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Groups with twisted *p*-periodic cohomology

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Abstract. We give a characterization of groups with twisted p-periodic cohomology in terms of group actions on mod p homology spheres. An equivalent algebraic characterization of such groups is also presented.

1. Introduction

We will consider groups with twisted p-periodic cohomology (p a prime) in the following sense. Write $\hat{Z}_p(\omega)$ for the group of p-adic integers, equipped with a G-action via a homomorphism $\omega : G \to \hat{Z}_p^\times$. For M a $\mathbb{Z}G$-module, we write M_ω for the $\mathbb{Z}G$-module $M \otimes \hat{Z}_p(\omega)$ with diagonal G action.

Definition 1.1. A group G is said to have twisted p-periodic cohomology, if there are a $k > 0$, a homomorphism $\omega : G \to \hat{Z}_p^\times$ and a cohomology class $e_\omega \in H^n(G, \hat{Z}_p(\omega))$ for some $n > 0$, such that

$$e_\omega \cup - : H^i(G, M) \to H^{i+n}(G, M_\omega)$$

is an isomorphism for all $i \geq k$ and all p-torsion $\mathbb{Z}G$-modules M of finite exponent. In case the twisting ω can be chosen to be trivial, we say that G has p-periodic cohomology.

By replacing e_ω with e_ω^2 we see that for G with twisted p-periodic cohomology one can assume, if one wishes to, that the degree n of the periodicity generator is even. In case of a finite group G, we infer, by replacing e_ω by a suitable cup power, that if G has twisted p-periodic cohomology, it also has p-periodic cohomology. A classical theorem states that a finite group has p-periodic cohomology if and only if all its abelian p-subgroups are cyclic. Moreover, the finite groups with p-periodic cohomology have the following characterization in terms of actions on $\mathbb{Z}/p\mathbb{Z}$-homology spheres.

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Theorem 1.2 (Swan [12])**.** *A finite group G has p-periodic cohomology if and only if there exists a finite, simply connected free G-CW-complex, which has the same $\mathbb{Z}/p\mathbb{Z}$-homology as some sphere.*

Our goal is to find a similar characterization for arbitrary groups with (twisted) p-periodic cohomology.

Definition 1.3. A CW-complex X is called a $\mathbb{Z}/p\mathbb{Z}$-homology n-sphere, if $H_*(X, \mathbb{Z}/p\mathbb{Z}) \cong H_*(S^n, \mathbb{Z}/p\mathbb{Z})$.

In Section 5 we will prove the following generalization of Theorem 1.2.

Theorem 1.4. *A group G has twisted p-periodic cohomology if and only if there exists a simply connected $\mathbb{Z}/p\mathbb{Z}$-homology sphere X, which is a free G-CW-complex satisfying $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}}(X/G) < \infty$.*

For the definition of the cohomological dimension $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}}$ of a space see Section 2.

As we will see in Section 7, there are groups which have twisted p-periodic cohomology but which do not have p-periodic cohomology. For groups with p-periodic cohomology we prove the following characterization.

Theorem 1.5. *A group G has p-periodic cohomology if and only if there exists a free G-CW-complex X with homotopically trivial G-action such that X is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere satisfying $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}}(X/G) < \infty$.*

We will also consider groups with $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology in the following sense.

Definition 1.6. A group G is said to have $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology, if there is a cohomology class $e \in H^n(G, \mathbb{Z}/p\mathbb{Z})$ for some $n > 0$ and an integer $k > 0$, such that for every $\mathbb{Z}/p\mathbb{Z}[G]$-module M the map

$$e \cup - : H^i(G, M) \to H^{i+n}(G, M)$$

is an isomorphism for all $i \geq k$.

The following is a simple observation.

Lemma 1.7. *Suppose that G has twisted p-periodic cohomology. Then G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology.*

Indeed, if $e_\omega \in H^n(G, \hat{Z}_p(\omega))$ gives rise to twisted periodicity as above and $e_\omega(p) \in H^n(G, (\mathbb{Z}/p\mathbb{Z})_\omega)$ denotes the mod p reduction of e_ω, then G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology with periodicity generator the $(p-1)$-fold cup product $e := e_\omega(p)^{p-1} \in H^{n(p-1)}(G, \mathbb{Z}/p\mathbb{Z})$.

If M is a fixed $\mathbb{Z}G$-module which is p-torsion of finite exponent p^{k+1}, then the $p^k(p-1)$-fold twisted module

$$M_{\omega^{p^k(p-1)}} := ((\cdots(M_\omega)\cdots)_\omega)_\omega$$

is naturally isomorphic as a $\mathbb{Z}G$-module to M. Therefore, if G has twisted p-periodic cohomology of some period n, its cohomology with M coefficients will actually be periodic in high dimensions $d \geq d_0(M)$, with period $n \cdot p^k(p-1)$. In general, it is not possible to choose the dimensions $d_0(M)$ so that they are bounded by a number independent of M. This observation leads to an example of a group with twisted p-periodic cohomology but not having p-periodic cohomology (cp. Example 7.3).

It is this example together with the fundamental paper [1] by Adem and Smith which inspired our work. For background on groups acting freely on finite-dimensional homology spheres, see [10] and [13].

2. $\mathbb{Z}/p\mathbb{Z}$-dimension for spaces and $\mathbb{Z}/p\mathbb{Z}$-localization

Similarly to the definition of the $\mathbb{Z}/p\mathbb{Z}$-cohomological dimension of groups, one defines the $\mathbb{Z}/p\mathbb{Z}$-cohomological dimension for spaces as follows.

Definition 2.1. Let X be a connected CW-complex and $k > 0$. The $\mathbb{Z}/p\mathbb{Z}$-cohomological dimension $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}}(X)$ of X is the smallest integer n such that $H^i(X, M) = 0$ for all $\mathbb{Z}/p\mathbb{Z}[\pi_1(X)]$-modules M and all $i > n$; if there is no such n, we write $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}}(X) = \infty$.

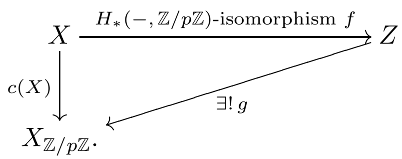
A simple induction on k shows that if $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} X < \infty$, then there exists an $i > 0$ such that for all k and all $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(X)]$-modules M, $H^j(X, M) = 0$ for all $j > i$.

In [3], Bousfield constructed, on the homotopy category of CW-complexes, the localization with respect to $H_*(-, \mathbb{Z}/p\mathbb{Z})$, which we call the $\mathbb{Z}/p\mathbb{Z}$-localization and which consists of a functorial $H_*(-, \mathbb{Z}/p\mathbb{Z})$-isomorphism

$$c(X) : X \to X_{\mathbb{Z}/p\mathbb{Z}},$$

which is characterized by the following universal property.

For every $H_*(-, \mathbb{Z}/p\mathbb{Z})$-isomorphism $f : X \to Z$ there is a unique map (up to homotopy) $g : Z \to X_{\mathbb{Z}/p\mathbb{Z}}$ such that $g \circ f \simeq c(X)$:



If X is simply connected (or nilpotent) and of finite type, then $X_{\mathbb{Z}/p\mathbb{Z}}$ agrees with Sullivan's p-completion $\hat{X}_p$ (cp. [11]), and $X \to \hat{X}_p$ is profinite p-completion on the level of homotopy groups.

Note that if X is simply connected, then one has $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} X = \mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} X_{\mathbb{Z}/p\mathbb{Z}}$, but for instance

$$\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} S^1 = 1 < \mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} S^1_{\mathbb{Z}/p\mathbb{Z}} = \infty$$

(because $\pi_1(S^1_{\mathbb{Z}/p\mathbb{Z}})$ contains a free abelian subgroup of infinite rank).

By the *standard* $\mathbb{Z}/p\mathbb{Z}$-homology n-sphere we mean $S^n_{\mathbb{Z}/p\mathbb{Z}}$.

Lemma 2.2. *Let X be a $\mathbb{Z}/p\mathbb{Z}$-homology n-sphere. Then $X_{\mathbb{Z}/p\mathbb{Z}}$ is homotopy equivalent to $S^n_{\mathbb{Z}/p\mathbb{Z}}$.*

Proof. Assume that $H_*(X, \mathbb{Z}/p\mathbb{Z}) \cong H_*(S^n, \mathbb{Z}/p\mathbb{Z})$. We first consider the case of $n = 1$. It follows that $\pi_1(X)_{ab} \otimes \mathbb{Z}/p\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. Choose an $f : S^1 \to X$ mapping to a generator of $\pi_1(X)_{ab} \otimes \mathbb{Z}/p\mathbb{Z}$. Then f induces an isomorphism in homology with $\mathbb{Z}/p\mathbb{Z}$-coefficients. It follows that f induces a homotopy equivalence $S^1_{\mathbb{Z}/p\mathbb{Z}} \to X_{\mathbb{Z}/p\mathbb{Z}}$. Now assume that $n > 1$. Since

$$H_1(X, \mathbb{Z}/p\mathbb{Z}) \cong H_1(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z}) = 0,$$

we also have $H_1(\pi_1(X_{\mathbb{Z}/p\mathbb{Z}}), \mathbb{Z}/p\mathbb{Z}) = 0$. But $\pi_1(X_{\mathbb{Z}/p\mathbb{Z}})$ is an $H\mathbb{Z}/p\mathbb{Z}$-local group, thus $\pi_1(X_{\mathbb{Z}/pZ}) = 0$ (see [3, Thm. 5.5]). We proceed by showing that $X_{\mathbb{Z}/p\mathbb{Z}}$ is $(n-1)$-connected. Let $\pi_i(X_{\mathbb{Z}/p\mathbb{Z}})$ be the first non-vanishing homotopy group of $X_{\mathbb{Z}/p\mathbb{Z}}$, $i > 1$. Because an $H_*(-, \mathbb{Z}_{(p)})$-isomorphism is also an $H_*(-, \mathbb{Z}/p\mathbb{Z})$-isomorphism, $X_{\mathbb{Z}/p\mathbb{Z}}$ is $H\mathbb{Z}_{(p)}$-local and therefore its homology groups with $\mathbb{Z}$-coefficients are uniquely q-divisible for q prime to p. Moreover, for $n > i > 1$, multiplication by p is bijective on $H_i(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z})$, because $H_j(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z}) = 0$ for $j = i-1, i$. Thus $H_i(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z})$ is a $\mathbb{Q}$-vector space for $1 < i < n$. Since the only $\mathbb{Q}$-vector space, which is $H\mathbb{Z}/p\mathbb{Z}$-local as an abelian group, is the trivial one, and because the homotopy groups of $X_{\mathbb{Z}/p\mathbb{Z}}$ are $H\mathbb{Z}/p\mathbb{Z}$-local, we conclude from the Hurewicz Theorem that $X_{\mathbb{Z}/p\mathbb{Z}}$ must be $(n-1)$-connected. It follows that the natural maps

$$\pi_n(X_{\mathbb{Z}/p\mathbb{Z}}) \to H_n(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}) \to H_n(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}, p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

are both surjective. Choose an $f : S^n \to X_{\mathbb{Z}/p\mathbb{Z}}$ which maps to a generator of $H_n(X_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z})$ and it follows that f induces a homotopy equivalence

$$S^n_{\mathbb{Z}/p\mathbb{Z}} \to X_{\mathbb{Z}/p\mathbb{Z}}. \qquad \square$$

There is also a fiberwise version of $\mathbb{Z}/p\mathbb{Z}$-localization (see [8] for details). If

$$X \to E \to B$$

is a fibration of connected CW-complexes, one can construct a new fibration

$$X_{\mathbb{Z}/p\mathbb{Z}} \to E^f_{\mathbb{Z}/p\mathbb{Z}} \to B,$$

together with a map $E \to E^f_{\mathbb{Z}/p\mathbb{Z}}$ over B, which restricts on the fibers to $\mathbb{Z}/p\mathbb{Z}$-localization $X \to X_{\mathbb{Z}/p\mathbb{Z}}$. Using the Serre spectral sequence, we conclude the following. If $F \to E \to B$ is a fibration of connected CW-complexes with F simply connected, then

$$\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} E = \operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} E^f_{\mathbb{Z}/p\mathbb{Z}}.$$

Also, if the fiber F is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere, then fiberwise $\mathbb{Z}/p\mathbb{Z}$-localization yields a fibration with fiber a standard $\mathbb{Z}/p\mathbb{Z}$-homology sphere

$$S^n_{\mathbb{Z}/p\mathbb{Z}} \to E^f_{\mathbb{Z}/p\mathbb{Z}} \to B.$$

3. Fibrations, orientations and Euler classes

If $F \to E \to B$ is a fibration of connected CW-complexes, then $\pi_1(E) \to \pi_1(B)$ is surjective and lifting of loops defines a natural map $\theta : \pi_1(B) \to [F, F]$, a *homotopy action* of $\pi_1(B)$ on F.

Definition 3.1. Let $F \to E \to B$ be a fibration of connected CW-complexes. The fibration is called orientable, if the associated homotopy action $\pi_1(B) \to [F, F]$ is trivial. We call the fibration $H\mathbb{Z}/p^k\mathbb{Z}$-orientable, if $\pi_1(B)$ acts trivially on $H_*(F, \mathbb{Z}/p^k\mathbb{Z})$.

Clearly, if a fibration is orientable, it is $H\mathbb{Z}/p^k\mathbb{Z}$-orientable for all k.

Definition 3.2. Let $F \to E \to B$ be a fibration of connected CW-complexes. We call such a fibration *$\mathbb{Z}/p\mathbb{Z}$-spherical* in case F is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere (or, equivalently, if $F_{\mathbb{Z}/p\mathbb{Z}} \simeq S^n_{\mathbb{Z}/p\mathbb{Z}}$ for some $n > 0$).

We will make use of the following observation.

Lemma 3.3. *For a group G the following conditions are equivalent.*

(a) *There exists a simply connected free G-CW-complex X which is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere satisfying* $\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} X/G < \infty$.
(b) *There exists a $\mathbb{Z}/p\mathbb{Z}$-spherical fibration $F \to E \to K(G, 1)$ with F simply connected and* $\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} E < \infty$.

Proof. Let X be as in (a) and $f : X/G \to K(G, 1)$ the classifying map for the universal cover X of X/G. Then the homotopy fiber of f is G-homotopy equivalent to X, thus (b) holds. If $F \to E \to K(G, 1)$ is as in (b), the universal cover of E is G-homotopy equivalent to F, thus (a) holds. □

Note that if X is any $\mathbb{Z}/p\mathbb{Z}$-homology sphere, it is also a $\mathbb{Z}/p^k\mathbb{Z}$-homology sphere for $k > 1$ as one easily sees by induction on k. Thus, for a $\mathbb{Z}/p\mathbb{Z}$-spherical fibration $F \to E \to B$ as in Definition 3.2, the $\pi_1(B)$-module $H_n(F, \mathbb{Z}/p^k\mathbb{Z})$ is isomorphic to a twisted module $(\mathbb{Z}/p^k\mathbb{Z})_\omega$, where $\omega : \pi_1(B) \to (\mathbb{Z}/p^k\mathbb{Z})^\times$ corresponds to the action of $\pi_1(B)$ on $H_n(F, \mathbb{Z}/p^k\mathbb{Z})$. (If we need to emphasize the dependence of ω on k, we write $\omega(k)$ in place of ω). We call the twisted module $(\mathbb{Z}/p^k\mathbb{Z})_\omega$ the k-orientation module. The fibration is $H\mathbb{Z}/p^k\mathbb{Z}$-orientable in the sense of Definition 3.1, if the k-orientation module is the trivial $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-module $\mathbb{Z}/p^k\mathbb{Z}$. We write $\bar{\omega}$ for the map $\pi_1(B) \to (\mathbb{Z}/p^k\mathbb{Z})^\times$ given by $\bar{\omega}(x) = \omega(x^{-1})$, and more generally ω^n for the map with $\omega^n(x) = \omega(x^n)$, $n \in \mathbb{Z}$. For any $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-module M we write M_ω for $M \otimes (\mathbb{Z}/p^k\mathbb{Z})_\omega$ with diagonal $\pi_1(B)$-action $x \cdot (m \otimes z) = xm \otimes \omega(x)z$. Similarly, we consider the diagonal action on $\operatorname{Hom}_{\mathbb{Z}/p^k\mathbb{Z}}((\mathbb{Z}/p^k\mathbb{Z})_\omega, M)$ given by

$$(xf)(z) = x \cdot f(\bar{\omega}(x)z).$$

Therefore, there is a natural isomorphism of $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-modules

$$H^n(F, M) \cong \operatorname{Hom}(H_n(F, \mathbb{Z}/p^k\mathbb{Z}), M) \cong \operatorname{Hom}((\mathbb{Z}/p^k\mathbb{Z})_\omega, M) \cong M_{\bar{\omega}}.$$

In the case of a $\mathbb{Z}/p\mathbb{Z}$-spherical fibration $F \to E \to B$, the only possibly nonzero differential in the Serre spectral sequence with coefficients in a $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(E)]$-module K,

$$E_2^{i,\ell} = H^i(B, H^\ell(F,K)) \Longrightarrow H^{i+\ell}(E,K),$$

is the transgression differential

$$d_{n+1} : E_2^{i,n} = E_{n+1}^{i,n} \to E_{n+1}^{i+n+1,0} = E_2^{i+n+1,0}.$$

Taking for K the k-orientation module $(\mathbb{Z}/p^k\mathbb{Z})_\omega$ and choosing $i = 0$, this yields

$$d_{n+1} : \mathbb{Z}/p^k\mathbb{Z} = H^0(B, \mathbb{Z}/p^k\mathbb{Z}) \to H^{n+1}(B, (\mathbb{Z}/p^k\mathbb{Z})_\omega),$$

and the image of $1 \in \mathbb{Z}/p^k\mathbb{Z}$,

$$d_{n+1}(1) =: e(k)_\omega \in H^{n+1}(B, (\mathbb{Z}/p^k\mathbb{Z})_\omega),$$

is called the *twisted* $\mathbb{Z}/p^k\mathbb{Z}$-Euler class of the given $\mathbb{Z}/p\mathbb{Z}$-spherical fibration. Let now M be an arbitrary $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-module and choose $K = M_\omega$. Thus $H^n(F, M_\omega) = M$ and

$$d_{n+1} : E_2^{i,n} = H^i(B,M) \to H^{i+n+1}(B, M_\omega) = E_2^{i+n+1,0}$$

is given by the cup product with $e(k)_\omega$. The kernel and image of d_{n+1} are determined as

$$E_\infty^{i,n} = \ker d_{n+1} \subset E_2^{i,n} = H^i(B,M) \xrightarrow{d_{n+1}} H^{i+n+1}(B, M_\omega)$$

and

$$H^i(B,M) \xrightarrow{d_{n+1}} H^{i+n+1}(B, M_\omega) = E_2^{i+n+1,0} \twoheadrightarrow \operatorname{coker} d_{n+1} = E_\infty^{i+n+1,0},$$

respectively. The natural surjection $\sigma : H^{i+n}(E,M) \to E_\infty^{i,n}$ has as kernel the subgroup $E_\infty^{i+n,0}$ and, by splicing things together, one gets the *Gysin-sequence*

$$\to H^{i+n}(E,M) \xrightarrow{\sigma} H^i(B,M) \xrightarrow{e(k)_\omega \cup -} H^{i+n+1}(B, M_\omega)$$
$$\to H^{i+n+1}(E, M_\omega) \to .$$

One concludes that for large values of i and all $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-modules M, the cup product with $e(k)_\omega$ induces for all k isomorphisms

$$e(k)_\omega \cup - : H^i(B,M) \xrightarrow{\cong} H^{i+n+1}(B, M_\omega)$$

if and only if there exists a j_0 such that for all $j > j_0$, $H^j(E,M) = 0$ for all $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-modules M and all k (here M is viewed as $\pi_1(E)$-module via $\pi_1(E) \to \pi_1(B)$). In case F is simply connected, this amounts to $\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} E < \infty$.

Corollary 3.4. *Let $F \to E \to B$ be a $\mathbb{Z}/p\mathbb{Z}$-spherical fibration of CW-complexes with B connected and F simply connected, with twisted $\mathbb{Z}/p^k\mathbb{Z}$-Euler classes $e(k)_{\omega(k)} \in H^n(B, (\mathbb{Z}/p^k\mathbb{Z})_{\omega(k)})$, $k \geq 1$. Then the following conditions are equivalent.*

(1) $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} E < \infty$.
(2) *There exists* i_0 *such that, for all* $i > i_0$ *and all* $k \geq 1$,

$$e(k)_{\omega(k)} \cup - : H^i(B, M) \to H^{i+n}(B, M_{\omega(k)})$$

is an isomorphism for all $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$*-modules* M.

In the situation of Corollary 3.4, it follows from the naturality of the Serre spectral sequence that the twisted $\mathbb{Z}/p^k\mathbb{Z}$-Euler classes $e(k)_{\omega(k)}$ are the reduction mod p^k of a class $e_\omega \in H^n(B, \hat{Z}_p(\omega))$, where $\hat{Z}_p(\omega)$ is isomorphic to $\pi_{n-1}(F_{\mathbb{Z}/p\mathbb{Z}}) \cong \pi_{n-1}(S^{n-1}_{\mathbb{Z}/p\mathbb{Z}})$ as a $\pi_1(B)$-module. Therefore, the following holds.

Corollary 3.5. *If there exists a* $\mathbb{Z}/p\mathbb{Z}$*-spherical fibration of CW-complexes* $F \to E \to K(G,1)$ *with* F *simply connected and* $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} E < \infty$*, then* G *has twisted* p*-periodic cohomology.*

The following lemma permits us to pass from $\mathbb{Z}/p\mathbb{Z}$-spherical fibrations to $H\mathbb{Z}/p\mathbb{Z}$-orientable ones.

Lemma 3.6. *Let* $F_1 \to E_1 \to B$ *be a* $\mathbb{Z}/p\mathbb{Z}$*-spherical fibration of CW-complexes with* B *connected and* F_1 *simply connected, such that* $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} E_1 < \infty$*. Then the* $(p-1)$*-fold fiberwise join yields an* $H\mathbb{Z}/p\mathbb{Z}$*-orientable* $\mathbb{Z}/p\mathbb{Z}$*-spherical fibration* $F_2 \to E_2 \to B$ *over the same base, with* $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} E_2 < \infty$.

Proof. Let $e_\omega \in H^n(B, (\mathbb{Z}/p\mathbb{Z})_\omega)$ be the twisted Euler class of the fibration $F_1 \to E_1 \to B$. Because $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} E_1 < \infty$, we infer from Corollary 3.4 that there exists i_0 such that

$$e_\omega \cup - : H^i(B, M) \to H^{i+n}(B, M_\omega)$$

is an isomorphism for all $i > i_0$ and all $\mathbb{Z}/p\mathbb{Z}[\pi_1(B)]$-modules M. We then perform a fiberwise $(p-1)$-fold join to obtain a new $\mathbb{Z}/p\mathbb{Z}$-spherical fibration $F_2 \to E_2 \to B$ with Euler class $e = e_\omega^{p-1}$. This new fibration is $H\mathbb{Z}/p\mathbb{Z}$-orientable, because the $(p-1)$-fold tensor product of $(\mathbb{Z}/p\mathbb{Z})_\omega$ with diagonal action is the trivial $\mathbb{Z}/p\mathbb{Z}[\pi_1(B)]$-module $\mathbb{Z}/p\mathbb{Z}$. Moreover,

$$e \cup - : H^i(B, M) \to H^{i+(p-1)n}(B, M)$$

is an isomorphism for $i > i_0$ and all $\mathbb{Z}/p\mathbb{Z}[\pi_1(B)]$-modules M. Note that e is the reduction mod p of the twisted $\mathbb{Z}/p^k\mathbb{Z}$-Euler class

$$e(k)_{\omega(k)} \in H^n(B, (\mathbb{Z}/p^k\mathbb{Z})_{\omega(k)})$$

of the $\mathbb{Z}/p\mathbb{Z}$-spherical fibration $F_2 \to E_2 \to B$. Induction on k then shows that

$$e(k)_{\omega(k)} \cup - : H^i(B, L) \to H^i(B, L_{\omega(k)})$$

is an isomorphism for all $\mathbb{Z}/p^k\mathbb{Z}[\pi_1(B)]$-modules L. We infer from Corollary 3.4 that $\mathrm{cd}_{\mathbb{Z}/p\mathbb{Z}} E_2 < \infty$. □

4. Partial Euler classes

For a connected CW-complex X we write P_qX for its q-th Postnikov section, with canonical map $X \to P_qX$ such that

(1) $\pi_i(P_q(X)) = 0$ for $i > q$,

(2) $\pi_j(X) \xrightarrow{\cong} \pi_j(P_qX)$ for $j \leq q$.

In case that X is a $\mathbb{Z}/p\mathbb{Z}$-homology n-sphere, we have $X_{\mathbb{Z}/p\mathbb{Z}} \simeq S^n_{\mathbb{Z}/p\mathbb{Z}}$. Therefore, $P_q(X_{\mathbb{Z}/p\mathbb{Z}}) = \{*\}$ for $q < n$ and $P_n(X_{\mathbb{Z}/p\mathbb{Z}}) \simeq K(\hat{Z}_p, n)$. Adapting the terminology of [1], we define *k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler classes* as follows.

Definition 4.1. Let B be a connected CW-complex and $k \geq 0$. Then $\epsilon \in H^n(B, \mathbb{Z}/p\mathbb{Z})$ is a *k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class* if there exists a fibration

$$(\Phi) : P_{n-1+k}(S^{n-1}_{\mathbb{Z}/p\mathbb{Z}}) \to E \to B$$

such that $\pi_1(B)$ acts trivially on $H^{n-1}(P_{n-1+k}(S^{n-1}_{\mathbb{Z}/p\mathbb{Z}}), \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$ and there is a generator of that group which transgresses to ϵ in the Serre spectral sequence with $\mathbb{Z}/p\mathbb{Z}$-coefficients for the fibration (Φ). The k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class ϵ is called *orientable*, if the fibration (Φ) can be chosen to be orientable in the sense of Definition 3.1.

Lemma 4.2. *Let B be a connected CW-complex and $\epsilon \in H^n(B, \mathbb{Z}/p\mathbb{Z})$ a k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class. Then for all $\ell > 0$, ϵ^ℓ is a k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class. If ϵ is orientable in the sense of Definition 4.1, then so is ϵ^ℓ.*

Proof. Let

$$P := P_{n-1+k}(S^{n-1}_{\mathbb{Z}/p\mathbb{Z}}) \to E \to B$$

be a fibration such that $\pi_1(B)$ acts trivially on $H^{n-1}(P, \mathbb{Z}/p\mathbb{Z})$ and let $\alpha \in H^{n-1}(P, \mathbb{Z}/p\mathbb{Z})$ be an element which transgresses to ϵ. By forming fiberwise the ℓ-fold join and applying $\mathbb{Z}/p\mathbb{Z}$-localization, we obtain a fibration

$$(*^\ell P)_{\mathbb{Z}/p\mathbb{Z}} \to E(\ell) \to B.$$

In the Serre spectral sequence with $\mathbb{Z}/p\mathbb{Z}$-coefficients for this new fibration, $(\alpha * \cdots * \alpha)_{\mathbb{Z}/p\mathbb{Z}}$ transgresses to ϵ^ℓ. Since $*^\ell S^{n-1} \simeq S^{n\ell-1}$, we have

$$P_{n\ell-1+k}((*^\ell P)_{\mathbb{Z}/p\mathbb{Z}}) = P_{n\ell-1+k}(S^{n\ell-1}_{\mathbb{Z}/p\mathbb{Z}})$$

and we obtain, by taking fiberwise Postnikov sections, a fibration

$$P_{n\ell-1+k}(S^{n\ell-1}_{\mathbb{Z}/p\mathbb{Z}}) \to E^f(\ell) \to B$$

for which the image of $(*^\ell \alpha)_{\mathbb{Z}/p\mathbb{Z}}$ under the natural map

$$H^{n\ell-1}((*^\ell P)_{\mathbb{Z}/p\mathbb{Z}}, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\cong} H^{n\ell-1}(P_{n\ell-1+k}(S^{n\ell-1}_{\mathbb{Z}/p\mathbb{Z}}), \mathbb{Z}/p\mathbb{Z})$$

transgresses to ϵ^ℓ. It is obvious that ϵ^ℓ is orientable if ϵ is. □

Lemma 4.3. *Let $(\Phi_0) : S^n_{\mathbb{Z}/p\mathbb{Z}} \to E \to B$ be a fibration with B connected and $n > 0$. By taking fiberwise Postnikov sections, we obtain fibrations*

$$(\Phi_k) : P_{n+k}S^n_{\mathbb{Z}/p\mathbb{Z}} \to E_k \to B, \quad k \geq 0.$$

The fibrations (Φ_k), $k \geq 0$, *are all orientable if and only if* $\pi_1(B)$ *acts trivially on* $\pi_n(S^n_{\mathbb{Z}/p\mathbb{Z}}) \cong \hat{Z}_p$.

Proof. This follows from the functoriality of P_{n+k} and the fact that homotopy classes $S^n_{\mathbb{Z}/p\mathbb{Z}} \to S^n_{\mathbb{Z}/p\mathbb{Z}}$ correspond naturally to elements of $\pi_n(S^n_{\mathbb{Z}/p\mathbb{Z}})$. □

Definition 4.4. Let X be a connected CW-complex with fundamental group G. An element $x \in H^n(X, \mathbb{Z}/p\mathbb{Z})$ is called ω-p-integral, if there exists an action $\omega : G \to \hat{Z}_p{}^\times$ such that G acts trivially on $\hat{Z}_p(\omega)/p\hat{Z}_p(\omega) \cong \mathbb{Z}/p\mathbb{Z}$ and x lies in the image of the natural coefficient homomorphism $H^n(X, \hat{Z}_p(\omega)) \to H^n(X, \mathbb{Z}/p\mathbb{Z})$. In case the action ω can be chosen to be trivial, x is called p-integral.

To deal with non-orientable fibrations, we recall the following fact. Let

$$(F) : K(M, m) \to E \to B$$

be a fibration with connected base B, $m > 0$ and induced action of $\pi_1(B) = G$ on M corresponding to the homomorphism $\phi : G \to \mathrm{Aut}(M)$. Such fibrations are classified by cohomology elements with local coefficients as follows. There is a universal fibration

$$K(M, m+1) \to L_\phi(M, m+1) \to K(G, 1)$$

such that fibrations of type (F) correspond to homotopy classes of maps $f : B \to L_\phi(M, m+1)$ over $K(G, 1)$. The homotopy class over $K(G, 1)$ of such an f corresponds to an element in the cohomology with local coefficients $H^{m+1}(B, M)$, see [2] or [6].

The following lemma is a variation of [1, Lem. 2.5].

Lemma 4.5. *Let* $x \in H^{2n}(X, \mathbb{Z}/p\mathbb{Z})$ *be an* ω*-*p*-integral element. Then some cup power of* x *is a* k*-partial* $\mathbb{Z}/p\mathbb{Z}$*-Euler class and this* k*-partial* $\mathbb{Z}/p\mathbb{Z}$*-Euler class is orientable (in the sense of Definition 4.1) in case* x *is* p*-integral.*

Proof. Let G be the fundamental group of X. Since x is ω-p-integral, there exist $\omega : G \to \hat{Z}_p^\times$ and $\tilde{x} \in H^{2n}(X, \hat{Z}_p(\omega))$ mapping to x under reduction mod p. Let $\mu : X \to L_\omega(\hat{Z}_p, 2n)$ correspond to $\tilde{x}$. It classifies a fibration

$$K(\hat{Z}_p(\omega), 2n-1) \to E \to X$$

with

$$\begin{aligned} H^{2n-1}(K(\hat{Z}_p(\omega), 2n-1), \mathbb{Z}/p\mathbb{Z}) &\cong H^{2n-1}(K(\hat{Z}_p/p\hat{Z}_p, 2n-1), \mathbb{Z}/p\mathbb{Z}) \\ &\cong \mathbb{Z}/p\mathbb{Z} \end{aligned}$$

having trivial G-action. This shows that x is a 0-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class. Suppose now that $k > 0$ is given and that x^m is a $(k-1)$-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class. Thus there is a fibration

$$P_{2nm-1+k-1}(S^{2nm-1}_{\mathbb{Z}/p\mathbb{Z}}) =: P(k-1) \to E(k-1) \to X$$

with a generator of $H^{2nm-1}(P(k-1), \mathbb{Z}/p\mathbb{Z})^G = \mathbb{Z}/p\mathbb{Z}$ transgressing to $y := x^m$. By Lemma 4.2, for all j, the power y^j is a $(k-1)$-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class too. Thus there are fibrations

$$P_{2nmj-1+k-1}(S^{2nmj-1}_{\mathbb{Z}/p\mathbb{Z}}) =: Q(k-1) \to F(k-1) \to X$$

with a generator of $H^{2nmj-1}(Q(k-1), \mathbb{Z}/p\mathbb{Z})^G = \mathbb{Z}/p\mathbb{Z}$ transgressing to $y^j = x^{mj}$. To show that for a suitable j, the power y^j gives rise to a k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class, we need to check that the classifying map

$$\theta : Q(k-1) \to K(\pi, 2nmj + k)$$

for the fibration $Q(k) \to Q(k-1)$ factors through $F(k-1)$. Note that

$$\pi := \pi_{2nmj+k-1}(Q(k)) = \pi_{2nmj+k-1}(S^{2nmj-1}_{\mathbb{Z}/p\mathbb{Z}})$$

is a finite p-group on which $\pi_1(X) = G$ acts via

$$\omega^{mj} : G \to \hat{Z}_p^\times = \mathrm{HoAut}(S^{2nmj-1}_{\mathbb{Z}/p\mathbb{Z}}).$$

We write $\underline{\pi}$ for π with that action. Because of the naturality of the Postnikov section functor, the homotopy fibration

$$Q(k) \to Q(k-1) \xrightarrow{\theta} K(\underline{\pi}, 2nmj + k)$$

is compatible with the homotopy G-action via ω^{mj} on these spaces. Therefore,

$$[\theta] \in H^{2nmj+k}(Q(k-1), \underline{\pi})$$

is G-invariant with respect to the diagonal G-action on this cohomology group. In the Serre spectral sequence for $Q(k-1) \to F(k-1) \to X$ with $\underline{\pi}$ coefficients,

$$H^s(X, H^t(Q(k-1), \underline{\pi})) \Rightarrow H^{s+t}(F(k-1), \underline{\pi}),$$

the cohomology class $[\theta]$ lies thus in

$$E_2^{0,2nmj+k} = H^{2nmj+k}(Q(k-1), \underline{\pi})^G.$$

To show that $[\theta]$ is the restriction of a class in the cohomology of $F(k-1)$ with $\underline{\pi}$-coefficients amounts to showing that $[\theta]$ is a permanent cycle. The same argument as in [1, Lem. 2.5] shows that this is the case for j a large enough p-power. It follows that some power of x is a k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class. In case x is p-integral, the argument shows that the k-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class we obtained is orientable. □

5. Proof of Theorems 1.4 and 1.5

We will give the proof of Theorem 1.4. The proof of Theorem 1.5 is analogous but simpler.

Suppose that G has twisted p-periodic cohomology. Then there exist for some $n > 0$ an ω-p-integral class $\epsilon \in H^{2n}(G, \mathbb{Z}/p\mathbb{Z})$ and $\epsilon_\omega \in H^{2n}(G, \hat{Z}_p(\omega))$, whose reduction mod p is ϵ, such that there is an $\ell_0 > 0$ with the property that the cup product with ϵ_ω induces isomorphisms $H^i(G, M) \to H^{i+2n}(G, M_\omega)$ for all $i \geq \ell_0$ and all p-torsion $\mathbb{Z}G$-modules M of finite exponent. By Lemma 4.5

we can find a cup power ϵ^m which is an ℓ_o-partial $\mathbb{Z}/p\mathbb{Z}$-Euler class. Therefore, we have a fibration

$$F(\ell_0) : P_{2nm-1+\ell_0}(S^{2nm-1}_{\mathbb{Z}/p\mathbb{Z}}) \to E(\ell_0) \to K(G,1),$$

with the property that a generator of

$$H^{2nm-1}(P_{2nm-1+\ell_0}(S^{2nm-1}_{\mathbb{Z}/p\mathbb{Z}}), \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$$

transgresses to ϵ^m in the Serre spectral sequence for $F(\ell_0)$. We want to show inductively that ϵ^m is a k-partial Euler class for all $k \geq \ell_0$. Write $P(j)$ for $P_{2nm-1+j}(S^{2nm-1}_{\mathbb{Z}/p\mathbb{Z}})$. We will inductively construct fibrations

$$F(k) : P(k) \to E(k) \to K(G,1)$$

for $k > \ell_0$ with the property that a generator of $H^{2nm-1}(P(k), \mathbb{Z}/p\mathbb{Z})$ transgresses to ϵ^m. To pass from $F(k-1)$ to $F(k)$ we argue as follows. We have a diagram

$$\begin{array}{lccccc} F(k-1): & P(k-1) & \longrightarrow & E(k-1) & \longrightarrow & K(G,1) \\ & \uparrow & & \vdots\uparrow & & \uparrow = \\ F(k): & P(k) & \dashrightarrow & E(k) & \dashrightarrow & K(G,1) \end{array}$$

in which the fibration $P(k) \to P(k-1)$ has fiber $K(\pi(\omega), 2nm-1+k)$ and is classified by a map

$$\theta : P(k-1) \to K(\pi(\omega), 2nm+k),$$

where $\pi(\omega)$ stands for the finite p-group $\pi := \pi_{2nm-1+k}(S^{2nm-1}) \otimes \hat{Z}_p$ with G-action induced by

$$\omega^m : G \to \hat{Z}_p^\times \cong \mathrm{Aut}(\pi_{2nm-1}(S^{2nm-1}_{\mathbb{Z}/p\mathbb{Z}})).$$

To construct the fibration $F(k)$ and the dotted arrows depicted above, we need to show that θ factors through $E(k-1)$. This amounts to showing that $[\theta]$, which lies in $H^{2nm+k}(P(k-1), \pi(\omega))$, is in the image of the restriction map

$$H^{2nm+k}(E(k-1), \pi(\omega)) \to H^{2nm+k}(P(k-1), \pi(\omega)).$$

As argued in the proof of Lemma 4.5, $[\theta] \in H^{2nm+k}(P(k-1), \pi(\omega))$ is G-invariant with respect to the diagonal G-action via ω^m on this cohomology group. The restriction map in question corresponds to an edge homomorphism in the Serre spectral sequence with $\pi(\omega)$-coefficients for the fibration $P(k-1) \to E(k-1) \to K(G,1)$:

$$\begin{aligned} H^{2nm+k}(E(k-1), \pi(\omega)) &\twoheadrightarrow E_\infty^{0,2nm+k} \\ &\subset E_2^{0,2nm+k} = H^{2nm+k}(P(k-1), \pi(\omega))^G. \end{aligned}$$

We need therefore to check that $[\theta]$ is a permanent cycle in the Serre spectral sequence. The only differentials on $[\theta]$ which could be nonzero are, for dimension reasons, the differential

$$d_{k+2} : E_2^{0,2nm+k} = E_{k+2}^{0,2nm+k} \to E_{k+2}^{k+2,2nm-1}$$

which takes values in

$$\ker\big(\epsilon_\omega^m \cup - : H^{k+2}(G, \pi(\omega)_{\bar{\omega}^m}) \to H^{k+2+2nm}(G, \pi(\omega))\big),$$

respectively the differential

$$d_{2nm+k+1} : E_{2nm+k+1}^{0,2nm+k} \to E_{2nm+k+1}^{2nm+k+1,0},$$

which takes values in

$$\operatorname{coker}\big(\epsilon_\omega^m \cup - : H^{k+1}(G, \pi(\omega)) \to H^{2nm+k+1}(G, \pi(\omega)_{\omega^m})\big).$$

Because $k > \ell_0$, we know that for any p-torsion module M of bounded exponent,

$$\epsilon_\omega^m \cup - : H^s(G, M) \to H^{s+2nm}(G, M_{\omega^m})$$

is an isomorphism for $s = k+1$, respectively $s = k+2$. The differentials d_{k+2}, respectively $d_{2nm+k+1}$ depicted above are therefore equal to 0. We conclude that the fibrations in the diagram above can be constructed as displayed. Passing to homotopy limits in the towers $\{F(k)\}_{k \geq 0}$ of that diagram, one obtains a fibration

$$F(\infty) : S_{\mathbb{Z}/p\mathbb{Z}}^{2n-1} \to E \to K(G, 1),$$

as desired. To check that $\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}}(E) < \infty$, one considers the Serre spectral sequence of the fibration $F(\infty)$ with coefficients in a $\mathbb{Z}/p\mathbb{Z}[G]$-module L and finds that $H^j(E, L) = 0$ for j large enough, independent of L, finishing the first part of the proof.

Suppose now conversely that X is a simply connected free G-CW-complex which is a $\mathbb{Z}/p\mathbb{Z}$-homology sphere satisfying $\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} X/G < \infty$. By Lemma 3.3 there exists a $\mathbb{Z}/p\mathbb{Z}$-spherical fibration $F \to E \to K(G, 1)$ with F simply connected and $\operatorname{cd}_{\mathbb{Z}/p\mathbb{Z}} E < \infty$. Corollary 3.5 then implies that G has twisted p-periodic cohomology, completing the proof of Theorem 1.4.

6. Algebraic characterization

Let $x \in H^n(G, \mathbb{Z}/p\mathbb{Z})$ and consider a $\mathbb{Z}/p\mathbb{Z}[G]$-projective resolution

$$\mathcal{P}_* : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \cdots \to P_0 \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

Denote for $i > 0$ the image $\partial_i P_i$ by K_i and let $\iota_i : K_i \to P_{i-1}$ be the natural injection. A cocycle representative of x corresponds to a map $\theta : K_n \to \mathbb{Z}/p\mathbb{Z}$. Form the diagram

$$\begin{array}{ccccccccccccc}
K_n & \xrightarrow{\iota_n} & P_{n-1} & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0 \\
\downarrow{\scriptstyle\theta} & & \downarrow & & \downarrow{\scriptstyle=} & & & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle=} & & \\
\tilde{x} : \mathbb{Z}/p\mathbb{Z} & \longrightarrow & A & \longrightarrow & P_{n-2} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & 0
\end{array}$$

where the square on the left is a push-out square. Then the class of the n-fold extension $\tilde{x}$, $[\tilde{x}] \in \operatorname{Ext}^n_{\mathbb{Z}/p\mathbb{Z}G}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$, corresponds to $x \in H^n(G, \mathbb{Z}/p\mathbb{Z})$.

Lemma 6.1. *Let $e \in H^n(G, \mathbb{Z}/p\mathbb{Z})$ and consider the associated n-extension*

$$\tilde{e} : 0 \to \mathbb{Z}/p\mathbb{Z} \to A \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z}/p\mathbb{Z} \to 0$$

as above. Then the following conditions are equivalent.

(1) *G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology via the cup product with e.*
(2) *The $\mathbb{Z}/p\mathbb{Z}[G]$-projective dimension of A, $\operatorname{proj.dim}_{\mathbb{Z}/p\mathbb{Z}[G]} A$, is finite.*

Proof. Let $\mathcal{P}_*$ be a $\mathbb{Z}/p\mathbb{Z}[G]$-projective resolution of $\mathbb{Z}/p\mathbb{Z}$ and choose a map $\theta(e) : K_n \to \mathbb{Z}/p\mathbb{Z}$ to represent e as above, giving rise to the n-extension $\tilde{e}$. It is known that the cup product with e is induced by a chain map $\Theta : \mathcal{P}_* \to \mathcal{P}_*$ of degree $-n$ which extends $\theta(e)$. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_n(e) & \longrightarrow & P_{n-1} & \longrightarrow & K_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \mathbb{Z}/p\mathbb{Z} & \longrightarrow & A & \longrightarrow & K_{n-1} & \longrightarrow & 0.
\end{array}$$

From the corresponding commutative diagram of long exact Ext-sequences

$$\begin{array}{ccccccc}
\longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(K_{n-1}, -) & \longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(P_{n-1}, -) & \longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(K_n, -) & \longrightarrow \\
 & {\scriptstyle =}\uparrow & & \uparrow & & {\scriptstyle \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(\theta(e), -)}\uparrow & \\
\longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(K_{n-1}, -) & \longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(A, -) & \longrightarrow & \operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(\mathbb{Z}/p\mathbb{Z}, -) & \longrightarrow
\end{array}$$

follows that $\operatorname{Ext}^i_{\mathbb{Z}/p\mathbb{Z}[G]}(\theta(e), -)$ is an isomorphism for large i if and only if A satisfies $\operatorname{proj.dim}_{\mathbb{Z}/p\mathbb{Z}[G]} A < \infty$. □

Corollary 6.2. *Let G be a group with $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology. There exists $k > 0$ such that for all $i \geq k$ and all projective $\mathbb{Z}/p\mathbb{Z}[G]$-modules P, $H^i(G, P) = 0$.*

Proof. By Lemma 6.1, there is a monomorphism $\iota : \mathbb{Z}/p\mathbb{Z} \to A$ with A a $\mathbb{Z}/p\mathbb{Z}[G]$-module of finite projective dimension d over $\mathbb{Z}/p\mathbb{Z}[G]$. Let I be an injective $\mathbb{Z}/p\mathbb{Z}[G]$-module. I injects into $A \otimes_{\mathbb{Z}/p\mathbb{Z}} I$ via $x \mapsto \iota(1) \otimes x$ and, as I is injective, I is a retract of $A \otimes_{\mathbb{Z}/p\mathbb{Z}} I$. For any projective $\mathbb{Z}/p\mathbb{Z}[G]$-module P, $P \otimes_{\mathbb{Z}/p\mathbb{Z}} I$ (with diagonal G-action) is projective too. It follows that

$$\operatorname{proj.dim}_{\mathbb{Z}/p\mathbb{Z}[G]} A \otimes_{\mathbb{Z}/p\mathbb{Z}} I \leq d$$

and, because I is a retract of that module,

$$\operatorname{proj.dim}_{\mathbb{Z}/p\mathbb{Z}[G]} I \leq d.$$

We conclude that the supremum of the projective length of injective $\mathbb{Z}/p\mathbb{Z}[G]$-modules is at most d, i.e.,

$$\operatorname{spli} \mathbb{Z}/p\mathbb{Z}[G] \leq d.$$

This implies that the supremum of the injective length of projective $\mathbb{Z}/p\mathbb{Z}[G]$-modules is also no greater than d, i.e.,

$$\operatorname{silp} \mathbb{Z}/p\mathbb{Z}[G] \leq d,$$

see [5, Thm. 2.4]. We infer that $H^i(G, P) = 0$ for $i > d$ and all projective $\mathbb{Z}/p\mathbb{Z}[G]$-modules P. □

The following is an algebraic characterization of groups with twisted p-periodic cohomology.

Lemma 6.3. *A group G has twisted p-periodic cohomology if and only if there exist an $n > 1$ and an exact sequence of $\mathbb{Z}/p\mathbb{Z}[G]$-modules*

$$\epsilon : 0 \to \mathbb{Z}/p\mathbb{Z} \to A \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z}/p\mathbb{Z} \to 0$$

with P_i projective for $0 \leq i \leq n-2$ and $\operatorname{proj.dim}_{\mathbb{Z}/pZ[G]} A < \infty$*, such that $[\epsilon] \in \operatorname{Ext}^n_{\mathbb{Z}/p\mathbb{Z}[G]}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) = H^n(G, \mathbb{Z}/p\mathbb{Z})$ is ω-p-integral. G has p-periodic cohomology if and only if there is an ϵ as above with $[\epsilon]$ being p-integral (for the definition of ω-p-integral and p-integral elements see Definition 4.4).*

Proof. Suppose G has twisted p-periodic cohomology. By definition, there exist $k > 0$ and $m > 0$ and $\sigma : G \to \hat{Z}_p^\times$ and $e_\sigma \in H^m(G, \hat{Z}_p(\sigma))$ such that the cup product with e_σ induces isomorphisms $H^i(G, M) \to H^{i+m}(G, M_\sigma)$ for all p-torsion $\mathbb{Z}[G]$-modules M of finite exponent and all $i \geq k$. It follows from the proof of Lemma 1.7 that G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology via the cup product with $e := e_\sigma(p)^{p-1}$, where $e_\sigma(p)$ denotes the mod p reduction of e_σ. Putting $n = m(p-1)$, it follows that $e \in H^n(G, \mathbb{Z}/p\mathbb{Z})$ is ω-p-integral with respect to $\omega = \sigma^{p-1}$ and can be represented (cp. Lemma 6.1) by an n-extension

$$\tilde{e} : 0 \to \mathbb{Z}/p\mathbb{Z} \to A \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z}/p\mathbb{Z} \to 0$$

with P_i projective for $0 \leq i \leq n-2$ and $\operatorname{proj.dim}_{\mathbb{Z}/pZ[G]} A < \infty$. Conversely, if we are given an n-extension

$$\epsilon : 0 \to \mathbb{Z}/p\mathbb{Z} \to A \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z}/p\mathbb{Z} \to 0$$

with P_i projective for $0 \leq i \leq n-2$ and $\operatorname{proj.dim}_{\mathbb{Z}/pZ[G]} A < \infty$ representing an ω-p-integral class $[\epsilon] = e \in H^n(G, \mathbb{Z}/p\mathbb{Z})$, then we choose $\tilde{e} \in H^n(G, \hat{Z}_p(\omega))$, an element with mod p reduction equal to e. Let M be a p-torsion $\mathbb{Z}/p\mathbb{Z}[G]$-module of finite exponent. Induction with respect to the exponent of M shows that the cup product

$$\tilde{e} \cup - : H^i(G, M) \to H^{i+n}(G, M_\omega)$$

is an isomorphism for i large and all such M. It follows that G is twisted p-periodic. The untwisted version of the lemma corresponds to the case where we can choose for ω the trivial homomorphism. □

7. Some remarks and examples

In general, one cannot expect a group G to have $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology even if all its finite subgroups do. For instance, if G contains a free abelian subgroup S of infinite rank, G does not have $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology, because S does not have $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology. We will display below a large class of groups, which do have $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology, if all their finite subgroups do. For the proofs, we will make use of Tate cohomology $\hat{H}^*(G,-)$ for arbitrary groups G, as defined in [9]. In case G admits a finite-dimensional classifying space for proper actions $\underline{E}G$, there is a finitely convergent stabilizer spectral sequence

$$E_1^{m,n} = \prod_{\sigma\in\Sigma_m} \hat{H}^n(G_\sigma, M) \Longrightarrow \hat{H}^{m+n}(G,M),$$

where Σ_m is a set of representatives of m-cells of $\underline{E}G$ and M a $\mathbb{Z}G$-module. For G a group, M a $\mathbb{Z}G$-module and $\mathcal{F}$ the set of finite subgroups of G, we write

$$\mathcal{H}^q(G,M) \subset \prod_{H\in\mathcal{F}} \hat{H}^q(H,M)$$

for the set of *compatible* families $(u_H)_{H\in\mathcal{F}}$ with respect to restriction maps of finite subgroups of G, induced by embeddings given by conjugation by elements of G.

There are many results on groups G which imply the existence of a finite-dimensional $\underline{E}G$. For instance, groups of cohomological dimension 1 over $\mathbb{Q}$ do: they act on a tree with finite stabilizers. Also, if there is a short exact sequence $H \to G \to Q$ of groups and H as well as Q admit a finite-dimensional $\underline{E}$ and there is a bound on the order of the finite subgroups of Q, then there exists a finite-dimensional model for $\underline{E}G$ (cp. Lück [7, Thm. 3.1]).

Lemma 7.1. *Suppose G admits a finite-dimensional $\underline{E}G$. Then the following holds.*

(i) *The natural map induced by restricting to finite subgroups*

$$\rho : \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z}) \to \mathcal{H}^*(G,\mathbb{Z}/p\mathbb{Z})$$

has the property that every element in the kernel of ρ is nilpotent, and that for every $u \in \mathcal{H}^*(G,\mathbb{Z}/p\mathbb{Z})$ there is a k such that u^{p^k} lies in the image of ρ.

(ii) *If $\dim \underline{E}G = t$ and the order of every finite p-subgroup of G divides p^s, then for every $\mathbb{Z}_{(p)}G$-module M and all i, we have*

$$p^{s(t+1)} \cdot \hat{H}^i(G,M) = 0.$$

(iii) *If there is a bound on the order of the finite p-subgroups of G, then the natural map*

$$\alpha : \hat{H}^*(G,\mathbb{Z}_{(p)}) \to \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z})$$

has the property that every element in the kernel of α is nilpotent and for any $u \in \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z})$ there exists k such that u^{p^k} lies in the image of α.

Proof. Statement (i) is [10, Cor. 3.3]. For (ii) we observe that for every $\mathbb{Z}_{(p)}G$-module M, the E_1-term of the stabilizer spectral sequence is annihilated by p^s. Since $\underline{E}G$ has dimension t, this implies that $p^{s(t+1)}$ annihilates all groups $\hat{H}^*(G,M)$. For (iii) we first use (ii) to conclude that $p^{s(t+1)}$ annihilates the groups $\hat{H}^*(G,\mathbb{Z}_{(p)})$. One then argues as in the proof of [4, Chap. X, Lem. 6.6] that for any $\ell > 0$ and $x \in \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z})$, x^{p^ℓ} lies in the image I_ℓ of

$$\hat{H}^*(G,\mathbb{Z}/p^{\ell+1}\mathbb{Z}) \to \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z}),$$

and that for ℓ large enough, I_ℓ equals the image of the natural map

$$\alpha : \hat{H}^*(G,\mathbb{Z}_{(p)}) \to \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z}),$$

implying one part of (iii). If y lies in the kernel of α, the long exact coefficient sequence associated with the short exact sequence

$$\mathbb{Z}_{(p)} \xrightarrow{p} \mathbb{Z}_{(p)} \to \mathbb{Z}/p\mathbb{Z}$$

shows that $y = pz$ for some z and therefore $y^{s(t+1)} = p^{s(t+1)}z^{s(t+1)} = 0$, finishing the proof of (iii). □

Theorem 7.2. *Let G be a group which admits a finite-dimensional $\underline{E}G$. Then the following holds.*

(a) *G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology if and only if all its finite subgroups do.*
(b) *G has p-periodic cohomology if all its finite subgroups do and there is a bound on the order of the finite p-subgroups of G.*

Proof. (a): If G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology and $e \in H^n(G,\mathbb{Z}/p\mathbb{Z})$ is a periodicity generator, then every subgroup $H < G$ has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology, with periodicity generator the restriction $e_H \in H^n(H,\mathbb{Z}/p\mathbb{Z})$. This follows from the natural isomorphism

$$H^*(H,M) \cong H^*(G,\operatorname{Coind}_H^G M)$$

(Shapiro Lemma). If all finite subgroups of G have $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology, there exists a unit $u \in \hat{H}^n(G,\mathbb{Z}/p\mathbb{Z})$ for some $n > 0$ (cp. [10, Thm. 4.4]). Since $\dim \underline{E}G$ is finite, there is a $k > 0$ such that the natural map $\theta : H^j(G,M) \to \hat{H}^j(G,M)$ is an isomorphism for all $j \geq k$ and all $\mathbb{Z}G$-modules M. Choose ℓ such that the degree of u^ℓ is larger than k and choose $e \in H^{n\ell}(G,\mathbb{Z}/p\mathbb{Z})$ such that $\theta(e) = u^\ell$. Then G has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology with periodicity generator e, finishing the proof of (a).

(b): We assume that all finite subgroups of G have p-periodic cohomology and that there is a bound on the order of the finite p-subgroups of G. From [10, Thm. 4.4] we conclude that there exists a unit $u \in \hat{H}^n(G,\mathbb{Z}/p\mathbb{Z})$ for some $n > 0$. Let $v = u^{-1}$. By Lemma 7.1 we can find $k > 0$ and $\tilde{u},\tilde{v} \in \hat{H}^*(G,\mathbb{Z}_{(p)})$ such that

$$\alpha(\tilde{u}) = u^{p^k} \quad \text{and} \quad \alpha(\tilde{v}) = v^{p^k},$$

where $\alpha : \hat{H}^*(G,\mathbb{Z}_{(p)}) \to \hat{H}^*(G,\mathbb{Z}/p\mathbb{Z})$ is the natural map. From Lemma 7.1 we conclude that $1 - \tilde{u}\tilde{v}$ is nilpotent, thus $\tilde{u}\tilde{v}$ is invertible, and we conclude that $\tilde{u} \in \hat{H}^{np^k}(G,\mathbb{Z}_{(p)})$ is a unit. Since G admits a finite-dimensional $\underline{E}G$, the supremum of the injective length of projective $\mathbb{Z}G$-modules, $\operatorname{silp}\mathbb{Z}G$, is finite. Therefore, there is an n_0 such that $H^n(G,P)=0$ for all $n>n_0$ and all projective $\mathbb{Z}G$-modules P. By a basic property of Tate cohomology, this implies that there exists $m>0$ such that the canonical map $\lambda : H^i(G,L)\to \hat{H}^i(G,L)$ is an isomorphism for all $i>m$ and all $\mathbb{Z}G$-modules L. By choosing an $r>0$ such that $\tilde{u}^r$ has degree larger than m, it follows that there is an $\epsilon\in H^{np^kr}(G,\mathbb{Z}_{(p)})$ with $\lambda(\epsilon)=\tilde{u}^r$. Let

$$\beta : H^{np^kr}(G,\mathbb{Z}_{(p)})\to H^{np^kr}(G,\hat{Z}_p)$$

be the canonical map and put $e=\beta(\epsilon)$. Then the cup product with e induces isomorphisms

$$e\cup - : H^j(G,M)\to H^{j+np^kr}(G,\hat{Z}_p\otimes M)=H^{j+np^kr}(G,M)$$

for all $j>m$ and all p-torsion $\mathbb{Z}G$-modules M of bounded exponent, proving that G has p-periodic cohomology. □

Note that we made use of the bound condition in (b) of Theorem 7.2 to prove the result, but that bound is not a necessary condition. For instance, the Prüfer group $\mathbb{Z}_{p^\infty} := \mathbb{Q}/\mathbb{Z}_{(p)}$ has p-periodic cohomology, but no bound on the order of its finite p-subgroups. On the other hand, the following is an example of a group G which admits a finite-dimensional $\underline{EG}$ and with all finite subgroups having p-periodic cohomology, but with no p-periodic cohomology. Let $\alpha\in\hat{Z}_p^\times$ be a p-adic unit and define $G(\alpha)$ to be the semi-direct product $\mathbb{Z}_{p^\infty}\rtimes_\alpha\mathbb{Z}$, where we have identified $\operatorname{Aut}(\mathbb{Z}_{p^\infty})$ with $\hat{Z}_p^\times$.

Example 7.3. Let p be an odd prime and put $G(1+p)=\mathbb{Z}_{p^\infty}\rtimes_{1+p}\mathbb{Z}$.

(a) $G(1+p)$ has $\mathbb{Z}/p\mathbb{Z}$-periodic cohomology of period 2.
(b) $G(1+p)$ does not have p-periodic cohomology.
(c) $G(1+p)$ has twisted p-periodic cohomology.
(d) $G(1+p)$ acts freely on a simply connected 7-dimensional $G(1+p)$-CW-complex which is a $\mathbb{Z}/p\mathbb{Z}$-homology 3-sphere.

Proof. (a): Let $\mathbb{Z}$ act on $\mathbb{Q}$ via $\phi:\mathbb{Z}\to\operatorname{Aut}(\mathbb{Q})$ defined by $\phi(n)q=(1+p)^nq$, $q\in\mathbb{Q}$. Form the semi-direct product $H=\mathbb{Q}\rtimes_\phi\mathbb{Z}$. There is a natural surjective map $H\to G(1+p)$ with kernel isomorphic to $\mathbb{Z}_{(p)}$. Note that H has cohomological dimension 3. Choose Y to be a 3-dimensional model for $K(H,1)$ and X the covering space corresponding to $\mathbb{Z}_{(p)}<H$. X is a free $G(1+p)$-CW-complex and $X\simeq K(\mathbb{Z}_{(p)},1)$, thus X is a $\mathbb{Z}/p\mathbb{Z}$-homology 1-sphere. We then have homotopy fibration

$$X\to X/G(1+p)\to BG(1+p),\quad X_{\mathbb{Z}/p\mathbb{Z}}=S^1_{\mathbb{Z}/p\mathbb{Z}}$$

which is $H\mathbb{Z}/p\mathbb{Z}$-orientable, because multiplication by $1+p$ is the identity on $\mathbb{Z}/p\mathbb{Z}$. It follows that the associated $\mathbb{Z}/p\mathbb{Z}$-Euler class $e\in H^2(G(1+p),\mathbb{Z}/p\mathbb{Z})$ induces, via the cup product, isomorphisms

$$H^i(G(1+p),M)\xrightarrow{e\cup -}H^{i+2}(G(1+p),M)$$

for all $i>3$ and all $\mathbb{Z}/p\mathbb{Z}[G]$-modules M, which proves (a).

(b): We consider the subgroups

$$G_n = \mathbb{Z}/p^n\mathbb{Z} \rtimes_{1+p} \mathbb{Z} < G(1+p)$$

and observe that the minimal p-period for $H^*(G_n, \mathbb{Z}/p^n\mathbb{Z})$ is at least $2p^{n-1}$, because multiplication by $1+p$ on $H^2(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{Z}/p^n\mathbb{Z}) = \mathbb{Z}/p^n\mathbb{Z}$ is an automorphism of order p^{n-1} for odd p. Thus, the minimal p-period for G_n goes to ∞ as n tends to ∞. Therefore, $G(1+p)$ does not have p-periodic cohomology.

(c): We observe that the twisted $\hat{Z}_p$-Euler class $\tilde{e} \in H^2(G(1+p), \hat{Z}_p(\omega))$ of the homotopy $\mathbb{Z}/p\mathbb{Z}$-spherical fibration constructed in (a), with $\omega : G(1+p) \to \hat{Z}_p^\times$ given by $(x, y) \mapsto (1+p)^y$ for $(x, y) \in \mathbb{Z}_{p^\infty} \rtimes \mathbb{Z}$, has reduction mod p equal to the $\mathbb{Z}/p\mathbb{Z}$-Euler class e of (a). It follows that $G(1+p)$ has twisted p-periodic cohomology of period 2, with twisted p-periodicity induced by the cup product with $\tilde{e}$.

(d): We again look at the free $G(1+p)$-CW-complex X as constructed in (a). The join $X * X$ is a simply connected free $G(1+p)$-CW-complex of dimension 7, which is a $\mathbb{Z}/p\mathbb{Z}$-homology 3-sphere, completing the proof. □

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