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Profinite Etale Cobordism

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Profinite Etale Cobordism

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Zusammenfassung

In der vorliegenden Arbeit wird eine neue Kohomologietheorie, der proendliche étale Kobordismus, für glatte Schemata von endlichem Typ über einem Körper entwickelt. Die Motivation dieser Theorie besteht darin, für den algebraischen Kobordismus von Levine-Morel und Voevodsky eine étale topologische Theorie zu entwickeln, die einfacher auszurechnen ist und interessante Vergleichsabbildungen vom algebraischen Kobordismus in diese neue Theorie zulässt. Zudem soll damit der Zugang zu arithmetischen Fragen mit Hilfe des algebraischen Kobordismus erleichtert werden. Eine wichtige Eigenschaft des étalen Kobordismus liegt in der Existenz einer konvergenten Atiyah-Hirzebruch Spektralsequenz ausgehend von étaler Kohomologie.

Für die Entwicklung dieser Theorie wird zum einen gezeigt, dass es auf proendlichen Spektren eine stabile Modellstruktur gibt. Zum anderen konstruieren wir eine étale Realisierung der stabilen motivischen Kategorie der $\mathbb{P}^1$-Spektren.

Es wird gezeigt, dass die natürlichen Transformationen vom algebraischen Kobordismus in den étalen Kobordismus über einem separabel abgeschlossenen Körper surjektiv sind. Dieses Ergebnis und die Atiyah-Hirzebruch Spektralsequenz führen zu der Vermutung, dass über einem separabel abgeschlossenen Körper algebraischer und proendlicher étaler Kobordismus mit endlichen Koeffizienten nach Invertieren eines Bottelementes isomorph sind.

Contents

1 Introduction

1.1 Motivation and main results

The aim of this paper is the construction of a new cohomology theory for smooth schemes over a field, called profinite étale cobordism.

Since the 1990s two approaches to the theory of algebraic cobordism for smooth schemes have been made. On the one hand, Morel and Voevodsky have developed $\mathbb{A}^1$-homotopy theory for schemes. Voevodsky used it for the construction of cohomology theories on schemes. In particular, he showed the existence of an algebraic cobordism theory $MGL^{*,*}$ in analogy to Thom's homotopical definition of complex cobordism in topology.

On the other hand, Levine and Morel used Quillen's insight for a geometric construction of complex cobordism for proving that there is another possible definition of algebraic cobordism via a purely geometric construction. They proved that this Ω^* is the universal object for oriented cohomology theories on smooth schemes over a field. It is conjectured that Ω^* is in fact a geometrical description of the $MGL^{2*,*}$-part of Voevodsky's theory. Hopkins and Morel recently proved that the canonical map $\Omega^* \to MGL^{2*,*}$ is surjective over a field of characteristic zero.

Both approaches have been used to prove famous results. Voevodsky used his construction for the proof of the Milnor Conjecture. Levine and Morel proved Rost's Degree Formula. Nevertheless, both theories are still hard to compute and most of their features have only been proved over fields of characteristic zero.

The purpose of this paper is the construction of an étale topological version of cobordism for smooth schemes that is easier to compute since it is closely related to the étale cohomology. The idea is due to Eric Friedlander who constructed in [Fr1] a first version of an étale topological K-theory for schemes that turned out to be a powerful tool for the study of algebraic K-theory with finite coefficients. In particular, Thomason proved in his famous paper [Th] that algebraic K-theory with finite coefficients agrees with étale K-theory after inverting a Bott element. The aim of this paper is also to construct a candidate for an analogous statement for algebraic cobordism, see Conjecture 1.5.

At the end of the 1970s, in [Sn] Victor P. Snaith has already constructed a p-adic cobordism theory for schemes. His approach is close to the definition algebraic K-theory by Quillen. He defines for every scheme V over $\mathbb{F}_q$, $q = p^n$, a topological cobordism spectrum $\underline{A\mathbb{F}}_{q,V}$. The homotopy groups of this spectrum are the p-adic cobordism groups of V. He has calculated these

groups for projective bundles, Severi-Brauer schemes and other examples.

In this paper, we do not follow Snaith's construction, but we provide Friedlander's idea with a general setting. We consider the profinite completion $\hat{\mathrm{Et}}$ of Friedlander's étale topological type functor of [Fr2] with values in the category of simplicial profinite sets $\hat{\mathcal{S}}$. We construct a stable homotopy category $\hat{\mathcal{SH}}$ for $\hat{\mathcal{S}}$ and define general cohomology theories for profinite spaces. We apply these cohomology theories to $\hat{\mathrm{Et}}\,X$ for a scheme X over a field k. This yields a general foundation for different étale topological cohomology theories.

When we apply this construction to the cohomology theory represented by the profinitely completed $\hat{MU}$-spectrum of complex cobordism, we get an étale topological cobordism theory $\hat{MU}^*_{\text{ét}}$ for schemes of finite type over a field, which we call profinite étale cobordism. Note that this theory depends on a fixed prime number ℓ, which must be different from the characteristic of the base field k.

The main feature of $\hat{MU}^*_{\text{ét}}$ is the existence of an Atiyah-Hirzebruch spectral sequence, see Proposition 7.14 and Theorem 7.15:

Theorem 1.1 *1. Let k be a separably closed field. There are isomorphisms*

$$\hat{MU}^*_{\text{ét}}(k) \cong MU^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \text{ and } \hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu) \cong MU^* \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^\nu.$$

2. Let k be a field. For every smooth scheme X over k, there is a convergent spectral sequence $\{E^{p,q}_r\}$ with

$$E^{p,q}_2 = H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q) \implies \hat{MU}^{p+q}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu).$$

These properties gives rise to several applications.

On the one hand, based on the results of Panin [Pa], they enable us to prove, see Theorem 7.26:

Theorem 1.2 *Let k be a separably clsoed field. Profinite étale cobordism $\hat{MU}^*_{\text{ét}}(-)$, resp. $\hat{MU}^*_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$, is an oriented cohomology theory on* Sm/k

This implies that we have a unique natural morphism $\theta_{\hat{MU}} : \Omega^*(X) \to \hat{MU}^{2*}_{\text{ét}}(X)$ for every X in Sm/k. We prove, see Theorem 8.5:

Theorem 1.3 *The canonical map $\theta_{\hat{MU}} : \Omega^*(X; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ is surjective for every X in* Sm/k.

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In fact, once we have set up the theory, this result follows easily from [LM]. On the other hand, via a stable étale realization functor of motivic spectra, we construct a natural map $\phi : MGL^{*,*}(X) \to \hat{MU}^*_{\text{ét}}(X)$, see Section 8.2, Theorem 8.12 and Corollary 8.13:

Theorem 1.4 *This natural map ϕ fits into a commutative triangle for every X in* Sm/k

$$
\begin{array}{ccc}
\Omega^*(X) & \overset{\theta_{MGL}}{\Longrightarrow} & MGL^{2*,*}(X) \\
\theta_{\hat{MU}} \searrow & & \nearrow \phi \\
 & \hat{MU}^{2*}_{\text{ét}}(X). &
\end{array}
$$

There is a similar triangle for $\mathbb{Z}/\ell^\nu$-coefficients.

In particular, the map $\phi : MGL^{2*,*}(X; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ is surjective for every X in Sm/k.

For a field k, we start the study of the absolute Galois group $G_k = \text{Gal}(k^s/k)$ on étale cobordism, where k^s denotes a separable closure of k. We deduce from the previous results that the action of G_k on $\hat{MU}^*_{\text{ét}}(k^s)$, resp. $\hat{MU}^*_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$, is trivial, see Theorem 8.17. Together with the Atiyah-Hirzebruch spectral sequence applied to Galois cohomology, this enables us to determine the étale cobordism of a finite field $k = \mathbb{F}_q$, $\ell \nmid q$, see Theorem 8.19.

Finally, the Atiyah-Hirzebruch spectral sequence together with the results of Levine [Le2] on motivic cohomology with inverted Bott element yield good reasons for the following conjecture, see Section 8.4 and Conjecture 8.20:

Conjecture 1.5 *Let X be a smooth scheme of finite type over a separably closed field k of characteristic different from ℓ. Suppose that ℓ is odd or that $\ell^\nu \geq 4$. Let $\beta \in MGL^{0,1}(k; \mathbb{Z}/\ell^\nu)$ be the Bott element. The induced morphism*

$$\phi : MGL^{*,*}(X; \mathbb{Z}/\ell^\nu)[\beta^{-1}] \to \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$$

is an isomorphism.

At the end of Section 8, we will explain a strategy to prove this conjecture, which is close to Levine's new proof of Thomason's K-theory theorem in [Le1]. In particular, we would like to use the Atiyah-Hirzebruch spectral sequence of Hopkins and Morel for algebraic cobordism in [HM].

1.2 Survey of the paper

The outline of the paper is as follows. Towards the construction of profinite étale cobordism, there are several technical problems to solve. The first main result of this paper is that it is possible to establish a stable model structure on profinite spectra. It is based on Morel's $\mathbb{Z}/\ell$-cohomological model structure on $\hat{\mathcal{S}}$ of [Mo2], see Theorem 3.2:

Theorem 1.6 *There is a stable model structure on profinite spectra* $\mathrm{Sp}(\hat{\mathcal{S}})$ *such that the suspension functor* $S^1 \wedge -$ *is an equivalence on the corresponding homotopy category* $\hat{\mathcal{SH}}$.

This result cannot be deduced in the same way as the stable structure on simplicial spectra, since $\hat{\mathcal{S}}$ is not proper. We use a generalized theorem on left Bousfield localization of model categories similar to a result of Hirschhorn [Hi] and a generalized result on stable model structure on spectra similar to the work of Hovey in [Ho2]. We have discussed these theorems in the appendix.

The next step is the study of generalized cohomology theories on profinite spaces. Our knowledge on profinite completion of spectra and the results of Bousfield-Kan on ℓ-completion of spaces enables us to determine the coefficients of $\hat{M}U$. We also construct an Atiyah-Hirzebruch spectral sequence for profinite spaces. The proof of the existence of this spectral sequence is mainly due to Dehon in [De]. Profinitely completed cobordism has already been used for other purposes. In particular, Francois-Xavier Dehon uses the completed cobordism theory $\hat{M}U^*(-)$ for the study of Lannes' T-functor in [De]. Many of his ideas were fruitful for this paper.

With the stable category on $\hat{\mathcal{S}}$ we give this approach a general setting. Hence the discussions of the first sections may already be interesting on their own. In order to clarify the argument and the relevance of the discussions of Sections 2-4 for the reader, we mention the following point.

Since our definitions are all compatible with étale cohomology in a certain sense, see Remark 5.3, we can deduce from these facts about profinite cohomology theories the existence of the mentioned Atiyah-Hirzebruch spectral sequence starting from étale cohomology and converging to profinite étale cobordism. Furthermore, we can determine the coefficients $\hat{M}U^*_{\text{ét}}(k)$ for a separably closed field k.

In Sections 5 and 6, we dedicate our attention to the study of the completed étale topological type functor $\hat{\text{Ét}}$ on schemes of finite type over a field with values in $\hat{\mathcal{S}}$. This functor is due to Artin-Mazur and Friedlander. It has been

extended to the $\mathbb{A}^1$-homotopy category of schemes independently by Daniel Isaksen [I3] and Alexander Schmidt [Sch].

We use this functor to construct a stable étale realization of motivic spectra, see Section 6 and Theorem 6.8:

Theorem 1.7 *The functor* $\hat{\text{Ét}}$ *yields an étale realization of the stable motivic homotopy category of* $\mathbb{P}^1$-*spectra:*

$$\text{L}\hat{\text{Ét}} : \mathcal{SH}^{\mathbb{P}^1}(k) \to \hat{\mathcal{SH}}.$$

This is another main result of the paper and may be interesting on its own. Although we do not know if $\hat{\text{Ét}}\, MGL$ is isomorphic to $\hat{MU}$ in $\hat{\mathcal{SH}}$ over an arbitrary base field, this realization is the key ingredient in the construction of the map $\phi : MGL^{*,*}(X) \to \hat{MU}^*_{\text{ét}}(X)$ from algebraic to profinite étale cobordism.

In Sections 7 and 8 we start the study of étale cohomology theories and the comparison with algebraic cobordism. These two sections contain the study of profinite étale cobordism and give the proofs for the main results that we mentioned above.

The reader could even start with Section 7 if one accepts first the existence of a stable category of profinite spectra, second the existence of the functor $\hat{\text{Ét}}$ on schemes and motivic spectra and third the existence of an Atiyah-Hirzebruch spectral sequence for profinite generalized cohomology theories and the isomorphism $\hat{MU}^* \cong MU^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$. For the last fact, we even give another proof in Section 7.

The purpose of the appendix is to prove a general stabilization theorem for left proper fibrantly generated model categories, see Theorem B.11:

Theorem 1.8 *Let* $\mathcal{C}$ *be a left proper fibrantly generated simplicial model category with all small limits and let* T *be a left adjoint endofunctor on* $\mathcal{C}$. *There is a model structure on the category of spectra* $\text{Sp}(\mathcal{C}, T)$ *such that* T *becomes a Quillen equivalence on* $\text{Sp}(\mathcal{C}, T)$.
This model structure is again a left proper fibrantly generated simplicial model structure.

We apply this theorem in Section 3 to the left proper fibrantly generated model category of profinite spaces. This stabilization is the only purpose of the appendix. But since we gave the proofs in a very general setting, we have postponed them to the end of the paper.

In Appendix 1, we prove first a theorem on left Bousfield localization of fibrantly generated model categories, based on [Hi].

In Appendix 2, this theorem is applied to an intermediate model structure on spectra in order to get a stable category, based on [Ho2]. The ideas for the proofs are due to the papers [Ho2] of Hovey and [Hi] of Hirschhorn plus an idea of Christensen-Isaksen in [CI]. Nevertheless, our results are slightly more general and might be of interest for other situations.

2 Profinite spaces

We introduce the category of (pointed) profinite spaces $\hat{\mathcal{S}}$ (resp. $\hat{\mathcal{S}}_*$) and its model structure defined by Morel [Mo2] for a fixed prime number ℓ. An important construction is the Bousfield-Kan-ℓ-completion of profinite spaces, which is an explicit fibrant replacement functor in this model structure.

Via this fibrant replacement functor, we introduce homotopy groups of profinite spaces and study their behavior under the profinite completion functor $\mathcal{S} \to \hat{\mathcal{S}}$. In particular, we show that these homotopy groups are always pro-

ℓ-groups.

Then we define a completion functor $\text{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ from the category of pro-simplicial sets to profinite spaces and show that the respective $\mathbb{Z}/\ell$-homotopy categories are equivalent. This compares our approach to the one of [I4].

We finish this section with a technical statement. The model structure of $\hat{\mathcal{S}}$ is fibrantly generated. This is necessary in order to apply the localization and stabilization theorems of the appendix to $\hat{\mathcal{S}}$. At this point, the category $\hat{\mathcal{S}}$ is less well behaved than $\mathcal{S}$ and $\text{pro} - \mathcal{S}$, but it has properties that are good enough for our purposes.

2.1 The profinite ℓ-completion of Bousfield-Kan

Let $\mathcal{E}$ denote the category of sets and let $\mathcal{F}$ be the full subcategory of $\mathcal{E}$ whose objects are finite sets. Let $\hat{\mathcal{E}}$ be the category of compact and totally disconnected topological spaces. We may identify $\mathcal{F}$ with a full subcategory of $\hat{\mathcal{E}}$ in the obvious way. The limit functor $\lim : \text{pro} - \mathcal{F} \to \hat{\mathcal{E}}$, which sends a pro-object X of $\mathcal{F}$ to the limit in $\hat{\mathcal{E}}$ of the diagram corresponding to X, is an equivalence of categories.

We denote by $\mathcal{S}$ (resp. $s\mathcal{F}$, resp. $\hat{\mathcal{S}}$) the category of simplicial sets (resp. simplicial finite sets, resp. simplicial profinite sets). The objects of $\mathcal{S}$ (resp. $\hat{\mathcal{S}}$) will be called *spaces* (resp. *profinite spaces*). There is a subtle distinction between the two categories pro-$s\mathcal{F}$ and $\hat{\mathcal{S}}$. The obvious functor between them is not an equivalence. See [I2] for more details and a counter example.

For a profinite space X we define the ordered set $\mathcal{Q}(X)$ of simplicial open equivalence relations on X. For every element Q of $\mathcal{Q}(X)$ the quotient X/Q is a simplicial finite set and the map $X \to X/Q$ is a map of profinite spaces. In fact, when we consider the limit $\lim_{Q \in \mathcal{Q}(X)} X/Q$ in $\hat{\mathcal{S}}$, the map $X \to \lim_{Q \in \mathcal{Q}(X)} X/Q$ is an isomorphism, cf. [Qu2] , Lemma 2.3.

The forgetful functor $\hat{\mathcal{E}} \to \mathcal{E}$ admits a left adjoint $(\hat{-}) : \mathcal{E} \to \hat{\mathcal{E}}$. It induces dimensionwisely a functor $(\hat{-}) : \mathcal{S} \to \hat{\mathcal{S}}$, which we call profinite completion. It is left adjoint to the forgetful functor $|\cdot| : \hat{\mathcal{S}} \to \mathcal{S}$ which sends a profinite space X to its underlying simplicial set $|X|$. By adjunction, the profinite completion of a simplicial set Z may be identified with the filtered colimit in $\hat{\mathcal{S}}$ of the simplicial finite subsets of Z.

Let X be a profinite space. The continuous cohomology $H^*(X; \pi)$ of X with coefficients in the profinite abelian group π is defined as the cohomology

of the complex $C^*(X; \pi)$ of continuous cochains of X with values in π, i.e. $C^n(X; \pi)$ denotes the set $\mathrm{Hom}_{\hat{\mathcal{E}}}(X_n, \pi)$ of continuous maps $X_n \to \pi$.

Remark 2.1 ([Mo2], §1.2.)
1. Let X be a simplicial profinite set. If we denote by $C_*(X)$ the chain complex defined in degree n to be the free profinite abelian group on the profinite set X_n, then the universal property of the free profinite group yields for every profinite abelian group π an isomorphism $C^n(X) = \mathrm{Hom}_{\hat{\mathcal{E}}}(X_n, \pi) \cong \mathrm{Hom}(C_n(X), \pi)$ for each n where the last Hom denotes the morphisms of profinite abelian groups. This shows the relation to the classical definition of the cohomology of simplicial sets, cf. [Ma1] p. 5.
2. Let π be a finite abelian group. There is a natural isomorphism between the continuous cohomology with coefficients in π of the profinite completion $\hat{Y}$ of a simplicial set Y and its ordinary cohomology with coefficients in π, i.e.

$$H^*(Y; \pi) \cong H^*(\hat{Y}; \pi).$$

This isomorphism exists already on the level of complexes, since there is a natural bijection $\mathrm{Hom}_{\hat{\mathcal{E}}}(\hat{X}_n, \pi) \cong \mathrm{Hom}_{\mathcal{E}}(X_n, \pi)$. We will explain later how to generalize this result to an arbitrary abelian pro-ℓ-group π.
3. Let π be a finite abelian group and let $\{X_s\}$ be a cofiltered diagram of profinite spaces. The canonical map

$$\mathrm{colim}_s H^*(X_s; \pi) \overset{\cong}{\longrightarrow} H^*(\lim_s X_s; \pi)$$

is an isomorphism, cf. [De], Lemme 1.1.1.
4. Let π be a finite abelian group, the complex $C^*(X; \pi)$ is naturally identified with the filtered colimit of complexes $\mathrm{colim}_Q C^*(X/Q; \pi)$, with Q running through $\mathcal{Q}(X)$, and the cohomology $H^*(X; \pi)$ is naturally isomorphic to the colimit of the cohomologies $H^*(X/Q; \pi)$.
5. There is a Künneth formula, i.e. for two profinite spaces X and Y the map $H^*(X; \mathbb{Z}/\ell) \otimes H^*(Y; \mathbb{Z}/\ell) \to H^*(X \times Y; \mathbb{Z}/\ell)$ is an isomorphism, cf. [Mo2].

Let ℓ be a fixed prime number. Fabien Morel has shown in [Mo2] that the category $\hat{\mathcal{S}}$ can be given the structure of a closed model category in the sense of [Qu1]. The weak equivalences are the maps inducing isomorphisms in continuous cohomology with coefficients $\mathbb{Z}/\ell$; the cofibrations are the degree-wise monomorphisms and the fibrations are the maps that have the right lifting property with respect to the cofibrations that are also weak equivalences. One should note that there are functorial factorizations for any map

in $\hat{\mathcal{S}}$ into a trivial cofibration followed by a fibration, respectively into a cofibration followed by a trivial fibration. For example, we get a functorial factorization by fixing the construction of the proof of Proposition 1 on page 355 of [Mo2] on the existence of factorizations. All the constructions done there are functorial. Furthermore, all small limits and products exist in $\hat{\mathcal{S}}$.

Morel has also given an explicit construction of fibrant replacements in $\hat{\mathcal{S}}$, cf. [Mo2], 2.1. It is based on the $\mathbb{Z}/\ell$-completion functor of [BK]. Let Y be a simplicial set. We denote by $\text{Res}^\bullet Y$ its cosimplicial $\mathbb{Z}/\ell$-resolution, by $\text{Tot}_s(\text{Res}^\bullet Y)$ its s-th total space and by $P^t \text{Tot}_s \text{Res}^\bullet Y$ its t-th Postnikov decomposition, cf. [BK], part I. If Y is a finite set in each degree, then $\text{Tot}_s \text{Res}^\bullet Y$ is also finite in each degree for all $s \geq 0$. Hence the total space $\text{TotRes}^\bullet Y$, i.e. the ℓ-completion of Bousfield-Kan of Y, which is the limit of the tower $\text{Tot}_s \text{Res}^\bullet Y$, has a natural structure of a profinite space, which we denote by $\hat{Y}^\ell$. Now let X be a profinite space. We denote by $\hat{X}^\ell$ the limit in $\hat{\mathcal{S}}$ of the $\widehat{X/Q}^\ell$, Q running through $\mathcal{Q}(X)$. We call this space the *ℓ-completion of Bousfield-Kan* of the profinite space X. For a simplicial set Y we denote by $\hat{Y}^\ell \in \hat{\mathcal{S}}$ its *profinite ℓ-completion* defined as the limit $\lim_{Q,s,t} P^t \text{Tot}_s \text{Res}^\bullet(X/Q)$ in $\hat{\mathcal{S}}$, where X/Q are the simplicial finite quotients of X. Its underlying simplicial set is isomorphic in $\text{Ho}(\mathcal{S})$ to the ℓ-completion in $\mathcal{S}$ of Sullivan [Su], which is the limit of the pro-Artin-Mazur-ℓ-completion, cf. [Mo1].

Morel shows that the natural map $\theta_X : X \to \hat{X}^\ell$ defines a functorial fibrant replacement in $\hat{\mathcal{S}}$, [Mo2], 2.1, Prop. 2. The spaces $P^t \text{Tot}_s(\text{Res}^\bullet X/Q)$ are in fact ℓ-finite spaces, i.e. fibrant simplicial finite sets whose homotopy groups for every choice of basepoint are finite ℓ-groups, trivial except for a finite number of them. The profinite space $\hat{X}^\ell$ may be identified with the filtered limit in Q, s and t of the ℓ-spaces $P^t \text{Tot}_s(\text{Res}^\bullet X/Q)$. Hence $\hat{X}^\ell$ is a pro-ℓ-space in the terminology of [Mo2].

For the study of generalized cohomology theories on profinite spaces it will be crucial to check the compatibility of the ℓ-completion with other constructions in $\hat{\mathcal{S}}$, resp. $\hat{\mathcal{S}}_*$, such as products, smash products and function spaces.

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2.2 Pointed profinite spaces and the simplicial structure

The category $\hat{\mathcal{S}}$ has a pointed analogue $\hat{\mathcal{S}}_*$ whose objects are maps $* \to X$ in $\hat{\mathcal{S}}$, where $*$ denotes the constant simplicial set equal to a point. Its morphisms are maps in $\hat{\mathcal{S}}$ that respect the basepoints. The forgetful functor $\hat{\mathcal{S}}_* \to \hat{\mathcal{S}}$ has a left adjoint, which consists in adding a disjoint basepoint $X \mapsto X_+$. The ℓ-completion of a pointed profinite space is naturally pointed and $X \mapsto X_+$ is compatible with ℓ-completion. $\hat{\mathcal{S}}_*$ has the obvious induced closed model category structure.

The product for two pointed profinite spaces X and Y in $\hat{\mathcal{S}}_*$ is the smash product $X \wedge Y \in \hat{\mathcal{S}}_*$, defined in the usual way as the quotient $(X \times Y)/(X \vee Y)$ in $\hat{\mathcal{S}}$. For a pointed profinite space X we define its reduced cohomology with coefficients in the profinite abelian group π, denoted $\tilde{H}^*(X; \pi)$, to be the kernel of the induced morphism $H^*(X; \pi) \to H^*(pt; \pi)$. It is clear that $\tilde{H}^*(X_+; \pi) \cong H^*(X; \pi)$. We also have a Künneth formula.

Lemma 2.2 *For two pointed profinite spaces X and Y, the canonical map $\tilde{H}^*(X; \mathbb{Z}/\ell) \otimes \tilde{H}^*(Y; \mathbb{Z}/\ell) \to \tilde{H}^*(X \wedge Y; \mathbb{Z}/\ell)$ is an isomorphism.*

Proof The canonical map is an isomorphism in the case that X and Y are simplicial finite sets. Hence it is an isomorphism in general since the tensor product commutes with filtered colimits. $\square$

Proposition 2.3 *Let X and Y be simplicial sets whose $\mathbb{Z}/\ell$-cohomology is finite in each degree. Then the map $\widehat{X \times Y}^\ell \to \hat{X}^\ell \times \hat{Y}^\ell$ is a weak equivalence in $\hat{\mathcal{S}}$.*

Proof We have $\widehat{X \times Y}^\ell = \lim_Q (\widehat{X \times Y})/Q^\ell = \lim_{Q_1, Q_2} (X/\widehat{Q_1 \times Y}/Q_2)^\ell$, which is due to the fact that the finite quotients $(X \times Y)/Q$ are in 1-1-correspondence with finite quotients $X/Q_1 \times Y/Q_2$; and we have $\hat{X}^\ell \times \hat{Y}^\ell = \lim_{Q_1} \widehat{X/Q_1}^\ell \times \lim_{Q_1} \widehat{Y/Q_2}^\ell$. The hypothesis implies that X and Y are nilpotent spaces in the sense of [BK]. Thus the same is true for the finite quotients X/Q_1 and Y/Q_2. Then [BK] VI, §6 Proposition 6.5, says that

$$H^*((X/\widehat{Q_1 \times Y}/Q_2)^\ell; \mathbb{Z}/\ell) \to H^*(\widehat{X/Q_1}^\ell \times \widehat{Y/Q_2}^\ell; \mathbb{Z}/\ell)$$

is an isomorphism. We conclude by taking colimits and by Remark 2.1. $\square$

Proposition 2.4 *Let X and Y be pointed simplicial sets whose $\mathbb{Z}/\ell$-cohomology is finite in each degree. Then the map $\widehat{X \wedge Y}^\ell \to \hat{X}^\ell \wedge \hat{Y}^\ell$ is a weak equivalence in $\hat{\mathcal{S}}_*$.*

Proof This follows as the previous proposition from [BK] VI, §6, Proposition 6.6. $\qquad\square$

The categories $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}_*$ have natural simplicial structures. There is a simplicial set $\mathrm{Map}_{\hat{\mathcal{S}}_*}(X,Y) = \mathrm{Map}_*(X,Y)$ for all profinite spaces X and Y. It is naturally pointed by the map $X \to * \to Y$. It is characterized by the bijection

$$\mathrm{Hom}_{\mathcal{S}_*}(Z, \mathrm{Map}_*(X,Y)) \cong \mathrm{Hom}_{\hat{\mathcal{S}}_*}(X \wedge Z, Y)$$

for every pointed simplicial finite set Z. The space $\mathrm{Map}_*(X,Y)$ is given in the n-th dimension by the set $\mathrm{Hom}_{\hat{\mathcal{S}}_*}(X \wedge \Delta[n]_+, Y)$, where $\Delta[n]_+$ is the well-known pointed simplicial finite set. For a pointed simplicial finite set Z the space $\mathrm{Map}_*(Z,Y)$ has a natural structure of a profinite space considered as the filtered limit in $\hat{\mathcal{S}}_*$ of the simplicial finite sets $\mathrm{Map}_*(\mathrm{sk}_n Z, Y/Q)$. We denote this profinite space by $\hom_*(Z,Y) \in \hat{\mathcal{S}}_*$. For an arbitrary simplicial set Z we define the profinite space $\hom_*(Z,Y)$ by taking the filtered limit in $\hat{\mathcal{S}}_*$ of the profinite spaces $\hom_*(Z_s, Y)$ over all simplicial finite subsets Z_s of Z. This defines a right adjoint to the functor $X \mapsto Z \wedge X$, $\hat{\mathcal{S}}_* \to \hat{\mathcal{S}}_*$. We define the functor $X \otimes \cdot := X \wedge \cdot : \mathcal{S}_* \to \hat{\mathcal{S}}_*$ for a profinite space X by taking the filtered limit in $\hat{\mathcal{S}}_*$ of the profinite spaces $X/Q \wedge Z_s$ for all simplicial finite subsets Z_s of the simplicial set Z and all finite quotients X/Q of X. These functors define a simplicial structure on $\hat{\mathcal{S}}_*$ and similarly on $\hat{\mathcal{S}}$.

Let $Z \leftarrow X \to Y$ be a diagram of profinite spaces and let $X \to Y$ be a cofibration. We denote its colimit by $Z \cup_X Y$. On the level of continuous cochains we get that $C^*(X, \mathbb{Z}/\ell)$ is the colimit of the induced diagram $C^*(Z, \mathbb{Z}/\ell) \leftarrow C^*(Z \cup_X Y, \mathbb{Z}/\ell) \to C^*(Y, \mathbb{Z}/\ell)$. This yields a Mayer-Vietoris long exact sequence in cohomology

$$\ldots \to H^n(Z \cup_X Y; \mathbb{Z}/\ell) \to H^n Y \oplus H^n Z \to H^n X \to H^{n+1}(Z \cup_X Y; \mathbb{Z}/\ell) \to \ldots$$

Using this sequence Dehon shows that $\hat{\mathcal{S}}$ and $\hat{\mathcal{S}}_*$ carry in fact a *simplicial* model structure, cf. [De] §1.4.

As an application, we define for $X \in \hat{\mathcal{S}}_*$ the profinite space $\hat{\Omega}X \in \hat{\mathcal{S}}_*$. Let S^1 be the simplicial finite set $\Delta[1]/\partial\Delta[1]$. We set $\hat{\Omega}X := \hom_*(S^1, X)$. There is a natural isomorphism $\mathrm{Hom}_{\hat{\mathcal{S}}_*}(S^1 \wedge X, Y) \cong \mathrm{Hom}_{\hat{\mathcal{S}}_*}(X, \hat{\Omega}Y))$. In addition, $\hat{\Omega}X$ is fibrant if X is fibrant.

This adjunction may be extended to the homotopy category $\text{Ho}(\hat{\mathcal{S}}_*)$ of $\hat{\mathcal{S}}_*$. For every profinite space X and every fibrant profinite space Y, we have a natural bijection $\text{Hom}_{\text{Ho}(\hat{\mathcal{S}}_*)}(S^1 \wedge X, Y) \cong \text{Hom}_{\text{Ho}(\hat{\mathcal{S}}_*)}(X, \hat{\Omega}Y))$. Since $S^1 \wedge X$ is a cogroup object and $\hat{\Omega}X$ a group object in $\text{Ho}(\hat{\mathcal{S}}_*)$, we conclude that the previous bijection is in fact an isomorphism of groups.

The following proposition has been proved by Francois-Xavier Dehon in [De] using its Eilenberg-Moore spectral sequence for profinite spaces. We give an alternative proof using the results of Bousfield and Kan [BK].

Proposition 2.5 *Let X be a simply connected simplicial set whose $\mathbb{Z}/\ell$-cohomology is finite in every degree. Then the map $\widehat{\Omega X}^\ell \to \hat{\Omega}(\hat{X}^\ell)$ is a weak equivalence in $\hat{\mathcal{S}}_*$. This map is in fact a homotopy equivalence.*

Proof Since S^1 is a simplicial finite set, the finite quotients $\hat{X}/Q$ of $\hat{X}$ and $\text{hom}_*(S^1, X)/Q$ of $\text{hom}_*(S^1, X)$ are in 1-1 correspondence. Hence we may write

$$\hat{\Omega}(\hat{X}^\ell) = \text{hom}_*(S^1, \lim_Q \widehat{X/Q}^\ell) = \lim_Q \text{hom}_*(S^1, \widehat{X/Q}^\ell) = \lim_Q \hat{\Omega}(\widehat{X/Q}^\ell);$$

and

$$\widehat{\Omega X}^\ell = \lim_Q \left(\text{hom}_*\widehat{(S^1, X)}/Q\right)^\ell = \lim_Q (\widehat{\hat{\Omega}(X/Q)}).$$

By our hypothesis X is a nilpotent space in the sense of [BK] and hence X/Q is nilpotent, too; by [BK] VI, §6, Prop. 6.5, we know that

$$H^*((\widehat{\hat{\Omega}(X/Q)})^\ell; \mathbb{Z}/\ell) \cong H^*(\hat{\Omega}(\widehat{X/Q}^\ell); \mathbb{Z}/\ell).$$

Hence we get

$$\begin{aligned} H^*((\widehat{\hat{\Omega}(X)})^\ell; \mathbb{Z}/\ell) &\cong \text{colim}_Q H^*((\widehat{\hat{\Omega}(X/Q)})^\ell); \mathbb{Z}/\ell) \\ &\cong \text{colim}_Q H^*(\hat{\Omega}(\widehat{X/Q}^\ell); \mathbb{Z}/\ell) \cong H^*(\hat{\Omega}(\hat{X}^\ell); \mathbb{Z}/\ell), \end{aligned}$$

which is what we had to prove.

The last statement follows from the fact that the spaces in question are cofibrant and fibrant and the fact that weak equivalences between such objects are homotopy equivalences. $\square$

2.3   Homotopy groups of profinite spaces

Definition 2.6 *Let $X \in \hat{\mathcal{S}}$ be a pointed profinite space and let $\hat{X}^{\ell}$ be its ℓ-completion. We denote by $\pi_0 X$ the coequalizer in $\hat{\mathcal{E}}$ of the diagram $d_0, d_1 : X_1 \overset{\rightarrow}{\rightarrow} X_0$. We define $\pi_n X$ for $n \geq 1$ to be the group $\pi_n |\hat{X}^{\ell}|$ where $|\hat{X}^{\ell}|$ is the underlying fibrant simplicial set of $\hat{X}^{\ell}$. We call $\pi_n X$ the n-th homotopy group of X.*

Proposition 2.7 *A map $f : X \to Y$ in $\hat{\mathcal{S}}_*$ is a weak equivalence if and only if $\pi_*(f)$ is an isomorphism.*

Proof The maps $X \to \hat{X}^{\ell}$ and $Y \to \hat{Y}^{\ell}$ are weak equivalences. Hence f is a weak equivalence if and only if $\hat{f}^{\ell}$ is a weak equivalence. Since all objects in $\hat{\mathcal{S}}_*$ are cofibrant, the map $\hat{f}^{\ell}$ is a weak equivalence if and only if $\hat{f}^{\ell}$ is a homotopy equivalence. This finishes the proof. □

We can already conclude from the definition $\pi_n X = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S}_*)}(S^n, |\widehat{X}^{\ell}|)$ that $\pi_n X$ has a natural structure of a profinite group. The following proposition shows that our definition coincides with the one given by Dehon in [De].

Proposition 2.8 *1. For every profinite space X, the natural map*

$$\mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}})}(*, X) \to \pi_0 X$$

is an isomorphism.
2. For every pointed profinite space X and every $n \geq 0$ we have

$$(1) \qquad\qquad \pi_n(\hat{\Omega} X) \cong \pi_{n+1}(X).$$

In particular, we have $\pi_n X \cong \pi_0(\hat{\Omega}^n X)$.

Proof The first statement is Proposition 1.3.4 a) of [De].
The second statement follows from the fact that S^n is a simplicial finite set for all $n \geq 1$. For $n = 0$, we use the first statement to conclude by adjunction

$$
\begin{aligned}
\pi_0(\hat{\Omega} X) \quad & \cong \quad \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(*, \hat{\Omega} X) \cong \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(*, \hat{\Omega}\hat{X}^{\ell}) \\
& \cong \quad \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(S^1, \hat{X}^{\ell}) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S}_*)}(S^1, |\hat{X}^{\ell}|) \\
& = \quad \pi_1 |\hat{X}^{\ell}| = \pi_1 X.
\end{aligned}
$$

For $n \geq 1$, we have the following sequence of isomorphisms using Proposition 2.5 and adjunction

$$
\begin{aligned}
\pi_n(\hat{\Omega}X) &= \pi_n(|\widehat{\hat{\Omega}X}^\ell|) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S}_*)}(S^n, |\widehat{\hat{\Omega}X}^\ell|) \cong \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(\widehat{S^n}, \widehat{\hat{\Omega}X}^\ell) \\
&= \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(S^n, \widehat{\hat{\Omega}X}^\ell) \cong \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(S^n, \hat{\Omega}(\hat{X}^\ell)) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(S^{n+1}, \hat{X}^\ell) \cong \mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}}_*)}(S^{n+1}, \widehat{X}^\ell) \\
&\cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{S}_*)}(S^{n+1}, |\widehat{X}^\ell|) \\
&= \pi_{n+1}(X).
\end{aligned}
$$

$\square$

As a corollary, we see that $\pi_n X$ has the natural structure of a pro-ℓ-group for every $n \geq 1$, see also [Mo1], Cor. 1.4.1.3.

Proposition 2.9 *For every pointed profinite space X and every $n \geq 0$, we have an isomorphism $\pi_n X \cong \lim_{Q,s,t} \pi_n(P^t \mathrm{Tot}_s(\mathrm{Res}^\bullet X/Q))$.*
In particular, for every $n \geq 1$ the group $\pi_n X$ has the structure of a pro-ℓ-group.

Proof We know that there is an isomorphism $\hat{X}^\ell \cong \lim_{Q,s,t} P^t \mathrm{Tot}_s \mathrm{Res}^\bullet(X/Q)$. Since π_0 commutes with limits and sends weak equivalences to homeomorphisms in $\hat{\mathcal{E}}$, we conclude that we have an isomorphism

$$
\pi_n X \cong \lim_{Q,s,t} \pi_n(P^t \mathrm{Tot}_s \mathrm{Res}^\bullet(X/Q))
$$

for every $n \geq 0$. The description of the homotopy groups of the simplicial finite sets $P^t \mathrm{Tot}_s \mathrm{Res}^\bullet(X/Q)$ finishes the proof. $\square$

We conclude this subsection with a collection of results on the homotopy groups of the ℓ-completion of a simplicial set, see also [BK] VI, §5.

Proposition 2.10 *Let X be a pointed connected simplicial set.*
1. We have an isomorphism

$$
\pi_1(\widehat{X}^\ell) \cong \widehat{\pi_1(X)}^\ell.
$$

In particular, if $\pi_1 X$ is a finitely generated abelian group, then $\pi_1(\widehat{X}^\ell) \cong \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \pi_1 X$.
2. If X is in addition simply connected, and its higher homotopy groups are finitely generated, then we have for all $n \geq 2$

$$
\pi_n \hat{X}^\ell \cong \widehat{\pi_n X}^\ell \cong \mathbb{Z}_\ell \otimes_{\mathbb{Z}} \pi_n X,
$$

where $\mathbb{Z}_\ell$ denotes the ℓ-adic integers.

Proof We deduce this proposition from the methods of [BK]. The hypothesis on X implies that X is a nilpotent space in the sense of [BK], i.e. $\pi_1 X$ acts nilpotently on all $\pi_i X$. We denote by $\mathbb{Z}/\ell_\infty X$ the Bousfield-Kan $\mathbb{Z}/\ell$-completion in $\mathcal{S}$ of X. The assumptions imply that X is a $\mathbb{Z}/\ell$-good space, i.e. $H^*(X; \mathbb{Z}/\ell) \cong H^*(\mathbb{Z}/\ell_\infty X; \mathbb{Z}/\ell)$, cf. [BK] VI, §5, Proposition 5.3. Considering the isomorphism $H^*(X; \mathbb{Z}/\ell) \cong H^*(\hat{X}; \mathbb{Z}/\ell) \cong H^*(|\hat{X}^\ell|; \mathbb{Z}/\ell)$, this implies that $|\hat{X}^\ell|$ is $\mathbb{Z}/\ell$-complete by [BK] V, §3, Proposition 3.3. Then Proposition 5.4 of [BK], VI, §5, says that the two ℓ-completions $|\hat{X}^\ell|$ and $\mathbb{Z}/\ell_\infty X$ are weakly equivalent. Then [BK] VI, §5, Proposition 5.2 yields the result.

Alternatively, one could deduce the result from [Su], Theorem 3.1, and the fact that $|\widehat{X}^\ell|$ is isomorphic in $\mathrm{Ho}(\mathcal{S})$ to the ℓ-completion of Sullivan. $\square$

Example 2.11 *In the special case of the simplicial circle S^1, viewed as a profinite space, one has*

$$\pi_1(S^1) = \pi_1(\widehat{S^1}^\ell) \cong \widehat{\mathbb{Z}}^\ell = \mathbb{Z}_\ell.$$

The last proposition is crucial for our studies of cohomology theories on $\hat{\mathcal{S}}$, since it will allow us to compute the coefficients for some cohomology theories, e.g. for profinite cobordism $\widehat{MU}$.

2.4   Comparison with the category of pro-simplicial sets

Let $\mathrm{pro} - \mathcal{S}$ be the category of pro-objects in $\mathcal{S}$. Its objects are cofiltered diagrams $X(-) : I \to \mathcal{S}$ and its morphisms are defined by

$$\mathrm{Hom}_{\mathrm{pro}-\mathcal{S}}(X(-), Y(-)) := \lim_{t \in J} \mathrm{colim}_{s \in I} \mathrm{Hom}_{\mathcal{S}}(X(s), Y(t)),$$

see [AM] for more details. The cohomology with $\mathbb{Z}/\ell$-coefficients is defined to be

$$H^*(X(-), \mathbb{Z}/\ell) := \mathrm{colim}_s H^*(X(s), \mathbb{Z}/\ell).$$

Isaksen has constructed several model structures on $\mathrm{pro} - \mathcal{S}$, cf. [I1] and [I4]. We are interested in the $\mathbb{Z}/\ell$-cohomological model structure of [I4] in which the weak equivalences are morphisms inducing isomorphisms in $\mathbb{Z}/\ell$-cohomology and the cofibrations are levelwise monomorphisms.

We want to compare $\mathrm{pro} - \mathcal{S}$ with $\hat{\mathcal{S}}$. These two categories are not equivalent, but we will see that completion induces an equivalence on the level of

homotopy categories. We define a completion functor $(\hat{-}) : \mathrm{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ as the composite of two functors. First we apply $(\hat{-}) : \mathcal{S} \to \hat{\mathcal{S}}$ levelwise to get a functor $\mathrm{pro} - \mathcal{S} \to \mathrm{pro} - \hat{\mathcal{S}}$. Then we take the limit in $\hat{\mathcal{S}}$ of the underlying diagram. The next lemma shows that the functor $(\hat{-})$ has good properties with respect to the model structures on $\mathrm{pro} - \mathcal{S}$ and $\hat{\mathcal{S}}$.

Lemma 2.12 *1. Let $X \in \mathrm{pro} - \mathcal{S}$ be a pro-simplicial set and let π be a finite abelian group. Then we have a natural isomorphism of cohomology groups*

$$H^*(X; \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(\hat{X}; \mathbb{Z}/\ell).$$

2. A morphism $f : X \to Y$ of pro-simplicial sets induces an isomorphism in $\mathbb{Z}/\ell$-cohomology if and only if the morphism $\hat{f} : \hat{X} \to \hat{Y}$ in $\hat{\mathcal{S}}$ induces an isomorphism in continuous $\mathbb{Z}/\ell$-cohomology.
3. The functor $(\hat{-}) : \mathrm{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ preserves monomorphisms.
4. The functor $(\hat{-}) : \mathrm{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ preserves finite limits and finite colimits.

Proof 1. This follows from the definition of $(\hat{-})$ and Remark 2.1.
2. Suppose $f : X \to Y$ in $\mathrm{pro} - \mathcal{S}$ induces an isomorphism in $\mathbb{Z}/\ell$-cohomology. We have the natural sequence of commutative squares

$$
\begin{array}{ccc}
H^*(X; \mathbb{Z}/\ell) & \xrightarrow{f^*} & H^*(Y; \mathbb{Z}/\ell) \\
{=}\downarrow & & \downarrow{=} \\
\mathrm{colim}_s H^*(X_s; \mathbb{Z}/\ell) & \longrightarrow & \mathrm{colim}_t H^*(Y_t; \mathbb{Z}/\ell) \\
\cong\downarrow & & \downarrow\cong \\
\mathrm{colim}_s H^*(\widehat{X_s}; \mathbb{Z}/\ell) & \longrightarrow & \mathrm{colim}_t H^*(\widehat{Y_t}; \mathbb{Z}/\ell) \\
\cong\downarrow & & \downarrow\cong \\
H^*(\hat{X}; \mathbb{Z}/\ell) & \xrightarrow{\hat{f}^*} & H^*(\hat{Y}; \mathbb{Z}/\ell)
\end{array}
$$

which proves the second assertion.
3. This is clear.
4. Since $(\hat{-}) : \mathcal{S} \to \hat{\mathcal{S}}$ is a left adjoint functor it preserves all colimits and it commutes with finite limits. The functor $\lim : \mathrm{pro} - \hat{\mathcal{S}} \to \hat{\mathcal{S}}$ commutes with all limits and finite colimits. Hence the last statement holds, too. $\square$

Corollary 2.13 *The functor $(\hat{-}) : \mathrm{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ induces an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathrm{pro} - \mathcal{S}) \xrightarrow{\sim} \mathrm{Ho}(\hat{\mathcal{S}}),$$

where the left hand side is the $\mathbb{Z}/\ell$-cohomological homotopy category of [I4].

Isaksen has considered a different functor $\mathrm{pro}-\mathcal{S} \to \hat{\mathcal{S}}$ in [I4] and has shown that it induces an equivalence on the level of homotopy categories, too. Our approach fits better in the later picture for the comparison of generalized cohomology theories.

2.5   The model structure on $\hat{\mathcal{S}}$ is fibrantly generated

For our studies of generalized cohomology theories and a stable structure on profinite spaces, we have to show the technical point that the model structure on $\hat{\mathcal{S}}$, hence also on $\hat{\mathcal{S}}_*$, is fibrantly generated. The necessity of this point becomes clear from the localization and stabilization results of the appendix, cf. Theorem A.3.

We recall some notations and constructions from [Mo2]. Let $n \geq 0$ be a non-negative integer and S be a profinite set. The functor $\hat{\mathcal{S}}^{\mathrm{op}} \to \mathcal{E}$, $X \mapsto \mathrm{Hom}_{\hat{\mathcal{E}}}(X_n, S)$ is representable by a simplicial profinite set, which we denote by $L(S, n)$. It is given by the formula

$$L(S, n) : \Delta^{\mathrm{op}} \to \hat{\mathcal{E}}, \ [k] \mapsto S^{\mathrm{Hom}_\Delta([n],[k])}.$$

If M is a profinite abelian group, then $L(M, n)$ has a natural structure of a simplicial profinite abelian group and the abelian group $\mathrm{Hom}_{\hat{\mathcal{S}}}(X, L(M, n))$ can be identified with the group $C^n(X; M)$ of continuous n-cochains of X with values in M. Furthermore, for every k $L(M, *)([k])$ may be considered in the usual way as an abelian cochain complex. For a profinite abelian group M, let $Z^n(X; M)$ denote the abelian group of n-cocycles of the complex $C^*(X; M)$. The functor $\hat{\mathcal{S}}^{\mathrm{op}} \to \mathcal{E}$, $X \mapsto Z^n(X; M)$ is also representable by a simplicial profinite abelian group, which we denote by $K(M, n)$, called the profinite Eilenberg-MacLane space of type (M, n). We may define $K(M, n)([k])$ to be the subgroup of all cocycles of $L(M, n)([k])$; see also [Ma1] §23, for these constructions.
The natural homomorphism $C^n(X; M) \to Z^{n+1}(X; M)$ given by the differential defines a natural morphism of simplicial profinite abelian groups $L(M, n) \to K(M, n + 1)$.
Consider the two sets of morphisms

$$P := \{L(M, n) \to K(M, n+1), \ K(M, n) \to *|M \text{ abelian pro} - \ell - \mathrm{group}, n \geq 0\}$$

and

$$Q := \{L(M, n) \to *|M \text{ abelian pro} - \ell - \mathrm{group}, n \geq 0\}$$

of Lemme 2 of [Mo2].

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Theorem 2.14 *The simplicial model structure on $\hat{\mathcal{S}}$, in which the weak equivalences are the $\mathbb{Z}/\ell$-cohomological isomorphisms and the cofibrations are the dimensionwise monomorphisms, is left proper and fibrantly generated with P as the set of generating fibrations and Q as the set of generating trivial fibrations.*

Proof The left properness follows from the fact that all objects in $\hat{\mathcal{S}}$ are cofibrant, cf. Corollary 13.1.3 of [Hi].

We write Fib and Cof for the classes of fibrations and cofibrations of the model structure on $\hat{\mathcal{S}}$ of [Mo2]. Note that the subcategory Q-cocell of relative Q-cocell complexes consists here of limits of pullbacks of elements of Q and look at Definition 2.1.7 of [Ho1] for the notations $P - \mathrm{proj}$ and $Q - \mathrm{fib}$ used below. In order to prove the second statement we check the six conditions of the dual of Theorem 2.1.19 of [Ho1]. We only assume that the category has all small limits but only finite colimits. But since we use cosmall instead of small objects, the theorem holds for this kind of fibrantly generated model categories as well. We check now the six conditions of [Ho1], Theorem 2.1.19.

1. It is clear that the weak equivalences satisfy the two-out-of-three axioms.

2. and 3. We have to show that the codomains of the maps in P and Q are cosmall relative to P-cocell and Q-cocell, respectively. Note that the terminal object $*$ is cosmall relative to all maps. Hence it remains to show that the objects $K(M, n)$ are cosmall relative to Q-cocell. It suffices to show that they are cosmall relative to a filtered sequence of maps $\ldots \to Y_1 \to Y_0$ of maps in $\hat{\mathcal{S}}$. Consider the canonical map $f : \mathrm{colim}_\alpha \mathrm{Hom}_{\hat{\mathcal{S}}}(Y_\alpha, K(M, n)) \to \mathrm{Hom}_{\hat{\mathcal{S}}}(\lim_\alpha Y_\alpha, K(M, n))$. We have to show that it is an isomorphism. By the definition of the spaces $K(M, n)$ this map is equal to the map

$$\mathrm{colim}_\alpha Z^n(Y_\alpha, M) \to Z^n(\lim_\alpha Y_\alpha, M).$$

But this map is already an isomorphism on the level of complexes $\mathrm{colim}_\alpha C^n(Y_\alpha, M) \cong C^n(\lim_\alpha Y_\alpha, M)$ as in [Se], Proposition 8; hence f is an isomorphism.

4. We know that $L(M, n) \to *$ is a trivial fibration. Since trivial fibrations are preserved under pullbacks and limits, we get $Q - \mathrm{cocell} \subseteq W \cap \mathrm{Fib} \subseteq W \cap P - \mathrm{fib}$ which is what we had to show.

5. Given a diagram

(2)
$$\begin{array}{ccc} A & \to & L(M, n) \\ \downarrow & & \downarrow \\ B & \to & * \end{array}$$

we have to show that there is a lift if the map $f : A \to B$ is in $P - \mathrm{proj}$, i.e. has the left lifting property with respect to P. But if F is in $P - \mathrm{proj}$, then

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we get a lifting $B \to K(M, n+1)$ for any map $A \to K(M, n+1)$. Hence the diagram (2) yields a diagram

$$
(3) \qquad
\begin{array}{ccc}
A & \to & L(M, n) \\
\downarrow & & \downarrow \\
B & \to & K(M, n+1) \\
\| & & \downarrow \\
B & \to & *
\end{array}
$$

and we know that there is a lifting $B \to L(M, n)$ in the upper rectangle which is also the lift of the diagram (2) above. Hence $f \in Q - \mathrm{proj}$.

It remains to show that $P - \mathrm{proj} \subseteq W$. Let $f : A \to B$ be a map in $P - \mathrm{proj}$. By definition of the spaces $K(\mathbb{Z}/\ell, n)$ and the definition of $P - \mathrm{proj}$ we get that $f^* : Z^n(B, \mathbb{Z}/\ell) \to Z^n(A, \mathbb{Z}/\ell)$ is surjective for all $n \geq 0$. Hence it is enough to show that $f^*(\mathrm{Im}\,(C^n A \to Z^{n+1} A)) \subseteq \mathrm{Im}\,(C^n B \to Z^{n+1} B)$. The other lifting property of maps in $P - \mathrm{proj}$ is equivalent to the surjectivity of the map $C^n B \to C^n A \times_{Z^{n+1} A} Z^{n+1} B$ from which we get the desired result that $f^* : H^n(B, \mathbb{Z}/\ell) \to H^n(A, \mathbb{Z}/\ell)$ is an isomorphism for all $n \geq 0$.

6. We show that $W \cap Q - \mathrm{proj} \subseteq P - \mathrm{proj}$. Let $f : A \to B$ be a map that belongs to W and $Q - \mathrm{proj}$. Since f^* is an isomorphism on $\mathbb{Z}/\ell$-cohomology, it can be shown that f induces an isomorphism on cohomology with coefficients in any abelian pro-ℓ-group M. This implies already the surjectivity of the map $f^* : Z^n(B, M) \to Z^n(A, M)$ for all $n \geq 0$. The lifting property of maps in $Q - \mathrm{proj}$ implies the surjectivity of the induced map $f^* : C^n(B, M) \to C^n(A, M)$ for all $n \geq 0$. Using the isomorphism on cohomology it follows that $C^n B \to C^n A \times_{Z^{n+1} A} Z^{n+1} B$ is surjective which is equivalent to the other desired lifting property of maps in $P - \mathrm{proj}$. Hence $f \in P - \mathrm{proj}$.

Now we have proved using theorem 2.1.19 of [Ho1] that the quadruple $(\hat{\mathcal{S}}, W, Q - \mathrm{proj}, P - \mathrm{fib})$ is a fibrantly generated model category. It remains to show that it coincides with the given structure $(\hat{\mathcal{S}}, W, \mathrm{Cof}, \mathrm{Fib})$. Since $L(M, n) \to *$ is a trivial fibration, we know already $\mathrm{Cof} \subseteq Q - \mathrm{proj}$. It remains to show that every map in $Q - \mathrm{proj}$ is a monomorphism in each dimension. Then the cofibrations and weak equivalences of the two structures coincide and hence the fibrations coincide as well.

Let $f : X \to Y$ be a map in $Q - \mathrm{proj}$. Hence every map $X \to L(M, n)$ can be lifted to a map $Y \to L(M, n)$. This means by using the definition of the spaces $L(M, n)$ that the map $C^n(Y, M) \to C^n(X, M)$ induced by f is surjective for all $n \geq 0$. This is equivalent to the surjectivity of the maps $\mathrm{Hom}_{\hat{\mathcal{E}}}(Y_n, M) \to \mathrm{Hom}_{\hat{\mathcal{E}}}(X_n, M)$ for all $n \geq 0$ and all abelian pro-ℓ-groups M. If we choose $M = F(X_n)$ to be the free abelian pro-ℓ-group on the set X_n defined in [Se], then we see that the map $X_n \to F(X_n)$, sending x to

its class in $F(X_n)$, is in the image of $\mathrm{Hom}_{\hat{\mathcal{E}}}(Y_n, M) \to \mathrm{Hom}_{\hat{\mathcal{E}}}(X_n, M)$ only if $X_n \hookrightarrow Y_n$ is a monomorphism for each n. Hence f is also a cofibration. $\quad\square$

3 Profinite spectra

We introduce the usual notion of spectra on $\hat{\mathcal{S}}$. The results of the appendix show that there is a stable model structure on $\mathrm{Sp}(\hat{\mathcal{S}})$. This is the main technical result for the construction of generalized cohomology theories on schemes via the profinite étale topological type functor.

We study the behavior of the profinite completion functor $(\hat{-}) : \mathrm{Sp}(\mathcal{S}) \to \mathrm{Sp}(\hat{\mathcal{S}})$ and show that it factorizes through stable equivalences and is in fact a left Quillen functor with right adjoint the forgetful functor.

The following subsection is dedicated to stable homotopy groups and their relation to stable equivalences. We show that these homotopy groups behave well under completion of spaces. In particular, we study them for profinite Eilenberg-MacLane spectra and the completed complex cobordism spectrum $\hat{MU}$. In this last case, we get $\pi_*(\hat{MU}) \cong \mathbb{Z}_\ell \otimes \pi_*(MU)$.

Furthermore, we give an explicit construction of a fibrant replacement functor for spectra satisfying some conditions among which is MU. These results are similar to those of [De].

At the end of this section we deduce the existence of Postnikov-decompositions for profinite spectra from general model category theory.

3.1 The stable structure of profinite spectra

In order to stabilize the category $\hat{\mathcal{S}}$ of profinite spaces we begin with the usual definition of spectra.

Definition 3.1 *A* spectrum X *of simplicial profinite sets consists of a sequence* $X_n \in \hat{\mathcal{S}}_*$ *of pointed profinite spaces for* $n \geq 0$ *and maps* $\sigma_n : S^1 \wedge X_n \to X_{n+1}$ *in* $\hat{\mathcal{S}}_*$, *where* $S^1 = \Delta^1/\partial\Delta^1 \in \hat{\mathcal{S}}_*$.
A map $f : X \to Y$ *of spectra consists of maps* $f_n : X_n \to Y_n$ *in* $\hat{\mathcal{S}}_*$ *for* $n \geq 0$ *such that* $\sigma_n(1 \wedge f_n) = f_{n+1}\sigma_n$.
We denote by $\mathrm{Sp}(\hat{\mathcal{S}})$ *the corresponding category and call it the category of profinite spectra.*

We apply the results of Appendix B to the category $\hat{\mathcal{S}}_*$. By Theorem 2.14 its model structure is simplicial, left proper and fibrantly generated. Note

that $S^1 \wedge \cdot$ is a left Quillen endofunctor since it takes monomorphisms to monomorphisms and preserves cohomological equivalences by Lemma 2.2.

Theorem 3.2 *There is a* stable model structure on $\mathrm{Sp}(\hat{\mathcal{S}})$ *for which the prolongation* $S^1 \wedge \cdot : \mathrm{Sp}(\hat{\mathcal{S}}) \to \mathrm{Sp}(\hat{\mathcal{S}})$ *is a Quillen equivalence.*
In particular, the stable equivalences *are the maps that induce an isomorphism on all generalized cohomology theories, represented by profinite* $\hat{\Omega}$-*spectra; the* stable cofibrations *are the maps* $i : A \to B$ *such that* i_0 *and the induced maps* $j_n : A_n \amalg_{S^1 \wedge A_{n-1}} S^1 \wedge B_{n-1} \to B_n$ *are monomorphisms; the* stable fibrations *are the maps with the right lifting property with respect to all maps that are both stable equivalences and stable cofibrations.*

Proof By Theorem 2.14, we know that the model structure on $\hat{\mathcal{S}}_*$ is fibrantly generated, left proper and simplicial. Hence we can apply Theorem B.11 to $\hat{\mathcal{S}}_*$, $S^1 \wedge \cdot$. The fact that $S^1 \wedge \cdot$ is a Quillen equivalence for this model structure is implied by Theorem B.13 for $T = S^1 \wedge \cdot$. $\qquad\square$

Remark 3.3 *The stable fibrant objects are exactly the* $\hat{\Omega}$-*spectra, i.e. spectra* E *such that each* E_n *is a fibrant object and the adjoint structure maps* $E_n \to \hat{\Omega}E_{n+1}$ *are weak equivalences in* $\hat{\mathcal{S}}_*$ *for all* $n \geq 0$.

Define the functor $\hat{\Omega} : \mathrm{Sp}(\hat{\mathcal{S}}) \to \mathrm{Sp}(\hat{\mathcal{S}}), E \mapsto \hat{\Omega}(E)$, where $\hat{\Omega}(E)_n := \hat{\Omega}(E_n)$. As usual one can prove the following lemma.

Lemma 3.4 *1. The functor*

$$\hat{\mathcal{SH}} \to \hat{\mathcal{SH}}, \ E \mapsto S^1 \wedge E$$

is an equivalence of categories.
2. The homotopy category $\hat{\mathcal{SH}}$ *of profinite spectra is an additive category.*

Proof Choose a functorial stable replacement

$$E \mapsto E^f.$$

Then $E \mapsto \hat{\Omega}(E^f)$ is an inverse to $E \mapsto S^1 \wedge E$ using the two stable equivalences

$$E \to \hat{\Omega}((S^1 \wedge E)^f)$$

and
$$S^1 \wedge \hat{\Omega}(E^f) \to E^f$$
which are induced by the adjunction morphisms.

As S^1 is a cogroup object in $\mathrm{Ho}(\hat{\mathcal{S}}_*)$, its codiagonal $\phi : S^1 \to S^1 \vee S^1$ induces, via first assertion of the lemma and $\phi \wedge \mathrm{id}_E : S^1 \wedge E \to S^1 \wedge E \vee S^1 \wedge E$, for any spectrum E a morphism $E \to E \vee E$. This yields a natural structure of a cogroup object on E. Being natural in E, this structure has to be abelian. Thus the category $\hat{\mathcal{SH}}$ is additive. $\square$

Remark 3.5 *Since* $\mathrm{Sp}(\hat{\mathcal{S}})$ *is a pointed model category we may conclude by the general theory of [Qu1] I, §2 and §3, that* $\mathrm{Sp}(\hat{\mathcal{S}})$ *has fiber and cofiber sequences, which satisfy the axioms of a triangulation. Since the above theorem tells us that* Σ *is an equivalence on* $\hat{\mathcal{SH}}$, *we may call* $\mathrm{Sp}(\hat{\mathcal{S}})$ *a stable category.*

3.2 Profinite completion of spectra

Lemma 3.6 *The functor* $(\hat{-}) : \mathcal{S}_* \to \hat{\mathcal{S}}_*$ *commutes with smash products. Furthermore, the two functors* Ω *and* $\hat{\Omega}$ *agree after applying the forgetful functor* $|\cdot| : \hat{\mathcal{S}}_* \to \mathcal{S}_*$, *i.e.* $\Omega(|Z|) \cong |\hat{\Omega}(Z)|$ *for every pointed profinite space* $Z \in \hat{\mathcal{S}}_*$.

Proof Since $(\hat{-})$ is left adjoint to the forgetful functor it preserves colimits. Hence it remains to check that it commutes with finite products. But this follows from the fact that the finite quotients of a product $X \times Y$ of two spaces are in bijective correspondence with pairs of finite quotients of X and Y respectively. Since $(\hat{-})$ is defined by applying a limit we see immediately that it commutes with finite products.

For every $X \in \mathcal{S}_*$ we have natural bijections coming from adjunction

$$
\begin{aligned}
\mathrm{Hom}_{\mathcal{S}}(S^1 \wedge X, |\hat{\Omega}Z|) &\cong \mathrm{Hom}_{\hat{\mathcal{S}}}(\hat{X}, \hat{\Omega}Z) &\cong \mathrm{Hom}_{\hat{\mathcal{S}}}(S^1 \wedge \hat{X}, Z) \\
&\cong \mathrm{Hom}_{\hat{\mathcal{S}}}(\widehat{S^1 \wedge X}, Z) &\cong \mathrm{Hom}_{\mathcal{S}}(S^1 \wedge X, |Z|) \\
&\cong \mathrm{Hom}_{\mathcal{S}}(X, \Omega|Z|).
\end{aligned}
$$

Hence the two Hom-functors defined by $\Omega(|Z|)$ and $|\hat{\Omega}(Z)|$ agree on $\mathcal{S}$ and hence by Yoneda the two spaces $\Omega(|Z|)$ and $|\hat{\Omega}(Z)|$ are isomorphic for every profinite space Z. $\square$

Let $\mathrm{Sp}(\mathcal{S})$ be the stable model structure of simplicial spectra defined in [BF]. We may extend the profinite completion to spectra. Let $(\hat{-}) : \mathrm{Sp}(\mathcal{S}) \to \mathrm{Sp}(\hat{\mathcal{S}})$

be the profinite completion applied levelwise that takes the spectrum X to the profinite spectrum $\hat{X}$ whose structure maps are given, using Lemma 3.6, by

$$S^1 \wedge \hat{X}_n \cong \widehat{S^1 \wedge X_n} \xrightarrow{\hat{\sigma}} \hat{X}_{n+1}.$$

Since the two functors $(\hat{-})$ and $|\cdot|$ on $\hat{\mathcal{S}}_*$ commute with smash products we get that commutative diagrams

$$
\begin{array}{ccc}
X_{n+1} & \rightarrow & |Y_{n+1}| \\
\uparrow & & \uparrow \\
S^1 \wedge X_n & \rightarrow & S^1 \wedge |Y_n|
\end{array}
$$

are in bijective correspondence with commutative diagrams

$$
\begin{array}{ccc}
\hat{X}_{n+1} & \rightarrow & Y_{n+1} \\
\uparrow & & \uparrow \\
S^1 \wedge \hat{X}_n & \rightarrow & S^1 \wedge Y_n
\end{array}
$$

for every $X \in \mathrm{Sp}(\mathcal{S})$ and $Y \in \mathrm{Sp}(\hat{\mathcal{S}})$. Let $|\cdot| : \mathrm{Sp}(\hat{\mathcal{S}}) \rightarrow \mathrm{Sp}(\mathcal{S})$ be the levelwise applied forgetful functor. Hence $(\hat{-})$ and $|\cdot|$ form an adjoint pair of functors.

Proposition 3.7 *The functor $(\hat{-}) : \mathrm{Sp}(\mathcal{S}) \rightarrow \mathrm{Sp}(\hat{\mathcal{S}})$ preserves weak equivalences and cofibrations.*
The functor $|\cdot| : \mathrm{Sp}(\hat{\mathcal{S}}) \rightarrow \mathrm{Sp}(\mathcal{S})$ preserves fibrations and weak equivalences between fibrant objects.
In particular, $(\hat{-})$ induces a functor on the homotopy categories and the adjoint pair $((\hat{-}), |\cdot|)$ is a Quillen pair of adjoint functors.

Proof Let $i : A \rightarrow B$ be a cofibration in $\mathrm{Sp}(\mathcal{S})$. Since $(\hat{-}) : \mathcal{S}_* \rightarrow \hat{\mathcal{S}}_*$ preserves cofibrations and pushouts as a left Quillen functor, the maps $\hat{i}_0$ and $\hat{j}_n$ are cofibrations in $\hat{\mathcal{S}}_*$. Hence $\hat{i}$ is a cofibration in $\mathrm{Sp}(\hat{\mathcal{S}})$.
Now let $f : X \rightarrow Y$ be a weak equivalence in $\mathrm{Sp}(\mathcal{S})$. We want to show that $\hat{f}$ is a weak equivalence in $\mathrm{Sp}(\hat{\mathcal{S}})$. By [Hi], Theorem 9.7.4, this is equivalent to the statement that $\mathrm{Map}(\tilde{\hat{f}}, E) : \mathrm{Map}(\tilde{\hat{Y}}, E) \rightarrow \mathrm{Map}(\tilde{\hat{X}}, E)$ is a weak equivalence of simplicial sets for every fibrant object E and some cofibrant approximation $\tilde{\hat{f}}$ of $\hat{f}$ in $\mathrm{Sp}(\hat{\mathcal{S}})$. Since $(\hat{-})$ and $X \wedge \cdot$ are left adjoints, we have natural isomorphisms extending the adjunction of $(\hat{-})$ and $|\cdot|$ for every $U \in \mathrm{Sp}(\mathcal{S})$ and $V \in \mathrm{Sp}(\hat{\mathcal{S}})$

$$(4) \qquad \mathrm{Map}_{\mathrm{Sp}(\mathcal{S})}(U, |V|) \cong \mathrm{Map}_{\mathrm{Sp}(\hat{\mathcal{S}})}(\hat{U}, V).$$

Hence $\mathrm{Map}_{\mathrm{Sp}(\hat{\mathcal{S}})}(\hat{f}, E)$ is a weak equivalence of simplicial sets if and only if $\mathrm{Map}_{\mathrm{Sp}(\mathcal{S})}(f, |E|)$ is a weak equivalence. Since $\hat{(-)}$ preserves cofibrations, the completion of $\tilde{f}$ is a cofibrant approximation of $\hat{f}$. We conclude the first part by [Hi], Theorem 9.7.4, and the following lemma, reminding the fact that the fibrant objects of $\mathrm{Sp}(\mathcal{S})$ are exactly the Ω-spectra.

Lemma 3.8 *If E is an $\hat{\Omega}$-spectrum, then $|E|$ is an Ω-spectrum.*

This is the analogue of the fact that the underlying simplicial set $|X|$ of a fibrant profinite space is a Kan simplicial set, cf. [Mo2], §2.1, Proposition 1.

Proof Let E be an $\hat{\Omega}$-spectrum. This implies that each E_n is fibrant and $E_n \to \hat{\Omega} E_{n+1}$ is a weak equivalence for each $n \geq 0$. By [Mo2], § 2, Proposition 1, this implies that $|E_n|$ is fibrant and $|E_n| \to \Omega|E_{n+1}|$ is a weak equivalence for each n. Hence $|E|$ is an Ω-spectrum. $\qquad\square$

We continue the proof of the second statement of the proposition. The fact that $|\cdot|$ preserves fibrations follows now from adjunction since $\hat{(-)}$ preserves trivial cofibrations. Now let $f : E \to F$ be a weak equivalence between fibrant objects in $\mathrm{Sp}(\hat{\mathcal{S}})$. Again by [Hi], Theorem 9.7.4, we have to show that $\mathrm{Map}_{\mathrm{Sp}(\mathcal{S})}(W, |f|)$ is a weak equivalence for every cofibrant object W of $\mathrm{Sp}(\mathcal{S})$. By the isomorphism (4) this is equivalent to that $\mathrm{Map}_{\mathrm{Sp}(\hat{\mathcal{S}})}(\hat{W}, f)$ is a weak equivalence for every such W. But since $\hat{W}$ is also cofibrant in $\mathrm{Sp}(\hat{\mathcal{S}})$ and since f is a weak equivalence in $\mathrm{Sp}(\hat{\mathcal{S}})$, we get that $\mathrm{Map}_{\mathrm{Sp}(\hat{\mathcal{S}})}(\hat{W}, f)$ is a weak equivalence of simplicial sets.

The last statement follows from general model category theory. $\qquad\square$

3.3 Stable homotopy groups and fibrant replacements

For a level fibrant replacement functor of a profinite spectrum E we may use the explicit construction of the ℓ-completion functor $X \mapsto \widehat{X}^{\ell}$ in $\hat{\mathcal{S}}_*$. Since all construction in [BK] and [Mo2] are compatible with products there are natural maps

$$X \times \widehat{Y}^{\ell} \to \widehat{X}^{\ell} \times \widehat{Y}^{\ell} \xrightarrow{\sim} \widehat{X \times Y}^{\ell}$$

and similarly maps

$$X \wedge \widehat{Y}^{\ell} \to \widehat{X}^{\ell} \wedge \widehat{Y}^{\ell} \xrightarrow{\sim} \widehat{X \wedge Y}^{\ell}.$$

For a simplicial spectrum E this yields structure maps

$$S^1 \wedge \widehat{E_n}^\ell \to (\widehat{S^1 \wedge E_n})^\ell \overset{\widehat{\sigma_n}^\ell}{\to} \widehat{E_{n+1}}^\ell,$$

which ensure that we may associate to every profinite spectrum E a level fibrant spectrum E^{lf} such that $E \to E^{lf}$ is a level equivalence.

We may also define homotopy groups of profinite spectra. For $n \in \mathbb{Z}$ we set

(5) $$\pi_n(E) := \pi_n(E^{lf}) = \mathrm{colim}_k \pi_{n+k}(E_k^{lf}).$$

For morphisms of spectra we set $\pi_n(g) := \pi_n(g^{lf})$.

Remark 3.9 *In order to calculate the stable homotopy groups of a spectrum E, we may replace it by a fibrant spectrum E^f. Then its homotopy groups are the homotopy groups of the infinite loop space E_0^f. By Proposition 2.9 we know that the homotopy groups of a profinite space have a natural structure of pro-ℓ-groups. Hence we conclude that the stable homotopy groups of any profinite spectrum have the structure of pro-ℓ-groups.*

We may describe stable equivalences via stable homotopy groups.

Proposition 3.10 *A map of profinite spectra $g : E \to F$ is a stable equivalence if and only if $\pi_n(g)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Proof Since we are interested in the homotopy type of the map, we may consider a stable fibrant replacement $g^f : E^f \to F^f$ of g. The map g is a stable equivalence if and only if g^f is a stable equivalence. But a map between $\hat{\Omega}$-spectra is a stable equivalence if and only if it is a level equivalence, e.g. see [Ho2]. But this means that all maps $\pi_n(g_k^f)$ are isomorphisms for all n and k, which finishes the proof by Proposition 2.7. $\square$

Corollary 3.11 *Let $f : X \to Y$ be a map in $\mathrm{Sp}(\hat{\mathcal{S}})$ such that $|f| : |X| \to |Y|$ is a stable equivalence of $\mathrm{Sp}(\mathcal{S})$. Then f is a stable equivalence of $\mathrm{Sp}(\hat{\mathcal{S}})$.*

Proof This follows immediately from the proposition above. $\square$

From model category theory we know that fibrant replacements with respect to the stable model structure exist for every $E \in \mathrm{Sp}(\hat{\mathcal{S}})$. We would like to construct explicit stable fibrant replacements in $\mathrm{Sp}(\hat{\mathcal{S}})$. We will employ the

Bousfield-Kan-ℓ-completion functor defined on profinite spaces. Since the ℓ-completion behaves well only under certain conditions, cf. [BK], we give a construction only for profinite spectra satisfying the following hypotheses.

The idea for the following construction is inspired by Dehon [De]. Let E be a (-1)-connected profinite spectrum, i.e. each profinite space E_n is $(n-1)$-connected for all $n \geq 0$ and let $|E| \in \mathrm{Sp}(\mathcal{S})$ be its underlying simplicial spectrum. We may consider an Ω-spectrum $R(|E|)$ stable equivalent to $|E|$, i.e. a stable fibrant replacement of $|E|$ in $\mathrm{Sp}(\mathcal{S})$. After the application of the levelwise ℓ-completion we get maps $\widehat{(R|E|)}_n^{\,\ell} \to \Omega(\widehat{R|E|})_{n+1}^{\,\ell}$ that are weak equivalences in $\hat{\mathcal{S}}_*$ for all n. Then Proposition 2.5 implies that all the composite maps

$$\widehat{(R|E|)}_n^{\,\ell} \to (\Omega\widehat{R|E|}_{n+1})^{\ell} \to \hat{\Omega}(\widehat{R|E|})_{n+1}^{\,\ell}$$

are weak equivalences in $\hat{\mathcal{S}}_*$ for all $n \geq 1$. It is clear that the resulting profinite spectrum is also level fibrant. From Corollary 3.11, we deduce that it is also stable equivalent to E. Hence we have constructed an explicit stable fibrant replacement of E in $\mathrm{Sp}(\hat{\mathcal{S}})$.

Definition and Proposition 3.12 *Let E be a (-1)-connected profinite spectrum. We suppose that the $\mathbb{Z}/\ell$-cohomology of each E_n, $n \geq 1$, is finite dimensional in each degree. We consider the profinite spectrum $\hat{E}^\ell \in \mathrm{Sp}(\hat{\mathcal{S}})$ whose spaces are defined by $\hat{E}_0^\ell := \hat{\Omega}(\widehat{R|E|})_1^{\,\ell}$ for $n = 0$ and by $\widehat{(R|E|)}_n^{\,\ell}$ for all $n \geq 1$ with structure maps adjoint to the above defined maps $\widehat{(R|E|)}_n^{\,\ell} \to \hat{\Omega}(\widehat{R|E|})_{n+1}^{\,\ell}$ for $n \geq 1$ and the obvious map for $n = 0$. By the preceding discussion, the profinite spectrum $\hat{E}^\ell$ is a* stable fibrant replacement *of E in* $\mathrm{Sp}(\hat{\mathcal{S}})$ *and we call it the ℓ-completion of E.*

The reason why we use this definition for $\hat{E}_0^\ell$ is that if E_0 has infinitely many connected components, the natural structure map will not be a weak equivalence any more.

This discussion also leads to an ℓ-completion of spectra $E \in \mathrm{Sp}(\mathcal{S})$ which is inspired by [De], §1.6.

Definition 3.13 *Let E be a (-1)-connected spectrum in $\mathrm{Sp}(\mathcal{S})$. We suppose that the $\mathbb{Z}/\ell$-cohomology of each E_n, $n \geq 1$, is finite dimensional in each degree. Let RE be an equivalent Ω-spectrum. Then we define the* profinite ℓ-completion $\hat{E}^\ell$ *to be the profinite spectrum whose spaces are defined by*

$\hat{E}_0^\ell := \hat{\Omega}(\widehat{R(E)_1})^\ell$ *for all* $n = 0$ *and by* $(\widehat{R(E)_n})^\ell$ *for all* $n \geq 1$ *with structure maps adjoint to the above defined maps* $\widehat{R(E)_n}^\ell \to \hat{\Omega}\widehat{R(E)_n}^\ell$ *for* $n \geq 1$ *and the obvious map for* $n = 0$.

Remark 3.14 *As above, for* $E \in \mathrm{Sp}(\mathcal{S})$*, it is clear that* $\hat{E}^\ell$ *is stable equivalent to* $\hat{E}$ *in* $\mathrm{Sp}(\hat{\mathcal{S}})$*. From the homotopy theoretic point of view we may consider either* $\hat{E}^\ell$ *or* $\hat{E}$*. The point is that we want to apply the comparison results of Proposition 2.10 to the situation of spectra. The previous definition gives us the form of the* $\hat{E}$ *that fits well with the results in Proposition 2.10.*

Similar versions of the following facts have also been proved by Dehon [De].

Proposition 3.15 *Let* E *be a* (-1)*-connected spectrum and suppose that the* $\mathbb{Z}/\ell$*-cohomology of each* E_n*,* $n \geq 1$*, is finite dimensional in each degree. Then the stable homotopy groups of the profinite* ℓ*-completion* $\hat{E}^\ell$ *are given by the following isomorphism for all* n

$$\pi_n\hat{E}^\ell \cong \widehat{\pi_n E}^\ell \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} \pi_n E.$$

Proof Since $\hat{E}^\ell$ is an $\hat{\Omega}$-spectrum, the stable homotopy group $\pi_n\hat{E}^\ell$ is equal to the homotopy group $\pi_{n+1}((\hat{E}^\ell)_1)$ for all n. The hypothesis on E and the definition of $\hat{E}^\ell$ imply that the spaces $\hat{E}_n^\ell$ satisfy the conditions of Proposition 2.10. This completes the proof. $\qquad\square$

Corollary 3.16 *Let* MU *be the simplicial spectrum representing complex cobordism. For the profinite* ℓ*-completion* $\hat{MU}^\ell$ *of* MU *there is an isomorphism*

$$\pi_*\hat{MU} = \pi_*\hat{MU}^\ell \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} \pi_*MU \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} \mathbb{L}_*$$

where $\mathbb{L}*$ *denotes the Lazard ring, cf. [Ad].*

Proof Since MU is (-1)-connected and its cohomology groups are finitely generated in each degree, cf. [Ad], MU satisfies the hypothesis of Proposition 3.15 above. $\qquad\square$

We finally consider the profinite completion $\hat{KU}$ of the Ω-spectrum representing complex K-theory, see e.g [Ma2]: $\hat{KU}_{2i} = \widehat{BU \times \mathbb{Z}}$ and $\hat{KU}_{2i+1} = \hat{U}$ for all $i \geq 0$. By Lemma 2.8 and Proposition 2.10 we get

Corollary 3.17 $\pi_{2i}(\hat{KU}) \cong \mathbb{Z}_\ell$ and $\pi_{2i+1}(\hat{KU}) = 0$ for all i.

4 Generalized cohomology theories on profinite spaces

We define generalized cohomology theories on $\hat{\mathcal{S}}$ via the stable category of profinite spectra in the classical way. Such profinitely completed cohomology theories have already been studied by Dehon in [De].

It is the main advantage of the stable category of profinite spectra that it gives a canonical and general setting for cohomology theories on profinite spaces and the profinite completion of cohomology theories on simplicial sets. In fact, our main objective are the cohomology theories represented by the completion of simplicial spectra such as $\hat{MU}$. The results of the previous section enable us to calculate the coefficients of this theory.

An important tool for the calculation of cohomology groups is the profinite Atiyah-Hirzebruch spectral sequence. The construction of this spectral sequence is the analog of the topological Atiyah-Hirzebruch spectral sequence. It has already been constructed by Dehon in a more restricted setting. We slightly generalize his proof.

At the end we will prove a Künneth formula for $\hat{MU}$ with $\mathbb{Z}/\ell^\nu$-coefficients, similar to and inspired by the results of [De].

4.1 Generalized cohomology theories

In analogy to the stable homotopy theory of simplicial spectra the above construction enables us to consider generalized cohomology theories for profinite spaces. We define them to be the functors represented by profinite spectra. Let E be a spectrum in $\mathrm{Sp}(\hat{\mathcal{S}})$, we set

$$(6) \qquad E^n(X) := \mathrm{Hom}_{\hat{\mathcal{SH}}}(X, E[n]),$$

where $\mathrm{Hom}_{\hat{\mathcal{SH}}}(X, E[n])$ denotes the set of maps that lower the dimension by n, and call this the n-th cohomology group of X with values in E. We set $E^*(X) := \bigoplus_n E^n(X)$. For a pointed profinite space X we define its n-th cohomology groups for the spectrum E by

$$(7) \qquad E^n(X) := \mathrm{Hom}_{\hat{\mathcal{SH}}}(\Sigma^\infty(X), E \wedge S^n)$$

For a profinite space X without a chosen basepoint, let X_+ be the profinite space $X \coprod pt$ with additional basepoint. We define the cohomology of X to be the one of X_+.

For a pair (X, A) of profinite spaces we define the relative cohomology by

$$(8) \qquad E^n(X, A) := E^n(X/A).$$

We have the identity $E^*(X, pt) = E^*(X)$. Just as for simplicial spectra [Swi], 8.21, one can prove the following isomorphism

$$(9) \qquad \mathrm{Hom}_{\hat{\mathcal{SH}}}(\Sigma^\infty(X), E \wedge S^n) \cong \mathrm{colim}_k \mathrm{Hom}_{Ho(\hat{\mathcal{S}}_*)}(\Sigma^k(X), E_{n+k}).$$

In particular, for an $\hat{\Omega}$-spectrum E, we get:

$$(10) \qquad E^n(X) \cong \mathrm{Hom}_{Ho(\hat{\mathcal{S}}_*)}(X, E_n).$$

4.2 Continuous cohomology theories

Definition 4.1 *Let E be a spectrum in $\mathrm{Sp}(\mathcal{S})$. We define the* continuous *cohomology represented by E to be the theory $\hat{E}^*(\cdot)$ on $\hat{\mathcal{S}}_*$.*

Remark 4.2 *1. Note that since profinite completion commutes with smash-products by Lemma 3.6, the completion of a multiplicative cohomology theory is multiplicative, too.*
2. Let E be a (-1)-connected spectrum. We suppose that the $\mathbb{Z}/\ell$-cohomology of each E_n, $n \geq 1$, is finite dimensional in each degree. Since the spectra $\hat{E}$ and $\hat{E}^\ell$ are equivalent in $\mathrm{Sp}(\hat{\mathcal{S}})$, the continuous ℓ-completed cohomology theory on $\hat{\mathcal{S}}_*$ represented by $\hat{E}^\ell$ is isomorphic to $\hat{E}$, i.e. for all $X \in \hat{\mathcal{S}}_*$ we have $\hat{E}^{\ell*}(X) = \hat{E}^*(X)$.

Example 4.3 *By abuse of notations we write MU for the simplicial spectrum $\mathrm{Sing}(MU)$ and we will write KU for $\mathrm{Sing}(KU)$. There are the obvious examples for profinite spectra representing generalized cohomology theories:*
1. The Eilenberg-MacLane spectra $H\pi$ given by $H\pi_n = K(\pi, n)$ for an abelian pro-ℓ-group π.
2. The profinite completion $\hat{KU}$ of the simplicial spectrum representing complex K-theory.
3. The profinite completion $\hat{MU}$ of the simplicial spectrum representing complex cobordism.

Proposition 4.4 *Let E be a multiplicative (-1)-connected spectrum and suppose that the $\mathbb{Z}/\ell$-cohomology of each E_n, $n \geq 1$, is finite dimensional*

in each degree. Then the coefficient ring $\hat{E}^* := \hat{E}^*(pt)$ of the profinite completion satisfies the following isomorphism

$$\hat{E}^*(pt) \cong \widehat{E^*(pt)}^\ell \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} E^*(pt).$$

For the homotopy groups of $\hat{E}$ we get

$$\pi_*(\hat{E}) \cong \widehat{\pi_*(E)}^\ell \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} \pi_*(E).$$

Proof This is a corollary of Proposition 3.15, since $\hat{E}$ and $\hat{E}^\ell$ are equivalent objects in $\mathrm{Sp}(\hat{\mathcal{S}})$. $\qquad\qquad\square$

Since the spectrum MU satisfies the hypotheses of the previous proposition, we get the following result.

Corollary 4.5 *For the homotopy groups of $\hat{MU}$ there is an isomorphism*

$$\pi_*(\hat{MU}) \cong \widehat{\pi_*(MU)}^\ell \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} \pi_*(MU).$$

For the coefficient ring of the profinite cobordism $\hat{MU}^*$ we have the following isomorphism

$$\hat{MU}^* = \hat{MU}^{\ell*} \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} MU^*.$$

Let G be a finite group. There is a simplicial Moore spectrum MG. For a simplicial spectrum E we define $EG := E \wedge MG$ to be the spectrum with coefficients in G corresponding to E, cf. [Ad], Chapter III.

Definition 4.6 *Let E be a simplicial spectrum and G a finite abelian group. We define the continuous cohomology theory with G-coefficients to be*

$$\hat{E}^*(\cdot; G) := \widehat{EG}^*(\cdot).$$

By [Ad] III, Prop. 6.6, there are exact sequences

(11) $\qquad 0 \to \pi_n(E) \otimes G \to \pi_n(EG) \to \mathrm{Tor}_1^\mathbb{Z}(\pi_{n-1}(E), G) \to 0$

(12) $\qquad 0 \to E^n(X) \otimes G \to (EG)^n(X) \to \mathrm{Tor}_1^\mathbb{Z}(E^{n+1}(X), G) \to 0$

for all spectra E and all spaces X.

Corollary 4.7 *For the homotopy groups of the profinite cobordism with $\mathbb{Z}/\ell^\nu$-coefficients $\hat{M}U\mathbb{Z}/\ell^\nu$ there is an isomorphism*

$$\pi_*(\hat{M}U\mathbb{Z}/\ell^\nu) \cong \pi_*(\widehat{MU\mathbb{Z}/\ell^\nu})^\ell \cong \mathbb{Z}/\ell^\nu \otimes_\mathbb{Z} \pi_*(MU).$$

For the coefficient ring of the profinite cobordism $\hat{M}U^*$ we have the following isomorphism

$$(\hat{M}U\mathbb{Z}/\ell^\nu)^* = (\widehat{MU\mathbb{Z}/\ell^\nu}^\ell)^* \cong \mathbb{Z}_\ell \otimes_\mathbb{Z} (MU\mathbb{Z}/\ell^k)^*.$$

Proof Since $\pi_*(MU)$ and MU^* have no torsion, the assertions follow from the previous exact sequences and Proposition 4.4. □

For the Eilenberg-MacLane spectra recall the construction of the spaces $K(\pi, n)$ in the previous section. We have the following results.

Proposition 4.8 *Let π be an abelian pro-ℓ-group.*
1. The Eilenberg-MacLane spaces $K(\pi, n)$ represent continuous cohomology in $\hat{\mathcal{H}}$, i.e. for every profinite space X and every $n \geq 0$ there is an isomorphism

$$H^n(X; \pi) \cong \mathrm{Hom}_{\hat{\mathcal{H}}}(X, K(\pi, n)).$$

2. The Eilenberg-MacLane spectrum $H\pi = H(\pi)$ represents continuous cohomology with coefficients in π, i.e. $[\Sigma^\infty(X), H\pi[n]] \cong H^n(X; \pi)$ for every n and every $X \in \hat{\mathcal{S}}$.
3. Let Y be a simplicial set. We denote by $|\pi|$ the underlying abstract group. There is a natural isomorphism

$$H^*(Y; |\pi|) \cong H^*(\hat{Y}; \pi).$$

In particular, if G is a finitely generated abelian group, there are isomorphisms

(13) $$H^*(Y; \mathbb{Z}_\ell \otimes_\mathbb{Z} G) \cong H^*(\hat{Y}; \hat{G}^\ell)$$

and

(14) $$H^*(Y; \mathbb{Z}/\ell^k \otimes_\mathbb{Z} G) \cong H^*(\hat{Y}; \mathbb{Z}/\ell^k \otimes_\mathbb{Z} G)$$

for every k.

Proof 1. Since every space in $\hat{\mathcal{S}}$ is cofibrant and since the spaces $K(\pi, n)$ are fibrant if π is an abelian pro-ℓ-group, cf. [Mo2], §1.4, Lemme 2, we have

$$\mathrm{Hom}_{\mathrm{Ho}(\hat{\mathcal{S}})}(X, K(\pi, n)) \cong \mathrm{Hom}_{\hat{\mathcal{S}}}(X, K(\pi, n))/\sim$$

where $\sim$ denotes simplicial homotopy. Then the proof of the analogue result [Ma1], Theorem 24.4, works here as well.

2. We know that $K(\pi, n) \to \hat{\Omega}K(\pi, n+1)$ is a homotopy equivalence, hence it is a $\mathbb{Z}/\ell$-equivalence and $H\pi$ is an $\hat{\Omega}$-spectrum. From (10) and $H^n(X; \pi) \cong \mathrm{Hom}_{\hat{\mathcal{H}}}(X, K(\pi, n))$ the assertion follows.

3. Since π is an abelian pro-ℓ-group, $K(\pi, n)$ is a fibrant profinite space for each n. Hence by adjunction we have natural isomorphisms $\mathrm{Hom}_{\mathcal{H}}(Y, |K(\pi, n)|) \cong \mathrm{Hom}_{\hat{\mathcal{H}}}(\hat{Y}, K(\pi, n))$. But by the construction of Eilenberg-MacLane spaces we have $|K(\pi, n)| \cong K(|\pi|, n)$ in $\mathcal{S}$ and we deduce the desired result from the first point above. $\qquad\square$

Remark 4.9 *1. The third statement (13) of the proposition is a useful generalization of Remark 2.1. The case (14) is in fact a special case of (13), but also of Remark 2.1 with $\pi = \mathbb{Z}/\ell^k \otimes_{\mathbb{Z}} G$, since this is a finite abelian group if G is finitely generated abelian.*

2. Since the model structures on $\hat{\mathcal{S}}$ and $\mathrm{Sp}(\hat{\mathcal{S}})$ depend on the chosen prime number ℓ, it is clear that one cannot represent continuous cohomology with arbitrary coeffiecients in $\hat{\mathcal{SH}}$. The homotopy groups of profinite spaces are pro-ℓ-groups and hence only cohomology with pro-ℓ-coefficients is representable in $\hat{\mathcal{SH}}$.

3. It follows from the proposition that the mod ℓ-Steenrod algebra of continuous cohomology operations is identical to the usual one.

4.3 Postnikov decomposition

We show the existence of a Postnikov decomposition for every profinite spectrum. Although some books use the existence of arbitrary colimits in the usual category of spectra for this purpose, there is another way to do so without referring to colimits. This enables us to use this construction in our profinite setting.

First we note that by general nonsense on pointed model categories we get the existence of fiber and cofiber sequences in $\mathrm{Sp}(\hat{\mathcal{S}})$. We can construct for every profinite spectrum E a connective covering $E_{\geq 0} \to E$ which induces an isomorphism $\pi_k(E_{\geq n}) = \pi_k(E)$ for all $k \geq 0$ and with $\pi_k(E_{\geq 0}) = 0$ for all $k < 0$. We may construct it by considering the diagram

$$
\begin{array}{ccc}
& & P(H\pi_0(E)[1]) \\
(15) & & \downarrow \\
& E \quad \longrightarrow & H\pi_0(E)[1]
\end{array}
$$

where $H\pi$ denotes the Eilenberg-MacLane spectrum for a profinite abelian group π and $P(H(\pi_0(E)[1])$ is the path object of $H\pi_0(E)[1]$. We define $E_{\leq 0}$ to be the fiber product of this diagram. Then we obtain inductively spectra $E_{\leq n}$ and maps $E \to E_{\leq n}$ for every n and we call $E_{\leq n}$ the *Postnikov n-truncation of E*. It has the property $\pi_k(E_{\leq n}) = \pi_k(E)$ for $k \leq n$ and $\pi_k(E_{\leq n}) = 0$ for $k > n$. We get in fact a *Postnikov tower* $\ldots \to E_{\leq n} \to E_{\leq n-1} \to \ldots$ for every E. The cofibre of each morphism $E_{\leq n} \to E_{\leq n-1}$ is identified with the Eilenberg-MacLane spectrum $H(\pi_n(E))[n+1]$.

Again by general nonsense [Qu1] I,§3, we know that the morphism $E \to E_{\leq n-1}$ fits into a fiber sequence $E_{\geq n} \to E \to E_{\leq n-1}$ and we call the spectrum $E_{\geq n}$ the *n-connective covering of E* which has the property $\pi_k(E_{\geq n}) = \pi_k(E)$ for $k \geq n$ and $\pi_k(E_{\geq n}) = 0$ for $k < n$.

4.4 Atiyah-Hirzebruch spectral sequence

Let X be a profinite space. For every integer p we denote by $\mathrm{sk}_p X$ the profinite subspace of X which is generated by the simplices of degree less or equal p. We call $\mathrm{sk}_p X$ the *p-skeleton* of X. For every k the set of k-simplices of $\mathrm{sk}_p X$ is closed in X_k such that $\mathrm{sk}_p X$ is simplicial profinite subset of X, cf. [De]. In addition, $\mathrm{sk}_p X$ is equal to the colimit of the diagram

$$
\begin{array}{ccc}
X_p \times \mathrm{sk}_{n-1}\Delta[p] & \longrightarrow & \mathrm{sk}_{p-1}X \\
\downarrow & & \downarrow \\
X_p \times \Delta[p] & \longrightarrow & \mathrm{sk}_p X
\end{array}
$$
(16)

which is induced by the obvious map $X_p \times \Delta[p] \to X$, where we consider X_p as a constant simplicial profinite set. This description implies that we get morphisms $\mathrm{sk}_{p-1}X \to \mathrm{sk}_p X$, which are cofibrations; and it implies that X is equal to the colimit in $\hat{\mathcal{S}}$ of the sequence $\ldots \mathrm{sk}_{p-1}X \to \mathrm{sk}_p X \to \ldots$ One should note that if $X/-$ denotes the diagram of the simplicial finite quotients of X, then the limit of the diagram $\mathrm{sk}_p(X/-)$ is equal to $\mathrm{sk}_p X$. With this knowledge in hands we may continue to construct in the classical way the Atiyah-Hirzebruch spectral sequence for profinite cohomology theories. In the following we will write X^p for $\mathrm{sk}_p X$.

Let π be an abelian profinite group. We define the cochain complex

$$D^p(X, \pi) := \tilde{H}^p(X^p/X^{p-1}; \pi)$$

with differential $d : D^p(X; \pi) \to D^{p+1}(X; \pi)$ defined as the composite

$$\tilde{H}^p(X^p/X^{p-1}; \pi) \to H^{p+1}(X^{p+1}; \pi) \to \tilde{H}^{p+1}(X^{p+1}/X^p; \pi).$$

One verifies easily that this defines a cochain complex. For the construction of the profinite Atiyah-Hirzebruch spectral sequence the following result will be important in order to identify the $E_2^{p,q}$-term.

Proposition 4.10 *For every profinite space X, the cohomology of X is given by the above cochain complex, i.e.*

$$H(D^*(X; \pi), d) \cong H^*(X; \pi).$$

Proof The proof is essentially the same as for CW-complexes. We consider the exact couple arising from the long exact sequence for the pair (X^p, X^{p-1}):

$$
(17) \qquad
\begin{array}{ccc}
H^{p+q}(X^{p-1}; \pi) & \longleftarrow & H^{p+q}(X^p; \pi) \\
\partial \searrow & & \nearrow \\
& \tilde{H}^{p+q}(X^p/X^{p-1}; \pi). &
\end{array}
$$

Via Diagram (16) we see that X^p/X^{p-1} is isomorphic to a limit of joints of p-spheres and that hence $\tilde{H}^{p+q}(X^p/X^{p-1}; \pi) = 0$ for $q > 0$. Since $d_1 : E_1^{p,0} \to E_1^{p+1,0}$ is equal to $d : D^p(X; \pi) \to D^{p+1}(X; \pi)$ we have

$$E_2^{*,0} = E_\infty^{p,0} = H(D^*(X; \pi), d).$$

From the definition of continuous cohomology for profinite spaces and from $(\mathrm{sk}_{p+r} X)_p = X_{p+r}$ for $r \geq 1$, it is clear that

$$H^p(X^{p+r}; \pi) = H^p(X; \pi) \text{ for } r \geq 1.$$

By induction on p, we show in addition that

$$H^{p+r}(X^p; \pi) = 0 \text{ for } r \geq 1.$$

This is clear for $p = 0$. Using (16) we get the exact sequence

$$0 = \tilde{H}^{p+r}(X^p/X^{p-1}; \pi) \to \tilde{H}^{p+r}(X^p; \pi) \to \tilde{H}^{p+r}(X^{p-1}; \pi) \to \tilde{H}^{p+r+1}(X^p/X^{p-1}; \pi) = 0.$$

By induction, $H^{p+r}(X^{p-1}; \pi) = 0$ for $r \geq 1$ and, by exactness, $H^{p+r}(X^p; \pi) = 0$ for $r \geq 1$, too. One continues just as in topology to conclude the desired isomorphism $E_2^{*,0} \cong H^*(X; \pi)$, see e.g. [Mc], Theorem 4.11. $\qquad \square$

Let E be a profinite spectrum. We want to construct an Atiyah-Hirzebruch spectral sequence for E. We consider the skeletal filtration $\mathrm{sk}_0 X \subset \ldots \subset \mathrm{sk}_p X \subset \mathrm{sk}_{p+1} X \subset \ldots \subset X$. This filtration yields a filtration on $E^* X$ defined by $F^p E^* X := \mathrm{Ker}(E^* X \to E^* X^{p-1})$.

The proof of the following theorem is essentially the one given by Adams in [Ad]. We use the special consideration of the profinite setting by Dehon in [De], Proposition 2.1.9. But we are able to prove a slightly more general statement than Dehon for any profinite spectrum.

By Remark 3.9 we know that the homotopy groups $\pi_q E$ have a natural profinite structure and hence also the coefficient groups E^q. Thus we may consider continuous cohomology with coefficients in E^q.

Theorem 4.11 *For any profinite spectrum E and for any profinite space X there is a spectral sequence $\{E_r^{p,q}\}$ with $E_2^{p,q} \cong H^p(X; E^q) \Longrightarrow E^{p+q}(X)$. The spectral sequence converges strongly, in the sense of [Ad] III, §8.2, to the graded term $e_\infty^{p,q} = F^p E^* X / F^{p+1} E^* X$ of the filtration on $E^* X$ if $\lim_r^1 E_r^{p,q} = 0$. In particular, the spectral sequence converges if $H^p(X; E^q) = 0$ for p sufficiently large or if $H^p(X; E^q)$ is finite for all p. We call it the* profinite Atiyah-Hirzebruch spectral sequence.

Proof The construction of the spectral sequence is a standard argument. The cofibre sequence $X_+^{p-1} \to X_+^p \to X_+^p / X_+^{p-1}$ induces an exact couple in cohomology

(18)
$$\bigoplus_p E^*(X^{p-1}) \longleftarrow \bigoplus_p E^*(X^p)$$
$$\partial \searrow \qquad \nearrow$$
$$\bigoplus_p E^*(X^p / X^{p-1})$$

which leads to a spectral sequence with $E_1^{p,q} = E^{p+q}(X^p / X^{p-1}) \Longrightarrow E^{p+q}(X)$, the differentials of which are of bidegree $(r, 1 - r)$. The convergence properties are described in and follow from [Ad] III, §8.2 or [Fr1], Proposition A.2. In particular, if $H^p(X; E^q) = 0$ for $p \geq n + 1$, the spectral sequence degenerates at the E_{n+1}-term and $\oplus_p E_{n+1}^{p,*}$ is isomorphic to the quotient $F^p E^* X / F^{p+1} E^* X$. Hence it remains to identify the E_2-term.

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Claim: We have a natural isomorphism

$$E_1^{p,q} = E^{p+q}(X^p/X^{p-1}) \cong \tilde{H}^p(X^p/X^{p-1}; \pi_p E_{p+q}).$$

If we can prove this claim, we get, together with Proposition 4.10 and the fact $\pi_p E_{p+q} = E^q$, that $E_2^{p,q} \cong H^p(X; E^q)$ and we are done.

Hence it remains to prove the claim. In the following we suppose that E is a fibrant profinite spectrum, i.e. an $\hat{\Omega}$-spectrum. This identification is due to [De]. We follow his argumentation. By the characterizing diagram (16) of the pth-skeleton X^p of X we see that $X^p/X^{p-1} \cong \lim_{F \in \mathcal{Q}(X_p)}(\vee_F S^p)$ is a limit in $\hat{\mathcal{S}}_*$ of joints of p-spheres. The limit comes from the profinite structure of X_p; F runs through the finite sets equal to quotients of X_p. Since E is fibrant we can identify E_{p+q} with the limit $\lim_t E_{p+q}(t)$ where $E_{p+q}(t)$ is a finite ℓ-space for every t, i.e. $\pi_i E_{p+q}(t)$ is a finite ℓ-group for every i and t and $\pi_i E_{p+q}(t) \neq 0$ only for a finite number of i. Since $\vee_F S^p$ is a simplicial finite set and since $\pi_i E_{p+q}(t)$ is finite, we conclude that we have an isomorphism

$$\lim_t \mathrm{colim}_{F \in \mathcal{Q}(X_p)} \tilde{H}^p(\vee_F S^p; \pi_p E_{p+q}(t)) \cong \tilde{H}^p(X^p/X^{p-1}; \pi_p E_{p+q}).$$

By [Swi], Proposition 6.39, we have

$$
\begin{aligned}
\tilde{H}^p(\vee_F S^p; \pi_p E_{p+q}(t)) &\cong \mathrm{Hom}_{\hat{\mathcal{H}}_*}(\vee_F S^p, K(\pi_p E_{p+q}(t), p)) \\
&\cong \mathrm{Hom}_{\mathrm{Ab}}(\pi_p(\vee_F S^p), \pi_p E_{p+q}(t)).
\end{aligned}
$$

Hence we get an isomorphism

$$\tilde{H}^p(X^p/X^{p-1}; \pi_p E_{p+q}) \cong \mathrm{Hom}_{\mathrm{pro-Ab}}(\pi_p(\vee_- S^p), \pi_p E_{p+q}(-)).$$

On the other hand we have the natural map

$$
\begin{aligned}
E^{p+q}(X^p/X^{p-1}) &= \mathrm{Hom}_{\hat{\mathcal{H}}_*}(X^p/X^{p-1}, E_{p+q}) \\
&\to \mathrm{Hom}_{\mathrm{pro-}\mathcal{H}_*}(X^p/X^{p-1}, E_{p+q}(-)) \\
&\to \mathrm{Hom}_{\mathrm{pro-Ab}}(\pi_p(\vee_- S^p), \pi_p E_{p+q}(-)).
\end{aligned}
$$

Hence in order to prove the claim, it suffices to show the following lemma.

Lemma 4.12 Cf. Lemma 2.1.10 of [De]. *Let $S(-)$ be a filtered diagram of finite joints of p-spheres with limit S in $\hat{\mathcal{S}}_*$ and let E be a fibrant profinite spectrum. Then the composite*

$$\mathrm{Hom}_{\hat{\mathcal{H}}_*}(S, E_{p+q}) \to \mathrm{Hom}_{\mathrm{pro-}\mathcal{H}_*}(S(-), E_{p+q}(-)) \to \mathrm{Hom}_{\mathrm{pro-Ab}}(\pi_p(\vee_- S^p), \pi_p E_{p+q}(-))$$

is an isomorphism of abelian groups.

Proof The proof is the same as for Lemma 2.1.10 of [De]. For the sake of completeness, we reformulate it here in more detail. For every pair of indices F and t we have a sequence of isomorphisms using the facts that $\pi_p S^p \cong \mathbb{Z}$, S^p considered in $\mathcal{S}_*$, and that $E_{p+q}(t)$ is fibrant:

$$
\begin{aligned}
\operatorname{Hom}_{\mathcal{H}_*}(\vee_F S^p, E_{p+q}(t)) &\cong \prod_F \operatorname{Hom}_{\mathcal{H}_*}(S^p, E_{p+q}(t)) \\
&\cong \prod_F \pi_p(E_{p+q}(t)) \cong \prod_F \operatorname{Hom}_{\mathrm{Ab}}(\pi_p(S^p), \pi_p(E_{p+q}(t))) \\
&\cong \operatorname{Hom}_{\mathrm{Ab}}(\pi_p(\vee_F S^p), \pi_p(E_{p+q}(t)))
\end{aligned}
$$

from which we deduce that the map

$$
\operatorname{Hom}_{\mathrm{pro}-\mathcal{H}_*}(\vee_- S^p, E_{p+q}(-)) \to \operatorname{Hom}_{\mathrm{pro}-\mathrm{Ab}}(\pi_p(\vee_- S^p), \pi_p(E_{p+q}(-)))
$$

is an isomorphism. Hence it remains to show that the left hand side is isomorphic to $\operatorname{Hom}_{\hat{\mathcal{H}}_*}(\lim_{F \in \mathcal{Q}(X_p)}(\vee_F S^p), E_{p+q})$. Since $E_{p+q}(t)$ is a finite ℓ-space for every t and since $\vee_F S^p$ is a simplicial finite set, Proposition 1.3.2 of [De] implies that

$$
\operatorname{colim}_{F \in \mathcal{Q}(X_p)} \operatorname{Hom}_{\mathcal{H}_*}(\vee_F S^p, E_{p+q}(t)) \overset{\cong}{\longrightarrow} \operatorname{Hom}_{\hat{\mathcal{H}}_*}(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t))
$$

is an isomorphism for every t. Furthermore, Corollary 1.4.3 of [De] implies that

$$
\operatorname{Hom}_{\hat{\mathcal{H}}_*}(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, \lim_t E_{p+q}(t)) \cong \lim_t \operatorname{Hom}_{\hat{\mathcal{H}}_*}(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t))
$$

is an isomorphism if $\lim_t^1 \pi_1(\operatorname{hom}_*(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t)))$ vanishes. If we could show that $\lim_t^1 \pi_1(\operatorname{hom}_*(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t))) = 0$ we would get

$$
\lim_t \operatorname{colim}_{F \in \mathcal{Q}(X_p)} \operatorname{Hom}_{\mathcal{H}_*}(\vee_F S^p, E_{p+q}(t)) \cong \operatorname{Hom}_{\hat{\mathcal{H}}_*}(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t)).
$$

Hence we have to show that $\lim_t^1 \pi_1(\operatorname{hom}_*(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t)))$ vanishes. Again since $\vee_F S^p$ is a simplicial finite set and since $E_{p+q}(t)$ is fibrant we get for every t

$$
\begin{aligned}
\pi_1(\operatorname{hom}_*(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t))) &\cong \operatorname{colim}_{F \in \mathcal{Q}(X_p)} \pi_1(\operatorname{hom}_*(\vee_F S^p, E_{p+q}(t))) \\
&\cong \operatorname{colim}_{F \in \mathcal{Q}(X_p)} \pi_1(\vee_F \Omega^p(E_{p+q}(t))) \\
&\cong \operatorname{colim}_{F \in \mathcal{Q}(X_p)} \oplus_F \pi_{p+1}(E_{p+q}(t)).
\end{aligned}
$$

Since $\pi_{p+1} E_{p+q}(t)$ is a finite group, the pro-abelian group $\{\operatorname{colim}_{F \in \mathcal{Q}(X_p)} \oplus_F \pi_{p+1}(E_{p+q}(t))\}_t$ is pro-isomorphic to a tower of surjections. Hence the $\lim^1$-term of the tower $\{\pi_1(\operatorname{hom}_*(\lim_{F \in \mathcal{Q}(X_p)} \vee_F S^p, E_{p+q}(t))\}_t$ vanishes. $\qquad\square$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

4.5 A Künneth formula for profinite cobordism

We finish this section with a discussion of a Künneth isomorphism in $\hat{M}U\mathbb{Z}/\ell^\nu$ for the product $X \times Y$ in $\hat{\mathcal{S}}$ for profinite spaces X and Y satisfying certain conditions. The whole argumentation is inspired by the work of Dehon [De], where he proves the corresponding result for $\hat{M}U$. We say that $Y \in \hat{\mathcal{S}}$ is *without ℓ-torsion* if the reduction map $\mathbb{Z}/\ell^n \to \mathbb{Z}/\ell$ induces a surjection $H^*(Y; \mathbb{Z}/\ell^n) \to H^*(Y; \mathbb{Z}/\ell)$ for every positive n, cf. [De], Definition 2.1.5. This is equivalent to the fact that $H^*(Y; \mathbb{Z}_\ell)$ has no torsion. For the application to étale cohomology theories in later sections, we only need the following version of a Künneth isomorphism for finite dimensional profinite spaces.

Theorem 4.13 *Let X and Y be finite dimensional profinite spaces and let X be without ℓ-torsion. Then we have for every ν Künneth isomorphisms*

$$\hat{M}U^*(X) \otimes_{\hat{M}U^*} \hat{M}U^*(Y) \overset{\cong}{\longrightarrow} \hat{M}U^*(X \times Y)$$

and

$$\hat{M}U^*(X; \mathbb{Z}/\ell^\nu) \otimes_{(\hat{M}U\mathbb{Z}/\ell^\nu)^*} \hat{M}U^*(Y; \mathbb{Z}/\ell^\nu) \overset{\cong}{\longrightarrow} \hat{M}U^*(X \times Y; \mathbb{Z}/\ell^\nu).$$

Proof The first assertion is a restricted version of [De], Proposition 4.3.2. The part for $\mathbb{Z}/\ell^\nu$-coefficients follows in a similar way. I am very grateful to Francois-Xavier Dehon for an explanation of this point. We will prove the assertion following his argument.

We suppose first that X and Y are finite dimensional without ℓ-torsion. The point is that the morphism of coefficients $(\hat{M}U\mathbb{Z}/\ell^\nu)^* \to (H\mathbb{Z} \wedge MU\mathbb{Z}/\ell^\nu)^{\wedge *}$ is injective and, since X has no ℓ-torsion, the map $E_2^{*,*}(X; MU\mathbb{Z}/\ell^\nu) \to E_2^{*,*}(X; H\mathbb{Z} \wedge MU\mathbb{Z}/\ell^\nu)$ is also injective. Since the spectral sequence for $H\mathbb{Z} \wedge MU\mathbb{Z}/\ell^\nu$ degenerates at the E_2-stage, the same is true for the spectral sequence for $MU\mathbb{Z}/\ell^\nu$. One can deduce from this fact that we have an isomorphism $\hat{M}U^*(X; \mathbb{Z}/\ell^\nu)/f^1 \cong H^*(X; \mathbb{Z}/\ell)$ as described in the proof of [De], Proposition 2.1.8, where f^* is an ℓ-adic filtration on MU^*-modules defined in [De], §2.1.

Since Y and $X \times Y$ have no ℓ-torsion, the same argument is valid for them as well. By Corollaire 2.1.4 of [De], it suffices to check that the map

$$\hat{M}U^*(X; \mathbb{Z}/\ell^\nu)/f^1 \otimes_{(\hat{M}U\mathbb{Z}/\ell^\nu)^*} \hat{M}U^*(Y; \mathbb{Z}/\ell^\nu)/f^1 \to \hat{M}U^*(X \times Y; \mathbb{Z}/\ell^\nu)/f^1$$

is an isomorphism. But this follows from the previous discussion and the Künneth isomorphism for $\mathbb{Z}/\ell$-cohomology. This finishes the case that both

X and Y have no ℓ-torsion.

For the case that Y has ℓ-torsion, we use induction on the skeletal filtration of Y. Let Y^n be the n-th skeleton of Y. If $\hat{MU}^*(X; \mathbb{Z}/\ell^\nu) \otimes_{(\hat{MU}\mathbb{Z}/\ell^\nu)^*}$ $\hat{MU}^*(Z; \mathbb{Z}/\ell^\nu) \to \hat{MU}^*(X \times Z; \mathbb{Z}/\ell^\nu)$ is an isomorphism for $Z = Y^n$ and for $Z = Y^{n+1}/Y^n$, then it is also an isomorphism for $Z = Y^{n+1}$. This follows from the long exact sequence associated to the cofibre sequence $Y^n \to Y^{n+1} \to Y^n/Y^{n+1}$ and the exactness of the tensor product with $\hat{MU}^*(X; \mathbb{Z}/\ell^\nu)$. The exactness of $\hat{MU}^*(X; \mathbb{Z}/\ell^\nu) \otimes -$ is due to the fact that X has no ℓ-torsion. We deduce the assertion of the theorem by induction on n from the proof of Proposition 4.3.2 of [De] using the point that we consider only the n-th skeleton of Y instead of the whole space. $\qquad\square$

Remark 4.14 *There is a version of the previous theorem with the assumption that Y is an arbitrary profinite space and X is a profinite space without ℓ-torsion whose $\mathbb{Z}/\ell$-cohomology is finite in each degree.*
One proves the $\mathbb{Z}/\ell^\nu$-coefficient case by taking the limit over the skeletal filtration.

4.6 Comparison with generalized cohomology theories of pro-spectra

Isaksen constructs in [I5] a stable model structure on the category of pro-spectra and defines generalized cohomology theories of pro-spectra. If E is a pro-spectrum then the r-th cohomology $E^r_{\mathrm{pro}}(X)$ of a pro-spectrum X with coefficients in E is the set $[X, E]^{-r}_{\mathrm{pro}}$ maps that lower the degree by r in the stable homotopy category of pro-spectra. In addition, for two pro-spectra X and Y he shows the existence of an Atiyah-Hirzebruch spectral sequence $E^{p,q}_2 = H^{-p}(X; \pi_{-q}E) \Rightarrow [X, E]^{p+q}_{\mathrm{pro}}$ which is in particular convergent if E is a constant pro-spectrum and X is the suspension spectrum of a finite dimensional pro-space. We show that this definition of cohomology of pro-spectra gives in the cases of interest the same groups as our definition via profinite spectra.

First we construct a functor $(\hat{-}) : \mathrm{pro} - \mathrm{Sp}(\mathcal{S}) \to \mathrm{Sp}(\hat{\mathcal{S}})$ that induces morphisms on cohomology theories. As for pro-spaces we define $(\hat{-})$ to be the composition of the profinite completion on spectra followed by taking the homotopy limit in $\mathrm{Sp}(\hat{\mathcal{S}})$ of the corresponding diagram in $\mathrm{Sp}(\hat{\mathcal{S}})$. Note that the homotopy limit exists by [Hi], Chapter 18, keeping in mind that all limits exist, see Convention A.1 on limits. Since the completion commutes

with suspension this functor obviously agrees with our previous definition for the case of the suspension spectrum of a space. A map $f : E \to F$ in $\mathrm{pro} - \mathrm{Sp}(\mathcal{S})$ is a weak equivalence if f is a levelwise n-equivalence for all n. This implies that f induces an isomorphism $\pi_k f$ on pro-homotopy groups, cf. [I5], Theorem 8.4. We want to show that $\hat{(-)}$ sends a weak equivalence f to a stable equivalence in $\mathrm{Sp}(\hat{\mathcal{S}})$. For this we may suppose that f is a level map $\{f_s : E_s \to F_s\}$ and that $\pi_k f_s$ is an isomorphism of stable homotopy groups for every k and every s. We know that the completion $\hat{(-)} : \mathrm{Sp}(\mathcal{S}) \to \mathrm{Sp}(\hat{\mathcal{S}})$ sends stable equivalences to stable equivalences. Since the homotopy limit is by construction well behaved with respect to levelwise weak equivalences $\mathrm{holim}_s \hat{f}_s$ is also a stable equivalence. Hence the functor induces maps $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{pro}-\mathrm{Sp}(\mathcal{S}))}(X, E) \to \mathrm{Hom}_{\hat{\mathcal{SH}}}(\hat{X}, \hat{E})$ and hence also maps $E^*_{\mathrm{pro}}(X) \to \hat{E}^*(\hat{X})$. These maps yield morphisms of Atiyah-Hirzebruch spectral sequences.

Theorem 4.15 *For any k, consider the constant pro-spectrum $MU\mathbb{Z}/\ell^\nu$. Let $\{X_s\}$ be a finite dimensional pro-space. Then we have an isomorphism*

$$MU^*_{\mathrm{pro}}(\{X_s\}; \mathbb{Z}/\ell^\nu) \cong \hat{MU}^*(\hat{X}; \mathbb{Z}/\ell^\nu).$$

Proof As pointed out above the completion functor induces a map of spectral sequences. But for a constant pro-spectrum the completion is just the completion defined in Section 3 on profinite spectra.
Hence our previous results on the coefficients of $MU\mathbb{Z}/\ell^\nu$ and $\hat{MU}\mathbb{Z}/\ell^\nu$ imply that it induces an isomorphism of coefficients $(MU\mathbb{Z}/\ell^\nu)^* \cong (\hat{MU}\mathbb{Z}/\ell^\nu)^*$ and hence an isomorphism of E_2-terms. Since both spectral sequences converge for the given X, the abutments are also isomorphic. □

5 Profinite étale realization on the unstable $\mathbb{A}^1$-homotopy category

The construction of the étale topological type functor Et from locally noetherian schemes to pro-simplicial sets is due to Artin-Mazur and Friedlander. The construction of the $\mathbb{A}^1$-homotopy category of schemes makes naturally arise the question if one could enlarge this functor to the category of spaces and if this functor behaves well with respect to the model structure. These questions have been answered by Isaksen and Schmidt. The first step into this direction was the construction of a model structure on $\mathrm{pro}-\mathcal{S}$. In view of

Et , there are in fact at least two interesting structures. It turns out that the
$\mathbb{Z}/\ell$-cohomological model structure of $\mathrm{pro}-\mathcal{S}$ fits better with the $\mathbb{A}^1$-localized
homotopy category. This is due to the fact that over a field of positive char-
acteristic $p > 0$, the étale fundamental group of $\mathbb{A}^1$ is highly non-trivial. To
avoid this obstruction one has to complete away from the characteristic of
the ground field using étale cohomology. The projection $X \times \mathbb{A}^1 \to X$ in-
duces an isomorphism in étale cohomology $H_{\mathrm{ét}}^*(X; \mathbb{Z}/\ell) \to H_{\mathrm{ét}}^*(X \times \mathbb{A}^1; \mathbb{Z}/\ell)$
for every prime $\ell \neq p$.

We will adapt the constructions of Isaksen [I3] and Dugger [D] to $\hat{\mathcal{S}}$ and check
that we still get an induced left derived functor on the homotopy categories.
We will calculate Êt for some examples.

5.1   The functor Êt

Definition 5.1 Definition 4.4 of [Fr2].
Let $X.$ be a locally noetherian simplicial scheme. The étale topological type
of $X.$ is defined to be the following pro-simplicial set

$$\mathrm{Et}\, X = \pi \circ \Delta : HRR(X.) \to \mathcal{S}$$

*sending a hypercovering $U_{.,.}$ of $X.$ to the simplicial set of connected compo-
nents of the diagonal of $U_{.,.}$. If $f : X. \to Y.$ is a map of locally noetherian
simplicial schemes, then the* strict étale topological type *of f is the strict
map $\mathrm{Et}\, f : \mathrm{Et}\, X. \to \mathrm{Et}\, Y.$ given by the functor $f^* : HRR(Y.) \to HRR(X.)$
and the natural transformation $\mathrm{Et}\, X. \circ \mathrm{Et}\, f \to \mathrm{Et}\, Y..$*

We refer the reader to [Fr2] and [I3] for a detailed discussion of the category
$HRR(X.)$ of rigid hypercoverings of $X.$ and rigid pullbacks.

Isaksen now uses the insight of Dugger [D] that one can construct the $\mathbb{A}^1$-
homotopy category in a universal way. Starting from an almost arbitrary
category $\mathcal{C}$, Dugger constructs an enlargement of $\mathcal{C}$ that carries a model
structure and is universal for this property. He also shows how to enlarge
functors $\mathcal{C} \to \mathcal{M}$ from $\mathcal{C}$ to a model category $\mathcal{M}$. In [I3], Isaksen takes
Dugger's model and extends the functor Et via this general method to the
$\mathbb{A}^1$-homotopy category. The idea is that $\mathrm{Et}\, X$ should be the above $\mathrm{Et}\, X$ on
a representable presheaf X and should preserve colimits and the simplicial
structure.

For our purpose, we would like to define Êt directly in the way Dugger
suggests. But the problem is that $(\hat{-}) : \mathrm{pro}-\mathcal{S} \to \hat{\mathcal{S}}$ is not a left adjoint
functor and does not preserve all small colimits. Therefore, we define Êt to

be the composition of Et followed by completion. To be precise we make the following definition:

Definition 5.2 *If X is a representable presheaf, then $\operatorname{Et} X$ is the étale topological type of X. If P is a discrete presheaf, i.e. each simplicial set $P(U)$ is 0-dimensional, i.e. P is just a presheaf of sets, then P can be written as a colimit $\operatorname{colim}_i X_i$ of representables and we define $\operatorname{Et} P := \operatorname{colim}_i \operatorname{Et} X_i$. Finally, an arbitrary simplicial presheaf can be written as the coequalizer of the diagram*

$$\coprod_{[m]\to[n]} P_m \otimes \Delta^n \rightrightarrows \coprod_{[n]} P_n \otimes \Delta^n,$$

where each P_n is discrete. Define $\operatorname{Et} P$ to be the coequalizer of the diagram

$$\coprod_{[m]\to[n]} \operatorname{Et} P_m \otimes \Delta^n \rightrightarrows \coprod_{[n]} \operatorname{Et} P_n \otimes \Delta^n.$$

We define the profinite étale topological type functor $\hat{\operatorname{Et}}$ *to be the composition of* Et *and the profinite completion functor* $\operatorname{pro} - \mathcal{S} \to \hat{\mathcal{S}}$:

$$(19) \qquad \hat{\operatorname{Et}} := \widehat{(\cdot)} \circ \operatorname{Et} : \Delta^{\operatorname{op}}\operatorname{PreShv}(\operatorname{Sm}/S) \to \hat{\mathcal{S}}.$$

For computations the following remark is crucial.

Remark 5.3 *1. By [Fr2], Proposition 5.9, we know that $H^*_{\text{ét}}(X; M) \cong H^*(\operatorname{Et} X, M)$ where $H^*(Z; M)$ denotes the cohomology of a pro-simplicial set Z with coefficients in the local coefficient system M corresponding to the sheaf M. For a finite abelian group π, we have in addition a natural isomorphism by Lemma 2.12: $H^*(Z; \pi) \cong H^*(\hat{Z}; \pi)$ for every pro-simplicial set Z. Hence we get as well*

$$H^*_{\text{ét}}(X; \pi) \cong H^*(\hat{\operatorname{Et}} X, \pi)$$

for every locally noetherian scheme X and every finite abelian group π.
2. For a morphism $g : Z \to X$ of schemes over k, there are relative cohomology groups $H^*(\hat{\operatorname{Et}} X, \hat{\operatorname{Et}} Z; \mathbb{Z}/\ell)$ fitting in a natural long exact sequence

$$\ldots \to H^*(\hat{\operatorname{Et}} X, \hat{\operatorname{Et}} Z; \mathbb{Z}/\ell) \to H^*(\hat{\operatorname{Et}} X; \mathbb{Z}/\ell) \to H^*(\hat{\operatorname{Et}} Z; g^*\mathbb{Z}/\ell) \to \ldots$$

For a closed immersion $Z \hookrightarrow X$ with open complement U, in [Fr2], Corollary 14.5, Friedlander has shown that these relative cohomology groups coincide with the étale cohomology of X with support in Z, i.e. we have a natural isomorphism

$$H^*_{\text{ét},Z}(X; \mathbb{Z}/\ell) \cong H^*(\hat{\operatorname{Et}} X, \hat{\operatorname{Et}} U; \mathbb{Z}/\ell).$$

Although Êt does not commute with products we have a weaker compatibility.

Proposition 5.4 *Let X and Y be smooth schemes of finite type over a separably closed field of characteristic $p \neq \ell$. Then the canonical map is a weak equivalence in $\hat{\mathcal{S}}$*

$$\hat{\mathrm{Et}}\,(X \times Y) \xrightarrow{\simeq} \hat{\mathrm{Et}}\,X \times \hat{\mathrm{Et}}\,Y.$$

Proof This follows from the Künneth formula for smooth schemes over a separably closed field proved in [SGA4$\frac{1}{2}$] [Th. finitude], Corollaire 1.11. It provides an isomorphism

$$H^*_{\text{ét}}(X \times Y; \mathbb{Z}/\ell) \cong H^*_{\text{ét}}(X; \mathbb{Z}/\ell) \otimes H^*_{\text{ét}}(Y; \mathbb{Z}/\ell).$$

On the other hand, we have a Künneth formula for the cohomology of profinite spaces which yields an isomorphism

$$H^*(\hat{\mathrm{Et}}\,X \times \hat{\mathrm{Et}}\,Y; \mathbb{Z}/\ell) \cong H^*(\hat{\mathrm{Et}}\,X; \mathbb{Z}/\ell) \otimes H^*(\hat{\mathrm{Et}}\,Y; \mathbb{Z}/\ell).$$

This implies that $\hat{\mathrm{Et}}\,(X \times Y) \longrightarrow \hat{\mathrm{Et}}\,X \times \hat{\mathrm{Et}}\,Y$ is a weak equivalence in $\hat{\mathcal{S}}$. $\square$

In the following example, we consider objects in the category $\Delta^{\mathrm{op}}\mathrm{PreShv}(\mathrm{Sm}/k)$ of simplicial presheaves over Sm/k.

Example 5.5 *Let R be a strict local henselian ring, i.e. a local henselian ring with separably closed residue field. Then $\mathrm{Spec}\,R$ has no nontrivial étale covers and the étale topological type of $\mathrm{Spec}\,R$ is a contractible space, i.e. $\hat{\mathrm{Et}}\,(\mathrm{Spec}\,R) \cong *$.*

Example 5.6 *Let k be a separably closed field with $\mathrm{char}(k) \neq \ell$. Let $c : \mathcal{S} \to \Delta^{\mathrm{op}}\mathrm{PreShv}(\mathrm{Sm}/k)$ be the functor that sends a simplicial set Z to the constant presheaf defined by Z. For every n, $cZ([n])$ is isomorphic to a disjoint union of copies of $\mathrm{Spec}\,k$. Hence, since Et commutes with coproducts and since $\mathrm{Et}\,(\mathrm{Spec}\,k) = *$ by Example 5.5 above, its étale topological type $\mathrm{Et}\,cZ$ is just the simplicial set Z itself.*
This shows that the pro-simplicial set $\mathrm{Et}\,Z$ is in fact just a simplicial set. The completion $\widehat{(\cdot)} : \mathcal{S} \to \hat{\mathcal{S}}$ preserves all colimits. Hence the same argument holds for $\hat{\mathrm{Et}}$, $\hat{\mathrm{Et}}\,(cZ) = \widehat{\mathrm{Et}\,cZ}$ and $\hat{\mathrm{Et}}\,(cZ) = Z$ for every $Z \in \hat{\mathcal{S}}$.
In particular, if S^1 denotes the simplicial circle, then $S^1([n]) = \coprod_{S^1([n])} \mathrm{Spec}\,k$ and $(S^1 \wedge \mathcal{X})([n]) = \coprod_{S^1([n])} \mathcal{X}([n])$. This yields an isomorphism in $\hat{\mathcal{S}}_*$

$$(20) \qquad S^1 \wedge \hat{\mathrm{Et}}\,(\mathcal{X}) \xrightarrow{\cong} \hat{\mathrm{Et}}\,(S^1) \wedge \hat{\mathrm{Et}}\,(\mathcal{X}) \xrightarrow{\cong} \hat{\mathrm{Et}}\,(S^1 \wedge \mathcal{X}).$$

Furthermore, Friedlander proved in [Fr2] that the étale fundamental group $\pi_1^{\text{ét}}(X)$ of a scheme is isomorphic as pro-group to the pro-fundamental group of $\text{Et}\,X$. From Proposition 2.10 we deduce the following

Proposition 5.7 *Let X be a connected locally noetherian scheme. The fundamental group of $\hat{\text{Et}}\,X$ is isomorphic to the ℓ-completion of the étale fundamental group of X, i.e.*

$$\pi_1(\hat{\text{Et}}\,X) \cong \widehat{\pi_1^{\text{ét}}(X)}^{\ell}.$$

Example 5.8 *Let k be a separably closed field with $\text{char}(k) \neq \ell$. Since $H_{\text{ét}}^*(\mathbb{A}_k^1; \mathbb{Z}/\ell) \cong H_{\text{ét}}^*(k; \mathbb{Z}/\ell)$ we know that $\hat{\text{Et}}\,\mathbb{A}_k^1 \simeq *$, i.e. that $\hat{\text{Et}}\,\mathbb{A}_k^1$ is contractible in $\hat{\mathcal{S}}$ and*

$$\pi_*^{\ell}(\hat{\text{Et}}\,\mathbb{A}_k^1) = *.$$

Example 5.9 *Let k be a separably closed field with $\text{char}(k) \neq \ell$. The space $\mathbb{G}_m$ is connected and its étale fundamental group is $\hat{\mathbb{Z}}$. Its ℓ-completion is hence equal to $\mathbb{Z}_\ell$ and we get $\hat{\text{Et}}\,\mathbb{G}_m \cong K(\mathbb{Z}_\ell, 1)$. Hence $\hat{\text{Et}}\,\mathbb{G}_m$ is isomorphic to S^1 in $\hat{\mathcal{H}}_*$ by (2.11).*

Example 5.10 *Let k be a separably closed field with $\text{char}(k) \neq \ell$. Let $\mathbb{P}_k^1$ be the projective line over k. Since $\mathbb{P}_k^1$ is connected and $\pi_1^{\text{et}}(\mathbb{P}_k^1) = 0$ is trivial, cf. for example [Mi] I, Example 5.2 f), $\hat{\text{Et}}\,\mathbb{P}_k^1$ is a simply connected space. Apart from H^0 its only nonzero étale cohomology group is $H_{\text{ét}}^2(\mathbb{P}, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$, see [Mi] VI, Example 5.6 with a chosen isomorphism $\mathbb{Z}/\ell \to \mathbb{Z}/\ell(1)$. Hence $\hat{\text{Et}}\,\mathbb{P}_k^1$ is isomorphic in $\hat{\mathcal{S}}_*$ to the simplicial finite set S^2.*

5.2 Unstable profinite étale realization

We want to show that $\hat{\text{Et}}$ induces a functor on the homotopy category. As indicated above we choose the universal model of [D] for the unstable $\mathbb{A}^1$-homotopy category of smooth schemes over a base field k.

Dugger showed that after some localization the projective model structure on $U(\text{Sm}/k) := \Delta^{\text{op}}\text{PreShv}(\text{Sm}/k)$ is a model for the $\mathbb{A}^1$-homotopy category $\mathcal{H}_{\mathbb{A}^1}(k)$ of [MV]. In this projective model structure the weak equivalences (fibrations) are objectwise weak equivalences (fibrations) of simplicial sets. The cofibrations are the maps having the left lifting property with respect to all trivial fibrations. Then one takes the left Bousfield localization of this model structure at the set S of maps:

1. for every finite collection $\{X^a\}$ of schemes with disjoint union X, the map $\coprod X^a \to X$ from the coproduct of the presheaves represented by each X^a to the presheaf represented by X;

2. every étale (Nisnevich) hypercover $U \to X$;

3. $X \times \mathbb{A}^1 \to X$ for every scheme X.

We call this the *étale (Nisnevich) $\mathbb{A}^1$-local projective model structure* according to [I3], and denote it by $L_{\text{ét}}U(k) = L_{S,\text{ét}}U(\mathrm{Sm}/k)$ (resp. $L_{\mathrm{Nis}}U(k) = L_{S,\mathrm{Nis}}U(\mathrm{Sm}/k)$).

Proposition 8.1 of [D] states that $L_{\mathrm{Nis}}U(k)$ is Quillen equivalent to the Nisnevich $\mathbb{A}^1$-localized model category $\mathcal{MV}_k$ of [MV] and the analogue holds for the étale case.

The following lemma is clear from the theory of Bousfield localization.

Lemma 5.11 *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories. Let S be a set of maps in $\mathcal{C}$ and let QC denote a fibrant replacement for objects C of $\mathcal{C}$. Suppose that the total left derived functor LF of F and that the left Bousfield localizations $\mathcal{C}/S$ and $\mathcal{D}/FQ(S)$ exist. Then F induces a functor*

$$F/S : \mathcal{C}/S \to \mathcal{D}/FQ(S)$$

and if F sends the maps in S into weak equivalences in $\mathcal{D}$, F induces a total left derived functor on the localized category $\mathrm{Ho}(\mathcal{C}/S) \to \mathrm{Ho}(\mathcal{D})$.

Since Et is a left Quillen functor on $U(\mathrm{Sm}/k)$ by [D], Proposition 2.3, in order to show that Et induces a left Quillen functor on $L_T U(\mathrm{Sm}/k)$, it suffices to show that Et takes the relations defined above into weak equivalences in $\mathrm{pro} - \mathcal{S}$.

Theorem 5.12 *Let ℓ be a prime different from the characteristic of k. With respect to the étale (Nisnevich) $\mathbb{A}^1$-local projective model structure on simplicial presheaves on Sm/k, the functor $\hat{\mathrm{Et}}$ induces a functor $\mathrm{L}\hat{\mathrm{Et}}$ from the étale (Nisnevich) $\mathbb{A}^1$-homotopy category to the $\mathbb{Z}/\ell$-cohomological homotopy category of $\hat{\mathcal{S}}$. In particular, $\mathrm{L}\hat{\mathrm{Et}}\,X$ is just $\hat{\mathrm{Et}}\,X$ for every scheme in Sm/k, and hence $\hat{\mathrm{Et}}$ preserves $\mathbb{A}^1$-weak equivalences between smooth schemes over k.*

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Proof Since every Nisnevich hypercover is also an étale hypercover it suffices to check this for the étale case. The first part of the proof is the one of [I3], Theorem 2.6. The functor Et above is exactly the functor Re : $U(\mathrm{Sm}/k) = \Delta^{\mathrm{op}}\mathrm{Pre}(\mathrm{Sm}/k) \to \mathrm{pro}-\mathcal{S}$ of [D], Proposition 2.3, hence it is a left Quillen functor. In order to see that Et is in fact a left Quillen functor on $L_S(U(\mathrm{Sm}/k))$, by Theorem 3.1.6 in [Hi] it remains to show that it takes cofibrant approximations of maps in S to weak equivalences of $\mathrm{pro}-\mathcal{S}$. This is done in [I3].

Note that the condition on ℓ is needed to ensure that the projection $X \times \mathbb{A}^1 \to X$ induces an isomorphism on étale cohomology $H^*_{\text{ét}}(X; \mathbb{Z}/\ell) \xrightarrow{\cong} H^*_{\text{ét}}(X \times \mathbb{A}^1; \mathbb{Z}/\ell)$. The functor Et takes weak equivalences between cofibrant objects to weak equivalences.

By Lemma 2.12 the completion functor $\widehat{(\cdot)}$ preserves weak equivalences and cofibrations. Hence the composition Êt sends weak equivalences between cofibrant objects into weak equivalences and the total left derived functor

$$\mathrm{L}\hat{\mathrm{E}}\mathrm{t} : \mathcal{H}^{\text{ét}}_{\mathbb{A}^1}(k) \to \hat{\mathcal{H}}$$

exists.

The last statement of the theorem follows from the definition of the total left derived functor and the fact that all representable presheaves are cofibrant in $L_S(U(\mathrm{Sm}/k))$, cf. [D]. $\qquad\square$

Remark 5.13 *Since every Nisnevich hypercover is also an étale hypercover, and since* Et *and hence also* Êt *send all étale hypercovers to weak equivalences, the functor from the Nisnevich $\mathbb{A}^1$-homotopy category factors through the étale $\mathbb{A}^1$-homotopy category:*

$$(21) \qquad \mathrm{L}\hat{\mathrm{E}}\mathrm{t} : \mathcal{H}^{\text{Nis}}_{\mathbb{A}^1}(k) \to \mathcal{H}^{\text{ét}}_{\mathbb{A}^1}(k) \xrightarrow{\mathrm{L}\hat{\mathrm{E}}\mathrm{t}} \hat{\mathcal{H}}$$

where the first functor corresponds to sending a presheaf to its associated sheaf in the étale topology.

We conclude this section with a result of [I3] on distinguished square. We will deduce from this result that we get a Mayer-Vietoris sequence for profinite étale cohomology theories.

We recall the definition of an *elementary distinguished square* of [MV]. It is a diagram

$$(22) \qquad \begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{i} & X \end{array}$$

of smooth schemes in which i is an open inclusion, p is étale and $p : p^{-1}(X - U) \to X - U$ is an isomorphism, where $X - U$ and $p^{-1}(X - U)$ are given the reduced structure. In particular, the maps i and p form a Nisnevich cover of X. As in [I3], Theorem 2.10, we can prove the following excision theorem for elementary distinguished squares. A similar result has already been proved by Friedlander in [Fr2], Lemma 14.10.

Theorem 5.14 *Given an elementary distinguished square of smooth schemes over k. Then the square*

(23)
$$
\begin{array}{ccc}
\hat{\mathrm{Et}}\,(U \times_X V) & \longrightarrow & \hat{\mathrm{Et}}\,V \\
\downarrow & & \downarrow \\
\hat{\mathrm{Et}}\,U & \longrightarrow & \hat{\mathrm{Et}}\,X
\end{array}
$$

is a homotopy pushout square of profinite spaces.

Proof The proof is the exact analog of the proof of Theorem 2.10 of [I3] with LEt replaced by $\hat{\mathrm{LEt}}$. $\qquad\square$

6 Profinite étale realization on the stable $\mathbb{A}^1$-homotopy category

We extend the results of the previous section to the stable $\mathbb{A}^1$-homotopy category. One of the reasons why we consider the model $\hat{\mathcal{S}}$ of $\mathrm{Ho}(\hat{\mathcal{S}})$, instead of $\mathrm{pro}-\mathcal{S}$, is that it seems to be easier to extend the functor $\hat{\mathrm{Et}}$ to the category of spectra.

We consider S^1-spectra and motivic $\mathbb{P}^1$-spectra separately. The first point for both will be to choose the correct model for the stable $\mathbb{A}^1$-homotopy theory. Then we deduce the existence of an étale realization as a left derived functor on the stable homotopy category from [Ho2].

The problem for the motivic version is that Et and hence also $\hat{\mathrm{Et}}$ do not commute with products in general. Fortunately, $\hat{\mathrm{Et}}$ commutes with the smash product by the simplicial circle S^1. But for $\mathbb{P}^1$ we have to construct an intermediate category and then we will show by a zig-zag of functors that we get a functor on the homotopy level.

This étale realization of the stable motivic category is the technical key point for the construction of a transformation from algebraic cobordism given by the MGL-spectrum to the profinite étale cobordism of the next section. We

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see in this and the following sections that the general case of an étale topological realization of the stable motivic category and the corresponding cohomology theories is more complicated than the complex realization for varieties over $\mathbb{C}$.

6.1   Etale realization of S^1-spectra

Let k be a separably closed field. The advantage of S^1-spectra in our context is that by Formula (20) we have $\hat{\text{Et}}\, S^1 \wedge \hat{\text{Et}}\, \mathcal{X} \cong \hat{\text{Et}}\, (S^1 \wedge \mathcal{X})$ for every simplicial presheaf $\mathcal{X}$ on Sm/k and $\hat{\text{Et}}\, S^1 = S^1$ in $\hat{\mathcal{S}}_*$, i.e. $\hat{\text{Et}}$ commutes with the simplicial suspension. Hence we get a functor

$$\hat{\text{Et}} : \text{Sp}^{S^1}(k) \to \text{Sp}(\hat{\mathcal{S}})$$

by sending a spectrum E to the profinite spectrum $\hat{\text{Et}}\, E$ given in degree n by $(\hat{\text{Et}}\, E)_n := \hat{\text{Et}}\, (E_n)$ with structure maps

$$S^1 \wedge \hat{\text{Et}}\, E_n \overset{\cong}{\to} \hat{\text{Et}}\, (S^1 \wedge E_n) \overset{\hat{\text{Et}}\,(\sigma)}{\to} \hat{\text{Et}}\, E_{n+1}$$

where σ is the structure map of the given spectrum E. The aim of this section is to show the following theorem.

Theorem 6.1 *The functor $\hat{\text{Et}}_{\text{Sp}} : \text{Sp}^{S^1}(k) \to \text{Sp}(\hat{\mathcal{S}})$ admits a left derived functor $\text{L}\hat{\text{Et}}_{\text{Sp}} : \mathcal{SH}^{S^1}(k) \to \hat{\mathcal{SH}}$ on the stable $\mathbb{A}^1$-homotopy category of S^1-spectra. We call this functor the S^1-stable étale realization functor.*

We will prove this theorem in the following steps. First, we have to choose the right model for the stable homotopy category of S^1-spectra. By [D], we know that $LU(k)$ is a left proper cellular simplicial model category which allows us to apply the methods of [Ho2].

Proposition 6.2 *The obvious functors*

$$\text{Sp}^{S^1}(k) \to \text{Sp}(\mathcal{MV}_k, S^1 \wedge \cdot) \to \text{Sp}(LU(k), S^1 \wedge \cdot)$$

are Quillen equivalences and we get

$$\mathcal{SH}^{S^1}(k)_s \simeq \text{Ho}^{\text{stable}}(\text{Sp}(\mathcal{MV}_k, S^1 \wedge \cdot)) \simeq \text{Ho}^{\text{stable}}(\text{Sp}(LU(k), S^1 \wedge \cdot)).$$

Proof The first equivalence follows from [Ho2], Corollary 3.5, and the obvious fact that that the cofibrations are mapped to cofibrations and the fibrant objects correspond to each other in both categories.

The second equivalence follows from [Ho2], Theorem 5.7, taking into account that $LU(k)$ is Quillen equivalent to $\mathcal{MV}_k$ and that all objects in $\mathcal{MV}_k$ are cofibrant. $\square$

Hence it suffices to show the following

Proposition 6.3 *The functor* $\hat{\mathrm{Et}}_{\mathrm{Sp}} : \mathrm{Sp}(LU(k), S^1 \wedge \cdot) \to \mathrm{Sp}(\hat{\mathcal{S}})$ *induces a total left derived functor*

$$\mathrm{L}\hat{\mathrm{Et}}_{\mathrm{Sp}} : \mathcal{SH}^{S^1}(k) \simeq \mathrm{Ho}^{\mathrm{stable}}(\mathrm{Sp}(LU(k), S^1 \wedge \cdot)) \to \hat{\mathcal{SH}}.$$

Proof The first part of the proof is analogous to the one of [Ho2], Prop. 5.5. We know that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ sends level equivalences between cofibrant objects to level equivalences in $\mathrm{Sp}(\hat{\mathcal{S}})$ since $\hat{\mathrm{Et}}$ sends weak equivalences between cofibrant objects to weak equivalences in $\hat{\mathcal{S}}_*$. Hence it induces a total left derived functor on the projective structure of $\mathrm{Sp}(LU(k), S^1 \wedge \cdot)$.

In order to show that this is a functor on the stable structure, it suffices to show that $\hat{\mathrm{Et}}_{\mathrm{Sp}}(\zeta_n^A)$ is a stable equivalence for maps $\zeta_n^A : \Sigma_{n+1} S^1 \wedge A \to \Sigma_n A$ in $\mathrm{Sp}(LU(k), S^1 \wedge \cdot)$ for all cofibrant objects $A \in LU(k)$, since these are the maps at which we localize for the stabilization.

We use the notation $\Sigma_n : LU(k) \to \mathrm{Sp}(LU(k), S^1 \wedge \cdot)$ for the left adjoint to the n-th evaluation functor. It is given by $(\Sigma_n \mathcal{X})_m = S^{m-n} \mathcal{X}$ if $m \geq n$ and $(\Sigma_n \mathcal{X})_m = 0$ otherwise. We denote by $\hat{\Sigma}_n : \hat{\mathcal{S}}_* \to \mathrm{Sp}(\hat{\mathcal{S}})$ the corresponding functor on $\hat{\mathcal{S}}_*$. Since $\hat{\mathrm{Et}}_{\mathrm{Sp}} \circ \Sigma^\infty = \hat{\Sigma}^\infty \circ \hat{\mathrm{Et}}$ and $\hat{\mathrm{Et}}_{\mathrm{Sp}} \circ \Sigma_n = \hat{\Sigma}_n \circ \hat{\mathrm{Et}}$ and by considering the commutative diagram

$$
\begin{array}{ccc}
\zeta_n^{\hat{\mathrm{Et}}\,A} : & \hat{\Sigma}_{n+1}(S^1 \wedge \hat{\mathrm{Et}}\,A) & \longrightarrow & \hat{\Sigma}_n(\hat{\mathrm{Et}}\,A) \\
& \hat{\Sigma}_{n+1}(\mathrm{id}) \uparrow & & \uparrow \mathrm{id} \\
& \hat{\Sigma}_{n+1}(\hat{\mathrm{Et}}\,(S^1 \wedge A)) & \longrightarrow & \hat{\Sigma}_n(\hat{\mathrm{Et}}\,A) \\
& \| & & \| \\
\hat{\mathrm{Et}}_{\mathrm{Sp}}\,\zeta_n^A : & \hat{\mathrm{Et}}_{\mathrm{Sp}}(\Sigma_{n+1}(S^1 \wedge A)) & \longrightarrow & \hat{\mathrm{Et}}_{\mathrm{Sp}}(\Sigma_n A)
\end{array}
$$

we see that $\zeta_n^{\hat{\mathrm{Et}}\,A}$ and $\hat{\mathrm{Et}}_{\mathrm{Sp}}\,\zeta_n^A$ differ by the stable equivalence $\hat{\Sigma}_{n+1}(\mathrm{id})$. Since $\zeta_n^{\hat{\mathrm{Et}}\,A}$ is a stable equivalence in $\mathrm{Sp}(\hat{\mathcal{S}})$ by definition, we have shown that $\hat{\mathrm{Et}}_{\mathrm{Sp}}\,\zeta_n^A$ is in fact a stable equivalence.

The last point is to show that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ factors through $\mathbb{A}^1$-weak equivalences. By Lemma 5.11 and [Mo3] it suffices to show that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ sends the maps

$\Sigma^\infty((X \times \mathbb{A}^1)_+)[n] \to \Sigma^\infty(X_+)[n]$ to stable equivalences in $\mathrm{Sp}(\hat{\mathcal{S}})$ for all $X \in \mathrm{Sm}/k$ and all $n \in \mathbb{Z}$. But we already know that $\hat{\mathrm{Et}}\,(X \times \mathbb{A}^1) \to \hat{\mathrm{Et}}\,(X)$ is a weak equivalence in $\hat{\mathcal{S}}$ and since $\widehat{\Sigma^\infty} \circ \hat{\mathrm{Et}} = \hat{\mathrm{Et}}_{\mathrm{Sp}} \circ \Sigma^\infty$ we see that $\hat{\mathrm{Et}}_{\mathrm{Sp}}\, \Sigma^\infty((X \times \mathbb{A}^1)_+)[n] \to \hat{\mathrm{Et}}_{\mathrm{Sp}}\, \Sigma^\infty(X_+)[n]$ is in fact a level weak equivalence in $\mathrm{Sp}(\hat{\mathcal{S}})$. This shows that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ induces a left derived functor on the category $\mathcal{SH}^{S^1}(k) = \mathcal{SH}^{S^1}_{\mathbb{A}^1}(k)$ of $\mathbb{A}^1$-localized S^1-spectra. $\qquad\square$

6.2   Etale realization of motivic spectra

Let k be a fixed separably closed field. There is a serious drawback when we want to extend arguments of the previous section to the situation of $\mathbb{P}^1$-spectra, since the profinite étale homotopy type does not commute with smash products in general. However the projections to each factor induce a canonical map

$$(24) \qquad\qquad \hat{\mathrm{Et}}\,(\mathbb{P}^1 \wedge \mathcal{X}) \to \hat{\mathrm{Et}}\,(\mathbb{P}^1) \wedge \hat{\mathrm{Et}}\,(\mathcal{X}).$$

Lemma 6.4 *The canonical map* $\hat{\mathrm{Et}}\,(\mathbb{P}^1 \wedge \mathcal{X}) \to \hat{\mathrm{Et}}\,(\mathbb{P}^1) \wedge \hat{\mathrm{Et}}\,(\mathcal{X})$ *is a weak equivalence in* $\hat{\mathcal{S}}_*$.

Proof First we consider a pointed smooth scheme X over k. In this case Proposition 5.4 says that

$$(25) \qquad\quad H^*(\hat{\mathrm{Et}}\,\mathbb{P}^1 \wedge \hat{\mathrm{Et}}\,X; \mathbb{Z}/\ell) \xrightarrow{\cong} H^*(\hat{\mathrm{Et}}\,(\mathbb{P}^1 \wedge X); \mathbb{Z}/\ell)$$

is an isomorphism.

Since the $\mathbb{Z}/\ell$-cohomology of $\hat{\mathrm{Et}}\,X$ and $\mathrm{Et}\,X$ agree for every scheme X by Remark 5.3, we may consider Et in order to prove the lemma. The advantage is that Et commutes with all colimits by definition.

Since the smash product gives rise to a left adjoint functor on the category $LU(k)$, it commutes with all colimits, too. If $\mathcal{X}$ is presented as a colimit $\mathcal{X} \cong \mathrm{colim}_s Y_s$ with schemes Y_s smooth over k, we have $\mathrm{Et}\,(\mathbb{P}^1 \wedge \mathrm{colim}_s Y_s) = \mathrm{colim}_s \mathrm{Et}\,(\mathbb{P}^1 \wedge Y_s)$ and hence, by Formula (25),

$$\begin{aligned}
H^*(\mathrm{Et}\,\mathbb{P}^1 \wedge \mathrm{Et}\,\mathcal{X}; \mathbb{Z}/\ell) &\cong \lim_s H^*(\mathrm{Et}\,\mathbb{P}^1 \wedge \mathrm{Et}\,Y_s; \mathbb{Z}/\ell) \\
&\cong \lim_s H^*(\mathrm{Et}\,\mathbb{P}^1; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*(\mathrm{Et}\,Y_s; \mathbb{Z}/\ell).
\end{aligned}$$

On the other hand, the cohomology of the right hand side of the canonical map in the lemma is isomorphic to

$$H^*(\mathrm{Et}\,\mathbb{P}^1; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} \lim_s H^*(\mathrm{Et}\,Y_s; \mathbb{Z}/\ell).$$

Since $H^i(\text{Et}\,\mathbb{P}^1; \mathbb{Z}/\ell) \cong H^i_{\text{ét}}(\mathbb{P}^1; \mathbb{Z}/\ell)$ is isomorphic to a rank one vector space over $\mathbb{Z}/\ell$ for $i = 0, 2$ and vanishes otherwise by [Mi] VI, Example 5.6, and since $H^i(\text{Et}\,Y_s; \mathbb{Z}/\ell) \cong H^i_{\text{ét}}(Y_s; \mathbb{Z}/\ell)$ is always a finite $\mathbb{Z}/\ell$-vector space, we conclude that, in this special case, $H^*_{\text{ét}}(\mathbb{P}^1; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} \lim_s H^*_{\text{ét}}(Y_s; \mathbb{Z}/\ell)$ is isomorphic to $\lim_s H^*_{\text{ét}}(\mathbb{P}^1; \mathbb{Z}/\ell) \otimes_{\mathbb{Z}/\ell} H^*_{\text{ét}}(Y_s; \mathbb{Z}/\ell)$. This completes the proof of the lemma. $\qquad\square$

It remains the problem to define a functor from $\mathbb{P}^1$-spectra to profinite spectra. The previous discussion shows that we cannot proceed as in the case of S^1-spectra. One idea to solve this problem could be to choose a fibrant replacement for $\hat{\text{Et}}\,E_n$ for every n. Then Lemma 6.4 implies that we have an isomorphism $\hat{\text{Et}}\,\mathbb{P}^1 \wedge \hat{\text{Et}}\,E_n \cong \hat{\text{Et}}\,(\mathbb{P}^1 \wedge E_n)$ in the homotopy category $\hat{\mathcal{H}}_*$. This yields an inverse map in $\hat{\mathcal{H}}_*$ and a structure map $\hat{\text{Et}}\,\mathbb{P}^1 \wedge \hat{\text{Et}}\,E_n \to \hat{\text{Et}}\,E_{n+1}$. Then one can use the fact that maps in $\hat{\mathcal{H}}_*$ with a fibrant target can be lifted to a map in $\hat{\mathcal{S}}_*$.

The problem is that this map is unique only up to homotopy and that we have to choose a lifting for every $\mathbb{P}^1$-spectrum. But we have to ensure that this choice is functorial.

We solve this problem in the way that we do not pretend to construct a functor on the level of $\mathbb{P}^1$-spectra but only on the level of homotopy categories. We summarize the information of all possible liftings in an intermediate category $\mathcal{C}$ whose homotopy category is canonically equivalent to the homotopy category of S^2-spectra $\hat{\mathcal{SH}}_2$. Hence the introduction of $\mathcal{C}$ is just one possible way to avoid the problem of a functorial choice of liftings which may not even exist.

Therefore, in order to construct a functor on the homotopy category of motivic $\mathbb{P}^1$-spectra, we consider a zig-zag of functors

$$\text{Sp}^{\mathbb{P}^1}(k) \xrightarrow{\tilde{\text{Et}}} \mathcal{C} \xhookleftarrow{i} \text{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot)$$

with an intermediate category $\mathcal{C}$ whose objects are given by sequences

$$\{E_n, E_n'; S^2 \wedge E_n \xleftarrow{p_n} E_n' \xrightarrow{q_n} E_{n+1}\}_{n \in \mathbb{Z}}$$

where all E_n and maps are in $\hat{\mathcal{S}}_*$, p_n is a weak equivalence in $\hat{\mathcal{S}}_*$; its morphisms are levelwise morphisms of $\hat{\mathcal{S}}_*$ which make the obvious diagrams commute, where the map $S^2 \wedge E_n \to S^2 \wedge F_n$ is the map induced by $E_n \to F_n$. The functor

$$\text{Sp}(\hat{\mathcal{S}})_2 := \text{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot) \xhookrightarrow{i} \mathcal{C}$$

denotes the full embedding which sends $\{F_n, S^2 \wedge F_n \xrightarrow{\sigma_n} F_{n+1}\}$ to $\{F_n, S^2 \wedge F_n \xleftarrow{\text{id}} S^2 \wedge F_n \xrightarrow{\sigma_n} F_{n+1}\}$.

The functor $\tilde{\hat{\mathrm{Et}}}$ is defined via the canonical map (24) and the image of the structure maps of the $\mathbb{P}^1$-spectra. Lemma 6.4 ensures that this is a well-defined functor into $\mathcal{C}$. We have to consider S^2 instead of S^1, since $\hat{\mathrm{Et}}\,\mathbb{P}^1$ is isomorphic in $\hat{\mathcal{S}}$ to S^2.

We define a set W of maps in $\mathcal{C}$ as the image of the stable equivalences of $\mathrm{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot)$ under the embedding i. It is a set since the stable equivalences form a set in $\mathrm{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot)$. Since the maps in W come from a model structure and since i is a full embedding, it is clear that W admits a calculus of fractions. This ensures that we may form the localized category $\mathrm{Ho}(\mathcal{C}) := \mathcal{C}[W^{-1}]$ in which the maps on W become isomorphisms. We call the maps in W weak equivalences or stable equivalences, by abuse of notation. We will call a map in W a level equivalence if it is in the image of the level equivalences of $\mathrm{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot)$ under i.

Furthermore, i sends stable equivalences into weak equivalences by definition and hence it induces a functor on the homotopy categories, which we also denote by i.

Proposition 6.5 *The induced functor* $i : \hat{\mathcal{SH}}_2 \to \mathrm{Ho}(\mathcal{C})$ *is an equivalence of categories.*

Proof We need to construct an inverse functor $j : \mathrm{Ho}(\mathcal{C}) \to \hat{\mathcal{SH}}_2$. Since i is a full embedding we have to show that for every $F \in \mathrm{Ho}(\mathcal{C})$ there is in a functorial way a spectrum $E \in \hat{\mathcal{SH}}_2$ such that $i(E) \cong F$ in $\mathrm{Ho}(\mathcal{C})$. The crucial point is to construct the structure map of a spectrum out of the given data of F.

Let RS^2 be the image of S^2 under a fixed fibrant replacement functor R in $\hat{\mathcal{S}}_*$. We consider $\mathrm{Sp}(\hat{\mathcal{S}}_*, RS^2)$ as a Quillen equivalent model for $\hat{\mathcal{SH}}_2$. Then we apply R on each level and get the following sequence

$$(26) \qquad RS^2 \wedge RF_n \overset{p_n}{\underset{\sim}{\leftarrow}} RF_n' \overset{q_n}{\to} RF_{n+1}$$

where the first map is still a weak equivalence by the Künneth formula for the continuous cohomology of profinite spaces. Hence considering all n, the resulting object is isomorphic to the initial F in $\mathrm{Ho}(\mathcal{C})$. Since all spaces in (26) are cofibrant and fibrant there is a left inverse s_n of p_n in $\hat{\mathcal{S}}_*$ such that $s_n p_n = \mathrm{id}_{RS^2 \wedge RF_n}$ and a homotopy $p_n s_n \sim \mathrm{id}_{RF_n'}$. Unfortunately, the choice of s_n for all n may not be functorial.

But we get the following: Let s_n and s_n' be two choices of such left inverses for each n. If we start with $n = 0$, the given homotopies $p_0 s_0 \sim \mathrm{id}_{RF_0'}$ and

$p_0 s_0' \sim \mathrm{id}_{RF_0'}$ yield homotopy equivalences f_0 and f_1 making the following diagram

(27)
$$\begin{array}{ccc} RS^2 \wedge RF_0 & \stackrel{\mathrm{id} \wedge f_0}{\to} & RS^2 \wedge RF_0 \\ q_0 s_0 \downarrow & & \downarrow q_0 s_0' \\ RF_1 & \stackrel{f_1}{\to} & RF_1 \end{array}$$

commutative. Continuing this construction inductively we get a homotopy equivalence between the two resulting spectra via s and s'. Hence they are canonically isomorphic in $\hat{\mathcal{SH}}_2$. $\qquad \square$

In order to get a functor $\mathcal{SH}(k) \to \hat{\mathcal{SH}}_2$, we have to show that $\tilde{\hat{\mathrm{Et}}}_{\mathrm{Sp}} :$ $\mathrm{Sp}^{\mathbb{P}^1}(k) \to \mathcal{C}$ has a total left derived functor. Then we may compose the resulting functors $\mathrm{L}\hat{\mathrm{Et}}$ and j of the above proof to get a functor

$$\mathrm{L}\hat{\mathrm{Et}} := j \circ \mathrm{L}\tilde{\hat{\mathrm{Et}}} : \mathcal{SH}(k) \to \hat{\mathcal{SH}}_2$$

on the homotopy categories. As before we have to choose the right model for the stable motivic category $\mathcal{SH}^{\mathbb{P}^1}(k)$.

Proposition 6.6 *The obvious functors*

$$\mathrm{Sp}^{\mathbb{P}^1}(k) \to \mathrm{Sp}(\mathcal{MV}_k, \mathbb{P}^1 \wedge \cdot) \to \mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)$$

are in fact Quillen equivalences and we get

$$\mathcal{SH}^{\mathbb{P}^1}(k) = \mathrm{Ho}^{\mathrm{stable}}(\mathrm{Sp}(\mathcal{MV}_k, \mathbb{P}^1 \wedge \cdot)) \simeq \mathrm{Ho}^{\mathrm{stable}}(\mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)).$$

Proof The proof is identical to the one of Proposition 6.2. $\qquad \square$

In the following we will use $\mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)$ as a model for $\mathcal{SH}^{\mathbb{P}^1}(k)$. We need to show

Proposition 6.7 *The functor* $\tilde{\hat{\mathrm{Et}}} : \mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot) \to \mathcal{C}$ *induces a total left derived functor*

$$\mathrm{L}\tilde{\hat{\mathrm{Et}}} : \mathcal{SH}^{\mathbb{P}^1}(k) \simeq \mathrm{Ho}^{\mathrm{stable}}(\mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)) \to \mathrm{Ho}(\mathcal{C}).$$

Proof The proof is similar to the one of Proposition 6.3, but we have to be careful at some points. Therefore, we give the proof in detail.

As in the proof of Proposition 6.3, we know that $\tilde{\hat{\mathrm{Et}}}_{\mathrm{Sp}}$ sends level weak equivalences between cofibrant objects to weak equivalences in $\mathcal{C}$, since $\hat{\mathrm{Et}}$ sends weak equivalences between cofibrant objects to weak equivalences in $\hat{\mathcal{S}}_*$. Hence it induces a total left derived functor on the projective structure of $\mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)$.

We use the notation $\Sigma_n^{\mathbb{P}^1} : LU(k) \to \mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)$ for the left adjoint to the n-th evaluation functor. It is given by $(\Sigma_n^{\mathbb{P}^1} \mathcal{X})_m = (\mathbb{P}^1)^{m-n} \mathcal{X}$ if $m \geq n$ and $(\Sigma_n^{\mathbb{P}^1} \mathcal{X})_m = 0$ otherwise.

We denote by $F_n : \hat{\mathcal{S}}_* \to \mathcal{C}$ the composition of the corresponding functor $\hat{\Sigma}_n : \hat{\mathcal{S}}_* \to \mathrm{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot)$ followed by the embedding $i : \mathrm{Sp}(\hat{\mathcal{S}}, S^2 \wedge \cdot) \to \mathcal{C}$. One should note that the image of F_n in $\mathcal{C}$ lies in the image of i by definition. In order to show that there exists a total derived functor on the stable structure it suffices to show that $\hat{\mathrm{Et}}_{\mathrm{Sp}}(\zeta_n^A)$ is a stable equivalence for maps $\zeta_n^A : \Sigma_{n+1}\mathbb{P}^1 \wedge A \to \Sigma_n A$ in $\mathrm{Sp}(LU(k), \mathbb{P}^1 \wedge \cdot)$ for all cofibrant objects $A \in LU(k)$, since these are the maps at which we localize for the stabilization.

Since $\hat{\mathrm{Et}} \, \mathbb{P}^1 \cong S^2$, there are canonical maps $F_{n+1}(\hat{\mathrm{Et}}(\mathbb{P}^1 \wedge A) \to F_{n+1}(S^2 \wedge A)$ and $\hat{\mathrm{Et}}_{\mathrm{Sp}}(\Sigma_n^{\mathbb{P}^1}(A)) \to F_n(\hat{\mathrm{Et}}(A))$. These maps are level weak equivalences by Lemma 6.4. We consider the commutative diagram in $\mathcal{C}$

$$
\begin{array}{ccc}
\zeta_n^{\hat{\mathrm{Et}}\, A} : & F_{n+1}(S^2 \wedge \hat{\mathrm{Et}}\, A) & \overset{\text{stable e.}}{\longrightarrow} & F_n(\hat{\mathrm{Et}}\, A) \\
& \text{level w. e.} \uparrow & & \| \\
& F_{n+1}(\hat{\mathrm{Et}}(\mathbb{P}^1 \wedge A)) & \longrightarrow & F_n(\hat{\mathrm{Et}}\, A) \\
& \text{level w. e.} \uparrow & & \uparrow \text{level w. e.} \\
\hat{\mathrm{Et}}_{\mathrm{Sp}}\, \zeta_n^A : & \hat{\mathrm{Et}}_{\mathrm{Sp}}(\Sigma_{n+1}^{\mathbb{P}^1}(\mathbb{P}^1 \wedge A)) & \longrightarrow & \hat{\mathrm{Et}}_{\mathrm{Sp}}(\Sigma_n^{\mathbb{P}^1} A).
\end{array}
$$

We see that $\zeta_n^{\hat{\mathrm{Et}}\, A}$ and $\hat{\mathrm{Et}}_{\mathrm{Sp}}\, \zeta_n^A$ differ only by level weak equivalences in $\mathcal{C}$ which are also stable equivalences. Since $\zeta_n^{\hat{\mathrm{Et}}\, A}$ is a stable equivalence in $\mathcal{C}$ by definition, the two-out-of-three axiom for weak equivalences implies that $\hat{\mathrm{Et}}_{\mathrm{Sp}}\, \zeta_n^A$ is in fact a stable equivalence.

The last point is to show that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ factors through $\mathbb{A}^1$-weak equivalences. By Lemma 5.11 and [Mo3] it suffices to show that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ sends the maps $\Sigma_{\mathbb{P}^1}^\infty((X \times \mathbb{A}^1)_+)[n] \to \Sigma_{\mathbb{P}^1}^\infty(X_+)[n]$ to stable equivalences for all $X \in \mathrm{Sm}/k$ and all $n \in \mathbb{Z}$. But we already know that $\hat{\mathrm{Et}}(X \times \mathbb{A}^1) \to \hat{\mathrm{Et}}(X)$ is a weak equivalence in $\hat{\mathcal{S}}$. Hence $\hat{\mathrm{Et}}_{\mathrm{Sp}} \Sigma_{\mathbb{P}^1}^\infty((X \times \mathbb{A}^1)_+)[n] \to \hat{\mathrm{Et}}_{\mathrm{Sp}} \Sigma_{\mathbb{P}^1}^\infty(X_+)[n]$ is a level weak equivalence in $\mathcal{C}$. This shows that $\hat{\mathrm{Et}}_{\mathrm{Sp}}$ induces a total left derived functor on the category $\mathcal{SH}^{\mathbb{P}^1}(k) = \mathcal{SH}_{\mathbb{A}^1}^{\mathbb{P}^1}(k)$ of $\mathbb{A}^1$-localized $\mathbb{P}^1$-spectra. $\quad\square$

As in Theorem 5.12, since the suspension spectrum of a smooth scheme is a stable cofibrant object, we have $L\hat{\mathrm{Et}} = \hat{\mathrm{Et}}$ on such spectra and $\hat{\mathrm{Et}}$ sends

stable $\mathbb{A}^1$-equivalences to the image in $\mathcal{C}$ of stable equivalences in $\hat{\mathcal{SH}}_2$. We summarize this discussion in the following

Theorem 6.8 *The functor* $\hat{\mathrm{Et}}$ *yields via the above composition an étale realization of the stable motivic homotopy category of $\mathbb{P}^1$-spectra:*

$$\mathrm{L}\hat{\mathrm{Et}} : \mathcal{SH}^{\mathbb{P}^1}(k) \to \hat{\mathcal{SH}}_2.$$

In particular, the functor $\mathrm{L}\hat{\mathrm{Et}}$ *is equal to* $\hat{\mathrm{Et}}$ *on the suspension spectra of smooth schemes and* $\hat{\mathrm{Et}}$ *sends stable $\mathbb{A}^1$-equivalences between suspension spectra of smooth schemes to isomorphisms in* $\hat{\mathcal{SH}}_2$.

Remark 6.9 *We could have proven a similar étale realization for symmetric $\mathbb{P}^1$-spectra in essentially the same way. But we have not yet been able to prove that the stable model structure on symmetric spectra over $\hat{S}$ does respect the monoidal structure, see Appendix B.2. Hence until now, we would not get any extra structure.*

7   Profinite étale cobordism

The study of profinite étale cohomology theories is the main purpose of this paper. The idea is to apply generalized profinite cohomology theories represented by a profinite spectrum to the functor $\hat{\mathrm{Et}}$.

We show that every such profinite étale cohomology theory is in fact a cohomology theory on Sm/k in the sense of [Pa]. One should note that in order to get the $\mathbb{A}^1$-invariance, we have to complete away from characteristic of k and cannot use a π_*-model structure, since $\pi_1^{\text{ét}}(\mathbb{A}^1_k)$ is non-trivial over a field of positive characteristic, see [Sch].

We show that the naturally arising profinite étale K-theory represented by $\hat{KU}$ for schemes over a separably closed field is isomorphic to the étale K-theory of Friedlander [Fr1].

The second theory we study is profinite étale cobordism represented by $\hat{MU}$. The coefficients of this theory are determined by our knowledge about profinite cohomology theories. We deduce an Atiyah-Hirzebruch spectral sequence from Theorem 4.11 for étale cobordism with finite coefficients $\hat{MU}\mathbb{Z}/\ell^\nu$ starting with étale cohomology.

We show in the last section that $\hat{MU}_{\text{ét}}$ is in fact an oriented cohomology theory on Sm/k if k is separably closed. In order to prove this, we have to

use non-trivial results on $\hat{M}U$ to be able to reduce the problems to questions in étale cohomology.

The ideas for the proof that $\hat{M}U_{\text{ét}}$ is an oriented cohomology theory are not new. For our purpose we use the techniques of [Pa]. We define an orientation of $\hat{M}U_{\text{ét}}$ with respect to smooth schemes over k. We use this orientation and a projective bundle formula to construct a Chern structure on $\hat{M}U_{\text{ét}}$. The key ingredient for the proof of the projective bundle formula is the Atiyah-Hirzebruch spectral sequence of $\hat{M}U$. From facts about étale cohomology and the results of [Pa], we deduce that étale cobordism is an oriented cohomology theory on smooth schemes over a separably closed field k in the sense of [Pa] and hence in the sense of [LM].

This result will enable us to compare profinite étale cobordism with algebraic cobordism in the next section.

For the convenience of the reader we recall some notations and definitions: we denote by $\hat{\mathcal{S}}$, resp. $\hat{\mathcal{S}}_*$, the category of simplicial profinite sets, resp. pointed simplicial profinite sets; $\hat{\mathcal{H}}$, resp. $\hat{\mathcal{H}}_*$, is the corresponding homotopy category; $\text{Sp}(\hat{\mathcal{S}})$ is the category of profinite spectra and $\hat{\mathcal{SH}}$ denotes the stable homotopy category of profinite spectra; we write $\Sigma^\infty(Y)$ for the suspension spectrum of a profinite space Y, see Sections 2 and 3.

Every profinite spectrum E determines a generalized cohomology theory $E^*(-)$ on profinite spaces, e.g. $\hat{M}U$ and $\hat{M}U\mathbb{Z}/\ell^\nu$ yield profinite cobordism theories $\hat{M}U^*(-)$ and $\hat{M}U^*(-;\mathbb{Z}/\ell^\nu)$, see Section 4.

The functor $\hat{\text{Ét}} : \text{Sch}/k \to \hat{\mathcal{S}}$ is the profinitely completed étale topological type functor, see Section 5; we denote by $\text{L}\hat{\text{Ét}} : \mathcal{SH}^{\mathbb{P}^1}(k) \to \hat{\mathcal{SH}}$ the stable étale realization functor, see Section 6 and Theorem 6.8.

7.1 Profinite étale cohomology theories

Let k be a fixed base field and let ℓ be a fixed prime different from the characteristic of k. We consider the category Sch/k of schemes of finite type over $\text{Spec}\, k$. This implies in particular that all schemes are noetherian and we may apply the étale topological type functor.

Definition 7.1 *Let $E \in \text{Sp}(\hat{\mathcal{S}})$ be a profinite spectrum and let $X \in \text{Sch}/k$. 1. We define the profinite étale cohomology of X in E to be the profinite cohomology theory represented by E applied to the profinite space $\hat{\text{Ét}}\, X$, i.e.*

$$E^n_{\text{ét}}(X) := E^n(\hat{\text{Ét}}\, X) = \text{Hom}_{\hat{\mathcal{SH}}}(\Sigma^\infty(\hat{\text{Ét}}\, X), E[n]),$$

where we add a base-point if X is not already pointed.

2. We define relative étale cohomology groups $E_{\text{ét}}^n(X, U)$ for an open subscheme $U \subset X$ by

$$E_{\text{ét}}^n(X, U) := E^n(\hat{\text{Et}}\,(X)/\hat{\text{Et}}\,(U)).$$

3. We define the étale cohomology group

$$E_{\text{ét}}^n(\mathbb{P}_k^\infty) := \lim_V E_{\text{ét}}^n(\mathbb{P}(V))$$

to be the limit over all finite dimensional vector spaces $V \hookrightarrow k^\infty$ of the just defined groups $E_{\text{ét}}^n(\mathbb{P}(V))$.

Remark 7.2 *Since $\hat{\text{Et}}$ is also a functor on spaces, we could have defined $E_{\text{ét}}^n(\mathbb{P}_k^\infty)$ just as for schemes to be the group $E^n(\hat{\text{Et}}\,\mathbb{P}_k^\infty)$. But we will use results that are based on the first definition. Since $\hat{\text{Et}}$ does not commute with arbitrary colimits, the two definitions differ in general. But there is a canonical map $E^*(\hat{\text{Et}}\,\mathbb{P}^\infty) \to E_{\text{ét}}^*(\mathbb{P}^\infty)$.*

Proposition 7.3 *If π is a finite abelian group, then the étale cohomology $H^*(X; \pi)$ with constant coefficients π is equal to the profinite étale cohomology theory represented by the Eilenberg-MacLane spectrum $H\pi$.*

Proof This follows directly from Remark 5.3 and Proposition 4.8. $\qquad\square$

Proposition 7.4 *Let $U \overset{i}{\hookrightarrow} X \overset{p}{\leftarrow} V$ induce an elementary distinguished square, cf. (22). Let E be a profinite spectrum. Then there is a Mayer-Vietoris long exact sequence of graded groups*

$$\cdots E_{\text{ét}}^n(X) \to E_{\text{ét}}^n(U) \oplus E_{\text{ét}}^n(V) \to E_{\text{ét}}^n(U \times_X V) \to E_{\text{ét}}^{n+1}(X) \cdots .$$

Proof This follows directly from Theorem 5.14. $\qquad\square$

Proposition 7.5 *Let E be a profinite spectrum. Let $U \subset X$ be an open subscheme of X. We get a long exact sequence of cohomology groups*

$$E^*(\hat{\text{Et}}\,X) \overset{j^*}{\to} \quad E^*(\hat{\text{Et}}\,U) \overset{\partial}{\to} \quad E^*(\hat{\text{Et}}\,(X)/\hat{\text{Et}}\,(U)) \overset{i^*}{\to}$$
$$\overset{i^*}{\to} E^{*+1}(\hat{\text{Et}}\,X) \overset{j^*}{\to} E^{*+1}(\hat{\text{Et}}\,U)$$

where $j : \hat{\text{Et}}\,U \hookrightarrow \hat{\text{Et}}\,X$ and $i : (\hat{\text{Et}}\,X, \oslash) \hookrightarrow (\hat{\text{Et}}\,X, \hat{\text{Et}}\,U)$ denote the natural induced inclusions.

Proof Since $\hat{\mathrm{Et}}\, U \hookrightarrow \hat{\mathrm{Et}}\, X \to \hat{\mathrm{Et}}\,(X)/\hat{\mathrm{Et}}\,(U)$ is isomorphic to a cofiber sequence in $\hat{\mathcal{S}}_*$, this is just the usual long exact sequence of Hom-groups induced by a cofiber sequence in a simplicial model category, see [Qu1]. $\qquad\square$

On the category Sm/k of smooth schemes of finite type over k, every étale cohomology theory satisfies homotopy invariance.

Proposition 7.6 *Let E be a profinite spectrum. Let $X \in \mathrm{Sm}/k$. The projection $p : X \times \mathbb{A}^1 \to X$ induces an isomorphism $p^* : E_{\mathrm{ét}}^*(X) \xrightarrow{\cong} E_{\mathrm{ét}}^*(X \times \mathbb{A}^1)$.*

Proof This is clear since p induces a weak equivalence in $\hat{\mathcal{S}}$, see Theorem 5.12, and hence induces isomorphisms on cohomology theories. $\qquad\square$

More generally, this implies

Corollary 7.7 *Let E be a profinite spectrum. Let $V \to X$ be an $\mathbb{A}^n$-bundle over X in Sm/k. Then $p^* : E_{\mathrm{ét}}^*(X) \xrightarrow{\cong} E_{\mathrm{ét}}^*(V)$ is an isomorphism.*

Proof This follows immediately from the Mayer-Vietoris sequence of Proposition 7.4 and the previous proposition. $\qquad\square$

In [Pa], Definition 2.0.1, Panin gives the definition of a cohomology theory on Sm/k. The following theorem states that we have constructed a way to define such cohomology theories starting with a profinite spectrum and applying its associated cohomology theory to $\hat{\mathrm{Et}}$ on Sm/k.

Theorem 7.8 *Let E be a profinite spectrum. The étale cohomology theory $E_{\mathrm{ét}}^*(-)$ represented by E satisfies the axioms of a cohomology theory on Sm/k of [Pa].*

Proof We check the axioms of a cohomology theory in the sense of [Pa], Definition 2.0.1.
1. Localization: This is clear from Remark 7.5.
2. Excision: Let $e : (X', U') \to (X, U)$ be a morphism of pairs of schemes in Sm/k such that e is étale and for $Z = X - U$, $Z' = X' - U'$ one has $e^{-1}(Z) = Z'$ and $e : Z' \to Z$ is an isomorphism. By [Mi] III, Proposition 1.27, we know that the morphism e induces an isomorphism in étale cohomology $H^*(\hat{\mathrm{Et}}\,(X)/\hat{\mathrm{Et}}\,(U); \mathbb{Z}/\ell) \cong H^*(\hat{\mathrm{Et}}\,(X')/\hat{\mathrm{Et}}\,(U'); \mathbb{Z}/\ell)$. Hence $\hat{\mathrm{Et}}\,(X')/\hat{\mathrm{Et}}\,(U') \to$

$\hat{\mathrm{E}}\mathrm{t}\,(X)/\hat{\mathrm{E}}\mathrm{t}\,(U)$ is an isomorphism in $\hat{\mathcal{H}}_*$. Therefore, it induces the desired isomorphism

$$E^*(\hat{\mathrm{E}}\mathrm{t}\,(X)/\hat{\mathrm{E}}\mathrm{t}\,(U); \mathbb{Z}/\ell^\nu) \cong E^*(\hat{\mathrm{E}}\mathrm{t}\,(X')/\hat{\mathrm{E}}\mathrm{t}\,(U'); \mathbb{Z}/\ell^\nu)$$

for every étale cohomology theory.
3. Homotopy invariance: This is the content of Proposition 7.6. □

Later we will prove that profinite étale cobordism is in fact an *oriented* cohomology theory in the sense of [Pa] and of [LM].

7.2   Profinite étale K-theory

As a first application, we consider a comparison statement for profinite K-theory and the pro-K-theory of [Fr1], and hence for profinite étale K-theory and for Friedlander's étale K-theory. Let $\{X_s\} \in \mathrm{pro}-\mathcal{S}$ be a pro-object in $\mathcal{S}$ and let BU be the simplicial set representing complex K-theory. Friedlander defines the K-theory of $\{X_s\}$ for $\epsilon = 0, 1$ and $k > 0$ by

$$K^\epsilon(\{X_s\}; \mathbb{Z}/\ell^\nu) = \mathrm{Hom}_{\mathrm{pro}-\mathcal{H}}(\{\Sigma^\epsilon X_s\}, \{P^n BU \wedge \mathbb{Z}/\ell^\nu\})$$

where $\{P^n BU\}$ denotes the Postnikov tower of BU considered as a proobject in $\mathcal{S}$. For a locally noetherian scheme X, Friedlander defines the étale K-theory by
$$K_{\text{ét}}^\epsilon(X; \mathbb{Z}/\ell^\nu) := K^\epsilon(\mathrm{Et}\, X; \mathbb{Z}/\ell^\nu).$$

Friedlander also establishes an Atiyah-Hirzebruch spectral sequence, [Fr1], Proposition 1.4,

$$E_2^p(\mathbb{Z}/\ell^\nu) = H^n(\{X_s\}; \mathbb{Z}/\ell^\nu) \Longrightarrow K^*(\{X_s\}; \mathbb{Z}/\ell^\nu)$$

which is convergent provided that $E_2^p(\mathbb{Z}/\ell^\nu) = 0$ for n sufficiently large or that $E_2^p(\mathbb{Z}/\ell^\nu)$ is finite for each p.
Following Definition 7.1 we set

Definition 7.9 *We define the* profinite K-theory *of a profinite space X to be the cohomology theory represented by the profinitely completed spectrum $\hat{KU}$ and define* profinite K-theory with $\mathbb{Z}/\ell^\nu$-coefficients *to be represented by $\hat{KU} \wedge \mathbb{Z}/\ell^\nu$. The* profinite étale K-theory *of a scheme X over k, is hence defined by*
$$\hat{KU}_{\text{ét}}^i(X) := \hat{KU}^i(\hat{\mathrm{E}}\mathrm{t}\, X)$$

and profinite étale K-theory with $\mathbb{Z}/\ell^\nu$-coefficients *by*

$$\hat{KU}^i_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) := \hat{KU}^i(\hat{\text{Et}}\, X; \mathbb{Z}/\ell^\nu).$$

Theorem 7.10 *Let k be a separably closed field. For any scheme X of finite type over k and any ν there is a strongly convergent spectral sequence $\{E_r^{p,q}\}$:*

$$E_2^{p,q} = H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \Longrightarrow \hat{KU}^{p+q}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu).$$

Proof This is the spectral sequence of Theorem 4.11 where we use the isomorphism

$$H^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \cong H^*(\hat{\text{Et}}\, X; \mathbb{Z}/\ell^\nu)$$

of Remark 5.3 for the groups $(\hat{KU}\mathbb{Z}/\ell^\nu_{\geq 0})^q$ which are isomorphic to $\mathbb{Z}/\ell^\nu$, where $\hat{KU}\mathbb{Z}/\ell^\nu_{\geq 0}$ denotes a connective covering. Since the étale cohomology groups $H^i_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ vanish for large i, see e.g. [Mi] VI, Theorem 1.1, the convergence also follows from Theorem 4.11. $\square$

Since $\hat{KU}$ is an $\hat{\Omega}$-spectrum, we have for every profinite space X

$$\hat{KU}^i(X; \mathbb{Z}/\ell^\nu) \cong \text{Hom}_{\hat{\mathcal{H}}_*}(X_+, \hat{KU}_i \wedge \mathbb{Z}/\ell^\nu).$$

Hence the functor $(\hat{-}) : \text{pro} - \mathcal{S} \to \hat{\mathcal{S}}$ yields natural maps

$$K^\epsilon(\{X_s\}; \mathbb{Z}/\ell^\nu) \to \hat{KU}^\epsilon(\hat{X}; \mathbb{Z}/\ell^\nu)$$

for every $\{X_s\} \in \text{pro} - \mathcal{S}$. Furthermore, by Corollary 3.17 and (11) we have $\pi_q(\widehat{KU} \wedge \mathbb{Z}/\ell^k) \cong \mathbb{Z}/\ell^\nu$ for q even and $\cong 0$ for q odd. Now Proposition 2.12 implies that we have, on the one hand, for every $\{X_s\} \in \text{pro} - \mathcal{S}$ an isomorphism of the E_2-terms of the corresponding Atiyah-Hirzebruch spectral sequences and, on the other hand, a natural map between the abutments. Both maps are compatible. Hence we have the diagram

$$\begin{array}{ccccc}
E_2^p(\mathbb{Z}/\ell^\nu) & = & H^p(\{X_s\}; \mathbb{Z}/\ell^\nu) & \Longrightarrow & K^*(\{X_s\}; \mathbb{Z}/\ell^\nu) \\
& & \downarrow \cong & & \downarrow \\
E_2^p(\mathbb{Z}/\ell^\nu) & = & H^p(X; \mathbb{Z}/\ell^\nu) & \Longrightarrow & \hat{KU}^*(\hat{X}; \mathbb{Z}/\ell^\nu).
\end{array}$$

The two sequences are in particular both convergent if $H^p(\{X_s\}; \mathbb{Z}/\ell^\nu) \cong H^p(\hat{X}; \mathbb{Z}/\ell^\nu)$ either equals 0 for p sufficiently large or is finite for all p. Since the étale types of schemes of finite type over a separably closed field satisfy the convergence conditions of the theorem, we get the following

Theorem 7.11 *For every scheme X of finite type over a separably closed field k the étale K-theory groups $K^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ of [Fr1] are isomorphic to the profinite étale K-theory groups $\hat{KU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ defined above.* $\square$

7.3 Profinite étale cobordism

Following Definition 7.1 we set

Definition 7.12 *1. We define the* profinite étale cobordism *of a scheme X of finite type over k, to be the cohomology theory represented by the profinitely completed spectrum $\hat{MU}$ applied to $\hat{\mathrm{Et}}\, X$, i.e.*

$$\hat{MU}^*_{\text{ét}}(X) := \hat{MU}^*(\hat{\mathrm{Et}}\, X),$$

and profinite étale cobordism with $\mathbb{Z}/\ell^\nu$-coefficients *to be the cohomology theory represented by the profinitely completed spectrum $\hat{MU}\mathbb{Z}/\ell^\nu$ applied to $\hat{\mathrm{Et}}$, i.e.*

$$\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) := (\hat{MU}\mathbb{Z}/\ell^\nu)^*(\hat{\mathrm{Et}}\, X).$$

2. We also have relative étale cobordism groups $\hat{MU}^*_{\text{ét}}(X, U)$, resp. $\hat{MU}^*_{\text{ét}}(X, U; \mathbb{Z}/\ell^\nu)$, for an open subscheme $U \subset X$ defined by

$$\hat{MU}^*_{\text{ét}}(X, U) := \hat{MU}^*(\hat{\mathrm{Et}}\,(X)/\hat{\mathrm{Et}}\,(U))$$

and

$$\hat{MU}^*_{\text{ét}}(X, U; \mathbb{Z}/\ell^\nu) := \hat{MU}^*(\hat{\mathrm{Et}}\,(X)/\hat{\mathrm{Et}}\,(U); \mathbb{Z}/\ell^\nu).$$

3. We define the étale cobordism group

$$\hat{MU}^*_{\text{ét}}(\mathbb{P}^\infty_k) := \lim_V \hat{MU}^*_{\text{ét}}(\mathbb{P}(V))$$

and similarly for $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$ as in Definition 7.1.
4. We define ℓ-adic étale cobordism to be the limit

$$\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}_\ell) := \lim_\nu \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu).$$

Remark 7.13 *1. The exterior product $MU_n \times MU_m \to MU_{m+n}$ on MU together with the diagonal map $X \to X \times X$ yield a commutative ring structure on $\hat{MU}^*_{\text{ét}}(X)$ and $\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$.*
2. We have introduced the notion of ℓ-adic cobordism, since it is the étale cohomology theory to which ℓ-adic étale cohomology converges, see Theorem 7.15. We will not give any further comment on this theory, since most of its properties follow from those of $\hat{MU}\mathbb{Z}/\ell^\nu$.
One should note that it is not true in general that $\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}_\ell) = \hat{MU}^*_{\text{ét}}(X)$.

Proposition 7.14 *Let R be a strict local henselian ring, i.e. a local henselian ring with separably closed residue field. Then*

$$\hat{MU}^*_{\text{ét}}(\operatorname{Spec} R) \cong MU^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell.$$

In particular, for a separably closed field k we get

$$\hat{MU}^*_{\text{ét}}(k) \cong MU^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \text{ and } \hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu) \cong MU^* \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^\nu.$$

Proof This follows from Corollary 4.5 and Example 5.5. The last assertion follows from the exact sequence (12), since MU^* has no torsion. □

Theorem 7.15 *Let k be a field. For every scheme X in Sm/k, there is a convergent spectral sequence $\{E_r^{p,q}\}$ with*

$$E_2^{p,q} = H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q) \Longrightarrow \hat{MU}^{p+q}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu).$$

In addition, we have also a convergent spectral sequence for ℓ-adic cobordism

$$E_2^{p,q} = \lim_\nu H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q) = H^p_{\text{ét}}(X; \mathbb{Z}_\ell \otimes MU^q) \Longrightarrow \hat{MU}^{p+q}_{\text{ét}}(X; \mathbb{Z}_\ell).$$

Proof This is the spectral sequence of Theorem 4.11 together with the isomorphisms

$$H^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q) \cong H^*(\hat{\text{Ét}}\, X; \mathbb{Z}/\ell^\nu \otimes MU^q)$$

of Remark 5.3 for the finite group $M = \mathbb{Z}/\ell^\nu \otimes MU^q = \mathbb{Z}/\ell^\nu \otimes \hat{MU}^q$. The condition on the cohomology of X for strong convergence in Theorem 4.11 is satisfied since $H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q)$ are finite groups for all p by [Mi] VI, Corollary 5.5.

For the second sequence, the fact that $H^p_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes MU^q)$ are finite groups for all p implies that exact couples remain exact after taking limits and hence that $E_2^{p,q} = \lim_\nu E_2^{p,q}(\mathbb{Z}/\ell^\nu) = H^p_{\text{ét}}(X; \mathbb{Z}_\ell \otimes MU^q)$ converges to $\lim_\nu \hat{MU}^{p+q}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) = \hat{MU}^{p+q}_{\text{ét}}(X; \mathbb{Z}_\ell)$. □

Remark 7.16 *By Theorem 4.11, we get a spectral sequence*

$$E_2^{p,q} = H^p(\hat{\text{Ét}}\, X; \hat{MU}^q) \Longrightarrow \hat{MU}^{p+q}_{\text{ét}}(X).$$

But the problem is that, in general, we do not have an isomorphism $H^p(\hat{\text{Ét}}\, X; \hat{MU}^q) \cong H^p_{\text{ét}}(X; \hat{MU}^q)$, since $\hat{MU}^q$ is infinite. In addition, the étale cohomology with $\mathbb{Z}_\ell$-coefficients does not have good properties.

Theorem 7.17 *Let X be an algebraic variety over $\mathbb{C}$. Let $X(\mathbb{C})$ be the topological space of complex points. For every ν, there is an isomorphism*

$$\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \cong MU^*(X(\mathbb{C}); \mathbb{Z}/\ell^\nu).$$

Proof By Corollary 3.16 we know that

$$(\hat{MU}\mathbb{Z}/\ell^\nu)^q \cong (MU\mathbb{Z}/\ell^\nu)^q \cong MU^q \otimes \mathbb{Z}/\ell^\nu.$$

Since $\pi_q(MU)$ is a finitely generated abelian group, we get that $MU^q \otimes \mathbb{Z}/\ell^k$ is a finite group. We denote by $Sing(X(\mathbb{C}))$ the singular simplicial set of the topological space $X(\mathbb{C})$. By Theorem 8.4 and Corollary 8.5 of [Fr2], by Remark 5.3 and Corollary 3.16 we get a map of spectral sequences and an isomorphism

$$H^p(Sing(X(\mathbb{C})); (MU\mathbb{Z}/\ell^\nu)^q) \cong H^p(\hat{\text{Ét}}\, X; (\hat{MU}; \mathbb{Z}/\ell^\nu)^q)$$

between singular and profinite étale cohomology for all ν, p and q. Since X is finite dimensional, the corresponding complex, respectively étale Atiyah-Hirzebruch spectral sequences are convergent with isomorphic E_2-terms. □

7.4 Profinite étale cobordism is an oriented cohomology theory

We prove that $\hat{MU}^{2*}_{\text{ét}}(-)$ and $\hat{MU}^{2*}_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$ are oriented cohomology theories in the sense of [LM] on the category Sm/k of smooth quasi-projective schemes of finite type over a separably closed field k with char $k \neq \ell$.
We have to check the axioms given in [LM]. The axioms of a cohomology theory have already been shown for every étale cohomology theory. In order to check the axioms of an oriented cohomology theory for $\hat{MU}_{\text{ét}}$, our strategy is as follows. We define an orientation class for $\hat{MU}_{\text{ét}}$ and use it to define the first Chern class of a line bundle. Using the Chern class of the canonical line bundle on the projective n-space, we construct an explicit isomorphism for the cobordism of $\mathbb{P}^n_k$. The method to prove this isomorphism relies on the Atiyah-Hirzebruch spectral sequence for profinite cobordism. We use the spectral sequence to reduce the computation of the cobordism of $\mathbb{P}^n$ to the computation of the étale cohomology of $\mathbb{P}^n$ with $\mathbb{Z}/\ell$-coefficients. This reduction uses non-trivial results on profinite cobordism and is the key point for the proof that $\hat{MU}^*_{\text{ét}}$, resp. $(\hat{MU}\mathbb{Z}/\ell^\nu)^*_{\text{ét}}$, yields an oriented cohomology theory on Sm/k.

This computation and the Künneth theorem for $\hat{M}U$, resp. $\hat{M}U\mathbb{Z}/\ell^\nu$, yield a projective bundle formula. Then we use Grothendieck's method to define Chern classes for all vector bundles.

The general theory of cohomology theories with Chern classes given in [Pa] then implies all the remaining properties.

But one should note that the fact that $\hat{M}U_{\text{ét}}$ is an oriented cohomology theory on Sm/k is a nontrivial result and does not just follow from the fact that MU is an oriented theory on topological spaces. This is due to the fact that we may not use an embedding of k in $\mathbb{C}$. We need an orientation with respect to smooth schemes over any separably closed field.

To deal with the twists in étale cohomology we choose a primitive ℓ^ν-th-root of unity which yields an isomorphism $\mathbb{Z}/\ell^\nu \cong \mathbb{Z}/\ell^\nu(1)$, see Remarks 7.31 and 8.14.

Definition 7.18 *Let $\mathbb{P}_k^\infty$ be the infinite projective space in the category $LU(k)$ of spaces over k and let E be a commutative ring spectrum in $\text{Sp}(\hat{\mathcal{S}})$. Since we have an isomorphism $\hat{\text{Ét}}\,\mathbb{P}_k^1 \cong S^2$ in $\hat{\mathcal{H}}$ by Example 5.10, we have a canonical map*

$$E^2(\hat{\text{Ét}}\,\mathbb{P}_k^\infty) \to E^2(\hat{\text{Ét}}\,\mathbb{P}_k^1) \xrightarrow{\cong} E^2(S^2) \xrightarrow{\cong} \pi_0(E).$$

An orientation *for E with respect to Sm/k is a class $x_E \in \tilde{E}^2(\hat{\text{Ét}}\,\mathbb{P}^\infty)$ that maps to 1 under the above map.*

Example 7.19 *1. For $\hat{M}U$, resp. $\hat{M}U\mathbb{Z}/\ell^\nu$, we choose the orientation $x_{\hat{M}U} \in \hat{M}U^2(\hat{\text{Ét}}\,\mathbb{P}_k^\infty)$, resp. $x_{\hat{M}U\mathbb{Z}/\ell^\nu} \in \hat{M}U^2(\hat{\text{Ét}}\,\mathbb{P}_k^\infty; \mathbb{Z}/\ell^\nu)$, as the image of the orientation x_{MGL} under $\hat{\text{Ét}}$. This means that we define the map $x_{\hat{M}U} : L\hat{\text{Ét}}\,(\mathbb{P}^\infty) \to \hat{M}U \wedge S^2$ in the preimage of $1 \in \pi_0(\hat{M}U)$ such that the following diagram*

$$
\begin{array}{ccc}
\hat{\text{Ét}}\,(\mathbb{P}^\infty) & \xrightarrow{\tilde{x}_{\hat{M}U}} & \hat{M}U \wedge S^2 \\
L\hat{\text{Ét}}\,(x_{MGL}) \downarrow & \nearrow \phi & \\
L\hat{\text{Ét}}\,(MGL) \wedge \mathbb{P}^1 & &
\end{array}
$$

is commutative, where ϕ is the morphism defined in the next section using an étale Thom class.

Note that $x_{\hat{M}U} \in \hat{M}U^2(\hat{\text{Ét}}\,\mathbb{P}_k^\infty)$ also defines an element $\tilde{x}_{\hat{M}U} \in \hat{M}U_{\text{ét}}^2(\mathbb{P}_k^\infty)$ via the canonical map $\hat{M}U^*(\hat{\text{Ét}}\,\mathbb{P}^\infty) \to \hat{M}U_{\text{ét}}^*(\mathbb{P}^\infty)$, recall Definition 7.1 and Remark 7.2.

If $k = \mathbb{C}$, this orientation $x_{\hat{M}U}$ is mapped to the classical complex orientation of MU via the isomorphism of Proposition 7.17. This is due to the fact

that x_{MGL} corresponds to the complex orientation of MU under the complex realization functor, [Vo1].

2. For $H\mathbb{Z}/\ell^\nu$ an orientation $x_{H\mathbb{Z}/\ell^\nu}$ is given by a generator of $H^2(\hat{E}t\,\mathbb{P}_k^\infty; \mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu$. This last isomorphism is deduced from

$$H^2(\hat{E}t\,\mathbb{P}_k^\infty; \mathbb{Z}/\ell^\nu) \cong H^2(\mathrm{Et}\,\mathbb{P}_k^\infty; \mathbb{Z}/\ell^\nu) \cong \varprojlim_n H^2(\mathrm{Et}\,\mathbb{P}_k^n; \mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu,$$

where we use the fact that Et *commutes with colimits.*

The importance of an orientation of $\hat{M}U$, resp. $\hat{M}U\mathbb{Z}/\ell^\nu$, is that it enables us to define the first Chern class of a line bundle.

A line bundle L/X over a scheme X of dimension n over k corresponds to a morphism $X \to \mathbb{P}_k^{n+1}$ unique up to $\mathbb{A}^1$-weak equivalences. Since $L\hat{E}t$ agrees with $\hat{E}t$ on smooth schemes by Theorem 5.12, this also defines a unique map $\lambda : \hat{E}t\,X \to \hat{E}t\,\mathbb{P}_k^{n+1}$ in $\hat{\mathcal{H}}$.

It is clear from the definition that the orientation $x_{\hat{M}U}$ restricted to $\hat{E}t\,\mathbb{P}_k^{n+1}$ corresponds to a map $x_{\hat{M}U} : \hat{\Sigma}^\infty(\hat{E}t\,\mathbb{P}_k^{n+1}) \to \hat{M}U \wedge S^2$. The composition of these two maps is the first Chern class of L/X.

Definition 7.20 *A line bundle defines via composition a morphism in $\hat{\mathcal{SH}}$*

$$c_1(L) : \hat{\Sigma}^\infty(\hat{E}t\,X) \xrightarrow{\lambda} \hat{\Sigma}^\infty(\hat{E}t\,\mathbb{P}_k^{n+1}) \xrightarrow{x_{\hat{M}U}} \hat{M}U \wedge S^2,$$

i.e. an element $c_1(L) \in \hat{M}U_{\text{ét}}^2(X)$ which we call the first Chern class of L. *Similarly, we define the first Chern class of a line bundle in $\hat{M}U_{et}^2(X; \mathbb{Z}/\ell^\nu)$.*

Now we have to show that this Chern class for line bundles yields a Chern structure on étale cobordism. Therefore, we have to calculate the étale cobordism of the projective n-space $\mathbb{P}_k^n$ over k. The method we use is indirectly based on the Atiyah-Hirzebruch spectral sequence for profinite cobordism. We use the Atiyah-Hirzebruch spectral sequence to reduce the computation of $\hat{M}U_{\text{ét}}^*(\mathbb{P}^n)$ to the well-known case of étale cohomology $H_{\text{ét}}^*(\mathbb{P}^n; \mathbb{Z}/\ell)$ with $\mathbb{Z}/\ell$- coefficients. A crucial point is the isomorphism

(28) $$\hat{M}U_{\text{ét}}^*(X)/\ell \cdot \hat{M}U^{*\leq 0} \cong H_{\text{ét}}^*(X; \mathbb{Z}/\ell)$$

for every X in Sm/k.

Proposition 7.21 *Let k be a separably closed field. Let $\mathbb{P}_k^n$ be the projective space of dimension n over k. Let $\mathcal{O}(1) \to \mathbb{P}_k^n$ be the canonical quotient line*

bundle. 1. Set $\xi := c_1(\mathcal{O}(1)) \in \hat{MU}^2_{\text{ét}}(\mathbb{P}^n_k)$. The étale cobordism of $\mathbb{P}^n_k$ is given by the isomorphism

$$\hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k) \cong \hat{MU}^*[u]/(u^{n+1}), \ \xi \mapsto u.$$

2. Set $\xi := c_1(\mathcal{O}(1)) \in \hat{MU}^2_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell^\nu)$. The étale cobordism with $\mathbb{Z}/\ell^\nu$-coefficients of $\mathbb{P}^n_k$ is given by the isomorphism

$$\hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell^\nu) \cong (\hat{MU}\mathbb{Z}/\ell^\nu)^*)[u]/(u^{n+1}), \ \xi \mapsto u.$$

Proof We deduce both results from general facts about filtered MU^*-modules in [De]. The crucial point is that $H^*(\hat{\text{Ét}}\,\mathbb{P}^n_k; \mathbb{Z}/\ell^m) \cong H^*_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell^m)$ surjects onto $H^*(\hat{\text{Ét}}\,\mathbb{P}^n_k; \mathbb{Z}/\ell)$ for every m, cf. [Mi] VI, 5.6. Hence $\hat{\text{Ét}}\,\mathbb{P}^n_k$ is a profinite space of finite dimension without ℓ-torsion, see the discussion of Theorem 4.13.

By Proposition 2.1.8 of [De] respectively by definition, both MU^*-modules $M := \hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k)$ and $N := \hat{MU}^*[u]/(u^{n+1})$ are objects of the category $\hat{\mathcal{L}}$ of free MU^*-modules that are complete with respect to the ℓ-adic filtration f^* of [De], §2.1.

By Corollaire 2.1.4 of [De], it suffices to show that $M/f^1 \cong N/f^1$ as $\mathbb{Z}/\ell$-vector spaces. But since $\hat{\text{Ét}}\,\mathbb{P}^n_k$ is a finite dimensional profinite space, the isomorphism $M/f^1 \cong N/f^1$ follows from the isomorphism (28), proved in Proposition 2.1.8 of [De] together with Remark 5.3, and the isomorphism of étale cohomology $H^*_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[u]/(u^{n+1})$. This proves the assertion for $\hat{MU}$. One should remark that the proof of Proposition 2.1.8 of [De] is based on the Atiyah-Hirzebruch spectral sequence for $\hat{MU}$.

The second assertion follows from the first one using the fact that for a space X without ℓ-torsion, we have $\hat{MU}^*(X; \mathbb{Z}/\ell^\nu) \cong \hat{MU}^*(X) \otimes \mathbb{Z}/\ell^\nu$, see [Ad] III, Proposition 6.6.

We could deduce the second assertion also directly from the Atiyah-Hirzebruch spectral sequence for $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$ as in [Ad] I, Lemma 2.5, using the determination of $H^*_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell^\nu)$, since all differentials vanish. □

Proposition 7.22 For the projective space $\mathbb{P}^n_k$ over a separably closed field k and every $X \in \text{Sm}/k$, there are Künneth isomorphisms for the étale cobordism

$$\hat{MU}^*_{\text{ét}}(X) \otimes_{\hat{MU}^*} \hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k) \overset{\cong}{\longrightarrow} \hat{MU}^*_{\text{ét}}(X \times \mathbb{P}^n_k)$$

and

$$\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \otimes_{(\hat{MU}\mathbb{Z}/\ell^\nu)^*} \hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell^\nu) \overset{\cong}{\longrightarrow} \hat{MU}^*_{\text{ét}}(X \times \mathbb{P}^n_k; \mathbb{Z}/\ell^\nu).$$

Proof Again, the crucial point is that $H^*(\hat{\mathrm{Et}}\,\mathbb{P}^n_k; \mathbb{Z}/\ell^m) \cong H^*_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell^m)$ surjects onto $H^*(\hat{\mathrm{Et}}\,\mathbb{P}^n_k; \mathbb{Z}/\ell)$ for every m. Hence $\hat{\mathrm{Et}}\,\mathbb{P}^n_k$ is a profinite space without ℓ-torsion. Over the separably closed field k, $H^i_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ and $H^i_{\text{ét}}(\mathbb{P}^n_k; \mathbb{Z}/\ell)$ vanish for large i and hence the profinite spaces $\hat{\mathrm{Et}}\,X$ and $\hat{\mathrm{Et}}\,\mathbb{P}^n_k$ are finite dimensional spaces in $\hat{\mathcal{S}}$. This enables us to deduce the proposition from Theorem 4.13. $\qquad\square$

Theorem 7.23 Projective Bundle Formula
Let $E \to X$ be a rank n vector bundle over X in Sm/k, $\mathcal{O}(1) \to \mathbb{P}(E)$ the canonical quotient bundle with zero section $s : \mathbb{P}(E) \to \mathcal{O}(1)$. Set $\xi := c_1(\mathcal{O}(1)) \in \hat{MU}^2_{\text{ét}}(\mathbb{P}(E))$. Then $\hat{MU}^*_{\text{ét}}(\mathbb{P}(E))$ is a free $\hat{MU}^*_{\text{ét}}(X)$-module with basis $(1, \xi, \ldots, \xi^{n-1})$.
Similarly, $\hat{MU}^*_{\text{ét}}(\mathbb{P}(E); \mathbb{Z}/\ell^\nu)$ is a free $\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$-module with basis $(1, \xi, \ldots, \xi^{n-1})$, where $\xi := c_1(\mathcal{O}(1)) \in \hat{MU}^2_{\text{ét}}(\mathbb{P}(E))$.

Proof We consider $\hat{MU}_{\text{ét}}$. We prove this theorem first for the case of a trivial bundle on X. We have to show that the map $\hat{MU}^*_{\text{ét}}(X \times \mathbb{P}^n_k) \xrightarrow{\cong} \hat{MU}^*_{\text{ét}}(X)[u]/(u^n)$, sending ξ to u, is an isomorphism. By Proposition 5.4, we know that the canonical map $\hat{\mathrm{Et}}\,(X) \times \hat{\mathrm{Et}}\,(\mathbb{P}^n_k) \to \hat{\mathrm{Et}}\,(X \times \mathbb{P}^n_k)$ is a weak equivalence in $\hat{\mathcal{S}}$. Proposition 7.21 yields an isomorphism

$$\hat{MU}^*_{\text{ét}}(X)[u]/(u^n) \xrightarrow{\cong} \hat{MU}^*_{\text{ét}}(X) \otimes \hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k).$$

Hence it suffices to prove that the projections induce an isomorphism

$$\hat{MU}^*_{\text{ét}}(X) \otimes \hat{MU}^*_{\text{ét}}(\mathbb{P}^n_k) \xrightarrow{\cong} \hat{MU}^*_{\text{ét}}(X \times \mathbb{P}^n_k).$$

This follows from the previous proposition and finishes the proof of the theorem in the case that E is a trivial bundle on X.
A similar argument using the second assertion of Proposition 7.21 proves the case of a trivial bundle for $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$.
For the general case, since E is by definition locally trivial for the Zariski topology on X, it suffices to show that the theorem holds for X if it holds for open subsets X_0, X_1 and $X_0 \cap X_1$, with $X = X_0 \cup X_1$. This point now follows from comparing the two Mayer-Vietoris-sequences

$$\cdots \to \hat{MU}^i_{\text{ét}}(\mathbb{P}(E_0)) \oplus \hat{MU}^i_{\text{ét}}(\mathbb{P}(E_1))$$
$$\to \hat{MU}^i_{\text{ét}}(\mathbb{P}(E_{01})) \to \hat{MU}^{i+1}_{\text{ét}}(\mathbb{P}(E)) \to \cdots$$

$$\hat{MU}^*[u]/(u^n) \otimes (\cdots \to \hat{MU}^i_{\text{ét}}(X_0) \oplus \hat{MU}^i_{\text{ét}}(X_1)$$
$$\to \hat{MU}^i_{\text{ét}}(X_{01}) \to \hat{MU}^{i+1}_{\text{ét}}(X) \to \cdots)$$

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where we have written X_{01} for $X_0 \cap X_1$, E_0, E_1 and E_{01} for the restrictions of E to X_0, X_1 and X_{01}.

Again, a similar argument shows the general case for $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$. $\qquad\square$

We use Grothendieck's idea to introduce higher Chern classes for vector bundles.

Definition 7.24 *Let E be a vector bundle of rank n on X in Sm/k and let $\xi := c_1(\mathcal{O}(1))$ be as above. Then we define the ith Chern class of E to be the element $c_i(E) \in \hat{MU}^{2i}_{\text{ét}}(E)$ such that $c_0(E) = 1$, $c_i(E) = 0$ for $i > n$ and $\Sigma_{i=0}^n (-1)^i c_i(E) \xi^{n-i} = 0$. By the projective bundle formula of Theorem 7.23 these elements are unique.*

Similarly, we define $c_i(E) \in \hat{MU}^{2i}_{\text{ét}}(E; \mathbb{Z}/\ell^\nu)$ by Theorem 7.23.

Lemma 7.25 *1. For each line bundle over a smooth X, the class $c_1(L) \in \hat{MU}^2_{\text{ét}}(X)$, resp. $c_1(L) \in \hat{MU}^2_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$, is nilpotent.*

2. The assignment of Chern classes is functorial with respect to the pullback, i.e. $f^*(c_i(E)) = c_i(f^*E)$ for a bundle E on X and a morphism $f : Y \to X$ in Sm/k.

Proof 1. This may be proved exactly as in [Pa], Lemma 3.6.4.
2. This follows immediately from Theorem 7.23. $\qquad\square$

Theorem 7.26 *The étale cobordism theories satisfy the axioms of an oriented ring cohomology theory on Sm/k of [Pa], Definition 3.1.1. This implies that for every projective morphism $f : Y \to X$ of codimension d there are natural transfer maps*

$$f_* : \hat{MU}^*_{\text{ét}}(Y) \to \hat{MU}^{*+2d}_{\text{ét}}(X)$$

and

$$f_* : \hat{MU}^*_{\text{ét}}(Y; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{*+2d}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$$

which satisfy all the axioms of a push-forward in an oriented cohomology theory of [LM].

Proof We have proved in Theorem 7.8 that $\hat{MU}_{\text{ét}}(-)$ and $\hat{MU}_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$ satisfy the axioms of a cohomology theory in the sense of [Pa], Definition 2.0.1. By Remark 7.13, we know that $\hat{MU}_{\text{ét}}(-)$, resp. $\hat{MU}_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$, is

also a ring cohomology theory.

Secondly, the orientation on $\hat{MU}_{ét}(-)$, resp. $\hat{MU}_{ét}^*(-;\mathbb{Z}/\ell^\nu)$, and Theorem 7.23 show that we have in fact an oriented theory with Chern classes in the sense of [Pa], Definition 3.1.1, using Theorem 3.2.4 of [Pa].

Now there is a general procedure, described in 4.2 and 4.3 of [Pa], to define natural transfer maps for every projective morphism in Sm/k. Note that since we require transfer maps only for projective maps, we only have to construct transfers for closed immersions of smooth schemes and the projection of the projective space $\mathbb{P}_X^n \to X$ over X. In Theorem 4.7.1 of [Pa] it is proved that the push-forward is independent of the decomposition of f.

Furthermore, also the compatibility of pullbacks and push-forwards for transversal morphisms is proved in Theorem 4.7.1 of [Pa].

That f_* shifts the degree by $2d$ comes from the fact that Chern classes live in even degrees and the fact that the construction of f_* depends on Chern classes.

This finishes the proof of the theorem. □

We state a general consequence of this extra structure on étale cobordism.

Corollary 7.27 *For every closed embedding of smooth varieties $i : Z \hookrightarrow X$ with open complement $j : U \hookrightarrow X$ there are exact sequences*

$$\hat{MU}_{ét}^*(Z) \xrightarrow{\ i_*\ } \hat{MU}_{ét}^*(X) \xrightarrow{\ j^*\ } \hat{MU}_{ét}^*(U)$$

and

$$\hat{MU}_{ét}^*(Z;\mathbb{Z}/\ell^\nu) \xrightarrow{\ i_*\ } \hat{MU}_{ét}^*(X;\mathbb{Z}/\ell^\nu) \xrightarrow{\ j^*\ } \hat{MU}_{ét}^*(U;\mathbb{Z}/\ell^\nu).$$

We call them the Gysin sequences *for $\hat{MU}_{ét}^*(-)$, resp. $\hat{MU}_{ét}^*(-;\mathbb{Z}/\ell^\nu)$.*

Proof This is proved in [Pa] 4.4.2. □

As another corollary of the last theorem we get

Corollary 7.28 *Let k be a separably closed field and ℓ a prime number different from the characteristic of k. Profinite étale cobordism $\hat{MU}_{ét}^{2*}(-)$, resp. $\hat{MU}_{ét}^{2*}(-;\mathbb{Z}/\ell^\nu)$, is an oriented cohomology theory on Sm/k in the sense of [LM].* □

Using Theorem 7.23 we can identify $\hat{MU}_{ét}^{2*}(\mathbb{P}^n)$ with the ring $\hat{MU}^*[u]/(u^{n+1})$, identifying $c_1(\mathcal{O}(1))$ with u, and in the same way

$$\hat{MU}_{ét}^*(\mathbb{P}^n \times \mathbb{P}^n) = \hat{MU}^*[u,v]/(u^{n+1}, v^{n+1}).$$

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As Quillen pointed out in [Qu3], Proposition 2.7, we can associate to the theory $\hat{MU}_{\text{ét}}$, resp. $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$, a formal group law, i.e. a power series $F(u,v) \in \hat{MU}^*[[u,v]]$, $F(u,v) = \Sigma_{i,j\geq 0} a_{ij} u^i v^j$ satisfying some axioms with $a_{ij} \in \hat{MU}^{2-2i-2j}$, to an oriented cohomology theory such that

$$c_1(L \otimes M) = F(c_1(L), c_1(M))$$

for any two line bundles L, M over X.

Let $\mathbb{L}_*$ be the Lazard ring. There is Quillen's famous theorem that $\mathbb{L}_* \to \pi_{2*}(MU)$ is an isomorphism.

We have already determined the coefficients of $\hat{MU}_{\text{ét}}$ over a separably closed field to be $\hat{MU}^* \cong MU^* \otimes_\mathbb{Z} \mathbb{Z}_\ell$, resp. $(\hat{MU}\mathbb{Z}/\ell^\nu)^* \cong MU^* \otimes_\mathbb{Z} \mathbb{Z}/\ell^\nu$, see Proposition 7.14. But we have to check that the natural morphism $\Phi : \mathbb{L}^* \to \hat{MU}^*_{\text{ét}}$ induced by the formal group law on $\hat{MU}_{\text{ét}}$ agrees with this isomorphism.

Proposition 7.29 *Let k be a separably closed field. The map $\Phi_\ell : \mathbb{L}^* \otimes \mathbb{Z}_\ell \to \hat{MU}^{2*}_{\text{ét}}(k)$ induced by the natural map of formal group laws is an isomorphism. Similarly, the induced map $\Phi/\ell^\nu : \mathbb{L}^* \otimes \mathbb{Z}/\ell^\nu \to \hat{MU}^{2*}_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$ is an isomorphism.*

Proof We consider the induced map $\Phi_\ell : \mathbb{L}^* \otimes \mathbb{Z}_\ell \to \hat{MU}^{2*}_{\text{ét}}(k)$. It is clear that this map Φ_ℓ is a morphism in the category $\hat{\mathcal{L}}$ of [De], §2.1, that we already used in the proof of Proposition 7.21. Furthermore, it is clear that this map induces an isomorphism $\mathbb{Z}/\ell = \mathbb{L}^* \otimes \mathbb{Z}_\ell/f^1 \overset{\cong}{\to} \hat{MU}^{2*}_{\text{ét}}(k)/f^1 \cong \mathbb{Z}/\ell$ modulo the filtration $f^{/ast}$ of [De], §2.1. By [De], Corollaire 2.1.4, this implies that the Φ_ℓ is an isomorphism.

The second assertion follows from the first one by the coefficient exact sequence. $\square$

Remark 7.30 *The proof of the last proposition is similar to the proof of $\Phi : \mathbb{L}^* \cong MGL^{2*,*}(k)$ by Hopkins and Morel [HM]. The idea is to construct a spectrum $MGL/(x_1, x_2, \ldots)$, where the x_i's are the generators of $\mathbb{L}^* \cong \mathbb{Z}[x_1, x_2, \ldots]$. The crucial point for the proof of the surjectivity of the map $\Phi : \mathbb{L}^* \to MGL^{2*,*}(k)$ is to show that the map of spectra*

$$MGL/(x_1, x_2, \ldots) \to H\mathbb{Z}$$

is an isomorphism, where $H\mathbb{Z}$ denotes the motivic Eilenberg-MacLane spectrum. In the proof of Proposition 7.29, this corresponds to the isomorphism

$\mathbb{Z}/\ell = \mathbb{L}^* \otimes \mathbb{Z}_\ell / f^1 \xrightarrow{\cong} \hat{MU}^{2*}_{\text{ét}}(k)/f^1 \cong H^*(\hat{\text{Ét}}\, k; \mathbb{Z}/\ell) = \mathbb{Z}/\ell.$

Then one shows by induction on n that $\mathbb{L}^n \to MGL^{2n,n}(k)$ is surjective.

The injectivity of Φ can be shown as for $\mathbb{L}^* \to \Omega^*(k)$ in [LM].

Remark 7.31 *The proof of Theorem 7.26 is based on two results. On the one hand, we used Panin's discussion and results on oriented cohomology theories to deduce the missing structures from the existence of an orientation and the projective bundle bundle formula. On the other hand, we had to prove the projective bundle formula. For this purpose, we used the Atiyah-Hirzebruch spectral sequence to reduce the problem to the projective bundle formula for étale cohomology.*

Although we worked over a separably closed base field, étale cohomology $\oplus_q H^*_{\text{ét}}(-; \mu_{\ell^\nu}^{\otimes q})$ is an oriented theory over any field k with char $k \neq \ell$. The problem for our argument arises with the twist of the coefficients. Nevertheless, it seems to be very likely that the above proof also works over any field with characteristic prime to ℓ. Maybe one needs to modify the definition of étale cobordism for this purpose.

One could start with a naive version of twisted coefficients for $\hat{MU}_{\text{ét}}$ as follows. For any field k, let μ_{ℓ^ν} be the group of ℓ^νth roots of unity in k. Since we can form a Moore spectrum MG for every finite abelian group, we may consider the twisted Moore spectrum $M(\mu_{\ell^\nu}^{\otimes q}) = M\mathbb{Z}/\ell^\nu(q)$. Since μ_{ℓ^ν} is finite, we can consider the profinite completion $\hat{MU}\mathbb{Z}/\ell^\nu(q)$ of MU $\wedge$ $M\mathbb{Z}/\ell^\nu(q)$. We could define profinite étale cobordism with twisted coefficients $\hat{MU}^*_{\text{ét}}(-; \mathbb{Z}/\ell^\nu(*))$ to be the profinite cohomology represented by the profinite spectrum $\hat{MU}\mathbb{Z}/\ell^\nu(*)$ applied to the functor $\hat{\text{Ét}}$ on schemes over k.

If k is a separably closed field and if $X \in \mathrm{Sm}/k$, we have a canonical isomorphism

$$\hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu(r)) \cong \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \otimes \mathbb{Z}/\ell^\nu(r)$$

for every r.

For, we have a canonical isomorphism for étale cohomology for every r

$$E_2^{p,q} = H^p_{\text{ét}}(X; MU^q \otimes \mathbb{Z}/\ell^\nu(r)) \cong H^p_{\text{ét}}(X; MU^q \otimes \mathbb{Z}/\ell^\nu) \otimes \mathbb{Z}/\ell^\nu(r) =: \tilde{E}_2^{p,q}.$$

The two converging Atiyah-Hirzebruch spectral sequences imply the result. But the geometry of $\hat{\mathcal{SH}}$ does not seem to reflect this extra structure. It is more likely, that one has to find another way to define twisted étale cobordism to handle this problem.

8 Algebraic versus Profinite Etale Cobordism

Let ℓ be fixed prime number different from the characteristic of the base field k. Using the results of the previous sections, we compare étale cobordism with the algebraic cobordism theories of Levine/Morel and Voevodsky over a separably closed field. The first important result is that $\Omega^*(X; \mathbb{Z}/\ell^\nu)$ surjects onto $\hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ for every smooth scheme over a separably closed field. We deduce from this that $\hat{MU}^{2*}_{\text{ét}}(X)$, resp. $\hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$, is generated as an $\mathbb{L}^* \otimes \mathbb{Z}_\ell$-module, resp. $\mathbb{L}^* \otimes \mathbb{Z}/\ell^\nu$-module, by elements of positive degree. Furthermore, we construct a map from the MGL-theory to profinite étale cobordism via the results of the previous section on the stable étale realization functor. The construction is not as straightforward as one would like, since we do not know if $\hat{\text{Et}} \, MGL$ is isomorphic to $\hat{MU}$ in $\hat{\mathcal{SH}}$. Nevertheless, the stable étale realization of Theorem 6.8 is the main technical advantage of the use of $\hat{\text{Et}}$ instead of Et and is a necessary ingredient for the construction of the map from algebraic to profinite étale cobordism. The other key point is an étale Thom class for vector bundles. This Thom class yields the map $\hat{\text{Et}} \, MGL \to \hat{MU}$.

In an intermediate section, we use these facts to study the action of the absolute Galois group G_k on $\hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$ for an arbitrary field k. We show that G_k acts trivially on $\hat{MU}^*_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$, where k^s denotes a separable closure of k. The Atiyah-Hirzebruch spectral sequence applied to the étale cohomology of k enables us to determine the étale cobordism of the finite field $k = \mathbb{F}_q$.

At the end, we conjecture that our map from algebraic cobordism to profinite étale cobordism is an isomorphism for $\mathbb{Z}/\ell^\nu$-coefficients after inverting a Bott element. We explain a strategy to prove this conjecture and give arguments for the truth of the conjecture.

8.1 Comparison with Ω^*

We consider the algebraic cobordism theories $\Omega^*(-)$ and $\Omega^*(-; \mathbb{Z}/\ell^\nu)$. As a corollary of Theorem 7.26 we get

Theorem 8.1 *By universality of Ω^*, the structure of an oriented cohomology theory on $\hat{MU}_{\text{ét}}$ and $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$ yields canonical morphisms*

$$\theta : \Omega^*(X) \to \hat{MU}^{2*}_{\text{ét}}(X)$$

and

$$\theta : \Omega^*(X; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$$

defined by sending a generator $[f : Y \to X] \in \Omega^*(X)$, $f : Y \to X$ a projective morphism between smooth schemes, to the element $f_*(1_Y) \in \hat{MU}^{2*}_{\text{ét}}(X)$; similarly with $\mathbb{Z}/\ell^\nu$-coefficients.

If k is a field of characteristic zero, Theorem 4 of [LM] states that $\Omega^*(k) \cong \mathbb{L}^*$. But it is conjectured that this isomorphism holds for every field, see Conjecture 2 of [LM]. We refer to this conjecture as

Assumption 8.2 *For the field k, the canonical morphism $\mathbb{L}^* \xrightarrow{\cong} \Omega^*(k)$ is an isomorphism.*

If we suppose that this isomorphism holds for k, we may conclude

Proposition 8.3 *Let k be a separably closed field and suppose that Assumption 8.2 is true for k.*
Then θ becomes an isomorphism on coefficient rings

$$\theta : \Omega^*(k; \mathbb{Z}/\ell^\nu) \cong \hat{MU}^{2*}_{\text{ét}}(k; \mathbb{Z}/\ell^\nu).$$

Furthermore, θ yields an isomorphism

$$\theta : \Omega^*(k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \hat{MU}^{2*}_{\text{ét}}(k).$$

In particular, for every separably closed field k, the morphisms

$$\theta : \Omega^*(k; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$$

and

$$\theta \otimes \mathbb{Z}_\ell : \Omega^*(k) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \to \hat{MU}^{2*}_{\text{ét}}(k)$$

are surjective.

Proof The first part follows from the assumption and Proposition 7.14.
For the surjectivity, consider the canonical map $\Phi : \mathbb{L}^* \otimes \mathbb{Z}/\ell^\nu \to \Omega^*(k; \mathbb{Z}/\ell^\nu)$. Levine and Morel have shown that it is injective for every field, see [LM], Corollary 12.3. When we compose this map with $\theta : \Omega^*(k; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$, we get the canonical map

$$\mathbb{L}^* \otimes \mathbb{Z}/\ell^\nu \xrightarrow{\Phi} \Omega^*(k; \mathbb{Z}/\ell^\nu) \xrightarrow{\theta} \hat{MU}^{2*}_{\text{ét}}(k; \mathbb{Z}/\ell^\nu).$$

Since this map is unique, it must be the identity. Hence $\theta : \Omega^*(k; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$ is surjective.

A similar argument works for $\theta : \Omega^*(k) \otimes \mathbb{Z}_\ell \to \hat{MU}^{2*}_{\text{ét}}(k)$.

Note that for the surjectivity assertion we did not need to suppose that Assumption 8.2 is true. $\square$

Proposition 8.4 *Over a separably closed field, the oriented cohomology theories $\hat{MU}^{2*}_{\text{ét}}(-)$ and $\hat{MU}^{2*}_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$ satisfy the generalized degree formula of Theorem 13.6 and Corollary 13.7 of [LM].*

Proof It is clear that $\hat{MU}^{2*}_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$ is generically constant in the sense of [LM] for trivial reasons since k is separably closed. Moreover, it satisfies the localization property of Proposition 7.27. This implies that $\hat{MU}^{2*}_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$ satisfies the cohomology version of the generalized degree formula of [LM], Theorem 13.6 and Corollary 13.7, since the proofs only use the two additional properties, generically constant and the Gysin exact sequence, of an oriented cohomology theory. $\square$

In particular, we may define a degree map $\deg : \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu) \to \hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$ as in [LM]. Again, given the conjecture, we can prove the following

Theorem 8.5 *Let k be a separably closed field. Then:*
1. $\theta : \Omega^*(X; \mathbb{Z}/\ell^\nu) \to \hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$ is surjective for every X in Sm/k.
2. $\theta : \Omega^*(X) \otimes_\mathbb{Z} \mathbb{Z}_\ell \to \hat{MU}^{2*}_{\text{ét}}(X)$ is surjective for every X in Sm/k.

Proof We prove this theorem by induction on $\dim_k X$. For $\dim_k X = 0$ the surjectivity is proved in Proposition 8.3.

Let $\dim_k X = n$ and $\alpha \in \hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$. Following the argument in the proof of Theorem 13.6 in [LM], there is an open subscheme $j : U \hookrightarrow X$ such that U contains all generic points of X and $\alpha - \deg(\alpha)_{|U} = 0$. Let $W \hookrightarrow X$ be the closed complement of U. We already know that $\deg(\alpha)$ lies in the image of $\Omega^*(X; \mathbb{Z}/\ell^\nu)$ by the case of $\dim_k X = 0$. By the Gysin sequence, there is an element $\alpha' \in \hat{MU}^{2*}_{\text{ét}}(W; \mathbb{Z}/\ell^\nu)$ such that $\alpha - \deg(\alpha) - i_*(\alpha') = 0$ in $\hat{MU}^{2*}_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$. By the induction hypothesis and $\text{codim}_X W > 0$, the element α' is in the image of $\Omega^*(W; \mathbb{Z}/\ell^\nu)$. Hence we deduce that α is in the image of $\Omega^*(X; \mathbb{Z}/\ell^\nu)$.

For the second assertion the same proof works. $\square$

From the proof of Theorem 8.5 and the properties of $\Omega^*(\cdot)$ we deduce the following

Corollary 8.6 *Let k be a separably closed field.*
1. For any irreducible X in Sm/k, $\hat{M}\hat{U}_{\text{ét}}^*(X; \mathbb{Z}/\ell^\nu)$ is generated as an $\mathbb{L}^* \otimes_\mathbb{Z}$ $\mathbb{Z}/\ell^\nu$-module by $1 \in \hat{M}\hat{U}_{\text{ét}}^0(X; \mathbb{Z}/\ell^\nu)$ and elements of $\hat{M}\hat{U}_{\text{ét}}^{*>0}(X; \mathbb{Z}/\ell^\nu)$.
2. For any irreducible X in Sm/k, $\hat{M}\hat{U}_{\text{ét}}^*(X)$ is generated as an $\mathbb{L}^* \otimes_\mathbb{Z} \mathbb{Z}_\ell$-module by $1 \in \hat{M}\hat{U}_{\text{ét}}^0(X)$ and elements of $\hat{M}\hat{U}_{\text{ét}}^{*>0}(X)$.

8.2 Comparison with $MGL^{*,*}$

Now we dedicate our attention to the comparison with the algebraic cobordism theory represented by the MGL-spectrum in the stable motivic $\mathbb{A}^1$-homotopy category. The technical advantage of $\hat{\mathrm{Et}}$ compared to Et now becomes apparent since we could construct a stable realization of $\mathbb{P}^1$-spectra in the previous section. We will use this functor to define a map from MGL-theory to profinite étale cobordism. First we define Thom classes which will enable us to define canonical maps of oriented cohomology theories. We will do this in essentially the same way in which one shows that MGL is the universal oriented $\mathbb{P}^1$-spectrum.
Let k be a separably closed field. Let V be a vector bundle of rank d over X in Sm/k. We recall that the Thom space $\mathrm{Th}(V) \in LU(k)$ of V is defined to be the quotient
$$\mathrm{Th}(V) = \mathrm{Th}(V/X) = V/(V - i(X))$$
where $i : X \to V$ denotes the zero section of V. We reformulate a lemma from $\mathbb{A}^1$-homotopy theory.

Proposition 8.7 *Let V be a vector bundle over X and $\mathbb{P}(V) \to \mathbb{P}(V \oplus \mathcal{O})$ be the closed embedding at infinity. Then the canonical morphism of pointed sheaves: $\mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V) \to \mathrm{Th}(V)$ induces a weak equivalence in $\hat{\mathcal{S}}_*$ via $\hat{\mathrm{Et}}$.*

Proof This is the same proof as for Proposition 3.2.17 of [MV] where we use the fact that $\hat{\mathrm{Et}}$ preserves $\mathbb{A}^1$-weak equivalences between smooth schemes, see Theorem 5.12, and commutes with quotients, i.e $\hat{\mathrm{Et}}(X/A) = \hat{\mathrm{Et}} X/\hat{\mathrm{Et}} A$.
$\square$

Corollary 8.8 *Let $T = \mathbb{A}^1/(\mathbb{A}^1 - 0)$ in $LU(k)$. Then we have an isomorphism in $\hat{\mathcal{H}}_*$ of pointed profinite spaces $\hat{\mathrm{Et}}(T) \cong S^2$.*

Proof This follows from the previous proposition using the fact that $T = \text{Th}(\mathcal{O})$ and $\hat{\text{Et}}\,\mathbb{P}^1 \cong S^2$ in $\hat{\mathcal{H}}$ by Example 5.10. $\qquad\square$

Now we define the Thom class of V in $\hat{MU}^{2d}_{\text{ét}}(\text{Th}(V))$. From the isomorphism $\mathbb{P}(V \oplus \mathcal{O})/\mathbb{P}(V) \cong \text{Th}(V)$ of the previous proposition we deduce an exact sequence induced by the cofiber sequence

$$\hat{MU}^*_{\text{ét}}(\text{Th}(V)) \to \hat{MU}^*_{\text{ét}}(\mathbb{P}(V \oplus \mathcal{O})) \to \hat{MU}^*_{\text{ét}}(\mathbb{P}(V)).$$

Using the projective bundle formula of Theorem 7.23 this sequence is isomorphic to the exact sequence

$$\hat{MU}^*_{\text{ét}}(\text{Th}(V)) \to \hat{MU}^*[1, u, \ldots, u^d] \to \hat{MU}^*[1, u, \ldots, u^{d-1}].$$

The element $u^d - c_1(V)u^{d-1} + \ldots + (-1)^d c_d(V) \in \hat{MU}^*[1, u, \ldots, u^d]$ is sent to $u^d - c_1(V)u^{r-1} + \ldots + (-1)^d c_d(V) \in \hat{MU}^*[1, u, \ldots, u^{d-1}]$ which is 0 by the definition of Chern classes via the projective bundle formula. By exactness, there is an element $\text{th}_{\text{ét}}(V) \in \hat{MU}^{2d}_{\text{ét}}(\text{Th}(V))$ that is sent to the above element $u^d + \ldots + (-1)^d c_d(V) \in \hat{MU}^*[1, u, \ldots, u^d]$ which is homogeneous of degree $2d$ since u has degree 2. We call $\text{th}_{\text{ét}}(V)$ an *étale Thom class of V*. Hence the Thom class of V is a morphism

$$\text{th}_{\text{ét}}(V) : \hat{\Sigma}^\infty(\hat{\text{Et}}\,\text{Th}(V)) \to \hat{MU} \wedge S^{2d}$$

in $\hat{\mathcal{SH}}$.

In the same way, we define an étale Thom class for $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$.

We apply this argument to the tautological n-bundle γ_n over the infinite Grassmannian. Since $MGL_n = \text{Th}(\gamma_n)$ we get via $\hat{\text{Et}}$ a map of profinite spaces

$$\hat{\text{Et}}\,(MGL_n) \to \hat{MU}_{2n}.$$

In the proof of Theorem 6.8 we have seen that $\hat{\text{Et}}$ yields a map of spectra on the level of homotopy categories. Hence the above map yields a morphism in $\hat{\mathcal{SH}}_2$

(29) $$\phi : \text{L}\hat{\text{Et}}\,(MGL) \longrightarrow \hat{MU}$$

where $\text{L}\hat{\text{Et}}$ denotes the total left derived functor of $\hat{\text{Et}}$ on $\mathbb{P}^1$-spectra.

Since smooth schemes $X \in \text{Sm}/k$ are cofibrant objects, $\text{L}\hat{\text{Et}}$ and $\hat{\text{Et}}$ agree on Sm/k by Theorem 5.12. Hence this morphism in $\hat{\mathcal{SH}}_2$ yields a natural morphism of cohomology theories

$$(\text{L}\hat{\text{Et}}\,MGL)^*(\hat{\text{Et}}\,X) \to \hat{MU}^*_{\text{ét}}(X)$$

for every scheme X in Sm/k. It is clear that this construction is natural in X.

Recall the definition of algebraic cobordism for a scheme X in Sm/k

$$MGL^{p,q}(X) := \mathrm{Hom}_{\mathcal{SH}^{\mathbb{P}^1}(k)}(\Sigma_{\mathbb{P}^1}^{\infty}(X), MGL[p-2q] \wedge (\mathbb{P}^1)^{\wedge q}),$$

where $E[n] := E \wedge (S_s^1)^{\wedge n}$, S_s^1 denoting the simplicial circle. We have seen in Section 6 that $S_s^1 \wedge -$ commutes with $\hat{\mathrm{Et}}$. Hence the elements in degree p,q are sent to elements in degree p via ϕ. We summarize this discussion in the following

Theorem 8.9 *The obvious induced map of Theorem 6.8 yields a natural map for every X in Sm/k*

(30) $$\phi : MGL^{*,*}(X) \to \hat{MU}_{\mathrm{\acute{e}t}}^{*}(X),$$

which is $\phi : MGL^{p,q}(X) \to \hat{MU}_{\mathrm{\acute{e}t}}^{p}(X)$ on the p,q-level.
In the same way we get a map for MGL with $\mathbb{Z}/\ell^{\nu}$-coefficients

(31) $$\phi : MGL^{*,*}(X; \mathbb{Z}/\ell^{\nu}) \to \hat{MU}_{\mathrm{\acute{e}t}}^{*}(X; \mathbb{Z}/\ell^{\nu}).$$

Recall that a morphism of oriented cohomology theories is a natural transformation that commutes with Chern classes and transfer maps. In fact, Theorem 4.1.4 of [Pa] states that we have to check only the compatibility with one of these structures. It is clear from the construction that we get the following

Theorem 8.10 *The natural transformations $\phi : MGL^{*,*}(-) \to \hat{MU}_{\mathrm{\acute{e}t}}^{*}(-)$ and $\phi : MGL^{*,*}(-; \mathbb{Z}/\ell^{\nu}) \to \hat{MU}_{\mathrm{\acute{e}t}}^{*}(-; \mathbb{Z}/\ell^{\nu})$ are morphisms of oriented cohomology theories on Sm/k.*

Proof We have to check that ϕ respects Chern classes in both theories. Then Theorem 4.1.4 of [Pa] implies that the transfer maps in both theories agree. But the construction of Chern classes in $\hat{MU}_{\mathrm{\acute{e}t}}$ is the exact analog of the construction of Chern classes in MGL only involving the functor $\hat{\mathrm{Et}}$ at the appropriate places, i.e. the orientation $x_{\hat{MU}}$ of $\hat{MU}_{\mathrm{\acute{e}t}}$ is just $\hat{\mathrm{Et}}(x_{MGL})$, i.e. the image of the orientation x_{MGL} of MGL under $\hat{\mathrm{Et}}$. Hence we have $\phi(c_1^{MGL}(L)) = c_1^{\hat{MU}}(L)$ in $\hat{MU}_{\mathrm{\acute{e}t}}^{2}(X)$. Then Theorem 4.1.4 of [Pa] implies that ϕ respects the oriented structures and pushforwards, i.e. ϕ is a morphism of oriented cohomology theories.

The same argument holds for $\mathbb{Z}/\ell^{\nu}$-coefficients. $\qquad\square$

Remark 8.11 *We could have deduced from this theorem that $\hat{MU}_{\text{ét}}$ carries in fact the formal group law induced by $\mathbb{L}^* \to \mathbb{L}^* \otimes \mathbb{Z}_\ell$. Since Hopkins and Morel proved in [HM] that $\Phi : \mathbb{L}^* \to MGL^{2*,*}(k)$ is an isomorphism, the above theorem implies that the formal group law on $\hat{MU}_{\text{ét}}$ is the one induced by $MGL^{2*,*}$.*

As a corollary we get the following

Theorem 8.12 *Let k be a separably closed field. The canonical map of oriented cohomology theories $\theta : \Omega^*(-) \to \hat{MU}_{\text{ét}}^{2*}(-)$ factors through $MGL^{2*,*}(-)$, i.e. the following diagram*

$$\begin{array}{ccc}
\Omega^*(X) & \overset{\theta_{MGL}}{\longrightarrow} & MGL^{2*,*}(X) \\
{\scriptstyle \theta_{\hat{MU}}} \searrow & & \swarrow {\scriptstyle \phi} \\
& \hat{MU}_{\text{ét}}^{2*}(X) &
\end{array}$$

is commutative for every X in Sm/k.
Furthermore, the diagram

$$\begin{array}{ccc}
\Omega^*(X; \mathbb{Z}/\ell^\nu) & \overset{\theta_{MGL}}{\longrightarrow} & MGL^{2*,*}(X; \mathbb{Z}/\ell^\nu) \\
{\scriptstyle \theta_{\hat{MU}}} \searrow & & \swarrow {\scriptstyle \phi} \\
& \hat{MU}_{\text{ét}}^{2*}(X; \mathbb{Z}/\ell^\nu) &
\end{array}$$

is commutative for every X in Sm/k.

Proof By universality of Ω^*, $\theta_{\hat{MU}}$ is the unique morphism of oriented cohomology theories on Sm/k from $\Omega^*(-)$ to $\hat{MU}_{\text{ét}}^{2*}(-)$. Since the composition of $\theta_{MGL} : \Omega^*(-) \to MGL^{2*,*}(-)$ with ϕ is also a morphism of oriented cohomology theories by Theorem 8.10, the triangle commutes.
The same argument holds for $\mathbb{Z}/\ell^\nu$-coefficients. $\qquad\square$

Corollary 8.13 *Let k be a separably closed. Then the maps*

$$\phi : MGL^{2*,*}(X) \otimes \mathbb{Z}_\ell \to \hat{MU}_{\text{ét}}^{2*}(X)$$

and

$$\phi : MGL^{2*,*}(X; \mathbb{Z}/\ell^\nu) \to \hat{MU}_{\text{ét}}^{2*}(X; \mathbb{Z}/\ell^\nu)$$

are surjective for every X in Sm/k.

Proof This follows immediately from the previous theorem and Theorem 8.5. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 8.14 *1. Hopkins and Morel have proved that $\Omega^*(X) \to MGL^{2*,*}(X)$ is surjective for every X in Sm/k over every field of characteristic zero in [HM].*

2. Let $MGL_{\text{ét}}$ be the $\mathbb{P}^1$-Thom-spectrum of algebraic cobordism in $\mathcal{SH}^{\mathbb{P}^1}(k)_{\text{ét}}$, i.e. instead of using the Nisnevich topology, we consider MGL and smooth schemes as sheaves for the étale topology.

The map ϕ actually factors through $MGL_{\text{ét}}^{*,*}(X)$, since $\hat{\mathrm{Et}}$ factors through the étale version of $\mathbb{A}^1$-homotopy theory, see Remark 5.13.

3. Suppose that one has constructed an Atiyah-Hirzebruch spectral sequence from étale cohomology with coefficients in $\mathbb{Z}/\ell^\nu \otimes \mathbb{L}^*$ to $MGL_{\text{ét}}(-; \mathbb{Z}/\ell^\nu)$. It is clear from the construction of ϕ and the isomorphism $H^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu \otimes \mathbb{L}^*) \xrightarrow{\cong} H^*(\hat{\mathrm{Et}}\, X; \mathbb{Z}/\ell^\nu \otimes \mathbb{L}^*)$ that this spectral sequence would imply an isomorphism

$$\phi_{\text{ét}} : MGL_{\text{ét}}^{*,*}(X; \mathbb{Z}/\ell^\nu) \xrightarrow{\cong} \hat{MU}_{\text{ét}}^*(X; \mathbb{Z}/\ell^\nu)$$

whenever both spectral sequences converge.

Such an Atiyah-Hirzebruch spectral sequence for motivic cohomology and Nisnevich MGL has been constructed by Hopkins and Morel [HM]. We will discuss an application of it in the following section.

4. We have seen that this definition of profinite étale cobordism is not good enough to reflect questions of the q-twist of the MGL-theory, see also Remark 7.31. This point should be clarified in future work.

8.3 The Galois action on étale cobordism

The comparison with $\Omega^*(-; \mathbb{Z}/\ell^\nu)$ gives us the opportunity to study the action of the Galois group of a field on $\hat{MU}_{\text{ét}}^*$.

Let k be a field of characteristic $p \neq \ell$ and let k^s be a separable closure of k. If X is a scheme over k, there is a natural action on $X_{k^s} = X \otimes_k k^s$ of the Galois group $G_k := \mathrm{Gal}(k^s/k)$ of k. The definition of $\hat{\mathrm{Et}}$ immediately implies an action of G_k on $\hat{\mathrm{Et}}\, X_{k^s}$.

Definition 8.15 *Let X be defined over a field k. We define the action of the Galois group $G_k := \mathrm{Gal}(k^s/k)$ on $\hat{MU}_{\text{ét}}^*(X_{k^s})$, resp. $\hat{MU}_{\text{ét}}^*(X_{k^s}; \mathbb{Z}/\ell^\nu)$, to be the one induced by the action of G_k on $\hat{\mathrm{Et}}\, X_{k^s}$.*

In order to study the Galois action on $\hat{MU}^*_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$, we use Proposition 8.3. We deduce the following result from [LM], §3.2. Let $H_{m,n} \hookrightarrow \mathbb{P}^n \times \mathbb{P}^m$ denote a smooth closed subscheme defined by the vanishing of a transverse section of the line bundle $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$. The smooth projective k^s-schemes $H_{m,n}$ are known as Milnor hypersurfaces.

From the definition of the map $\Omega^*(k^s) \to \hat{MU}^{2*}_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$, sending the class $[Y \to \operatorname{Spec} k^s]$ to $\pi_*(1_Y)$, we deduce the following

Proposition 8.16 *Let k^s be a separably closed field. Then $\hat{MU}^*_{\text{ét}}(k^s)$, resp. $\hat{MU}^*_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$, is generated by the classes of projective n-spaces and the classes of smooth Milnor hypersurfaces, i.e by the elements $\pi_*(1_{\mathbb{P}^n_{k^s}})$ for the projections $\pi : \mathbb{P}^n_{k^s} \to \operatorname{Spec} k^s$ and the elements $\pi_*(1_{H_{m,n}})$ with projection $\pi : H_{m,n} \to \operatorname{Spec} k^s$.*

Proof This follows from Proposition 8.3 and the fact that $\Omega^*(k^s)$ is generated by the classes $[\mathbb{P}^n_{k^s}]$ and $[H_{m,n}]$ for all m, n, see [LM], Remark 3.7. □

Hence, in order to determine the action of the absolute Galois group on the group $\hat{MU}_{\text{ét}}(k^s)$, resp. $\hat{MU}_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$, it suffices to know the action of the Galois group on the projective space $\pi : \mathbb{P}^n_{k^s} \to k^s$ and the hypersurface $H_{m,n} \to k^s$.

By [Mi] VI, 5.6, the étale cohomology of Y, for $Y = \mathbb{P}^n_{k^s}$ or $Y = H_{m,n}$, is given by $H^i_{\text{ét}}(Y; \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell(-i/2)$ if i is even, $0 \leq i \leq 2n$, resp. $0 \leq i \leq 2(m+n-1)$, and vanishes otherwise, where the twist is given by the cyclotomic character. Consider $\sigma \in G_k$. Since k^s is separably closed, this implies that the induced morphism $\sigma : Y \to Y^\sigma$ induces an isomorphism on étale cohomology with $\mathbb{Z}/\ell$-coefficients, where Y^σ is the twisted k^s-scheme with structure morphism $\sigma \circ \pi$. Hence σ induces an isomorphism in $\hat{\mathcal{H}}$:

$$\sigma : \hat{\text{Êt}}\, Y^n_{k^s} \xrightarrow{\cong} \hat{\text{Êt}}\, Y^\sigma$$

for $Y = \mathbb{P}^n_{k^s}$ or $Y = H_{m,n}$. This implies the following

Theorem 8.17 *The action of G_k on $\hat{MU}^*_{\text{ét}}(k^s; \mathbb{Z}/\ell^\nu)$ is trivial.*

Remark 8.18 *It is clear from the construction of the Atiyah-Hirzebruch spectral sequence in Theorem 4.11 that the corresponding spectral sequence for $\hat{MU}\mathbb{Z}/\ell^\nu_{\text{ét}}$ is equivariant under the Galois action. This can be seen as follows. By definition, the simplicial structure of $\hat{\text{Êt}}\, X$ is given by the connected components $\pi(U_i)$ of the étale hypercoverings $\ldots \to U_1 \to U_0 \to X$. Hence*

the action of G_k on $\hat{\mathrm{E}}\mathrm{t}\,X$ is in fact an action on every simplicial dimension. In particular, this implies that the action of G_k preserves the skeletal filtration on $\hat{\mathrm{E}}\mathrm{t}\,X_{k^s}$ and induces an action on every exact couple. Since both actions on $H^*_{\text{ét}}(X;\mathbb{Z}/\ell^\nu)$ and on $\hat{MU}^*_{\text{ét}}(X_{k^s};\mathbb{Z}/\ell^\nu)$ are induced by the one on $\hat{\mathrm{E}}\mathrm{t}\,X_{k^s}$, the assertion follows.

We apply this knowledge to the calculation of $\hat{MU}^*_{\text{ét}}(k;\mathbb{Z}/\ell^\nu)$ for a field k that may not be separably closed. We have the equality of the Galois cohomology $H^*(G_k;(\hat{MU}\mathbb{Z}/\ell^\nu)^t)$ with the étale cohomology $H^*_{\text{ét}}(k;(\hat{MU}\mathbb{Z}/\ell^\nu)^t)$. Hence the spectral sequence of Theorem 7.15 becomes

$$(32) \qquad E_2^{s,t} = H^s(G_k;(\hat{MU}\mathbb{Z}/\ell^\nu)^t) \Longrightarrow \hat{MU}^{s+t}_{\text{ét}}(k;\mathbb{Z}/\ell^\nu).$$

Let k be the finite field $k = \mathbb{F}_q$, $q = p^n$ and $p \neq \ell$. By [Se], we know the cohomology of $G_{\mathbb{F}_q} = \hat{\mathbb{Z}}$. The cohomology groups $H^s(G_{\mathbb{F}_q};A)$ vanish for all $s \geq 2$ for every finite $G_{\mathbb{F}_q}$-module A. Hence the spectral sequence (32) for $G_{\mathbb{F}_q}$ degenerates at the E_2-stage and that there is only one nonzero group for every $n = s + t$, in position $s = 0,1$ and $t \leq 0$, t even. This implies that the group $\hat{MU}^n_{\text{ét}}(k;\mathbb{Z}/\ell^\nu)$ is the unique group $E_2^{s,t}$ with $s + t = n$, i.e.

$$\hat{MU}^n_{\text{ét}}(\mathbb{F}_q;\mathbb{Z}/\ell^\nu) = \begin{cases} H^0(G_{\mathbb{F}_q};(\hat{MU}\mathbb{Z}/\ell^\nu)^n) & : \quad n \text{ even} \\ H^1(G_{\mathbb{F}_q};(\hat{MU}\mathbb{Z}/\ell^\nu)^{n-1}) & : \quad n \text{ odd.} \end{cases}$$

By [Se], we know that $H^0(G_{\mathbb{F}_q};(\hat{MU}\mathbb{Z}/\ell^\nu)^n) = ((\hat{MU}\mathbb{Z}/\ell^\nu)^n)^{G_{\mathbb{F}_q}}$ is the fix-module, and $H^1(G_{\mathbb{F}_q};(\hat{MU}\mathbb{Z}/\ell^\nu)^n) = (\hat{MU}\mathbb{Z}/\ell^\nu)^n/(1-F)(\hat{MU}\mathbb{Z}/\ell^\nu)^n$, where F is a generator of $G_{\mathbb{F}_q} = \hat{\mathbb{Z}}$. But since $G_{\mathbb{F}_q}$ acts trivially on $(\hat{MU}\mathbb{Z}/\ell^\nu)^n$ by Proposition 8.17, both cohomology groups are equal to $(\hat{MU}\mathbb{Z}/\ell^\nu)^n = \mathbb{Z}/\ell^\nu \otimes \mathbb{L}^n$. This implies the following

Theorem 8.19 *The profinite étale cobordism with $\mathbb{Z}/\ell^\nu$-coefficients of a finite field $k = \mathbb{F}_q$, $\ell \nmid q$, is given by*

$$\hat{MU}^n_{\text{ét}}(\mathbb{F}_q;\mathbb{Z}/\ell^\nu) = \begin{cases} \mathbb{Z}/\ell^\nu \otimes MU^n & : \quad n \text{ even} \\ \mathbb{Z}/\ell^\nu \otimes MU^{n-1} & : \quad n \text{ odd.} \end{cases}$$

In particular, $\hat{MU}^n_{\text{ét}}(\mathbb{F}_q;\mathbb{Z}/\ell^\nu)$ does not vanish in odd degrees.

In future work we plan to use these ideas to analyze the étale cobordism of other fields, e.g. a p-adic local field k, i.e. a finite extension of $\mathbb{Q}_p$,

$p \neq \ell$. Such a field has cohomological dimension 2. Hence the associated Atiyah-Hirzebruch spectral sequence from the Galois cohomology of k to $\hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$ is still easy to describe. It collapses at the E_2-stage since $E_2^{p,q} \neq 0$ if and only if $p = 0, 1, 2$ and q even.

If n is odd, $\hat{MU}^n_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$ is isomorphic to $H^1(G_k; (\hat{MU}\mathbb{Z}/\ell^\nu)^{n-1})$. If n is even, there are two terms occurring: $H^0(G_k; (\hat{MU}\mathbb{Z}/\ell^\nu)^n)$ and $H^2(G_k; (\hat{MU}\mathbb{Z}/\ell^\nu)^{n-2})$. By [Se] II, §5, Theorem 2, the two cohomology groups $H^2(G_k; (\hat{MU}\mathbb{Z}/\ell^\nu)^q)$ and $H^0(G_k; ((\hat{MU}\mathbb{Z}/\ell^\nu)^q)')$ are dual to each other via the cup-product pairing, where we write $((\hat{MU}\mathbb{Z}/\ell^\nu)^q)'$ for $\text{Hom}((\hat{MU}\mathbb{Z}/\ell^\nu)^q, \mathbb{G}_m)$. Since G_k acts trivially on $(\hat{MU}\mathbb{Z}/\ell^\nu)^q$, we even have $H^i(G_k; (\hat{MU}\mathbb{Z}/\ell^\nu)^q) \cong H^i(G_k; \mathbb{Z}/\ell^\nu) \otimes_{\mathbb{Z}/\ell^\nu} \hat{MU}^q$ for every i.

Although these information should suffice for a concrete statement on $\hat{MU}^*_{\text{ét}}(k; \mathbb{Z}/\ell^\nu)$, for time reasons we have not yet been able neither to formulate nor to prove it.

8.4 A conjecture

At the end of this section we would like to explain and give arguments for a conjecture on the relation of algebraic and profinite étale cobordism after inverting a Bott element.

First we recall a theorem of Levine for cohomology. Let $H^p(X; \mathbb{Z}/n(q))$ denote the motivic cohomology of a smooth scheme X over a field k. For $\text{Spec}\, k$ there is an isomorphism $H^0(\text{Spec}\, k; \mathbb{Z}/n(1)) \cong \mu_n(k)$ with the group of n-th roots of unity in k. We suppose that k contains an n-th root of unity ζ, we have a corresponding *motivic Bott element* $\beta_n \in H^0(\text{Spec}\, k; \mathbb{Z}/n(1))$.

Levine has shown in [Le2] that motivic $\mathbb{Z}/n$-cohomology of a smooth scheme over k agrees with étale $\mathbb{Z}/n$-cohomology after inverting the Bott element. More precisely, we form the bigraded ring

$$H^*(X; \mathbb{Z}/n(*))[\beta_n^{-1}] := (\oplus_{p,q} H^p(X; \mathbb{Z}/n(q)))[\beta_n^{-1}].$$

There is a cycle class map $cl^{*,*} : H^*(X; \mathbb{Z}/n(*)) \to H^*_{\text{ét}}(X; \mu_n^{\otimes *})$ with $cl^{0,1}(\beta_n) = \zeta$. Since k contains ζ, the cup product with ζ is an isomorphism on étale cohomology. Hence we get an induced map

$$(33) \qquad\qquad cl^{*,*} : H^*(X; \mathbb{Z}/n(*))[\beta_n^{-1}] \to H^*_{\text{ét}}(X; \mu_n^{\otimes *}).$$

This map is an isomorphism for every smooth scheme X over k if n is odd or if $4 | n$ or if k contains a square root of -1, cf. Theorem 1.1 of [Le2].

Furthermore, Hopkins and Morel [HM] have constructed an Atiyah-Hirzebruch spectral sequence for algebraic cobordism

$$(34) \qquad\qquad E_2^{*,*,*} = H^{*;*}(X, \mathbb{L}^*) \Longrightarrow MGL^{*,*}(X),$$

where $H^{*,*}(X; \mathbb{L}^*)$ denotes motivic cohomology with coefficients in the Lazard ring. We still suppose that k contains an n-th root of unity. We consider the spectral sequence for $\operatorname{Spec} k$. Since $E_2^{p,1,q}(\operatorname{Spec} k)$ is concentrated in degrees $p = 0$ and $p = 1$ we deduce an isomorphism $MGL^{0,1}(X) \cong k^\times$. For $\mathbb{Z}/n$-coefficients, the exact sequence for coefficients implies that we get an isomorphism

$$MGL^{0,1}(X; \mathbb{Z}/n) \cong \mu_n(k)$$

and, via the spectral sequence, the motivic Bott element defined above, more precisely $\beta_n \cdot 1_{\mathbb{L}/n}$, is sent to an induced Bott element $\beta_n \in MGL^{0,1}(X; \mathbb{Z}/n)$.

Let us now suppose that k is separably closed, $n = \ell^\nu$, and that the characteristic of k is not equal to ℓ. In particular, k contains an ℓ^ν-th root of unity ζ. It defines an element $\zeta \in H^0_{\text{ét}}(\operatorname{Spec} k; \mu_{\ell^\nu})$. The element $\zeta \cdot 1_{\mathbb{L}/\ell^\nu}$ induces via the Atiyah-Hirzebruch spectral sequence an element $\zeta_{\hat{MU}} \in \hat{MU}^0_{\text{ét}}(\operatorname{Spec} k; \mathbb{Z}/\ell^\nu)$ or equivalently a unit element in $\hat{MU}^0_{\text{ét}}(\operatorname{Spec} k; \mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu$. The multiplication with ζ yields an isomorphism of spectral sequences and hence multiplication with $\zeta_{\hat{MU}}$ is an isomorphism on étale cobordism. Furthermore, the map ϕ of (30) maps the element β_{ℓ^ν} to the element $\zeta_{\hat{MU}}$. This implies that ϕ also yields a map $\phi : MGL^{*,*}(X; \mathbb{Z}/\ell^\nu)[\beta^{-1}] \to \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$.

Conjecture 8.20 *Let X be a smooth scheme of finite type over a separably closed field k of characteristic different from ℓ. Suppose that ℓ is odd or that $\ell^\nu \geq 4$. Let $\beta \in MGL^{0,1}(k; \mathbb{Z}/\ell^\nu)$ be the Bott element defined above. We conjecture that the morphism*

$$\phi : MGL^{*,*}(X; \mathbb{Z}/\ell^\nu)[\beta^{-1}] \to \hat{MU}^*_{\text{ét}}(X; \mathbb{Z}/\ell^\nu)$$

is an isomorphism.

In fact, one should conjecture that ϕ can be defined for every field and yields an isomorphism over every field that satisfies the hypothesis. For our construction of ϕ in the previous section, we have used that k was separably closed, since we used results that we have proved only over a separably closed field. But it seems to be very likely, that ϕ does not depend on that fact. On the other hand, we have argued in the previous section, see Remarks 7.31

and 8.14, that $\hat{MU}_{\text{ét}}$ may not be the best theory over a non-separably closed field. We intend to ameliorate the theory for this purpose.

Nevertheless, the proof of the conjecture over a separably closed field would already be a big step towards the proof of a more general conjecture. Thomason [Th] and Levine [Le2] proved that algebraic K-theory, respectively motivic cohomology, satisfies étale descent after inverting the Bott element. Then they showed that the statement holds over an algebraically closed field and reduced the general case via descent to this special case.

Hence for algebraic cobordism one could try the same, i.e. one could try to prove that $MGL^{*,*}(-;\mathbb{Z}/\ell^{\nu})[\beta^{-1}]$ satisfies étale descent and then reduce to the special case of a separably closed field.

For a separably closed field k, the surjectivity of the map in the even degree part follows from the surjectivity of the map in Corollary 8.13, since β lives in bidegree $(0,1)$. Hence the surjectivity holds over a separably closed field for the even degree part.

The above discussion of Atiyah-Hirzebruch spectral sequences indicates a possible strategy to prove that ϕ becomes an isomorphism. It remains to show that ϕ fits into a morphism of spectral sequences together with the cycle class map and the inverted Bott element. Marc Levine used the same strategy to give a new proof of Thomason's result on K-theory via two Atiyah-Hirzebruch spectral sequences, cf. [Le1].

In order to translate the cycle class map to our homotopy setting, we use the fact that motivic cohomology can be represented by a motivic spectrum $H\mathbb{Z}/\ell^{\nu}_{\text{mot}}$ in $\mathcal{SH}^{\mathbb{P}^1}(k)$. The stable étale realization functor yields a morphism

$$
\begin{aligned}
H^p(X;\mathbb{Z}/\ell^{\nu}(q)) = \ & \operatorname{Hom}_{\mathcal{SH}^{\mathbb{P}^1}(k)}(\Sigma^{\infty}_{\mathbb{P}^1}(X), H\mathbb{Z}/\ell^{\nu}_{\text{mot}}[p-2q] \wedge (\mathbb{P}^1)^{\wedge q}) \\
\to\ & \operatorname{Hom}_{\hat{\mathcal{SH}}}(\hat{\Sigma}^{\infty}(X), \hat{\text{LEt}}\, H\mathbb{Z}/\ell^{\nu}(q)_{\text{mot}}).
\end{aligned}
$$

The cycle class translates into a morphism of profinite spectra in $\hat{\mathcal{SH}}$

$$
\hat{\text{LEt}}\, (H\mathbb{Z}/\ell^{\nu}_{\text{mot}}) \to H\mathbb{Z}/\ell^{\nu}(q)_{\text{ét}}.
$$

A Existence of left Bousfield localization of fibrantly generated model categories

The aim of the first part of this appendix is the proof of the localization Theorem A.3 for the special situation of a left proper fibrantly generated model category $\mathcal{M}$, e.g. $\hat{\mathcal{S}}$. The dual version of this theorem is the left localization theorem of [Hi], but our result is more general. We only assume that the given model structure is fibrantly generated whereas in [Hi] it is used that the model structure is cellular, which is stronger than cofibrantly generated. The outline and most of the lemmas used to prove the theorem are just dual version of the ones in [Hi]. We will give the proof only of those statements that are not exact dual assertions. In particular, we point out at which places we use left properness and fibrantly generated and how we avoid the dual of cellularity.

We use this theorem in the next appendix to prove that there is a stable modle structure on the category of spectra for every fibrantly generated left proper model category. This enables us to stabilize the category of profinite spaces.

We assume the reader to be familiar with the notions of model categories, as explained e.g. in [Hi] or [Ho1].

Convention A.1 *We make the convention that a model category has by definition all finite limits and finite colimits as in [Qu1]. If they need to have more than finite limits or finite colimits we will indicate this. In our applications the categories have all small limits and all results we need only use the existence of small limits.*

In this subsection we prove an analogue of Theorem 4.1.1 of [Hi]. We recall the definition of local objects and local equivalences.

Definition A.2 *Let $\mathcal{M}$ be a simplicial model category. Let K be a set of objects in $\mathcal{M}$.*

1. *A map $f : A \to B$ is called a K-local equivalence if for every element X of K the induced map of simplicial mapping spaces $f^* : \operatorname{Map}(B, X) \to \operatorname{Map}(A, X)$ is a weak equivalence of simplicial sets.*

2. *Let $\mathcal{C}$ denote the class of K-local equivalences. An object X is called K-local if it is $\mathcal{C}$-local, i.e. if X is fibrant and for every K-local equivalence*

$f : A \to B$ the induced map $f^* : \mathrm{Map}(B, X) \to \mathrm{Map}(A, X)$ is a weak equivalence.

Theorem A.3 *Let $\mathcal{M}$ be a left proper fibrantly generated simplicial model category with all small limits. Let K be a set of fibrant objects in $\mathcal{M}$ and let $\mathcal{C}$ be the class of K-local equivalences.*

1. *The left Bousfield localization of $\mathcal{M}$ with respect to $\mathcal{C}$ exists, i.e. there is a model category structure $L_{\mathcal{C}}\mathcal{M}$ on the underlying category $\mathcal{M}$ in which*

    (a) *the class of weak equivalences of $L_{\mathcal{C}}\mathcal{M}$ equals the class of K-local equivalences of $\mathcal{M}$,*

    (b) *the class of cofibrations of $L_{\mathcal{C}}\mathcal{M}$ equals the class of cofibrations of $\mathcal{M}$,*

    (c) *the class of fibrations of $L_{\mathcal{C}}\mathcal{M}$ is the class of maps with the right lifting property with respect to those maps that are both cofibrations and K-local equivalences.*

2. *The fibrant objects of $L_{\mathcal{C}}\mathcal{M}$ are the K-local objects of $\mathcal{M}$.*

3. *$L_{\mathcal{C}}\mathcal{M}$ is left proper. It is fibrantly generated if every object of $\mathcal{M}$ is cofibrant.*

4. *The simplicial structure of $\mathcal{M}$ gives $L_{\mathcal{C}}\mathcal{M}$ the structure of a simplicial model category.*

Remark A.4 *1. Although* fibrantly generated *is the dual notion of* cofibrantly generated *the proof of theorem A.3 is not the dual of Theorem 4.1.1 of [Hi] for left Bousfield localization. But we can use a dual proof of Theorem 5.1.1 of [Hi] for right Bousfield localization where the cofibrations and fibrations change their role plus an idea of [CI] in order to avoid the dual of Proposition 5.2.3 the proof of which uses the cellularization. The fact that we can construct a* left *localization is due to the left properness.*
2. The theorem does not depend on the simplicial structure of $\mathcal{M}$. In fact, it is also true for a non-simplicial $\mathcal{M}$ but we would have to use the machinery of homotopy function complexes instead of just simplicial mapping spaces.
3. Note that no result that we need below makes use of the fact that more than finite colimits exist in $\mathcal{M}$. Only in order to verify the factorization model axiom we need the existence of small limits since we use a variant of the cosmall object argument.

We split the proof into some lemmas that we collect from [Hi] and [CI]. Let P be the set of generating fibrations in $\mathcal{M}$ and let Q denote the set of generating trivial fibrations. Let K be a fixed set of fibrant objects in $\mathcal{M}$. We define the two sets of maps

$$\Lambda(K) := \{f : X^{\Delta^n} \to X^{\partial \Delta^n}, X \in K, n \geq 0\}$$

and

$$\overline{\Lambda(K)} := \Lambda(K) \cup Q.$$

Lemma A.5 *Let $\mathcal{M}$ be fibrantly generated model category and let K be a set of objects of $\mathcal{M}$. If B is a cofibrant object of $\mathcal{M}$ then a map $g : A \to B$ is $\overline{\Lambda(K)}$-projective if and only if it is both a cofibration and a K-local equivalence.*

Proof The proof is dual to the one of Proposition 5.2.4 of [Hi]. □

Lemma A.6 *Let $\mathcal{M}$ be fibrantly generated model category and let K be a set of objects of $\mathcal{M}$. Then a map that is a transfinite tower of pullbacks of maps in $\overline{\Lambda(K)}$ is a K-local fibration.*

Proof The proof is dual to the one of Proposition 5.2.5 of [Hi]. Note that it does not make use of the cellularity of $\mathcal{M}$. □

Lemma A.7 *Let $\mathcal{M}$ be a model category and K a set of objects. A map $g : X \to Y$ is both a K-local fibration and a K-local weak equivalence if and only if it is a trivial fibration.*

Proof This is a dual of Lemma 5.3.2 of [Hi] whose proof does not depend on the cellularity or right properness of $\mathcal{M}$. Hence the dual proof of Lemma 5.3.2 gives a proof for $\mathcal{M}$ left proper and fibrantly generated. □

Lemma A.8 *Let $\mathcal{M}$ be a model category and K a set of objects. Then there is a functorial factorization for every map $g : X \to Y$ in $\mathcal{M}$ as $X \xrightarrow{q} W \xrightarrow{p} Y$ in which q is a K-local cofibration and p is both a K-local weak equivalence and a K-local fibration.*

Proof This follows from Lemma A.7 and the functorial factorization into a cofibration followed by a trivial fibration. □

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Lemma A.9 *Let $\mathcal{M}$ be a left proper fibrantly generated model category and K a set of objects. If $g : X \to Y$ is a weak equivalence, $h : Y \to Z$ is a fibration and the composition $hg : X \to Z$ is a K-local fibration, then h is a K-local fibration.*

Proof This is the dual proof to Lemma 5.3.4 in [Hi], but we point out where we use left properness and how the diagram looks like.

If $i : A \to B$ is a map in $K - W \cap K - \mathrm{Cof} = W \cap \mathrm{Cof}$ (last equality holds by Lemma A.7), then one can choose by Proposition 8.1.23 of [Hi] a cofibrant approximation $\tilde{i} : \tilde{A} \to \tilde{B}$ to i such that $\tilde{i} \in \mathrm{Cof}$. By Propositions 3.1.5 and 3.2.3 of [Hi] $\tilde{i}$ is a K-local equivalence. Since $\mathcal{M}$ is left proper, Proposition 13.2.1 of [Hi] implies that it is sufficient to show that h has the right lifting property with respect to $\tilde{i}$.

Suppose we have a diagram

$$
\begin{array}{ccc}
 & & X \\
 & & \downarrow g \\
\tilde{A} & \overset{s}{\to} & Y \\
\tilde{i} \downarrow & & \downarrow h \\
\tilde{B} & \overset{t}{\to} & Z.
\end{array}
$$

Now we continue as in the proof of Lemma 5.3.4 in [Hi] by constructing lifts $j : \tilde{A} \to X$ and $k : \tilde{B} \to X$, consider the composition $u := gk : \tilde{B} \to Y$ which has the property $ui \simeq s$ in the category of objects over Z and conclude by the homotopy extension property of cofibrations in this case (Proposition 7.3.10 of [Hi]) that there is a map $v : \tilde{B} \to Y$ such that $v \simeq u$, $v\tilde{i} = s$ and hence $hv = t$. $\qquad\square$

In [Hi] the cellularity of the given model category is used for the existence of factorizations. We have to avoid this point. In order to do so, we need a dual version of Lemma 2.5 of [CI].

Lemma A.10 *Let $\mathcal{M}$ be a left proper fibrantly generated simplicial model category and let K be a set of fibrant objects. Then every map $g : A \to B$ has a factorization into a cofibration $s : A \to W$ that has the left lifting property with respect to all maps $X^{\Delta^n} \to X^{\partial\Delta^n}$, $X \in K$, $n \geq 0$ followed by a K-fibration.*

Proof This is a variation of the small object argument, dual to the proof of Lemma 2.5 of [CI]. We prove this lemma by transfinite induction of length κ

where κ is a regular cardinal such that the codomains of the generating trivial fibrations Q and the objects of K are κ-cosmall relative to the fibrations. Let J_o be the set of all squares

$$
\begin{array}{ccc}
A & \to & X^{\Delta^n} \\
g \downarrow & & \downarrow \\
B & \to & X^{\partial \Delta^n}
\end{array}
$$

for all objects $X \in K$. Define Z_0 to be the pullback $\prod_{J_0} X^{\Delta^n} \times_{\prod_{J_0} X^{\partial \Delta^n}} B$ and let $j_0 : A \to Z_0$, $q_0 : Z_0 \to B$ be the obvious maps. Now we factor j_0 into a cofibration $i_0 : A \to W_0$ followed by a trivial fibration $p_0 : W_0 \to Z_0$. This is the first step of the induction.

If β is a limit ordinal, set $W_\beta := \lim_{\alpha < \beta} W_\alpha$ and set $i_\beta := \operatorname{colim}_{\alpha < \beta} i_\alpha$.

If β is a successor ordinal, define J_β to be the set of all squares

$$
\begin{array}{ccc}
A & \to & X^{\Delta^n} \\
g \downarrow & & \downarrow \\
W_{\beta-1} & \to & X^{\partial \Delta^n}
\end{array}
$$

for all $X \in K$. Define Z_β to be the pullback $\prod_{J_\beta} X^{\Delta^n} \times_{\prod_{J_\beta} X^{\partial \Delta^n}} W_{\beta-1}$ and let $j_\beta : A \to Z_\beta$, $q_\beta : Z_\beta \to Z_\beta$ be the obvious maps. Now we factor j_β into a cofibration $i_\beta : A \to W_\beta$ followed by a trivial fibration $p_\beta : W_\beta \to Z_\beta$.

By transfinite induction we get a κ-tower

$$
\ldots \to W_\beta \to \ldots \to W_1 \to W_0 \to B.
$$

Note that there are compatible maps $i_\beta : A \to W_\beta$. Since filtered limits exist in $\mathcal{M}$ we may continue with the following definition. Let W be $\lim_\beta W_\beta$ and let $i : A \to W$ be $\lim_\beta i_\beta$. By construction, each i_β is a cofibration for every successor ordinal β. Since $\mathcal{M}$ is fibrantly generated and κ is such that the codomains of Q and the objects of K are κ-cosmall relative to the fibrations we get that i is a cofibration. Furthermore one can show that i has the left lifting property with respect to all maps $X^{\Delta^n} \to X^{\partial \Delta^n}$, $X \in K$, $n \geq 0$ by the cosmall object argument as follows:

Suppose we have a diagram

$$
\begin{array}{ccc}
A & \xrightarrow{k} & X^{\Delta^n} \\
i \downarrow & & \downarrow f \\
W & \xrightarrow{h} & X^{\partial \Delta^n}
\end{array}
$$

for some $X \in K$ and $n \geq 0$. By the theory of simplicial model categories if X is a fibrant object then the map $f : X^{\Delta^n} \to X^{\partial \Delta^n}$ is a fibration. Since the

objects in K are κ-cosmall relative to the fibrations, there is a $\beta < \kappa$ such that h is the composite $W \to W_\beta \overset{h_\beta}{\to} X^{\partial \Delta^n}$. By construction, there is a map $k_\beta : W_{\beta+1} \to X^{\Delta^n}$ such that $fk_\beta = h_\beta j$ and $k = k_\beta i_{\beta+1}$ where j is the map $W_{\beta+1} \to W_\beta$. The composition $W \to W_{\beta+1} \overset{k_\beta}{\to} X^{\Delta^n}$ is the required lift in the diagram.

It remains to show that $p : W \to Y$ is a K-fibration. Since K-fib is defined by a right lifting property and p is a transfinite tower of maps $W_{\beta+1} \to W_\beta$ it suffices to show that each $W_{\beta+1} \to W_\beta$ is a K-fibration. The map $W_{\beta+1} \to W_\beta$ is the composition $W_{\beta+1} \to Z_{\beta+1} \to W_\beta$ where the first map is a trivial fibration and hence also a map in $K - W \cap K - \text{fib}$. The second map is a product of maps of the form $X^{\Delta^n} \to X^{\partial \Delta^n}$ each of which is a K-fibration by Lemma A.5. Hence $Z_{\beta+1} \to W_\beta$ is a K-fibration. $\qquad \square$

Lemma A.11 *Let $\mathcal{M}$ be a left proper fibrantly generated model category and let K be a set of objects. Then there is functorial factorization for every map $g : X \to Y$ in $\mathcal{M}$ as $X \overset{q}{\to} W \overset{p}{\to} Y$ in which q is both a K-local weak equivalence and a K-local cofibration and p is a K-local fibration.*

Proof Again we dualize the proof of Proposition 5.3.5 of [Hi] point out how the diagrams must look like and where we use the left properness. Let $j : \tilde{X} \to X$ be a fibrant cofibrant approximation to X. Lemma A.10 implies that there is a functorial factorization of the composition gj as $\tilde{X} \overset{s}{\to} \tilde{W} \overset{r}{\to} X$ such that s is a cofibration and in $\overline{\Lambda(K)}$-projective and r is a K-local fibration. Let $Z := X \amalg_{\tilde{X}} \tilde{W}$. We factor the induced map $Z \to Y$ into $Z \overset{u}{\to} W \overset{p}{\to} Y$ where u is a trivial cofibration and p is a fibration. This yields the following diagram

$$
\begin{array}{ccc}
\tilde{X} & \overset{j}{\longrightarrow} & X \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle v} \\
\tilde{W} & \underset{t}{\longrightarrow} & Z \overset{u}{\to} W
\end{array}
$$

Let $q := uv$. Now since j is a weak equivalence and s is a cofibration the left properness of $\mathcal{M}$ implies that t is also a weak equivalence. Hence ut is a weak equivalence and s is a cofibrant approximation to q. Lemma A.5 implies that s is a K-local equivalence.

By Lemma A.5, since s is in $\overline{\Lambda(K)}$-proj it is in particular a K-local equivalence. Hence $q \in K - W$ since $j \in K - W$ and $qj = ts \in K - W$ (two

out of three holds for $K - W$). Now since $q = uv$ is the composition of two cofibrations and the K-local cofibrations equal the cofibrations in $\mathcal{M}$, we have $q \in K - W \cap K - Cof$. By Lemma A.6 this q is a weak equivalence and a cofibration.

Since u, t and hence also ut are weak equivalences and $r = put$ is by construction a K-local fibration, we conclude by Lemma A.9 that p is a K-local fibration. $\qquad\square$

Proof of Theorem A.3 1. Axiom M1 is satisfied since it is satisfied in $\mathcal{M}$. Axiom M2 follows from Proposition 3.2.3 of [Hi]. Axiom M3 follows from Proposition 3.2.4 and Lemma 7.2.8 of [Hi]. The right lifting property of K-local fibrations is given by definition. The left lifting property of K-local cofibrations follows from Lemma A.7. The factorization of a map into a K-local cofibration followed by a trivial K-local fibration follows from Lemma A.8 and the factorization into a trivial K-local cofibration followed by a K-local fibration follows from Lemma A.11.

2. This follows from Proposition 3.4.1 of [Hi].

3. The left properness follows from Proposition 3.4.4 of [Hi]. If every object in $\mathcal{M}$ is cofibrant, then Lemma A.10 and lemma A.6 imply that $L_{\mathcal{C}}\mathcal{M}$ is fibrantly generated with generating fibrations $\overline{\Lambda(K)}$ and generating trivial fibrations Q.

4. This is the analogue proof as for part 4 of Theorem 5.1.1 in [Hi]. $\qquad\square$

B Stable model structure on spectra

The well known stabilization of the category of simplicial spectra of [BF] uses the fact that $\mathcal{S}$ is proper in an essential way in order to construct functorial factorizations. Since the category $\hat{\mathcal{S}}$ is only left but not right proper, we may not use their localization method.

Hovey [Ho2] has pointed out a general way to stabilize a left proper cellular model category with respect to a left Quillen endofunctor T. Since we do have left properness but not the property of being cofibrantly generated we have to modify this construction.

The aim is to construct a stable structure on $\mathcal{C}$ with respect to an endofunctor via the well known construction of spectra on $\mathcal{C}$. We reformulate the results of [Ho2] for the situation of a left proper fibrantly generated model category $\mathcal{C}$, e.g. $\hat{\mathcal{S}}$. The projective and stable model structure is the same as in [Ho2], but we have to give different proofs for the key points of the construction. We also have to choose different generating fibrations. But the idea for the

construction and the proof that the structure in fact becomes stable are due to [Ho2].

In the first part, we construct a stable structure on the usual classical notion of spectra in analogy to the one of [BF].

In the second part, we show, following [Ho2], that this works also for symmetric spectra on a symmetric monoidal model structure that is left proper and fibrantly generated. Unfortunately, we have not yet been able to show that the resulting stable structure is again monoidal. Since this is the point why symmetric spectra are better than usual spectra, we have not used symmetric spectra for our stable structure on $\hat{\mathcal{S}}$. But there should be a monoidal structure on it.

B.1 Spectra

In this section we assume always that $\mathcal{C}$ is a left proper fibrantly generated simplicial model category with all small limits. Let $T : \mathcal{C} \to \mathcal{C}$ be a left Quillen endofunctor on $\mathcal{C}$. Let $U : \mathcal{C} \to \mathcal{C}$ be its right adjoint.

Definition B.1 *A* spectrum *X is a sequence $(X_n)_{n\geq 0}$ of objects of $\mathcal{C}$ together with structure maps $\sigma : TX_n \to X_{n+1}$ for all $n \geq 0$. A* map of spectra *$f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ commuting with the structure maps. We denote the category of spectra by $\mathrm{Sp}(\mathcal{C}, T)$.*

We begin by defining an intermediate strict structure, see [BF] or [Ho2]. But note that we are in a dual situation of a fibrantly generated model structure. Consider the following functors:

Definition B.2 *Given $n \geq 0$, the* evaluation functor *$\mathrm{Ev}_n : \mathrm{Sp}(\mathcal{C}, T) \to \mathcal{C}$ takes X to X_n. It has a left adjoint $F_n : \mathcal{C} \to \mathrm{Sp}(\mathcal{C}, T)$ defined by $(F_n A)_m = T^{m-n} A$ if $m \geq n$ and $(F_n A)_m = *$ otherwise. The structure maps are the obvious ones.*
The evaluation functor also has a right adjoint $R_n : \mathcal{C} \to \mathrm{Sp}(\mathcal{C}, T)$ defined by $(R_n A)_i = U^{n-i} A$ if $i \leq n$ and $(R_n A)_i = *$ otherwise. The structure map $T U^{n-i} A \to U^{n-i} A$ is adjoint to the identity map of U^{n-i} when $i < n$.

Note that Ev_n commutes with limits and colimits, where limits and colimits are constructed levelwise in $\mathrm{Sp}(\mathcal{C}, T)$. If $G : I \to \mathrm{Sp}(\mathcal{C}, T)$ is a functor from

a small category to $\mathrm{Sp}(\mathcal{C}, T)$, then, for limits, the structure maps are defined as the composites

$$T(\lim \mathrm{Ev}_n \circ G) \to \lim(T \circ \mathrm{Ev}_n \circ G) \stackrel{\lim(\sigma \circ G)}{\to} \lim \mathrm{Ev}_{n+1} \circ G.$$

For colimits one uses the fact that T is left adjoint and hence preserves colimits to get the obvious structure maps.

Definition B.3 *A map f in $\mathrm{Sp}(\mathcal{C}, T)$ is a* projective weak equivalence *(resp.* projective fibration*) if each map f_n is a weak equivalence (resp. fibration). A map i is a* projective cofibration *if it has the left lifting property with respect to all projective trivial fibrations.*

We will show that the projective structure is in fact a fibrantly generated model structure. We denote the set of generating fibrations of $\mathcal{C}$ by P and the set of generating trivial fibrations by Q. Inspired by Proposition B.5, we set

$$\tilde{P} := \bigcup_n R_n P \cup \{g_n : R_n R \to R_n S \times_{R_{n-1} U S} R_{n-1} U R, \text{ for all } f : R \to S \text{ in } P\}$$

and

$$\tilde{Q} := \bigcup_n R_n Q \cup \{g_n : R_n R \to R_n S \times_{R_{n-1} U S} R_{n-1} U R, \text{ for all } f : R \to S \text{ in } Q\}$$

where R_n is the right adjoint of the evaluation functor Ev_n and g_n is the map induced by the commutative diagram

$$
\text{(35)} \qquad
\begin{array}{ccc}
R_n R & \to & R_n S \\
\downarrow & & \downarrow \\
R_{n-1} U R & \to & R_{n-1} U S.
\end{array}
$$

We will show that the maps in $\tilde{P}$ are the generating fibrations and the maps in $\tilde{Q}$ are the generating trivial fibrations for the projective model structure on $\mathrm{Sp}(\mathcal{C}, T)$.

Lemma B.4 *If $f : X \to Y$ is a fibration (resp. trivial fibration) in $\mathcal{C}$, then the maps $R_n f$ and $g_n : R_n X \to R_n Y \times_{R_{n-1} U Y} R_{n-1} U X$ are fibrations (resp. trivial fibrations) in $\mathrm{Sp}(\mathcal{C}, T)$.*

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Proof Since R_n is defined via the right Quillen functor U which preserves fibrations and trivial fibrations, it is clear that the map $(R_n f)_i = U^{n-i}(f)$ on the i-th level is a fibration (resp. trivial fibration). In diagram (35), the maps on the i-th level are either identities or equal to $U^{n-i}(f)$. Hence the induced map g_n is also a fibration (resp. trivial fibration). $\qquad\square$

The definition of $\tilde{P}$ and $\tilde{Q}$ becomes clear after the following

Proposition B.5 *A map* $i : A \to B$ *in* $\mathrm{Sp}(\mathcal{C}, T)$ *is a projective (trivial) cofibration if and only if* $i_0 : A_0 \to B_0$ *and the induced maps* $j_n : A_n \amalg_{TA_{n-1}} TB_{n-1}$ *for* $n \geq 1$ *are (trivial) cofibrations in* $\mathcal{C}$.

Proof We prove the cofibration case, the case of trivial cofibrations is similar. We use the standard idea of [Ho2], Proposition 1.14. Let $i : A \to B$ be a map with the left lifting property with respect to projective trivial fibrations. Since projective fibrations are defined levelwise, it is clear that $i_0 : A_0 \to B_0$ is a cofibration. Let $f : X \to Y$ be a trivial fibration in $\mathcal{C}$. We have to show that there is a lift $B_n \to X$ for any commutative diagram

$$
\begin{array}{ccc}
A_n \amalg_{TA_{n-1}} TB_{n-1} & \to & X \\
j_n \downarrow & & \downarrow f \\
B_n & \longrightarrow & Y.
\end{array}
$$

By adjunction, this diagram has a lift if and only if the induced diagram

$$
\begin{array}{ccc}
A & \longrightarrow & R_n X \\
i \downarrow & & \downarrow g_n \\
B & \to & R_n Y \times_{R_{n-1}UY} R_{n-1}UX
\end{array}
$$

has a lift. Now R_n is defined via the right Quillen functor U, see Definition B.2. This implies that all maps in Diagram (35) are projective trivial fibrations and hence the induced map is one, too. So g_n is a trivial fibration if f is one, see Lemma B.4. Hence there is a lift in the last diagram.
Now suppose that i_0 and j_n are cofibrations in $\mathcal{C}$ for all $n > 0$. Let

$$
\begin{array}{ccc}
A & \overset{a}{\to} & X \\
i \downarrow & & \downarrow f \\
B & \overset{b}{\to} & Y
\end{array}
$$

be a commutative diagram in $\mathrm{Sp}(\mathcal{C}, T)$ and let f be a trivial fibration. We construct a lift $h_n : B_n \to X_n$, compatible with the structure maps, by induction on n. There is a lift h_0 since i_0 is a cofibration and f_0 is a trivial

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fibration in $\mathcal{C}$. Suppose h_j is constructed for all $j < n$. Then we can consider the diagram

$$
\begin{array}{ccc}
A_n \amalg_{TA_{n-1}} TB_{n-1} & \xrightarrow{(f_n,\sigma \circ Th_{n-1})} & X \\
j_n \downarrow & & \downarrow f_n \\
B_n & \xrightarrow{b_n} & Y_n.
\end{array}
$$

The lift $h_n : B_n \to X_n$ of this diagram is the desired map. $\qquad\square$

Theorem B.6 *Let $\mathcal{C}$ be a left proper fibrantly generated simplicial model category with all small limits. The projective weak equivalences, projective fibrations and projective cofibrations define a left proper fibrantly generated simplicial model structure on $\mathrm{Sp}(\mathcal{C},T)$ with set of generating fibrations $\tilde{P}$ and set of generating trivial fibrations $\tilde{Q}$.*

We will prove this theorem in a series of lemmas. The existence of finite colimits and limits is clear since they are defined levelwise. The two-out-of-three- and the retract-axiom are immediate as well. The crucial point is the factorization axiom for which we use the cosmall object argument. For the notations $\tilde{P} - \mathrm{proj}$, $\tilde{P} - \mathrm{fib}$, $\tilde{Q} - \mathrm{proj}$, $\tilde{Q} - \mathrm{fib}$ and the cosmall object argument, which is the exact dual of the small object argument, we refer the reader to the book of Hovey [Ho1]: in $\tilde{P} - \mathrm{proj}$ are all the maps having the left lifting property with respect to all maps in $\tilde{P}$; in $\tilde{P} - \mathrm{fib}$ are all the maps having the right lifting property with respect to maps in $\tilde{P} - \mathrm{proj}$.

Proposition B.7 *The projective cofibrations are exactly the maps which have the left lifting property with respect to all maps in $\tilde{Q}$. The projective trivial cofibrations are exactly the maps which have the left lifting property with respect to all maps in $\tilde{P}$.*

Proof By Proposition B.5, $i : A \to B$ is a projective cofibration if and only if i_0 and j_n are cofibrations in $\mathcal{C}$ for all $n > 0$. But since $\mathcal{C}$ is a fibrantly generated model category with Q the set of generating trivial fibrations, i_0 and j_n are cofibrations if and only if they have the left lifting property with respect to all maps in Q, i.e. if and only if i_0 and j_n are in $Q - \mathrm{proj}$. We have seen that a lift in a diagram

$$
\begin{array}{ccc}
A_n \amalg_{TA_{n-1}} TB_{n-1} & \rightarrow & R \\
j_n \downarrow & & \downarrow f \\
B_n & \longrightarrow & S
\end{array}
$$

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is by adjunction equivalent to a lift in the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & R_n R \\
i \downarrow & & \downarrow g_n \\
B & \rightarrow & R_n S \times_{R_{n-1}US} R_{n-1}UR.
\end{array}
$$

Hence i_0 and j_n are in $Q - \mathrm{proj}$ if and only if the map i is an element of $\tilde{Q} - \mathrm{proj}$. This completes the proof in the cofibration case. The trivial cofibration case is similar. $\square$

Corollary B.8 *The projective fibrations are exactly the maps in $\tilde{P} - \mathrm{fib}$. The projective trivial cofibrations are exactly the maps in $\tilde{Q} - \mathrm{fib}$.*

Lemma B.9 *If Z is a cosmall object in $\mathcal{C}$ relative to the fibrations, then $R_n Z$, $n \geq 0$, is cosmall relative to the level fibrations in $\mathrm{Sp}(\mathcal{C}, T)$. Similarly, if Z is a cosmall object in $\mathcal{C}$ relative to the trivial fibrations, then $R_n Z$, $n \geq 0$, is cosmall relative to the level trivial fibrations in $\mathrm{Sp}(\mathcal{C}, T)$. In addition, the codomains of the maps in $\tilde{P}$ (resp. $\tilde{Q}$) are cosmall relative to the level fibrations (resp. trivial fibrations) in $\mathrm{Sp}(\mathcal{C}, T)$.*

Proof This is clear since Ev_n commutes with all limits. $\square$

We can now finish the proof of Theorem B.6. It remains to show the factorization axiom. Note that by hypothesis, $\mathcal{C}$ and hence $\mathrm{Sp}(\mathcal{C}, T)$ have all small limits. This enables us to apply the cosmall object argument. By Lemma B.9 and since we have shown in Corollary B.8 that the class of projective fibrations (resp. trivial fibrations) is equal to $\tilde{P} - \mathrm{fib}$ (resp. $\tilde{Q} - \mathrm{fib}$), we can apply the cosmall object argument to the class $\tilde{P}$ (resp. $\tilde{Q}$). In the first case, this yields a factorization into a map in $\tilde{P} - \mathrm{proj}$ followed by a map in $\tilde{P} - \mathrm{cocell}$, where $\tilde{P} - \mathrm{cocell}$ consists of maps which are transfinite compositions of pullbacks of maps in $\tilde{P}$. But we have shown that $\tilde{P} - \mathrm{proj}$ is equal to the class of projective trivial cofibrations and $\tilde{P} - \mathrm{cocell} \subseteq \tilde{P} - \mathrm{fib}$, [Ho1] dual to Lemma 2.1.10. Since we have shown that $\tilde{P} - \mathrm{fib}$ is equal to the class of projective fibrations, this is the first desired factorization.

The cosmall object argument applied to $\tilde{Q}$ yields a factorization into a map in $\tilde{Q} - \mathrm{proj}$ followed by a map in $\tilde{Q} - \mathrm{cocell}$. Since we know that $\tilde{Q} - \mathrm{proj}$ is equal to the class of projective cofibrations and $\tilde{Q} - \mathrm{cocell} \subseteq \tilde{Q} - \mathrm{fib}$ and since $\tilde{Q} - \mathrm{fib}$ is equal to the class of projective trivial fibrations, we have the second factorization.

This proves that the projective structure is in fact a model structure on

$\mathrm{Sp}(\mathcal{C}, T)$. But we have also shown that this structure satisfies the axioms of a fibrantly generated model category, see [Ho1], Definition 2.1.17.

The left properness is clear since coproducts are defined levelwise in $\mathrm{Sp}(\mathcal{C}, T)$. This completes the proof of Theorem B.6. □

It remains to modify this structure in order to get a stable structure, i.e. one in which the prolongation of T is a Quillen equivalence. We will do this by applying the localization Theorem A.3 to the projective model structure on spectra. We want the stable weak equivalences to be the maps that induce isomorphisms on all generalized cohomology theories. A generalized cohomology theory is represented by the analogue of an Ω-spectrum. Since the K-local objects are the fibrant objects in the localized model structure, we have to choose the set K to consist exactly of these analogues of Ω-spectra.

Definition B.10 *A spectrum $E \in \mathrm{Sp}(\mathcal{C}, T)$ is defined to be an Ω-spectrum if each E_n is fibrant and the adjoint structure maps $E_n \to UE_{n+1}$ are weak equivalences for all $n \geq 0$.*

By Corollary 9.7.5 in [Hi], since each E_n is fibrant and since the right Quillen functor U preserves fibrations, we have that $E_n \to UE_{n+1}$ is a weak equivalence in $\mathcal{C}$ if and only if the induced map $\mathrm{Map}(A, E_n) \to \mathrm{Map}(A, UE_{n+1})$ is a weak equivalence in S for every cofibrant object A. By adjunction this is equivalent to $\mathrm{Map}(F_nA, E) \to \mathrm{Map}(F_{n+1}TA, E)$ being a weak equivalence in S for every cofibrant object A. Note that by [Hi], Corollary 9.7.5, this is what the weak equivalences have to be in a model structure.

So in order to get the Ω-spectra as stable fibrant objects and the maps inducing isomorphisms on cohomology theories as stable equivalences we have to choose the maps $F_{n+1}TA \to F_nA$ adjoint to the identity map of TA to be the K-local equivalences. Hence we define the set K to consist of all Ω-spectra.

Using the fact that the projective model structure on $\mathrm{Sp}(\mathcal{C}, T)$ is fibrantly generated, left proper and simplicial, we may deduce the following

Theorem B.11 *Let $\mathcal{C}$ be a left proper fibrantly generated simplicial model category with all small limits. Let K be the set of Ω-spectra. There is a stable model structure on $\mathrm{Sp}(\mathcal{C}, T)$ which is defined to be the K-localized model structure $L_S\mathrm{Sp}(\mathcal{C}, T)$ of Theorem A.3 where S is the class of all K-local equivalences.*

Remark B.12 *In particular, the* stable equivalences *are the K-local equivalences; the* stable cofibrations *are the projective cofibrations; the* stable fi-

brations *are the maps that have the right lifting property with respect to all stable trivial cofibrations; the stable fibrant objects are the Ω-spectra.*

Define the *prolongation of T* to be the functor $T : \mathrm{Sp}(\mathcal{C}, T) \to \mathrm{Sp}(\mathcal{C}, T)$, defined by $(TX)_n = TX_n$ with structure maps $T(TX_n) \overset{T\sigma}{\to} TX_{n+1}$ where σ is the structure map of X. The adjoint Ω is defined in the analog way. Just as in [Ho2] one can prove that T is a Quillen equivalence on the stable structure.

Theorem B.13 *Let $\mathcal{C}$ be a left proper fibrantly generated simplicial model category with a left Quillen endofunctor T. Then the prolongation $T : \mathrm{Sp}(\mathcal{C}, T) \to \mathrm{Sp}(\mathcal{C}, T)$ is a Quillen equivalence with respect to the stable structure.*

Proof This is the proof of Theorem 3.9 in [Ho2]. $\square$

B.2 Symmetric spectra

We can transfer these results also to symmetric spectra on a left proper fibrantly generated simplicial model category $\mathcal{C}$ with a symmetric monoidal structure. The ideas and definitions are due to [Ho2]. We extend his results to our unusual and slightly more general situation. For the definition of these notions, we refer the reader to [Ho1].

For the whole section, let $\mathcal{C}$ be a left proper fibrantly generated simplicial closed symmetric monoidal model category with all small limits. For simplicity, we suppose that its unit object S is cofibrant. Since any endofunctor $T : \mathcal{C} \to \mathcal{C}$ is of the form $T(L) = L \otimes K$ for $K = T(S)$, we will consider in this section the functor $- \otimes K$ instead of $T-$. The right adjoint of T is then given by $L \to L^K$. All the definitions and ideas for the construction of symmetric spectra are taken from [Ho2].

Definition B.14 *A* symmetric sequence *in $\mathcal{C}$ is a sequence $X_0, X_1, \ldots, X_n, \ldots$ of objects in $\mathcal{C}$ with an action of Σ_n on X_n, where Σ_n is the symmetric group of n letters. A map of symmetric sequences is a sequence of Σ_n-equivariant maps $X_n \to Y_n$. We denote the resulting category by $\mathcal{C}^\Sigma$.*
We define the object $\mathrm{Sym}(K)$ in $\mathcal{C}^\Sigma$ to be the symmetric sequence $(S, K, K \otimes K, \ldots, K^{\otimes n}, \ldots)$ where Σ_n acts on $K^{\otimes n}$ by permutation, using the commutativity and associativity isomorphisms.

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Definition B.15 *A symmetric spectrum X is a sequence of Σ_n-objects X_n in $\mathcal{C}$ with Σ_n-equivariant structure maps $X_n \otimes K \to X_{n+1}$, such that the composite*

$$X_n \otimes K^{\otimes p} \to X_{n+1} \otimes K^{\otimes p-1} \to \ldots \to X_{n+p}$$

is $\Sigma_n \times \Sigma_p$-equivariant for all $n, p \geq 0$. A map of symmetric spectra is a collection of Σ_n-equivariant maps $X_n \to Y_n$ compatible with the structure maps. We denote the category of symmetric spectra on $\mathcal{C}$ by $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$.
We define the symmetric spectrum $\overline{\mathrm{Sym}(K)}$ in $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ to be the initial object 0 in degree 0 and $K^{\otimes n}$ in degree n for $n > 0$ with the obvious structure maps.

One can show as in [HSS] that $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is again a closed symmetric monoidal category with $\mathrm{Sym}(K)$ as the unit. We denote the monoidal structure by $X \wedge Y = X \otimes_{\mathrm{Sym}(K)} Y$, and the closed structure by $\mathrm{Hom}_{\mathrm{Sym}(K)}(X, Y)$, where we consider X and Y as $\mathrm{Sym}(K)$-modules.

Definition B.16 *1. Given $n \geq 0$, the evaluation functor $\mathrm{Ev}_n : \mathrm{Sp}^\Sigma(\mathcal{C}, K) \to \mathcal{C}$ takes X to X_n.*
2. The evaluation functor has a left adjoint $F_n : \mathcal{C} \to \mathrm{Sp}^\Sigma(\mathcal{C}, K)$, defined by $F_n A = \tilde{F}_n A \otimes \mathrm{Sym}(K)$, where $\tilde{F}_n A$ is the symmetric sequence $(0, \ldots, 0, \Sigma_n \times A, 0, \ldots)$.
3. The evaluation functor has a right adjoint $R_n : \mathcal{C} \to \mathrm{Sp}^\Sigma(\mathcal{C}, K)$, defined by $R_n A = \mathrm{Hom}(\mathrm{Sym}(K), \tilde{R}_n A)$, where $\tilde{R}_n A$ is the symmetric sequence that is the terminal object in dimensions other than n and is the cofree Σ_n-object $\mathrm{Hom}_{\mathcal{C}}(\Sigma_n, A)$, i.e. $n!$-product of A in $\mathcal{C}$ together with the Σ_n-action defined by $(\rho f)(\rho') = f(\rho' \rho)$ for $\rho \in \Sigma_n$ and $f \in \mathrm{Hom}_{\mathcal{C}}(\Sigma_n, A)$.

Definition B.17 *A map f in $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is a projective weak equivalence (resp. projective fibration) if each map f_n is a weak equivalence (resp. fibration). A map i is a projective cofibration if it has the left lifting property with respect to all projective trivial fibrations.*

Again we will show that the projective structure is in fact a fibrantly generated model structure. We recall that the set of generating fibrations of $\mathcal{C}$ is denoted by P and the set of generating trivial fibrations by Q. Inspired by the forthcoming proposition we set

$$\tilde{P} := \bigcup_n R_n P \cup \{q_n : R_n X \to R_n Y \times_{R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)}, Y))} R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)}, X)),$$

$$\text{for all } p : X \to Y \text{ in } P\}$$

and

$$\tilde{Q} \; := \; \bigcup_n R_n Q \cup \{q_n : R_n X \to R_n Y \times_{R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)},Y))} R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)},X)),$$

$$\text{for all } p : X \to Y \text{ in } Q\}$$

where R_n is the right adjoint of the evaluation functor Ev_n and q_n is the induced map in the commutative diagram

$$(36) \qquad \begin{array}{ccc} R_n X & \to & R_n Y \\ \downarrow & & \downarrow \\ R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)},X)) & \to & R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)},Y)). \end{array}$$

We will show that the maps in $\tilde{P}$ are the generating fibrations and the maps in $\tilde{Q}$ are the generating trivial fibrations for the projective model structure on $\mathrm{Sp}(\mathcal{C},T)$. First we describe the projective cofibrations in $\mathrm{Sp}^\Sigma(\mathcal{C},K)$. This is slightly more complicated than in the case of usual spectra. The reason for this comes from the fact that we have to ensure that all maps are equivariant with respect to the actions of the groups Σ_n. The following definition and the idea how to solve this problem is taken from [Ho2], Definition 8.4.

The following proposition is well-known, see [Ho2], Proposition 8.5, or [HSS], Proposition 5.2.2. But we will give a different proof similar to the one of Proposition B.5, which enables us to understand the definition of the generating fibrations and trivial fibrations in the projective model structure on $\mathrm{Sp}^\Sigma(\mathcal{C},K)$. It avoids the assumption that we had already shown that the projective structure is in fact a model structure. Furthermore, we do not have to assume that $\mathcal{C}$ is cofibrantly generated, which is not the case in our situation.

Definition B.18 *We define the n-th latching space of $A \in \mathrm{Sp}^\Sigma(\mathcal{C},K)$ by $L_n A := \mathrm{Ev}_n(A \wedge \overline{\mathrm{Sym}(K)})$. The obvious map $\overline{\mathrm{Sym}(K)} \to \mathrm{Sym}(K)$ induces a map of spectra $i : A \wedge \overline{\mathrm{Sym}(K)} \to A \wedge \mathrm{Sym}(K)$ and a map $\mathrm{Ev}_n(i) : L_n A \to A_n$.*

Note that the latching space is a Σ_n-object in $\mathcal{C}$. There is a model structure on the category of Σ_n-objects of $\mathcal{C}$, where the fibrations and weak equivalences are the underlying ones.
We recall that for given maps $f : A \to B$ and $g : C \to D$ in $\mathcal{C}$ one defines the pushout product $f \square g$ of f and g to be the map

$$f \square g : (A \otimes D) \amalg_{A \otimes C} (B \otimes C) \to B \otimes D$$

induced by the obvious commutative diagram.

Proposition B.19 *A map $f : A \to B$ in $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is a projective cofibration if and only if the induced map $\mathrm{Ev}_n(f\Box i) : A_n \amalg_{L_n A} L_n B \to B_n$ is a Σ_n-cofibration in $\mathcal{C}$ for all n. Similarly, f is a projective trivial cofibration if and only if $\mathrm{Ev}_n(f\Box i)$ is a trivial Σ_n-cofibration in $\mathcal{C}$ for all n.*

Proof As usual we prove only the cofibration case, since the case of trivial cofibrations is similar. Suppose first that each map $\mathrm{Ev}_n(f\Box i)$ is a Σ_n-cofibration in $\mathcal{C}$. The following argument is exactly the one in the proof of Proposition 8.5 in [Ho2].

Now suppose that f is a projective cofibration of symmetric spectra. We have to show that each map $\mathrm{Ev}_n(f\Box i)$ is a Σ_n-cofibration in $\mathcal{C}$, i.e. each map $\mathrm{Ev}_n(f\Box i)$ has the left lifting property with respect to every trivial Σ_n-fibration $p : X \to Y$ in $\mathcal{C}$. We recall that this just means that p is an underlying trivial fibration in $\mathcal{C}$. Suppose we have a commutative diagram

$$
\begin{array}{ccc}
A_n \amalg_{L_n A} L_n B & \to & X \\
\mathrm{Ev}_n(f\Box i) \downarrow & & \downarrow p \\
B_n & \longrightarrow & Y.
\end{array}
$$

By adjunction, this diagram has a lift if and only if the induced diagram

$$
\begin{array}{ccc}
A & \longrightarrow & R_n X \\
f \downarrow & & \downarrow q_n \\
B & \to & R_n Y \times_{R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)},Y))} R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)}, X)).
\end{array}
$$

Since all the constructions R_n, $\mathrm{Hom}(\overline{\mathrm{Sym}(K)}, -)$ and pullbacks preserve trivial fibrations, the map q_n is a projective trivial fibration in $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$. Hence there is a lift in the last diagram, since we have supposed that f is a projective cofibration. $\qquad \square$

Proposition B.20 *The projective cofibrations are exactly the maps which have the left lifting property with respect to all maps in $\check{Q}$. The projective trivial cofibrations are exactly the maps which have the left lifting property with respect to all maps in $\tilde{P}$.*

Proof By Proposition B.19 $f : A \to B$ is a projective cofibration if and only if $\mathrm{Ev}_n(f\Box i)$ are Σ_n-cofibrations in $\mathcal{C}$ for all n. Since $\mathcal{C}$ is a fibrantly generated model category with Q the set of generating trivial fibrations, $\mathrm{Ev}_n(f\Box i)$ are Σ_n-cofibrations if and only if they have the left lifting property with respect

to all maps in Q, i.e. if and only if $\mathrm{Ev}_n(f \square i)$ are in $Q - \mathrm{proj}$. We have seen that a lift in a diagram

$$
\begin{array}{ccc}
A_n \amalg_{L_n A} L_n B & \to & X \\
\mathrm{Ev}_n(f \square i) \downarrow & & \downarrow p \\
B_n & \longrightarrow & Y
\end{array}
$$

is by adjunction equivalent to a lift in the diagram

$$
\begin{array}{ccc}
A & \longrightarrow & R_n X \\
f \downarrow & & \downarrow q_n \\
B & \to & R_n Y \times_{R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)}, Y))} R_n(\mathrm{Hom}(\overline{\mathrm{Sym}(K)}, X)).
\end{array}
$$

Hence the maps $\mathrm{Ev}_n(f \square i)$ are in $Q - \mathrm{proj}$ if and only if the map f is an element of $\tilde{Q} - \mathrm{proj}$. This completes the proof in the cofibration case. The trivial cofibration case is similar. $\qquad\square$

Corollary B.21 *The projective fibrations are exactly the maps in $\tilde{P} - \mathrm{fib}$. The projective trivial cofibrations are exactly the maps in $\tilde{Q} - \mathrm{fib}$.*

As in the case of ordinary spectra we deduce the following

Theorem B.22 *Let $\mathcal{C}$ be a left proper fibrantly generated simplicial closed symmetric monoidal model category with all small limits. The projective weak equivalences, projective fibrations and projective cofibrations define a left proper fibrantly generated simplicial closed symmetric monoidal model structure on $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ with set of generating fibrations $\tilde{P}$ and set of generating trivial fibrations $\tilde{Q}$ and unit object $\mathrm{Sym}(K)$.*

Proof The model structure part may be deduced exactly as in Theorem B.6. The only thing we have to explain is the closed symmetric monoidal structure on $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$. This is just Theorem 8.3 of [Ho2]. $\qquad\square$

In order to obtain a stable model structure on $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ we have to localize the projective structure at an appropriate set of fibrant objects.

Definition B.23 *A symmetric spectrum $X \in \mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is a symmetric Ω-spectrum if X is level fibrant and the adjoint $X_n \to X_{n+1}^K$ of the structure map of X is a weak equivalence for all n.*

We want the symmetric Ω-spectra to be the stable fibrant objects in $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$. As in the previous section we have to choose the set maps

$$\mathcal{S} := \{\zeta_n^A : F_{n+1}(A \otimes K) \to F_n A; A \text{ cofibrant in } \mathcal{C}\}$$

to be the S-local equivalences, where the map ζ_n^A is adjoint to the map

$$A \otimes K \to \mathrm{Ev}_{n+1} F_n A = \Sigma_{n+1} \times (A \otimes K)$$

corresponding to the identity of Σ_{n+1}. With the set S consisting of all symmetric Ω-spectra we obtain the desired set $\mathcal{S}$ of S-local equivalences.

Definition B.24 *Let $\mathcal{C}$ be a left proper fibrantly generated simplicial model category with all small limits. Let S be the set of all symmetric Ω-spectra. We define the* stable model structure *on $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ to be the S-localized model structure $L_\mathcal{S}\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ of Theorem A.3 where $\mathcal{S}$ is the class of all S-local equivalences.*

Remark B.25 *In particular, the* stable equivalences *are the S-local equivalences; the* stable cofibrations *are the projective cofibrations; the* stable fibrations *are the maps that have the right lifting property with respect to all stable trivial cofibrations; the* stable fibrant objects *are the symmetric Ω-spectra.*

Theorem B.26 *The functor $- \otimes K$ is a Quillen equivalence with respect to the stable structure of $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$.*

Proof The proof is given by [Ho2], Theorem 8.10. □

Unfortunately, we are not yet able to prove an analogue of Theorem 8.11 of [Ho2] in our setting stating that the monoidal structure is compatible with the stable model structure. The problem is that we have only left properness together with generating fibrations. We plan to investigate this problem in future work.

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