The Topology of a Bubble in dry Foam

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Abstract: The topological structure of such a bubble requires two necessary conditions. Firstly, Euler's relation between the numbers of its (surface) areas, its (Plateau) borders, and its nodal points (vertices) must be fulfilled. Secondly, always three borders on its surface meet at each nodal point. These conditions allow certain sets of areas with different numbers of borders. However, only when those areas can form a correct net of borders connected by nodal points at a sphere then a bubble topology for such a set has been realized. With forced T1 and inverse T2 processes and by the use of computer programs real topologies have been obtained for bubbles with area numbers up to 16. Their construction out of the computer data is illustrated. A special classification scheme among bubbles of equal area number as well as the appearance of topological isomers are discussed.

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1.Introduction

Foam appears in various formations. Always it is a liquid or solid component that is homogeneously mixed with gas bubbles of different shapes and seizes. Everybody knows the foam of beer or of a soap solution. A recent presentation of the physics of foams is given in the book of Weaire and Hutzler [1]. They also report about topological changes in a fresh foam caused by so called T1 processes that aspire to a minimum of its free energy at unaltered number of bubbles. In a later phase of aging so called T2 processes also cause topological changes. Smaller bubbles have higher pressure than their neighbors. So they loose their gas by diffusion through their skin to the neighbors until they disappear. This article deals with a quasi stable state of the foam before onset of T2 processes and the still later following rupture of single soap films.

1.1 A bubble in dry foam

The liquid content of foam can be so high, that each bubble is shaped like a sphere without contacting other bubbles. By withdrawing more and more soap solution the bubbles touch each other and finally soap films form the borders between neighboring bubbles. Following Plateau's rules [2] there are always three films that meet under equal angles of 120 degrees along a line, named Plateau border. And always exactly four Plateau borders meet at one nodal point and form a tetrahedral vertex with mutual angles of about 109 degrees. We speak of dry foam, if the nodal points can be treated as mathematical points, the Plateau borders as lines, and the soap films as areas in space. Let us exclude external forces (gravitation etc.). Then the mean curvature of such an area is constant and proportional to the pressure difference of the neighbor bubbles of that area (Laplace-Young equation [3, 4]). The surface of a stable bubble in dry foam consists of a certain number of such areas, and each of these areas is bordered by a certain number of lines and the same number of nodal points.

1.2 Topology of a bubble

We presume, that a stable bubble is only simply-connected to each of its neighbor bubbles. Otherwise both bubbles would reach a more stable state by performing T1 processes. The same would hold for a bubble that touches itself. For the topological structure of the bubble the shape of its areas is irrelevant. The single elements of the bubble may be bent, turned, stretched, or shifted, as long as a continuous one-to-one correlation of all its points to those of the original positions is conserved. So, the volume and the pressure of the bubble, as well as the temperature do not matter. Also the Plateau angles may be changed.

1.3 Topological equivalence of 3D foam bubbles and 2D bubble clusters

In 3D foam each (nodal) point connects four lines (Plateau borders). On a bubble three of these lines run at its surface and the forth line is directed outwards. In order to describe the topology of that bubble, we do not need this forth line. This net of lines knotted by points already represents the topology of this bubble. It remains unchanged, if this net covers the surface of a sphere. Another representation of the topology is obtained, when the net is spread across a plane (according to a sphere of infinite radius). When no line goes to infinity, then one of the areas covers the infinitely far point of the plane. It quasi surrounds the other areas of the bubble. This 2D presentation of the topology of a bubble shall be called its graph.

Often one studies a 2D foam instead of the 3D foam, because it is easier to collect the data. A dry foam between parallel glass plates provides a network of soap films normal to the two plates when their distance is sufficiently small. In the limit of a true 2D foam there remain lines, instead of the films, and nodal points, instead of the Plateau borders. Each line has the shape of a circular segment (Laplace-Young equation). Exactly three lines end in each nodal point. By this way the whole plane is divided into areas each having a certain number of sides and the same number of nodal points. We have a 2D bubble cluster, when the 2D foam covers a finite range of the plane. Its topological structure can easily be seen. We complete the 2D bubble cluster by the external part of the plane, to recognize the equivalence we have looked for: Each 2D bubble cluster is equivalent to a 3D foam bubble with the same topology.

2. Definitions

2.1 Graphs and dual Graphs

In Fig.1(above) the graphs for the topologies of the simplest bubbles are shown. These are all the possible topologies one gets for bubbles with four to six areas. For later usage the areas of each graph are numbered. With more areas the amount of different topologies will rise very fast.

Besides the graphs a dual presentation of the topology can be useful (Fig.1 below). Instead of a nodal point in the graph the dual graph has a dual area. Instead of an area in the graph the dual graph has a dual nodal point (here extended to a circle). The line between neighboring areas in the graph corresponds to the connecting line between dual nodal points. As a consequence the dual plane is divided into triangular areas only. A dual nodal point joins as many lines as the respective area has sides. Here the area numbers in the circles cover the dual nodal points. The infinite nodal point is marked by the external arms, all having the same number.

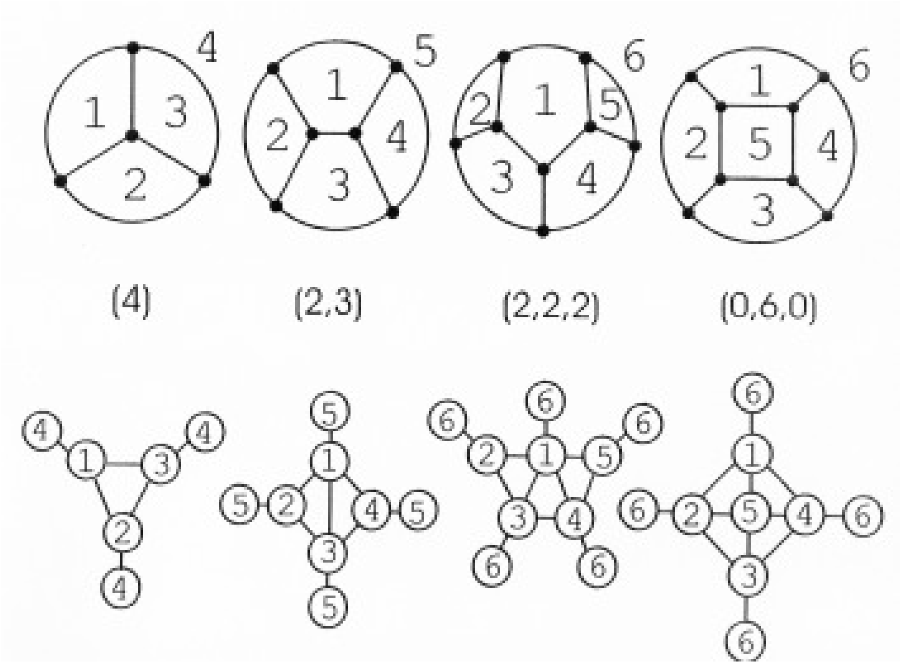


Fig.1 Above: Graphs for the topology of simple bubbles.
Below: Dual graphs for the topology of these bubbles.

2.2 Gross formulas

To subdivide bubbles having the same number of areas let us sort these areas according to the number of their sides. For each bubble the number of areas having *n* sides is named $N_2(n)$. (Index 2 means the dimension of the areas). Then we define a gross formula (Gf), similar to the gross formulas used for organic molecules, here in form of a list ordered by the number *n* :

$$\left(N_2(3), N_2(4), \ldots, N_2\left(N_2^0 - 1\right)\right) \tag{1}$$

All gross formulas begin at *n*=3 and end at an *n*, that is by 1 less than the total number N_2^0 of the areas of a bubble. The reasons for that are given in section 4.1. In Fig.1 the gross formulas are added to the graphs. As we know from organic chemistry also foam bubbles with the same gross formula can have isomer topologies. To construct the graph from the gross formula alone is difficult at larger N_2^0. Then more specifications are required.

2.3 Relations between the numbers of areas, lines, and nodal points

The total number of areas of a bubble is N_2^0. We define N_1^0 as the number of its lines and N_0^0 the number of its points. Then Euler's relation [5] says:

$$N_0^0 - N_1^0 + N_2^0 = 2 \tag{2}$$

With those parts of Plateau's rules being also valid for the topology, we have:

$$3N_0^0 = 2N_1^0 \tag{3}$$

The reason for that is: When 3 lines end at each nodal point, then with $3N_0^0$ each line is counted twice. With eq.(2) we extend eq.(3) to

$$3N_0^0 = 2N_1^0 = 6\left(N_2^0 - 2\right) \tag{4}$$

N_2^0 results from the gross formula by summing up all the area numbers:

$$N_2^0 = \sum_n N_2(n) \tag{5}$$

Multiply each item of the sum of eq.(5) by *n* before summing up and you get

$$N_2^1 = \sum_n nN_2(n) \tag{6}$$

This, however, is two times the number of sides N_1^0, because each border belongs to two areas and is counted twice. So we finally we get:

$$3N_0^0 = 2N_1^0 = N_2^1 = 6\left(N_2^0 - 2\right) \tag{7}$$

3. Lists of gross formulas

By definition the gross formula (Gf) of a bubble with N_2^0 areas consists of $\left(N_2^0 - 3\right)$ integers between null and N_2^0 . Eqs.(5) to (7) provide necessary conditions, that must be fulfilled for each Gf. We shall see that the number of these Gf grows nearly exponentially with N_2^0. Therefore, a search for all Gf that fulfill the above conditions requires computer help. All computer programs used in this paper have been written with Mathematica 2.0.

3.1 Program for lists of Gf that fulfill eqs.(5) to (7)

A program, deposited in Appendix A, will help to find all those Gf of N_2^0, that satisfy eqs.(5) to (7). For $N_2^0 = 4$ and $N_2^0 = 5$ one gets only one Gf (see Fig.1). For $N_2^0 = 6$ you find four Gf, for $N_2^0 = 7$ they are ten. For $N_2^0 = 13$ there are already 1498 different Gf.

Increasing N_2^0 beyond 13 provided problems concerning storage capacity and computing time. However, in Appendix B an other program is presented, that allowed to go up to $N_2^0 = 18$. Here you have already 33.327 different Gf. Bubbles with 18 surface areas are very rare in bulk dry foam. After Matzke [6] the experimental average of N_2^0 is 13.70.

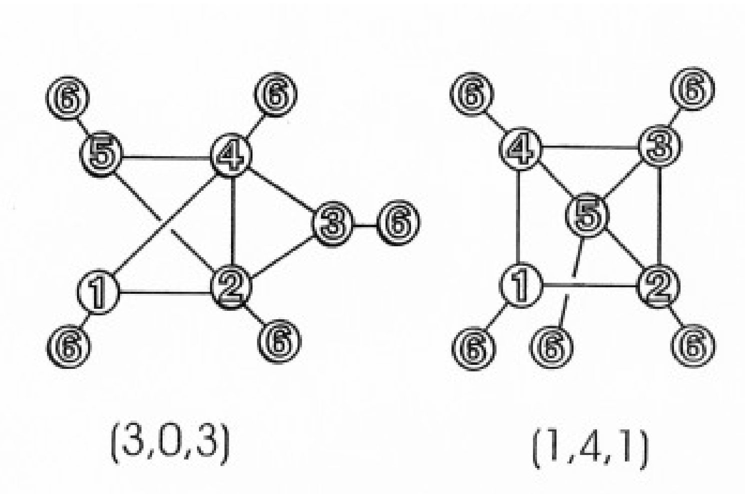


Fig.2 Two examples for imaginary gross formulas (iGf): The dual graphs show forbidden crossing lines for the neighborhood of areas of a 2D cluster. Area No.6 contains the infinitely distant point of the plane.

3.2 Real and imaginary gross formulas

Our main goal are those Gf, for which a plane graph provides the topology of a bubble. Then we call it a real Gf (rGf). The conditions in eqs.(5) to (7) are necessary, but not sufficient for that. Let us consider the simple examples of $N_2^0 = 6$. In addition to the two cases in Fig.1 there are also (1,4,1) and (3,0,3). For them it is impossible to gain a plane graph. The tentative dual graphs presented in Fig.2 always have inadmissible crossings of the connecting lines. We call those cases imaginary Gf (iGf). For $N_2^0 = 8$ in Tab.1 you get a list of all iGf and of all real gross formulas (rGf). Moreover the values of N_2^2 are given (see eq.(B1) in Appendix B). For the following we mainly consider the rGf.

iGf	N_2^2	rGf	N_2^2
(5,0,0,0,3)	192	(4,0,0,4,0)	180
(4,1,0,1,2)	186	(3,1,2,1,1)	178
(4,0,2,0,2)	184	(3,1,1,3,0)	176
(4,0,1,2,1)	182	(2,4,0,0,2)	180
(3,2,1,0,2)	182	(2,3,1,1,1)	176
(3,2,0,2,1)	180	(2,2,3,0,1)	174
(3,0,4,0,1)	176	(2,2,2,2,0)	172
(3,0,3,2,0)	174	(2,1,4,1,0)	170
(2,3,0,3,0)	174	(2,0,6,0,0)	168
(1,5,0,1,1)	174	(1,4,1,2,0)	170
(1,4,2,0,1)	172	(1,3,3,1,0)	168
(1,2,5,0,0)	166	(0,6,0,2,0)	168
(0,6,1,0,1)	170	(0,4,4,0,0)	164
(0,5,2,1,0)	166		

Tab.1. List of all imaginary gross formulas (iGf) and real gross formulas (rGf) of bubbles having 8 areas ($N_2^0 = 8$) with data for N_2^2 (see text).

4. Construktive ways to real gross formulas

In order to sort out the rGf together with their graph, we have two constructive ways. Either we add a new area to a graph already known. This is the inverse of a 2-dimensional (2D) T2 process. Or we perform a 2D T1 process on a plane graph and gain new rGf with the same number of areas. Both ways shall now be used.

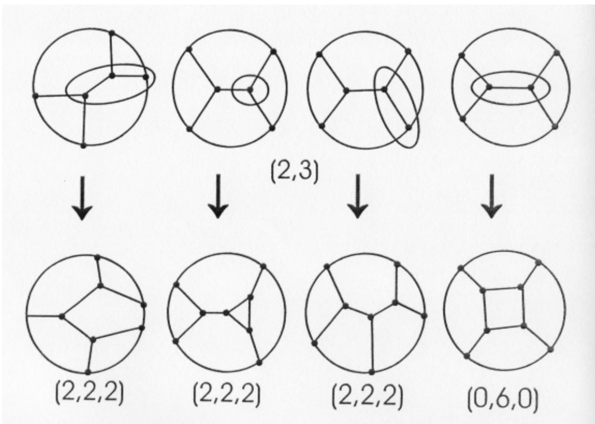
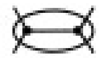
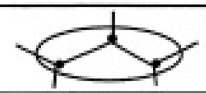
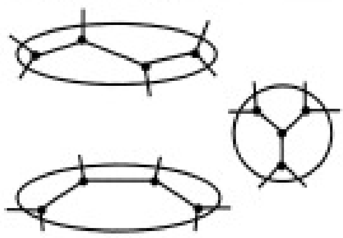
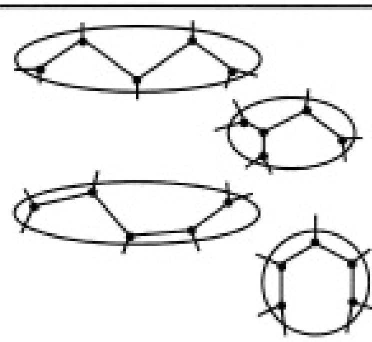


Fig.3 The inverse T2 process, performed on a simple 2D cluster, increases the number of areas by 1.

4.1 Inverse T2 process in 2D

In the inverse T2 process one increases the number of areas by one. Fig.3 shows different variants on the graph for $N_2^0 = 5$. You place a simply closed line (loop) around one, two, or more nodal points (maximal $N_0^0 / 2$). Hereby no area of the graph must be entered more than once. Otherwise two areas would have contact at more than one side afterwards. Also, the number of intersection points with the borders must be two more than the number of nodal points within the loop (maximal N_2^0). Otherwise at least one area of the existing graph would be within the loop. If now all the internal part of the loop is erased, we get a new area with nodal points at the former intersection points. Again the maximum number of sides of the resulting graph remains by one less than the new N_2^0. When N_2^0 increases by 1 then N_1^0 increases by 3 and N_0^0 by 2 (see eq.(7)). Tab.2 collects the variants for including a new n-cornered area. Each list contains (in a cyclic way) by how much the neighbors increase the number of their corners. Their sum always equals $(6-n)$.

n	loop around (n-2) nodes	$(\ldots,\Delta_k,\ldots)$
3		(1,1,1)
4		(1,0,1,0)
5		(1,0,0,1,-1)
6		(1,0,-1,1,0,-1) (1,-1,1,-1,1,-1) (1,0,0,0,1,-2)
7		(1,0,-1,0,1,-1,-1) (1,-1,0,1,-2,1,-1) (1,0,-2,1,0,0,-1) (1,0,0,0,0,1,-3)

Tab.2. Variants for including a new area by applying the inverse (2D) T2 process.

Why are two areas not allowed to have contact at more than one side? This question concerns the stability of a bubble respectively of an equivalent 2D cluster. In Fig. 4 (above) two areas (No.1 and No.2) within a 2D cluster have contact at two borders. Both are circular segments with equal curvature and equal center because of the same pressure difference between both areas. This also holds when more than two areas are in the middle part. A simple proof can be found by using the relations in [7]. Fig. 4 is drawn for equal pressure on No.1

and No.2. In any case we only get an indifferent equilibrium: Without changing the free energy the whole middle part may be shifted until one of the two borders between No.1 and No.2 disappears. Then by a T1 process (see Section 4.2) the free energy is reduced. When this more stable state is reached then the two areas have contact at only one border.

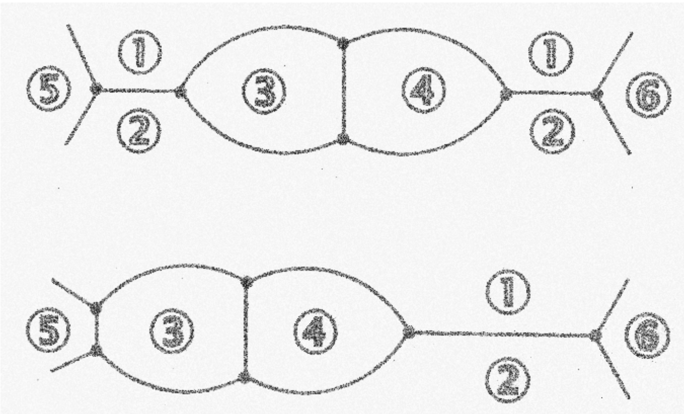


Fig.4 In a stable 2D cluster two areas must not have more than one common border. The position of areas 3 and 4 is energetically indifferent (above). It will become stable after the T1 process (below).

Therefore, after the 2D cluster has reached a local minimum of its free energy, also two-sided areas have disappeared. We conclude that a stable 2D cluster must not have areas with less than three sides. (The only exception would be a 2D two-bubble “cluster“ with three 2-cornered areas ($N_2^0 = 3$)). We may further conclude: A stable 2D cluster must not have areas with more than $N_2^0 - 1$ sides. An area with N_2^0 sides needs N_2^0 different neighbors. That gives an inconsistancy.

For 3D foam bubbles it is not so easy, to generally exclude the existence of areas with only two sides. One could think of a bubble consisting of three 2-cornered areas, that is shifted along a Plateau border. Whether without change of energy, that is the question. Because of the existence of non-spherical soap skins in 3D foam a proof is here more difficult. It would already be helpful to know if all Plateau borders are of circular shape. Until one offers a concrete example for a 2-cornered area in stable 3D foam we will exclude this possibility for now.

The construction of a graph of certain topology does not require that the way used takes place in reality. That opens an alternative procedure to form graphs with one more area. Consider the graph of a bubble with N_2^0 areas and choose one area with n sides. From one of its sides you draw a line to one of the $(n-1)$ other sides. That cuts this area into two parts. So you have formed a new graph. The number of its nodal points increases by 2, the number of its sides by 3. From the n-cornered area we get (with $k = 1,\dots,n-1$) a $(k+2)$-cornered and a $(n+2-k)$-cornered area. And two neighboring areas get one more nodal point. The number of different cuts $s(n)$ on this area is $s(n) = n(n-1)/2$.

The total number S of possible cuts made on all areas of the bubble is

$$S = \sum_{n=3}^{N_2^0-1} N_2(n)s(n) = \tfrac{1}{2}\left(N_2^2 - N_2^1\right), \tag{8}$$

where (as defined in (B.1)) we have

$$N_2^2 = \sum_{n=3}^{N_2^0-1} n^2 N_2(n) \quad . \tag{9}$$

4.2 T1 process in 2D

In Fig. 5 we present this topological change within a graph and its dual graph. Hereby the number of areas of the bubble remains unchanged. Two neighboring nodal points are forced to come into a transient contact and then separate again into an other direction. Thereafter the areas No.2 and 4 have one side more. That excludes areas with $\left(N_2^0-1\right)$ corners before. The areas No.1 and 3 loose one side. That excludes 3-cornered areas before.

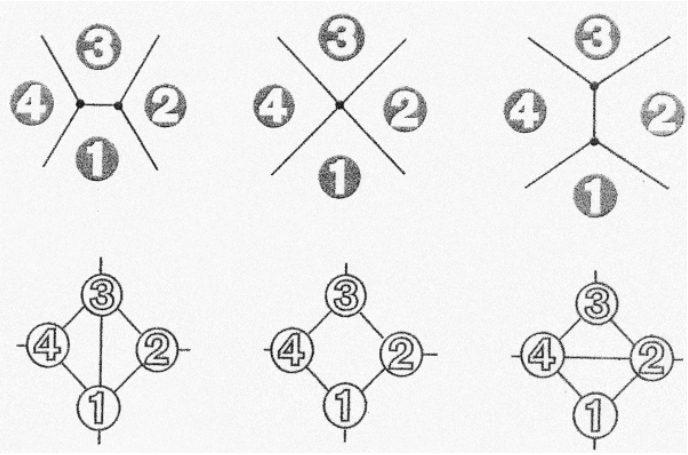


Fig.5 The T1 process: Local topological change within a graph (above) and its dual graph (below).

4.3 Isomer topologies of the same gross formula

Topological isomers exist, when for a certain rGf the areas allow different connections. In Fig. 6 we bring three examples. The first example shows for $N_2^0 = 7$ two graphs forming a topological mirror symmetry. It remains a question of definition, whether they can be seen as topological isomers. We will deny this here. The second example belongs to $N_2^0 = 8$. The same rGf allows two true (structural) isomers. The last example provides three structural isomers of the same rGf. We start from the graph of a bubble containing twelve 5-cornered areas with the rGf (0,0,12,0,0,0,0,0,0), presented three times in Fig.6. Then we include by the inverse T2 process two 3-cornered areas at separate places (marked by larger nodal points) having different distances. The rGf for all three isomers is (2,0,6,6,0,0,0,0,0,0,0,0). Only when the distance of both triangles is too small, then another rGf results.

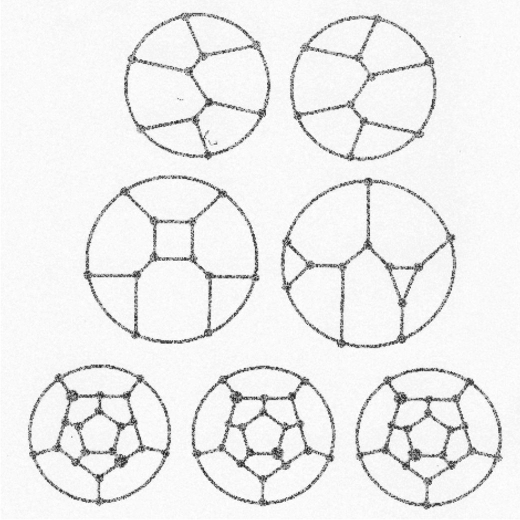


Fig.6 Examples of isomer graphs. Middle: Typical (structure) isomer. Above: Mirror symmetry, no (structure) isomer. Below: Three isomer graphs derived from the dodecahedron by adding two triangular via T2 process at the two marked nodal points.

4.4 Homologeous series for 3D-bubbles

Similar to the usage in the organic chemistry one finds a whole series of graphs by repeatedly adding the same area (or group of areas). So one easily reaches graphs with larger N_2^0. We choose two examples, that may help us when starting computer programs. Their graphs shown in Fig.7 (above and below) follow from a repeated inverse T2-process that successively adds a 4-cornered area. Two areas increase their *n* by 1. For $N_2^0 > 6$ the single $N_2(n)$ are $N_2(4) = N_2^0 - 2;$ and $N_2(N_2^0 - 2) = 2;$ respectively, $N_2(3) = 2;$ $N_2(4) = N_2^0 - 4;$ and $N_2(N_2^0 - 1) = 2$. All other $N_2(n)$ are zero. From here T1 processes lead to other graphs with equal N_2^0.

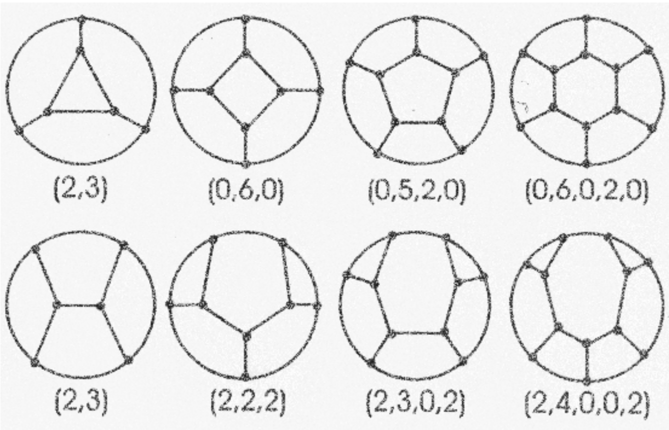


Fig.7 Two series of graphs with their gross formulas.

5. Programs for real gross formulas and their nodal lists

The numbering of the areas for a graph has already been proved true in describing the figures. They now render indispensable for developing the computer programs, that change a bubble or its graph the way wanted. The rGf of the bubble alone will not disclose its topology. The information required for it shall be found in the so called nodal list.

5.1 The nodal list of the bubble

The nodal list (Noli) provides a straightforward way for constructing the graph of the bubble. The Noli informs about the three neighbor areas of each of the $N_0^0 = 2(N_2^0 - 2)$ nodal points. All areas of the bubble are numbered from 1 to N_2^0. Hereby the succession is unimportant. Then for each nodal point you form a sub list of the numbers of its three neighboring areas. Here the succession of the three numbers defines a sense of rotation. It must be uniform for all nodal points, whereas a cyclic rotation of these three numbers does not matter. As an example we choose the Noli of the simple rGf (2,2,2) (see Fig.1). It could look like that: {{1,6,2},{2,6,3},{3,6,4},{4,6,5},{5,6,1},{1,2,3},{1,3,4},{1,4,5}}. Each number in it appears as often as its area has corners. With this knowledge we easily construct the graph. Favorable, but not mandatory, is an external area with the highest number of corners. This has influence on the image of the graph, but not on the topology of the bubble. Inverting the sense of rotation of all sub lists leads to the mirror image of the graph, but does not change the (structural) topology.

5.2 Programs for inverse T2 processes in 2D

You may systematically try by hand (quasi with paper and pencil) to detect all rGf having one more area out of the complete set of the rGf for N_2^0. However, soon there are limits. From $N_2^0 = 8$ to $N_2^0 = 9$ was the last step done by this way. Then computer programs may help. In Appendix C you find two such programs. They both need a list of Noli belonging to N_2^0 we start with.

The first program generates a 3-cornered area at any allowed nodal point of each Noli. So one gains new rGf with one more area. Each new rGf is added to a list, together with its Noli, in form of a (rGf,Noli) pair. By adding 3-cornered areas, however, we do not obtain the complete set of rGf having 1 more area. Certainly we miss any new rGf with $N_2(3) = 0$.

Therefore, we need at least one other program. We may include a 4-cornered area at two neighboring nodal points (see Tab.2). The second program in Appendix C can do this task. From each Noli it makes a list of pairs of neighboring nodal points. They characterize the $3(N_2^0 - 2)$ lines of the graph. Then each side is replaced by a 4-cornered area and the new (rGf,Noli) pair has conserved the $N_2(3) = 0$ in all rGf. However we now miss any rGf with $N_2(4) = 0$. The inclusion of an area with more than 4 corners as well as the alternative way of cutting an area into two pieces, as depicted before, requires larger programs that are not needed, when we also apply T1 processes.

5.3 Programs for T1 processes

Such a program requires, like the second program in Appendix C, a list of pairs of neighboring nodal points. The programs laid down in Appendix D for the T1 process again deliver for each new rGf a Noli from which its graph can be formed. Two versions are presented. The first version allows to gain new (rGf,Noli) pairs from a single (rGf,Noli) pair and by repeated applying the same program to all new pairs one continues until new pairs are no more found. The second version is used, when a larger number of (rGf,Noli) pairs is already available and one wants to detect the missing rest. To give an example, you find in Tab.3 the list of (rGf,Noli) pairs belonging to $N_2^0 = 8$. It contains all 13 possible rGf, but only one of the two isomer Noli with the same rGf (2,2,2,2), that we know from Fig.6. The search for all isomers of a certain rGf shall be postponed to a later task.

```
{{{0,4,4,0,0},
  {{1,5,6},{1,6,2},{7,4,3},{7,5,4},{7,6,5},{7,2,6},
   {4,5,8},{5,1,8},{4,8,3},{3,8,1},{7,3,2},{2,3,1}}},
 {{0,6,0,2,0},
  {{4,8,3},{5,2,1},{3,5,4},{6,3,7},{7,1,6},{6,1,2},
   {3,8,7},{7,8,1},{5,1,4},{4,1,8},{5,3,2},{2,3,6}}},
 {{1,3,3,1,0},
  {{1,3,4},{1,5,6},{1,6,2},{7,4,3},{7,5,4},{7,6,5},
   {7,2,6},{1,4,8},{4,5,8},{5,1,8},{7,3,2},{2,3,1}}},
 {{1,4,1,2,0},
  {{1,5,6},{1,6,2},{7,4,3},{7,5,4},{7,6,5},{7,2,6},
   {2,7,1},{1,7,3},{4,5,8},{5,1,8},{4,8,3},{3,8,1}}},
 {{2,0,6,0,0},
  {{4,8,3},{6,2,5},{5,2,1},{3,5,4},{5,3,6},{6,3,7},
   {7,1,6},{6,1,2},{3,8,7},{5,1,4},{8,4,7},{7,4,1}}},
 {{2,1,4,1,0},
  {{7,6,5},{7,2,6},{2,7,1},{1,7,3},{4,5,8},{4,8,3},
   {3,8,1},{6,2,5},{7,5,3},{3,5,4},{5,2,8},{8,2,1}}},
 {{2,2,2,2,0},
  {{1,3,4},{1,6,2},{7,4,3},{7,5,4},{7,6,5},{7,2,6},
   {2,7,1},{1,7,3},{1,4,8},{4,5,8},{1,8,6},{6,8,5}}},
 {{2,2,3,0,1},
  {{1,5,6},{1,6,2},{7,5,4},{7,6,5},{7,2,6},{2,7,1},
   {4,5,8},{5,1,8},{4,8,3},{3,8,1},{3,1,4},{4,1,7}}},
 {{2,3,1,1,1},
  {{1,3,4},{1,5,6},{1,6,2},{7,6,5},{7,2,6},{2,7,1},
   {1,7,3},{1,4,8},{4,5,8},{5,1,8},{7,5,3},{3,5,4}}},
 {{2,4,0,0,2},
  {{1,3,4},{1,5,6},{1,6,2},{7,4,3},{7,6,5},{7,2,6},
   {2,7,1},{1,7,3},{1,4,8},{5,1,8},{5,8,7},{7,8,4}}},
 {{3,1,1,3,0},
  {{1,3,4},{7,4,3},{7,5,4},{7,6,5},{7,2,6},{2,7,1},
   {1,7,3},{1,4,8},{4,5,8},{5,1,8},{6,2,5},{5,2,1}}},
 {{3,1,2,1,1},
  {{1,3,4},{1,5,6},{1,6,2},{7,4,3},{7,2,6},{2,7,1},
   {1,7,3},{1,4,8},{4,5,8},{5,1,8},{7,6,4},{4,6,5}}},
 {{4,0,0,4,0},
  {{6,2,5},{5,2,1},{5,3,6},{7,1,6},{6,1,2},{7,8,1},
   {5,1,4},{4,1,8},{3,5,8},{8,5,4},{3,8,6},{6,8,7}}}}
```

Tab.3. List of the 13 (rGf, Noli) pairs of $N_2^0 = 8$ (see text).

5.4 The construction of the graph out of its nodal list

You will learn how to construct the graph, if its Noli is known. In Fig.8 nodal lists (Noli) of two selected examples together with their rGf are shown. They belong to $N_2^0 = 11$ and $N_2^0 = 13$. By counting how often the number of a certain area number appears in the Noli we get its number *n* of borders. In the table below you find the numbers of all areas sorted after *n*. For a control, the rGf tells us how many areas must belong to each *n*. As outer area of the graphs we choose No.3 (or No.8); both have maximum *n*. So we have 6 nodal points at the external border of the graph. Each such point delivers a line going inward to an other nodal point. From the Noli we collect all 6 nodal triplets containing No.3 (or No.8) and sort them in a consecutive order to be transferred into the graph. To each nodal point in the graph for which two area numbers are known the third area number is found in the Noli. Hereby we always have to observe the right sense of rotation.

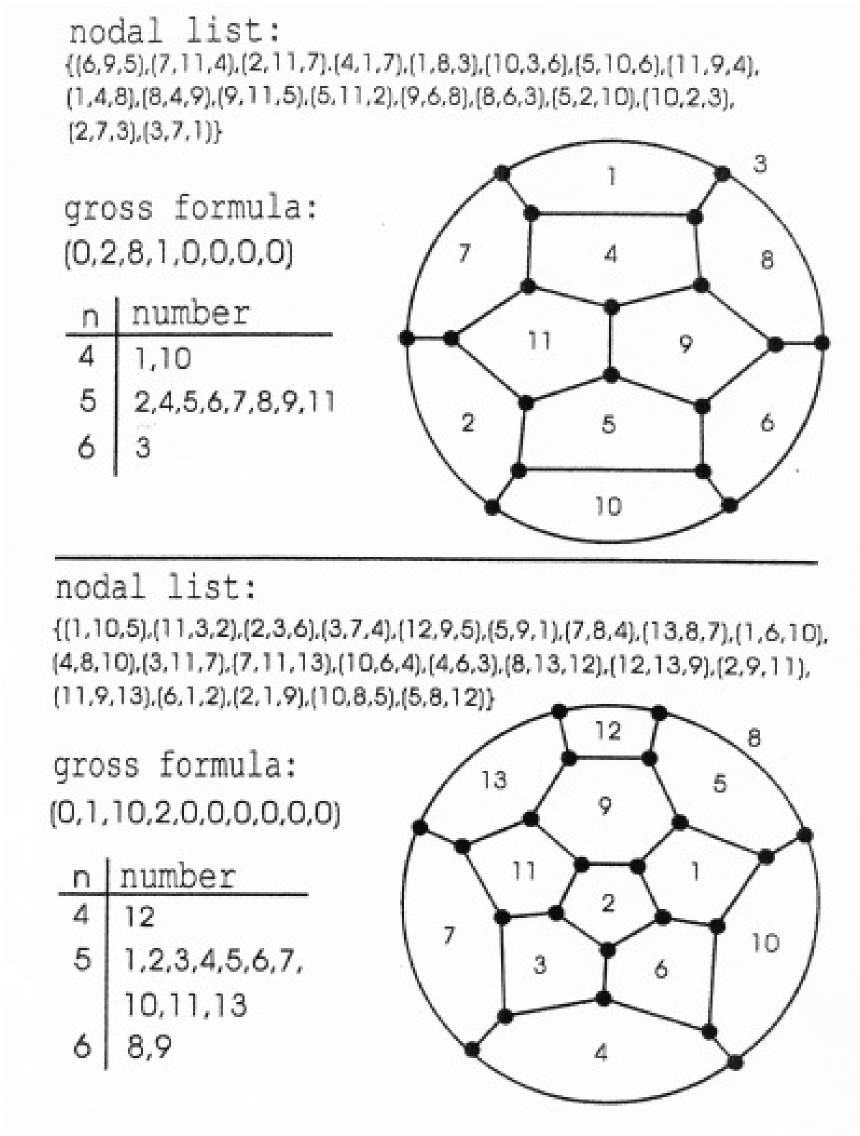


Fig.8 Construction of the graph out of the nodal list of rGf (2 examples).

6. Results and discussions

We begin with Tab.4 by comparing the number Z_{Gf} of all Gf and its part Z_{rGf} of all rGf, that have been found for $N_2^0 = 4,\ldots,16$. For $N_2^0 > 13$ we still may miss a few rGf. In the right column you find the ratio Z_{rGf}/Z_{Gf} . With increasing N_2^0 it runs through a minimum and reaches 0.67 at $N_2^0 = 16$. Where will it converge with growing N_2^0? We only know for sure, that it cannot be larger than 1.

N_2^0	Z_{rGf}	Z_{Gf}	$\frac{Z_{rGf}}{Z_{Gf}}$
4	1	1	1.00
5	1	1	1.00
6	2	4	0.50
7	5	10	0.50
8	13	27	0.48
9	33	66	0.50
10	85	157	0.54
11	199	346	0.58
12	445	738	0.60
13	947	1498	0.63
14	1909	2951	0.65
15	3711	5615	0.66
16	6934	10419	0.67

Tab.4. *Z(rGf)* is the number of all rGf. *Z(Gf) = Z(rGf) + Z(iGf)* is the number of all Gf. N_2^0 runs from 4 to 16.

For $N_2^0 = 4,\dots,16$ we have collected the sorted lists containing the Z_{rGf} sub lists of (rGf,Noli) pairs, the same way as presented for $N_2^0 = 8$ in Tab.3. The vast extent of all data excludes their presentation here. They are useful for various statistical analyses. An example is given in Tab.5. For $N_2^0 = 12$ we have plotted the frequency $H(n,z)$, with which $N_2(n) = z$ appears in the $Z_{rGf} = 445$ (Tab.4) rGf. For a check-up: $\sum_z H(n,z) = 445$ must hold for each *n=3,...,11*. Above the step-like envelope all $H(n,z)$ are zero. At $H(5,12) = 1$ the envelope looks like a resonant peak that we do not find for other N_2^0. This stems from the rGf of the dodecahedron. The $H(11,2) = 1$ points to (2,8,0,0,0,0,0,0,2), described in section 4.4 and Fig.7 (below). It is the only one with $N_2(11) = 2$.

H(n,z)

z=									
12			1						
11			0						
10		1	0						
9		0	0						
8		6	3	1					
7		19	8	1					
6	13	35	18	8	1				
5	58	51	30	12	2				
4	97	62	50	29	13	3			
3	99	67	64	50	30	13			
2	87	75	85	90	74	49	31	8	1
1	56	69	96	120	128	129	110	85	49
0	35	60	90	134	197	251	304	352	395
	n=3	4	5	6	7	8	9	10	11

Tab.5. Table of the rate *H(n,z)* for $N_2^0 = 12$. *H(n,z)* is the numberof times you find z = $N_2(n)$ in all the 445 rGf of $N_2^0 = 12$ (see eq.1 and capt.3.2).

All N_2^1 are divisible by 6, what we know from eq.(7). Together with eq.(6) we derive two tests for all rGf:

1) The sum of all $N_2(n)$ with *n*= (2 mod 1) is divisible by two:

$$\sum_k N_2(2k+1) = 2N; N = \text{int} \tag{10}$$

2) The sum of all $N_2(n)$ with *n*=(3 mod 1) plus two times the sum of all $N_2(n)$ with *n*=(3 mod 2) is divisible by three:

$$\sum_k N_2(3k+1) + 2\sum_k N_2(3k+2) = 3N; N = \text{int} \tag{11}$$

These tests also hold for iGf because they fulfill the same eqs.(6) and (7).

An other relation that uses N_2^1 defines the average number n_{av} of corners (or sides) among the areas of a certain bubble. One gets

$$n_{av} = N_2^1 / N_2^0 = 6\left(N_2^0 - 2\right) / N_2^0 = 6 - 12 / N_2^0 \quad . \tag{12}$$

This formula shows that n_{av} only depends on N_2^0. It runs from 3 at $N_2^0 = 4$, over 4 at $N_2^0 = 6$, over 5 at $N_2^0 = 12$, up to 6 at $N_2^0 = \text{inf}$.

Now we shall discuss the properties of N_2^2. While N_2^1 is constant for all Gf of a certain N_2^0, this does not hold for the N_2^2. They cover a whole range of values, and they all are even. A proof for that is easily done, if we remember the $(+1,-1)$-shift, by which all Gf have been found (see Appendix B). Hereby N_2^2 is reduced by $2(u-1)$, with $u = 2,..., N_2^0 - 4$, depending on the distance *u* between the places to be shifted. For the starting Gf with $N_2(3) = N_2^0 - 3$ and $N_2(N_2^0 - 1) = 3$ (with all other $N_2(n) = 0$) we get $N_2^2 = 3((N_2^0 + 1)N_2^0 - 8)$. That is an even number for all N_2^0. The starting Gf does not have a predecessor for a $(+1,-1)$-shift. Therefore, all N_2^2 are even. Also we may conclude that the odd terms in each Gf always appear in pairs.

As for all Gf with $N_2^2 = \max$, we also have a homologous series for rGf with $N_2^2 = \max$. It has been formulated in section 4.4 and shown in Fig.7 (below). In Tab.6 these rGf with $N_2^2 = \max$ are listed for $N_2^0 = 4,...,16$, together with their value of N_2^2. Except of N_2^0 equal to 4 and 5 the maximal N_2^2 of the rGf is always smaller than that of the Gf. And except of N_2^0 equal to 7 and 8 (**) there is only one rGf with maximum value. To construct the exceptional graphs may be a good exercise.

The right side of Tab.6 contains the rGf with $N_2^2 = \min$ and their values of N_2^2 in the row to the right. Except of N_2^0 equal to 11 and 13 (***) the rGf and Gf for $N_2^2 = \min$ are the same. The examples chosen in Fig.8 show the graphs of these exceptions. Neither a T1 step performed on these graphs nor a (normal or inverse) T2 step performed on the graph of the dodecahedron yields a rGf of the form (0,1,10,0,...) for $N_2^0 = 11$ or (0,0,12,1,0,...) for $N_2^0 = 13$. The reason for these exceptions is the high symmetry of the dodecahedron of $N_2^0 = 12$. Only two other rGf have also areas with unique corner numbers (*). Their topology is equal to that of a cube and a tetrahedron. This list could be completed by a plane honeycomb lattice presenting the graph of a bubble with infinite number of hexagons.

New Noli gained in the programs for T1 and T2 processes cannot be discerned right off as belonging to a graph already known or to an isomer graph. That comes mainly from the freedom of numbering the N_2^0 areas. Therefore, a future program has to find a way to distinguish topological isomerism more easily.

What can we say already now about the occurrence of (structural) isomers? The first example was shown in Fig.6 for $N_2^0 = 8$. Certainly, the more areas with differing numbers *n* of their corners a bubble has, the more alternative graphs are thinkable.

N_2^0	rGf with $N_2^2 = \max$		N_2^2	rGf with $N_2^2 = \min$		N_2^2
4	(4)		36	(4)	*	36
5	(2,3)		66	(2,3)		66
6	(2,2,2)		100	(0,6,0)	*	96
7	(2,3,0,2),(3,0,3,1)	**	138	(0,5,2,0)		130
8	(2,4,0,0,2),(4,0,0,4,0)	**	180	(0,4,4,0,0)		164
9	(2,5,0,0,0,2)		226	(0,3,6,0,0,0)		198
10	(2,6,0,0,0,0,2)		276	(0,2,8,0,0,0,0)		232
11	(2,7,0,0,0,0,0,2)		330	(0,2,8,1,0,0,0,0)	***	268
12	(2,8,0,0,0,0,0,0,2)		388	(0,0,12,0,0,0,0,0,0)	*	300
13	(2,9,0,0,0,0,0,0,0,2)		450	(0,1,10,2,0,0,0,0,0,0)	***	338
14	(2,10,0,0,0,0,0,0,0,0,2)		516	(0,0,12,2,0,0,0,0,0,0,0)		372
15	(2,11,0,0,0,0,0,0,0,0,0,2)		586	(0,0,12,3,0,0,0,0,0,0,0,0)		408
16	(2,12,0,0,0,0,0,0,0,0,0,0,2)		660	(0,0,12,4,0,0,0,0,0,0,0,0,0)		444

Tab.6. Left: rGf for $N_2^2 = \max$. Right: rGf for $N_2^2 = \min$. Both with their respective N_2^2 . The number of bubble areas runs from 4 to 16 (see text).

So let us now confine to the search for graphs that have the limiting N_2^2 given in Tab.6, extended to $N_2^0 > 16$. All graphs of rGf with $N_2^2 = \max$ are devoid of isomers. This also holds for sure with $N_2^2 = \min$ up to $N_2^0 = 15$. With each additional area the number of hexagons steps up by 1, while the number of pentagons remains twelve. You easily find graphs having a symmetrical arrangement of few hexagons. For $N_2^0 = 14$ to $N_2^0 = 16$ they have, in a topological sense, maximum distance from each other. From $N_2^0 = 17$ upwards at least 2 hexagons touch each other. For $N_2^0 = 18$ they form two compact groups of three with maximum distance. For $N_2^0 = 19$ you may get one group of three and one of four, both with a threefold symmetry. $N_2^0 = 20$ possesses an equatorial ring out of six hexagons and one hexagon at each pole, separated by two rings of six pentagons. A graph with the topology of a soccer ball (or a Fullerene molecule out of 60 carbon atoms) also has minimum N_2^2 with 12 pentagons and 20 hexagons. Here the pentagons are already in the minority. Each pentagon is surrounded by five hexagons, while each hexagon is connected to three pentagons. The same dodecahedral symmetry is found in two other series of graphs. With z=0,1,2,... the one has $N_2(6) = 10(2+3z(2+z))$, the other has $N_2(6) = 10(3+z(4+z))$.

For $N_2^0 = 16$ we have the first case of isomerism among those with $N_2^2 = \min$. Alternatively to the case given above, the 4 hexagons form two clusters of two hexagons in maximum distance. With larger N_2^0 and appropriate distribution of the 12 pentagons among the many hexagons more isomers are obtained. In Fig.9 (above) you see the arrangement of two pentagons and two hexagons. With the T1 process you obtain an isomer graph. When finally the pentagons unite into compact clusters these T1 steps come to an end.

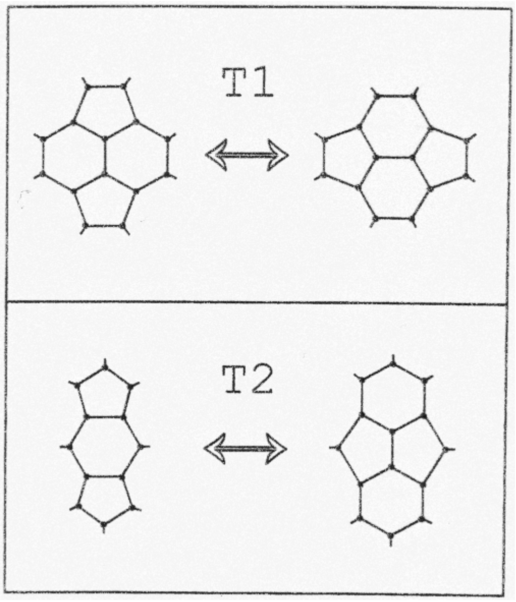


Fig.9 Search for graphs at the limit $N_2^2 = \min$. Above: rGf-invariant T1 step. Below: T2 step and inverse T2 step.

Not all isomers of a rGf with $N_2^2 = \min$ can be reached by such rGf-invariant T1 steps. For instance in the two series of $N_2(6)$ above, when *z>0,* all pentagons are too far apart. Other examples may be described. We divide the (regular spherical 3D) graph of the dodecahedron into two semi-spheres. Then we add two (or more) rings, each consisting of 5 hexagons, between the semi-spheres and close the whole. For large N_2^0 you get long tube-like bubbles that hardly exist in real foams. But certain single wall nanotubes with closed ends, each containing five pentagons, may have such topologies. One can build other tube-like bubbles with more than five hexagons around the tube circuit. Let the tubes consist of two (or more) rings, each containing 6 hexagons. Then the closure is formed by 6 pentagons with one hexagon at each end. Again, the special arrangement, as required in Fig. 9, cannot be found in the given examples. Therefore, no rGf-invariant T1 process can reach these isomers.

Fig.9(below) shows how a T2 step increases (or decreases) the areas by one hexagon under the condition that $N_2^2 = \min$ is conserved. This enables us to construct such graphs with higher N_2^0. We suppose that with T1 and T2 steps, that conserve $N_2^2 = \min$, you can form a graph of any N_2^0 you want.

Finally we formulate the following conjecture: $G(N_2^0)$ be the set of all possible (rGf,Noli) pairs for N_2^0 >5 including each isomer by one (rGf,Noli) pair. Then all (allowed) T1 steps form a unique net connecting all elements of $G(N_2^0)$. This conjecture lives from the fact, that each T1 step is reversible. We know that each element of $G(N_2^0)$ allows at least one T1 step. It remains to proof that there are no two or more separate nets of T1 steps. Then we could derive all elements of $G(N_2^0)$ out of only one graph or its Noli, presumed the isomers of a rGf can easily be distinguished.

Concluding the discussions let me remark that obviously the topology of a bubble tells nothing about its shape, size, or frequency of appearance in a dry foam. However, the results of this study offer some help in the statistical analysis of dry foams. One next could study the various topological forms of a 3D cluster. How many topologies exist for a 3D cluster out of a given number of bubbles? The goal is to solve that puzzle in a systematic way.

Appendix A

The following program (C:\math\asolu08) provides all 27 solutions of real and imaginary gross formulas (Gf) of a 2D cluster having $N_2^0 = 8$ areas.

C:\math\asolu08

```
allsol8={sol={};
m8={{1,1,1,1,1},{3,4,5,6,7},{0,0,1,0,0},{0,0,0,1,0},
{0,0,0,0,1}};
t8=Table[LinearSolve[m8,{8,36,x,y,z}],
{x,0,8},{y,0,8},{z,0,8}];sol={};
For[a=1,a<=8,a++,For[b=1,b<=8,b++,For[c=1,c<=8,c++,
For[k=1;en=0,k<=8-3,k++,
If[t8[[a,b,c,k]]>=0,en++;
If[en==8-3,AppendTo[sol,t8[[a,b,c]]]]]]]]];
sol8=Sort[sol]}
```

In chapter 2 the eqs.(5) and (6) are two linear equations for the $(N_2^0 - 3)$ unknown $N_2(n)$. The square matrix *m8* contains additional rows for $N_2(5) = x$, $N_2(6) = y$ and $N_2(7) = z$, where the parameters *x*, *y*, *z* are integers. The remaining lines are needed, to reject all solutions, that contain negative $N_2(n)$.

The following program for $N_2^0 = 9$ (C:\math\asolu09) shows what has to be changed if we increase (or decrease) the number of areas by 1. So we easily get programs for 7, 6, and for more than 9 areas. The number 36 or 42 within the *LinearSolve*-command is N_2^1, obtained from eqn.(7).

C:\math\asolu09

```
allsol9={sol={};
m9={{1,1,1,1,1,1},{3,4,5,6,7,8},{0,0,1,0,0,0},
{0,0,0,1,0,0},{0,0,0,0,1,0},{0,0,0,0,0,1}};
t9=Table[LinearSolve[m9,{9,42,w,x,y,z}],
{w,0,9},{x,0,9},{y,0,9},{z,0,9}];
For[a=1,a<=9,a++,For[b=1,b<=9,b++,
For[c=1,c<=9,c++,For[d=1,d<=9,d++,
For[k=1;en=0,k<=9-3,k++,
If[t9[[a,b,c,d,k]]>=0,en++;
If[en==9-3,AppendTo[sol,t9[[a,b,c,d]]]]]]]]]];
```

```
sol9=Sort[sol]}
```

Appendix B

There is a faster path to the complete list of all (real and imaginary) gross formulas (Gf). One can show, that from each of these Gf you reach a new one, if you increase the number of corners of one area by 1 and decrease the number of corners of a second area by 1. When the second area had only one corner more than the first, then the Gf remains the same. So it suffices to demand, that the second area has at least two corners more than the first. Such a process shall be called a $(+1,-1)$ shift. Let us begin with a Gf, for which only $N_2(3)$ and $N_2(N_2^0-1)$ are larger than zero. Then with each allowed $(+1,-1)$ shift we reach a new Gf.

The $(+1,-1)$ shift program is
C:\math\bf

```
bru=b;brx={};
For[k=1,k<=Dimensions[b][[1]],k++,a=b[[k]];
                                                       (*k-th gross formula*)
For[j=1,j<=z-2,j++,
For[i=j+2,i<=z,i++,
If[a[[j]]>0&&a[[i]]>0,c=a;
c[[j]]=c[[j]]-1;c[[j+1]]=c[[j+1]]+1;                 (* +1 shift *)
c[[i]]=c[[i]]-1;c[[i-1]]=c[[i-1]]+1;                 (* -1 shift *)
If[c!=a,
AppendTo[brx,c]]]]]];br=Union[brx]];
bri=Union[bru,br];
b=Complement[bri,bru];
bru=bri;all=Union[all,b];          (* all collects all Gf *)
Print["b=",b];                          (* b contains new Gf *)
Print[Dimensions[all]];
                                   (* Repeat "<<bf", til printout b={} *)
```

With eqs.(5) to (7) we derive the starting Gf: $N_2(3)=N_2^0-3$ and $N_2(N_2^0-1)=3$.
The program is started (depending on $z=N_2(3)$) with the following input data:

```
         z=4;b={{4,0,0,3}};all=b;<<bf
 (*or    z=5;b={{5,0,0,0,3}};all=b;<<bf   ,
   or    z=6;b={{6,0,0,0,0,3}};all=b;<<bf    , etc.*)
```

For each N_2^0 the program ends with a Gf, that doesn't allow another $(+1,-1)$ shift. Then either only one of the $N_2(n)$ is different from zero. That holds for $N_2^0=4$ with $N_2(3)=4$, for $N_2^0=6$ with $N_2(4)=6$ and for $N_2^0=12$ with $N_2(5)=12$. Or only two $N_2(n)$ are different from zero and their area number *n* must differ by just 1. For $N_2^0=5$, and $N_2^0=4$, start- and goal-Gf are the same. For $6<N_2^0<12$ we get: $N_2(4)=12-N_2^0$ and $N_2(5)=2N_2^0-12$. For $12<N_2^0$ we get: $N_2(5)=12$ and $N_2(6)=N_2^0-12$.

All Gf of a certain N_2^0 can be classified by another number. We define it as:

$$N_2^2=\sum_n n^2 N_2(n) \tag{B1}$$

We can show, that with each $(+1,-1)$ shift N_2^2 will decrease. From that we conclude: For each $N_2^0 > 5$ there is exact one Gf with maximum N_2^2 and exact one Gf with minimum N_2^2 . If you exchange start and stop of the search for Gf, such $(-1,+1)$ shifts form the same set of Gf, that also is identical to the set of Gf obtained from appendix A. That means, our program *bf* delivers all Gf that fulfill the necessary conditions in Gl.(5) to (7).

Appendix C

C.1 Insertion of a 3-cornered area by the inverse T2 process

For an area number $N_2^0 = z$ we start with a ready list of (rGf,Noli) pairs. The program inserts at one of the 2(z-2) nodes of a Noli a 3-cornered area by the inverse T2-process. The same is done in turn for all nodes of the Noli and for all Noli of the list. The T2 program is

C:\math\t2_3prg

```
ac3p[z_, t_] := {albrutt={}; alknoli={}; rbfknoli={};
For[h = 1, h <= hm, h++,                      (*all hm clusters*)
For[i = 1, i <= 2*(z - 2), i++,
                              (*all 2(z-2) nodes of a cluster*)
ap = Append[t[[h,i]],t[[h,i,1]]];
pa = Partition[ap, 2, 1]; fla = Flatten[pa];
in = Insert[fla, z + 1, {{3}, {5}, {7}}];
j2 = Partition[in, 3];
j1 = Delete[t[[h]], i]; tp = Join[j1, j2]; fl = Flatten[tp];
ze = Table[Count[fl, k], {k, z + 1}];
n2 = Table[Count[ze, k], {k, 3, z}];
AppendTo[albrutt,n2];
AppendTo[alknoli,tp];
AppendTo[rbfknoli,{n2,tp}]]]};
```

To search (rGf,Noli) pairs for $N_2^0 = 9$ use the following command group:

```
z=08; bk=(*include list of (rGf,Noli) pairs of N_2^0=8*);
hm=Length[bk];t=Table[bk[[i,2]],{i,hm}];
<<t2_3prg
ac3p[z,t];
```

The files albrutt and rbfknoli are still unsorted. A sorted list of rGf is built with the command brutto=Union[albrutt]. To each rGf you find in rbfknoli a (rGf,Noli) pair.

C.2 Insertion of a 4-cornered area by the inverse T2 process

The following program consists of two parts. They are laid down under C:\math\t2_4prg and \t2q. In the first part we use one of the known (rGf,Noli) pairs belonging to $N_2^0 = z$. The inverse T2 process inserts a 4-cornered area in turn at each of the 3(z-2) sides. In the second part the same procedure is repeated in turn on each cluster of a list bk of already known (rGf,Noli) pairs.

C:\math\t2_4prg

```
t2a4[z_,t_]:={np={};t1np={};mtt1={};mnt1={};
flat=Flatten[t];tc=Table[Count[flat,k],{k,z}];
For[i=1,i<=2z-5,i++,For[j=i+1,j<=2z-4,j++,        (*start ForFor*)
alpa={t[[i]],t[[j]]};fl=Flatten[alpa];uno=Union[fl];
```

```
If[Dimensions[uno]=={4},AppendTo[np,alpa],Null]]];   (*end ForFor*)
For[k=1,k<=3(z-2),k++,paar=np[[k]];                    (*start For*)
in=Intersection[paar[[1]],paar[[2]]];
un=Union[Flatten[paar]];c=Complement[un,in];
AppendTo[t1np,np[[k]]]];                                  (*end For*)
For[h=1,h<=Dimensions[t1np][[1]],h++,par=t1np[[h]];   (*start For*)
int=Intersection[par[[1]],par[[2]]];po1=Position[t,par[[1]]][[1]];
po2=Position[t,par[[2]]][[1]];j1=Delete[t,{po1,po2}];
c1=Complement[par[[1]],int][[1]];c2=Complement[par[[2]],int][[1]];
p1=Position[par[[1]],c1][[1,1]];p2=Position[par[[2]],c2][[1,1]];
wh={p1,p2};sw=Switch[wh,
{1,1},{1,2,3,5,6,4},{1,2},{1,2,3,6,4,5},{1,3},{1,2,3,4,5,6},
{2,1},{2,3,1,5,6,4},{2,2},{2,3,1,6,4,5},{2,3},{2,3,1,4,5,6},
{3,1},{3,1,2,5,6,4},{3,2},{3,1,2,6,4,5},{3,3},{3,1,2,4,5,6}];
flp=Flatten[par];nt1=Part[flp,sw];pa=Part[nt1,{1,2,2,6,6,3,3,1}];
pati=Partition[pa,2];dpati=Dimensions[pati][[1]];
For[i=1,i<=dpati,i++,f[i]=Join[pati[[i]],{z+1}]];
j2=Table[f[i],{i,dpati}];t0=Join[j1,j2];vec=0;
For[es=1,es<=2z-3,es++,For[te=es+1,te<=2z-2,te++,  (*start ForFor*)
If[Union[t0[[es]]]==Union[t0[[te]]],vec++]]];       (*end ForFor*)
If[vec==0,seli={};
For[r=1,r<=2z-2,r++,                                   (*start For*)
ta={{t0[[r,1]],t0[[r,2]]},
{t0[[r,2]],t0[[r,3]]},{t0[[r,3]],t0[[r,1]]}};
seli=Union[seli,ta]];                                     (*end For*)
If[Dimensions[seli][[1]]==6z-6,tt1=t0;
fla=Flatten[tt1];
zt1=Table[Count[fla,k],{k,z+1}];n2t1=Table[Count[zt1,k],{k,3,z}];
AppendTo[mtt1,tt1];AppendTo[mnt1,n2t1]]]]]}               (*end For*)
```

C:\math\t2q

```
t2real[x_,xmin_,xmax_]:=
{Do[{t=bk[[x,2]];
t2a4[z,t];
bralt=brutto;brutto=Union[mnt1,bralt];
If[Dimensions[brutto][[1]]>Dimensions[bralt][[1]],
cobra=Complement[brutto,bralt];
For[i=1,i<=Dimensions[cobra][[1]],i++,
poco=Position[mnt1,cobra[[i]]][[1,1]];
AppendTo[mt1,{cobra[[i]],mtt1[[poco]]}]]]]},{x,xmin,xmax}];
Print[Dimensions[brutto]]}
```

To start the search for (rGf,Noli) pairs of $N_2^0 = 9$ use the following command group:

```
z=08; bk=(*include list of (rGf,Noli) pairs for N_2^0=8*);
dbk=Length[bk]
xmin=1; xmax=dbk; brutto={}; mt1={};
<<t2_4prg
<<t2q
t2real[x,xmin,xmax];          (*delivers Dimensions[brutto]*)
```

In the list *brutto* you find the new rGf. The list *mt1* contains sorted (rGf,Noli) pairs for all rGf of *brutto*. These (rGf,Noli) pairs with $N_2^0 = 9$ can be used as starting list for getting (rGf,Noli) pairs of $N_2^0 = 10$:

```
z=09; bk=mt1; dbk=Length[bk]
xmin=1; xmax=dbk; brutto={}; mt1={};
```

```
t2real[x,xmin,xmax];          (*delivers Dimensions[brutto]*)
```

And by that way we could go on and gain a first list of (rGf,Noli) pairs for higher and higher N_2^0 .

Appendix D

For a given number of areas $N_2^0 = z$, and for a given (rGf,Noli) pair we apply a T1 process at each of the 3(z-2) sides of the respective graph to obtain new (rGf,Noli) pairs. They are collected in the lists mnt1 and mtt1. The program is C:\math\t1n

```
napaa[z_,t_]:={np={};t1np={};mtt1={};mnt1={};
flat=Flatten[t];tc=Table[Count[flat,k],{k,z}];
For[i=1,i<=2z-5,i++,For[j=i+1,j<=2z-4,j++,          (*start ForFor*)
alpa={t[[i]],t[[j]]};fl=Flatten[alpa];un=Union[fl];
If[Dimensions[un]=={4},AppendTo[np,alpa],Null]]];     (*end ForFor*)
For[k=1,k<=3(z-2),k++,paar=np[[k]];                     (*start For*)
in=Intersection[paar[[1]],paar[[2]]];
un=Union[Flatten[paar]];c=Complement[un,in];
If[(tc[[in[[1]]]]>3&&tc[[in[[2]]]]>3&&tc[[c[[1]]]]<(z-1)
&&tc[[c[[2]]]]<(z-1)),AppendTo[t1np,np[[k]]]]];          (*end For*)
For[h=1,h<=Dimensions[t1np][[1]],h++,par=t1np[[h]];    (*start For*)
int=Intersection[par[[1]],par[[2]]];po1=Position[t,par[[1]]][[1]];
po2=Position[t,par[[2]]][[1]];j1=Delete[t,{po1,po2}];
c1=Complement[par[[1]],int][[1]];c2=Complement[par[[2]],int][[1]];
p1=Position[par[[1]],c1][[1,1]];p2=Position[par[[2]],c2][[1,1]];
wh={p1,p2};sw=Switch[wh,
{1,1},{2,4,1,1,4,5},{1,2},{2,5,1,1,5,6},{1,3},{2,6,1,1,6,4},
{2,1},{3,4,2,2,4,5},{2,2},{3,5,2,2,5,6},{2,3},{3,6,2,2,6,4},
{3,1},{1,4,3,3,4,5},{3,2},{1,5,3,3,5,6},{3,3},{1,6,3,3,6,4}];
flp=Flatten[par];nt1=Part[flp,sw];j2=Partition[nt1,3];
t0=Join[j1,j2];vec=0;seli={};
For[es=1,es<=2z-5,es++,For[te=es+1,te<=2z-4,te++,   (*start ForFor*)
If[Union[t0[[es]]]==Union[t0[[te]]],vec++]]];        (*end ForFor*)
If[vec==0,seli={};
For[r=1,r<=2z-4,r++,                                    (*start For*)
ta={{t0[[r,1]],t0[[r,2]]},
{t0[[r,2]],t0[[r,3]]},{t0[[r,3]],t0[[r,1]]}};
seli=Union[seli,ta]];                                     (*end For*)
If[Dimensions[seli][[1]]==6z-12,tt1=t0;
fla=Flatten[tt1];
zt1=Table[Count[fla,k],{k,z}];n2t1=Table[Count[zt1,k],{k,3,z-1}];
AppendTo[mtt1,tt1];AppendTo[mnt1,n2t1]]]]]}              (*end For*)
```

For starting and manifold repeating the program t1n we use the following program:
C:\math\t1k

```
t1real[wdh_]:=
{Do[{ran=Random[Integer,{1,Dimensions[mtt1][[1]]}];
t=mtt1[[ran]];napaa[z,t];
bralt=brutto;brutto=Union[mnt1,bralt];
If[Dimensions[brutto][[1]]>Dimensions[bralt][[1]],
cobra=Complement[brutto,bralt];
For[i=1,i<=Dimensions[cobra][[1]],i++,
poco=Position[mnt1,cobra[[i]]][[1,1]];
```

```
AppendTo[mt1,{cobra[[i]],mtt1[[poco]]}]]]},{wdh}];
Print[Dimensions[brutto]]}
```

The simple case of $N_2^0 = 8$ is used as an example for getting from a known (rGf,Noli) pair more such pairs. To start the program you may use the following command group:

```
z=8; mt1={{{2,4,0,0,2},
          {{1,8,2},{1,2,3},{1,3,4},{1,4,5},{1,5,6},{1,6,7},
           {8,1,7},{8,3,2},{8,4,3},{8,5,4},{8,6,5},{8,7,6}}}};
brutto={mt1[[1,1]]}; mtt1={mt1[[1,2]]}; t=mtt1[[1]];
cobra={1};
<<t1n
<<t1k
t1real[1]
```

An outprint shows the total number tn of newly found rGf and the number $5 = z - 3$. To continue you repeat the command t1real[k] with k=1,2,4,8,16,..., until z does not grow any more. In our example that happens, depending on the starting data, at the last for tn=13. The same number of rGf resulted with the inverse T2 process done with pencil and paper on the 5 graphs of $N_2^0 = 7$. All (rGf,Noli) pairs are collected in *mt1*, all rGf in *brutto*. Other (rGf,Noli) pairs chosen for starting the program may yield less. Only when the same maximum tn is obtained with different starting pairs, you may be sure to a certain extent, that all possible rGf have been found. The reason for that is the following. Only one of the isomer graphs belonging to the same rGf will be collected as an (rGf,Noli) pair. The missing Noli-isomers however, could open the way to further rGf. A first step to remove this problem was the *Random* command used in C:\math\t1k.

The following program (C:\math\t1q) is an alternative to t1k, after we already have a file with a larger number of (rGf,Noli) pairs.

C:\math\t1q

```
t1real[x_,xmin_,xmax_]:=
{Do[{t=bk[[x,2]];
napaa[z,t];
bralt=brutto;brutto=Union[mnt1,bralt];
If[Dimensions[brutto][[1]]>Dimensions[bralt][[1]],
cobra=Complement[brutto,bralt];
For[i=1,i<=Dimensions[cobra][[1]],i++,
poco=Position[mnt1,cobra[[i]]][[1,1]];
AppendTo[mt1,{cobra[[i]],mtt1[[poco]]}]]]},{x,xmin,xmax}];
Print[Dimensions[brutto]]}
```

To start the program in case of $N_2^0 = 8$ you may use the list of (rGf,Noli) pairs shown in Tab.3. Then apply the following command group:

```
z=8; bk=<<(*list of (rGf,Noli) pairs from Tab.3 *);
dbk=Length[bk]; xmin=1; xmax=dbk; brutto={}; mt1={};
<<t1n
<<t1q
```

```
t1real[x,xmin,xmax]
```

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