mathfrak{S}$ that fix the base circle C_0. It is a non-uniform lattice in $\mathrm{PSL}(2, \mathbb{R})$ (this group was used crucially in Sarnak's work [12] on Apollonian packings). Using geometric considerations, we reduce the spacing problem to counting elements $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of the Fuchsian group Γ such that the lower row $(c, d) \in \mathbb{R}^2$ lies in a dilated region defined by the intersection of certain quadratic inequalities, and such that a/c lies in a given interval. After that, we use the spectral theory of automorphic forms, specifically the work of Anton Good [5], to show that the number of lattice points is asymptotically a multiple of the area of this dilated region.

1.5. **Plan of the paper.** In Section 2 we give details how to construct $\mathcal{P}$ from the initial configuration $\mathcal{K}$. We will also construct related groups and prove their geometric properties. In Sections 3, 4 and 5 we will prove the uniform
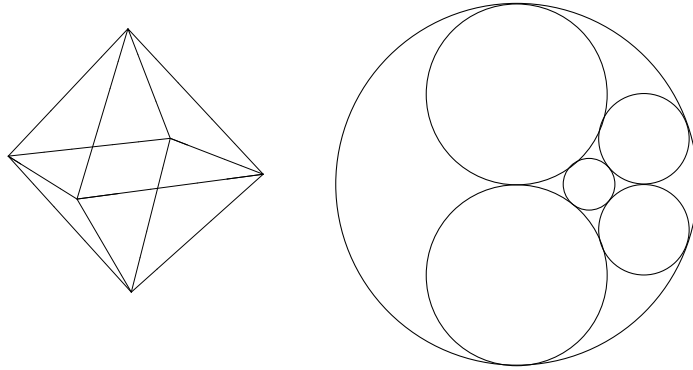
Figure 4. An initial configuration in the plane, which we call the Apollonian-3 configuration, for which the associated tangency graph is the octahedron.

distribution of tangencies (1) and Theorem 1.3 in the special case that $\mathcal{P}$ is a generalized Ford packing, and C_0 is one of the two lines from $\mathcal{P}$. In Section 6 we show how to deduce the general case of (1) and Theorem 1.3 from the results of Section 3.4 and Section 4. In Section 7 we compute some examples. We reduce the computational effort involved by using conformal invariance, observing that the packings in the examples admit extra symmetries. For instance, in the case of Apollonian-9 packings, the associated symmetry group is the non-arithmetic Hecke triangle group $\mathbb{G}_5$.

2. Constructions

2.1. **Constructing initial configurations.** We start with a (finite) triangulation G of the sphere. By the circle packing theorem of Koebe, Andreev and Thurston [13], there is a circle packing $\mathcal{K}$ of S^2 having G as its tangency graph, that is, a collection of disks with disjoint interiors on S^2 corresponding to the vertices of G (there must be at least four vertices), where two disks are tangent if and only if the corresponding vertices are connected in G. The interstices, or gaps, between the disks, are the connected components of the complement $S^2 \setminus \bigcup_{C \in \mathcal{K}} C$ of the disks in S^2. Because we assume that G is a triangulation, i.e., each face of G is a triangle, this means that the gaps are triangular.

We may identify S^2 with the Riemann sphere $\widehat{\mathbb{C}} = \mathbb{C} \cup \infty$, and then by picking a point which will correspond to ∞ and performing a stereographic projection onto the plane, we obtain a circle packing $\mathcal{K}$ in the finite plane $\mathbb{C}$. If ∞ is contained in the interior of one of the disks, then $\mathcal{K}$ is realized as a configuration of finitely many circles, with one circle containing the other ones, see Figure 4. If ∞ is a point of tangency of two disks, then $\mathcal{K}$ is a configuration consisting of two lines and several other circles in between; we call this type of configuration a generalized Ford configuration, see Figure 7.
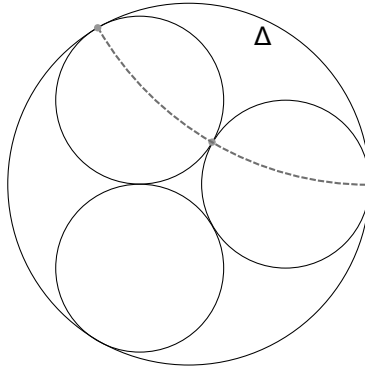
Figure 5. An initial configuration, which generates a classical Apollonian packing. The dual circle corresponding to the triangular gap labeled Δ is the dashed circle.

2.2. **Construction of the packing.** We start with a circle packing $\mathcal{K}$ as above, of H tangent circles $C_1, C_2, \ldots, C_H$ in the plane, so that the gaps between circles are curvilinear triangles. We build up a circle packing $\mathcal{P}$ from $\mathcal{K}$ by refections: for each triangular gap, we draw a dual circle which passes through the vertices of the triangle, see Figure 5. We can generate new circles in the gap by reflecting $\mathcal{K}$ through the dual circles. Each of the circles forming the gap is fixed under this reflection, and the remaining circles are reflected into the gap. Denote these reflections by $\mathcal{S} = \{S_1, S_2, \ldots, S_k\}$ and the corresponding triangles by $\mathcal{T} = \{\triangle_{S_1}, \triangle_{S_2}, \ldots, \triangle_{S_k}\}$, where k is the number of gaps. Note that all these reflections are anti-holomorphic maps in $\mathbb{C}$ (throughout this paper, by triangle we mean solid triangle with vertices removed).

Let $\mathfrak{S} = \langle S_1, S_2, \ldots, S_k \rangle$ be the group generated by the reflections. The reflection group $\mathfrak{S}$ has only the relations $S_1^2 = S_2^2 = \cdots = S_k^2 = \mathrm{id}$. This can be seen from the pingpong lemma: $\triangle_{S_i}$ are disjoint sets, and S_j maps every $\triangle_{S_i}$ into $\triangle_{S_j}$ when $j \neq i$. If we apply $\mathfrak{S}$ to the configuration $\mathcal{K}$, we obtain a circle packing $\mathcal{P}$ of infinitely many circles.

Pick a circle C_0 from $\mathcal{P}$, and let $\mathcal{P}_0$ be the circles tangent to C_0 and let $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{P}_0 = \{C_1, \ldots, C_h\}$. We are interested in studying the counting and spacing problems on $\mathcal{P}_0$. The first step is to place $\mathcal{P}$ in an ambient hyperbolic space with C_0 being the boundary, and show that there is a finite covolume Fuchsian group Γ acting on $\mathcal{P}_0$ and having finitely many orbits.

2.3. **Construction of Γ.** A general element in $\mathfrak{S}$ is given by $T_1 \cdot \ldots \cdot T_m$ with $T_i \in \mathcal{S}$. Therefore each circle $C \in \mathcal{P}$ can be represented by $T_1 \cdot \ldots \cdot T_m C_l$, where $C_l \in \mathcal{K}$. We denote the triangular gap corresponding to T_i by $\triangle_{T_i}$. We define the length of the above word to be m. We say the word above is *minimal* for C if we can not find any shorter expression, and we say C is in m-th generation.

We first show that there is a canonical way to express a circle with minimal length:

Lemma 2.4. *The minimal word for a given circle $C \in \mathcal{P}$ is unique. Moreover, $T_1 T_2 \ldots T_m C_l = C$ is minimal if and only if we have*

$$(2) \quad \begin{cases} C_l \cap \triangle_{T_m} = \varnothing, \\ T_m C_l \subset \triangle_{T_m}, \quad T_{m-1} T_m C_l \subset \triangle_{T_{m-1}}, \quad \ldots, \quad T_1 \ldots T_m C_l \subset \triangle_{T_1}. \end{cases}$$

Proof. We first prove the "only if" direction by induction on the generation of C. For the base case first generation, if $T_1 C_l$ is minimal, then $\triangle_{T_1} \cap C_l = \varnothing$, otherwise T_1 will fix C_l and then the word is not minimal. As a result, $T_1 C_l \subset \triangle_{T_1}$. Suppose the "only if" direction holds for generations up to $m-1$ and $C = T_1 T_2 \ldots T_m C_l$ is a minimal word for C, then $T_2 \ldots T_m C_l$ has to be a minimal word for some circle. By our induction hypothesis, $T_2 \ldots T_m C_l \in \triangle_{T_2}, \ldots, T_m C_l \subset \triangle_{T_m}$ and $C_l \cap \triangle_{T_m} = \varnothing$. Since $T_1 \neq T_2$ (otherwise $T_1 T_2 = \mathrm{id}$ and the word is not minimal), we must have $T_1 T_2 \cdots T_m C_l \subset T_1 \triangle_{T_2} \subset \triangle_{T_1}$.

From (2) we see that T_1 corresponds to the unique triangle in $\mathcal{T}$ where $T_1 C$ sits in, T_2 corresponds to the unique triangle in $\mathcal{T}$ where T_2 sits in, etc. Finally, T_k corresponds to the unique triangle in $\mathcal{T}$ where $T_{k-1} \cdots T_1 C$ sits in. This canonical description shows the uniqueness of minimal expression for a circle.

We prove the "if" direction by induction on the length of expression (2). If $\triangle_{T_1} \cap C_l = \varnothing$, then $T_1 C_l$ is a circle lying in $\triangle_{T_1}$. The only circles with shorter words are the ones in $\mathcal{K}_0$, all of which do not lie in $\triangle_{T_1}$, so $T_1 C_l$ is minimal. Now suppose $C = T_1' T_2' \ldots T_m' C_{l'}$ is the minimal word for C. Then by our discussion above, T_1' is the unique reflection that corresponds to the triangle in $\mathcal{T}$ where C sits in, so $T_1' = T_1$. Then $T_1 C = T_2 \ldots T_m C_l$ satisfies condition (2) and the minimal word for $T_1 C$ is $T_1 C = T_2' \ldots T_m' C_{l'}$. By our induction hypothesis, we have $m' = m$, $l' = l$, $T_i' = T_i$ for each $1 \leq i \leq m$. $\square$

If $m \geq 1$, we write $\triangle_{T_1 \ldots T_m} = T_1 \cdot \ldots \cdot T_{m-1} \triangle_{T_m}$, and define $\triangle_C = \triangle T_1 \cdots T_m$ to be the *minimal* triangle to C. Clearly C is contained in $\triangle_C$.

Now let Γ_0 be the subgroup of $\mathfrak{S}$ consisting of elements that fix a given circle C_0. By relabeling let $\mathcal{K}_0 = \{C_1, C_2, \ldots, C_h\} \subset \mathcal{K}$ be the set of circles that are tangent to C_0, and their tangencies are $\alpha_1, \alpha_2, \ldots, \alpha_h$. Let $\mathcal{S}_0 = \{S_1, \ldots, S_h\} \subset \{S_1, \ldots, S_k\}$ be the set of generators of $\mathfrak{S}$ that fix C_0, where $S_i \in \mathcal{S}_0$ is the reflection corresponding to the triangle formed by C_0, C_i, C_{i+1}. For our convenience, at this point we make the convention that $C_{i+h} = C_i$, $\alpha_{i+h} = \alpha_i$ and $S_{i+h} = S_i$ for $i \in \mathbb{Z}$, because the C_i from $\mathcal{K}_0$ form a loop around C_0.

Lemma 2.5. $\Gamma_0 = \langle \mathcal{S}_0 \rangle$ *is generated by the reflections in $\mathcal{S}_0$.*

Proof. If $\gamma = T_1 \ldots T_q$ where $T_1, \ldots, T_q \in \mathcal{S}$ and $\gamma C_0 = C_0$, we first absorb the trivial identities $T_i T_{i+1} = \mathrm{id}$ if $T_i = T_{i+1}$, then we cut the first and last few consecutive letters of the word $T_1 \ldots T_q$ that are from $\mathcal{S}_0$ (which may not exist). We continue this procedure several times until it stabilizes, then we get a word for C_0, which has to satisfy condition (2) so is minimal. But the
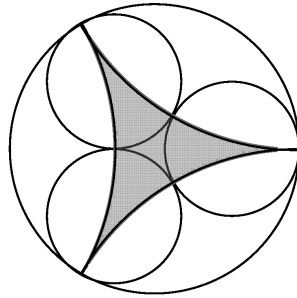
Figure 6. The fundamental domain $\mathcal{F}_0$ for Γ_0.

minimal word for C_0 is just C_0. Since each cutting corresponds to multiplying an element in $\langle \mathcal{S}_0 \rangle$, we have $\gamma \in \langle \mathcal{S}_0 \rangle$. □

Lemma 2.6. Γ_0 *acts on* $\mathcal{P}_0$, *and has finitely many orbits on* $\mathcal{P}_0$.

Proof. Γ_0 acts on $\mathcal{P}_0$ because $\mathfrak{S}$ sends pairs of tangent circles to pairs of tangent circles. Since Γ_0 fixes C_0, it will send any circle tangent to C_0 to some circle tangent to C_0. Now suppose $C \in \mathcal{P}_0$ and the minimal word for C is $C = T_1 \ldots T_m C_l$. From the argument in Lemma 2.4, T_1 corresponds to a triangle $\triangle_{T_1} \in \mathcal{T}$, which has to be a triangle that contains part of C_0 where C lies in, so T_1 has to be from $\mathcal{S}_0$. By the same argument, one can show that $T_2, \ldots, T_m \in \mathcal{S}_0$, and $C_l \in \mathcal{K}_0$. Therefore, the number of orbits is h, the cardinality of $\mathcal{K}_0$. □

2.7. **Geometric properties of** Γ. Now we put a hyperbolic structure associated to Γ. Without loss of generality we assume that C_0 is the bounding circle of radius 1, whose interior is the unit disk $\mathbb{D}$, and let g_i be the geodesic connecting α_i and α_{i+1} (again we extend the definition for all $i \in \mathbb{Z}$ by setting $g_i = g_{i+h}$), so that the region bounded by g_i and C_0 contains the triangle $\triangle_{S_i}$. Each $S_i \in \mathcal{S}_0$ preserves the metric in $\mathbb{D}$, so $\Gamma_0 \subset \mathrm{Isom}(\mathbb{D})$, the isometry group of $\mathbb{D}$. Let $\mathcal{F}_0$ be the open region bounded by g_i's (see Figure 6). The following theorem, due to Poincaré (see [11, Thm. 7.1.3]), tells us that Γ_0 acts discontinuously on $\mathbb{D}$ with fundamental domain $\mathcal{F}_0$.

Theorem 2.8. *Let* P *be a finite-sided, convex polyhedron in* $\mathbb{D}$ *of finite volume all of whose dihedral angles are submultiples of* π. *Then the group generated by the reflections in the sides of* P *is a discrete reflection group with respect to the polyhedron* P.

As corollaries, Γ_0 has $\alpha_1, \ldots, \alpha_h$ as cusps, and the fundamental domain for Γ_0 has volume $V(\Gamma_0) = \pi(h-2)$ by Gauss–Bonnet.

For our purpose, we want to take the orientation preserving subgroup of Γ_0, which we denote by Γ. The group Γ is a free, index-2 subgroup of Γ_0 generated by $S_2 S_1, \ldots, S_h S_1$. To see that Γ is free, we apply the pingpong

lemma to $S_i S_1$, which maps $\triangle_{S_j}$ to $\triangle_{S_i}$ when $j \neq i$. From the properties of Γ_0, we immediately get the following result.

Proposition 2.9. Γ *has cusps at* α_i, *with stabilizer* $\Gamma_{\alpha_i} = \langle S_i S_{i+1} \rangle$. *The area of a fundamental domain is* $\text{area}(\Gamma) = 2\pi(h - 2)$.

3. The uniform distribution of tangencies in circle packings

We now study the distribution of tangencies on a fixed circle from a circle packing. Recall that $\mathcal{P}$ is generated from a finite circle packing $\mathcal{K}$, which is associated to a triangulated spherical graph. We pick a circle $C_0 \in \mathcal{P}$ and let $\mathcal{P}_0$ be the collections of circles in $\mathcal{P}$ which are tangent to C_0. Given a bounded arc $\mathcal{I} \subseteq C_0$, let $\mathcal{A}_{T,\mathcal{I}}$ be the collection of tangencies of circles from $\mathcal{P}_0$, whose curvatures are bounded by T. Let l be the standard arclength measure. We will show the following theorem.

Theorem 3.1. *As* $T \to \infty$,

$$\#\mathcal{A}_{T,\mathcal{I}} \sim l(\mathcal{I}) c_{\mathcal{P},C_0} T$$

with $c_{\mathcal{P},C_0}$ *independent of* $\mathcal{I}$ *given by*

$$(3) \qquad c_{\mathcal{P},C_0} = \frac{D}{2\pi^2(h-2)}, \quad D = \sum_{i=1}^{h} D_i,$$

where D_i *is the hyperbolic area of the region bounded by* C_i *and the two geodesics linking* α_i *to* α_{i+1} *and* α_i *to* α_{i-1} *(see Figure 8). Moreover,* $c_{\mathcal{P},C_0}$ *is conformal invariant: if* $M \in \text{SL}(2,\mathbb{C})$, $M(\mathcal{P}) = \tilde{\mathcal{P}}$, *and* $M(C_0) = \tilde{C}_0$, *then* $c_{\tilde{\mathcal{P}},\tilde{C}_0} = c_{\mathcal{P},C_0}$.

We deduce Theorem 3.1 from some old results of A. Good [5], but it is in principle known; for instance it is a special case of [10].

We will first prove Theorem 3.1 for a generalized Ford packing, when the initial configuration contains two parallel lines, one of them being the base circle C_0. The general case will follow by conformal invariance (Theorem 6.1) whose proof is deferred to Section 6. For an explicit formula for the areas D_i in the case of a Ford configuration, see (7) at the end of the present section.

3.2. **Ford configurations.** We first show that any of our initial configurations may be brought into a standard shape, that of a (generalized) Ford configuration, see Figure 7.

Lemma 3.3. *We can map the initial configuration* $\mathcal{K}$ *by using a Möbius transformation and possibly a reflection, to one which includes two parallels* $C_0 = \mathbb{R}$ *and* $C_1 = \mathbb{R} + \mathbf{i}$, *two circles tangent to both of* $\mathbb{R}$ *and* $\mathbb{R} + \mathbf{i}$ *with their tangencies on* $\mathbb{R}$ *being* 0 *and* $\frac{t}{2}$, *respectively, and some circles in between.*

Proof. We may use a Möbius transformation of the Riemann sphere to map the configuration to the interior of the strip $0 \leq \Im z \leq 1$ as follows: Pick a triangle in the tangency graph G one of whose vertices corresponds to C_0,
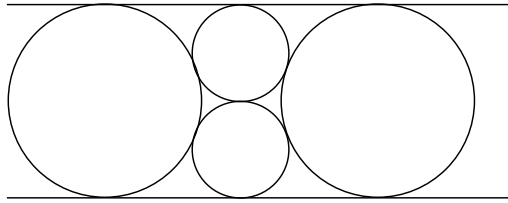
FIGURE 7. The generalized Ford configuration corresponding to the Apollonian-3 configuration of Figure 4.

and whose other vertices correspond to two other circles C_1 and C_2, so that C_0, C_1, C_2 form a triple of mutually tangent circles. Setting $\alpha_1 = C_0 \cap C_1$, $\alpha_2 = C_0 \cap C_2$, $\beta = C_1 \cap C_2$ to be the tangency points, corresponding to the edges of the triangle, we see that there is a (unique) Möbius transformation $M \in \mathrm{SL}(2, \mathbb{C})$ with

$$M : \alpha_1 \mapsto \infty, \quad \alpha_2 \mapsto 0, \quad \beta \mapsto \mathbf{i}.$$

Then M will map C_0 and C_1 to a pair of tangent lines passing through 0 and $\mathbf{i}$, respectively, and C_2 will be mapped to a circle tangent to both C_0 and C_1 at these points. Hence M will map C_0 to the real axis $\tilde{C}_0 = \mathbb{R}$, C_1 to the line $\tilde{C}_1 = \mathbb{R} + \mathbf{i}$ and C_2 to the circle $\tilde{C}_2 = C(\frac{\mathbf{i}}{2}, \frac{1}{2})$ centered at $\frac{\mathbf{i}}{2}$ and having radius $\frac{1}{2}$, and the remaining circles all lie in the strip $0 \leq \Im(z) \leq 1$.

The triangular gap $\triangle$ between C_0, C_1, C_2 is mapped into one of the two triangular gaps forming the complement in the strip $0 \leq \Im(z) \leq 1$ of the circle $\tilde{C}_2 = C(\frac{\mathbf{i}}{2}, \frac{1}{2})$. After applying if necessary the reflection $z \mapsto -\overline{z}$, we may assume that $\triangle$ is mapped to the gap in the half plane $\Re(z) < 0$, so that all the other circles in $\mathcal{K}$ are mapped into the gap with $\Re(z) > 0$. Since any triangulation of the sphere has at least four vertices, there must be at least one such circle distinct from C_0, C_1, C_2. Since in the resulting configuration the gaps are still triangular, that means that there is one other circle tangent to both $\mathbb{R}$ and $\mathbb{R} + \mathbf{i}$, say $\tilde{C}_h = C(\frac{t}{2} + \frac{\mathbf{i}}{2}, \frac{1}{2})$ $(t \geq 2)$, and that the images of all the other circles are mapped to circles lying between $\tilde{C}_2$ and $\tilde{C}_h$, as in Figure 7.   □

3.4. **Uniform distribution of tangencies for generalized Ford packings.** We carry over all previous notations. The initial configuration $\mathcal{K}$ includes two parallel lines $C_0 = \mathbb{R}$ and $C_1 = \mathbb{R} + \mathbf{i}$, two circles $C_2 = C(\frac{\mathbf{i}}{2}, \frac{1}{2})$ and $C_h = C(h + \frac{\mathbf{i}}{2}, \frac{1}{2})$ tangent to both of $\mathbb{R}$ and $\mathbb{R} + \mathbf{i}$, and some circles in between. Our base circle C_0 is $\mathbb{R}$, so $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{P}_0$ and $\mathcal{P}_0$ is the collection of all circles tangent to $\mathbb{R}$. The reflection S_h corresponds to the triangle formed by $\mathbb{R}$, $\mathbb{R} + \mathbf{i}$ and C_h. Applying S to $\mathcal{K}$, the fundamental domain for Γ is formed by geodesics connecting tangencies of $\mathcal{K}$ and $S\mathcal{K}$ on $\mathbb{R}$. Each tangency is a

parabolic point for Γ, and since $S_h S_1 0 = t$, we have

$$\Gamma_\infty = \langle S_h S_1 \rangle \pm \begin{pmatrix} 1 & t\mathbb{Z} \\ & 1 \end{pmatrix},$$

from Proposition 2.9.

We record a computation which will be used at several places:

Lemma 3.5. *Let* $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$.

(i) *If* $c \neq 0$, *then under the Möbius transform* M, *a circle* $C(x + y\mathbf{i}, y)$ *will be mapped to the circle*

$$C\Big(\frac{ax + b}{cx + d} + \frac{r\mathbf{i}}{(cx + d)^2}, \frac{r}{(cx + d)^2} \Big)$$

if $cx + d \neq 0$, *and to the line* $\Im z = 1/2c^2 y$ *if* $cx + d = 0$. *When* $c = 0$, *the image circle is*

$$C\Big(\frac{ax + b}{d}, \frac{r}{d^2} \Big).$$

(ii) *If* $c \neq 0$, *then the lines (degenerate circles)* $C = \mathbb{R} + \mathbf{i}y$ *are mapped to*

$$C\Big(\frac{a}{c} + \mathbf{i}\frac{1}{2c^2 y}, \frac{1}{2c^2 y} \Big),$$

and to the line $\mathbb{R} + a^2 y\mathbf{i}$ *if* $c = 0$.

Proof. For both (i) and (ii), if $c \neq 0$, we use the Bruhat decomposition to write M as

$$M = \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c^{-1} & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix}.$$

For each factor above, the transformation formula is simple. Then the curvature of the image circle is obtained by composing these simple formulae together. The case when $c = 0$ is a simple check. $\qquad\square$

To deduce Theorem 3.1, we need to calculate the contribution from each $C_i \in \mathcal{K}_0$. For $i \neq 1$, write $C_i = C(\alpha_i + r_i \mathbf{i}, r_i)$. From Lemma 3.5 it follows that

$$\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

sends C_i, $i \neq 0, 1$, to a circle of curvature $(c_\gamma \alpha_i + d_\gamma)^2 / r_i$ and the line $C_1 = \mathbb{R} + \mathbf{i}$ to a circle of curvature $2c_\gamma^2$. Therefore, for $i \neq 1$, we need to calculate

$$(4) \qquad \#\Big\{ \gamma \in \Gamma / \Gamma_{\alpha_i} \mid \frac{(c_\gamma \alpha_i + d_\gamma)^2}{r_i} \leq T, \ \frac{a_\gamma \alpha_i + b_\gamma}{c_\gamma \alpha_i + d_\gamma} \in \mathcal{I} \Big\},$$

and for $i = 1$, we need to calculate

$$(5) \qquad \#\Big\{ \gamma \in \Gamma / \Gamma_\infty \mid 2c_\gamma^2 \leq T, \ \frac{a_\gamma}{c_\gamma} \in \mathcal{I} \Big\}.$$

This counting problem is a special case of a theorem of A. Good (see the corollary on [5, p. 119]), which we quote here:[1]

Theorem 3.6 (Good, 1983). *Let Γ be a lattice in $\mathrm{PSL}(2, \mathbb{R})$. Suppose ∞ is a cusp of Γ with stabilizer Γ_∞ and ξ is any cusp of Γ with stabilizer Γ_ξ. Choose M_ξ so that $M_\xi(\xi) = \infty$. Any $\gamma \notin \Gamma_\infty$ can be written uniquely in the form*

$$\pm \begin{pmatrix} 1 & \theta_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -\nu^{-\frac{1}{2}} \\ \nu^{\frac{1}{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & \theta_2 \\ 0 & 1 \end{pmatrix}$$

with $\nu > 0$, so this determines functions θ_1, θ_2, ν on $\mathrm{PSL}(2, \mathbb{R})$. Let $\mathcal{I}, \mathcal{J}$ be two bounded intervals in $\mathbb{R}$. Then as $T \to \infty$,

$$\#\big\{\gamma' \in \Gamma M_\xi^{-1} \mid \nu(\gamma') \leq T,\ \theta_1(\gamma') \in \mathcal{I},\ \theta_2(\gamma') \in \mathcal{J}\big\} \sim \frac{l(\mathcal{I})l(\mathcal{J})T}{\pi\,\mathrm{area}(\Gamma)}.$$

Now apply Theorem 3.6 to deduce Theorem 3.1. For $i \neq 1$, we set $\xi = \alpha_i$ and

$$M_{\alpha_i} = \begin{pmatrix} a_0 & b_0 \\ -s & s\alpha_i \end{pmatrix}, \quad s := \left| \frac{2(\alpha_{i+1} - \alpha_{i-1})}{(\alpha_{i+1} - \alpha_i)(\alpha_{i-1} - \alpha_i)} \right|^{1/2},$$

so that $|M_{\alpha_i}(\alpha_{i+1}) - M_{\alpha_i}(\alpha_{i-1})| = \frac{1}{2}$. Therefore,

$$M_{\alpha_i} \Gamma_{\alpha_i} M_{\alpha_i}^{-1} = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\} := B$$

because $|M_{\alpha_i}(\alpha_{i+1}) - M_{\alpha_i}(\alpha_{i-1})|$ contributes half of the fundamental period of the stabilizer at ∞ of $M_{\alpha_i}\Gamma M_{\alpha_i}^{-1}$; the other half comes from $|M_{\alpha_i}S_{i+1}(\alpha_{i-1}) - M_{\alpha_i}(\alpha_{i+1})|$.

The map $\gamma \to \gamma' = \gamma M_{\alpha_i}^{-1}$ is a bijection of the cosets Γ/Γ_{α_i} and $\Gamma M_{\alpha_i}^{-1}/B$. Write

$$\gamma' = \begin{pmatrix} a_{\gamma'} & b_{\gamma'} \\ c_{\gamma'} & d_{\gamma'} \end{pmatrix}.$$

Then

$$a_{\gamma'} = (a_\gamma \alpha_i + b_\gamma)s, \quad c_{\gamma'} = (c_\gamma \alpha_i + d_\gamma)s$$

(when either α_{i-1} or α_{i+1} is ∞, this expression is interpreted as a limit). Then the condition

$$\frac{a_\gamma \alpha_i + b_\gamma}{c_\gamma \alpha_i + d_\gamma} \in \mathcal{I}$$

from (4) is the same as $\frac{a_{\gamma'}}{c_{\gamma'}} \in \mathcal{I}$ or $\theta_1(\gamma') \in \mathcal{I}$. The condition

$$\frac{(c_\gamma \alpha_i + d_\gamma)^2}{r_i} \leq T$$

is the same as $c_{\gamma'}^2 \leq r_i s^2 T$ or $\nu(\gamma') \leq r_i s^2 T$.

Therefore, counting (4) is the same as counting

$$\#\big\{\gamma' \in \Gamma M_{\alpha_i}^{-1} \mid \nu(\gamma') \leq r_i s^2 T,\ \theta_1(\gamma') \in \mathcal{I},\ \theta_2(\gamma') \in [0, 1)\big\}.$$

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[1]The notation in [5] is particularly cumbersome; the reader is advised to consult Good's index of notations on [5, p. 125] to convert to our more standard usage.
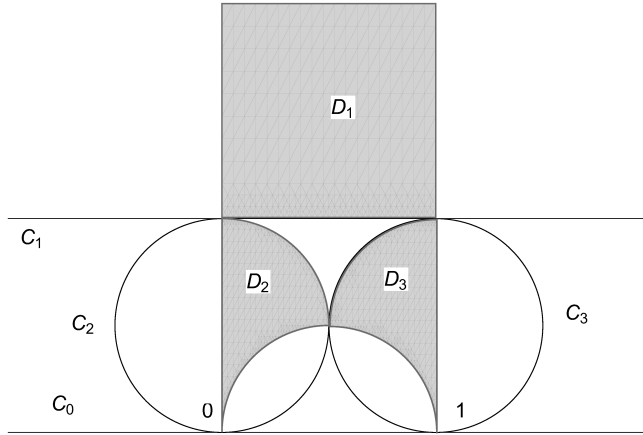
FIGURE 8. The regions D_i (shaded) bounded by the circles C_i in the initial configurations, and by the two geodesics linking α_i to α_{i-1} and to α_{i+1}. Displayed is an initial configuration, generating the classical Ford circles, consisting of two parallel lines $C_0 = \mathbb{R}$ and $C_1 = \mathbb{R} + \mathbf{i}$, and two mutually tangent circles $C_2 = C(\frac{\mathbf{i}}{2}, \frac{1}{2})$ and $C_3 = C(1 + \frac{\mathbf{i}}{2}, \frac{1}{2})$. Here $\alpha_1 = \infty$, $\alpha_2 = 0$, $\alpha_3 = 1$.

Recall that the area of Γ is $2\pi(h - 2)$. Then by setting $\mathcal{J} = [0, 1)$, from Theorem 3.6 we obtain

$$(4) \sim \frac{l(\mathcal{I}) r_i s^2}{2\pi^2 (h - 2)} T \quad \text{as } T \to \infty.$$

We interpret this in conformally invariant terms: Let D_i be the area of the region bounded by C_i and the two geodesics g_{i-1}, g_i (see Figure 8). Then M_{α_i} will translate this region to a triangle with vertices ∞, $M_{\alpha_i}(\alpha_{i+1})$, and $M_{\alpha_i}(\alpha_{i-1})$. Note that $M_{\alpha_i}(\alpha_i + 2r\mathbf{i})$ is a point on the segment connecting $M_{\alpha_i}(\alpha_{i-1})$ and $M_{\alpha_i}(\alpha_{i+1})$, and that $\Im M_{\alpha_i}(\alpha_i + 2r_i\mathbf{i}) = \frac{1}{2s^2 r_i}$. We also have $|M_{\alpha_i}(\alpha_{i+1}) - M_{\alpha_i}(\alpha_{i-1})| = \frac{1}{2}$. Since M_{α_i} is area preserving, we have

$$D_i = |M_{\alpha_i}(\alpha_{i+1}) - M_{\alpha_i}(\alpha_{i-1})| \cdot 2s^2 r_i = s^2 r_i = \left| \frac{2r_i(\alpha_{i+1} - \alpha_{i-1})}{(\alpha_{i+1} - \alpha_i)(\alpha_{i-1} - \alpha_i)} \right|.$$

Hence we find

$$(5) \sim \frac{l(\mathcal{I}) D_i}{2\pi^2 (h - 2)} T \quad \text{as } T \to \infty.$$

The case of $i = 1$ is simpler. Counting (5) is the same as counting

$$(6) \qquad \#\left\{ \gamma \in \Gamma \mid 2c_\gamma^2 \leq T, \ \frac{a_\gamma}{c_\gamma} \in \mathcal{I}, \ \frac{d_\gamma}{c_\gamma} \in \mathcal{J} \right\},$$

where $\mathcal{J}$ is a fundamental period of Γ_∞, the length of which is twice the distance of α_h and α_2, which is $2\alpha_h$. We obtain

$$(6) \sim \frac{l(\mathcal{I})2\alpha_h}{2\pi^2(h-2)}\frac{T}{2} \quad \text{as } T \to \infty.$$

Noting that the area of the region bounded by C_1 and the two geodesics $g_0 = g_h$, g_1 is $D_1 = \alpha_h$, we find

$$(6) \sim \frac{l(\mathcal{I})D_1}{2\pi^2(h-2)}T \quad \text{as } T \to \infty.$$

Adding up the contributions from all $C_i \in \mathcal{K}_0$, we obtain Theorem 3.1, with the constant

$$c_{\mathcal{P},C_0} = \frac{D}{2\pi^2(h-2)}, \quad D = \sum_{i=1}^{h} D_i,$$

where D_i is the hyperbolic area of the region bounded by C_i and the two geodesics g_{i-1} and g_i. Explicitly,

$$(7) \qquad D_1 = \alpha_h, \quad D_i = 2r_i\left|\frac{\alpha_{i+1} - \alpha_{i-1}}{(\alpha_{i+1} - \alpha_i)(\alpha_{i-1} - \alpha_i)}\right|, \quad i \neq 1$$

(when one of $\alpha_{i-1}, \alpha_{i+1}$ is ∞, we take the appropriate limiting value of D_i).

4. The gap distribution of tangencies in circle packings

We now study the gap distribution on C_0. As in Section 3, let $\mathcal{I} \subset C_0$ be an arc (or a bounded interval), and $\mathcal{A}_{T,\mathcal{I}} = \mathcal{A}_T \cap \mathcal{I}$ the tangencies in $\mathcal{I}$ whose circles have curvature at most T. Let $\{x_{T,\mathcal{I}}^i\}$ be the sequence of tangencies in $\mathcal{A}_{T,\mathcal{I}}$ ordered by counter-clockwise direction. The nearest-neighbor gaps, or spacings, between the tangencies are

$$d(x_{T,\mathcal{I}}^i, x_{T,\mathcal{I}}^{i+1})$$

and the mean spacing is

$$\langle d_{T,\mathcal{I}} \rangle := \frac{l(\mathcal{I})}{\#\mathcal{A}_{T,\mathcal{I}}}.$$

We define the gap distribution function on $\mathcal{I}$ to be

$$F_{T,\mathcal{I}}(s) = \frac{1}{\#\mathcal{A}_{T,\mathcal{I}}}\#\left\{x_{T,\mathcal{I}}^i \mid \frac{d(x_{T,\mathcal{I}}^i, x_{T,\mathcal{I}}^{i+1})}{\langle d_{T,\mathcal{I}} \rangle} \leq s\right\}.$$

We will find the limiting gap distribution of $\mathcal{A}_{T,\mathcal{I}}$:

Theorem 4.1. *There exists a continuous piecewise smooth function $F(s)$, which is independent of $\mathcal{I}$ such that*

$$\lim_{T \to \infty} F_{T,\mathcal{I}}(s) = F(s).$$

The limiting distribution $F(s)$ is conformal invariant: let $M, \tilde{\mathcal{P}}, \tilde{C}_0$ be as in Theorem 3.1, and $\tilde{F}$ be the gap distribution function of $\tilde{C}_0$ from $\tilde{\mathcal{P}}$, then $\tilde{F} = F$.
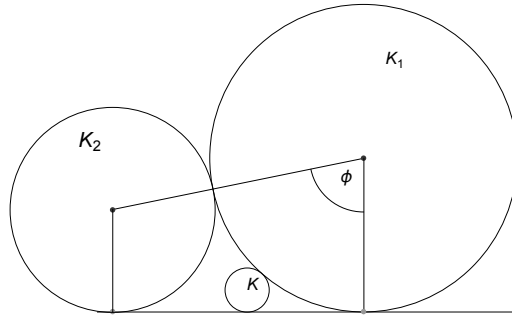
Figure 9. The radius of any circle contained in a triangular gap formed by two mutually tangent circles, both of which are tangent to the real line, is smaller than the radii of these two circles.

The explicit formula for F is given in Theorem 5.5.

This section is devoted to proving Theorem 4.1 in the case of generalized Ford packings.

4.2. **Geometric lemmas.** Recall that we assume the initial configuration $\mathcal{K}$ includes two parallel lines $\mathbb{R}$ and $\mathbb{R} + \mathbf{i}$, two circles tangent to both $\mathbb{R}$ and $\mathbb{R} + \mathbf{i}$ with their tangencies on $\mathbb{R}$ being 0 and $\frac{t}{2}$, respectively, and possibly some circles in between. Our base circle C_0 is $\mathbb{R}$, so $\mathcal{K}_0 = \mathcal{K} \cap \mathcal{P}_0$, and $\mathcal{P}_0$ is the collection of all circles tangent to $\mathbb{R}$.

Lemma 4.3. (i) *Let $K \in \mathcal{P}_0$ lie in a triangular gap $\triangle$ bounded by the real axis $\mathbb{R}$ and two mutually tangent circles $K_1, K_2 \neq \mathbb{R}$. Then both K_1 and K_2 have radius greater than that of K.*

  (ii) *If $C', C'' \in \mathcal{P}_0$ are such that C' lies in a triangular gap $\triangle$ bounded by the real axis, which is disjoint from C'', then there is another circle $C \in \mathcal{P}_0$ which separates C' and C'' and has smaller curvature (i.e. bigger radius) than that of C'.*

Proof. To see (i), we may move the circle K so that it is tangent to both the real line and the larger of the two initial circles K_1 and K_2, say it is K_1. We get a configuration as in Figure 9.

If we denote by r_1 the radius of the larger circle K_1, and by $\phi = \phi(K_1, K_2)$ the angle between the segment joining the centers of the two circles K_1, K_2, and the segment joining the point of tangency of the circle K_1 with real line and with the center of K_1, then a computation shows that the radius of the third mutually tangent circle K_2 is $r(\phi) = (\tan \frac{\phi}{2})^2 r_1$. This is an increasing function of ϕ. Since the angle $\phi(K_1, K)$ is smaller than $\phi(K_1, K_2)$ for any circle K contained in the triangular gap and tangent to both K_1 and the real line, it follows that the circle K has smaller radius than that of K_2 as claimed.
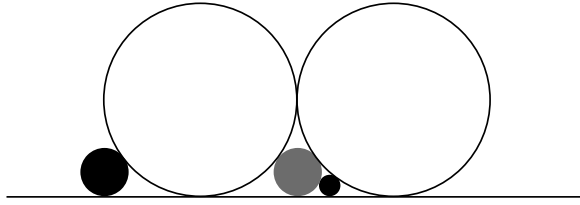
FIGURE 10. If $C', C'' \in \mathcal{P}_0$ (shaded black) are such that C' lies in a triangular gap $\triangle$ bounded by the real axis, with $\triangle$ disjoint from C'', then there is another circle $C \in \mathcal{P}_0$ (shaded gray), which is one of the bounding circles of $\triangle$, which separates C' and C'' and has smaller curvature than that of C'.

For (ii), note that if C'' is disjoint from the triangular gap $\triangle$ containing C', then one of the bounding circles C of $\triangle$ separates C' and C''. By part (i), C has bigger radius than C', see Figure 10. □

The following proposition is a crucial ingredient in understanding the gap distribution.

Proposition 4.4. *For any adjacent pair $\alpha'_1, \alpha'_2 \in \mathcal{A}_T$ with corresponding circles C'_1 and C'_2, there is an element $\gamma \in \Gamma_0$ and $C_i, C_j \in \mathcal{K}_0$ such that $\gamma(C_i) = C'_1$ and $\gamma(C_j) = C'_2$. The element $\gamma \in \Gamma_0$ is unique if C'_1, C'_2 are disjoint, and if they are tangent then the only other element with this property is $\gamma' = \gamma S_{i,j}$, where $S_{i,j} \in \mathcal{S}$ is the reflection in the dual circle corresponding to the triangle formed by C_i, C_j and $\mathbb{R}$.*

Proof. Let $C'_1 = T_1 \ldots T_m C_i$ be the minimal word for C'_1, so that the triangle associated to C'_1 is $\triangle_{C'_1} = T_1 \ldots T_{m-1} \triangle_{T_m}$. Let $C'_2 = T'_1 \ldots T'_n C_j$ be the minimal word for C'_2 and $\triangle_{C'_2} = T'_1 \ldots T'_{n-1} \triangle_{T'_n}$. Let us suppose $n \geq m$. Suppose there exists an integer $k \in [1, m]$ such that $T_k \neq T'_k$. We take k to be minimal, then $T_k \ldots T_m C_i \subset \triangle_{T_k}$ and $T'_k \ldots T'_n C_j \subset \triangle_{T'_k}$, and $\triangle_{T_k}$ and $\triangle_{T'_k}$ are two disjoint sets. Since $T_s = T'_s$ when $1 \leq s \leq k-1$, we see that $T_1 \ldots T_{k-1}$ maps $\triangle_{T_k}$ and $\triangle_{T'_k}$ to two disjoint triangles lying in $\triangle_{T_1}$. These two triangles contain C'_1 and C'_2, respectively. We have a contradiction here because by Lemma 4.3 (ii), there is another circle in $\mathcal{P}_0$ separating C'_1 and C'_2 which has smaller curvature, so C'_1 and C'_2 are not neighbors. Therefore $T_s = T'_s$ when $s \in [1, m]$, and

$$C'_1 = T_1 \ldots T_m C_i, \quad C'_2 = T_1 \ldots T_m T'_{m+1} \ldots T'_n C_j.$$

Now we claim that T'_s fixes C_i when $s = m+1, \ldots, n$. This will show that

$$C'_1 = T_1 \ldots T_m T'_{m+1} \ldots T'_n C_i, \quad C'_2 = T_1 \ldots T_m T'_{m+1} \ldots T'_n C_j$$

so that we may take $\gamma := T'_1 \ldots T'_n$, which satisfies $C'_1 = \gamma C_i$ and $C'_2 = \gamma C_j$ as required.

Otherwise, let $k \in [m+1, n]$ again be the smallest integer such that $T_k' C_i \neq C_i$, then $\triangle_{T_k'}$ contains $T_{k+1}' \ldots T_n' C_j$, and $\triangle_{T_k'}$ and C_i are disjoint. As a result $T_1 \ldots T_{k-1} = T_1' \ldots T_{k-1}'$ maps C_i and $\triangle_{T_k'}$ to two disjoint sets: C_1' and the triangle $\triangle = T_1' \ldots T_{k-1}' \triangle_{T_k}$, the latter of which contains C_2'. Therefore, by Lemma 4.3 (ii), between C_1' and C_2' there must be a circle whose curvature is smaller than that of C_2', so that C_1' and C_2' cannot be adjacent in $\mathcal{A}_T$, contradiction. Hence $T_s' C_i = C_i$ for $s \in [m+1, n]$ and we have constructed $\gamma := T_1' \ldots T_n'$.

To address uniqueness of γ, assume γ' also sends C_i to C_1', C_j to C_2', then $\gamma^{-1}\gamma'$ fixes C_i and C_j simultaneously. Therefore, if C_i and C_j are disjoint, then $\gamma^{-1}\gamma' = \mathrm{id}$; if C_i and C_j are tangent, then $\gamma^{-1}\gamma' = \mathrm{id}$ or $S_{i,j}$, where $S_{i,j} \in \mathcal{S}$ is the reflection corresponding to the triangle formed by C_i, C_j and $\mathbb{R}$. $\qquad \square$

4.5. Determining which cosets of Γ define neighbors in $\mathcal{A}_T$.

We now use Proposition 4.4 to relate the gap distribution problem to a question of counting certain cosets in Γ.

For each pair of circles $C_i, C_j \in \mathcal{K}_0$ in the initial configuration which are tangent to C_0, we will define regions $\Omega_T^{i,j}$ in the right cd-plane $\{(c, d) \mid c \geq 0\}$ as follows.

(a) If $\alpha_i, \alpha_j \neq \infty$, we require

$$(8) \qquad \frac{(c\alpha_i + d)^2}{r_i} \leq T, \quad \frac{(c\alpha_j + d)^2}{r_j} \leq T$$

and in addition:

(i) If C_i, C_j are tangent, then (c, d) satisfies

$$(9) \qquad \begin{cases} \text{if } (c\alpha_i + d)(c\alpha_j + d) < 0, \\ \text{then } 2c^2 > T \text{ and } \dfrac{(c\alpha_k + d)^2}{r_k} > T \text{ for all } k \neq i, j, \infty \end{cases}$$

and

$$(10) \qquad \begin{cases} \text{if } (c\alpha_i + d)(c\alpha_j + d) > 0, \\ \text{then } \dfrac{(c\alpha_{i,j}^{(k)} + d)^2}{r_{i,j}^{(k)}} > T \text{ for all } k \neq i, j. \end{cases}$$

where $S_{i,j} C_k = C(\alpha_{i,j}^{(k)} + \mathbf{i} r_{i,j}^{(k)}, r_{i,j}^{(k)})$, with $S_{i,j}$ being the reflection corresponding to the triangle formed by C_i, C_j and $\mathbb{R}$.

(ii) If C_i and C_j are disjoint (non-tangent), then (c, d) satisfies

$$(11) \qquad \begin{cases} \dfrac{(c\alpha_k + d)^2}{r_k} > T \quad \left(\text{or } 2c^2 > T \text{ if } \alpha_k = \infty\right), \\ \text{if either } \alpha_k \text{ lies between } \alpha_i \text{ and } \alpha_j, \text{ and } (c\alpha_i + d)(c\alpha_j + d) > 0, \\ \text{or } \alpha_k \text{ does not lie between } \alpha_i \text{ and } \alpha_j, \text{ and } (c\alpha_i + d)(c\alpha_j + d) < 0. \end{cases}$$

(b) If one of α_i, α_j is ∞, say $\alpha_j = \infty$, we require

$$(12) \qquad \frac{(c\alpha_i + d)^2}{r_i} \le T, \quad 2c^2 \le T$$

and in addition:

(i) If C_i, C_j are tangent, then (c, d) satisfies

$$(13) \qquad \begin{cases} \text{if } \alpha_i = 0,\, d > 0 \text{ or } \alpha_i = \dfrac{t}{2},\, \dfrac{ct}{2} + d < 0, \\[2mm] \text{then } \dfrac{(c\alpha_k + d)^2}{r_k} > T \text{ for all } k \neq i, j \end{cases}$$

and

$$(14) \qquad \begin{cases} \text{if } \alpha_i = 0,\, d < 0 \text{ or } \alpha_i = \dfrac{t}{2},\, \dfrac{ct}{2} + d > 0, \\[2mm] \text{then } \dfrac{(c\alpha_{i,j}^{(k)} + d)^2}{r_{i,j}^{(k)}} > T \text{ for all } k \neq i, j. \end{cases}$$

(ii) If C_i and C_j are disjoint (non-tangent), then (c, d) satisfies

$$(15) \qquad \begin{cases} \dfrac{(c\alpha_k + d)^2}{r_k} > T \\[2mm] \text{if either } \alpha_k > \alpha_i \text{ and } c\alpha_i + d > 0 \text{ or } \alpha_k < \alpha_j \text{ and } c\alpha_i + d < 0. \end{cases}$$

Note that $\Omega_T^{i,j}$ is a finite union of convex sets and

$$\Omega_T^{i,j} = \sqrt{T}\,\Omega_1^{i,j}.$$

The result is:

Proposition 4.6. *For any $C_i, C_j \in \mathcal{K}_0$ and $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma$, the circles $\gamma(C_i), \gamma(C_i)$ are neighbors in $\mathcal{A}_T$ if and only if $\langle c_\gamma, d_\gamma \rangle$ lies in some $\Omega_T^{i,j}$ in the right half plane, defined by the following conditions:*

(i) *If $\alpha_i, \alpha_j \neq \infty$, then we require (8), with (9) or (10) in the case that C_i, C_j are tangent, and if they are disjoint then we require (11).*
(ii) *If $\alpha_j = \infty$, then we require (12), with (13) or (14) in the case that C_i, C_j are tangent, and if they are disjoint then we require (15).*

Proof. Let $S \in \mathcal{S}$ be the reflection in the line $\mathrm{Re}(z) = t/2$, corresponding to the triangular gap bounded by the lines $\mathbb{R}$, $\mathbb{R} + \mathbf{i}$ and the circle with tangency $\frac{t}{2}$.

From Proposition 4.4, passing to the orientation preserving subgroup $\bar{\Gamma}$, every (unordered) pair of neighbor circles can be expressed as $(\gamma(C_i), \gamma(C_j))$ or $(\gamma S(C_i), \gamma S(C_j))$ for some $\gamma \in \Gamma$ and $C_i, C_j \in \mathcal{K}_0$ with $i \neq j$. The expression is unique when C_i and C_j are disjoint because the simultaneous stabilizer for C_i and C_j is trivial. However, we only need to calculate the contribution to F from (C_i, C_j) under Γ: the contribution from $(S(C_i), S(C_j))$ is the same as

that from (C_i, C_j) because they are mirror symmetric under the reflection of the line $\Re z = \frac{t}{2}$. Or more formally,

$$\Gamma(S(C_i), S(C_j)) = S\Gamma S(S(C_i), S(C_j)) = S\Gamma(C_i, C_j)$$

and S is metric preserving on $\mathbb{R}$.

When C_i and C_j are tangent, then $(S(C_i), S(C_j))$ is also in the orbit of (C_i, C_j) under Γ. In this case, every pair of circles from $\Gamma_0((C_i, C_j))$ can be uniquely expressed as $\gamma(C_i, C_j)$ for some $\gamma \in \Gamma$.

If two neighboring circles in $\mathcal{A}_T$ can be expressed as $\gamma(C_i), \gamma(C_j)$, then we need their curvatures to satisfy

$$\kappa(\gamma(C_i)) \leq T, \quad \kappa(\gamma(C_j)) \leq T.$$

By Lemma 3.5 we have $\kappa(\gamma C_j) = (c\alpha_j + d)^2/r_j$ and so we get (8).

All the circles in between $\gamma(C_i)$ and $\gamma(C_j)$ must have curvatures greater than T. We see from Lemma 4.3 that it suffices to check finitely many circles. A useful fact here is the following:

Observation 4.7. *If neither C_i nor C_j is the horizontal line from $\mathcal{K}_0$, which means $\alpha_i, \alpha_j \neq \infty$, then for another circle $C_l \in \mathcal{K}_0$, $\gamma(C_l)$ is in between $\gamma(C_i)$ and $\gamma(C_j)$ if and only if*

$$\mathrm{Sign}(c_\gamma \alpha_i + d_\gamma)(c_\gamma \alpha_j + d_\gamma) = -\mathrm{Sign}(\alpha_l - \alpha_i)(\alpha_l - \alpha_j).$$

If C_j is the horizontal line, so that $\alpha_j = \infty$, then $\gamma(C_l)$ is in between $\gamma(C_i)$ and $\gamma(C_j)$ if and only if

$$\mathrm{Sign}(c(c\alpha_i + d)) = \mathrm{Sign}(\alpha_l - \alpha_i).$$

Proof. The observation is that given real numbers a and b, a third real c lies between them if and only if

$$\mathrm{Sign}(c - a)(c - b) = -1.$$

Now take the numbers to be $a = \gamma \alpha_i$, $b \gamma \alpha_i$ and $c = \gamma \alpha_l$ and compute

$$\mathrm{Sign}(\gamma \alpha_l - \gamma \alpha_i) \cdot (\gamma \alpha_l - \gamma \alpha_j) = \mathrm{Sign} \frac{(\alpha_l - \alpha_i)(\alpha_l - \alpha_j)}{(c_\gamma \alpha_i + d_\gamma)(c_\gamma \alpha_j + d_\gamma)(c_\gamma \alpha_l + d)^2}$$
$$= -1$$

from which the claim follows.

In the case $\alpha_j = \infty$, we have

$$\mathrm{Sign}(\gamma \alpha_l - \gamma \alpha_i)(\gamma \alpha_l - \gamma \infty) = -\mathrm{Sign} \frac{\alpha_l - \alpha_i}{c(c\alpha_i + d)} = -1$$

which gives the claim. $\qquad\square$

There are two cases:

Case 1: C_i and C_j are tangent. Then α_k does not lie between α_i and α_j, for all k.

If $(c\alpha_i + d)(c\alpha_j + d) < 0$, then the circles C_k for $k \neq i, j$ are mapped by γ to some circles in between $\gamma(C_i)$ and $\gamma(C_j)$, and in this case we need

$$\text{if } (c\alpha_i + d)(c\alpha_j + d) < 0 \text{ then } \kappa(\gamma(C_k)) > T \text{ for all } k \neq i, j.$$

Using Lemma 3.5 which gives $\kappa(\gamma(C_k)) = (c\alpha_k + d)^2/r_k$, we obtain condition (9).

If $(c\alpha_i + d)(c\alpha_j + d) > 0$, then from Observation 4.7 none of the circles γC_k lie between γC_i and γC_j, but the circles $S_{i,j} C_k$ for $k \neq i, j$ are mapped by γ to some circles in between $\gamma(C_i)$ and $\gamma(C_j)$ (recall that $S_{i,j} \in \mathcal{S}$ is the reflection corresponding to the triangle formed by C_i, C_j and $\mathbb{R}$). In this situation we need

$$\text{if } (c\alpha_i + d)(c\alpha_j + d) > 0 \text{ then } \kappa(\gamma(S_{i,j} C_k)) > T \text{ for all } k \neq i, j.$$

By Lemma 3.5, if $S_{i,j} C_k = C(\alpha_{i,j}^{(k)} + \mathbf{i} r_{i,j}^{(k)}, r_{i,j}^{(k)})$, we have

$$\kappa(\gamma S_{i,j} C_k) = \frac{(c\alpha_{i,j}^{(k)} + d)^2}{r_{i,j}^{(k)}},$$

which gives condition (10).

Case 2: C_i and C_j are not tangent. We need to make sure that $\gamma(C_k)$ has curvature $> T$ whenever $\gamma(C_k)$ is in between $\gamma(C_i)$ and $\gamma(C_j)$.

Again from Observation 4.7, if $(c\alpha_i + d)(c\alpha_j + d) > 0$, then the circles whose tangencies lie between α_i and α_j will be mapped by γ to some circles between $\gamma(C_i)$ and $\gamma(C_j)$; if $(c\alpha_i + d)(c\alpha_j + d) < 0$, then the circles whose tangencies do not lie between α_i and α_j will be mapped in between $\gamma(C_i)$ and $\gamma(C_j)$. Thus we need $\kappa(\gamma(C_k)) > T$ if either

$$\alpha_k \text{ lies between } \alpha_i, \alpha_j \text{ and } (c\alpha_i + d)(c\alpha_j + d) > 0$$

or

$$\alpha_k \text{ does not lie between } \alpha_i, \alpha_j \text{ and } (c\alpha_i + d)(c\alpha_j + d) < 0.$$

This gives condition (11) once we use $\kappa(\gamma(C_k)) = (c\alpha_k + d)^2/r_k$ (Lemma 3.5).

From Lemma 4.3, once (10) or (9) hold when C_i, C_j are tangent, or (11) is satisfied if C_i and C_j are disjoint, then all circles between $\gamma(C_i)$ and $\gamma(C_j)$ have curvatures greater than T. Putting the above together, we have established Proposition 4.6. □

4.8. **The contribution of each pair of circles to the gap distribution function.** We need to calculate the contribution from each pair of circles (C_i, C_j) to the proportion $F_{T,\mathcal{I}}(s)$ of gaps of size at most s, and we denote this quantity by $F_{T,\mathcal{I}}^{i,j}(s)$:

$$(16) \qquad F_{T,\mathcal{I}}^{i,j}(s) = \frac{\#\{(x_{T,\mathcal{I}}^l, x_{T,\mathcal{I}}^{l+1}) \in \Gamma(\alpha_i, \alpha_j) \mid \frac{d(x_{T,\mathcal{I}}^l, x_{T,\mathcal{I}}^{l+1})}{\langle d_{T,\mathcal{I}} \rangle} \leq s\}}{\#\mathcal{A}_{T,\mathcal{I}}}.$$

We will later show that these have a limit as $T \to \infty$:

$$(17) \qquad F^{i,j}(s) := \lim_{T \to \infty} F_{T,\mathcal{I}}^{i,j}(s).$$

A direct computation shows that the distance between $\gamma(C_i)$ and $\gamma(C_j)$ is given by $|\frac{\alpha_i - \alpha_j}{(c\alpha_i + d)(c\alpha_j + d)}|$. The average gap is asymptotically $1/c_{\mathcal{P},C_0}T$. Therefore, the relative gap condition

$$\frac{|\frac{\alpha_i - \alpha_j}{(c\alpha_i + d)(c\alpha_j + d)}|}{1/c_{\mathcal{P},C_0}T(1 + o(1))} \leq s$$

in the definition of $F_{T,\mathcal{I}}^{i,j}$ can also be written as

$$(18) \qquad |(c\alpha_i + d)(c\alpha_j + d)| \geq \frac{|(\alpha_i - \alpha_j)|c_{\mathcal{P},C_0}T}{s}(1 + o(1)).$$

We will work with a simpler condition

$$(19) \qquad |(c\alpha_i + d)(c\alpha_j + d)| \geq \frac{|(\alpha_i - \alpha_j)|c_{\mathcal{P},C_0}T}{s}$$

which will finally lead to a limiting gap distribution function $F(s)$ which is continuous. Then in retrospect working with (18) will lead to the same limiting function $F(s)$, by the continuity of F.

If one of α_i and α_j is ∞ (say $\alpha_j = \infty$), the above is changed to

$$(20) \qquad c|(c\alpha_i + d)| \geq \frac{c_{\mathcal{P},C_0}T}{s}.$$

We define a region $\Omega_T^{i,j}(s)$ to be the elements $(c, d) \in \Omega_T^{i,j}$ satisfying (19) (or (20)). Then we have found that

$$(21) \quad F_{T,\mathcal{I}}^{i,j}(s)$$
$$= \frac{1}{\#\mathcal{A}_{T,\mathcal{I}}}\#\left\{\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma \mid \gamma\alpha_i, \gamma\alpha_j \in \mathcal{I}, (c_\gamma, d_\gamma) \in \Omega_T^{i,j}(s)\right\}.$$

The region $\Omega_T^{i,j}(s)$ is compact because condition (8) already gives a compact region bounded by two sets of parallel lines with different slopes. Note also that $\Omega_T^{i,j}(s)$ is a finite union of convex sets.

From the defining equation, it is clear that the region $\Omega_T^{i,j}(s)$ grows homogeneously with respect to T:

$$\Omega_T^{i,j}(s) = \sqrt{T}\Omega_1^{i,j}(s).$$

4.9. **Strong repulsion.** As a consequence of the analysis above, we find that the normalized gaps in our circle packings are all bounded away from zero:

Corollary 4.10. *There is some $\delta = \delta(\mathcal{P}, C_0) > 0$ so that all gaps satisfy*

$$d(x_T^i, x_T^{i+1})c_{\mathcal{P},C_0}T \geq \delta > 0.$$

Thus the limiting distribution $F(s)$ is supported away from the origin. This is a very strong form of level repulsion, familiar from the theory of the Farey sequence.

Proof. To prove the assertion, note that we expressed the distribution function F_T as a sum over all (unordered) pairs of distinct circles from the initial configuration $C_i, C_j \in \mathcal{K}_0$:

$$F_T(s) = \sum_{C_i, C_j \text{ tangent}} F_T^{i,j}(s) + \sum_{C_i, C_j \text{ disjoint}} F_T^{i,j}(s),$$

where

$$F_{T,\mathcal{I}}^{i,j}(s) = \frac{1}{\#\mathcal{A}_{T,\mathcal{I}}} \# \left\{ \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma \mid \frac{(c_\gamma, d_\gamma)}{\sqrt{T}} \in \Omega_1^{i,j}(s) \right\}.$$

Here $\Omega_1^{i,j}(s)$ is the region in the plane of points lying in the compact sets $\Omega_1^{i,j}$ of Section 4.5, satisfying $|Q_{i,j}(x,y)| > c_{\mathcal{P},C_0}/s$:

$$\Omega_1^{i,j}(s) = \left\{ (x,y) \in \Omega_1^{i,j} \mid |Q_{i,j}(x,y)| > \frac{c_{\mathcal{P},C_0}}{s} \right\}$$

with

$$Q_{i,j}(x,y) = \begin{cases} (x\alpha_i + y)(x\alpha_j + y) & \text{if } i, j \neq \infty, \\ x(x\alpha_j + y) & \text{if } i = \infty. \end{cases}$$

The sets $\Omega_1^{i,j}$ are compact, and therefore the functions $|Q_{i,j}(x,y)|$ are bounded on them. Thus if

$$\delta(\mathcal{P}, C_0) = \frac{c_{\mathcal{P},C_0}}{\max_{i,j} \max(|Q_{i,j}(x,y)| : (x,y) \in \Omega_1^{i,j})}$$

then $\delta > 0$ and $F_T(s) = 0$ if $0 \leq s \leq \delta$. $\qquad \square$

In the examples of Section 7, we have $\delta = 3/\pi^2 = 0.303964$ for the classical Apollonian packing in Section 7.2 (which reduces to the Farey sequence by conformal invariance), $\delta = 2\sqrt{2}/\pi^2 = 0.28658$ for the Apollonian-3 packing in Section 7.3, and $\delta = 5(1 + \sqrt{5})/(6\pi^2) = 0.273235$ for the Apollonian-9 packing in Section 7.4.

5. The limiting distribution $F(s)$

5.1. **Using Good's theorem.** We now pass to the limit $T \to \infty$, by relating the counting problem encoded in our formula (21) for $F_{T,\mathcal{I}}^{i,j}(s)$ to the area of the region $\Omega_1^{i,j}(s)$. This is done via an equidistribution theorem of A. Good. To formulate it, recall the Iwasawa decomposition

$$\mathrm{PSL}(2, \mathbb{R}) = N^+ A K,$$

where

$$N^+ = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}, \quad A = \left\{ \begin{pmatrix} y^{-\frac{1}{2}} & \\ & y^{\frac{1}{2}} \end{pmatrix} \mid y > 0 \right\},$$

and

$$K = \mathrm{PSO}_2 = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mid \theta \in [0, \pi) \right\}.$$

We can uniquely write any $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ as

$$(22) \qquad \gamma = \begin{pmatrix} 1 & x(\gamma) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y(\gamma)^{-\frac{1}{2}} & \\ & y(\gamma)^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta(\gamma) & -\sin\theta(\gamma) \\ \sin\theta(\gamma) & \cos\theta(\gamma) \end{pmatrix}.$$

We have the following joint equidistribution according to this decomposition, which is another special case of Good's theorem (see the corollary on [5, p. 119]):[2]

Theorem 5.2 (Good, 1983). *Let $\mathcal{I}$ be an bounded interval in $\mathbb{R}$, and let $\mathcal{J}$ be an interval in $[0, \pi)$. Then, as $T \to \infty$,*

$$\#\big\{\gamma \in \Gamma \mid x(\gamma) \in \mathcal{I},\ y(\gamma) < T,\ \theta(\gamma) \in \mathcal{J}\big\} \sim \frac{1}{\mathrm{area}(\Gamma)} l(\mathcal{I}) \frac{l(\mathcal{J})}{\pi} T.$$

Theorem 5.2 allows us to prove the following:

Proposition 5.3. *For a "nice" subset $\Omega \subset \{(c, d) \mid c \geq 0\}$,*

$$\#\left\{\gamma = \begin{pmatrix} * & * \\ c_\gamma & d_\gamma \end{pmatrix} \in \Gamma_\infty \setminus \Gamma \mid x(\gamma) \in \mathcal{I},\ (c_\gamma, d_\gamma) \in \sqrt{T}\Omega\right\} \sim \frac{2l(\mathcal{I})m(\Omega)}{\pi\,\mathrm{area}(\Gamma)} T$$

as $T \to \infty$, where m is the standard Lebesgue measure in $\mathbb{R}^2$. Here "nice" means bounded, convex with piecewise smooth boundary.

Proof. We note that if $\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$ then the Iwasawa decomposition (22) gives $d = y^{1/2}\cos\theta$, $c = y^{1/2}\sin\theta$ so that $(y^{1/2}, \theta)$ are polar coordinates in the (d, c) plane.

We prove Proposition 5.3 in the special case that Ω is bounded by two continuous and piecewise smooth curves $r_1(\theta), r_2(\theta)$ with $\theta \in [\theta_1, \theta_2]$, and $r_2(\theta) \geq r_1(\theta)$. Equality is obtained only if $\theta = \theta_1$ or θ_2. This special case suffices for what we need for $\Omega = \Omega_1^{i,j}(s)$. This is essentially a Riemann sum argument. First we can express $m(\Omega)$ as

$$m(\Omega) = \frac{1}{2}\int_{\theta_1}^{\theta_2} r_2^2(\theta) - r_1^2(\theta)d\theta.$$

Divide the interval $I = [\theta_1, \theta_2]$ into n equal subintervals $\{I_i \mid i = 1, \ldots, n\}$. For each subinterval I_i, pick $\theta_{1,i}^+, \theta_{1,i}^-, \theta_{2,i}^+, \theta_{2,i}^-$ at which r_1 and r_2 achieve their maximum and minimum, respectively. Let Ω_n^+ be the union of truncated sectors $I_i \times [r_1(\theta_{1,i}^-), r_2(\theta_{2,i}^+)]$, and Ω_n^- the union of $I_i \times [r_1(\theta_{1,i}^+), r_2(\theta_{2,i}^-)]$. We have

$$\Omega_n^- \subseteq \Omega \subseteq \Omega_n^+$$

and

$$\lim_{n \to \infty} m(\Omega_n^-) = \lim_{n \to \infty} m(\Omega_n^+) = m(\Omega).$$

We notice that the statement in Proposition 5.3 satisfies finite additivity. From Theorem 5.2, we know that Proposition 5.3 holds for sectors, thus it holds for a finite union of truncated sectors, which is the case for Ω_n^+ and Ω_n^-. Letting n go to infinity, we prove Proposition 5.3 for Ω. $\qquad\square$

---

[2]We state the result slightly differently than in [5]: We use $\mathrm{PSL}(2, \mathbb{R})$ instead Good's original form in $\mathrm{SL}(2, \mathbb{R})$; his parametrization for K differs from ours.

5.4. **The formula for $F(s)$.** We can now state our main result, the formula for the limiting gap distribution $F(s)$, along the way proving Theorem 4.1. We keep our previous notation.

Theorem 5.5. *For any interval $\mathcal{I} \subset C_0$, $\lim_{T\to\infty} F_{T,\mathcal{I}}(s) = F(s)$, where*

$$(23) \qquad F(s) = \frac{2}{D} \Big(\sum_{C_i, C_j \text{ tangent}} m(\Omega_1^{i,j}(s)) + 2 \sum_{C_i, C_j \text{ disjoint}} m(\Omega_1^{i,j}(s)) \Big)$$

with D given in (3).

Proof. We saw that

$$F_{T,\mathcal{I}}(s) = \sum_{C_i, C_j \text{ tangent}} F_{T,\mathcal{I}}^{i,j}(s) + 2 \sum_{C_i, C_j \text{ disjoint}} F_{T,\mathcal{I}}^{i,j}(s)$$

and we need to estimate $F_{T,\mathcal{I}}^{i,j}$. From Proposition 4.4 and Proposition 4.6 we can rewrite $F_{T,\mathcal{I}}^{i,j}$ as

$$\frac{\#\{\gamma \in \Gamma \mid \gamma\alpha_i, \gamma\alpha_j \in \mathcal{I}, (c_\gamma, d_\gamma) \in \Omega_T^{i,j}\}}{\#\mathcal{A}_{T,\mathcal{I}}}.$$

In terms of the Iwasawa coordinates (22), the condition $\gamma(\alpha_i), \gamma(\alpha_j) \in \mathcal{I}$ is essentially equivalent to $x(\gamma) \in \mathcal{I}$, when $y(\gamma)$ tends to $+\infty$, as we shall explain now. Writing $x = x(\gamma)$, $y = y(\gamma)$ and $\theta = \theta(\gamma)$, we have

$$\gamma(\alpha) = x + \frac{1}{y} \cdot \frac{\alpha \cos\theta - \sin\theta}{\alpha \sin\theta + \cos\theta}$$

and since $y = y(\gamma) \gg 1$, this will essentially be $x(\gamma)$ provided $|\alpha \sin\theta + \cos\theta|$ is bounded away from zero.

We first excise small sectors B_β of angle β containing as bisectors those θ's such that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \alpha_* = \infty,$$

where $* = i$ or j. Then for any $\epsilon > 0$, there exists T_0 such that if $(c_\gamma, d_\gamma) \in \Omega_T^{i,j}$ and $y(\gamma) > T_0$, then

$$\left| \frac{1}{y} \cdot \frac{\alpha_* \cos\theta - \sin\theta}{\alpha_* \sin\theta + \cos\theta} \right| < \epsilon.$$

Set

$$\tilde{\Omega} := \Omega_T^{i,j}(s) \setminus (\Omega_{T_0}^{i,j}(s) \cup B_\beta)$$

so that the area of $\tilde{\Omega}$ differs from that of $\Omega_T^{i,j}(s)$ by at most $O(\epsilon T)$. Then there are two intervals $\mathcal{I}_\epsilon^-, \mathcal{I}_\epsilon^+$ such that $\mathcal{I}_\epsilon^- \subset \mathcal{I} \subset \mathcal{I}_\epsilon^+$, and

$$l(\mathcal{I} - \mathcal{I}_\epsilon^-), l(\mathcal{I}_\epsilon^+ - \mathcal{I}) < 2\epsilon.$$

and so that

$$\begin{aligned} \#\big\{\gamma \mid x(\gamma) \in \mathcal{I}_\epsilon^-, (c_\gamma, d_\gamma) \in \tilde{\Omega}\big\} &\leq \#\big\{\gamma \mid \gamma\alpha_* \in \mathcal{I}, (c_\gamma, d_\gamma) \in \tilde{\Omega}\big\} \\ &\leq \#\big\{\gamma \mid x(\gamma) \in \mathcal{I}_\epsilon^+, (c_\gamma, d_\gamma) \in \tilde{\Omega}\big\}. \end{aligned}$$

Applying Proposition 5.3, which approximates the above cardinalities by

$$\frac{2l(\mathcal{I}_\epsilon^\pm)m(\tilde{\Omega})}{\pi \cdot 2\pi(h-2)} = \frac{2l(\mathcal{I})m(\Omega_1^{i,j}(s))T}{\pi \cdot 2\pi(h-2)} + O(\epsilon T)$$

(recall area$(\Gamma) = 2\pi(h-2)$), and dividing by

$$\#\mathcal{A}_{T,\mathcal{I}} \sim l(\mathcal{I})c_{\mathcal{P},C_0}T = l(\mathcal{I})\frac{D}{2\pi^2(h-2)}T,$$

we obtain

$$F_{T,\mathcal{I}}^{i,j}(s) = \frac{2m(\Omega_1^{i,j}(s))}{D}(1+o(1))(1+O(\epsilon)) + O(\beta).$$

Letting $\beta, \epsilon \to 0$, we get

(24)
$$\lim_{T\to\infty} F_{\mathcal{I},T}^{i,j}(s) = \frac{2}{D}m(\Omega_1^{i,j}(s)) := F^{i,j}(s),$$

with D as in (3), and $m(\Omega_1^{i,j}(s))$ being a piecewise smooth, continuous function of s. Summing over all pairs of circles in $\mathcal{K}_0$, we get

$$F(s) = \sum_{C_i,C_j \text{ tangent}} F^{i,j}(s) + 2 \sum_{C_i,C_j \text{ disjoint}} F^{i,j}(s)$$

$$= \frac{2}{D}\Big(\sum_{C_i,C_j \text{ tangent}} m(\Omega_1^{i,j}(s)) + 2 \sum_{C_i,C_j \text{ disjoint}} m(\Omega_1^{i,j}(s)) \Big).$$

The reason there is an extra factor of 2 for C_i, C_j disjoint is because $\Gamma(C_i, C_j)$ only parametrizes half of the gaps formed by the Γ_0-orbits of C_i, C_j, and the contribution from $S(C_i)$, $S(C_j)$ is identical to that from C_i, C_j. $\square$

6. Conformal invariance

Let $M \in \mathrm{SL}(2,\mathbb{C})$ be a Möbius transformation. In this section we show that if a circle packing $\mathcal{P}$ and a circle C_0 from $\mathcal{P}$ satisfy Theorem 3.1 with some constant $c_{\mathcal{P},C_0}$, and Theorem 4.1 with some piecewise smooth continuous function F, then the packing $M(\mathcal{P})$ and the circle $M(C_0)$ also satisfy Theorem 3.1 and Theorem 4.1 with the same constant and distribution function.

In fact, we will give a more refined statement: For any pair of distinct circles $C_i, C_j \in \mathcal{K}_0$ in the initial configuration, the densities $F_{\mathcal{I}}^{i,j}(s)$ defined in (16), (17) are conformally invariant. This is particularly useful in reducing the computations of the limiting densities in the examples of Section 7 to manageable length. The argument itself is routine: The claims are obvious if M is a dilation, and are proved for general $M \in \mathrm{SL}(2,\mathbb{C})$ by localizing.

Theorem 6.1. *Let $\mathcal{I} \subset C_0$ be an arc (if C_0 is a line then take a bounded interval), and $M \in \mathrm{SL}(2,\mathbb{C})$. Assume there exist $0 < b < B$ such that $b < |M'(x)| < B$ for any $x \in \mathcal{I}$.*

(i) *For the packing $M(\mathcal{P})$ and the base circle $M(C_0)$,*

$$\#\mathcal{A}_{M(\mathcal{I}),T} \sim c_{\mathcal{P},C_0} \cdot l(M(\mathcal{I})) \cdot T, \quad as \ T \to \infty.$$

(ii) *For any pair of distinct circles $C_i, C_j \in \mathcal{K}_0$ in the initial configuration, the densities $F_{\mathcal{I}}^{i,j}(s)$ are conformally invariant:*

$$F_{T,\mathcal{I}}^{i,j}(s) \sim F_{T,\mathcal{M}(\mathcal{I})}^{i,j}(s), \quad as \ T \to \infty,$$

where on the right-hand side, $F_{T,\mathcal{M}(\mathcal{I})}^{i,j}(s)$ refers to the gaps associated with the pair of circles $M(C_i)$, $M(C_j)$ and the packing $M(\mathcal{P})$ with base circle $M(C_0)$.

(iii) *As a consequence, for all M and subarcs $\mathcal{I} \subset C_0$,*

$$\lim_{T \to \infty} F_{M(\mathcal{I}),T}(s) = F(s).$$

Proof. We use the notation O and o throughout the proof related to the asymptotic growth as $T \to \infty$. Without further explanation, all the implied constants depend at most on M and $\mathcal{I}$.

(i) We first show that for a circle C with tangency α on $\mathcal{I}$, the curvature of $M(C)$ satisfies

$$(25) \qquad \kappa(M(C)) = \frac{\kappa(C)}{|M'(\alpha)|} + O_{M,\mathcal{I}}(1).$$

Choose a parametrization $z(t) = x(t) + \mathbf{i}y(t)$ for C in a neighborhood of α, with $z(0) = \alpha$. Then the curvature is given by

$$\kappa(C) = \frac{|\Im(\overline{z'(0)}z''(0))|}{|z'(0)|^3}.$$

Therefore, a direct computation shows that

$$\kappa(M(C)) = \frac{|\Im(\overline{(Mz)'(0)}(Mz)''(0))|}{|(Mz)'(0)|^3} = \frac{\kappa(C)}{|M'(\alpha)|} + O\Big(\Big|\frac{M''(\alpha)}{M'(\alpha)^2}\Big|\Big)$$

which gives (25).

We divide the arc $M(\mathcal{I})$ on $M(C_0)$ into u equal pieces $\mathcal{J}_1, \ldots, \mathcal{J}_u$ with their preimages $\mathcal{I}_1, \ldots, \mathcal{I}_u$ on C_0. We pick a point α_i from the interval $\mathcal{I}_i$ for each i, and let $\beta_i = M(\alpha_i)$. We have

$$(26) \qquad l(\mathcal{I}_i) = \frac{l(\mathcal{J}_i)}{|M'(\alpha_i)|}\Big(1 + O\Big(\frac{1}{u}\Big)\Big).$$

For any circle C on $\mathcal{J}_i$, we have

$$\kappa(M^{-1}(C)) = |M'(\alpha_i)|\kappa(C)\Big(1 + O\Big(\frac{1}{u}\Big)\Big).$$

Therefore, there exist $c_1, c_2 > 0$ such that

$$(27) \qquad \mathcal{A}_{\mathcal{I}_i, T|M'(\alpha_i)|(1-\frac{c_1}{u})} \subset M^{-1}(\mathcal{A}_{\mathcal{J}_i, T}) \subset \mathcal{A}_{\mathcal{I}_i, T|M'(\alpha_i)|(1+\frac{c_2}{u})}.$$

Dividing the above expression by T, and letting $T \to \infty$, we obtain

$$
c_{\mathcal{P},C_0} l(\mathcal{I}_i)|M'(\alpha_i)|\Big(1 - \frac{c_1}{u}\Big) \le \liminf_{T \to \infty} \frac{\#\mathcal{A}_{\mathcal{J}_i,T}}{T}
$$
$$
\le \limsup_{T \to \infty} \frac{\#\mathcal{A}_{\mathcal{J}_i,T}}{T}
$$
$$
\le c_{\mathcal{P},C_0} l(\mathcal{I}_i)|M'(\alpha_i)|\Big(1 + \frac{c_2}{u}\Big).
$$

Replace $l(\mathcal{I}_i)$ by (26), sum over all i, and let $u \to \infty$ to obtain

$$
\lim_{T \to \infty} \frac{\#\mathcal{A}_{M(\mathcal{I}),T}}{T} = c_{\mathcal{P},C_0} l(M(\mathcal{I})).
$$

Thus we prove the conformal invariance of the density of tangencies.

(ii) Now we show that for each pair of circles $C_p, C_q \in \mathcal{K}_0$,

$$
\lim_{T \to \infty} F_{M(\mathcal{I}),T}^{p,q}(s) = F^{p,q}(s).
$$

Let $\{y_T^{j,i}\}$ be an ordered sequence of $\mathcal{A}_{M(\mathcal{I}_i),T}$.

Let $\{x_{T|M'(\alpha_i)|}^{k,i}\}$ be the sequence of $\mathcal{A}_{\mathcal{I},|M'(\alpha_i)|T}$ ordered in a way such that $M(\{x_{T|M'(\alpha_i)|}^{k,i}\})$ has the same orientation as $\{y_T^{j,i}\}$.

Let $\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^{p,q}$ be the subset of $\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}$ consisting of those $x_{|M'(\alpha_i)|T}^{j,i}$ such that

$$
(x_{|M'(\alpha_i)|T}^{j,i}, x_{|M'(\alpha_i)|T}^{j+1,i}) \in \Gamma(\alpha_p, \alpha_q).
$$

Let $\mathcal{A}_{M(\mathcal{I}_i),T}^{p,q}$ be the subset of $\mathcal{A}_{M(\mathcal{I}_i),T}$ consisting of those $y_T^{j,i}$ such that

$$
(y_T^{j,i}, y_T^{j+1,i}) \in M\Gamma(\alpha_p, \alpha_q).
$$

Then

$$
F_{M(\mathcal{I}),T}^{p,q}(s) = \frac{\sum_{i=1}^{u} \#\{y_T^{j,i} \in \mathcal{A}_{M(\mathcal{I}_i),T}^{p,q} \mid \frac{d(y_T^{j,i},y_T^{j+1,i})}{l(M(\mathcal{I}))/\#\mathcal{A}_{M(\mathcal{I}),T}} \le s\}}{\sum_{i=1}^{u} \#\mathcal{A}_{M(\mathcal{I}_i),T}}.
$$

From (27), the symmetric difference

$$
\#\big(M^{-1}(\mathcal{A}_{M(\mathcal{I}_i),T}) \bigtriangleup \mathcal{A}_{\mathcal{I}_i,|M'(\alpha)|T}\big) = O\Big(\frac{T}{u^2}\Big).
$$

Therefore, if we let

$$
\mathcal{A}_{M(\mathcal{I}_i),T}^0 = \big\{y_T^{j,i} \mid M^{-1}(y_T^{j,i}) \text{ and } M^{-1}(y_T^{j+1,i}) \text{ are not neighbors}
$$
$$
\text{in } \mathcal{A}_{\mathcal{I}_i,|M'(\alpha)|T}\big\},
$$
$$
\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^0 = \big\{x_T^{k,i} \mid M(x_T^{j,i}) \text{ and } M(x_{|M'(\alpha_i)|T}^{k+1,i}) \text{ are not neighbors}
$$
$$
\text{in } \mathcal{A}_{\mathcal{I}_i,T}\big\},
$$

then

$$
\#\mathcal{A}_{M(\mathcal{I}_i),T}^0 = O\Big(\frac{T}{u^2}\Big) \quad \text{and} \quad \#\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^0 = O\Big(\frac{T}{u^2}\Big).
$$

Let $\{y_T^{j_v,i}\}$ and $\{x_{M'(\alpha_i)T}^{k_w,i}\}$ be the ordered sequence from $\mathcal{A}_{M(\mathcal{I}_i),T} \setminus \mathcal{A}_{M(\mathcal{I}_i),T}^0$ and $\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T} \setminus \mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^0$, respectively. Then M is a bijection on these two sequences. Let us say

$$M(x_{|M'(\alpha_i)|T}^{k_w,i}) = y_T^{j_v,i}.$$

Therefore,

$$\#\left\{ y_T^{j_v,i} \in \mathcal{A}_{M(\mathcal{I}_i),T}^{p,q} \setminus \mathcal{A}_{M(\mathcal{I}_i),T}^0 \mid \frac{d(y_T^{j_v,i}, y_T^{j_v+1,i})}{l(M(\mathcal{I}_i))/\#\mathcal{A}_{M(\mathcal{I}_i),T}} \leq s \right\}$$
$$= \#\left\{ y_T^{j_v,i} \in \mathcal{A}_{M(\mathcal{I}_i),T}^{p,q} \mid \frac{d(y_T^{j_v,i}, y_T^{j_v+1,i})}{l(M(\mathcal{I}_i))/\#\mathcal{A}_{M(\mathcal{I}_i),T}} \leq s \right\} + O\left(\frac{T}{u^2}\right)$$

and

$$\#\left\{ x_T^{k_w,i} \in \mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^{p,q} \setminus \mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^0 \mid \frac{d(x_T^{k_w,i}, y_T^{k_w+1,i})}{l(M(\mathcal{I}_i))/\#\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}} \leq s \right\}$$
$$= \#\left\{ y_T^{j_v,i} \in \mathcal{A}_{M(\mathcal{I}_i),T}^{p,q} \mid \frac{d(y_T^{j_v,i}, y_T^{j_v+1,i})}{l(M(\mathcal{I}_i))/\#\mathcal{A}_{M(\mathcal{I}_i),T}} \leq s \right\} + O\left(\frac{T}{u^2}\right).$$

Now

$$(28) \qquad d(y_T^{j_v,i}, y_T^{j+1,i}) = |M'(\alpha_i)| d(x_{|M'(\alpha_i)|T}^{k_w,i}, x_{|M'(\alpha_i)|T}^{k_w+1,i})\left(1 + O\left(\frac{1}{u}\right)\right),$$

$$(29) \qquad \frac{l(M(\mathcal{I}_i))}{\#\mathcal{A}_{M(\mathcal{I}_i),T}} = \frac{1}{c_{\mathcal{P},C_0}T}\left(1 + o(1)\right),$$

$$(30) \qquad \frac{l(I)}{\#\mathcal{A}_{\mathcal{I},|M'(\alpha_i)|T}} = \frac{1}{c_{\mathcal{P},C_0}|M'(\alpha_i)|T}\left(1 + o(1)\right).$$

Combining (28), (29), (30), we see that there exist c_3, c_4 which only depend on M and $\mathcal{I}$, and for any arbitrary small number ϵ_1, ϵ_2, there exists $T(\epsilon_1, \epsilon_2)$, such that when $T > T(\epsilon_1, \epsilon_2)$,

$$\#\left\{ x_{|M'(\alpha)|T}^{k_w,i} \in \mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^{p,q} \mid \frac{d(x_{|M'(\alpha)|T}^{k_w,i}, x_{|M'(\alpha)|T}^{k_w+1,i})}{l(\mathcal{I})/\#\mathcal{A}_{\mathcal{I},|M'(\alpha_i)|T}} \leq s(1-\epsilon_1)\left(1 - \frac{c_3}{u}\right) \right\}$$
$$\leq \#\left\{ y_T^{j_v,i} \in \mathcal{A}_{M(\mathcal{I}_i),T}^{p,q} \mid \frac{d(y_T^{j_v,i}, y_T^{j_v+1,i})}{l(M(\mathcal{I}))/\#\mathcal{A}_{M(\mathcal{I}),T}} \leq s \right\}$$
$$\leq \#\left\{ x_{|M'(\alpha)|T}^{k_w,i} \in \mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}^{p,q} \mid \frac{d(x_{|M'(\alpha)|T}^{k_w,i}, x_{|M'(\alpha)|T}^{k_w+1,i})}{l(\mathcal{I})/\#\mathcal{A}_{\mathcal{I},|M'(\alpha_i)|T}} \leq s(1+\epsilon_2)\left(1 + \frac{c_4}{u}\right) \right\}.$$

We also have

$$\#\mathcal{A}_{M(\mathcal{I}_i),T} = \#\mathcal{A}_{\mathcal{I}_i,|M'(\alpha_i)|T}\left(1 + O\left(\frac{1}{u}\right)\right).$$

Therefore, there exist $c_5, c_6 > 0$, such that

$$F^{p,q}_{\mathcal{I},|\mathcal{M}'(\alpha_i)|T}\Big(s(1-\epsilon_1)\Big(1-\frac{c_3}{u}\Big)\Big) - \frac{c_5}{u}$$

$$\leq \frac{\#\{y^{j,i}_T \in \mathcal{A}^{p,q}_{M(\mathcal{I}_i),T} \mid \frac{d(y^{j,i}_T, y^{j+1,i}_T)}{l(M(\mathcal{I}_i))/\#\mathcal{A}_{M(\mathcal{I}_i),T}} \leq s\}}{\#\mathcal{A}_{M(\mathcal{I}_i),T}}$$

$$\leq F^{p,q}_{\mathcal{I},|\mathcal{M}'(\alpha_i)|T}\Big(s(1+\epsilon_2)\Big(1+\frac{c_4}{u}\Big)\Big) + \frac{c_6}{u}.$$

As a result, as $T \to \infty$, we obtain

$$F^{p,q}\Big(s(1-\epsilon_1)\Big(1-\frac{c_3}{u}\Big)\Big) - \frac{c_5}{u} \leq \liminf_{T\to\infty} F^{p,q}_{M(\mathcal{I}_i),T}(s)$$

$$\leq \limsup_{T\to\infty} F^{p,q}_{M(\mathcal{I}_i),T}(s)$$

$$\leq F^{p,q}\Big(s(1+\epsilon_2)\Big(1+\frac{c_4}{u}\Big)\Big) + \frac{c_6}{u}.$$

Since $F^{p,q}_{M(\mathcal{I}),T}(s)$ is a convex combination of $F^{p,q}_{M(\mathcal{I}_i)}(s)$, if we let $u \to \infty$, $\epsilon_1, \epsilon_2 \to 0$ and use the continuity of $F^{p,q}$ (as follows from (24)), we obtain

$$\lim_{T\to\infty} F^{p,q}_{M(\mathcal{I}),T}(s) = F^{p,q}(s). \qquad \square$$

6.2. **Proof of Theorems 3.1 and 4.1.** Any bounded circle packing can be written as $M(\mathcal{P})$ for some generalized Ford packing $\mathcal{P}$. The only issue we need to deal with is ∞; if an arc $\mathcal{J}$ in $M(C_0)$ contains $M\infty$, the argument in the proof of Theorem 6.1 does not directly apply. However, this is easily solved by precomposing M with some $X \in \Gamma$, so that $(MX)^{-1}(\mathcal{J})$ is a bounded arc, and $b < |(MX)'(\alpha)| < B$ for some $b, B > 0$ and any $\alpha \in (MX)^{-1}(\mathcal{J})$. Then we can apply Theorem 6.1.

7. EXAMPLES

We compute the gap distribution function $F(s)$ for three examples: The classical Apollonian packing, where we start with a configuration of four mutually tangent circles, whose tangency graph is the tetrahedron; the Apollonian-3 packing in which one starts with three mutually tangent circles, and in either of the curvilinear triangles formed by three mutually tangent circles we pack three more circles, forming a sextuple whose tangency graph is the octahedron; and the Apollonian-9 packing introduced in [2] where the initial configuration consists of 12 circles, with the icosahedron as its tangency graph.

It would be interesting to apply the method to compute the gap distribution for other examples, such as the ball-bearing configurations of [6].

7.1. **The computational procedure.** We explain the procedure used in the computations: According to Theorem 5.5, the gap distribution $F(s)$ is given in terms of a sum of functions $F^{i,j}(s)$ over (unordered) pairs of distinct circles $C_i, C_j \in \mathcal{K}_0$ of areas of regions $\Omega^{i,j}(s) = \Omega^{i,j}_1(s)$ in the (c, d) plane, explicitly described in Section 4.5. So for instance in the case of the Apollonian-9 packing, there are $\binom{5}{2} = 10$ such pairs. We are able to cut down on the computational

effort involved by using conformal invariance, once we note that the packings we study below enjoy more symmetries than a generic packing. According to Theorem 6.1, if there is some $M \in \mathrm{SL}(2, \mathbb{C})$ which preserves $\mathcal{P}_0$ and C_0 (note that $F(S)$ only depends on $\mathcal{P}_0$, not on all of $\mathcal{P}$), and takes the pair of circles (C_i, C_i) in $\mathcal{K}_0$ to another pair $(MC_i, MC_j) = (C_k, C_\ell)$ in $\mathcal{K}_0$, then $F^{i,j}(s) = F^{k,\ell}(s)$.

In each of these packings, there exists a group $\tilde{\Gamma}$ which is larger than Γ that also acts on this given packing. In each of the three examples presented, the bigger group $\tilde{\Gamma}$ is one of the Hecke triangle groups $\mathbb{G}_q$, which is the group of Möbius transformations generated by the inversion $S : z \mapsto -1/z$ and the translation $T_q : z \mapsto z + \lambda_q$, where $\lambda_q = 2\cos\frac{\pi}{q}$:

$$\mathbb{G}_q = \left\langle T_q = \begin{pmatrix} 1 & 2\cos\frac{\pi}{q} \\ 0 & 1 \end{pmatrix}, \ S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

For $q = 3$ we recover the modular group: $\mathbb{G}_3 = \mathrm{PSL}(2, \mathbb{Z})$. For $q \neq 3, 4, 6$ the groups $\mathbb{G}_q$ are non-arithmetic.

In all three cases, we show that $\Gamma \trianglelefteq \mathbb{G}_q$ is a normal subgroup of the appropriate $\mathbb{G}_q$. In the first example, the classical Ford packing, we have $\Gamma \triangleleft \mathbb{G}_3 = \mathrm{PSL}(2, \mathbb{Z})$ which acts transitively on the Ford circles $\mathcal{P}_0$. In the second example, the Apollonian-3 packing, $\Gamma \triangleleft \mathbb{G}_4$. In the third example, the Apollonian-9 packing, $\Gamma \triangleleft \mathbb{G}_5$.

It is a simple check that in each of these cases, the generators $\mathbb{G}_q$ send circles in $\mathcal{K}_0$ to some circles in $\mathcal{P}_0$. So if we show $\mathbb{G}_q$ normalizes Γ, then it follows that $\mathbb{G}_q$ permutes the circles in $\mathcal{P}_0 = \Gamma\mathcal{K}_0$. We further show that $\mathbb{G}_q$ acts transitively on pairs of tangent circles in $\mathcal{K}_0$ for $q = 3, 4, 5$, and on pairs of disjoint circles for $q = 4, 5$. Hence by conformal invariance of the components $F^{i,j}(s)$, this gives an expression for $F(s)$ as a sum of areas of two regions for $q = 4, 5$ (only one for $q = 3$).

These regions are in turn expressed as a finite union of certain subregions $Z_k(s)$. For each subregion $Z_k \subseteq \Omega^{i,j}$, the first two equations give the conditions $\kappa(C_i) \leq 1, \kappa(C_j) \leq 1$, and the other equations except the last one gives the condition $\kappa(C) > 1$, where C exhausts circles in $\mathcal{K}_0$ such that $\gamma(C)$ is in between $\gamma(C_i)$ and $\gamma(C_j)$. The last quadratic equation corresponds to conditions (19) or (20), capturing relative gap information.

The areas of these regions can be explicitly computed in elementary terms, but the resulting formulae are too long to record. Instead we display plots of the density of the normalized gaps (the derivative of F).

7.2. **Classical Ford circles.** We start with a configuration of four mutually tangent circles in Figure 11: The base circle is $C_0 = \mathbb{R}$, and the other circles in the initial configuration are

$$C_1 = \mathbb{R} + \mathbf{i}, \quad C_2 = C\left(\frac{\mathbf{i}}{2}, \frac{1}{2}\right), \quad C_3 = C\left(1 + \frac{\mathbf{i}}{2}, \frac{1}{2}\right).$$

The resulting packing $\mathcal{P}$ is a classical Apollonian packing, generated by inscribing a unique circle into every curvilinear triangle produced by the four
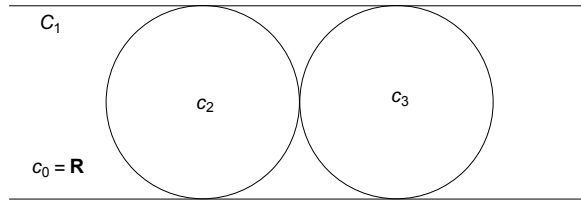
FIGURE 11. Ford configuration of Apollonian type.

mutually tangent circles and repeating. The circles $\mathcal{P}_0$ tangent to the real line are precisely the classical Ford circles $C(\frac{p}{q}, \frac{1}{2q^2})$, having as points of tangency the Farey sequence $\{p/q \mid q > 0, \gcd(p,q) = 1\}$. The gap distribution of the Farey sequence was found by Hall [7], and we carry out the computation here as a warm-up to illustrate our method.

The value of D is 3 according to (7), see Figure 8, and thus

$$c_{\mathcal{P}, C_0} = \frac{D}{2\pi^2(h-2)} = \frac{3}{2\pi^2}.$$

The group Γ_0 is generated by the three reflections

$$z \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \bar{z}, \quad z \to \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \bar{z}, \quad z \to \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \bar{z}$$

so Γ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, which are the generators of the principal congruence subgroup

$$\Gamma(2) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{Z}) \mid \gamma = I \bmod 2 \right\}$$

which is a normal subgroup of $\mathrm{PSL}(2, \mathbb{Z}) = \mathbb{G}_3$. Any two pairs of circles from $\{C_1, C_2, C_3\}$ are *conformally equivalent*, in the sense that there exists some $\gamma \in \mathbb{G}_3 = \mathrm{PSL}(2, \mathbb{Z})$ which preserves the packing and maps C_1, C_2 to this given pair of circles: Indeed,

$$T_3(C_1, C_2) = (C_1, C_3), \quad T_3 S(C_1, C_3) = (C_3, C_2).$$

Therefore we have $F^{1,3}(s) = F^{2,3}(s) = F^{1,2}(s)$, and so by Theorem 5.5,

$$F(s) = 3F^{1,2}(s) = \frac{2}{3} \cdot 2m(\Omega^{1,2}(s)).$$

It remains to compute the area of $\Omega^{1,2}(s)$.

We compute the region $\Omega_1^{1,2}(s)$ using (12), (13), (14). A direct computation gives

$$\kappa(\gamma(C_1)) = 2c^2, \quad \kappa(\gamma(C_2)) = 2d^2, \quad \kappa(\gamma(C_3)) = 2(c+d)^2.$$

The distance between $\gamma\alpha_1 = \gamma\infty$ and $\gamma\alpha_2 = \gamma 0$ is $1/|cd|$. When $d > 0$, C_3 will be mapped by γ to some circle in between $\gamma(C_1)$ and $\gamma(C_2)$ by Observation 4.7,
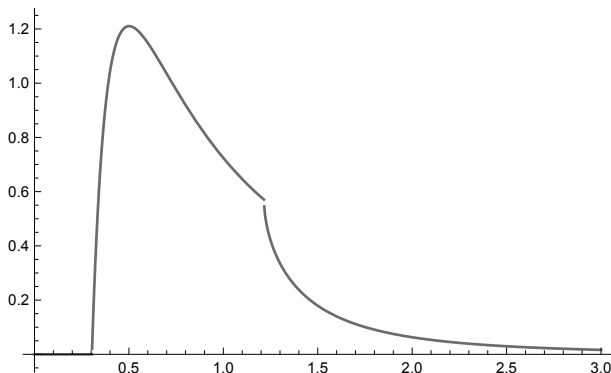
FIGURE 12. $P(s)$ for the classical Apollonian packing.

and the corresponding region is

$$Z_1(s) = \left\{ (c,d) \mid 0 < c \leq \frac{1}{\sqrt{2}}, \ 0 < d \leq \frac{1}{\sqrt{2}}, \ c + d \geq \frac{1}{\sqrt{2}}, \ cd \geq \frac{3}{2\pi^2 s} \right\}.$$

When $d < 0$, then $S_{1,2}(C_3) = C(-1 + \mathbf{i}/2, 1/2)$ will be mapped in between $\gamma(C_1)$ and $\gamma(C_2)$ by γ, and the relevant region is

$$Z_2(s) = \left\{ (c,d) \mid 0 < c \leq \frac{1}{\sqrt{2}}, \ 0 > d \geq -\frac{1}{\sqrt{2}}, \ c - d \geq \frac{1}{\sqrt{2}}, \ cd \leq -\frac{3}{2\pi^2 s} \right\}.$$

Note that $Z_1(s)$ and $Z_2(s)$ are symmetric about the c-axis, so we only need to calculate one area. Therefore, by Theorem 5.5,

$$F(s) = \frac{2}{3} \cdot 3m(\Omega^{1,2}(s)) = 4m(Z_1(s)).$$

The density function $P(s)$ of the normalized gaps (the derivative of $F(s)$) is given in Figure 12.

7.3. **The Apollonian-3 packing.** Our next example was discovered by Guettler and Mallows [6], in which one starts with three mutually tangent circles, and in either of the curvilinear triangles formed by three mutually tangent circles we pack three more circles, forming a sextuple $\mathcal{K}$, see Figure 13. The base circle is $C_0 = \mathbb{R}$ and the circles of $\mathcal{K}$ tangent to it are

$$C_1 = \mathbb{R} + \mathbf{i}, \quad C_2 = C\left(\frac{\mathbf{i}}{2}, \frac{1}{2}\right), \quad C_3 = C\left(\frac{\sqrt{2}}{2} + \frac{1}{4}\mathbf{i}, \frac{1}{4}\right), \quad C_4 = C\left(\sqrt{2} + \frac{1}{2}\mathbf{i}, \frac{1}{2}\right).$$

In this case,

$$D = 4\sqrt{2} \quad \text{and} \quad c_{\mathcal{P},C_0} = \frac{D}{2\pi^2(h-2)} = \frac{\sqrt{2}}{\pi^2}.$$

The group Γ_0 is generated by

$$z \to -\bar{z}, \ z \to \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & -1 \end{pmatrix} \bar{z}, \ z \to \begin{pmatrix} 3 & -2\sqrt{2} \\ 2\sqrt{2} & -3 \end{pmatrix} \bar{z}, \ z \to \begin{pmatrix} -1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix} \bar{z}.$$
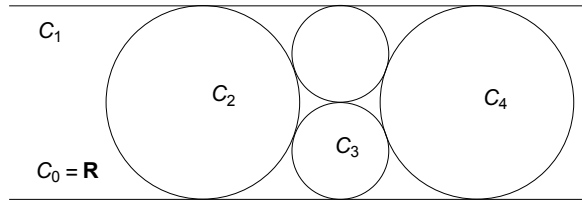
Figure 13. Ford configuration of Apollonian-3 type.

Therefore,

$$\Gamma = \left\langle \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix}, \begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix} \right\rangle.$$

The Hecke group $\mathbb{G}_4$ is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T_4 = \begin{pmatrix} 1 & \sqrt{2} \\ 0 & 1 \end{pmatrix}$. It contains Γ because

$$\begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix} = T_4^{-1} S T_4^{-2} S T_4^{-1}.$$

To see that Γ is normal in $\mathbb{G}_4$, we check that the conjugates of the generators of Γ by the generators S, T_4 of $\mathbb{G}_4$ still lie in Γ:

$$S \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix}^{-1},$$

$$S \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix}^{-1},$$

$$S \begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix} S^{-1} = \begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix}^{-1},$$

$$T_4 \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix} T_4^{-1} = \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix},$$

$$T_4 \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix} T_4^{-1} = - \begin{pmatrix} 1 & 2\sqrt{2} \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix},$$

$$T_4 \begin{pmatrix} 3 & -2\sqrt{2} \\ -2\sqrt{2} & 3 \end{pmatrix} T_4^{-1} = - \begin{pmatrix} 1 & 0 \\ 2\sqrt{2} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2\sqrt{2} \\ 0 & 1 \end{pmatrix}.$$

There are six (unordered) pairs of circles from $\mathcal{K}_0$: Four pairs of tangent circles (C_1, C_2), (C_2, C_3), (C_3, C_4), (C_4, C_1), and two pairs of disjoint circles (C_1, C_3), (C_2, C_4). Each of these cases gives one equivalence class under the action of $\mathbb{G}_4$:

$$T_4(C_1, C_2) = (C_1, C_4), \quad T_4 S(C_1, C_4) = (C_3, C_4), \quad T_4 S(C_3, C_4) = (C_2, C_3),$$

and

$$T_4 S(C_1, C_3) = (C_2, C_4).$$

Hence from formula (23),

$$F(s) = \frac{2}{4\sqrt{2}}\big(4m(\Omega^{1,2}(s)) + 4m(\Omega^{1,3}(s))\big) = \sqrt{2}\big(m(\Omega^{1,2}(s)) + m(\Omega_1^{1,3}(s))\big).$$

We compute the areas $\Omega_1^{1,2}$ and $\Omega_1^{1,3}$. Note that if $\gamma = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big)$ then

$$\kappa(\gamma(C_1)) = 2c^2, \qquad\qquad \kappa(\gamma(C_2)) = 2d^2,$$

$$\kappa(\gamma(C_3)) = 4\Big(\frac{\sqrt{2}}{2}c + d\Big)^2, \quad \kappa(\gamma(C_4)) = 2(\sqrt{2}c + d)^2.$$

First we consider $\Omega_1^{1,2}$: The distance between $\gamma(\alpha_1) = \gamma\cdot\infty$ and $\gamma(\alpha_2) = \gamma\cdot 0$ is $1/|cd|$. Therefore, we need to consider the cases $d > 0$ and $d < 0$ separately. If $d > 0$, then the circles C_3 and C_4 will be mapped by γ to some circles in between $\gamma(C_1)$ and $\gamma(C_2)$. The corresponding region Z_1 is

$$Z_1(s) = \Big\{(c,d) \mid 0 < \sqrt{2}c \le 1,\ 0 < \sqrt{2}d \le 1,\ \sqrt{2}c + 2d \ge 1,$$
$$2c + \sqrt{2}d \ge 1,\ cd \ge \frac{\sqrt{2}}{\pi^2 s}\Big\}.$$

If $d < 0$, then the circles

$$S_{1,2}C_3 = C\Big(-\frac{\sqrt{2}}{2} + \frac{\mathbf{i}}{4}, \frac{1}{4}\Big) \quad \text{and} \quad S_{1,2}C_4 = C\Big(-\sqrt{2} + \frac{\mathbf{i}}{2}, \frac{1}{2}\Big)$$

will be mapped in between $\gamma(C_1)$ and $\gamma(C_2)$. The corresponding region Z_2 is

$$Z_2(s) = \Big\{(c,d) \mid 0 < \sqrt{2}c \le 1,\ 0 > \sqrt{2}d \ge -1,\ \sqrt{2}c - 2d \ge 1,$$
$$2c - \sqrt{2}d \ge 1,\ cd \le -\frac{\sqrt{2}}{\pi^2 s}\Big\}.$$

The region $\Omega_1^{1,2}(s)$ is the union of $Z_1(s)$ and $Z_2(s)$ which are symmetric, hence the area is

$$m(\Omega_1^{1,2}(s)) = 2m(Z_1(s)).$$

Next we consider $\Omega_1^{1,3}(s)$. Here

$$\kappa(\gamma(C_3)) = (\sqrt{2}c + 2d)^2$$

and we need to split it into two cases. If $\sqrt{2}c + 2d > 0$, then γC_4 will be in between γC_1 and γC_3, and the corresponding region is

$$Z_3(s) = \Big\{(c,d) \mid 0 < \sqrt{2}c \le 1,\ 0 < \sqrt{2}c + 2d \le 1,\ 2c + \sqrt{2}d \ge 1,$$
$$c\Big(\frac{1}{\sqrt{2}}c + d\Big) \ge \frac{\sqrt{2}}{\pi^2 s}\Big\}.$$

Figure 14. $P(s)$ for the Apollonian-3 packing.

If $\sqrt{2}c + 2d < 0$, then γC_2 will be in between γC_1 and γC_3, and

$$Z_4(s) = \left\{ (c,d) \mid 0 < \sqrt{2}c \le 1,\ 0 > \sqrt{2}c + 2d \ge -1,\ \sqrt{2}d \le -1, \right.$$
$$\left. c\left(\frac{1}{\sqrt{2}}c + d \right) \le -\frac{\sqrt{2}}{\pi^2 s} \right\}.$$
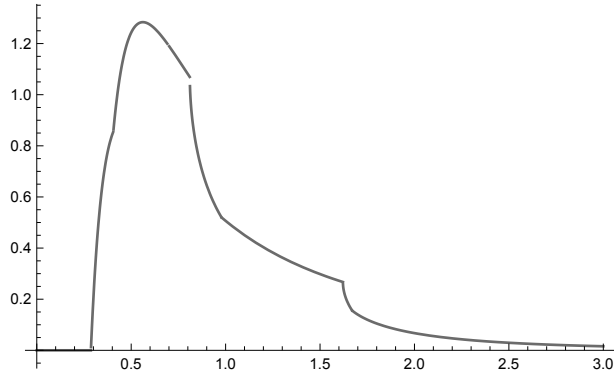
Then $\Omega_1^{1,3}(s)$ is the union of $Z_3(s)$ and $Z_4(s)$ (which have equal areas), and

$$m(\Omega_1^{1,3}(s)) = m(Z_3(s)) + m(Z_4(s)).$$

Thus

$$F_3(s) = \sqrt{2}\big(2m(Z_1(s)) + m(Z_3(s)) + m(Z_4(s))\big).$$

The density function $P(s)$ of the normalized gaps (the derivative of $F(s)$) is given in Figure 14.

7.4. **The Apollonian-9 packing.** This packing was introduced in [2]. The initial configuration $\mathcal{K}$, whose tangency graph is the icosahedron, is shown in Figure 15. The base circle is $C_0 = \mathbb{R}$ and the circles $\mathcal{K}_0$ tangent to it are

$$C_1 = \mathbb{R} + \mathbf{i}, \quad C_2 = C\Big(\frac{\mathbf{i}}{2}, \frac{1}{2} \Big), \quad C_3 = C\Big(\frac{\sqrt{5}-1}{2} + \frac{3-\sqrt{5}}{4}\mathbf{i}, \frac{3-\sqrt{5}}{4} \Big),$$
$$C_4 = C\Big(1 + \frac{3-\sqrt{5}}{4}\mathbf{i}, \frac{3-\sqrt{5}}{4} \Big), \quad C_5 = C\Big(\frac{\sqrt{5}+1}{2} + \frac{\mathbf{i}}{2}, \frac{1}{2} \Big).$$

The constant $c_{\mathcal{P},C_0}$ in this case is
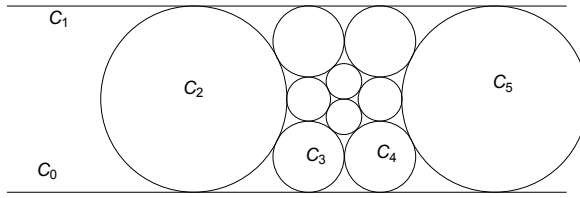
$$c_{\mathcal{P},C_0} = \frac{5(\sqrt{5}+1)}{12\pi^2}.$$

Figure 15. Ford configuration of Apollonian-9 type.

The group Γ_0 is generated by the reflections

$$z \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \bar{z}, \quad z \to \begin{pmatrix} 1 & 0 \\ \sqrt{5}+1 & -1 \end{pmatrix} \bar{z}, \quad z \to \begin{pmatrix} 2+\sqrt{5} & -1-\sqrt{5} \\ \sqrt{5}+3 & -2-\sqrt{5} \end{pmatrix} \bar{z},$$

$$z \to \begin{pmatrix} 2+\sqrt{5} & -3-\sqrt{5} \\ \sqrt{5}+1 & -2-\sqrt{5} \end{pmatrix} \bar{z}, \quad z \to \begin{pmatrix} -1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} \bar{z}.$$

Taking the products of the first matrix $z \mapsto -\bar{z}$ and each of the other four matrices, we get generators for Γ:

$$\begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1+\sqrt{5} & 1 \end{pmatrix}, \begin{pmatrix} 2+\sqrt{5} & 3+\sqrt{5} \\ 1+\sqrt{5} & 2+\sqrt{5} \end{pmatrix}, \begin{pmatrix} 2+\sqrt{5} & 1+\sqrt{5} \\ 3+\sqrt{5} & 2+\sqrt{5} \end{pmatrix}.$$

We claim that $\Gamma \subset \mathbb{G}_5$, the Hecke-5 group, which is generated by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T_5 = \begin{pmatrix} 1 & \frac{\sqrt{5}+1}{2} \\ 0 & 1 \end{pmatrix}.$$

We check that each generator of Γ can be written as word of S and T_5 and hence lies in $\mathbb{G}_5$: Clearly,

$$\begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} = T_5^2, \quad \begin{pmatrix} 1 & 0 \\ 1+\sqrt{5} & 1 \end{pmatrix} = ST_5^{-2}S^{-1}.$$

Moreover,

$$\begin{pmatrix} 2+\sqrt{5} & 3+\sqrt{5} \\ \sqrt{5}+1 & 2+\sqrt{5} \end{pmatrix} = T_5 S T_5^2 S T_5$$

and

$$\begin{pmatrix} 2+\sqrt{5} & 1+\sqrt{5} \\ \sqrt{5}+3 & 2+\sqrt{5} \end{pmatrix} = S T_5^{-1} S T_5^{-2} S T_5^{-1} S^{-1}$$

as claimed.

Since Γ is generated by elements of $\mathbb{G}_5$, it is a subgroup of that group, and since both groups have finite volume, Γ is a subgroup of finite index in $\mathbb{G}_5$, and in particular is non-arithmetic.

To see that Γ is normal in $\mathbb{G}_5$, we check that the conjugates of the generators of Γ by the generators S, T_5 of $\mathbb{G}_5$ still lie in Γ:

$$S \begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 0 \\ 1+\sqrt{5} & 1 \end{pmatrix}^{-1},$$

$$S \begin{pmatrix} 1 & 0 \\ 1+\sqrt{5} & 1 \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix}^{-1},$$

$$S \begin{pmatrix} 2+\sqrt{5} & 3+\sqrt{5} \\ 1+\sqrt{5} & 2+\sqrt{5} \end{pmatrix} S^{-1} = \begin{pmatrix} 2+\sqrt{5} & 1+\sqrt{5} \\ 3+\sqrt{5} & 2+\sqrt{5} \end{pmatrix}^{-1},$$

$$S \begin{pmatrix} 2+\sqrt{5} & 1+\sqrt{5} \\ 3+\sqrt{5} & 2+\sqrt{5} \end{pmatrix} S^{-1} = \begin{pmatrix} 2+\sqrt{5} & 3+\sqrt{5} \\ 1+\sqrt{5} & 2+\sqrt{5} \end{pmatrix}^{-1},$$

$$T_5 \begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} T_5^{-1} = \begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix},$$

$$T_5 \begin{pmatrix} 1 & 0 \\ 1+\sqrt{5} & 1 \end{pmatrix} T_5^{-1} = -\begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2+\sqrt{5} & 3+\sqrt{5} \\ 1+\sqrt{5} & 2+\sqrt{5} \end{pmatrix}^{-1},$$

and hence

$$T_5 \begin{pmatrix} 2+\sqrt{5} & 3+\sqrt{5} \\ 1+\sqrt{5} & 2+\sqrt{5} \end{pmatrix} T_5^{-1} = -\begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1+\sqrt{5} & 1 \end{pmatrix}^{-1},$$

$$T_5 \begin{pmatrix} 2+\sqrt{5} & 1+\sqrt{5} \\ 3+\sqrt{5} & 2+\sqrt{5} \end{pmatrix} T_5^{-1} = -\begin{pmatrix} 1 & 1+\sqrt{5} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2+\sqrt{5} & 1+\sqrt{5} \\ 3+\sqrt{5} & 2+\sqrt{5} \end{pmatrix}^{-1}.$$

The pairs (C_1, C_2), (C_1, C_3) are representatives of two conformally equivalent classes of pairs of circles from $\mathcal{K}_0$, characterized by the two circles being tangent or disjoint and $\mathbb{G}_5$ acts transitively on each class: Indeed,

$$(C_1, C_2) \xrightarrow{T_5} (C_1, C_5) \xrightarrow{T_5 S} (C_4, C_5) \xrightarrow{T_5 S} (C_3, C_4) \xrightarrow{T_5 S} (C_2, C_3)$$

and

$$(C_1, C_3) \xrightarrow{T_5 S} (C_2, C_5) \xrightarrow{T_5 S} (C_1, C_4) \xrightarrow{T_5 S} (C_3, C_5) \xrightarrow{T_5 S} (C_2, C_4).$$

Each equivalence class contains five pairs of circles. Therefore,

$$F(s) = \frac{4}{5(\sqrt{5}+1)} \big(5m(\Omega_1^{1,2}(s)) + 10m(\Omega_1^{1,3}(s))\big).$$

The curvatures of the transformed circles are

$$\kappa(\gamma(C_1)) = 2c^2, \quad \kappa(\gamma(C_2)) = 2d^2, \quad \kappa(\gamma(C_3)) = 2\Big(c + \frac{\sqrt{5}+1}{2}d\Big)^2,$$

$$\kappa(\gamma(C_4)) = 2\Big(\frac{\sqrt{5}+1}{2}c + \frac{\sqrt{5}+1}{2}d\Big)^2, \quad \kappa(\gamma(C_5)) = 2\Big(\frac{\sqrt{5}+1}{2}c + d\Big)^2.$$

Note that $\kappa(\gamma C_4)$ is greater than $\kappa(\gamma C_3)$ and $\kappa(\gamma C_5)$.

We first compute $\Omega_1^{1,2}(s)$. If $d > 0$, then by Observation 4.7, the circles C_3, C_4, C_5 will be mapped by γ in between $\gamma(C_1)$ and $\gamma(C_2)$. The corresponding region is, using (13),

$$Z_1 = \left\{ (c,d) \mid 0 < c \le \frac{\sqrt{2}}{2}, \ 0 < d \le \frac{\sqrt{2}}{2}, \ c + \frac{\sqrt{5}+1}{2}d \ge \frac{\sqrt{2}}{2}, \right.$$
$$\left. \frac{\sqrt{5}+1}{2}c + d \ge \frac{\sqrt{2}}{2}, \ cd \ge \frac{5(\sqrt{5}+1)}{12\pi^2 s} \right\}.$$

The condition $\kappa(\gamma(C_4)) \ge 1$ is redundant here.

If $d < 0$, then

$$S_{1,2}(C_3) = C\left(\frac{-\sqrt{5}+1}{2} + \frac{3-\sqrt{5}}{4}\mathbf{i}, \frac{3-\sqrt{5}}{4} \right),$$
$$S_{1,2}(C_4) = C\left(-1 + \frac{3-\sqrt{5}}{4}\mathbf{i}, \frac{3-\sqrt{5}}{4} \right),$$
$$S_{1,2}(C_5) = C\left(\frac{-\sqrt{5}-1}{2} + \frac{\mathbf{i}}{2}, \frac{1}{2} \right)$$

will be mapped in between $\gamma(C_1)$ and $\gamma(C_2)$. The corresponding region is, by (14),

$$Z_2 = \left\{ (c,d) \mid 0 < c \le \frac{\sqrt{2}}{2}, \ 0 > d \ge -\frac{\sqrt{2}}{2}, \ c - \frac{\sqrt{5}+1}{2}d \ge \frac{\sqrt{2}}{2}, \right.$$
$$\left. \frac{\sqrt{5}+1}{2}c - d \ge \frac{\sqrt{2}}{2}, \ -cd \le -\frac{5(\sqrt{5}+1)}{12\pi^2 s} \right\}.$$

The region $\Omega_1^{1,2}(s)$ is the union of Z_1 and Z_2, which are symmetric.

Now consider $\Omega_1^{1,3}(s)$. By (15), if $c + \frac{\sqrt{5}+1}{2}d > 0$, then the images γC_4 and γC_5 will be between γC_1 and γC_3, and the corresponding region is

$$Z_3 = \left\{ (c,d) \mid 0 < c \le \frac{\sqrt{2}}{2}, \ 0 < c + \frac{\sqrt{5}+1}{2}d \le \frac{\sqrt{2}}{2}, \ \frac{\sqrt{5}+1}{2}c + d \ge \frac{\sqrt{2}}{2}, \right.$$
$$\left. \frac{\sqrt{5}+1}{2}c + \frac{\sqrt{5}+1}{2}d \ge \frac{\sqrt{2}}{2}, \ c\left(\frac{\sqrt{5}-1}{2}c + d\right) \ge \frac{5(\sqrt{5}+1)}{12\pi^2 s} \right\}.$$

Similarly, if $c + \frac{\sqrt{5}+1}{2}d < 0$, then γC_2 will be in between γC_1 and γC_3, and the region is

$$Z_4 = \left\{ (c,d) \mid 0 < c \le \frac{\sqrt{2}}{2}, \ 0 > c - \frac{\sqrt{5}+1}{2}d \ge -\frac{\sqrt{2}}{2}, \ d \le -\frac{\sqrt{2}}{2}, \right.$$
$$\left. c\left(\frac{\sqrt{5}-1}{2}c + d\right) \le -\frac{5(\sqrt{5}+1)}{12\pi^2 s} \right\}.$$
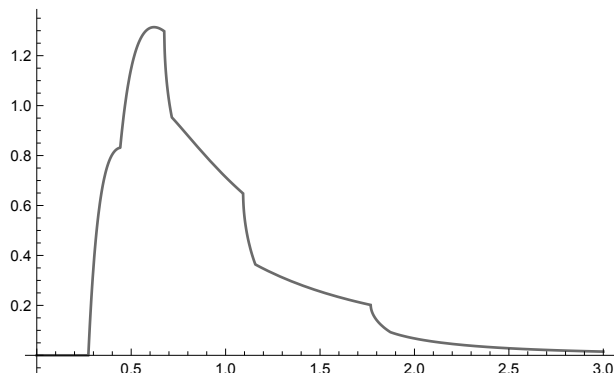
FIGURE 16. $P(s) = F'(s)$ for the Apollonian-9 packing.

The region $\Omega_1^{1,3}(s)$ is the union of Z_3 and Z_4. Therefore,

$$F(s) = \frac{4}{5(\sqrt{5}+1)}\big(5m(\Omega_1^{1,2}(s)) + 10m(\Omega_1^{1,3}(s))\big)$$

$$= \frac{8}{\sqrt{5}+1}\big(m(Z_1(s)) + m(Z_3(s)) + m(Z_4(s))\big).$$

The density function of the normalized gaps is given in Figure 16.

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Zeev Rudnick
Raymond and Beverly Sackler School of Mathematical Sciences,
Tel Aviv University, Tel Aviv 69978, Israel
E-mail: `rudnick@post.tau.ac.il`

Xin Zhang
Raymond and Beverly Sackler School of Mathematical Sciences,
Tel Aviv University, Tel Aviv 69978, Israel
E-mail: `mathzhangxin@gmail.com`