Mathematik

Mean Curvature Flow of Cones near Minimal Cones

Inaugural-Dissertation zur Erlangung des Doktorgrades der Naturwissenschaften im Fachbereich Mathematik und Informatik der Mathematisch-Naturwissenschaftlichen Fakultät der Westfälischen Wilhelms-Universität Münster

vorgelegt von

Marcel Wunderlich

aus

Münster

in

2019

Dekan:	Prof. Dr. Xiaoyi Jiang
Erster Gutachter:	Prof. Dr. Boris Vertman
Zweiter Gutachter:	Prof. Dr. Burkhard Wilking
Tag der mündlichen Prüfung(en):	5. Juni 2019
Tag der Promotion:	5. Juni 2019

Contents

Abstract

We investigate the short-time existence of the mean curvature flow of cones over compact manifolds in Euclidean space. For this we find a suitable linearization of the flow in terms of b-vector fields. This yields a parabolic operator which is essentially a shifted Laplacian. Its heat kernel has well understood asymptotics on an appropriate blow-up space, allowing us to prove mapping properties between certain weighted Hölder spaces. If the cross section of the cone is not a sphere and the initial cone is sufficiently close to a minimal cone, we can prove short-time existence of the mean curvature flow via a fixpoint argument. We show that its curvatures behave well and that, for possibly shorter duration, we have the classic preservation of mean convexity.

Zusammenfassung

Wir untersuchen die Kurzzeit-Existenz des mittleren Krümmungsflusses von Kegeln über kompakten Mannigfaltigkeiten im Euklidischen Raum. Hierfür linearisieren wir den Fluss in geeigneter Weise in Termen von b-Vektorfeldern. Dies resultiert in einem parabolischen Operator, welcher essentiell ein geshifteter Laplace-Operator ist. Die Asymptotik des zugehörigen Wärmeleitungskerns kann auf einem geeigneten Aufblasungsraum gut verstanden werden. Dies ermöglicht uns, Abbildungseigenschaften zwischen gewissen Hölder-Räumen zu beweisen. Falls die Grundfläche des Kegels keine Sphäre ist und der initiale Kegel hinreichend nah an einem Minimalkegel ist, können wir mittels eines Fixpunktarguments die Kurzzeitexistenz des mittleren Krümmungsflusses beweisen. Wir zeigen, dass sich die Krümmungen gut verhalten und, für möglicherweise kürzere Zeit, beweisen das klassische Ergebnis des Erhalts positiver mittlerer Krümmung.

Introduction

In this thesis we discuss the short time existence of the mean curvature flow in presence of conical singularities. The mean curvature flow, together with its extrinsic cousin, the Ricci flow, is to this day the most important geometric flow for the geometer. Many results and concepts can be transferred from one flow to the other. Of particular prominence in this context is Huisken's body of work, most notably the classification of mean convex surfaces by mean curvature flow with surgery.

One of the main concerns when studying geometric flows, i.e. evolutions of a manifold according to some curvature quantity, is the formation of singularities, where classical theory of partial differential equations fails to assure the existence of the flow beyond this point. Usually a curvature explodes as one approaches the singular time, and, in the case of aforementioned surgery, one can cut out the problematic area and replace it by a spherical cap, in order to reduce the curvature and restart the flow.

But surgery is not the only technique to deal with these singularities. Other options include weaker versions of the flow, like Brakke flow, or a recent approach by Sáez and Schnürer [SS14], where on interprets the evolving manifold as a projection of a non-singular flow.

In this thesis we consider the simplest geometric singularity a Riemannian manifold can develop: An isolated conical singularity. We establish, under suitable conditions, short-time existence of the flow, while preserving the singular structure.

While the relative simplicity of conical singularities is one part of their appeal to the researcher, they also occur rather naturally in the context of mean curvature flow, e.g. as the limit of shrinkers. Furthermore minimal surfaces, which are the constant solutions to the mean curvature flow, often exhibit conical points.

Our work is based on the work of Bahuaud and Vertman. In [BV14] they showed how to use the microlocal structure of the heat kernel on a compact manifold with isolated conical singularity to prove the short-time existence of the Yamabe flow on manifolds with conical singularities or even edges, and extended these results to show long time existence in [BV16]. Furthermore Vertman applied these techniques to the Ricci flow in [Ver16]. True to the aforementioned spirit of Ricci flow and mean curvature flow being cousins, we adapt these techniques to establish short-time existence for mean curvature flow of manifolds with isolated conical singularities, in a way that preserves the singular structure.

The conical setting

We say that a Riemannian manifold (M, g) is a cone, if M can be written as $M = (0, L) \times N$, where (N, g_N) is a compact Riemannian manifold and

g takes the form $dx^2 + x^2 g_N$, with x being the coordinate on the interval $(0, L)$. We usually consider the case $L = \infty$, but note that most of the arguments presented are local in nature and carry over to compact manifolds with isolated conical singularities, where around each singularity one finds a neighbourhood where the metric takes the above form.

Usually we work with local coordinates capturing the conical structure, i.e. x is the radial coordinate as above and $z = (z_1, \ldots, z_n)$ are local coordinates on N. Then it is convenient to work with the matrix representation of the metric w.r.t. these coordinates (x, z), which then has the block form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & x^2 g_N \end{pmatrix}.$$

Beyond this quite rigid geometry, we also call manifolds conical, where the metric in local coordinates has the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & x^2 g_N \end{pmatrix} + \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x^2) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x^2) \end{pmatrix}.$$

Here $\mathcal{O}$ is the Landau symbol as $x \to 0$, with the small modification that we insist on the argument occurring as a factor. In this situation we also refer to the manifold as a cone or as a manifold with an isolated conical singularity, respectively, and distinguish only the exact case.

Our setup will start with the cone over a compact submanifold of the unit sphere $\mathbb{S}^m \subset \mathbb{R}^{m+1}$, i.e. given an isometric immersion $\varphi_N : N \to \mathbb{S}^m$, the cone is given by $\varphi(x, z) = x\varphi_N(z)$. It is easy to see that in this case the induced metric is that of an exact cone. Similarly, when φ_N may vary with x, i.e. $\varphi_N : (0, L) \times N \to \mathbb{S}^m$, then the induced metric of $\varphi(x, z) = x\varphi_N(x, z)$ has an isolated, but not necessarily exact, conical singularity.

To establish existence not only for compact manifolds with isolated conical singularities, but also for "true" or "open" cones expanding to infinity, we have to talk about a manifold being *asymptotically conical*, i.e. looking like a cone at infinity. In this case one considers the boundary defining function $y = x^{-1}$ so that we reach infinity as $y \to 0$. Performing this simple change of coordinates one obtains that the metric of an exact cone is then locally given by $g = y^{-4} dy^4 + y^{-2} g_N$.

Structure of this thesis

Classically geometric flows lie at the intersection of geometry and the theory of partial differential equations. However, our main tools come from geometric and microlocal analysis, more specifically what some people call the "Melrose school". Each of these fields comes with its own language and concepts and one usually has to be aware of one's own mathematical background and, if possible, that of the audience or reader.

In particular the available introductory literature to Melrose's b-calculus, which we employ, usually has a strong analytic flavour. We begin this thesis with an informal introduction to these tools, written from a geometer's perspective, hopefully bridging the gap between these worlds. While the tools have been exploited quite extensively to obtain and describe analytic invariants, we think that at this point they are underutilized in a more geometric context.

Our goal is to obtain the solution via a fixpoint argument: We first have to find a linearization suitable to the b-calculus and define Hölder spaces adapted to the conical setting. Then we prove parabolic Schauder estimates for the heat kernel, which yield the necessary mapping properties to perform the fixpoint argument. Finally we can discuss the evolution of curvature quantities and even prove, for a possibly shorter time, preservation of mean convexity, i.e. non-negative mean curvature along the flow.

After the introduction to the tools from microlocal analysis in Section 1, we begin with the linearization of the mean curvature flow in Section 2. We use the parametric formulation of the mean curvature flow and represent the flow by a function $f : M \to \mathbb{R}$, which is essentially a section of the normal bundle of the initial cone. We linearize the resulting equation for the mean curvature flow by separating out the initial Laplacian of the manifold, as well as all linear terms. To account for the singularity, this linearization is done in terms of *complete b-vector fields*, which are generated by coordinate vector fields $x\partial_x, \partial_z$ and are explained in Section 1. We find that a solution to the mean curvature flow is equivalent to solving the partial differential equation

$$(\partial_t - \Delta')f = H_0 + Q(f), \ f(\cdot, 0) = 0,$$

where Δ' is the Laplacian of the initial manifold with a shift in the tangential operator, H_0 is the initial mean curvature and Q is a term quadratic in f and its first and second order spatial derivatives. Most of Section 2 is dedicated to this linearization. We can view Q as a mapping between weighted Hölder spaces $Q : x^2 C^{2+\alpha} \to C^\alpha$. These spaces are again adapted to the boundary geometry, with the admissible derivatives being again complete b-vector fields. For technical reasons we have to use slightly more complicated hybrid Hölder spaces, as explained in Section 2.4.

Standard functional analysis assures us the existence of an inverse operator for $(\partial_t - \Delta')$, the *heat kernel* H. We will then exploit the microlocal structure of this heat kernel, which we explicate in Section 1. The results of Vertman and Mazzeo [MV12] assure that this heat kernel has a certain asymptotic expansion on the so called *heat space*, allowing us to perform parabolic Schauder estimates, which comprise Section 3 and Section 4. Here we also need to impose lower bounds on the shift in Δ', which then translates into a lower bound for the norm of the second fundamental form of the cross section. This limits our results to cones whose cross section is not a topological sphere. However, this is no strong restriction, cf. Section 6.4, since cones over spheres can be viewed as Lipschitz graphs and are already well understood.

This way we establish that the heat kernel, acting as a convolution in time, admits the mapping property $H : C^\alpha \to x^2 C^{2+\alpha}$. Now we can define a mapping

$$F : x^2 C^{2+\alpha} \to x^2 C^{2+\alpha},\ f \mapsto H \circ Q(f).$$

Using a standard contracting mapping argument, we then find a non-trivial fixpoint f for F, i.e. $F(f) = f$. Then, as H is the inverse of $(\partial_t - \Delta')$, we have a solution to the mean curvature flow:

$$(\partial_t - \Delta')f = (\partial_t - \Delta')F(f) = (\partial_t - \Delta')H \circ Q(f) = Q(f)$$

This allows us to obtain the main theorem, which can be stated informally as follows:

Theorem (Theorem 6.2)**.** Let $\varphi_0 : M^m \to \mathbb{R}^{m+1}$ be a perturbation of an immersed minimal (open) cone with cross section N not a sphere, such that the initial mean curvature H_0 lies in a certain weighted $C^{1+\alpha}$-space s.t. $H_0 \to 0$ as one approaches the tip or infinity. Then there exists, for a small time T, a mean curvature flow $\varphi_t : M \times [0,T] \to \mathbb{R}^{m+1}$, such that the immersion stays conical in the above sense.

As noted, the locality of the arguments actually also give us the result for compact manifolds with isolated conical singularities, as long as these satisfy the conditions in the theorem.

After obtaining the main theorem, we follow-up with some further analysis of the curvatures along the solution. These can be read off rather easily from Hölder spaces in which the solutions live. We note that our solutions decay towards the singularity as well as towards infinity, allowing us to prove the preservation of mean convexity for small times:

Theorem (Theorem 7.5)**.** Let $\varphi_t : M \times [0,T] \to \mathbb{R}^{m+1}$ a solution to the conical mean curvature flow as above, such that the initial mean curvature H_0 satisfies $H_0 \geq 0$. Then there exists $0 < \tilde{T} \leq T$ such that $H_t \geq 0$ for all $t \in [0, \tilde{T}]$.

Finally we conclude this thesis with a small survey on related problems and in particularly investigate the relationship to the corresponding works by Bahuaud, Kröncke and Vertman for Yamabe flow and Ricci flow, to further explore the limitations and strengths of the techniques used.

Acknowledgements

First and foremost I want to thank my advisor Boris Vertman, for providing me with the opportunity to write a thesis on the exciting and promising topic of curvature flows in the singular setting, as well as for his guidance and patience in introducing me to the field of singular analysis.

Furthermore I want to thank the members and former members of the Arbeitsgruppe Differentialgeometrie in Münster, both for the great mathematical environment, in particular the Reading Seminar, as well as the good times outside the office. Special thanks go out to our secretary Sandra, for enabling us to research and teach, instead of dealing with the overhead of administration.

I am deeply grateful for Sira, who accompanied and supported me during my dissertation and probably had to suffer more from my mood and the pressure than any other person, myself included.

Finally I want to thank my family and, most of all, my grandmother, for her continuous encouragement and support during my studies. Without her I would not have had access to higher education, let alone the possibility of studying.

Vielen Dank Oma, dass Du Dich so sehr um mich gekümmert hast. Ohne Dich wäre das alles nicht möglich gewesen.

1 Heat Kernel Asymptotics and b-calculus

In order to perform the Schauder estimates in Section 3 and Section 4 we need to understand the asymptotic behaviour of the heat kernel of the shifted Laplacian near the conical singularity. The asymptotic behaviour can be captured in the notion of a *polyhomogeneous conormal distribution* on a certain *blow-up space*. We will begin by explaining what these notions exactly mean, as these originate in the field of geometric analysis and might be new to readers with a more geometric or topological background.

Readers already familiar with these concepts may very well skip the first subsection and begin with Theorem 1.4, or even only refer to the figures in that particular section, as the involved constructions are standard.

For everyone else the following serves as an introduction, or maybe even an invitation, to the world of boundary geometry.

1.1 Boundary geometry and b-calculus in the broad sense

In the following section we wish to give the reader an overview of the techniques and vocabulary used in obtaining the heat kernel asymptotics on the singular manifold M. The underlying framework is that of Melrose's boundary-calculus, or b-calculus for short, which is elaborated to great depth in his tome [Mel93]. In it Melrose also extends many concepts of differential geometry as covariant derivatives and curvature tensors to manifolds with corners. A lighter introduction to only the differentiable structure on such manifolds is provided by Joyce in [Joy12]. We introduce all concepts only for the case of real valued functions and refer the reader to the aforementioned sources, although they carry over to distributions, tensors and other definitions as one would expect.

Finally we show how these concepts can be used to analyze the singular behaviour of some functions, before we consider explicitly the heat kernel in the next section. The general motive is to make the underlying space more complicated in order to obtain an easier description of the heat kernel locally.

Technically the b-calculus only applies in the situation near the tip, and one would introduce the scattering-calculus or sc-calculus near infinity. But this distinction only appears at a level of detail we will not be able to cover in this thesis.

1.1.1 Manifolds with corners

A n-manifold with corners is modeled in the same way as a manifold with a boundary, i.e. in a chart near the boundary it looks like $[0, \infty)^k \times \mathbb{R}^{n-k}$, where, in contrast to the boundary case, k may take values greater than 1. This yields the inductive definition that a manifold with corners is the union of its interior and its boundary hypersurfaces, which again are manifolds with corners. The usual way to define smoothness in this context is the following: A function $f : M \to \mathbb{R}$ is said to be *smooth up to the boundary* if there exists an embedding $M \hookrightarrow \tilde{M}$ of M into a manifold $\tilde{M}$ without boundary, such that f extends to a smooth function on $\tilde{M}$. Using Seeley's extension theorem [See64], this is equivalent to say that f is smooth up to the boundary iff all derivatives of f are bounded on bounded subsets on the interior of M. We denote the space of functions which are smooth up to the boundary as usual by $C^\infty(M)$.

On the boundary the tangent space clearly loses dimensions. However, we can still give a basis of tangent vectors which is valid both in the interior, as well as on the boundary: In a chart as above with coordinates $(x_1, \ldots, x_k, z_1, \ldots, z_{n-k})$ on $[0,\infty)^k \times \mathbb{R}^{n-k}$ we can generate the tangent space by coordinate vector fields $\{x_1\partial_{x_1}, \ldots, x_k\partial_{x_k}, \partial_{z_1}, \ldots, \partial_{z_{n-k}}\}$. In the interior this is clearly equivalent to the usual coordinate vector fields, and as one approaches the boundary problematic directions get scaled down and vanish on the boundary. We call vector fields which are linear combinations with smooth coefficients of such a basis *b-vector fields* and denote the space of b-vector fields with the symbol $\mathcal{V}_\mathrm{b}$. Furthermore we will, by abuse of notation, write $\mathcal{V}_\mathrm{b} f$ when talking about the differentiation of the function f under any b-vector field.

1.1.2 Boundary defining functions

Consider for a moment the manifold $[0,\infty)$ with coordinate x. We can see x as the distance function of the boundary $\{0\}$. By writing the boundary as $\{t \in [0,\infty) \mid x(t) = 0\} = \{x = 0\}$, we see that it is defined by the function x and we consequently call x a *boundary defining function*, often abbreviated as *bdf* of the boundary. Extending the example to $(0,\infty) \times \mathbb{R}$ with coordinates (x, y), we again see that x is the boundary defining function.

The simplest manifold with corners is the quadrant $\mathbb{R}^2_+ = [0,\infty)^2$, which now has two boundary defining functions, with x being the bdf for the y-axis and y being the bdf for the x-axis. To make this rigid we have the following definition.

Definition 1.1 (Boundary defining functions)**.** Let M be a manifold with corners. A *boundary defining function (bdf)* of the boundary hypersurface $H \subset \partial M$ is a function $\rho : M \to [0,\infty)$ which is smooth up to the origin, has non-vanishing differential on H and $H = \rho^{-1}(0)$.

Very much like the construction of convex polyhedra in Euclidean space, boundary components and corners of the manifold can be realized as intersections of 0-level sets of boundary defining functions.

1.1.3 Polyhomogeneous conormal functions

We now introduce a class of functions, which are not necessarily smooth up to the boundary anymore, but can be approximated by a power series in terms of the boundary defining functions with controlled singular behaviour, similar to meromorphic functions. For this assume that ρ is the boundary defining function of a boundary component and choose a small coordinate patch near $H = \{\rho = 0\}$ with coordinates $(x = \rho, z_1, \dots, z_{n-1}) =: (x, z)$.

Furthermore we call $E \subset \mathbb{R}$ an *index set* if

(i) E is discrete and bounded from below

(ii) if $r \in E$ then also $r + n \in E$ for all $n \in \mathbb{N}$.

We call f an *homogeneous conormal function for the boundary face H with index set E* if we have in any coordinate patch as above

$$f(x, z) \sim \sum_{r \in E} x^r a_r(z) \text{ as } x \to 0,$$

where $\sim$ bears the following meaning: Let $f_N(x, z)$ denote the partial sum for $r < N$, then for all N there exists a uniform constant C_N so that on compact subsets of the boundary we have $|f(x, z) - f_N(x, z)| \leq C_N x^N$ and similar estimates for all $x\partial_x$ and ∂_z derivatives.

Iterating this definition we obtain that of a polyhomogeneous conormal function with an *index family* $(E_1, \dots, E_k)$, which is a k-tuple of index sets:

Definition 1.2 (Polyhomogeneous conormal functions)**.** Let M be a manifold with corners and boundary defining functions $\{\rho_i \mid i = 1, \dots, k\}$. A function $f : M \to \mathbb{R}$ is called *polyhomogeneous conormal with index family* $E = (E_1, \dots, E_k)$, if it is homogeneous conormal and near each boundary $H_j = \rho_j^{-1}(0)$ we have

$$f \sim \sum_{r \in E_j} a_{jr} \rho_j^r, \text{ as } \rho_j \to 0,$$

with coefficients a_{jr} conormal on H_j and polyhomogeneous with index family $(E_1, \dots, E_{j-1}, E_{j+1}, \dots, E_k)$ at any intersection $H_j \cap H_l$.

Remark 1.3. For the sake of simplicity we gave a more restrictive definition of polyhomogeneous conormal functions than the one appearing in the literature, e.g. as in [BV14, §2]. There one also allows log-terms to occur as well as complex powers. However, due to the results of Mazzeo and Vertman on the heat kernel asymptotics in [MV12], as well as those derived by Sher in [She13], the heat kernel is actually a polyhomogeneous conormal distribution in our sense on the appropriate blow-up spaces.

We actually do not need finer information about the index sets and are only interested in the leading behaviour, so we abbreviate

$$f \sim \rho_1^{\mu_1} \dots \rho_k^{\mu_k} g,$$

with g smooth, bounded up to the boundary and $\mu_j = \min E_j$, to indicate that f is polyhomogeneous conormal. In particular, when functions vanish to infinite order at the boundary, e.g. $e^{-1/t}$ as $t \to 0$ on $[0, \infty)$, one often writes $e^{-1/t} \sim t^\infty g$, or omits the corresponding boundary defining function from the expansion above and simply says that g vanishes to infinite order at the corresponding surface.

1.1.4 Blow-ups and blow-downs by example

While manifolds with corners are certainly interesting on their own, our goal is to analyze and resolve the singularities of the heat kernel (or other functions). The general idea is to find out on which set a function develops singular behaviour and modify the original manifold in such a way that the function becomes polyhomogeneous conormal on the new manifold. Borrowing the terminology from algebraic geometry, this is achieved by a *blow-up* of the submanifold on which the function displays singular behaviour, i.e. replacing it by an embedding of its inward pointing sphere bundle. This blow-up space is then equipped with a natural differentiable structure, in which polar and projective coordinates around this submanifold are smooth.

Often, and in particular in the case of the heat kernel, one has to perform multiple blow-ups and there is no hard and fast rule in which order one does this, as it depends on the application one has in mind.

The technical details can be found in Melrose's book [Mel93, §4]. However, these do not really illuminate the purpose of the process, so to introduce the nomenclature we will exercise this process on the standard toy example, $f : \mathbb{R}_+^2 \to \mathbb{R}$, $(x, y) \mapsto \sqrt{x^2 + y^2}$. Clearly f is smooth up to the boundary with the exception of the origin. While for any $y > 0$ we obtain an asymptotic expansion $f \sim \sum_{k=0}^\infty a_k(y) x^k$ as $x \to 0$, the coefficients $a_k(y)$ become more singular as $y \to 0$. However, we could try to force some homogeneity in the y-coordinate upon the function and factor out y, obtaining $f(x, y) = y\sqrt{(x/y)^2 + 1}$. If we treat $s = x/y$ as a new coordinate, we can develop $\sqrt{1 + s^2} \sim \sum_{k=0}^\infty c_k s^k$, and in total obtain a function $f(s, y) = y \sum_{k=0}^\infty c_k s^k$, which can be understood quite well as $s \to 0$ and $y \to 0$, as long as we may assume that s is bounded. Another approach would be to note that the behaviour of the differential is controlled by the slope of x and y, which can be expressed by $s = x/y$ conveniently.

In essence the slope measures the direction from which we approach the singularity and the goal of the blow-up is to create a space, which can capture these. For this we replace the origin by glueing in the inner pointing (unit) tangent sphere, and we denote the blow-up of $\mathbb{R}_+^2$ at the origin by

$[\mathbb{R}^2_+ : (0,0)]$. This yields a new boundary component, the so called *front face*, between the y-axis, which we now call *left face*, and the x-axis, the *right face*, with corresponding abbreviations $\mathrm{ff}, \mathrm{lf}, \mathrm{rf}$.

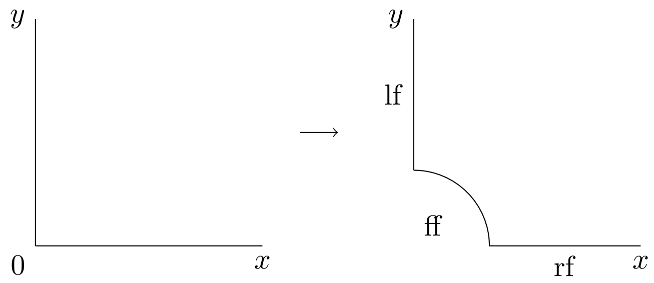

Figure 1: Blow-up $[\mathbb{R}^2_+ : (0,0)]$ of $\mathbb{R}^2_+$ at the origin.

We prefer to draw the resulting space as indicated in Fig. 1. We say that we are *away from the right face*, whenever the coordinate $s = x/y$ is bounded. Now away from the right face, we have y as the boundary defining function of the new front face, and, since s measures the slope between x and y, s as the boundary defining function of the left face lf. Vice versa, if $\tilde{s} = y/x$ is bounded, x is the boundary defining function of ff and $\tilde{s}$ that of rf. Each point on the front face now corresponds to a direction from which we approach the singularity in the origin. By prescribing a uniform bound $C > 1$ on s and $\tilde{s}$ we can now cover the manifold $[\mathbb{R}^2_+ : (0,0)]$ with two smooth charts, one with (y,s) as coordinates and one with $(x,\tilde{s})$ as coordinates. Later this uniform bound on the introduced coordinates will allow us to perform parabolic Schauder estimates.

Clearly we can always reverse this process and assign a point in the blow-up space one in the original. The so called blow-down map $\beta : [\mathbb{R}^2_+ : (0,0)] \to \mathbb{R}^2_+$ is in local coordinates given by $(y,s) \mapsto (ys, y)$ and $(x,\tilde{s}) \mapsto (x, x\tilde{s})$. Consequently we can pullback differentials and differential forms along β, by simply calculating the coordinate change, e.g. in the coordinates $(x,\tilde{s})$ we have

$$\beta^* \partial_x = \partial_x x \partial_x + \partial_x \tilde{s} \partial_{\tilde{s}} = \partial_x - x^{-1} \tilde{s} \partial_{\tilde{s}.}$$

In this particular case we also note that the b-differential $x\partial_x$ would still pullback to a b-differential $x\partial_x - \tilde{s}\partial_{\tilde{s}}$, since, as stated above, in the new coordinates both x and $\tilde{s}$ are boundary defining functions.

Coming back to the example of $f(x,y) = \sqrt{x^2+y^2}$, we now see that $\beta^* f : [\mathbb{R}^2_+ : (0,0)] \to \mathbb{R}$ is polyhomogeneous conormal with index sets $E_{\mathrm{ff}} = 1 + \mathbb{N}$, $E_{\mathrm{lf}} = E_{\mathrm{rf}} = \mathbb{N}$: Away from rf we have

$$f(y,s) = y\sqrt{1+s^2} \sim y \sum_{i=0}^{\infty} c_i s^{2i}$$

and away from lf we have

$$f(x,\tilde{s}) = x\sqrt{1+\tilde{s}^2} \sim y\sum_{i=0}^{\infty} \tilde{c}_i \tilde{s}^{2i}.$$

1.2 Heat kernel asymptotics

We now turn to the setting of a conical manifold. Let (M, g) be a cone, i.e. $M = (0,\infty)_x \times N$, with radial variable x, such that in local coordinates (x, z) the matrix representation of g is

$$g = \begin{pmatrix} 1 & 0 \\ 0 & x^2 g_N \end{pmatrix} + \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x^2) \end{pmatrix} = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x^2) \end{pmatrix}.$$

We are especially interested in the heat equation with a shifted Laplacian on the cross section. To be more specific, let Δ_N denote the Laplacian on the cross section for some fixed x, and $\Delta = \Delta_M$ the Laplacian of the whole manifold. Then we have

$$\Delta = \partial_x^2 + \frac{m-1}{x}\partial_x + \frac{1}{x^2}\Delta_N + \mathcal{E},$$

where $\mathcal{E}$ is an error term, given by the difference to the Laplacian of an exact cone metric. Then, for some constant $C > 0$, we call

$$\Delta' := \partial_x^2 + \frac{m-1}{x}\partial_x + \frac{1}{x^2}\left(\Delta_N - C\right) + \mathcal{E},$$

the *shifted Laplacian with shift C*. The corresponding parabolic differential operator of interest is then $\partial_t - \Delta'$. Note that this is the usual sign convention for geometric purposes; analytic papers, especially those concerned with asymptotics, usually consider the negative Laplacian, making its spectrum positive. The choice of the negative sign of C is made so it matches the linear term we obtain in the linearization of the mean curvature flow in Section 2. Consequently we have, as N is compact,

$$\begin{aligned} \operatorname{spec}(\Delta_N) &= \{0 > \lambda_1 > \lambda_2 > \dots\} \\ \operatorname{spec}(\Delta_N - C) &= \{0 > C > \lambda_1 > \lambda_2 > \dots\}. \end{aligned}$$

One of the analytical difficulties one encounters is, as the manifold M is not complete, that the Laplacian (and hence also the shifted Laplacian) may fail to be self-adjoint w.r.t. the standard L^2-scalar product on the core domain of bounded smooth functions. To overcome this, one considers self-adjoint extensions and corresponding domains on which the extension acts. This implicitly imposes boundary conditions and was, in the *exact* conical setting, first achieved by Cheeger in [Che83]. However the methods we now discuss are the bookkeeping one needs to make things work in the non-exact setting and to "sweep the error term $\mathcal{E}$ under the rug". We choose the Friedrichs extension, as its domain was characterized by Mazzeo and Vertman in [MV12] and is the extension for which the asymptotic expansions in this section hold. For this extension to exist we need that $-\Delta'$ is a symmetric operator bounded from below, which is fulfilled since $C > 0$. Without further comment we identify Δ' with its Friedrichs extension.

We denote the *heat kernel associated to the operator* $\partial_t - \Delta'$ by H or, if there is risk of confusion, by $e^{-t\Delta'}$. The heat operator acts as an integral convolution operator on functions u which are for each positive time t in the Friedrichs domain $\mathcal{D}(\Delta')$, i.e. $u(t,\cdot) \in \mathcal{D}(\Delta')$, via

$$e^{-t\Delta'} * u(t,p) = \int_0^t \int_M H(p,\tilde{p},\tilde{t}) u(\tilde{t},\tilde{p},t-\tilde{t}) \,\mathrm{dvol}_g(\tilde{p}) d\tilde{t}$$

and solves the inhomogeneous heat problem

$$\begin{cases} (\partial_t - \Delta')w(t,p) = u(t,p) \\ w(0,p) = 0. \end{cases}$$

Denote the Dirac delta distribution with mass in 0 by δ. Then standard functional analysis asserts that positing $H(p,\tilde{p},t) \to \delta(p-\tilde{p})$ as $t \to 0$ and that the range of H lies in the Friedrichs domain of Δ', determines the Friedrichs heat kernel uniquely.

The general idea of the construction is to assume that the heat kernel should roughly behave like the Euclidean heat kernel

$$H_{\mathbb{R}^n}(p,\tilde{p},t) = (4\pi t)^{-n/2} e^{-|p-\tilde{p}|^2/(4t)},$$

and one expects it to behave non-uniformly near the singularity. This yields a first picture of the occurring singularities, which are then resolved by appropriate blow-ups. Then one can iteratively build up a parametrix to the heat operator and finally it can be shown that this parametrix actually captures the asymptotic behaviour of the true heat kernel.

Next we will discuss these singularities, perform the appropriate blow-ups and introduce the local coordinates on the resulting blow-up spaces. Then we will recover the finer asymptotics. We explicitly do not perform the so called b-calculus (and sc-calculus) in the small sense, which is the actual book keeping ensuring that these asymptotics actually hold, and refer again to the underlying works of Mazzeo and Vertman [MV12], Albin [Alb07] and Sher [She13].

1.2.1 The heat space

In general there is no one algorithm to follow when trying to find the correct blow-ups. One way is, as in Section 1.1.4, to study the homogeneous behaviour of the equation and let this be one's guide to define the appropriate blow-ups. This however requires that one can actually determine the necessary homogeneity. Sometimes it is possible by analyzing the corresponding differential equation. This step is more of an heuristic and usually not found in the literature, however, in practice, one always checks the corresponding blow-ups by hand, to see if they are really appropriate.

We begin by naively assuming that

$$H(p, \tilde{p}, t) \sim t^{-m/2} e^{-d(p,\tilde{p})^2/t},$$

which is a distribution on $M^2 \times \mathbb{R}_+$. In general the term $d(p, \tilde{p})$ can not be given explicitly, but assume for the moment that it is given by the prototypical cone distance in local coordinates near the tip, i.e.

$$d(x, z, \tilde{x}, \tilde{z}) = \sqrt{|x - \tilde{x}|^2 + |x + \tilde{x}|^2 |z - \tilde{z}|^2},$$

which can be thought of as the leading term of the Taylor expansion of the distance function (cf. Appendix A for more details). We can see the space $M^2 \times \mathbb{R}_+$ as a manifold with corners with boundary defining functions $x = \rho_{\text{lf}}$, $\tilde{x} = \rho_{\text{rf}}$ and $t = \rho_{\text{tf}}$ of the corresponding faces lf, rf and tf, as pictured in Fig. 2.

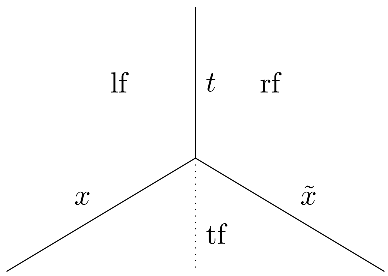

Figure 2: $M^2 \times \mathbb{R}_+$ seen as a manifold with corners.

The behaviour of this heat kernel is very well understood, unless both t and $d(p, \tilde{p})$ approach 0 simultaneously. In the Euclidean setting this singularity only occurs at the spatial diagonal

$$D = \left\{ (p, \tilde{p}, t) \in M^2 \times \mathbb{R}_+ \mid p = \tilde{p},\ t = 0 \right\},$$

indicated by the dotted line in Fig. 2. However, as the distance in the cross section collapses when both x and $\tilde{x}$ approach 0, we have another singular set, namely

$$A = \left\{ (p, \tilde{p}, t) \in M^2 \times \mathbb{R}_+ \mid x = \tilde{x} = 0,\ t = 0 \right\},$$

which is the corner of the space.

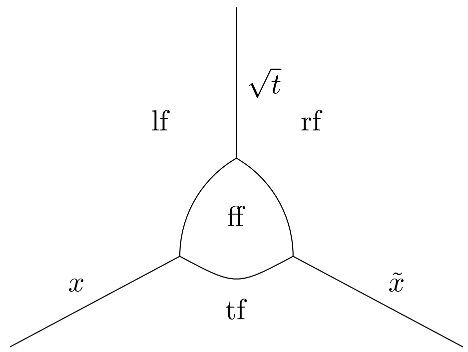

Figure 3: Single blow-up space $M' = [M^2 \times \mathbb{R}_+ : A]$ of the heat kernel

We remedy these singularities with a first blow-up of A, obtaining the intermediate heatspace $M' := [M^2 \times \mathbb{R}_+ : A]$. This gives us a new face, the front face, visualised in Fig. 3. This introduces new coordinates away from the faces lf, rf, tf. Similar to the toy example in Section 1.1.4 these measure the slope between the various coordinates, namely between x and $\tilde{x}$ and between $\sqrt{t}$ and x as well as between $\sqrt{t}$ and $\tilde{x}$, where the square root accommodates for the parabolic scaling property of the heat kernel. We describe the full sets of coordinates after the next blow-up, which is the one of D, as along the spatial diagonal the heat kernel still behaves non-uniformly. The resulting blow-up space, sometimes called *the heat space* $\mathcal{M} = [M' : D]$ is depicted in Fig. 4 and introduces an additional boundary face td.

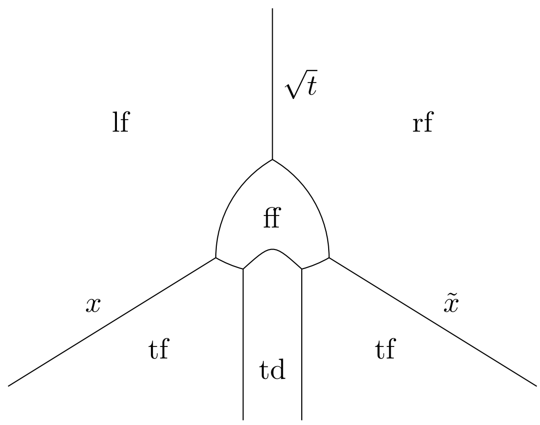

Figure 4: Double blow-up space $\mathcal{M} = [M' : D]$ of the heat kernel

Note that there is some freedom in choice of coordinates, as the usual condition on the validity of these coordinates is to be away from certain faces. For instance coordinates valid near the lower left corner will still be valid near the diagonal, as long as one is away from the right face, since the second blow-up does not interfere with any local coordinates near the corners of the front face. At points we will make use of this fact, but for now we introduce the most common choices.

Near the lower left corner we are away from the right face, which, as a subset of $M^2 \times \mathbb{R}_+$, was given by $\{x = 0\}$. Consequently the following coordinates are valid:

$$\rho_{\mathrm{lf}} = \tilde{s} = \frac{\tilde{x}}{x},\ \rho_{\mathrm{ff}} = x,\ \rho_{\mathrm{td}} = \tau = \frac{t}{x^2},\ z,\ \tilde{z}$$

Near the lower right corner the same coordinates are valid as near the lower left, just with the roles of x and $\tilde{x}$ exchanged, i.e.

$$\rho_{\mathrm{rf}} = s = \frac{x}{\tilde{x}},\ \rho_{\mathrm{ff}} = \tilde{x},\ \rho_{\mathrm{td}} = \tau = \frac{t}{\tilde{x}^2},\ z,\ \tilde{z}.$$

Near the top corner we are away from the temporal face, which in the initial space was given by $\mathrm{tf} = \{t = 0\} \subset M^2 \times \mathbb{R}_+$.

$$\rho_{\mathrm{ff}} = \tau = \sqrt{t},\ \rho_{\mathrm{rf}} = \xi = \frac{x}{\tau},\ \rho_{\mathrm{lf}} = \tilde{\xi} = \frac{\tilde{x}}{\tau},\ z,\ \tilde{z}$$

Near the intersection of front face and diagonal we begin by writing out the temporal diagonal in coordinates from the previous blow-up, say those from near the lower right. Then $D = \{s = 1, z = \tilde{z}, \tau = 0\}$. It is appropriate to once again rescale parabolically.

$$\rho_{\mathrm{td}} = \eta = \frac{\sqrt{t}}{x},\ S = \frac{x - \tilde{x}}{\sqrt{t}},\ Z = \frac{z - \tilde{z}}{\eta},\ \rho_{\mathrm{ff}} = x, z$$

In these coordinates tf lies in the limit $|(S, Z)| \to \infty$.

1.2.2 The heat space near infinity

Near infinity we make a change of coordinates and use the coordinates (y, z) with $y = x^{-1}$. Again we naively assume that

$$H(p, \tilde{p}, t) \sim t^{-m/2} e^{-d(p,\tilde{p})^2/t}.$$

In these coordinates the prototypical distance function is given by

$$d(y, z, \tilde{y}, \tilde{z}) = \sqrt{|y - \tilde{y}|^2 \left(y^{-1} + \tilde{y}^{-1}\right)^4 + |z - \tilde{z}|^2 \left(y^{-1} + \tilde{y}^{-1}\right)^2},$$

where we again refer to Appendix A for further details. So we can write the heat kernel as

$$H(y, z, \tilde{y}, \tilde{z}, t) \sim t^{-m/2} e^{-\left(\frac{1}{y} + \frac{1}{\tilde{y}}\right)^4 \frac{|y-\tilde{y}|^2}{t}} e^{-\left(\frac{1}{y} + \frac{1}{\tilde{y}}\right)^2 \frac{|z-\tilde{z}|^2}{t}}.$$

The first ostensible singularity now appears independent of time, as $(y, \tilde{y}) \to 0$, because then both $|y-\tilde{y}|^2$ and the leading factor of the distance function approach infinity. Consequently the first blow-up is that of

$$Y = \{y = 0,\ \tilde{y} = 0\}\,,$$

introducing two sets of coordinates involving $s = \frac{y}{\tilde{y}}, \tilde{y}$ and $\tilde{s} = \frac{\tilde{y}}{y}, y$, while leaving the other coordinates unchanged. We call this new first boundary face bb (Fig. 5). Plugging in these coordinates in the formula, it still does not suffice to understand the equation as $(y, \tilde{y}) \to 0$: We see that for the first exponential function in the first set of coordinates, with s bounded, the argument is then given by

$$\begin{aligned}
-\left(\frac{1}{y}+\frac{1}{\tilde{y}}\right)^4 \frac{|y-\tilde{y}|^2}{t} &= -\left(\frac{1}{s\tilde{y}^2}+\frac{1}{\tilde{y}}\right)^4 \frac{|s\tilde{y}^2+\tilde{y}|^2}{t} \\
&= -\frac{1}{\tilde{y}^2}\left(\frac{1}{s\tilde{y}}+1\right)^4 \frac{|s\tilde{y}+1|^2}{t} \to -\infty \text{ as } (y,\tilde{y}) \to 0.
\end{aligned}$$

Consequently we understand in these coordinates that

$$e^{-\left(\frac{1}{y}+\frac{1}{\tilde{y}}\right)\frac{|y-\tilde{y}|^2}{t}} \to 0 \text{ as } (y,\tilde{y}) \to 0.$$

However, when both $(y, \tilde{y}) \to 0$ and $z = \tilde{z}$, we have for the argument of the second factor

$$-\left(\frac{1}{y}+\frac{1}{\tilde{y}}\right)^4 \frac{|z-\tilde{z}|^2}{t} = -\frac{1}{\tilde{y}^2}\,(s\tilde{y}+1)^4\,\frac{|z-\tilde{z}|^2}{t},$$

so we still see some non-uniform behaviour. This motivates a second blow-up, namely of the intersection of the spatial diagonal with bb

$$D' = \operatorname{diag}(M) \cap \mathrm{bb} = \{y = 0, \tilde{y} = 0, z = \tilde{z}\}\,.$$

In this blow-up, $M'' = [M' : D']$, we have a new boundary face and call it, in accordance to the situation near the cone tip, ff (Fig. 6). Finally, we still have the classical singularity along the temporal diagonal $D = \operatorname{diag}(M) \cap \mathrm{td}$, leading to a third blow-up, where we arrive at similar coordinates as in the case near the tip along td, leading to the final heat space $\mathcal{M}_\infty = [M'' : D]$ (Fig. 7).

Near the outer left face the coordinates stem from the blow-up of $Y = \{y = \tilde{y}\}$. After the blow-up y is the boundary defining function of bb and we have the following coordinates.

$$\rho_{\mathrm{tf}} = \tau = \sqrt{t},\ \rho_{\mathrm{lf}} = \tilde{s} = \frac{\tilde{y}}{y},\ \rho_{\mathrm{bb}} = y,\ z,\ \tilde{z}$$

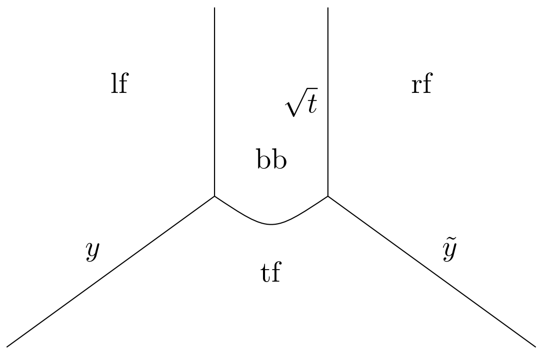

Figure 5: Single blow-up space $M' = [M^2 \times \mathbb{R}_+ : A]$ of the heat kernel near infinity

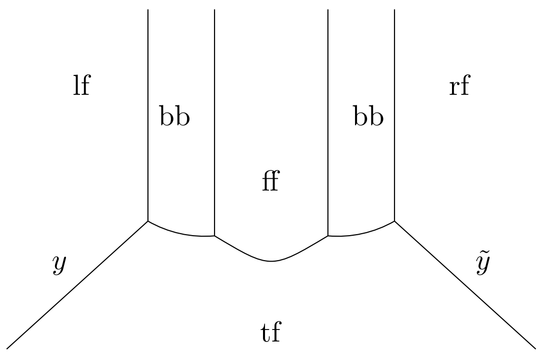

Figure 6: Double blow-up space $M'' = [M' : D']$ of the heat kernel near infinity

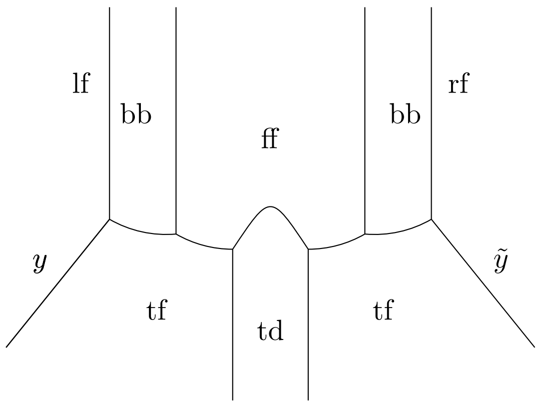

Figure 7: Heat space $\mathcal{M}_\infty$ of the heat kernel near infinity – a triple blow-up space

Near the outer right face we can use the same coordinates as near the outer left face, just with the roles of y and $\tilde{y}$ reversed.

$$\rho_{\mathrm{tf}} = \tau = \sqrt{t},\ \rho_{\mathrm{rf}} = s = \frac{y}{\tilde{y}},\ \rho_{\mathrm{bb}} = \tilde{y},\ z,\ \tilde{z}$$

Near the front face we obtain the new set of coordinates by writing the intersection D' of the spatial diagonal with bb in coordinates from the previous blow-up, say those with y as a boundary defining function of ff. Then $D' = \{y = 0, \tilde{s} = 1, z = \tilde{z}\}$. Consequently we obtain new projective coordinates

$$\rho_{\mathrm{td}} = \tau = \sqrt{t},\ \sigma = \frac{s-1}{y} = \frac{y-\tilde{y}}{y^2},\ \xi = \frac{z-\tilde{z}}{y},\ \rho_{\mathrm{ff}} = y,\ z,$$

as well as the corresponding set of coordinates using the coordinates from near the right face. Note that now bb lies in the limit as $|(\sigma, \xi)| \to \infty$, and both sets of coordinates are valid everywhere near ff (i.e. away from rf and lf).

Near the intersection of diagonal and front face we again rewrite the intersection in the previous coordinates. Then the intersection reads as $\mathrm{diag} \cap \mathrm{td} = \left\{\tau = 0,\ \sigma = 0,\ \tilde{\xi} = 0\right\}$ and we obtain corresponding coordinates

$$\rho_{\mathrm{td}} = \tau = \sqrt{t},\ \tilde{S} = \frac{\sigma}{\tau} = \frac{s-1}{y\tau},\ \tilde{Z} = \frac{\xi}{\tau} = \frac{\tilde{z}-z}{y\tau},\ \rho_{\mathrm{ff}} = y,\ z,$$

where now tf lies in the limit of $|(S, Z)| \to \infty$.

1.2.3 Obtaining the asymptotics

We now give a hands-on description on how to obtain the asymptotics. The general procedure is to restrict the operator $\partial_t - \Delta'$ to the front face in the blow-up space. There one can solve the heat equation in the model case explicitly, in our situation by either using the heat kernel of an exact cone, or the Euclidean heat kernel. Then the general theory tells us that these initial parametrices lie in a certain calculus and remaining error terms in the non-exact setting may be solved away with the von Neumann series, while preserving the asymptotic behaviour of the initial parametrix.

For our purposes it is important to understand the asymptotic behaviour of the heat kernel at all boundary faces of the corresponding heat spaces. We hope to illuminate the origin of these asymptotics a bit and sketch how one goes about in order to obtain the initial parametrices. For the bookkeeping we refer to the aforementioned works of Albin, and Mazzeo and Vertman.

Theorem 1.4 (Heat kernel asymptotics)**.** Let (M^m, g) be a conical manifold and let Δ' be the Laplace-Beltrami operator, with a possible shift by $C \geq 0$ in the tangential operator. Then the associated heat kernel lifts to a polyhomogeneous distribution on the heat space $\mathcal{M}$,

$$\beta^* H \sim \rho_{\mathrm{td}}^{-m} \rho_{\mathrm{ff}}^{-m} \rho_{\mathrm{lf}}^{\mu} \rho_{\mathrm{rf}}^{\mu} G,$$

where G is a bounded polyhomogeneous distribution on $\mathcal{M}$, vanishing to infinite order at tf, and μ is the minimum of the index set E at rf, given by

$$E = \left\{ \gamma \geq 0 \mid \gamma = -\frac{m-2}{2} + \sqrt{\frac{(m-2)^2}{4} + \lambda},\ \lambda \in -\operatorname{spec}(\Delta_N - C) \right\}.$$

Idea of proof. As alluded, the strategy is to first solve the heat equation in the exact case. Recall that in the conical situation, the matrix representation w.r.t. coordinates (x, z) of the metric has the form

$$g = \begin{pmatrix} 1 & 0 \\ 0 & x^2 g_N \end{pmatrix} + \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x^2) \end{pmatrix} =: g_0 + h.$$

For this consider the model Laplacian, i.e. that of g_0

$$\partial_x^2 + \frac{m-1}{x}\partial_x + \frac{1}{x^2}\Delta_N,$$

which we lift to the heat space $\mathcal{M}$ and examine it in coordinates which are valid near the front face, say those valid near the lower left (cf. Section 1.2.1).

$$\rho_{\mathrm{lf}} = \tilde{s} = \frac{\tilde{x}}{x},\ \rho_{\mathrm{ff}} = x,\ \rho_{\mathrm{td}} = \tau = \frac{t}{x^2},\ z,\ \tilde{z}$$

The lifts of the derivatives are now just calculated by the coordinate change, i.e.

$$\begin{aligned}\beta^*\partial_x &= \partial_x x \partial_x + \partial_x \tilde{s}\partial_{\tilde{s}} + \partial_x \tau \partial_\tau = \partial_x - \frac{1}{x}\tilde{s}\partial_{\tilde{s}} - \frac{1}{x}\tau\partial_\tau \\ \beta^*\partial_z &= \partial_z.\end{aligned}$$

As $x\partial_x$ is a b-vector field, we note that $\beta^*\partial_x = x^{-1}\mathcal{O}(\mathcal{V}_{\mathrm{b}})$. Here we also see the computational benefit of subsuming b-vector fields under $\mathcal{V}_{\mathrm{b}}$, as the calculation of $\beta^*\partial_x^2$ takes several lines, while we record, dropping the $\mathcal{O}$-notation in the computation,

$$\beta^*\partial_x^2 = x^{-1}\mathcal{V}_{\mathrm{b}}(x^{-1}\mathcal{V}_{\mathrm{b}}) = x^{-2}\mathcal{V}_{\mathrm{b}} + x^{-2}\mathcal{V}_{\mathrm{b}}^2,$$

since the application of $\mathcal{V}_{\mathrm{b}}$ does not alter x-behaviour. We subsume this in the notation $\beta^*\partial_x^2 = x^{-2}\mathcal{O}(\mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2)$, which is an extension of the Landau symbol as $x \to 0$, to denote linear combinations with smooth and bounded coefficients involving at least of the arguments.

In particular we note that the model Laplacian in these coordinates is still of conical form, i.e.

$$\Delta = \partial_x^2 + \frac{m-1}{x}\partial_x + \frac{1}{x^2}\left(\Delta_N + \mathcal{O}(\mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2)\right).$$

The main feature of the b-vector fields is that they vanish at the boundary surfaces. Hence the initial parametrix at the front face is chosen to be the exact heat kernel, which can be given explicitly. For this it is useful to consider a unitary transformation as in [BV16], namely

$$\begin{aligned}\Phi : L^2(M, \mathrm{dvol}(g_0) = dx^2 \oplus x^n\,\mathrm{dvol}_N) &\to L^2(M, dx^2 \oplus \mathrm{dvol}_N) \\ f &\mapsto x^{n/2} f,\end{aligned}$$

where $n = m - 1$ is the dimension of the cross section. A straightforward calculation shows that the model heat operator under this transformation has the form

$$\begin{aligned}\Phi \circ (\partial_t - \Delta') \circ \Phi^{-1} &= \partial_t - \Phi \circ \left(\partial_x^2 - \frac{m-1}{x}\partial_x - \frac{1}{x^2}(\Delta_N - C)\right) \circ \Phi^{-1} \\ &= \partial_t - \partial_x^2 - \frac{1}{x^2}(\Delta_N - C) + \frac{1}{x^2}\left(\left(\frac{n-1}{2}\right)^2 - \frac{1}{4}\right) \\ &=: \partial_t - \partial_x^2 + \frac{1}{x^2}\left(A - \frac{1}{4}\right)\end{aligned}$$

which is a so called Bessel-type equation.

As N is a compact manifold, Δ_N has a discrete spectrum $\mathrm{spec}(\Delta_N) = \{0, -\nu_0^2, -\nu_1^2, \dots\}$. This is also why in analysis often the negative Laplacian is preferred, and, in contrast to our, geometric, naming convention, called positive. Consequently $A = -\Delta_N + C + (n-1)^2/4$ has a discrete and positive spectrum. We can use the eigendecomposition of the spatial operator and for each $\lambda^2 \in \operatorname{spec}(A)$ obtain an ordinary differential operator l_λ, i.e.

$$-\Phi \circ \Delta' \circ \Phi^{-1} = \bigoplus_{\lambda^2 \in \operatorname{spec}(A)} -\partial_x^2 + \frac{1}{x^2}\left(\lambda^2 - \frac{1}{4}\right) =: \bigoplus_{\lambda^2 \in \operatorname{spec}(A)} l_\lambda.$$

The corresponding heat kernels are explicitly known (cf. [Les97, Section 2.3]) and given by

$$e^{-tl_\lambda}(x,\tilde{x}) = \frac{1}{2t}(x\tilde{x})^{1/2} I_\lambda\left(\frac{x\tilde{x}}{2t}\right) e^{-\frac{x^2+\tilde{x}^2}{4t}},$$

where

$$I_\lambda(z) = \left(\frac{1}{2}z\right)^\lambda \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k!\Gamma(\lambda+k+1)}$$

denotes the modified Bessel function of order λ, with λ always being the positive root of λ^2. Hence the heat kernel H^Φ of the transformed operator is given by

$$H^\Phi(t,x,\tilde{x},z,\tilde{z}) = \sum_\lambda \frac{1}{2t}(x\tilde{x})^{1/2} I_\lambda\left(\frac{x\tilde{x}}{2t}\right) e^{-\frac{x^2+\tilde{x}^2}{4t}} \phi_\lambda(z)\phi_\lambda(\tilde{z}),$$

where ϕ_λ is the eigenfunction to the eigenvalue λ^2 of A.

We have to relate the heat kernels $e^{t\Phi\Delta\Phi^{-1}} =: H^\Phi$ and $e^{t\Delta} =: H$. Both operators act as integral operators, and we note that, for a function supported near the singularity, say on $U = (0,1) \times N$, $u \in C_0^\infty(U)$, we have

$$\begin{aligned} H^\Phi * u &= \iint H^\Phi(t-\tilde{t},x,z,\tilde{x},\tilde{z})u(\tilde{x},\tilde{t})d\tilde{t}d\tilde{x}d\tilde{z} \\ &= \iint x^{n/2} H(t-\tilde{t},x,z,\tilde{x},\tilde{z})\tilde{x}^{-n/2}u(\tilde{x},\tilde{t})\tilde{x}^n d\tilde{t}d\tilde{x}d\tilde{z}, \\ &= X^{n/2} H * X^{-n/2} u, \end{aligned}$$

where X is the multiplication operator by x. Hence

$$H^\Phi(t,x,z,\tilde{x},\tilde{z}) = (x\tilde{x})^{n/2} H(t,x,z,\tilde{x},\tilde{z}).$$

Now we are in a position to simply read off the asymptotics, by evaluating

$$H(t,x,z,\tilde{x},\tilde{z}) = \frac{1}{2t}(x\tilde{x})^{\frac{-n+1}{2}} \sum_\lambda I_\lambda\left(\frac{x\tilde{x}}{2t}\right) e^{-\frac{x^2+\tilde{x}^2}{4t}} \varphi_\lambda(z)\varphi_\lambda(\tilde{z})$$

in the corresponding sets of coordinates, by assuming compact support in each of the regimes. Near the lower right we have coordinates

$$\rho_{\mathrm{rf}} = s = \frac{x}{\tilde{x}},\ \rho_{\mathrm{ff}} = \tilde{x},\ \rho_{\mathrm{td}} = \tau = \frac{t}{\tilde{x}^2},\ z,\ \tilde{z},$$

and consequently, as $n = m - 1$,

$$\begin{aligned} H(\tau, s, z, \tilde{x}, \tilde{z}) &= \frac{1}{2\tau\tilde{x}^2}(s\tilde{x})^{\frac{-n+1}{2}} \sum_\lambda I_\lambda\left(\frac{s\tilde{x}^2}{2\tau\tilde{x}^2}\right) e^{-\frac{(s^2+1)\tilde{x}^2}{4\tau\tilde{x}^2}} \varphi_\lambda(z)\varphi_\lambda(\tilde{z}) \\ &= \tilde{x}^{-m}\frac{1}{2\tau}s^{\frac{-n+1}{2}} \sum_\lambda I_\lambda\left(\frac{s}{2\tau}\right) e^{-\frac{(s^2+1)}{4\tau}} \varphi_\lambda(z)\varphi_\lambda(\tilde{z}). \end{aligned}$$

We immediately see from the formula that in this regime the heat kernel has leading behaviour at the front face of ρ_{ff}^{-m}, and vanishes to infinite order as $\tau = \rho_{\mathrm{td}} \to 0$. For the side face behaviour we have a contribution of s^λ by the modified Bessel function, so that we have a side face behaviour of $\rho_{\mathrm{rf}}^{\frac{-n+1}{2}+\lambda}$ and the right face index set is indeed given by

$$E = \left\{\gamma \geq 0 \mid \gamma = -\frac{n-1}{2} + \lambda \mid \lambda^2 \in \operatorname{spec} A\right\}.$$

As $A = -(\Delta_N - C) + \left(\frac{n-1}{2}\right)^2$ the claim follows in this regime.

As the expression is symmetric in x and $\tilde{x}$, we obtain near the lower left the same index set. Restricting to the top corner we have coordinates

$$\rho_{\mathrm{ff}} = \tau = \sqrt{t},\ \rho_{\mathrm{rf}} = \xi = \frac{x}{\tau},\ \rho_{\mathrm{lf}} = \tilde{\xi} = \frac{\tilde{x}}{\tau},\ z,\ \tilde{z},$$

so that the heat kernel reads as

$$\begin{aligned} H(\tau, \xi, z, \tilde{\xi}, \tilde{z}) &= \frac{1}{2\tau^2}(\xi\tau\tilde{\xi}\tau)^{\frac{-n+1}{2}} \sum_\lambda I_\lambda\left(\frac{\xi\tau\tilde{\xi}\tau}{2\tau^2}\right) e^{-\frac{\xi^2\tau^2+\tilde{\xi}^2\tau^2}{4\tau^2}} \varphi_\lambda(z)\varphi_\lambda(\tilde{z}) \\ &= \frac{1}{2}\tau^{-n+1-2}(\xi\tilde{\xi})^{\frac{-n+1}{2}} \sum_\lambda I_\lambda\left(\frac{\xi\tilde{\xi}}{2}\right) e^{-\frac{\xi^2+\tilde{\xi}^2}{4}} \varphi_\lambda(z)\varphi_\lambda(\tilde{z}). \end{aligned}$$

Again we immediately obtain a front face behaviour of order $-n+1-2 = -m$ and the side face index sets are now again given as before by E.

Assuming compact support near $\mathrm{td} \cap \mathrm{ff}$ the heat kernel asymptotics are captured in the coordinates

$$\rho_{\mathrm{td}} = \eta = \frac{\sqrt{t}}{x},\ S = \frac{x - \tilde{x}}{\sqrt{t}},\ Z = \frac{z - \tilde{z}}{\eta},\ \rho_{\mathrm{ff}} = x, z.$$

As in these coordinates $\tilde{x} = (1 - S\eta)x$ and $\tilde{z} = z - Z\eta$, the heat kernel reads after cancellations

$$\begin{aligned} &H(\eta, x, z, S, Z) \\ =&\frac{1}{2\eta^2}x^{-m}(1 - S\eta)^{\frac{-n+1}{2}} \sum_\lambda I_\lambda\left(\frac{(1 - S\eta)}{2\eta^2}\right) e^{-\frac{(1+(1-S\eta)^2)}{4\eta^2}} \varphi_\lambda(z)\varphi_\lambda(z - Z\eta). \end{aligned}$$

We immediately verify the expected front face behaviour as well as that the heat kernel vanishes to infinite order as $|S| \to \infty$. Furthermore the heat kernel shows some possibly singular behaviour at the temporal diagonal but of better order than the postulated ρ_{td}^{-m}. This behaviour actually is classical and appears also in the non-singular situation, as it comes from the initial condition

$$H(t, x, z, \tilde{x}, \tilde{z}) \to \delta(p, \tilde{p}) = \delta_0(x - \tilde{x})\delta_0(z - \tilde{z}) \text{ as } t \to 0,$$

since the k-dimensional δ-distribution is homogeneous of degree $-k$, so that in coordinates near the diagonal

$$\delta_0(S\eta x)\delta_0(Z\eta) = \eta^{-m}\delta_0(Sx)\delta_0(Z).$$

Finally we observe that, as we approach td only if simultaneously $z - \tilde{z}$ and t approach 0, that one actually approaches tf. Hence tf lies also in the limit $|Z| \to \infty$ and the heat kernel vanishes to infinite order in this situation.

It follows now from the proof of [MV12, Theorem 1.2] that these asymptotics of the heat kernel for the model Laplacian also hold for the heat kernel of the Laplacian Δ_g. □

The corresponding result near infinity follows more easily as everything reduces to the Euclidean situation and we will not carry out the evaluations in detail.

Theorem 1.5 (Heat kernel asymptotics near infinity)**.** Let (M^m, g) be an asymptotically conical manifold and let Δ' be the Laplace-Beltrami operator, with a possible shift by C in the tangential operator. Then the associated heat kernel lifts to a polyhomogeneous distribution on the heat space $\mathcal{M}_\infty$,

$$\beta^* H \sim \rho_{\mathrm{td}}^{-m}\rho_{\mathrm{ff}}^{0} G,$$

where G is a bounded polyhomogeneous distribution on $\mathcal{M}_\infty$, vanishing to infinite order at tf, the inner side face bb and the outer side faces rf and lf.

Idea of proof. Evaluating the model Laplacian in the coordinates near the front face, one sees that the Laplacian at the front face is actually the Euclidean one. As observed by Sher in [She13, Appendix A] the corresponding iterative heat kernel construction has been done by Albin [Alb07, Theorem 5.2], thus the asymptotics are given by the Euclidean heat kernel, which does not show singular behaviour as $y \to 0$ and $\tilde{y} \to 0$, but vanishes to infinite order as we move away from the diagonal, which again can be seen by evaluating the Euclidean heat kernel in all regimes.

The behaviour near the temporal diagonal again follows from the initial condition imposed. □

2 Setup and Linearization

In this section we first describe the parametric Ansatz for the mean curvature flow and introduce some further notation. We obtain a partial differential equation for a scalar function, whose asymptotic behaviour we then discuss as $x \to 0$ or $y = x^{-1} \to 0$. The resulting linearization then dictates the Hölder spaces and the mapping properties of the heat kernel, which we need to set up a fixpoint argument.

2.1 Setup for Mean Curvature Flow

We begin by following the very concise and accessible lecture notes by Mantegazza [Man11]. The mean curvature flow is the evolution of an immersed codimension-1-manifold proportionally to its mean curvature in direction of its normal vector.

Let M be a an orientable manifold of dimension m, which is immersed into $\mathbb{R}^{m+1}$ via $\varphi_0 : M \to \mathbb{R}^{m+1}$. Denoting the standard scalar product on $\mathbb{R}^{m+1}$ by $\langle \cdot, \cdot \rangle$, its pullback $g_0 = \varphi^* \langle \cdot, \cdot \rangle$ gives a Riemmanian metric on M so that the immersion becomes isometric. As usual in the extrinsic setting, given local coordinates $x_1, \ldots, x_n$ we identify the tangent vectors $\partial_i \varphi$ with their corresponding abstract derivations ∂_i whenever appropriate. Let ν be the inner pointing unit normal vector. Then the *second fundamental form* $A = (h_{ij})$ is given by $h_{ij} = \langle \partial_i \partial_j \varphi, \nu \rangle$. Its eigenvalues are called *principal curvatures* and their sum is the *mean curvature* H, or equivalently, it is the trace of A, i.e. $H = g^{ij} h_{ij}$, where we employed the Einstein summation convention, namely summing over indices appearing both as a sub- and as a superscript. The *mean curvature flow*, or *MCF* for short, now is a differentiable family of immersions $\varphi : [0, T] \times M \to \mathbb{R}^{m+1}$, such that

$$\begin{cases} \partial_t \varphi(t, p) = H_t(p) \nu_t(p) & \forall p \in M, \forall t \in [0, T] \\ \varphi(0, p) = \varphi_0(p) & \forall p \in M. \end{cases}$$

While the definition of the mean curvature depends on the choice of the unit normal, the *mean curvature vector* $H\nu$ does not, so that the the flow is well defined. The choice of the inner pointing unit normal is a purely aesthetic one, as this is the choice that makes spheres positively curved.

We now wish to see its relation to the heat equation. Note that both the inner pointing unit normal vector ν, as well as the immersion φ itself can be regarded as a smooth function $M \to \mathbb{R}^{m+1}$, or, as sections of a trivial line bundle over the manifold. The Gauß-Weingarten relations are given by

$$\partial_i \partial_j \varphi = \Gamma_{ij}^k \partial_k \varphi + h_{ij} \nu \quad \text{and} \quad \partial_i \nu = -h_{jl} g^{ls} \partial_s \varphi.$$

These represent the fact that the covariant derivative of the manifold is related to that of the ambient space via $\nabla^M = \nabla^{\mathbb{R}^{m+1}} - A\nu$. Interpreting smooth functions as sections of trivial line bundles, we have

$$\begin{aligned}\Delta\varphi &= g^{ij}\nabla^2_{ij}\varphi = g^{ij}\left(\nabla_{\partial_i}\nabla_{\partial_j}\varphi - \nabla_{\nabla_{\partial_i}\partial_j}\varphi\right)\\ &= g^{ij}(\partial_i\partial_j\varphi - \Gamma^k_{ij}\partial_k\varphi)\\ &= g^{ij}\left(h_{ij}\nu + \Gamma^k_{ij}\partial_k\varphi - \Gamma^k_{ij}\partial_k\varphi\right)\\ &= H\nu.\end{aligned}$$

Note that for any smooth function $\sigma : (M,g) \to \mathbb{R}^{m+1}$ the Laplacian $\Delta\sigma$ is actually the component-wise application of the Laplace-Beltrami operator of (M,g) on σ, i.e.

$$\Delta\sigma = (\Delta\sigma^1, \ldots, \Delta\sigma^{m+1}).$$

In particular this gives the mean curvature flow the appealing form

$$\partial_t\varphi = H_t\nu_t = \Delta_t\varphi,$$

where now also the Laplacian is time dependent, since the induced metric on M is. However this system is degenerate by the invariance of the Laplacian under diffeomorphisms. In order to overcome this limitation we use Fermi coordinates and represent any small perturbation of our initial surface in $\mathbb{R}^{m+1}$ via

$$\varphi = \varphi_0 + f\nu_0,$$

where $f : M \to \mathbb{R}$ is a smooth function on M and $\varphi_0 : M \to \mathbb{R}^{m+1}$ is the embedding of the initial surface. Consequently an evolution of the manifold can be modelled via a function $f : M \times [0,T] \to \mathbb{R}^{m+1}$ with $f(p,0) = 0$ for all $p \in M$.

In this thesis we only deal with short-time existence, so we will without further mention assume that φ stays immersed at all times.

As now the movement is restricted to the normal direction, we obtain a partial differential equation for f by projecting onto the time dependent normal ν, i.e.

$$\langle\partial_t\varphi, \nu\rangle = \langle\partial_t\varphi_0 + \partial_t(f\nu_0), \nu\rangle = \partial_t f\langle\nu_0, \nu\rangle,$$

where $\langle\cdot\,,\cdot\rangle$ is the standard scalar product on Euclidean space. Consequently

$$\partial_t f = \frac{\langle\Delta\varphi, \nu\rangle}{\langle\nu_0, \nu\rangle}.$$

In each component we still have the product rule of the Laplacian and it carries over with little abuse of notation to our situation. We have

$$\begin{aligned}\Delta\varphi &= \Delta(\varphi_0 + f\nu_0) = \Delta\varphi_0 + \Delta(f\nu_0)\\ &= \Delta\varphi_0 + f\Delta\nu_0 + (\Delta f)\nu_0 + 2g(\nabla f, \nabla\nu_0),\end{aligned}$$

where we denote by $2g(\nabla f, \nabla \nu_0)$ the vector with entries $2g(\nabla f, \nabla \nu_0^k)$, for $k = 1, \ldots, m+1$. In order to obtain a linearization w.r.t. the initial Laplacian, we separate $\Delta \equiv \Delta_t = \Delta_0 + (\Delta_t - \Delta_0)$ and obtain

$$\begin{aligned}\Delta\varphi \equiv \Delta_t\varphi = &\Delta_0\varphi_0 + f\Delta_0\nu_0 + (\Delta_0 f)\nu_0 + 2g(, \nabla\nu_0) \\ &+ (\Delta_t - \Delta_0)\varphi_0 + f(\Delta_t - \Delta_0)\nu_0 + ((\Delta_t - \Delta_0)f)\nu_0.\end{aligned}$$

As $\Delta_0\varphi_0 = H_0\nu_0$, we get, after cancellations, the following preliminary linearization

$$\begin{aligned}\partial_t f = &H_0 + \Delta_0 f + (\Delta_t - \Delta_0)f + f\frac{\langle \Delta_0\nu_0, \nu\rangle}{\langle \nu_0, \nu\rangle} + \frac{\langle 2g(\nabla f, \nabla\nu_0), \nu\rangle}{\langle \nu_0, \nu\rangle} \\ &+ \frac{\langle (\Delta_t - \Delta_0)\varphi_0, \nu\rangle}{\langle \nu_0, \nu\rangle} + f\frac{\langle (\Delta_t - \Delta_0)\nu_0, \nu\rangle}{\langle \nu_0, \nu\rangle}. \qquad \text{(MCF pre)}\end{aligned}$$

2.1.1 The conical setup near the tip

Before we continue examining this equation, we elaborate the conical setting a bit more. Let N be a $(m-1)$-dimensional, orientable and compact manifold, which immerses via φ_N into $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. Let $M = \mathbb{R}_+ \times N$ now be the topological cylinder over N. Let x be the coordinate on the ray $\mathbb{R}_+$ and $z = (z_1, \ldots, z_{m-1})$ be coordinates on N. In notation we will always pretend that N is 1-dimensional and write as coordinates (x, z).

We obtain a cone by scaling down the cross section linearly with the radial variable x, i.e. as an abstract manifold, we would equip M with a metric $g = dx^2 \oplus x^2 g_N$. Taking $g_N = \varphi_N^*\langle\cdot,\cdot\rangle$ as the induced metric, we obtain via $\varphi : M \to \mathbb{R}^{m+1}, (x, z) \mapsto x\varphi_N(z)$ an immersion of M. One readily checks that for the induced metric $g = \varphi^*\langle\cdot,\cdot\rangle$, as expected, we have $g = dx^2 \oplus x^2 g_N$ in local coordinates (x, z). As will become apparent later, we will need to consider slight perturbations of such conical manifolds and consequently do not assume that the rescaled cross section is independent of x, and only prescribe the scaling behaviour for the whole immersion. So in what follows we will always consider immersions of the form

$$\varphi : M \to \mathbb{R}^{m+1},\ (x, z) \mapsto x\varphi_N(x, z),$$

where $\varphi_N(x, \cdot) : N \to \mathbb{S}^m \subset \mathbb{R}^{m+1}$ is for each fixed x an immersion of the cross section.

As we use coordinates adapted to the conical situation, some of the formulae behave not as one might be accustomed to from classical geometry. The most obvious difference is that calculations are not performed in normal coordinates and that the metric at a point (x, z) has the form

$$
\begin{aligned}
g = (g_{ij}) &= \begin{pmatrix} \langle \partial_x \varphi, \partial_x \varphi \rangle & \langle \partial_x \varphi, \partial_z \varphi \rangle \\ \langle \partial_z \varphi, \partial_x \varphi \rangle & \langle \partial_z \varphi, \partial_z \varphi \rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle (\varphi_N + x\partial_x \varphi_N), (\varphi_N + x\partial_x \varphi_N) \rangle & \langle (\varphi_N + x\partial_x \varphi_N), x\partial_z \varphi_N \rangle \\ \langle x\partial_z \varphi_N, (\varphi_N + x\partial_x \varphi_N) \rangle & \langle x\partial_z \varphi_N, x\partial_z \varphi_N \rangle \end{pmatrix} \\
&= \begin{pmatrix} 1 & \mathcal{O}(x) \\ \mathcal{O}(x) & x^2 g_N \end{pmatrix}.
\end{aligned}
$$

For the second fundamental form we calculate

$$
\begin{aligned}
A = (h_{ij}) &= \begin{pmatrix} \langle \partial_x \partial_x x\varphi_N, \nu \rangle & \langle \partial_x \partial_z x\varphi_N, \nu \rangle \\ \langle \partial_z \partial_x x\varphi_N, \nu \rangle & \langle \partial_z \partial_z x\varphi_N, \nu \rangle \end{pmatrix} \\
&= \begin{pmatrix} \langle 2\partial_x \varphi_N + x\partial_x \partial_x \varphi_N, \nu \rangle & \langle \partial_z \varphi_N + x\partial_x \partial_z \varphi_N, \nu \rangle \\ \langle \partial_z \varphi_N + x\partial_x \partial_z \varphi_N, \nu \rangle & \langle x\partial_z \partial_z \varphi_N, \nu \rangle \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & xA_N \end{pmatrix},
\end{aligned}
$$

where one has to be careful, as $\langle \partial_x \varphi_N, \nu \rangle$ is in the non-exact setting not necessary 0, as opposed to $\langle \partial_z \varphi_N \rangle$ (as $\langle \partial_z \varphi, \nu \rangle = x \langle \partial_z \varphi_N, \nu \rangle = 0$).

Most of the following calculations amount to a counting game, weighing x- and z-coordinates up against each other. So the mean curvature is calculated as

$$
\begin{aligned}
H = g^{ij} h_{ij} &= \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x^{-1}) \\ \mathcal{O}(x^{-1}) & x^{-2} g_N^{-1} \end{pmatrix}^{ij} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & xA_N \end{pmatrix}_{ij} \\
&= \mathcal{O}(1) + \mathcal{O}(1) + \mathcal{O}(1) + x^{-1} H_N \\
&= \mathcal{O}(x^{-1}),
\end{aligned}
$$

where the first summand in the second line corresponds to the xx-index, the second and third to mixed indices and the last one to zz-indices. Unsurprisingly we note that for cones $H = \mathcal{O}(x^{-1})$.

In the extrinsic setting, the Riemann tensor, the Ricci tensor and the scalar curvature can be expressed by means of the second fundamental form:

$$
\begin{aligned}
\mathrm{Riem}_{ijkl} &= h_{ik} h_{jl} - h_{il} h_{jk} \\
\mathrm{Ric}_{ij} &= H h_{ij} - h_{il} g^{lk} h_{kj} \\
\mathrm{scal} &= H^2 - \|A\|^2.
\end{aligned}
$$

In particular we observe that the sectional curvatures behave as expected from the exact case:

$$K_{xz} = \frac{R_{xzxz}}{g_{xx}g_{zz} - g_{xz}^2} = \frac{h_{xz}h_{xz} - h_{xx}h_{zz}}{g_{xx}g_{zz} - g_{xz}^2} = \frac{\mathcal{O}(x)^2 - \mathcal{O}(1)\mathcal{O}(x)}{\mathcal{O}(1)\mathcal{O}(x^2) - \mathcal{O}(x)^2} = \mathcal{O}(x^{-1})$$
$$K_{zz'} = \frac{h_{zz'}^2 - h_{zz}h_{z'z'}}{g_{zz}g_{z'z'} - g_{zz'}^2} = \frac{\mathcal{O}(x)^2 - \mathcal{O}(x)^2}{\mathcal{O}(x^2)^2 - \mathcal{O}(x^2)^2} = \mathcal{O}(x^{-2})$$

2.1.2 Conical setup near infinity

Near infinity we will make a change of coordinates and use $y = x^{-1}$ instead of x as the radial coordinate, so that y is the boundary defining function of the boundary at infinity. Consequently now the embedding φ is given by

$$\varphi(y, z) \to y^{-1}\varphi_N(y, z).$$

We further impose that also in this situation $\varphi_N(y, z) = \mathcal{O}(1)$ as $y \to 0$. While near the tip terms like $\partial_x^2 x\varphi_N$ did not result in singular behaviour, we now usually have negative exponents so that any application of ∂_y results in worse asymptotics. As

$$\partial_y\varphi(y, z) = \partial_y(y^{-1}\varphi_N(y, z)) = -y^{-2}\varphi_N(y, z) + y^{-1}\partial_y\varphi_N(y, z)$$

we obtain for the metric

$$g = \begin{pmatrix} \mathcal{O}(y^{-4}) & \mathcal{O}(y^{-3}) \\ \mathcal{O}(y^{-3}) & \mathcal{O}(y^{-2}) \end{pmatrix}.$$

In contrast to the second fundamental form near the tip, near infinity it shares the asymptotic symmetries of the metric, since

$$\partial_y^2(\varphi(y, z)) = 2y^{-3}\varphi_N(y, z) - 2y^{-2}\partial_y\varphi_N(y, z) + y^{-1}\partial_y^2\varphi_N(y, z),$$

so that

$$A = \begin{pmatrix} \mathcal{O}(y^{-3}) & \mathcal{O}(y^{-2}) \\ \mathcal{O}(y^{-2}) & \mathcal{O}(y^{-1}) \end{pmatrix}.$$

Again we can calculate the most common curvatures in terms of the second fundamental form. In particular we observe for the sectional curvatures

$$K_{yz} = \frac{h_{yz}h_{yz} - h_{yy}h_{zz}}{g_{yy}g_{zz} - g_{yz}^2} = \frac{\mathcal{O}(y^{-2})^2 - \mathcal{O}(y^{-3})\mathcal{O}(y^{-1})}{\mathcal{O}(y^{-4})\mathcal{O}(y^{-2}) - \mathcal{O}(y^{-3})^2} = \mathcal{O}(y^2)$$
$$K_{zz'} = \frac{h_{zz'}^2 - h_{zz}h_{z'z'}}{g_{zz}g_{z'z'} - g_{zz'}^2} = \frac{\mathcal{O}(y^{-1})^2 - \mathcal{O}(y^{-1})^2}{\mathcal{O}(y^{-2})^2 - \mathcal{O}(y^{-2})^2} = \mathcal{O}(y^{-2}),$$

again confirming the expectations from the exact case.

2.2 Linearization near the tip

We now turn back to the Ansatz for mean curvature flow in Section 2.1. From now on we assume that $M = \mathbb{R}_+ \times N$ with compact cross section N is embedded via $\varphi_0(x,z) = x\varphi_N(x,z)$ into $\mathbb{R}^{m+1}$. We denote by ν_0 the inner pointing unit normal to $M_0 = \varphi_0(M)$ and consider

$$\varphi = \varphi_0 + f\nu_0$$

for a function $f : M \to \mathbb{R}$. Although later this function will describe the evolution as time passes, in this section we are interested in its spatial notions and ignore any temporal behaviour for now. In this step we determine the necessary regularity of f to carry out further arguments, so all derivatives are to be viewed as formal.

As we wish to preserve the conical structure, it is immediately clear that we have to impose a corresponding scaling onto f and assume

$$f(x,z) = xu(x,z).$$

Our goal now is to understand Eq. (MCF pre) and linearize it in terms of such an f, i.e. separating those terms which are purely given by the initial data, those terms which are linear in f and those which are of higher order in f. In light of the heat kernel asymptotics introduced in Theorem 1.4 the bookkeeping involves mainly counting the x^{-1}-factors and recording the differentials in terms of the edge differentials $\mathcal{V}_{\mathrm{b}} \in \{x\partial_x, \partial_z\}$.

In all estimates we will usually denote quantities of the initial metric by a zero in the index or, when there are already many indices involved, as an argument.

Remark 2.1. For the linearization it is more natural to work with u instead of f, which immediately becomes clear when calculating the inverse of the metric. This could be partly remedied by the use of so called *incomplete b-vector fields*, which are spanned by $\partial_x, x^{-1}\partial_z$ and correspond to the scaling of cone metrics. But on one hand this still leaves some linear factors which are more easily handled in the complete b-calculus, and on the other hand this allows us to calculate the linearization near infinity in almost exactly the same way. Afterwards we can simply go back to f, as $u = x^{-1}f$.

At this point it is convenient, almost necessary, to introduce further notations to ease the notational burden. For one, we expand the $\mathcal{O}$-notation to matrices. If we have matrices $A = (a_{ij}), B = (b_{ij})$, then we write $A = \mathcal{O}(B)$, when we have $a_{ij} = \mathcal{O}(b_{ij})$ for all i,j. Furthermore we write $f = \mathcal{O}_2(g_1,\dots,g_n)$ if f is quadratic in the arguments, i.e. using the more exact $\in$-notation of asymptotic classes, $f \in \bigcup_{i,j=1}^n \mathcal{O}(g_ig_j)$.

We begin by examining the metric and its inverse.

Lemma 2.2. For the induced metric, the following holds:

$$g = g_0 + u\mathcal{O}(g_0) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0) =: g + F$$
$$g^{-1} = g_0^{-1} + u\mathcal{O}(g_0^{-1}) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0^{-1}) =: g_0^{-1} + G$$

Proof. In local coordinates one has (cf. [Man11])

$$g_{ij} = g_{ij}(0) - 2xuh_{ij}(0) + x^2u^2h_{ik}(0)g^{kl}(0)h_{lj}(0) + (xu)_i(xu)_j.$$

As we have seen in the previous section

$$A_0 \sim \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x) \end{pmatrix},$$

so the first summand can immediately be recognized as

$$2xuA \sim u \begin{pmatrix} \mathcal{O}(x) & \mathcal{O}(x^2) \\ \mathcal{O}(x^2) & \mathcal{O}(x^2) \end{pmatrix}.$$

By ignoring the better behaviour in the entries involving x-coordinates, we can shorten this to $u\mathcal{O}(g_0)$. The asymptotics of the second term follow similarly:

$$\begin{aligned} x^2u^2h_{ik}g^{kl}h_{lj} &= x^2u^2 \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x) \end{pmatrix} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x^{-1}) \\ \mathcal{O}(x^{-1}) & \mathcal{O}(x^{-2}) \end{pmatrix} \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(x) \\ \mathcal{O}(x) & \mathcal{O}(x) \end{pmatrix} \\ &= x^2u^2 \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) \\ \mathcal{O}(1) & \mathcal{O}(1) \end{pmatrix} = u^2\mathcal{O}(x^2) \end{aligned}$$

For the third term we calculate $\partial_x(xu) = u + x\partial_x u = \mathcal{O}(u, \mathcal{V}_\mathrm{b} u)$ and $\partial_z(xu) = x\partial_z u = x\mathcal{O}(\mathcal{V}_\mathrm{b} u)$, so that

$$(xu)_i(xu)_j = \begin{pmatrix} \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u) & x\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u) \\ x\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u) & x^2\mathcal{O}_2(\mathcal{V}_\mathrm{b} u) \end{pmatrix} = \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0).$$

To unify the notation, we subsume the asymptotic behaviour of $u^2\mathcal{O}(x^2)$ under that of $\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0)$.

The asymptotics for the inverse metric follow now immediately by application of the von Neumann series

$$g^{-1} = g_0^{-1} + \sum_{k=1}^{\infty} \Big(g_0^{-1}(u + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u))\mathcal{O}(g_0)\Big)^k g_0^{-1} =: g_0^{-1} + G.$$

□

The previous lemma has sufficiently exact information for most of our calculations. Indeed, often we can even reduce its statement regarding the inverse of the metric to $g^{-1} - g_0^{-1} = G = u\mathcal{O}(g_0^{-1})$. However, the leading term of G is linear in u and actually leads to an improvement in the side face behaviour of the heat kernel. We record it in the following observation; it follows directly from looking at the von Neumann series.

Observation 2.3. The first order term of G is given by $-2g_0^{-1}xuA(0)g_0^{-1}$.

To discuss the linearization of the time-dependent Laplacian, we first establish the following elementary lemma.

Lemma 2.4 (Determinant lemma)**.** The determinant obeys

$$|g| = |g_0|(1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)).$$

Proof. Write $g = g_0 + F$. Note that

$$\begin{aligned}\mathcal{O}(F)_{ij} &= u\mathcal{O}(g_0)_{ij} + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0)_{ij}\\ &= \mathcal{O}(u)(g_0)_{ij} + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0)_{ij},\end{aligned}$$

as one can absorb the bounded terms of $\mathcal{O}(g_0)$ into the bounded terms of $\mathcal{O}(u, \mathcal{V}_\mathrm{b} u)$. Now the result follows by expanding the explicit formula for the determinant:

$$\begin{aligned}|g| &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)(g_0 + \mathcal{O}(F))_{1,\sigma(1)} \dots (g_0 + \mathcal{O}(F))_{m,\sigma(m)}\\ &= \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)(g_0)_{1,\sigma(1)} \dots (g_0)_{m,\sigma(m)}(1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b}))\\ &= |g_0|(1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)).\end{aligned}$$

□

Lemma 2.5 (Laplacian lemma)**.** For the time dependent Laplacian we have

$$\Delta_t = \Delta_0 + x^{-2}\left(\mathcal{O}(\mathcal{V}_\mathrm{b} u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u, \mathcal{V}_\mathrm{b}^2 u)\right)\left\{\mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\right\}.$$

Proof.

$$\begin{aligned}\Delta_t &= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g^{ij}\partial_j\right)\\ &= \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}g_0^{ij}\partial_j\right) + \frac{1}{\sqrt{|g|}}\partial_i\left(\sqrt{|g|}G^{ij}\partial_j\right)\\ &=: A + B\end{aligned}$$

For the first term we have

$$\begin{aligned}A &= \Delta_0 + g_0^{ij}\frac{\partial_i\sqrt{1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)}}{\sqrt{1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)}}\partial_j\\ &= \Delta_0 + g_0^{ij}\frac{1}{2}\frac{\partial_i(\mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u))}{1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)}\partial_j\\ &= \Delta_0 + (1 + \mathcal{O}(u))g_0^{ij}\partial_i(\mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u))\partial_j,\end{aligned}$$

where we made use of Lemma 2.4 and employed the geometric series to see that $\frac{1}{1+\mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u)} = (1+\mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u))$. We will now count the occurences of x^{-1}-factors coming from g_0^{-1} in order to determine the behaviour in terms of b-derivatives $\mathcal{V}_{\mathrm{b}}$. For this we introduce a new notation, where curly brackets make the case distinction whether the index equals to x or to z, so that we have

$$\begin{aligned}A &= \Delta_0 + \begin{Bmatrix} 1 \\ x^{-1} \end{Bmatrix}^i \begin{Bmatrix} 1 \\ x^{-1} \end{Bmatrix}^j \begin{Bmatrix} x^{-1}\mathcal{V}_{\mathrm{b}}(\mathcal{O}(u)+\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u)) \\ \mathcal{V}_{\mathrm{b}}(\mathcal{O}(u)+\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u)) \end{Bmatrix}_i \begin{Bmatrix} x^{-1}\mathcal{V}_{\mathrm{b}} \\ \mathcal{V}_{\mathrm{b}} \end{Bmatrix}_j \\ &= \Delta_0 + x^{-2}(\mathcal{O}(\mathcal{V}_{\mathrm{b}}u) + \mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u,\mathcal{V}_{\mathrm{b}}\mathcal{V}_{\mathrm{b}}u))\mathcal{V}_{\mathrm{b}}.\end{aligned}$$

We will now calculate the asymptotics of B. Expanding the terms,

$$B = G^{ij}\partial_i\partial_j + \frac{\partial_i|g|}{2|g|}G^{ij}\partial_j + \partial_i G^{ij}\partial_j,$$

we see that the latter two summands share the same asymptotics, so that

$$\begin{aligned}B &= G^{ij}\partial_i\partial_j + \begin{Bmatrix} x^{-1} \\ 1 \end{Bmatrix}_i \begin{Bmatrix} 1 \\ x^{-1} \end{Bmatrix}_i \begin{Bmatrix} 1 \\ x^{-1} \end{Bmatrix}_j (\mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u) + \mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u,\mathcal{V}_{\mathrm{b}}\mathcal{V}_{\mathrm{b}}u))\partial_j \\ &= G^{ij}\partial_i\partial_j + x^{-2}(\mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u) + \mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u,\mathcal{V}_{\mathrm{b}}\mathcal{V}_{\mathrm{b}}u))\mathcal{V}_{\mathrm{b}}\end{aligned}$$

Similarly we see that $G^{ij}\partial_i\partial_j = \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u)\mathcal{V}_{\mathrm{b}}^2$, so the claim follows. □

Note that actually the terms involving the second derivatives are given by $G^{ij}\partial_i\partial_j$, so in particular they are of the form

$$xug^{ik}h_{kl}g^{lj}\partial_i\partial_j + x^{-2}\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}})\mathcal{V}_{\mathrm{b}}^2.$$

Next we discuss the denominator $\langle\nu,\nu_0\rangle$ in Eq. (MCF pre). The normal can explicitly be calculated by

$$\nu = \frac{\hat{\nu}}{\|\hat{\nu}\|}, \text{ where } \hat{\nu} = \nu_0 - g^{ij}\langle\partial_i\varphi,\nu_0\rangle\partial_j\varphi.$$

Note that the norm $\|\hat{\nu}\|$ cancels out in the corresponding terms, so we only consider

$$\begin{aligned}\langle\nu_0,\hat{\nu}\rangle &= \langle\nu_0,\nu_0 - g^{ij}\langle\partial_i\varphi,\nu_0\rangle\partial_j\varphi\rangle \\ &= \langle\nu_0,\nu_0 - g^{ij}\langle\partial_{i\varphi_0} + \partial_i f\nu_0 + f\partial_i\nu_0\rangle(\partial_j\varphi_0 + \partial_j f\nu_0 + f\partial_j\nu_0)\rangle \\ &= 1 + g^{ij}\partial_i f\partial_j f.\end{aligned}$$

For the latter summand we record

$$\begin{aligned}g^{ij}\partial_i f\partial_j f &= \begin{Bmatrix} \mathcal{O}(1) \\ \mathcal{O}(x^{-1}) \end{Bmatrix}^i \begin{Bmatrix} \mathcal{O}(1) \\ \mathcal{O}(x^{-1}) \end{Bmatrix}^j \begin{Bmatrix} \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u) \\ x\mathcal{O}(\mathcal{V}_{\mathrm{b}}u) \end{Bmatrix}_i \begin{Bmatrix} \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u) \\ x\mathcal{O}(\mathcal{V}_{\mathrm{b}}u) \end{Bmatrix}_j \\ &= \mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u).\end{aligned}$$

Again by the geometric series we see that $\langle \nu, \nu_0 \rangle^{-1} \sim 1 + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)$. Consequently, we have

$$\partial_t f = H_0 + \Delta_0 f + (\Delta_t - \Delta_0) f + (L_1 + L_2 + L_3 + L_4)(1 + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)),$$

with the terms L_i given by

$$\begin{aligned} L_1 &= f \langle \Delta_0 \nu_0, \hat{\nu} \rangle \\ L_2 &= 2 \langle (g(\nabla f, \nabla \nu_0^k)_{k=1}^n), \hat{\nu} \rangle \\ L_3 &= \langle (\Delta_t - \Delta_0) \varphi_0, \hat{\nu} \rangle \\ L_4 &= f \langle (\Delta_t - \Delta_0) \nu_0, \hat{\nu} \rangle. \end{aligned}$$

We will now evaluate the asymptotics of these terms to obtain the complete linearization.

Asymptotics of $(\Delta_t - \Delta_0) f$

By Lemma 2.5 we have

$$(\Delta_t - \Delta_0) f = x^{-2} \left(\mathcal{O}(\mathcal{V}_{\mathrm{b}} u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u, \mathcal{V}_{\mathrm{b}}^2 u) \right) \left\{ \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2 \right\} f.$$

Using $f = xu$ we immediately obtain

$$(\Delta_t - \Delta_0) f = x^{-1} \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u, \mathcal{V}_{\mathrm{b}}^2 u).$$

Asymptotics of L_1

Lemma 2.6. For the Laplacian of the unit normal one has

$$\langle \Delta \nu, \nu \rangle = -\|A\|^2.$$

Proof. Straightforward calculation:

$$\langle \Delta \nu, \nu \rangle = \langle g^{ij} \nabla^2_{ij} \nu, \nu \rangle = \langle g^{ij} \partial_i \partial_j \nu - \Gamma^l_{ij} \partial_l \nu, \nu \rangle$$

As spatial derivatives of the unit normal are tangential, we have

$$\begin{aligned} \langle \Delta \nu, \nu \rangle &= g^{ij} \langle \partial_i \partial_j \nu, \nu \rangle \\ &= g^{ij} \partial_i \langle \partial_j \nu, \nu \rangle - g^{ij} \langle \partial_i \nu, \partial_j \nu \rangle \\ &= -g^{ij} \langle \partial_i \nu, \partial_j \nu \rangle \\ &= -g^{ij} \langle h_{il} g^{ls} \partial_s \varphi, h_{jk} g^{kr} \partial_r \varphi \rangle \\ &= -g^{ij} h_{il} g^{ls} h_{jk} g^{kr} g_{sr} \\ &= -g^{ij} h_{il} h_{jk} g^{kl} \\ &= -\|A\|^2. \end{aligned}$$

□

We now have that

$$L_1 = f\langle \Delta_0 \nu_0, \hat\nu\rangle = -\|A_0\|^2 f - f\langle \Delta_0\nu_0, g^{ij}\partial_i f \partial_j \varphi\rangle.$$

As the unit normal does not come with any x-weight, we simply record for its Laplacian that $\Delta_0\nu_0 = \mathcal{O}(x^{-2})$ and proceed with

$$\begin{aligned}
&\langle \Delta_0\nu_0, g^{ij}\partial_i f\partial_j\varphi\rangle = \mathcal{O}(x^{-2}) g^{ij}\partial_i f \mathcal{O}(\partial_j\varphi_0 + \partial_j f\nu_0 + f\partial_j\nu_0)\\
&= \mathcal{O}(x^{-2}) \begin{Bmatrix} \mathcal{O}(1) \\ \mathcal{O}(x^{-1}) \end{Bmatrix}^i \begin{Bmatrix} \mathcal{O}(1) \\ \mathcal{O}(x^{-1}) \end{Bmatrix}^j \begin{Bmatrix} \mathcal{O}(u, \mathcal{V}_\mathrm{b} u) \\ x\mathcal{O}(u) \end{Bmatrix}_i \begin{Bmatrix} \mathcal{O}(1) + \mathcal{O}(u, \mathcal{V}_\mathrm{b} u) \\ \mathcal{O}(x) + x\mathcal{O}(u, \mathcal{V}_\mathrm{b} u) \end{Bmatrix}_j\\
&= x^{-2}\mathcal{O}(u, \mathcal{V}_\mathrm{b} u).
\end{aligned}$$

As $f = xu$ it follows that

$$L_1 = -\|A_0\|^2 f - x^{-1}\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u).$$

Asymptotics of L_3

The expansion of the L_3-term is quite curious, as it contributes to the linear term, leading to an improvement of the side face index sets. To see this, first note that we can split up the normal vector $\hat\nu$ and obtain accordingly

$$\begin{aligned}
L_3 &= \langle (\Delta_t - \Delta_0)\varphi_0, \hat\nu\rangle\\
&= \langle (\Delta_t - \Delta_0)\varphi_0, \nu_0\rangle - \langle (\Delta_t - \Delta_0)\varphi_0, g^{ij}\langle \partial_i\varphi, \nu_0\rangle \partial_j\varphi\rangle\\
&=: K_1 - K_2.
\end{aligned}$$

For K_1 note that all first derivatives of φ_0 are tangential and consequently

$$K_1 = \langle G^{ij}\partial_i\partial_j\varphi_0, \nu_0\rangle = G^{ij} h_{ij}(0).$$

as all first derivatives of φ_0 are tangential. Recalling that the leading term of G^{ij} was given by

$$g^{ik}(-2fh_{kl} + f^2 h_{k\alpha} g^{\alpha\beta} h_{\beta l} + f_k f_l) g^{lj},$$

we see that the first summand is actually

$$-2f g^{ik} h_{kl} g^{lj} h_{ij}(0) = -2f\|A(0)\|^2,$$

whereas the higher order terms are given by

$$\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u)\mathcal{O}(g_0^{-1})\langle \mathcal{V}_\mathrm{b}^2\varphi_0, \nu_0\rangle.$$

For K_2 we use Lemma 2.5 and obtain

$$\begin{aligned}
K_2 &= \langle x^{-2}\mathcal{O}(u, \mathcal{V}_\mathrm{b} u, \mathcal{V}_\mathrm{b}^2 u) \left\{\mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\right\}\varphi_0, g^{ij}\partial_i f(\partial_j\varphi_0 + \partial_j f\nu_0 + f\partial_j\nu_0)\rangle\\
&= x^{-1}\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u, \mathcal{V}_\mathrm{b}^2 u).
\end{aligned}$$

Hence we record

$$L_3 = -\|A_0\|^2 f + x^{-1}\mathcal{O}_2(u, \mathcal{V}_\mathrm{b} u, \mathcal{V}_\mathrm{b}^2 u).$$

Asymptotics of L_4

As also $\partial_i \nu_0$ is tangential, we have that

$$L_4 = f\langle G^{ij}\partial_i\partial_j\nu_0, \nu_0\rangle = xu\mathcal{O}(G) = x^{-1}\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u).$$

Asymptotics of L_2

The calculation for L_2 is a bit more involved; first we simply write out the terms in local coordinates:

$$\begin{aligned} L_2 \sim & \langle \left\{ g(\nabla f, \nabla \nu_0^k \right\}_k, \nu\rangle \\ & = \langle g_{\beta\delta} g^{\alpha\beta}\partial_\alpha f g^{\gamma\delta}\partial_\gamma \nu_0, \nu\rangle \\ & \sim \partial_\alpha f g^{\alpha\beta} g_{\beta\delta} g^{\delta\gamma} g^{ij}\langle \partial_\gamma\nu_0, \partial_i f \partial_j \varphi_0\rangle \\ & = \partial_\alpha f g^{\alpha\gamma} g^{ij}\partial_i f \langle \partial_\gamma \nu_0, \partial_j\varphi_0\rangle \end{aligned}$$

Now we observe the asymptotics of the involved terms

$$\begin{aligned} L_2 \sim g^{\alpha\gamma} & \begin{Bmatrix} \mathcal{O}(u, \mathcal{V}_\mathrm{b}u) \\ x\mathcal{O}(u, \mathcal{V}_\mathrm{b}u) \end{Bmatrix}_\alpha \begin{Bmatrix} \mathcal{O}(u, \mathcal{V}_\mathrm{b}u) \\ x\mathcal{O}(u, \mathcal{V}_\mathrm{b}u) \end{Bmatrix}_i g^{ij} \begin{Bmatrix} 1 \\ x \end{Bmatrix}_j \\ = x^{-1} & \mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u). \end{aligned}$$

Finally it follows for the Linearization that

$$(\partial_t - \Delta_0)xu = H_0 - 3\|A(0)\|^2 xu + x^{-1}\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u).$$

Undoing the change to $f = xu$, we have the following linearization.

Proposition 2.7 (Linearization of the mean curvature flow)**.**

$$(\partial_t - \Delta_0)f = H_0 - 3\|A(0)\|^2 f + x^{-1}\mathcal{O}_2(x^{-1}f, \mathcal{V}_\mathrm{b}x^{-1}f, \mathcal{V}_\mathrm{b}^2 x^{-1}f). \quad \text{(MCF)}$$

It follows immediately that quadratic perturbations (i.e. $f = x^2u$) of minimal cones preserve the conical structure and have bounded initial mean curvature H_0, as it is – up to a factor – given by $\partial_t f$.

2.3 Linearization near infinity

In the previous section we dealt with the case that x is the boundary defining function for the cone tip. Now we want to discuss the linearization of the equation near $\{y = 0\}$, where $y = x^{-1}$. Naively one would simply use the result from Proposition 2.7 and apply a coordinate transformation. But we discarded terms of higher orders of x and absorbed them in the $\mathcal{O}$-notation. Near infinity these terms could translate into highly singular y-factors. However, as it turns out, we obtain the same asymptotics, which we will verify now. The calculations are a bit simpler than near the tip due to the second fundamental form having the same asymptotic pattern as the metric.

Again we look at the terms involved in Eq. (MCF pre) from Section 2.1 and make the Ansatz

$$\varphi = \varphi_0 + f\nu_0 = \varphi_0 + y^{-1}u\nu_0.$$

As elaborated in Remark 2.1 it is more natural to work with u instead of f. It should be noted that in this setting the alternative would be working in the so called *scattering calculus*, where the b-vector fields are replaced by the scattering fields $y^2\partial_y, y\partial_z$, which again correspond to the scaling of the asymptotically conical structure. First we recall from Section 2.1.2 the asymptotic behaviour of the first and second fundamental form:

$$g_0 = \begin{pmatrix} \mathcal{O}(y^{-4}) & \mathcal{O}(y^{-3}) \\ \mathcal{O}(y^{-3}) & \mathcal{O}(y^{-2}) \end{pmatrix}, \; A_0 = \langle \partial_i\partial_j\varphi_0, \nu_0\rangle = \begin{pmatrix} \mathcal{O}(y^{-3}) & \mathcal{O}(y^{-2}) \\ \mathcal{O}(y^{-2}) & \mathcal{O}(y^{-1}) \end{pmatrix}$$

The inverse of the metric is consequently given by

$$g_0^{-1} = \begin{pmatrix} \mathcal{O}(y^{4}) & \mathcal{O}(y^{3}) \\ \mathcal{O}(y^{3}) & \mathcal{O}(y^{2}) \end{pmatrix}.$$

We mimic the arguments from Section 2.2.

Lemma 2.8. For the induced metric, we have

$$\begin{aligned} g &= g_0 + u\mathcal{O}(g_0) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u)\mathcal{O}(g_0) \\ g^{-1} &= g_0^{-1} + u\mathcal{O}(g_0^{-1}) + \mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u)\mathcal{O}(g_0) =: g_0^{-1} + G. \end{aligned}$$

Proof. The proof is essentially that of Lemma 2.2. Recalling that the metric is given by

$$g = g(0) - (2y^{-1}uh_{ij}(0) + y^{-2}u^2(h_{il}g^{kl}h_{lj})(0) + (y^{-1}u)_i(y^{-1}u)_j),$$

the statement for the first order term is immediately clear. For the second term we have

$$\begin{aligned} h_{il}g^{kl}h_{lj} &= \begin{pmatrix} \mathcal{O}(y^{-3}) & \mathcal{O}(y^{-2}) \\ \mathcal{O}(y^{-2}) & \mathcal{O}(y^{-1}) \end{pmatrix}_{il} \begin{pmatrix} \mathcal{O}(y^{4}) & \mathcal{O}(y^{3}) \\ \mathcal{O}(y^{3}) & \mathcal{O}(y^{2}) \end{pmatrix}^{kl} \begin{pmatrix} \mathcal{O}(y^{-3}) & \mathcal{O}(y^{-2} \\ \mathcal{O}(y^{-2}) & \mathcal{O}(y^{-1}) \end{pmatrix}_{lj} \\ &= \begin{pmatrix} \mathcal{O}(y^{-2}) & \mathcal{O}(y^{-1}) \\ \mathcal{O}(y^{-1}) & \mathcal{O}(1) \end{pmatrix}, \end{aligned}$$

so that

$$y^{-2}u^2(h_{il}g^{kl}h_{lj})(0) = u^2\mathcal{O}(g_0).$$

The asymptotics of the last term follow once again simply by derivation. Finally we employ the von Neumann series to obtain the statement about g^{-1}. □

Again we have to separate the time dependent Laplacian from the initial Laplacian.

Lemma 2.9. For the time dependent Laplacian, it holds that

$$\Delta_t = \Delta_0 + y^2 \left(\mathcal{O}(\mathcal{V}_{\mathrm{b}} u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u, \mathcal{V}_{\mathrm{b}}^2 u) \right) \left\{ \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2 \right\}.$$

Proof. The proof is, mutatis mutandis, the same as the one for Lemma 2.5.

$$\begin{aligned}
\Delta_t &= \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j \right) \\
&= \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g_0^{ij} \partial_j \right) + \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} G^{ij} \partial_j \right) =: A + B.
\end{aligned}$$

From the first term we can now separate out the original Laplacian by using Lemma 2.4, which is proven ad verbatim in this situation:

$$\begin{aligned}
A &= \Delta_0 + g_0^{ij} \frac{\partial_i \sqrt{1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)}}{\sqrt{1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u}} \partial_j \\
&= \Delta_0 + \frac{1}{2} g_0^{ij} \frac{\partial_i (\mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u))}{1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)} \partial_j
\end{aligned}$$

Using the geometric series we obtain boundedness of the quotient and continue with

$$\begin{aligned}
A &= \Delta_0 + (1 + \mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)) g_0^{ij} \partial_i (\mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)) \partial_j \\
&= \Delta_0 + \begin{Bmatrix} \mathcal{O}(y^2) \\ \mathcal{O}(y) \end{Bmatrix}^i \begin{Bmatrix} \mathcal{O}(y^2) \\ \mathcal{O}(y) \end{Bmatrix}^j \begin{Bmatrix} y^{-1} \mathcal{V}_{\mathrm{b}} (\mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)) \\ \mathcal{V}_{\mathrm{b}} (\mathcal{O}(u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)) \end{Bmatrix}_i \begin{Bmatrix} y^{-1} \mathcal{V}_{\mathrm{b}} \\ \mathcal{V}_{\mathrm{b}} \end{Bmatrix}_j
\end{aligned}$$

We see that whenever i or j corresponds to the y-coordinate, the additional y-factor obtained from the inverse of the metric cancels with the singular y-factor from the derivative and we obtain

$$A = \Delta_0 + y^2 (\mathcal{O}(\mathcal{V}_{\mathrm{b}} u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u, \mathcal{V}_{\mathrm{b}}^2 u)) \mathcal{V}_{\mathrm{b}}.$$

Similarly we tackle B.

$$\begin{aligned}
B &= \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} G^{ij} \partial_j) \\
&= G^{ij} \partial_i \partial_j + \frac{\partial_i |g|}{|g|} G^{ij} \partial_j + \partial_j G^{ij} \partial_j
\end{aligned}$$

In order to shorten the argument, witness that the latter two summands share the same asymptotics. As before factors from the metric cancel with those of the derivatives, so that our calculation is analogous to the one seen before and can be applied ad verbatim. Finally observe that

$$\begin{aligned}
G^{ij}\partial_i\partial_j &= \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^i \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^i (\mathcal{O}(u)+\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u)) \begin{Bmatrix}y^{-1}\mathcal{V}_{\mathrm{b}}\\ \mathcal{V}_{\mathrm{b}}\end{Bmatrix}^i \begin{Bmatrix}y^{-1}\mathcal{V}_{\mathrm{b}}\\ \mathcal{V}_{\mathrm{b}}\end{Bmatrix}^j \\
&= y^2\left(\mathcal{O}(\mathcal{V}_{\mathrm{b}}u)+\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u)\right)\left\{\mathcal{V}_{\mathrm{b}},\mathcal{V}_{\mathrm{b}}^2\right\}
\end{aligned}$$

leading to the result. □

Recall that the time dependent unit normal ν is given by $\nu = \hat{\nu}/\|\hat{\nu}\|$, where $\hat{\nu} = \nu_0 - g^{ij}\langle\partial_i\varphi,\nu_0\rangle\partial_j\varphi$. Noting that $\|\hat{\nu}\|^{-1}$ is bounded, one checks by explicit calculation, cancelling all orthogonal factors, that

$$\langle\hat{\nu},\nu_0\rangle \sim 1 + g^{ij}\partial_i f\partial_j f.$$

As $f = y^{-1}u$ we obtain

$$\begin{aligned}
g^{ij}\partial_i f\partial_j f &= g^{ij}\partial_i(y^{-1}u)\partial_j(y^{-1}u) \\
&= \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^i \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j \begin{Bmatrix}y^{-2}\mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u)\\ y^{-1}\mathcal{O}(\mathcal{V}_{\mathrm{b}}u)\end{Bmatrix}_i \begin{Bmatrix}y^{-2}\mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u)\\ y^{-1}\mathcal{O}(\mathcal{V}_{\mathrm{b}}u)\end{Bmatrix}_j \\
&= \mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u).
\end{aligned}$$

Note that the additional terms of the inverse of the metric have even better asymptotic behaviour. For the tangential (w.r.t. the initial metric) part of $\hat{\nu}$ we note that

$$g^{ij}\partial_i f \sim \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u),$$

and consequently

$$\begin{aligned}
&g^{ij}\partial_i f(\partial_j\varphi_0 + f\partial_j\nu_0) \sim \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u)(\partial_j\varphi_0 + f\partial_j\nu_0) \\
&\sim \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j \begin{Bmatrix}\mathcal{O}(y^{-2})\\ \mathcal{O}(y^{-1})\end{Bmatrix}_j \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u) + \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j y^{-1}\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u) \\
&\sim \mathcal{O}(u,\mathcal{V}_{\mathrm{b}}u).
\end{aligned}$$

We have seen that $\langle\hat{\nu},\nu_0\rangle \sim 1 + \mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}})$, so the same is true for its inverse by the geometric series and we obtain from Eq. (MCF pre):

$$\begin{aligned}
\partial_t f &= H_0 + \Delta_0 f + (\Delta_t - \Delta_0)f + \Big(f\langle\Delta_0\nu_0,\hat{\nu}\rangle + 2\langle(g(\nabla f,\nabla\nu_0^k)_{k=1}^n),\hat{\nu}\rangle \\
&\quad + \langle(\Delta_t-\Delta_0)\varphi_0,\hat{\nu}\rangle + f\langle(\Delta_t-\Delta_0)\nu_0,\hat{\nu}\rangle\Big)(1+\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u)) \\
&=: H_0 + \Delta_0 f + (L_1+L_2+L_3+L_4)(1+\mathcal{O}_2(u,\mathcal{V}_{\mathrm{b}}u))
\end{aligned}$$

Asymptotics of $(\Delta_t - \Delta_0)f$

We argue exactly as in the situation near the tip and use Lemma 2.9 and $f = y^{-1}u$ to see that

$$\begin{aligned}(\Delta_t - \Delta_0)f &= y^2 \left(\mathcal{O}(\mathcal{V}_{\mathrm{b}}u) + \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2 u)\right) \left\{\mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2\right\} f \\ &= y^{-1}\mathcal{O} - 2(u, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2 u).\end{aligned}$$

Asymptotics of L_1

We claim that $L_1 = -\|A_0\|^2 f + y^2\mathcal{O}(u, \mathcal{V}_{\mathrm{b}}u)$. First we recall Lemma 2.6 which shows that $\langle \Delta_0\nu_0, \nu_0\rangle = -\|A_0\|^2$, where $\|A_0\|$ is the initial metric's norm of the second fundamental form. To examine the contribution $\Delta_0\nu_0$, we note that ν_0 and its differentials do not bear any y-weight, so it suffices to examine the minimal y-weight of Δ_0, which is y^2. Consequently we obtain

$$\begin{aligned}\langle \Delta_0\nu_0, g^{ij}\partial_i f \partial_j\varphi\rangle &= y^2 g^{ij}\partial_i f\partial_j\varphi \\ &= y^2 \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j \mathcal{O}(u, \mathcal{V}_{\mathrm{b}}u)\left(\partial_j\varphi_0 + \partial_j f\nu_0 + f\partial_j\nu_0\right) \\ &= y^2\mathcal{O}(u, \mathcal{V}_{\mathrm{b}}u) \begin{Bmatrix}\mathcal{O}(y^2)\\ \mathcal{O}(y)\end{Bmatrix}^j \begin{Bmatrix}\mathcal{O}(y^{-2}) + y^{-2}\mathcal{O}(u, \mathcal{V}_{\mathrm{b}}u)\\ \mathcal{O}(y^{-1}) + y^{-1}\mathcal{O}(u, \mathcal{V}_{\mathrm{b}}u)\end{Bmatrix}_j \\ &= y^2\mathcal{O}(u, \mathcal{V}_{\mathrm{b}}u),\end{aligned}$$

where we ignored better asymptotics from the $f\partial_j\nu_0$-term.

Asymptotics of L_2

To examine L_2, we calculate as in Section 2.2

$$L_2 = 2\langle (g(\nabla f, \nabla \nu_0^k)_{k=1}^n), \hat{\nu}\rangle = -2g^{\gamma\alpha}g^{ij}\partial_\alpha f\partial_i f\langle \partial_\gamma\nu_0, \partial_j\varphi\rangle.$$

Using the observations made before, we obtain

$$\begin{aligned}L_2 = g^{\gamma\alpha}\partial_\alpha f g^{ij}\partial_i f \begin{Bmatrix}\mathcal{O}(y^{-2})\\ \mathcal{O}(y^{-1})\end{Bmatrix}^j &= \begin{Bmatrix}y^2\\ y\end{Bmatrix}^\gamma \begin{Bmatrix}y^2\\ y\end{Bmatrix}^j \begin{Bmatrix}y^{-2}\\ y^{-1}\end{Bmatrix}_j \mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}}u) \\ = y\mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}}u).\end{aligned}$$

Asymptotics of L_3

We first split up L_3 according to the parts of the normal vector $\hat{\nu}$ and write

$$L_3 = \langle (\Delta_t - \Delta_0)\varphi_0, \nu_0\rangle + \langle (\Delta_t - \Delta_0)\varphi_0, g^{ij}\partial_i f\partial_j\varphi\rangle =: K_1 + K_2.$$

For K_1 note that only the second order differentials from $\Delta_t - \Delta_0$ contribute, and these are given exactly by $G^{ij}\partial_i\partial_j$. Furthermore recall the construction of g^{-1}, showing that the first order term in G^{ij} is given by $-2fg_0^{ik}h_{kl}(0)g_0^{lj}$:

$$\begin{aligned}
K_1 &= \langle G^{ij}\partial_i\partial_j\varphi_0, \nu_0\rangle = G^{ij}h_{ij}(0) \\
&= -2g_0^{ik}h_{kl}(0)g_0^{lj}h_{ij}(0)f + \mathcal{O}(g_0)h_{ij}(0)\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u) \\
&= -2\|A_0\|^2 f + y\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u).
\end{aligned}$$

For K_2 observe that by Lemma 2.9 $\Delta_t - \Delta_0 = y^2\mathcal{O}(u, \mathcal{V}_\mathrm{b}u, \mathcal{V}_\mathrm{b}^2u)\,\{\mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\}$ (neglecting for the moment that the second order differentials of u actually come in at least quadratically). Furthermore for the right hand side of the scalar product we have $g^{ij}\partial_i f\partial_j\varphi = \mathcal{O}(u, \mathcal{V}_\mathrm{b}u)$, so we have $K_2 = y\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u, \mathcal{V}_\mathrm{b}^2u)$ and conclude

$$L_3 = -\|A_0\|^2 f + y\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u, \mathcal{V}_\mathrm{b}^2u),$$

noting that second order differentials come in only in terms of higher order.

Asymptotics of L_4

Finally we turn to L_4. Again we make use of Lemma 2.9 and analogous to the asymptotics of L_3 arrive at

$$L_4 = y\mathcal{O}_2(u, \mathcal{V}_\mathrm{b}u, \mathcal{V}_\mathrm{b}^2u),$$

again noting that the second derivatives of u actually come in terms of higher order.

Proposition 2.10 (Linearization of Mean curvature flow near infinity)**.**

$$(\partial_t - \Delta_0)f = H_0 - 3\|A(0)\|^2 f + y\mathcal{O}_2(yf, \mathcal{V}_\mathrm{b}yf, \mathcal{V}_\mathrm{b}^2yf). \tag{MCF}$$

2.4 Hölder spaces

Towards proving the short-time existence of the mean curvature flow we prove mapping properties of the Heat kernel between certain Hölder spaces, which are adapted to the linearization. We first introduce Hölder spaces which do not allow further application of spatial derivatives. Using a standard partition of unity argument, these Hölder spaces are adapted to the particular situation near the cone tip and infinity. We then introduce the higher order Hölder spaces, by allowing b-derivatives and introducing weights in terms of the corresponding boundary defining function.

As $M = (0,\infty)\times N$, we write $\overline{M} := [0,\infty]\times N$ for its (two-ended) compactification. Consider functions $u : M\times I \to \mathbb{R}$, extending continuously to $\overline{M}\times I$, where $I \subset \mathbb{R}$ is an interval. We define the *α-th Hölder norm* by

$$\|u\|_\alpha = \|u\|_\infty + \sup\left\{\frac{|u(p,t) - u(p',t')|}{d_M(p,p')^\alpha + |t-t'|^{\frac{\alpha}{2}}}\right\},$$

and the corresponding Hölder space $C^\alpha(M \times I)$ as consisting of those functions with bounded α-th Hölder norm. For time independent functions $u : M \to \mathbb{R}$ the α-th Hölder norm is correspondingly defined without the time difference in the denominator.

For any finite cover $U_1, \ldots, U_k$ of M, with a subordinate partition of unity $\phi_1, \ldots, \phi_k$ we can define a topologically equivalent norm via

$$\|u\|_\alpha^\phi := \sum_{i=1}^k \|\phi_i u\|_\alpha.$$

By considering slices of the form $\mathbb{R}^+ \times U$ with $U \subset N$, we can produce a finite open cover of M, since N is compact. In these slices we can obtain an equivalent norm such that the distance terms are easily expressed in the local coordinates (x, z) and (y, z) respectively. This seems to be a rather standard approach near the cone tip, but we did not find any account of this. We refer the interested reader to Appendix A.

In the following subsections we will always assume to be in such a slice neighbourhood of the singularity or the boundary at infinity.

Although we use the explicit form of these coordinates for our calculations, the resulting spaces of course are coordinate invariant, cf. [Mel93].

2.4.1 Hölder spaces near the tip

In sufficiently small slices $(0, K) \times U$ near the cone tip the distance function d_M is equivalent to the prototypical distance function of a cone, i.e.

$$d_M(x, z, x', z') = \sqrt{(x - x')^2 + (x + x')^2 |z - z'|^2},$$

where $|z - z'|$ denotes the usual Euclidean distance.

We denote the space of functions with bounded α-th Hölder norm in this regime by

$$C_e^\alpha(M \times I) = C^\alpha(M \times I),$$

which is simply the usual Hölder space. The weighted Hölder space with weight x^γ is $x^\gamma C_e^\alpha(M \times I)$. This space can be equipped with a Hölder norm $\|u\|'_{\gamma,\alpha} = \|x^{-\gamma} u\|_\alpha$. We will need to consider the following hybrid Hölder space

$$C_{e,\gamma}^\alpha(M \times I) := x^\gamma C_e^\alpha(M \times I) \cap x^{\gamma+\alpha} C_e^0(M \times I),$$

where $C_e^0(M \times I)$ is the space of continuous functions with bounded norm on $M \times I$ and carries the usual supremum norm. We equip $C_{e,\gamma}^\alpha(M \times I)$ with

$$\|u\|_{\alpha,\gamma} = \|x^{-\gamma} u\|_\alpha + \|x^{-\gamma-\alpha} u\|_\infty.$$

On this basis we define the parabolic Hölder spaces of higher order. These should capture the behaviour of the heat equation, $(\partial_t - \Delta)H = \mathrm{id}$, where

H is the heat kernel. Note that the Laplacian in the conical setting is of the form $\Delta = x^{-2}\mathcal{O}(\mathcal{V}_b, \mathcal{V}_b^2)$, so that each application of ∂_t should equal $x^{-2}\mathcal{V}_b^2$ in regularity, or, equivalently, $x^2\partial_t$ should affect the regularity the same way as the application of $\mathcal{V}_b^2$ does. To be precise, we define $\mathcal{D}^k$ to be the space of admissible differentials of order k or less, i.e.

$$\mathcal{D}^k = \left\{ \mathcal{V}_b^i \circ (x^2\partial_t)^j \mid i, j \in \mathbb{N},\ i + 2j \leq k \right\},$$

so that the Hölder space of higher order is given by

$$C_{e,\gamma}^{k+\alpha}(M \times I) = \left\{ u \in C_{e,\gamma}^{\alpha} \mid Xu \in C_{e,\gamma}^{\alpha} \text{ for all } X \in \mathcal{D}^k \right\},$$

with norm

$$\|u\|_{\gamma,k+\alpha} = \sum_{X \in \mathcal{D}^k} \|Xu\|_{\gamma,\alpha}.$$

2.4.2 Hölder spaces near infinity

Near infinity we define the Hölder spaces the same way as near the tip, however we use $y = x^{-1}$ as the weight in place of x, and the corresponding prototypical distance function. In particular now the edge vector fields are combinations of $y\partial_y$ an ∂_z.

As now the boundary at infinity is approached as $y \to 0$, we can again consider small slices $(0, K) \times U$, such that we have for the distance function

$$d_M(y, z, y', z') \sim \sqrt{\left(\frac{1}{y} + \frac{1}{y'}\right)^4 |y - y'|^2 + \left(\frac{1}{y} + \frac{1}{y}\right)^2 |z - z'|^2}.$$

We now define the corresponding Hölder spaces exactly as before, with the main difference being the distance function and coordinate used for the weight, i.e. the hybrid Hölder space is defined as

$$C_{e,\gamma}^{\alpha}(M \times I) := y^{\gamma} C_e^{\alpha}(M \times I) \cap y^{\gamma+\alpha} C_e^0(M \times I).$$

Near infinity, the Laplacian is of the form $\Delta = y^2\mathcal{V}_b$, so that the admissible differentials of order k or less are now given by

$$\mathcal{D}^k = \left\{ \mathcal{V}_b^i \circ (y^{-2}\partial_t)^j \mid i, j \in \mathbb{N}, i + 2j \leq k \right\},$$

and the definition of the Hölder spaces of higher order can now be copied at verbatim.

2.4.3 Hölder spaces with prescribed behaviour at both ends

We now want to describe Hölder spaces, which have a certain asymptotic behaviour near the tip and a different behaviour near infinity. For this we consider a smooth cutoff function $\tilde{\phi} : \tilde{M} \to \mathbb{R}_{\geq 0}$ (i.e. $\tilde{\phi} + (1 - \tilde{\phi}) \equiv 1$), such that $\operatorname{supp}(\tilde{\phi})$ is connected, compact and contains a collar neighbourhood U of the boundary $\{x = 0\}$, such that $\tilde{\phi}\Big|_U \equiv 1$. Then its restriction $\phi := \tilde{\phi}\Big|_M$ to the cone is again a smooth cutoff function, with the same properties, except that the support is not compact anymore. We differentiate the spaces from Section 2.4.1 and Section 2.4.2 by an x and y in the index respectively. We then define the spaces

$$C^{k+\alpha}_{e,\gamma} \oslash C^{l+\beta}_{e,\delta}(M \times I) := \left\{ u \in C^0(M \times I) \;\middle|\; \begin{array}{r} \phi u \in C^{k+\alpha}_{e,\gamma}(M \times I)_x \\ (1-\phi)u \in C^{l+\beta}_{e,\delta}(M \times I)_y \end{array} \right\},$$

so that the left-hand argument to $\oslash$ denotes the prescribed regularity near the tip and the right-hand argument gives the behaviour near infinity. The corresponding Hölder norm is then given by

$$\|u\|_{(\gamma,k+\alpha),(\delta,l+\beta)} := \|\phi u\|_{\gamma,k+\alpha} + \|(1-\phi)u\|_{\delta,l+\alpha}.$$

This definition is actually independent of the choice of ϕ; as described in the first part of this section, the Hölder norms are defined locally on slices near the ends. Away from the ends we see that the given Hölder spaces are both the classical ones, as the distance functions involved are then just uniform equivalent to the Euclidean distance function and the weights are just bounded factors.

In the following sections we will prove Theorem 3.2 and Theorem 4.1, showing that

$$H : C^{\alpha}_{e,\gamma} \oslash C^{\alpha}_{e,\gamma'+3}(M \times I) \to C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I)$$

is a bounded mapping, for $\gamma + 2 \leq \mu$ and $\gamma' \geq 0$, where μ is the exponent of the lf and rf boundary defining functions of the polyhomogeneous lift of the heat kernel near the tip.

Remark 2.11 (Choice of weighted spaces)**.** Again we note that we chose our spaces in a way that allows for a unified treatment of the difficult and error prone calculations (i.e. the linearization and the Schauder estimates).

However, in application one might prefer to work with incomplete b-vector fields near the tip, i.e. fields of the form $\{\partial_x, x^{-1}\partial_z\}$ and scattering vector fields near infinity, i.e. fields of the form $\{y^2\partial_y, y\partial_z\}$. These vector fields then do not scale up or down with the cross section and local orthonormal frames would be of this form. Using these fields corresponds mostly to a weight shift, i.e. we can view $C^{2+\alpha}_{e,\gamma}(M \times I)_y$ as $C^{2+\alpha}_{sc,\gamma+2}(M \times I)_y$, where we replace applications of $y^{-2}\partial_t$ by ∂_t, i.e. the admissible differential operators near infinity would be given by $\mathcal{D}^k_{sc} = \left\{ \mathcal{V}^i_{sc} \circ \partial^j_t \mid i + 2j \leq k \right\}$. Analogously, near the tip, the space $C^{2+\alpha}_{e,\gamma}(M \times I)_x$ corresponds to $C^{2+\alpha}_{ie,\gamma-2}(M \times I)_x$, with admissible differentials $\mathcal{D}^k_{ie} = \left\{ \mathcal{V}^i_{ie} \circ \partial^j_t \mid i + 2j \leq k \right\}$. Note however that in both cases this correspondence is no equality.

3 Schauder estimates near the cone tip

In this section we wish to show certain mapping properties for the heat kernel H associated to the operator

$$\partial_t - \partial_x^2 - \frac{m-1}{x}\partial_x - \frac{1}{x^2}\left(\Delta_N - 3x^2\|A(0)\|^2\right),$$

which we obtained in the linearization Proposition 2.7. It is immediately clear from the formulae (or using a nicer geometric argument as in [Per02]) that $x^2\|A(0)\| = \mathcal{O}(1)$. We know from Theorem 1.4 that the heat kernel in this situation lifts to a polyhomogeneous distribution on the heat space $\mathcal{M}$ (recall Fig. 4) such that

$$H \sim \rho_{\mathrm{ff}}^{-m}\rho_{\mathrm{td}}^{-m}\rho_{\mathrm{lf}}^{\mu}\rho_{\mathrm{rf}}^{\mu}G$$

with G smooth up to all boundary faces and vanishing to infinite order at tf. Recall that μ was given as the minimum of the right face index set, which in this situation is

$$\left\{\gamma \geq 0 \mid \gamma = -\frac{m-2}{2} + \sqrt{\frac{(m-2)^2}{4} + \lambda}, \lambda \in -\operatorname{spec}(\Delta_N - x^2\|A(0)\|^2)\right\}.$$

In what follows we make the following assumption.

Assumption 3.1. We will assume that $\mu > 2$.

When setting up the fixpoint argument we need to make an additional assumption, namely that the underlying cone is close to a minimal cone. We postpone the discussion of the compatibility and consequences of these assumptions to Section 6.4 and continue stating and proving the main theorem of this section. The gist of it is, that as long as the underlying cone is not a cone over a sphere, we have lower bounds on the norm of the second fundamental form on minimal submanifolds of spheres, assuring that we can make this assumption.

Theorem 3.2 (Mapping properties of the Heat kernel)**.** Let M be a conical manifold such that the heat kernel H associated to

$$\partial_t - \partial_x\partial_x - \frac{m-1}{x}\partial_x - \frac{1}{x^2}\left(\Delta_N - 3\|A_N\|^2\right)$$

lifts to a polyhomogeneous distribution on the heat space $\mathcal{M}$, where the minimal element μ of the side face index sets is larger than 2. Then it admits the following mapping property:

$$H : C^{k+\alpha}_{e,\gamma}(M \times [0,T]) \to C^{k+2+\alpha}_{e,\gamma+2}(M \times [0,T])$$

is bounded, for $T > 0$ and any $\gamma \geq 0$ where

$$0 < \alpha \leq \mu - \gamma - 2.$$

In particular this mapping property will allow us to set up a standard fixpoint argument for short-time existence of (MCF) in our situation.

The proof of Theorem 3.2 amounts to showing that we have for $u \in C^{\alpha}_{e,\gamma}(M \times [0,T])$

$$\begin{aligned} Vx^{-\gamma-2}H[u] &\in C^{\alpha}_{e} \\ Vx^{-\gamma-2-\alpha}H[u] &\in C^{0}, \end{aligned}$$

where $V \in \{\mathrm{id}, \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2\}$. We do not need to check the mapping properties for $V = x^2\partial_t$ as we can then invoke the heat equation:

$$\begin{aligned} & x^2\partial_t x^{-\gamma-2}H[u] \in C^{\alpha}_{e} \\ \Longleftrightarrow\ & x^{-\gamma-2}x^2\partial_t H[u] \in C^{\alpha}_{e} \\ \Longleftrightarrow\ & x^{-\gamma-2}x^2(\Delta_0 + \|A_0\|)H[u] + x^{-\gamma-2}x^2 u \in C^{\alpha}_{e} \\ \Longleftrightarrow\ & x^{-\gamma-2}(\mathcal{V}_{\mathrm{b}}^2 + \mathrm{id})H[u] + x^{-\gamma}u \in C^{\alpha}_{e} \end{aligned}$$

Employing the triangle inequality, this amounts to the establishment of uniform bounds on the Hölder differences in space and in time, as well as on the supremum norms. More specifically, we have to find uniform bounds for

i) the spatial Hölder difference of $Vx^{-\gamma-2}H[u]$,

ii) the temporal Hölder difference of $Vx^{-\gamma-2}H[u]$,

iii) $\sup |Vx^{-\gamma-\alpha}H[u]|$,

where $V \in \{\mathrm{id}, \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2\}$. The philosophy for the estimates is as follows: If it works at the right face, it works everywhere, and if the spatial Hölder difference can be estimated, then also the temporal Hölder difference can be estimated. The sup-estimates are in any case nearly trivial – at least after having done the other estimates. Estimates near the diagonal are a bit more technical as they require a little integration trick, which we describe in detail at its first occurrence in Section 3.1.4.

Since the involved Hölder spaces, as explicated in Section 2.4, are defined locally, we can always assume that the heat kernel is compactly supported near one of the corners of the heat space $\mathcal{M}$.

Remark 3.3 (Notations and abuse thereof)**.** As the involved terms are, when explicitly written down, quite unwieldy, we abuse the notation in the following ways:

i) As always, we assume in notation the cross section to be 1-dimensional, so that we always work in local coordinates (x, z).

ii) We absorb any uniformly bounded factors in the constants or the heat kernel, and the constants and heat kernels may change from line to line.

iii) All integrals are understood as being written between absolute signs and we only use absolute signs on the leading factors or when they actually make a difference in the estimates.

iv) When substituting integrals we sometimes do not transform the integral boundaries and it is understood that the corresponding replacement in the integration variable still has to be made.

v) We only record the most problematic terms and disregard terms of higher order, e.g. those coming from the Leibniz rule, as higher order terms can be estimated ad verbatim.

vi) We subsume differentials and volume forms according to how they influence the asymptotic behaviour via the symbol $\sim$, and only record the worst case behaviour. For instance, the first two summands of $\partial_x^2 + x^{-1}\partial_x + \partial_x$ decrease leading x-factors by a power of 2, the last one only by a power of 1, so we write $\partial_x^2 + x^{-1}\partial_x + \partial_x \sim x^{-2}$, or similarly, $2\tau d\tau \sim \tau d\tau$.

The remainder of this section is devoted to prove the mapping properties and we will operate under the assumptions of Theorem 3.2.

3.1 Estimates on the spatial Hölder differences

Let $p = (x, z), p' = (x', z') \in M$. We wish to establish estimates of the form

$$|Vx^{-\gamma-2}H[u](p,t) - Vx'^{-\gamma-2}H[u](p',t)| \leq C\|u\|_{\gamma,\alpha} d_M(p,p')^\alpha,$$

for $V \in \{\mathrm{id}, \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2\}$, $u \in C^\alpha_{e,\gamma}(M \times I)$ and C a uniform constant. Recall that the distance function d_M is, after the localization argument outlined in Section 2.4.1, topologically equivalent to $\sqrt{(x-x')^2 + (x+x')^2|z-z'|^2}$.

Before we continue with the estimates, we observe the following simple consequence of the mean value theorem.

Lemma 3.4 (Fractional mean value theorem)**.** Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a differentiable function. Then for all $x < y \in I$ and any $\varepsilon > 0$ exists a $\delta \in (x, y)$, such that

$$|f(x) - f(y)| = |x^\varepsilon - y^\varepsilon|\varepsilon^{-1}\delta^{1-\varepsilon}f'(\delta).$$

Proof. Let a, b such that $a^{1/\varepsilon} = x$, $b^{1/\varepsilon} = y$ and set $g(t) := t^{1/\varepsilon}$. Then by the mean value theorem

$$|f(x) - f(y)| = |f \circ g(a) - f \circ g(b)| = |a - b|(f \circ g)'(\xi)$$

where $\xi \in (a, b) = (x^\varepsilon, y^\varepsilon)$. Applying the chain rule we have

$$|f(x) - f(y)| = |a - b|f'(g(\xi))g'(\xi) = |a - b|f'(\xi^{1/\varepsilon})N\xi^{1/\varepsilon-1}.$$

By letting $\delta = \xi^{1/\varepsilon} \in (x, y)$ we arrive at the statement. □

For the sake of the argument, let us abbreviate $\tilde{G} = Vx^{-\gamma-2}H$. We then establish

$$|\tilde{G}u(p,t) - \tilde{G}u(p',t)| \leq C\|u\|_\alpha \left(|x-x'|^\alpha + (x+x')^\alpha |z-z'|^\alpha\right).$$

To arrive at these estimates, we have to do a little bit of preparation. Writing $\tilde{p} = (\tilde{x}, \tilde{z})$, we have by the triangle inequality

$$\begin{aligned} |\tilde{G}u(p,t) - \tilde{G}u(p',t)| &= \left| \iint \left(\tilde{G}(p,\tilde{p},t-\tilde{t}) - \tilde{G}(p',\tilde{p},t-\tilde{t})\right) u(\tilde{p},\tilde{t}) d\tilde{t} d\tilde{p} \right| \\ \leq & \left| \iint \left(\tilde{G}(x,z,\tilde{x},\tilde{z},t-\tilde{t}) - \tilde{G}(x',z,\tilde{x},\tilde{z},t-\tilde{t})\right) u(\tilde{x},\tilde{z},\tilde{t}) d\tilde{t} d\tilde{x} d\tilde{z} \right| \\ + & \left| \iint \left(\tilde{G}(x',z,\tilde{x},\tilde{z},t-\tilde{t}) - \tilde{G}(x',z',\tilde{x},\tilde{z},t-\tilde{t})\right) u(\tilde{x},\tilde{z},\tilde{t}) d\tilde{t} d\tilde{x} d\tilde{z} \right|. \end{aligned}$$

Now we can apply Lemma 3.4 to each of the summands and obtain for X lying on the segment between x and x', and Z lying between z and z'

$$\begin{aligned} &|\tilde{G}u(p,t) - \tilde{G}u(p',t)| \\ \leq &|x-x'|^\alpha X^{1-\alpha} C \left| \iint \partial_X \tilde{G}(X,z,\tilde{x},\tilde{z},t-\tilde{t}) u(\tilde{x},\tilde{t}) d\tilde{t} d\tilde{x} d\tilde{z} \right| \qquad =: I_1 \\ + &|z-z'|^\alpha Z^{1-\alpha} C \left| \iint \partial_Z \tilde{G}(x,Z,\tilde{x},\tilde{z},t-\tilde{t}) u(\tilde{x},\tilde{t}) d\tilde{t} d\tilde{x} d\tilde{z} \right| \qquad =: I_2. \end{aligned}$$

Performing the estimates on the spatial Hölder difference now amounts to estimating I_1 and I_2 at the various corners of the heat space.

In most situations the results will simply follow from cancellations. Only near the diagonal the naive estimates will fail and leave a seemingly singular factor, which is then estimated with a different technique.

In the following, we write H for the heat kernel, $\tilde{G} = Vx^{-\gamma-\alpha}H$ as above and G for bounded distributions which vanish to infinite order at the temporal face, e.g. the heat kernel after factoring out the boundary defining functions coming from the polyhomogeneity, as well as derivatives of it.

3.1.1 Spatial estimates near the lower left corner

In this regime the asymptotic behaviour of the heat kernel is appropriately described in the following coordinates:

$$\rho_{\mathrm{lf}} = \tilde{s} = \frac{\tilde{x}}{x}, \ \tau = \frac{t-\tilde{t}}{x^2}, \ \rho_{\mathrm{ff}} = x, \ z, \ \tilde{z}, y$$

Accordingly the asymptotics of the heat kernel are given by $H = x^{-m}\tilde{s}^\mu G$, where G is vanishing to infinite order at the temporal face. In these coordinates we observe that $\beta^*\partial_x \sim x^{-1}\{\tilde{s}\partial_{\tilde{s}} + \tau\partial_\tau\}$ and $\beta^*\partial_z = \partial_z$. In particular, applying the b-vector fields to the heat kernel does not affect the asymptotics, so actually $\beta^*\partial_x \sim x^{-1}$. Consequently

$$\beta^* V x^{-\gamma-2} H = \beta^* \tilde{G} = x^{-m-\gamma-2} \tilde{s}^{\mu} G.$$

Furthermore we have

$$\beta^* \operatorname{dvol}(\tilde{p}) \sim \tilde{s}^{m-1} x^m d\tilde{s} d\tilde{z}, \ \beta^* dt \sim x^2 d\tau.$$

We begin by estimating I_1, noting that $u(\tilde{x}, \tilde{t}) = \tilde{x}^{\gamma+\alpha} v(\tilde{x}, \tilde{t})$.

$$\begin{aligned} I_1 =& |x - x'|^{\alpha} X^{1-\alpha} C \iint \partial_X \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\ =& |x - x'|^{\alpha} X^{1-\alpha} C \iint X^{-1} X^{-m-\gamma-2} \tilde{s}^{\mu} G (X\tilde{s})^{\gamma+\alpha} v X^{m+2} \tilde{s}^{m-1} d\tau d\tilde{s} d\tilde{z}. \end{aligned}$$

Now the X-factors cancel out exactly, so

$$I_1 \leq |x - x'|^{\alpha} C \iint \tilde{s}^{\mu+\gamma+m-1+\alpha} v d\tau d\tilde{s} d\tilde{z} \leq |x - x'|^{\alpha} C \|v\|_{\infty}.$$

Estimating I_2 is quite similar; note that due to differentiating in Z-direction instead of X-direction, we do not lose a power of X due to differentiation, netting a factor X^{α}, which we will convert into the leading factor of the distance function.

$$\begin{aligned} I_2 =& |z - z'|^{\alpha} Z^{1-\alpha} C \iint \partial_Z \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\ \leq& |z - z'|^{\alpha} Z^{1-\alpha} C \iint x^{-m-\gamma-2} \tilde{s}^{\mu} G (X\tilde{s})^{\gamma+\alpha} v X^{m+2} \tilde{s}^{m-1} d\tau d\tilde{s} d\tilde{z} \\ \leq& |z - z'|^{\alpha} Z^{1-\alpha} C \iint x^{\alpha} \tilde{s}^{\mu+\gamma+m-1+\alpha} v d\tau d\tilde{s} d\tilde{z} \end{aligned}$$

Since the cross section is compact, Z can be absorbed in the constant and as clearly $x^{\alpha} \leq (x + x')^{\alpha}$, we have

$$I_2 \leq (x + x')^{\alpha} |z - z'|^{\alpha} C \|v\|_{\infty}.$$

3.1.2 Spatial estimates near the lower right corner

Near the lower right corner of the heat space we employ the following coordinates:

$$\rho_{\mathrm{rf}} = s = \frac{x}{\tilde{x}}, \ \tau = \frac{t - \tilde{t}}{\tilde{x}^2}, \ \rho_{\mathrm{ff}} = \tilde{x}, \ z, \ \tilde{z}$$

Consequently the heat kernel is given by $H = \tilde{x}^{-m} s^{\mu} G$, where G vanishes to infinite order at tf. Again we do not see a decrease in front face behaviour due to application of b-vector fields, as $\beta^*\partial_x = \tilde{x}^{-1}\partial_s$, so that $\beta^* x\partial_x = s\partial_s$ and $\beta^*\partial_z = \partial_z$. Consequently we record for the asymptotics of the heat kernel

$$\beta^* V x^{-\gamma-2} H = \beta^* \tilde{G} = \tilde{x}^{-m-2-\gamma} s^{\mu-\gamma-2} G,$$

as well as for the volume form

$$\beta^* \operatorname{dvol}(\tilde{p}) \sim \tilde{x}^{m-1} d\tilde{x} d\tilde{z}, \; \beta^* d\tilde{t} \sim \tilde{x}^2 d\tau.$$

First we rewrite the integrals in these coordinates and factor out the boundary defining functions of $\tilde{G}$. For I_1 we have

$$\begin{aligned} I_1 &= |x-x'|^\alpha X^{1-\alpha} C \iint \partial_X \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\ &= |x-x'|^\alpha X^{1-\alpha} C \iint \tilde{x}^{-1} \partial_s (\tilde{x}^{-m-2-\gamma} s^{\mu-\gamma-2} G) \tilde{x}^{\gamma+\alpha} v \tilde{x}^{m+1} d\tau d\tilde{x} d\tilde{z} \\ &= |x-x'|^\alpha X^{1-\alpha} C \iint \tilde{x}^{-2+\alpha} s^{\mu-\gamma-3} G v d\tau d\tilde{x} d\tilde{z}. \end{aligned}$$

Now we integrate in the cross section and time, so this reduces to

$$I_1 \leq |x-x'|^\alpha C \|v\|_\infty X^{1-\alpha} \int \tilde{x}^{-2+\alpha} s^{\mu-\gamma-3} G d\tilde{x}.$$

Near the lower right the coordinate $s = \frac{X}{\tilde{x}}$ is bounded by some constant K, so that the integration region is actually $[XK^{-1}, \infty)$, and as we assume that G is compactly supported, it vanishes to infinite order as $\tilde{x} \to \infty$. Hence

$$\begin{aligned} I_1 \leq & |x-x'|^\alpha C \|v\|_\infty X^{1-\alpha} \int_{XK^{-1}}^\infty \tilde{x}^{-2+\alpha} s^{\mu-\gamma-3} G d\tilde{x} \\ = & |x-x'|^\alpha C \|v\|_\infty X^{\mu-\gamma-\alpha-2} \int_{XK^{-1}}^\infty \tilde{x}^{1+\gamma+\alpha-\mu} G d\tilde{x} \\ = & |x-x'|^\alpha C \|v\|_\infty X^{\mu-\gamma-\alpha-2} \left[\tilde{x}^{2-\mu+\gamma+\alpha} \right]_{XK^{-1}}^\infty \\ \leq & |x-x'|^\alpha C \|v\|_\infty. \end{aligned}$$

For I_2 we note that by differentiation in Z-direction, we gain x^α in contrast to I_1, which we will convert in the appropriate factor for the distance function again. Consequently I_2 is estimated as

$$\begin{aligned} I_2 = & |z-z'|^\alpha Z^{1-\alpha} C \iint \partial_Z \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\ \leq & |z-z'|^\alpha Z^{1-\alpha} C \iint \partial_Z \left(\tilde{x}^{-m-2-\gamma} s^{\mu-\gamma-2} G \right) \tilde{x}^{\gamma+\alpha} v \tilde{x}^{m+1} d\tau d\tilde{x} d\tilde{z} \\ \leq & |z-z'|^\gamma Z^{1-\gamma} C \iint \tilde{x}^{-1+\gamma} s^{\mu-\gamma-2} G v d\tau d\tilde{x} d\tilde{z}. \end{aligned}$$

The Z-factor is simply a bounded coordinate, so we absorb it in the constant. As before we reduce the problem by integrating in time and the cross section and then we use the boundedness of the s-coordinate to bound the integral.

$$\begin{aligned} I_2 \leq & |z-z'|^\alpha C \|v\|_\infty \int_{xK^{-1}}^\infty \tilde{x}^{-1+\alpha} s^{\mu-\gamma-2} G d\tilde{x} \\ \leq & |z-z'|^\alpha C \|v\|_\infty x^{\mu-\gamma-2} \int_{xK^{-1}}^\infty \tilde{x}^{1-\mu+\gamma+\alpha} G d\tilde{x} \\ \leq & |z-z'|^\alpha C \|v\|_\infty x^{\mu-\gamma-2} \left[\tilde{x}^{2-\mu+\gamma+\alpha} G \right]_{xK^{-1}}^\infty \leq |z-z'|^\alpha C \|v\|_\infty x^\alpha. \end{aligned}$$

As $x^\alpha \leq (x+x')^\alpha$ we obtain the desired bound:

$$I_2 \leq (x+x')^\alpha |z-z'|^\alpha C\|v\|_\infty.$$

3.1.3 Spatial estimates near the top corner

Estimates near the top corner are analogous to those near the lower right corner; we still include them for the sake of completeness. The appropriate coordinates in this regime are

$$\rho_{\mathrm{ff}} = \tau = \sqrt{t-\tilde{t}},\ \rho_{\mathrm{rf}} = \xi = \frac{x}{\tau},\ \rho_{\mathrm{lf}} = \tilde{\xi} = \frac{\tilde{x}}{\tau},\ z,\ \tilde{z},$$

and the heat kernel is given by $H = \tau^{-m}\xi^\mu\tilde{\xi}^\mu G$. We observe that $\beta^*\partial_x = \tau^{-1}\partial_\xi$, $\beta^*\partial_z = \partial_z$, so b-derivatives do not alter front or side face behaviour. For the volume form we observe

$$\beta^* \operatorname{dvol}(\tilde{p}) \sim \tilde{\xi}^{m-1}\tau^m d\tilde{\xi} d\tilde{z},\ \beta^* d\tilde{t} \sim \tau d\tau.$$

Finally we have

$$\beta^* V x^{-\gamma-2} H = \beta^*\tilde{G} \sim \tau^{-m-\gamma-2}\xi^{\mu-\gamma-2}\tilde{\xi}^\mu G.$$

Consequently we obtain

$$\begin{aligned}
I_1 =& |x-x'|^\alpha X^{1-\alpha} C \iint \partial_X \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\
\leq& |x-x'|^\alpha X^{1-\alpha} C \\
&\cdot \iint \tau^{-1}\partial_\xi(\tau^{-m-\gamma-2}\xi^{\mu-\gamma-2}\tilde{\xi}^\mu) G(\tilde{x}\tilde{\xi})^{\gamma+\alpha} v \tilde{\xi}^{m-1}\tau^{m+1} d\tau d\tilde{\xi} d\tilde{z} \\
\leq& |x-x'|^\alpha X^{1-\alpha} C \iint \tau^{-2+\alpha}\xi^{\mu-\gamma-3}\tilde{\xi}^{m-1+\mu+\gamma+\alpha} G v d\tau d\tilde{\xi} d\tilde{z}.
\end{aligned}$$

To simplify the argument, we first integrate in space, so that

$$I_1 \leq |x-x'|^\alpha C X^{1-\alpha}\|v\|_\infty \int \tau^{-2+\alpha}\xi^{\mu-\gamma-3} G d\tau,$$

where we again abuse notation and write G also for the integrated quantity. Again we use the boundedness of the coordinates, in this case $\xi < K$, so that in particular $\tau > XK^{-1}$. It follows that

$$\begin{aligned}
I_1 \leq& |x-x'|^\alpha C\|v\|_\infty X^{\mu-\gamma-\alpha-3} \int_{XK^{-1}}^\infty \tau^{2+\gamma+\alpha-\mu} G d\tau \\
=& |x-x'|^\alpha C\|v\|_\infty G X^{\mu-\gamma-\alpha-3} \left[\tau^{3+\gamma+\alpha-\mu} G\right]_{XK^{-1}}^\infty \\
\leq& |x-x'|^\alpha C\|v\|_\infty,
\end{aligned}$$

where we again assumed G to be compactly supported near the top corner. Estimating I_2 is the same game as before.

$$\begin{aligned}
I_2 = &|z-z'|^{\alpha} Z^{1-\alpha} C \iint \partial_Z \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\
\leq &|z-z'|^{\alpha} Z^{1-\alpha} C \iint \partial_Z \left(\tau^{-m-\gamma-2}\xi^{\mu-\gamma-2}\tilde{\xi}^{\mu} G\right)(\tilde{x}\tilde{\xi})^{\gamma+\alpha} v \tilde{\xi}^{m-1}\tau^{m+1} d\tau d\tilde{\xi} d\tilde{z} \\
\leq &|z-z'|^{\alpha} Z^{1-\alpha} C \iint \tau^{-1+\alpha}\xi^{\mu-\gamma-2}\tilde{\xi}^{m-1+\mu+\gamma+\alpha} G v d\tau d\tilde{\xi} d\tilde{z}
\end{aligned}$$

We again integrate in space and absorb the Z factor, so we are left with

$$\begin{aligned}
I_2 \leq &|z-z'|^{\alpha} C \|v\|_{\infty} \int_{xK^{-1}}^{\infty} \tau^{-1+\alpha}\xi^{\mu-\gamma} G d\tau \\
\leq &|z-z'|^{\alpha} C \|v\|_{\infty} x^{\gamma-\mu-2} \int_{xK^{-1}}^{\infty} \tau^{1-\mu+\gamma+\alpha} G d\tau \\
\leq &|z-z'|^{\alpha} C \|v\|_{\infty} x^{\gamma-\mu-2} \left[\tau^{2-\mu+\gamma+\alpha} G\right]_{xK^{-1}}^{\infty} \\
\leq &|z-z'|^{\alpha} C \|v\|_{\infty} x^{\alpha} \leq (x+x')^{\alpha}|z-z'|^{\alpha} C \|v\|_{\infty}.
\end{aligned}$$

3.1.4 Spatial estimates where the diagonal meets the front face

In this regime the “naive” estimates lead to an apparently singular factor, necessitating a more involved integration argument. To demonstrate, we will, at first, work in the following coordinates:

$$\rho_{\mathrm{td}} = \eta = \frac{\sqrt{t-\tilde{t}}}{x}, \ S = \frac{x-\tilde{x}}{\sqrt{t-\tilde{t}}}, \ Z = \frac{z-\tilde{z}}{\eta}, \ \rho_{\mathrm{ff}} = x, \ z$$

The heat kernel reads in these coordinates as $H = \eta^{-m} x^{-m} G$. The difficulty arises near the temporal diagonal, where even b-derivatives reduce the ff-asymptotics:

$$\begin{aligned}
\beta^* \partial_x &\sim x^{-1}\eta\partial_\eta + (x\eta)^{-1}\partial_S + x^{-1}\partial_Z + \partial_x \\
&\sim x^{-1} + x^{-1}\eta^{-1}\partial_S \sim x^{-1}\eta^{-1} \\
\beta^* \partial_z &\sim \eta^{-1}\partial_Z + \partial_z \sim \eta^{-1}\partial_Z
\end{aligned}$$

Consequently we obtain

$$\beta^* V x^{-\gamma-2} H \sim \rho^{-m-2} x^{-m-\gamma-2} G.$$

Remark 3.5. Take note that the problematic terms of the b-vector fields $\mathcal{V}_{\mathrm{b}}$ are of the form $\eta^{-1}\{\partial_S, \partial_Z\}$, i.e. the singular behaviour does stem from differentiation “around” the diagonal and not “along” it (Fig. 8). Then one usually can do partial differentiation once, as the boundary terms lie on tf, where the heat kernel vanishes to infinite order. However, to deal with the boundary terms, one has to rewrite the integrals in a different matter, and we do not know a way to alleviate this problem without stochastic completeness of the heat kernel. In [BV14] this fact could be exploited, allowing for the use of "non-hybrid" Hölder spaces. In our work, as well as in [BV16], we have to get along without this trick, and for this the hybrid Hölder spaces were introduced.

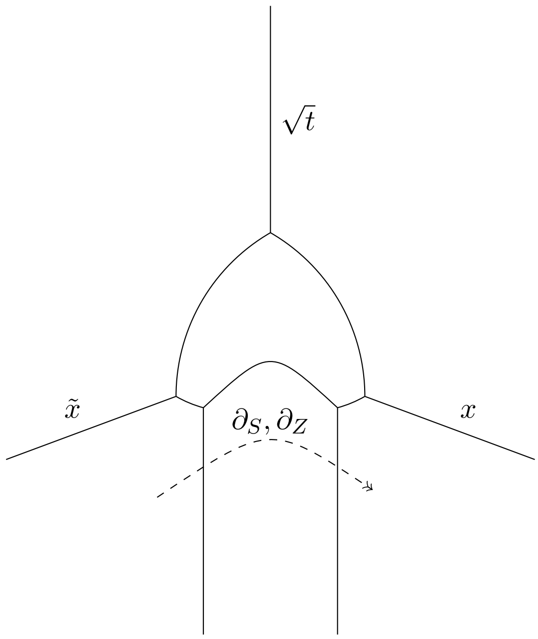

Figure 8: The problematic differentiation

For the volume form we compute

$$\beta^* \operatorname{dvol}(\tilde{p}) \sim \eta^m x^m (S\eta - 1)^{m-1} dSdZ, \;\; \beta^* d\tilde{t} \sim \eta x^2 d\eta.$$

We shortly demonstrate the arising problem for I_1, using that $\beta^* \partial_x \sim (x^{-1}\eta^{-1})$:

$$\begin{aligned} I_1 =& |x - x'|^\alpha X^{1-\alpha} C \iint \partial_X \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\ \leq& |x - x'|^\alpha X^{1-\alpha} C \iint (X\eta)^{-1} (\eta^{-m-2} X^{-m-\gamma-2} G) \\ &\cdot X^{\gamma+\alpha} (S\eta - 1)^{\gamma+\alpha} v \eta^{m+1} X^{m+2} (S\eta - 1)^{m-1} d\eta dS dZ \\ =& |x - x'|^\alpha X^{1-\alpha} C \iint \eta^{-2} X^{-1+\alpha} G v (S\eta - 1)^{m-1+\gamma+\alpha} d\eta dS dZ. \end{aligned}$$

If it were not for the η^{-2} coming from a possible application of two b-vector field, this would be easy to bound uniformly.

To deal with this issue, we go back to the coordinates from near the left, see Section 3.1.1. Although they do not capture the asymptotics of the heat kernel near the diagonal fully, they are valid away from rf and hence also near the diagonal. Note that now $\eta = \sqrt{\tau}$ is the boundary defining function of the diagonal. As a consequence,

$$d\tilde{t} = \frac{d(-\eta^2 x^2 + t)}{d\eta} d\eta \sim \eta x^2 d\eta.$$

Now we estimate:

$$\begin{aligned}
I_1 =& |x - x'|^{\alpha} X^{1-\alpha} C \iint \partial_X \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\
=& |x - x'|^{\alpha} X^{1-\alpha} C \iint \eta^{-m} X^{-1} X^{-m-\gamma-2} \tilde{s}^{\mu} G(X\tilde{s})^{\gamma+\alpha} v X^{m+2} \tilde{s}^{m-1} \eta d\eta d\tilde{s} d\tilde{z} \\
\leq& |x - x'|^{\alpha} C \|v\|_{\infty} \iint \eta^{-m+1} \tilde{s}^{\mu+\gamma+m-1+\alpha} v d\eta d\tilde{s} d\tilde{z}
\end{aligned}$$

In these coordinates we have that

$$d\left((x, z), (\tilde{x}, \tilde{z})\right) \sim x \sqrt{|1 - \tilde{s}|^2 + |1 + \tilde{s}|^2 |z - \tilde{z}|^2}.$$

Writing $r(\tilde{s}, z, \tilde{z}) = \sqrt{|1 - \tilde{s}|^2 + |z - \tilde{z}|^2}$, we note

$$x r(\tilde{s}, z, \tilde{z}) \leq d(x, \tilde{s}, z, \tilde{z}) \leq 2 x r(\tilde{s}, z, \tilde{z}).$$

Using this we transform $(\tilde{s}, \tilde{z})$ around $(1, z)$ into polar coordinates, with r as the radial variable. This gives us a factor of r^{m-1} in the integration, so that after integrating in the non-problematic angular coordinates

$$I_1 \leq |x - x'|^{\alpha} C \|v\|_{\infty} \iint \tau^{-m+1} r^{m-1} G d\eta dr.$$

Now we substitute $\sigma = \eta / r$; as $d\eta = r d\sigma$ this yields another factor of r and we have

$$I_1 \leq |x - x'|^{\alpha} C \|v\|_{\infty} \iint \sigma^{-m+1} r G d\sigma dr.$$

Observe that in the coordinates near the diagonal we have

$$\sigma^{-1} = r/\eta = \eta^{-1} \sqrt{|1 - \tilde{s}|^2 + |z - \tilde{z}|^2} = \sqrt{|S|^2 + |Z|^2}.$$

But in the limit of $|S|^2 + |Z|^2 \to \infty$ lies tf and G vanishes to infinite order there. Consequently we may estimate

$$I_1 \leq |x - x'|^{\alpha} C \int r dr \leq |x - x'|^2 C \|v\|_{\infty}.$$

Again the estimation of I_2 follows along the same lines; note that we here use Ξ as the variable from the mean value theorem.

$$
\begin{aligned}
I_2 =&|z-z'|^\alpha \Xi^{1-\alpha} C \iint \partial_\Xi \tilde{G} u d\tilde{t} d\tilde{x} d\tilde{z} \\
\leq &|z-z'|^\alpha \Xi^{1-\alpha} C \iint \eta^{-m} x^{-m-\gamma-2} \tilde{s}^\mu G(x\tilde{s})^{\gamma+\alpha} v x^{m+2} \tilde{s}^{m-1} \eta d\eta d\tilde{s} d\tilde{z} \\
\leq &|z-z'|^\alpha \Xi^{1-\alpha} C \|v\|_\infty \iint \eta^{-m+1} x^\alpha \tilde{s}^{\mu+\gamma+m-1+\alpha} d\tau d\tilde{s} d\tilde{z}
\end{aligned}
$$

Again we absorb the Ξ-factor in the constant, do the transformation to polar coordinates and integrate in the angular coordinates:

$$
\begin{aligned}
I_2 \leq &|z-z'|^\alpha x^\alpha C \|v\|_\infty \iint \eta^{-m+1} r^{m-1} d\eta dr \\
\leq &|z-z'|^\alpha x^\alpha C \|v\|_\infty \iint \sigma^{-m+1} r d\sigma dr \leq |z-z'|^\alpha |x+x'|^\alpha C \|v\|_\infty.
\end{aligned}
$$

3.2 Estimates on the temporal Hölder differences

We wish to establish

$$
x^{-\gamma-2} V H[u](p,\tilde{p},t') - x^{-\gamma-2} V H[u](p,\tilde{p},t) \leq |t'-t|^{\alpha/2} C \|u\|_{\gamma,\alpha},
$$

for any $V \in \{\mathrm{id}, \mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\}$. Again we abbreviate $\tilde{G} = V x^{-\gamma-2} H$.

For the estimates near the corners of the blow-up space, i.e. near the lower left, lower right and the top corner, we can use the rather naive estimate

$$
\begin{aligned}
&\iint_0^{t'} \tilde{G}(p,\tilde{p},t'-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) - \iint_0^{t} \tilde{G}(p,\tilde{p},t-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&= \iint_0^{t} \left(\tilde{G}(p,\tilde{p},t'-\tilde{t}) - \tilde{G}(p,\tilde{p},t-\tilde{t}) \right) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&+ \iint_t^{t'} \tilde{G}(p,\tilde{p},t'-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&=: T_1 + T_2.
\end{aligned}
$$

Using the mean value theorem the estimates for T_1 begin with

$$
T_1 \leq |t'-t| C \iint_0^t \partial_\theta \tilde{G}(p,\tilde{p},\theta-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}),
$$

for some $\theta \in [t,t']$.

These estimates however fail away from the lower corners. Near the top and diagonal we use a slightly more complicated estimate in order to use various algebraic arguments to obtain the leading Hölder factor of $|t'-t|^{\alpha/2}$. Near the diagonal we once again need a less naive argument and use partial integration.

We will first make the case distinction whether $2t \leq t'$ or not. If $2t \leq t'$, we have the inequalities

$$t < t' - t < t' < 2(t' - t),$$

allowing us to simply estimate the integrals

$$\begin{aligned}&\iint_0^{t'} \tilde{G}(p, \tilde{p}, t' - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) - \iint_0^{t} \tilde{G}(p, \tilde{p}, t - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ =:&K_1 - K_2.\end{aligned}$$

However, if $2t > t' \iff 2t - t' > 0$, we write

$$T_- = [0, 2t - t'],\ T_+ = [2t - t', t],\ T'_+ = [2t - t', t'].$$

Then over the common interval T_-, we can examine the difference between the heat kernels, so we rewrite the integrals as follows:

$$\begin{aligned}&\iint_0^{t'} \tilde{G}(p, \tilde{p}, t' - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) - \iint_0^{t} \tilde{G}(p, \tilde{p}, t - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ =&\iint_{T'_+} \tilde{G}(p, \tilde{p}, t' - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) - \iint_{T_+} \tilde{G}(p, \tilde{p}, t - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ &+ \iint_{T_-} \tilde{G}(p, \tilde{p}, t' - \tilde{t}) - \tilde{G}(p, \tilde{p}, t - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ =:&L_1 - L_2 + L_3\end{aligned}$$

Applying the mean value theorem to L_3 and adding $0 = -u(\tilde{p}, \theta) + u(\tilde{p}, \theta)$ for the $\theta \in [t, t']$ coming from the mean value theorem, we have

$$\begin{aligned}L_3 &\leq |t' - t|C \iint_{T_-} \partial_\theta \tilde{G}(p, \tilde{p}, \theta - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ &= |t' - t|C \iint_{T_-} \partial_\theta \tilde{G}(p, \tilde{p}, \theta - \tilde{t})(u(\tilde{p}, \tilde{t}) - u(\tilde{p}, \theta))d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ &+ |t' - t|C \iint_{T_-} \partial_\theta \tilde{G}(p, \tilde{p}, \theta - \tilde{t})u(\tilde{p}, \theta)d\tilde{t}\,\mathrm{dvol}(\tilde{p})\\ =:&L'_3 - L''_3.\end{aligned}$$

Note that we only need the mean value theorem near the top and right corner. Near the diagonal, we can estimate L'_3 using the Hölder continuity of u. For L''_3 we first make use of a simple consequence of the chain rule, allowing us to trade the ∂_θ-derivative with a $\partial_{\tilde{t}}$, followed by partial integration in time:

$$
\begin{aligned}
-L_3'' &= |t'-t|C \iint_{T_-} \partial_\theta \tilde{G}(p,\tilde{p},\theta-\tilde{t})u(\tilde{p},\theta)d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&= -|t'-t|C \iint_{T_-} \partial_{\tilde{t}} \tilde{G}(p,\tilde{p},\theta-\tilde{t})u(\tilde{p},\theta)d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&= -|t'-t|C \int \tilde{G}(p,\tilde{p},\theta-\tilde{t})u(\tilde{p},\theta)\,\mathrm{dvol}(\tilde{p})\Big|_0^{2t-t'} \\
&+ |t'-t|C \iint_{T_-} \tilde{G}(p,\tilde{p},\theta-\tilde{t})\,\underbrace{\partial_{\tilde{t}} u(\tilde{p},\theta)}_{=0}\,d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&= -|t'-t|C \int \tilde{G}(p,\tilde{p},\theta-\tilde{t})u(\tilde{p},\theta)\,\mathrm{dvol}(\tilde{p})\Big|_0^{2t-t'}
\end{aligned}
$$

In this situation we will make use of the following elementary estimate, which holds for any $\theta \in [t,t']$:

$$
\begin{aligned}
&|t'-t|\left|\int_0^{2t-t'} (\theta-\tilde{t})^{-2+\alpha/2} d\tilde{t}\right| \\
&= |t'-t|\frac{1}{1-\alpha/2}\,(\theta-\tilde{t})^{-1+\alpha/2}\Big|_0^{2t-t'} \\
&= |t'-t|C\left((\theta-2t+t')^{-1+\alpha/2}-\theta^{-1+\alpha/2}\right) \qquad \text{(ITE)} \\
&\le |t'-t|C(\theta-t+t'-t)^{-1+\alpha/2} \\
&\le |t'-t|C(t'-t)^{-1+\alpha/2} = |t'-t|^{\alpha/2}C.
\end{aligned}
$$

3.2.1 Temporal estimates near the lower left corner

We begin by estimating T_1. The time derivative transforms in the coordinates from near the lower left as follows:

$$
\beta^*\partial_\theta = \partial_\theta\tau\partial_\tau = \partial_\theta\sqrt{\theta-\tilde{t}}\partial_\tau \sim \tau^{-1}\partial_\tau.
$$

Consequently we estimate for T_1:

$$
\begin{aligned}
T_1 &\le |t'-t|C \iint_0^t \partial_\theta \tilde{G}(p,\tilde{p},\theta-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&= |t'-t|C \iint \tau^{-1}\partial_\tau(x^{-m-\gamma-2}\tilde{s}^\mu G)(x\tilde{s})^{\gamma+\alpha} v x^{m+2}\tilde{s}^{m-1}\tau d\tau d\tilde{s} d\tilde{z} \\
&\le |t'-t|C\|v\|_\infty \iint x^\alpha \tilde{s}^{\mu+m\gamma+\alpha-1} G d\tau d\tilde{s} d\tilde{z} \\
&\le |t'-t|C\|v\|_\infty
\end{aligned}
$$

For T_2 we skip the cancellation steps and note the absence of the time derivative:

$$\begin{aligned}
T_2 &\leq \iint_t^{t'} \tilde{G}(p,\tilde{p},t'-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&\leq C\|v\|_\infty \iint x^\alpha \tilde{s}^{\mu+m\gamma+\alpha-1}\tau G d\tau d\tilde{s} d\tilde{z} \\
&\leq C\|v\|_\infty \, \tau^2\Big|_t^{t'} \\
&= |t'-t|C\|v\|_\infty
\end{aligned}$$

3.2.2 Temporal estimates near the lower right corner

We use the same coordinates as in Section 3.1.2. We may estimate

$$\begin{aligned}
K_1 &= \iint_0^{t'} \tilde{G}(p,\tilde{p},t'-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&\leq \|v\|_\infty \iint \tilde{x}^{-m-\gamma-2}s^{\mu-\gamma-2}G\tilde{x}^{\gamma+\alpha}\tilde{x}^{m+1}d\tau d\tilde{x} d\tilde{z} \\
&= \|v\|_\infty \iint \tilde{x}^{-1+\alpha}s^{\mu-\gamma-2}G d\tau d\tilde{x} d\tilde{z}
\end{aligned}$$

Again we make use of the boundedness of the τ-coordinate, after transforming back to integration in $\tilde{t}$.

$$\begin{aligned}
K_1 &\leq \|v\|_\infty \iint \tilde{x}^{-3+\alpha}s^{\mu-\gamma-2}G d\tilde{t} d\tilde{x} d\tilde{z} \\
&= \|v\|_\infty x^{\mu-\gamma-2} \iint \tilde{x}^{-\mu+\gamma-1+\alpha}G d\tilde{t} d\tilde{x} d\tilde{z} \\
&= \|v\|_\infty x^{\mu-\gamma-2} \iint \tilde{x}^{-\mu+\gamma+1}G(t'-\tilde{t})^{-1+\alpha/2} d\tilde{t} d\tilde{x} d\tilde{z}
\end{aligned}$$

After integrating over the compact cross section, we take note that by the boundedness of s, $x < \tilde{x} < \infty$

$$\begin{aligned}
K_1 &\leq \|v\|_\infty C x^{\mu-\gamma-2} \int_0^{t'}\int_x^\infty \tilde{x}^{-\mu+\gamma+1}G(t'-\tilde{t})^{-1+\alpha/2} d\tilde{t} d\tilde{x} \\
&\leq \|v\|_\infty C x^{\mu-\gamma-2} \int_0^{t'} \tilde{x}^{-\mu+\gamma+2}\Big|_x^\infty (t'-\tilde{t})^{-1+\alpha/2} d\tilde{t} \\
&\leq \|v\|_\infty C \, (t'-\tilde{t})^{\alpha/2}\Big|_0^{t'} = \|v\|_\infty C t'^{\alpha/2} \\
&\leq \|v\|_\infty C |t'-t|^{\alpha/2}.
\end{aligned}$$

The integral K_2 is estimated exactly the same, just with t in place of t'. The integrals L_1 and L_2 follow mutatis mutandis, however we have to adjust the argument to the different integration region in time in the final step:

$$L_1 \leq \|v\|_\infty C\ (t' - \tilde{t})^{\alpha/2}\Big|_{2t-t'}^{t'} = \|v\|_\infty C 2^{\alpha/2}(t'-t)^{\alpha/2}$$
$$L_2 \leq \|v\|_\infty C\ (t - \tilde{t})^{\alpha/2}\Big|_{2t-t'}^{t} = \|v\|_\infty C(t'-t)^{\alpha/2}$$

For L_3 we first calculate

$$\beta^*(\partial_\theta) = \partial_\theta \tau \partial_\tau = \tilde{x}^{-2}\partial_\tau,$$

and begin by writing out the corresponding integral in the coordinates from near the lower right corner.

$$\begin{aligned} L_3 &\leq |t'-t| C \iint_{T_-} \partial_\theta \tilde{G}(p,\tilde{p},\theta-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\ &= |t'-t| C \iint \tilde{x}^{-2}\partial_\tau(\tilde{x}^{-m-\gamma-2}s^{\mu-\gamma-2}G)\tilde{x}^{\gamma+\alpha}v\tilde{x}^{m+1}d\tau d\tilde{x}d\tilde{z} \\ &\leq |t'-t| C\|v\|_\infty \iint \tilde{x}^{-3+\alpha}s^{\mu-\gamma-2}G d\tau d\tilde{x}d\tilde{z} \end{aligned}$$

We observe that the coordinate $\tau = \frac{\theta-\tilde{t}}{\tilde{x}}$ is uniformly bounded, so we can trade part of the singular $\tilde{x}$–factor for a singular $(\theta-\tilde{t})$-factor:

$$L_3 \leq |t'-t| C\|v\|_\infty \iint \tilde{x}^{-1}s^{\mu-\gamma-2}(\theta-\tilde{t})^{-1+\alpha/2}G d\tau d\tilde{x}d\tilde{z}$$

We now integrate the spatial coordinates, using $s = x/\tilde{x}$:

$$\begin{aligned} L_3 &\leq |t'-t| C\|v\|_\infty x^{\mu-\gamma-2} \iint \tilde{x}^{-\mu+\gamma+1}(\theta-\tilde{t})^{-1+\alpha/2}G d\tau d\tilde{x}d\tilde{z} \\ &\leq |t'-t| C\|v\|_\infty C \int (\theta-\tilde{t})^{-1+\alpha/2}G d\tau \end{aligned}$$

As we are in the situation of $2t - t' > 0$ we have

$$\theta - \tilde{t} \geq \theta - 2t + t' > t' - t$$

and can use a factor of $|t'-t|^{1-\alpha/2}$ to cancel out the singular time factor, leading to the desired estimate:

$$L_3 \leq |t'-t|^{\alpha/2} C\|v\|_\infty$$

3.2.3 Temporal estimates near the top corner

We first estimate K_1 and K_2. Both integrals are estimated the same, so we demonstrate the estimation for K_1, noting that for K_2 one only needs to replace the t' by t. As we are in the case $2t \leq t'$, we have $t < t' - t < t' < 2(t' - t)$ and, trivially, $t' \geq t' - \tilde{t}$, so we may estimate

$$\begin{aligned} K_1 &= \iint_0^{t'} \tilde{G}(p, \tilde{p}, t' - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t} \,\mathrm{dvol}(\tilde{p}) \\ &= (t')^{\alpha/2} \iint_0^{t'} (t')^{-\alpha/2} G(p, \tilde{p}, t' - \tilde{t}) d\tilde{t} \,\mathrm{dvol}(\tilde{p}) \\ &\leq |t' - t|^{\alpha/2} C \iint_0^{t'} (t' - \tilde{t})^{-\alpha/2} G(p, \tilde{p}, t' - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t} \,\mathrm{dvol}(\tilde{p}). \end{aligned}$$

In the coordinates from near the top coordinates and using that $u \in x^{\gamma+\alpha} C^0$ this reads as

$$\begin{aligned} K_1 &\leq |t' - t|^{\alpha/2} C \iint \tau^{-\alpha} (\tau^{-m-\gamma-2} \xi^{\mu-\gamma-2} \tilde{\xi}^{\mu} G)(\tilde{\xi}\tau)^{\gamma+\alpha} v \tilde{\xi}^{m-1} \tau^{m+1} d\tau d\tilde{\xi} d\tilde{z}. \\ &\leq |t' - t|^{\alpha/2} C \|v\|_\infty \iint \tau^{-1} \xi^{\mu-\gamma-2} \tilde{\xi}^{\mu+\gamma+m+\alpha-1} G d\tau d\tilde{\xi} d\tilde{z}. \end{aligned}$$

Integrating in the spatial coordinates, we quickly arrive at

$$\begin{aligned} K_1 &\leq |t' - t|^{\alpha/2} C \|v\|_\infty \int \tau^{-1} \xi^{\mu-\gamma-2} G d\tau \\ &\leq |t' - t|^{\alpha/2} C \|v\|_\infty x^{\mu-\gamma-2} \int_{xK^{-1}}^{\infty} \tau^{-\mu+\gamma+1} d\tau \\ &\leq |t' - t|^{\alpha/2} C \|v\|_\infty x^{\mu-\gamma-2} \; \tau^{-\mu+\gamma+2} \Big|_{xK^{-1}}^{\infty} \\ &\leq |t' - t|^{\alpha} C \|v\|_\infty, \end{aligned}$$

where we made use of the boundedness of the τ-coordinate, just as in Section 3.1.3.

To estimate L_1 and L_2 we begin by writing out the integrals and performing the usual cancellations:

$$\begin{aligned} L_1 &= \iint_{2t-t'}^{t'} \tilde{G}(p, \tilde{p}, t' - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t} \,\mathrm{dvol}(\tilde{p}) \\ &= \iint_{T'_+} \tau^{-m-\gamma-2} \tilde{\xi}^{\mu} \xi^{\mu-\gamma-2} (\tilde{\xi}\tau)^{\gamma+\alpha} G v \tau^{m+1} \tilde{\xi}^{m-1} d\tau d\tilde{\xi} d\tilde{z} \\ &\leq C \|v\|_\infty \iint_{T'_+} \tau^{-1+\alpha} G \xi^{\mu-\gamma-2} \tilde{\xi}^{\mu+\gamma+m+\alpha-1} d\tau d\tilde{\xi} d\tilde{z} \end{aligned}$$

Integrating the spatial coordinates and noting that the coordinates $\xi, \tilde{\xi}$ are uniformly bounded, we obtain

$$\begin{aligned} L_1 &\leq C \|v\|_\infty \iint_{T'_+} \tau^{-1+\alpha} G d\tau \\ &\leq C \|v\|_\infty \left| (t' - t)^{\alpha/2} \Big|_{2t-t'}^{t'} \right| \\ &= C \|v\|_\infty \left| -(t' - 2t + t') \right| \\ &\leq |t' - t|^{\alpha/2} C \|v\|_\infty. \end{aligned}$$

Now L_2 is estimated almost the same. The only difference lies in the last step, as $(t - 2t + t') = (t' - t)$.

Finally we turn to L_3. We note that $\beta^* \partial_\theta \sim \tau^{-1}\partial_\tau$ and begin estimating

$$\begin{aligned} L_3 &\leq |t' - t| C \iint_{T_-} \partial_\theta \tilde{G}(p, \tilde{p}, \theta - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\ &= |t' - t| C \iint_{T_-} \tau^{-1} \partial_\tau (\tau^{-m-\gamma-2} \xi^{\mu-\gamma-2} \tilde{\xi}^{\mu} G)(\tilde{\xi}\tau)^{\gamma+\alpha} v \tilde{\xi}^{m-1} \tau^{m+1} d\tau d\tilde{\xi} d\tilde{z} \\ &\leq |t' - t| C \|v\|_\infty \iint_{T_-} \tau^{-3+\alpha} \tilde{\xi}^{\mu+\gamma+m+\alpha-1} \xi^{\mu-\gamma-2} G d\tau d\tilde{\xi} d\tilde{z}. \end{aligned}$$

Integrating in the spatial coordinates we have

$$\begin{aligned} L_3 &\leq |t' - t| C \|v\|_\infty \int_{T_-} \tau^{-3+\alpha} G d\tau \\ &= |t' - t| C \|v\|_\infty \, \tau^{-2+\alpha} G \Big|_0^{2t-t'} \\ &\leq |t' - t| C \|v\|_\infty, \end{aligned}$$

where the last step is essentially Eq. (ITE), since $\tau^{-2+\alpha} = (\theta - \tilde{t})^{-1+\alpha/2}$.

3.2.4 Temporal estimates where the diagonal meets the front face

We begin by estimating K_1.

$$\begin{aligned} K_1 &= \iint_0^{t'} \tilde{G}(p, \tilde{p}, t' - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\ &= t'^{\alpha/2} \iint t'^{-\alpha/2} G(p, \tilde{p}, t' - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\ &\leq t'^{\alpha/2} \iint (t' - \tilde{t})^{-\alpha/2} G(p, \tilde{p}, t' - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \end{aligned}$$

After cancellations this would read in the coordinates from near the diagonal as

$$K_1 \leq t'^{\alpha/2} \iint \eta^{-1-\alpha/2} G v d\eta dS dZ,$$

so we continue estimating in the coordinates valid near the lower left, with $\eta = \sqrt{\tau}$ as boundary defining function of the diagonal:

$$\begin{aligned} K_1 &= t'^{\alpha/2} \iint (t' - \tilde{t})^{-\alpha/2} x^{-m-\gamma-2} \tilde{s}^{\mu} \eta^{-m} G u x^{m+2} \tilde{s}^{m-1} \eta d\eta d\tilde{s} d\tilde{z} \\ &= t'^{\alpha/2} \iint (t' - \tilde{t})^{-\alpha/2} x^{-\gamma} \tilde{s}^{\mu+m-1} \eta^{-m+1} G x^{\gamma+\alpha} \tilde{s}^{\gamma+\alpha} v d\eta d\tilde{s} d\tilde{z} \\ &= t'^{\alpha/2} \iint \eta^{-m+1-\alpha/2} \tilde{s}^{\mu+m+\gamma+\alpha-1} G v d\eta dS dZ. \end{aligned}$$

Transforming into polar coordinates with r as the radial variable, as in Section 3.1.4, and integrating the angular coordinates, we obtain

$$\begin{aligned}
K_1 &\le \|v\|_\infty t'^{\alpha/2} C \iint \eta^{-m+1-\alpha/2} r^{m-1} G d\eta dr \\
&= \|v\|_\infty t'^{\alpha/2} C \iint \sigma^{-m+1-\alpha/2} r^{1-\alpha/2} G d\sigma dr \\
&\le \|v\|_\infty t'^{\alpha/2} C \\
&\le \|v\|_\infty |t'-t|^{\alpha/2} C.
\end{aligned}$$

Now K_2 is estimated exactly the same, with the only difference being replacing t' by t.

For L_1 and L_2 we observe that for $\tilde{t} > 2t - t'$ we have

$$2(t'-t) = t' - 2t + t' \ge t' - \tilde{t} \ge t - \tilde{t},$$

so we can pull out the leading factor as in the estimates of K_1 and K_2:

$$\begin{aligned}
L_1 &\le (t'-t)^{\alpha/2} \iint (t'-\tilde{t})^{\alpha/2} \tilde{G}(p,\tilde{p},t'-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
L_2 &\le (t'-t)^{\alpha/2} \iint (t-\tilde{t})^{\alpha/2} \tilde{G}(p,\tilde{p},t-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p})
\end{aligned}$$

Now L_1 and L_2 are estimated just as K_1 and K_2.

The most difficulties arise when dealing with the L_3-integrals. First we calculate

$$\beta^* \partial_\theta \sim x^{-2} \eta^{-1} \partial_\eta + x^{-2} \eta^{-2} \{\partial_S, \partial_Z\}.$$

In particular we can treat the time-derivative as a $\eta^{-2} x^{-2}$-factor.

$$\begin{aligned}
L_3' &= |t'-t| C \iint_{T_-} \partial_\theta \tilde{G}(p,\tilde{p},\theta-\tilde{t})(u(\tilde{p},\tilde{t}) - u(\tilde{p},\theta)) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&\le |t'-t| C \|v\|_\alpha \iint_{T_-} x^{-2} \eta^{-2} (\eta^{-m-2} x^{-m-2-\gamma} G) \\
&\cdot |\theta - \tilde{t}|^{\alpha/2} \eta^{m+1} x^{m+2+\gamma} (1 - S\eta)^{m-1+\gamma+\alpha} d\eta dS dZ \\
&\le |t'-t| C \|v\|_\alpha \iint_{T_-} x^{-2} \eta^{-3} G (1-S\eta)^{m-1+\gamma+\alpha} |\theta - \tilde{t}|^{\alpha/2} d\eta dS dZ \\
&= |t'-t| C \|v\|_\alpha x^{-2+\alpha} \iint_{T_-} \eta^{-3+\alpha} G (1-S\eta)^{m-1+\gamma+\alpha} d\eta dS dZ
\end{aligned}$$

At this point we again notice that the $(1-S\eta)$-factor can be dropped in our estimates, as the heat kernel vanishes in the limit $|S| \to \infty$ to infinite order. Being in the case $2t > t'$, we can, after integration in time, estimate as in

Eq. (ITE):

$$
\begin{aligned}
L_3' &\leq |t'-t|C\|v\|_\alpha x^{-2+\alpha} \iint_{T_-} \eta^{-3+\alpha} G(1-S\eta)^{m-1+\gamma+\alpha} d\eta dS dZ \\
&\leq |t'-t|C\|v\|_\alpha x^{-2+\alpha}\ \eta^{-2+\alpha}\Big|_0^{2t-t'} \\
&= |t'-t|C\|v\|_\alpha\ (\theta-\tilde{t})^{-1+\alpha/2}\Big|_0^{2t-t'} \\
&\leq |t'-t|^{\alpha/2} C\|v\|_\alpha .
\end{aligned}
$$

Now L_3'' again needs a clever integration trick, because the following estimate only yields boundedness, but not the necessary leading factor of the Hölder distance:

$$
\begin{aligned}
L_3'' &= |t'-t|C \int \tilde{G}(p,\tilde{p},\theta-\tilde{t}) u(\tilde{p},\theta) dS dZ\Big|_0^{2t-t'} \\
\leq& |t'-t|\|v\|_\infty C \int \eta^{-m-2} x^{-m-\gamma-2+\alpha} G \eta^m x^{\gamma+m+\alpha}(1-\eta S)^{m+\gamma+\alpha} dS dZ\Big|_0^{2t-t'} \\
\leq& |t'-t|\|v\|_\infty C \int \eta^{-2} x^{-2+\alpha} G(1-\eta S)^{m+\gamma+\alpha} dS dZ\Big|_0^{2t-t'} \\
\leq& \|v\|_\infty C \int G(1-\eta S)^{m+\gamma+\alpha} dS dZ\Big|_0^{2t-t'}
\end{aligned}
$$

Using a similar argument as in Section 3.1.4, we first express L_3'' in the coordinates from the lower left, which are also valid near the diagonal, with $\eta = \frac{\sqrt{\theta-\tilde{t}}}{x}$ as the boundary defining function of td:

$$
\begin{aligned}
L_3'' &= |t'-t|C \int G\left(p,\tilde{p},\theta-\tilde{t}\right) u(\tilde{p},\theta)\, \mathrm{dvol}(\tilde{p})\Big|_0^{2t-t'} \\
&= |t'-t|C \int \eta^{-m} x^{-m-\gamma-2} \tilde{s}^\mu G \tilde{s}^{\gamma+\alpha} x^{\gamma+\alpha} \tilde{s}^{m-1} x^m v d\tilde{s} d\tilde{z}\Big|_0^{2t-t'} \\
&\leq |t'-t|C\|v\|_\infty \int \eta^{-m} x^{-2+\alpha} \tilde{s}^{\mu+\gamma+m+\alpha-1} G d\tilde{s} d\tilde{z}\Big|_0^{2t-t'}
\end{aligned}
$$

Recalling $r(\tilde{s},z,\tilde{z}) = \sqrt{|1-\tilde{s}|^2 + |z-\tilde{z}|^2}$, we transform $(\tilde{s},\tilde{z})$ around $(1,z)$ into polar coordinates and integrate in the angular coordinates. This again yields a factor of r^{m-1} from the transformation rule:

$$
L_3'' \leq |t'-t|C\|v\|_\infty \int \eta^{-m} x^{-2+\alpha} \tilde{s}^{\mu+\gamma+m+\alpha-1} r^{m-1} G dr\Big|_0^{2t-t'}
$$

We now substitute r by $\sigma = \eta/r$ so that we have $r^{m-1} = \sigma^{-m+1}\eta^{m-1}$ and

$dr = \sigma^{-2}\eta d\sigma$. Consequently we have

$$L_3'' \leq |t'-t|C\|v\|_\infty \int \eta^{-m}\sigma^{-m+1}\eta^{m-1}\sigma^{-2}\eta d\sigma\Big|_0^{2t-t'}$$
$$= |t'-t|C\|v\|_\infty \int \sigma^{-m-1}G d\sigma\Big|_0^{2t-t'},$$

and, since G vanishes to infinite order as $\sigma^{-1}\to\infty$, we obtain the desired bound

$$L_3'' \leq |t'-t|C\|v\|_\infty.$$

3.3 Estimates on the supremum

We establish that for $u \in C^\alpha_{e,\gamma}(M\times I)$ we have $Vx^{-\gamma-2-\alpha}H[u] \in C^0_e(M\times I)$, where $V \in \{\mathrm{id}, \mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\}$. This is considerably easier than the temporal and spatial estimates and we only perform these estimates for the sake of completeness. Estimates near the lower left and lower right corner follow from simple cancellations, and near the top corner we again use the boundedness of the coordinates. Also we have to resort to the integration trick near the intersection of front face and diagonal. This aside, the estimates are mainly a consequence of the assumption that u itself is already element of a weighted Hölder space, i.e. we have that $u(\tilde{x},\tilde{z},\tilde{t}) = \tilde{x}^{\gamma+\alpha}v(\tilde{x},\tilde{z},\tilde{t})$.

3.3.1 sup-estimates near the lower left corner

$$\begin{aligned}\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq& \sup\iint x^{-m-\gamma-2-\alpha}\tilde{s}^{\mu}Gu(\tilde{x},\tilde{z},\tilde{t})\tilde{s}^{m-1}x^{m+2}d\tau d\tilde{s}d\tilde{z}\\ \leq& \sup\iint x^{-\gamma-\alpha}\tilde{s}^{\mu+m-1}G(x\tilde{s})^{\gamma+\alpha}d\tau d\tilde{s}d\tilde{z}\\ \leq& \sup\iint \tilde{s}^{\gamma+m+\alpha-1}Gv d\tau d\tilde{s}d\tilde{z}\\ \leq& \|v\|_\infty C\end{aligned}$$

3.3.2 sup-estimates near the lower right corner

$$\begin{aligned}\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq& \sup\iint \tilde{x}^{-m-2-\gamma-\alpha}s^{\mu-\gamma-2-\alpha}Gu\tilde{x}^{m+2}d\tau d\tilde{x}d\tilde{z}\\ \leq& \sup\iint \tilde{x}^{-\gamma-\alpha}s^{\mu-\gamma-2-\alpha}G\tilde{x}^{\gamma+\alpha}v d\tau d\tilde{x}d\tilde{z}\\ \leq& \sup\iint s^{\mu-\gamma-2-\alpha}G d\tau d\tilde{x}d\tilde{z}\\ \leq& \|v\|_\infty C\end{aligned}$$

3.3.3 sup-estimates near the top corner

$$\begin{aligned}\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq & \sup \iint \tau^{-m-\gamma-2-\alpha}\xi^{\mu-\gamma-2-\alpha}\tilde{\xi}^{\mu}Gu\tilde{\xi}^{m-1}\tau^{m+1}d\tau d\tilde{\xi}d\tilde{z} \\ \leq & \sup \iint \tau^{-\gamma-1-\alpha}\xi^{\mu-\gamma-2-\alpha}\tilde{\xi}^{\mu+m-1}G(\tau\tilde{\xi})^{\gamma+\alpha}vd\tau d\tilde{\xi}d\tilde{z} \\ & \sup \iint \tau^{-1}\tilde{\xi}^{\mu+m+\gamma+a-1}\xi^{\mu-\gamma-2-\alpha}Gvd\tau d\tilde{\xi}d\tilde{z} \\ \leq & C\sup \int \tau^{-1}\xi^{\mu-\gamma-2-\alpha}Gvd\tau\end{aligned}$$

As before we have to make use of the fact that the coordinates are bounded and we may assume $\xi < K \iff xK^{-1} < \tau$ and consequently the integral reduces to

$$\begin{aligned}\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq & C\sup x^{\mu-\gamma-2-\alpha}\int_{xK^{-1}}^{\infty}\tau^{-\mu+\gamma+1+\alpha}Gvd\tau \\ \leq & C\sup\left[x^{\mu-\gamma-2-\alpha}\tau^{2-\mu+\gamma+\alpha}Gv\right]_{xK^{-1}}^{\infty} \\ \leq & \|v\|_\infty C.\end{aligned}$$

3.3.4 sup-estimates where the diagonal meets the front face

For the diagonal we note that it is again necessary to use the integration trick already employed in the previous estimates near the diagonal; we forego demonstrating the necessity and use directly the coordinates from near the lower left, with $\eta = \sqrt{\tau}$ being the boundary defining function of the diagonal.

$$\begin{aligned}\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq & \sup \iint \eta^{-m}x^{-m-\gamma-2-\alpha}\tilde{s}^{\mu}Gu\tilde{s}^{m-1}x^{m+2}\eta d\eta d\tilde{s}d\tilde{z} \\ = & \sup \iint \eta^{-m+1}x^{-\gamma-\alpha}\tilde{s}^{\mu+m-1}G(xs)^{\gamma+\alpha}vd\eta d\tilde{s}d\tilde{z} \\ = & \sup \iint \eta^{-m+1}\tilde{s}^{\mu+\gamma+m+\alpha-1}Gvd\eta d\tilde{s}d\tilde{z}\end{aligned}$$

Using $r(\tilde{s}, z, \tilde{z}) = \sqrt{|1-\tilde{s}|^2 + |z-\tilde{z}|^2}$ as radial variable, we transform $(\tilde{s}, \tilde{z})$ around $(1, z)$ into polar coordinates. We again obtain a factor of r^{m-1} and after integrating the angular coordinates, we arrive at

$$\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq \|v\|_\infty C\iint \eta^{-m-1}r^{m+1}Gd\eta dr.$$

Substituting $\sigma = \eta/r$, so that $d\eta = rd\sigma$ yields another factor of r and we have

$$\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq \|v\|_\infty C\iint \sigma^{-m+1}rGd\sigma dr.$$

As before, in the coordinates near the diagonal, $\sigma^{-1} = \sqrt{|S|^2 + |Z|^2}$, so that G vanishes to infinite order as $\sigma \to 0$ and we obtain

$$\|Vx^{-2-\gamma-\alpha}H[u]\|_\infty \leq \|v\|_\infty C\int rdr \leq \|v\|_\infty C.$$

4 Schauder Estimates near infinity

We now establish mapping properties for the heat kernel H near infinity, which is associated to the operator

$$\begin{aligned}&\partial_t - y^4\partial_y^2 + (m+1)y^3\partial_y - y^2\left(\Delta_N - 3y^{-2}\|A(0)\|^2\right)\\ &=: \partial_t - \Delta'\end{aligned}$$

obtained from the linearization in Proposition 2.10, by rearranging the terms and writing out $\Delta_0 = y^4\partial_y^2 - (m+1)y^3\partial_y + y^2\Delta_N$ in local coordinates. As before we have that $y^{-2}\|A(0)\| = \mathcal{O}(1)$. We do not need to impose any restriction on the minimum of $y^{-2}\|A(0)\|^2$ and have by virtue of Theorem 1.5 that

$$H \sim \rho_{\mathrm{td}}^{-m}\rho_{\mathrm{ff}}^{0}G,$$

with G vanishing to infinite order at the remaining boundary surfaces rf, lf, bb and tf (cf. Fig. 7).

We establish the following Theorem.

Theorem 4.1 (Mapping properties of the Heat kernel near infinity)**.** Let M be an asymptotically conical manifold. The heat kernel associated to

$$\partial_t - \Delta',$$

where Δ' is the Beltrami-Laplace operator, with a possible shift of the tangential operator, admits the following mapping property:

$$H : C^{\alpha}_{3+\gamma}(M\times[0,T]) \to C^{2+\alpha}_{\gamma}(M\times[0,T])$$

is bounded for $\gamma \geq 0$, $0<\alpha<1$.

The proof goes along the same lines as the one of Theorem 3.2. In particular we will abuse the notation as indicated in Remark 3.3, now considering powers of y instead of x.

Remark 4.2. These mapping properties do not seem to be optimal. We expect H to already map $C^{\alpha}_{e,2+\gamma}$ onto $C^{2+\alpha}_{e,\gamma}$, as this would match the mapping properties of $\Delta \sim y^2\mathcal{V}_{\mathrm{b}}^2$. Indeed, with the exception of the spatial estimates we could prove this. There the additional singular behaviour comes from differentiation by ∂_σ and ∂_ξ and it might be possible to adapt the integration trick outlined in Remark 3.5, but we were unable to do so, as we again lack a result about the stochastic completeness of the heat kernel.

Note that, since the heat kernel vanishes to infinite order as we approach $\mathrm{lf}, \mathrm{rf}, \mathrm{tf}$ and bb, estimates near the outer side faces are trivial and we only perform them exemplarily near the outer right corner in the spatial estimates.

Consequently only the estimates near the front face ff remain and reduce to two cases: The first one is near the intersection with bb. As the coordinates near the front face are valid across the whole front face, this is only one estimate and we do not need to distinguish between the right and left intersection. The second one is that near the intersection of td with ff and follows quite similarly to the estimates near the tip in this situation.

Finally the volume form of an exact asymptotically conical manifold is given by

$$\mathrm{dvol}(\tilde{p}) = |\det(g)| d\tilde{x} d\tilde{z} \sim \tilde{y}^{-m-1} d\tilde{x} d\tilde{z},$$

and we note that by Lemma 2.4 the same holds true for perturbations and non-exact asymptotically conical manifolds.

4.1 Estimates on the spatial Hölder differences near ∞

Let $p = (y, z), p' = (y', z') \in M$. We again wish to establish that

$$|Vy^{-\gamma} H[u](p, t) - Vy'^{-\gamma} H[u](p', t)| \leq C \|u\|_{\gamma,\alpha} d_M(p, p')^{\alpha},$$

for $V \in \{\mathrm{id}, \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2\}$. Again we use the heat equation to see that in this case $\partial_t \sim y^2 \mathcal{V}_{\mathrm{b}}^2$. As in the case near the cone tip, we make use of the localization argument provided in Section 2.4.2 and assume that heat kernel is supported in coordinate patches such that for the distance function

$$d(p, p') \sim \sqrt{\left(\frac{1}{y} + \frac{1}{y'}\right)^4 |y - y'|^2 + \left(\frac{1}{y} + \frac{1}{y'}\right)^2 |z - z'|^2}$$

holds. Then it suffices to show that in these coordinates

$$\begin{aligned} &|Vy^{-\gamma} H[u](p, t) - Vy'^{-\gamma} H[u](p', t)| \\ &\leq \left(\frac{1}{y} + \frac{1}{y'}\right)^{2\alpha} |y - y'|^{\alpha} \|u\|_{\alpha,\gamma} C + \left(\frac{1}{y} + \frac{1}{y'}\right)^{\alpha} |z - z'|^{\alpha} \|u\|_{\alpha,\gamma} C \end{aligned}$$

for $V \in \{\mathrm{id}, \mathcal{V}_{\mathrm{b}}, \mathcal{V}_{\mathrm{b}}^2\}$, $u \in C^{\alpha}_{e,2\gamma+3}(M \times [0, T])$ and some uniform constant C. As in Section 3.1 we first abbreviate $\tilde{G}(p, t) = (Vy^{-\gamma} H)(p, t)$ and then invoke Lemma 3.4 to obtain

$$\begin{aligned} |\tilde{G}(p, t)[u] - \tilde{G}(p', t)[u]| &\leq |y - y'|^{\alpha} Y^{1-\alpha} C \left| \partial_Y \tilde{G}(Y, z, t)[u] \right| \\ &\quad + |z - z'|^{\alpha} Z^{1-\alpha} C \left| \partial_Z \tilde{G}(y', Z, t)[u] \right|, \\ &=: I_1 + I_2, \end{aligned}$$

with $Y \in \mathbb{R}^+$ lying between y and y' and $Z \in \mathbb{R}^{m-1}$ lying on the straight path connecting z and z'. Again we rename y' to y in the estimates of the second term. We will now continue estimating these two terms in the various regimes of the heat kernel.

Note that the leading factor $\frac{1}{y}+\frac{1}{y'}$ is automatic, as its inverse is uniformly bounded by the compact support of the heat kernel. We can simply multiply by $1 = \left(y^{-1} + y'^{-1}\right)\left(y^{-1} + y'^{-1}\right)^{-1}$ and absorb the inverse in the uniform constant C.

In the following, we again distinguish the heat kernel by H, make use of the abbreviation $Vy^{-\gamma}H = \tilde{G}$ as above and denote by G any bounded distribution vanishing to infinite order at the faces $\mathrm{rf}, \mathrm{lf}, \mathrm{bb}$ and tf.

4.1.1 Estimates near the outer right corner

We only demonstrate the estimate of I_1 to clarify why we chose to omit the estimates near the outer corners for the remainder of this section. Near the outer right corner we have the coordinates

$$\rho_{\mathrm{tf}} = \tau = \sqrt{t},\ \rho_{\mathrm{rf}} = s = \frac{y}{\tilde{y}},\ \rho_{\mathrm{bb}} = \tilde{y},\ z,\ \tilde{z}.$$

As usual we calculate the pullbacks of the volume form and derivatives and obtain

$$\begin{aligned}
\beta^* d\tilde{t} &\sim \tau d\tau \\
\beta^* d\tilde{y} &= d\tilde{y} \\
\beta^* d\tilde{z} &= d\tilde{z},
\end{aligned}$$

as well as $\beta^*\partial_y = \frac{1}{\tilde{y}}\partial_s$, so we see that application of b-vector fields does not change the asymptotic behaviour. Consequently I_1 reads as

$$\begin{aligned}
I_1 &\le |y-y'|^\alpha Y^{1-\alpha} C \iint \partial_Y \tilde{G}(p,\tilde{p},t-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&\le |y-y'|^\alpha Y^{1-\alpha} C \iint \tilde{y}^{-\gamma-1} G y^{\gamma+3+\alpha} v \tilde{y}^{-m-1} \tau d\tau d\tilde{y} d\tilde{z} \\
&\le |y-y'|^\alpha Y^{1-\alpha} C \|v\|_\infty \iint \tilde{y}^{-m-1} G d\tau d\tilde{y} d\tilde{z}.
\end{aligned}$$

The seemingly singular behaviour in $\tilde{y}$ is now offset by the fact that G vanishes to infinite order as we approach the face bb, since $\tilde{y} = \rho_{\mathrm{bb}}$ and $\int \rho_{\mathrm{bb}}^{-N} G$ is bounded for any N. Consequently

$$I_1 \le |y-y'|^\alpha C \|v\|_\infty.$$

The other estimates follow along the same lines, as the heat kernel vanishes to infinite order at $\mathrm{td}, \mathrm{bb}, \mathrm{lf}$ and rf in these regimes. In the following we now only need to perform the estimates near the front face, both away from and near the temporal diagonal.

4.1.2 Estimates near the front face away from the diagonal

Near the front face, but away from the diagonal, we can use the coordinates

$$\rho_{\mathrm{td}} = \tau = \sqrt{t-\tilde{t}},\ \sigma = \frac{s-1}{y} = \frac{y-\tilde{y}}{y^2},\ \xi = \frac{z-\tilde{z}}{y},\ \rho_{\mathrm{ff}} = y,\ z.$$

We begin by calculating the pullbacks of volume form and derivatives in these coordinates and obtain for the former

$$\begin{aligned}\beta^* d\tilde{t} &\sim \tau d\tau \\ \beta^* d\tilde{y} &= \partial_\sigma (y - \sigma y^2) d\sigma \sim y^2 d\sigma \\ \beta^* d\tilde{z} &= \partial_\xi (z - \xi y) d\xi \sim y^{m-1} d\xi,\end{aligned}$$

where in the last line we again note that we suppress the $m-1$ dimensions of the cross section in notation.

The derivatives transform a bit unintuitively, as we observe for ∂_y

$$\beta^* \partial_y = \partial_y y \partial_y + \partial_y \sigma \partial_\sigma + \partial_y \xi \partial_\xi \sim \partial_y + \partial_y \sigma \partial_\sigma + \frac{1}{y} \xi \partial_\xi.$$

Now the $\partial_y \sigma$ term is calculated as

$$\partial_y \sigma = \partial_y \left(\frac{\tilde{y} - y}{y^2} \right) = -2 \frac{\tilde{y}}{y^3} + \frac{1}{y^2} = -2 \frac{1}{y} \sigma - \frac{1}{y^2}.$$

As the heat kernel vanishes to infinite order as $|(\sigma, \xi)| \to \infty$, the corresponding derivatives can be ignored for the estimate and only the singular y^2-factor remains:

$$\beta^* \partial_y \sim \partial_y + \frac{1}{y} \sigma \partial_\sigma + \frac{1}{y^2} \partial_\sigma + \frac{1}{y} \xi \partial_\xi \sim \frac{1}{y^2}$$

Similarly we have

$$\beta^* \partial_z = \partial_z \xi \partial_\xi \sim \frac{1}{y} \partial_\xi \sim \frac{1}{y}.$$

We conclude that each application of the b-vector fields decreases the power of the front face boundary defining function by one, i.e. $\beta^* \mathcal{V}_{\mathrm{b}} \sim \frac{1}{y}$. Moreover we note that in these coordinates, $\tilde{y} = y(1 - \sigma y)$. We drop the factor $(1 - \sigma y)$ from the following estimates, since it can be bounded uniformly.

Now I_1 follows as

$$\begin{aligned} I_1 &= |y - y'|^\alpha Y^{1-\alpha} C \iint \partial_Y \tilde{G}(p, \tilde{p}, t - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\ &\leq |y - y'|^\alpha Y^{1-\alpha} C \iint \frac{1}{Y^2} (Y^{-\gamma-2} G) Y^{3+\gamma+\alpha} v Y^{-m-1} Y^{m+1} \tau d\tau d\sigma d\xi \\ &\leq |y - y'|^\alpha Y^{1-\alpha} C \|v\|_\infty \iint Y^{-1+\alpha} \tau G d\tau d\sigma d\xi \\ &= |y - y'|^\alpha C \|v\|_\infty \iint \tau G d\tau d\sigma d\xi \\ &\leq |y - y'|^\alpha C \|v\|_\infty. \end{aligned}$$

In contrast to the estimates near the tip, now also the ∂_z-differentiation coming from the mean value theorem decreases front face behaviour, which can not be relieved by the additional Z-factor. Thus we record

$$\begin{aligned}
I_2 &= |z-z'|^\alpha Z^{1-\alpha} C \iint \partial_Z \tilde{G}(p,\tilde{p},t-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&\leq |z-z'|^\alpha Z^{1-\alpha} C \iint Y^{-\gamma-2} G Y^{3+\gamma+\alpha} v Y^{-m-1} Y^{m+1} \tau d\tau d\sigma d\xi \\
&\leq |z-z'|^\alpha C \|v\|_\infty \iint Y^{1+\alpha} \tau G d\tau d\sigma d\xi \\
&\leq |z-z'|^\alpha C \|v\|_\infty .
\end{aligned}$$

As discussed beforehand, the leading $\left(\frac{1}{y}+\frac{1}{y}\right)$-factors are automatic.

4.1.3 Estimates near the intersection of front face and diagonal

It will once again be necessary to employ an integration trick, as near the diagonal near the tip in Section 3.2.4. We forego demonstrating the necessity of this and immediately continue with the previous set of coordinates, which are still valid (but do not capture the asymptotics of the heat kernel completely), where τ is the boundary defining function of td.

The only difference to the previous estimates now is the additional factor of τ^{-m}. Repeating these estimates ad verbatim, we are left with

$$\begin{aligned}
I_1 &\leq |y-y'|^\alpha C \|v\|_\infty \iint \tau^{-m+1} G d\tau d\sigma d\xi \\
I_2 &\leq |z-z'|^\alpha C \|v\|_\infty \iint y^\alpha \tau^{-m+1} G d\tau d\sigma d\xi .
\end{aligned}$$

Again we search for a suitable radial function. We have that

$$\frac{1}{y}+\frac{1}{\tilde{y}} = \frac{1}{y} + \frac{1}{y(1-\sigma y)} = \frac{1}{y}\left(1+\frac{1}{1-\sigma y}\right).$$

As we are near the diagonal $\{s=0, z=\tilde{z}\}$ we may in particularly assume that $|s|$ is uniformly bounded away from 1. In particular we can bound the factor $(1-\sigma y)^{-1}$ uniformly and obtain

$$\frac{1}{y}+\frac{1}{\tilde{y}} \sim \frac{1}{y}.$$

We observe that in these coordinates we have the following uniform equivalence of the distance function

$$
\begin{aligned}
d^2 &\sim \left(\frac{1}{y}+\frac{1}{\tilde{y}}\right)^2 \left(\left(\frac{1}{y}+\frac{1}{\tilde{y}}\right)^2 |y-\tilde{y}|^2 + |z-\tilde{z}|^2\right) \\
&\sim \left(\frac{1}{y}\right)^2 \left(\left(\frac{1}{y}\right)^2 |y-\tilde{y}|^2 + |z-\tilde{z}|^2\right) \\
&= \left(\frac{1}{y}\right)^2 \left(\left(\frac{1}{y}\right)^2 |\sigma y^2|^2 + |\xi y|^2\right) \\
&= |\sigma|^2 + |\xi|^2.
\end{aligned}
$$

Letting $r = \sqrt{|\sigma|^2 + |\xi|^2}$ we also have $r/\tau = \sqrt{|S|^2 + |Z|^2}$, so r seems to be a good function for the integrating factor trick. We use it as a radial variable to transform (σ, ξ) around $(0,0)$ into polar coordinates. After integrating the angular coordinates, we have for I_1

$$
I_1 \leq |y-y'|^\alpha C \|v\|_\infty \iint \tau^{-m+1} r^{m-1} G d\tau dr.
$$

Substituting $\eta = r/\tau$ and consequently $dr = \tau d\eta$ we have

$$
I_1 \leq |y-y'|^\alpha C \|v\|_\infty \iint \tau \eta^{m-1} G d\tau d\eta \leq |y-y'|^\alpha C \|v\|_\infty,
$$

as G vanishes to infinite order as $\eta \to \infty$. The estimate of I_2 now follows ad verbatim. As before the leading factors of the distance function follow automatically.

4.2 Estimates on the temporal Hölder differences near ∞

Near infinity the estimates on the temporal Hölder differences parallel those near the tip.

For fixed p and $0 < t < t'$ we establish

$$
|Vy^{-\gamma} H[u](p,t') - Vy^{-\gamma} H[u](p,t)| \leq \|u\|_{\gamma,\alpha} C |t'-t|^{\alpha/2},
$$

for $u \in C^\alpha_{e,3+\gamma}(M \times [0,T])$ and $V \in \{\mathrm{id}, \mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\}$.

Using this notation we can set up the argument exactly as in Section 3.2. Near the front face and away from the temporal diagonal, we can proceed with the naive terms T_1 and T_2, while we have to mimic the rather elaborate argument near the diagonal.

4.2.1 Estimates near the front face away from the diagonal

Recall that for T_1 we had after application of the mean value theorem

$$
T_1 \leq |t'-t| C \iint_0^t \partial_\theta \tilde{G}(p, \tilde{p}, \theta - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t} \,\mathrm{dvol}(\tilde{p}),
$$

for some $\theta \in [t, t']$. The lift of the time derivative is given by $\beta^*\partial_\theta \sim \frac{1}{\tau}\partial_\tau$. Consequently T_1 is in these coordinates easily estimated as

$$\begin{aligned}
T_1 &\leq |t' - t|C \iint \frac{1}{\tau}\partial_\tau(y^{-\gamma-2}G)y^{\gamma+3+\alpha}vy^{-m-1}y^{m+1}\tau d\tau d\sigma d\xi \\
&\leq |t' - t|C\|v\|_\infty \iint y^\alpha G d\tau d\sigma d\xi \\
&= |t' - t|C\|v\|_\infty.
\end{aligned}$$

The estimates of T_2 follow along the same lines:

$$\begin{aligned}
T_2 &= \iint_t^{t'} \tilde{G}(p, \tilde{p}, t' - \tilde{t})u(\tilde{p}, \tilde{t})d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \\
&= \iint_t^{t'} (y^{-\gamma-2}G)y^{\gamma+3+\alpha}vy^{-m-1}y^{m+1}\tau d\tau d\sigma d\xi \\
&\leq C\|v\|_\infty \int_t^{t'} y^{1+\alpha}\tau G d\tau \\
&\leq |t' - t|C\|v\|_\infty.
\end{aligned}$$

4.2.2 Estimates near the intersection of front face and diagonal

Near the diagonal we will once again need to employ the integrating factor trick, but only when dealing with L_3''. The estimates of K_1, K_2, L_1, L_2 and L_3' can be done the naive way, by employing the coordinates

$$\rho_{\mathrm{td}} = \tau = \sqrt{t - \tilde{t}},\ S = \frac{\sigma}{\tau} = \frac{s-1}{y\tau},\ Z = \frac{\xi}{\tau} = \frac{\tilde{z} - z}{y\tau},\ \rho_{\mathrm{ff}} = y,\ z.$$

Again we calculate volume form and derivatives in these coordinates:

$$\begin{aligned}
&\beta^* d\tilde{t} \sim \tau d\tau \\
&\beta^* d\tilde{y} \sim \partial_S(y - S\tau y^2)dS = \tau y^2 dS \\
&\beta^* d\tilde{z} \sim \partial_Z(z - Z\tau y)dZ = \tau^{m-1}y^{m-1}dZ \text{ as } \dim N = m - 1
\end{aligned}$$

The derivatives are calculated as in Section 4.1.3, now incorporating an additional τ^{-1}-factor, so that we have

$$\begin{aligned}
\beta^*\partial_y &\sim \partial_y + \frac{1}{y}S\partial_S + \frac{1}{y^2\tau} + \frac{1}{y}\xi\partial_\xi \sim \frac{1}{y^2\tau} \\
\beta^*\partial_z &\sim \partial_z + \frac{1}{y\tau}\partial_Z,
\end{aligned}$$

and accordingly, as in the situation near the tip, each b-derivative now reduces both the ρ_{ff}- and ρ_{td}-exponents by 1.

If $2t \le t'$ we estimate K_1 and K_2; in this situation we have up to uniform constants $|t'-t| \le t \le t'$ and expand, as in Section 3.2 by $t'/t' = 1$ and $t/t = 1$ respectively.

$$\begin{aligned}
K_1 &\le |t'-t|^{\alpha/2} \iint_0^{t'} (t'-\tilde{t})^{-\alpha/2} \tilde{G}(p,\tilde{p},t'-\tilde{t}) u(\tilde{p},\tilde{t}) d\tilde{t} \, \mathrm{dvol}(\tilde{p}). \\
&= |t'-t|^{\alpha/2} \iint \tau^{-m-\alpha/2} y^{-\gamma-2} G y^{\gamma+3+\alpha} v y^{-m-1} y^{m+1} \tau^m d\tau dS dZ \\
&\le |t'-t|^{\alpha/2} C \|v\|_\infty \iint \tau^{-\alpha/2} y^{1+\alpha} d\tau dS dZ \\
&\le |t'-t|^{1+\alpha/2} C \|v\|_\infty.
\end{aligned}$$

As demonstrated near the tip, now K_2, L_1 and L_2 follow almost exactly the same. We now turn to L_3', which can be still estimated in these coordinates:

$$\begin{aligned}
L_3' &= |t'-t| C \iint_{T_-} \partial_\theta \tilde{G}(p,\tilde{p},\theta-\tilde{t})(u(\tilde{p},\tilde{t}) - u(\tilde{p},\theta)) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \\
&\le |t'-t| C \iint \frac{1}{\tau} \partial_\tau (\tau^{-m} y^{-\gamma-2} G) y^{\gamma+3} \tau^\alpha \|v\|_\alpha y^{-m-1} y^{m+1} \tau^m d\tau dS dZ \\
&\le |t'-t| C \|v\|_\alpha \iint \tau^{-2+\alpha} y G d\tau dS dZ
\end{aligned}$$

In T_- we have $\theta - \tilde{t} \le \theta - 2t + t' \le \theta - t' + t' - t \le t' - t$ and subsequently can trade $|t'-t|^{1-\alpha/2}$ for $\tau^{2-\alpha}$, to arrive at

$$L_3' \le |t'-t| C \|v\|_\alpha.$$

Finally we turn to the estimation of L_3'', for which we employ the same integrating factor trick as in Section 4.2.1:

$$\begin{aligned}
L_3'' &= |t'-t| C \int \tilde{G}(p,\tilde{p},\theta-\tilde{t}) u(\tilde{p},\theta) \, \mathrm{dvol}(\tilde{p}) \Big|_{\tilde{t}=0}^{\tilde{t}=2t-t'} \\
&\le |t'-t| C \int \tau^{-m} y^{-\gamma-2} G y^{\gamma+3+\alpha} \|v\|_\infty y^{-m-1} y^{m+1} d\sigma d\xi \Big|_{\tilde{t}=0}^{\tilde{t}=2t-t'} \\
&= |t'-t| C \|v\|_\infty \int \tau^{-m} y^{1+\alpha} G d\sigma d\xi \Big|_{\tilde{t}=0}^{\tilde{t}=2t-t'}
\end{aligned}$$

Again we transform (σ, ξ) around $(0,0)$ into polar coordinates with radial variable $r = \sqrt{|\sigma|^2 + |\xi|^2}$, followed by integration in the angular coordinates and substitution $\eta = r/\tau$:

$$\begin{aligned}
L_3'' &\le |t'-t| C \|v\|_\infty \int \tau^{-m} y G r^{m-1} dr \Big|_{\tilde{t}=0}^{\tilde{t}=2t-t'} \\
&\le |t'-t| C \|v\|_\infty \int \eta^m y G d\eta \Big|_{\tilde{t}=0}^{\tilde{t}=2t-t'}
\end{aligned}$$

Again $\eta = \sqrt{|S|^2 + |Z|^2}$ vanishes to infinite order at tf and we arrive at

$$\begin{aligned} L_3'' &\leq |t' - t| \; C \|v\|_\infty |_{\tilde{t}=0}^{\tilde{t}=2t-t'} \\ &\leq |t' - t| C \|v\|_\infty . \end{aligned}$$

4.3 Estimates on the supremum near ∞

Finally we establish for $u \in C^\alpha_{e,3+\gamma}(M \times I)$ that $V y^{-\gamma-\alpha} H[u] \in C^0_e(M \times I)$, where $V \in \{\mathrm{id}, \mathcal{V}_\mathrm{b}, \mathcal{V}_\mathrm{b}^2\}$. As near the tip these estimates are much easier and do not use any new ideas, relying only on the fact that u itself has a good y-weight, i.e. $u(\tilde{y}, \tilde{z}, \tilde{t}) = \tilde{y}^{3+\gamma+\alpha} v(\tilde{y}, \tilde{z}, \tilde{t})$, where $v \in C^0_e(M \times I)$.

We only demonstrate the estimate near the intersection of front face and temporal diagonal, as the estimates away from the temporal diagonal then follow exactly the same by replacing the coordinates and noticing that the heat kernel vanishes to infinite order at tf.

$$\begin{aligned} &\|V y^{-\gamma-\alpha} H[u]\|_\infty = \sup \iint \tilde{G}(p, \tilde{p}, t - \tilde{t}) u(\tilde{p}, \tilde{t}) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \\ &\leq \sup \iint \tau^{-m} y^{-\gamma-2-\alpha} G y^{\gamma+3+\alpha} v y^{-m-1} y^{m+1} \tau^m d\tau dS dZ \\ &= \sup \iint y G v d\tau dS dZ \\ &\leq \|v\|_\infty C. \end{aligned}$$

5 Further mapping properties of the Heat kernel

Having done all the estimates near the cone tip and near infinity, we now collect some additional mapping properties. We will not again go through all the integrals, but only note the necessary changes that have to be made for the proofs to work.

5.1 Time scaled spaces

For $\varepsilon > 0$ such that $\mu - \gamma - 2 - \varepsilon \geq 0$, we obtain the additional mapping property:

$$H : C^{1+\alpha}_{e,\gamma+\varepsilon}(M \times I)_x \to \sqrt{t}^{\varepsilon} C^{2+\alpha}_{e,\gamma+2}(M \times I)_x$$

The proof amounts to showing that for $u \in C^{1+\alpha}_{e,\gamma+\varepsilon}(M \times I)$, we have that $t^{-\varepsilon/2} H[u] \in C^{2+\alpha}_{e,\gamma+2}(M \times I)$. Consequently we have to deal with $t^{-\varepsilon/2}$ as additional factor in the estimates of Section 3.1, Section 3.2 and Section 3.3. Spatial and supremum estimates follow quickly by observing how the singular factor acts in the various regimes.

Near the lower left and lower right, we observe that

$$t^{-\varepsilon/2} \leq (t - \tilde{t})^{-\varepsilon/2} = (\tau x^2)^{-\varepsilon/2}$$

near the lower left, and $t^{-\varepsilon/2} \leq (\tau \tilde{x}^2)^{-\varepsilon/2}$ near the lower right. In these regimes the heat kernel vanishes to infinite order as $\tau \to 0$ and the singular ρ_{ff}-factor cancels out exactly with the additional x-weight in the starting space, so we can follow through with the estimates exactly as before.

Near the top corner we have, by the same reasoning, $t^{-\varepsilon/2} \leq \tau^{-\varepsilon}$. Recall that we integrated the singular τ-factors by observing, that due to the boundedness of the coordinates, the integration region for τ is actually $[xC, \infty) \cap \operatorname{supp} H$ for some constant C, converting singular τ-factors into singular x-factors. Consequently the additional negative τ-weight cancels with the better x-behaviour for u.

Near the diagonal we have $t^{-\varepsilon/2} \leq \eta^{-\varepsilon} x^{-\varepsilon}$. Our estimates where sharp with respect to the η-variable. This is why we start with higher regularity, so that we can use partial integration to shift one of the b-derivatives to the initial function u. Then, as we apply one less edge derivative to the heat kernel, one singular η-factor is freed up. The x-factor again cancels out exactly with the better x-behaviour of u.

The temporal estimates are slightly more difficult, because we use the mean value theorem on the temporal coordinates. First of all, we will now use the more involved estimates with K_1, K_2, L_1, L_2, L_3 also near the lower left corner. In the estimates of K_1, K_2, L_1 and L_2 the role of $t^{-\varepsilon/2}$ is as in the spatial case. In the estimates of L_3 (and L_3', L_3'' near the diagonal) we make use of the fact that

$$\theta - \tilde{t} \leq t' - \tilde{t} \leq t' - 2t + t' = 2(t' - t) \leq 2(t - 2t + t') \leq 2(t - \tilde{t}) \leq 2t.$$

Observing that

$$\begin{aligned}
&\iint_{T_-} t'^{-\varepsilon/2} G(p,\tilde{p},t'-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
&- \iint_{T_-} t^{-\varepsilon/2} G(p,\tilde{p},t-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) \\
\leq & t^{-\varepsilon/2} \iint_{T_-} G(p,\tilde{p},t'-\tilde{t}) - G(p,\tilde{p},t-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}),
\end{aligned}$$

we can then follow through with the estimates, as then $t^{-\varepsilon/2} \leq C(\theta - \tilde{t})$ for some uniform constant C.

Near infinity the analogous mapping property reads as

$$H : C^{1+\alpha}_{e,\gamma+3}(M \times I)_y \to \sqrt{t}^{\varepsilon} C^{2+\alpha}_{e,\gamma}(M \times I)_y,$$

where again $\varepsilon > 0$ is small. We do not need the additional y^{ε}-weight, since the boundary defining functions of tf do not involve y near infinity and the heat kernel again vanishes to infinite order as one approaches tf. Near the diagonal we once again apply one derivative less, freeing up the necessary leading factor for the td-boundary defining function.

We record the results of this section in the following theorem:

Theorem 5.1 (Time scaling)**.** Let $\gamma + 2 + \varepsilon \leq \mu$, $\gamma' \geq 0$, where μ is the minimal element of the right face index set of the heat kernel. Then the heat kernel admits the following mapping properties:

$$H : C^{1+\alpha}_{e,\gamma+\varepsilon} \oslash C^{1+\alpha}_{e,\gamma'+3}(M \times I) \to \sqrt{t}^{\varepsilon} C^{2+\alpha}_{e,\gamma+2} \oslash \sqrt{t}^{\varepsilon} C^{2+\alpha}_{e,\gamma'}(M \times I).$$

5.2 Allowing time derivatives

Similar to the previous Theorem 5.1 we can, by a slight modification of the arguments, obtain further mapping properties, which allow additional time derivatives. This will allow us in Theorem 7.5 to prove the preservation of mean convexity (mean curvature ≥ 0) along the flow. There we have to consider the time derivative of the mean curvature, which is up to a factor the time derivative of the solution. Let γ, γ' and μ as before and set

$$\begin{aligned}
\Lambda^{\alpha} &= \left\{ u \in C^{\alpha}_{e,\gamma} \oslash C^{\alpha}_{e,\gamma'+3}(M \times I) \mid t\partial_t \in C^{\alpha}_{e,\gamma} \oslash C^{\alpha}_{e,\gamma'+3}(M \times I) \right\} \\
\Lambda^{2+\alpha} &= \left\{ u \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \mid t\partial_t \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \right\}.
\end{aligned}$$

We claim that $H : \Lambda^{\alpha} \to \Lambda^{2+\alpha}$. For this we first observe that

$$t\partial_t \int_0^t \int_M H(p,\tilde{p},t-\tilde{t})u(\tilde{p},\tilde{t})d\tilde{t}\,\mathrm{dvol}(\tilde{p}) = t \int_M H(p,\tilde{p},t)u(\tilde{p},0)d\tilde{p},$$

as $H[u]$ is the solution to the inhomogeneous shifted heat equation with initial condition $H[u](p,0) = 0$.

Consequently our goal is to show that $t\int_M H(p,\tilde p,t)u(\tilde p,0)d\tilde p \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M\times I)$. Spatial estimates now follow in the various regimes as before, where, due to the absence of integration in time, lacking front face and temporal diagonal behaviour is (easily) cancelled with the leading t-factor. However, due to the lack of integration in time, we have to do the integrating factor trick near the diagonal as in the estimates of L_3''.

For the temporal estimates we have to modify our previous arguments. Again we will make the case distinction of $2t \leq t'$ and $2t > t'$, and use in the first case the integrals K_1 and K_2. In the second case, we write, by means of a 0-addition,

$$
\begin{aligned}
t'\int G(p,\tilde p,t')u(\tilde p,0)\,\mathrm{dvol}(\tilde p) - t\int G(p,\tilde p,t)u(\tilde p,0)\,\mathrm{dvol}(\tilde p))& \\
= t'\int (G(p,\tilde p,t') - G(p,\tilde p,t))u(\tilde p,0)\,\mathrm{dvol}(\tilde p)& \\
+ (t'-t)\int G(p,\tilde p,t)u(\tilde p,0)\,\mathrm{dvol}(\tilde p)& \\
=: P_1 + P_2.&
\end{aligned}
$$

By the mean value theorem we can estimate P_1 as

$$
P_1 \leq t'|t'-t|\int \partial_\theta G(p,\tilde p,\theta)u(\tilde p,0)\,\mathrm{dvol}(\tilde p),
$$

for some $\theta \in [t,t']$. For L_1, L_2 we can again estimate as before, cancelling the worse ff- and td-asymptotics with $t'^{1-\alpha/2}$ from the leading t' factor. For P_1 and P_2 we are in the situation of $2t > t'$ and can use $(t'-t)^{1-\alpha/2}$ to achieve the cancellation, where we note, that due to $t' < 2t \leq 2\theta$ the asymptotics of the leading factor t' cancel with those of $\beta^*\partial_\theta$. Near the diagonal we again employ the integrating factor trick as demonstrated in the estimates of L_3''.

Estimates near infinity follow as before along the same lines. Furthermore, using the modifications from the previous sections in these estimates, we obtain the corresponding version of Theorem 5.1.

6 Short-time existence

We now prove short-time existence of the conical mean curvature flow via a classical fixpoint argument. Fix for now again exponents γ, γ' and $\alpha, \varepsilon > 0$ such that $\gamma + 2 + \alpha + \varepsilon \leq \mu$, where μ was the minimal element of the right face index set.

Recall that finding a solution to the mean curvature flow was equivalent to Eq. (MCF). After careful analysis of the linearization in Section 2, we have by Proposition 2.7 and Proposition 2.10 that this equation is given by

$$(\partial_t - \Delta + 3\|A_0\|)f = H_0 + x^{-1}\mathcal{O}_2(\mathcal{V}_{\mathrm{b}}^{\{0,1,2\}}x^{-1}f) =: H_0 + Q(f).$$

Note that by Theorem 3.2 and Theorem 4.1

$$\begin{aligned} F : C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T]) &\to C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T]) \\ f &\mapsto e^{-t\Delta'}(H_0 + Q(f)), \end{aligned}$$

is a bounded mapping. We wrote the heat kernel for the operator

$$\Delta' = \Delta + 3\|A(0)\|^2$$

in its exponential form to not confuse it with the mean curvature. In this section we will prove that this mapping is actually a contraction. To see that F is well-defined, observe that due to the involved x- and y-weights, as H_0 and $Q(f)$ lie in $C^{\alpha}_{e,\gamma} \oslash C^{\alpha}_{e,\gamma'+3}(M \times I)$, where we also used the general consequence of the Leibniz rule that products of Hölder functions are again Hölder.

Let

$$Z_{B,T} := \left\{ u \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T]) \;\middle|\; \begin{array}{r} \|u\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)} < B \\ u(p,0) = 0 \end{array} \right\},$$

where the bound B and time T are to be determined during the following argument.

Observe that for $u, v \in Z_{B,T}$ and by the linearity of integral operators

$$\begin{aligned} &\|F(u) - F(v)\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)} \\ &= \left\| \int_0^T \int_M e^{-t\Delta'}(H_0 + Q(u) - H_0 + Q(v)) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \right\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)} \\ &= \left\| \int_0^T \int_M e^{-t\Delta'}(Q(u) - Q(v)) d\tilde{t}\, \mathrm{dvol}(\tilde{p}) \right\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)}. \end{aligned}$$

Using any cutoff function ϕ as in the definition of $C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$, this difference can be estimated as

$$\begin{aligned}
&\|F(u) - F(v)\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)} \\
\leq &\left\| \int_0^T \int_{\mathrm{supp}(\phi)} e^{-t\Delta'}(Q(u) - Q(v)) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \right\|_{\gamma+2,2+\alpha} \\
+ &\left\| \int_0^T \int_{\mathrm{supp}(1-\phi)} e^{-t\Delta'}(Q(u) - Q(v)) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \right\|_{\gamma',2+\alpha} \\
&=: D_1 + D_2,
\end{aligned}$$

where it is understood that the Hölder norms are defined as in Section 2.4.1 on the support of ϕ and as in Section 2.4.2 away from it. Similarly, we have for the norm of $F(u)$:

$$\begin{aligned}
&\|F(u)\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)} \\
= &\left\| \int_0^T \int_M (H_0 + Q(u)) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \right\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)} \\
\leq &\left\| \int_0^T \int_{\mathrm{supp}(\phi)} e^{-t\Delta'}(H_0 + Q(u)) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \right\|_{\gamma+2,2+\alpha} \\
+ &\left\| \int_0^T \int_{\mathrm{supp}(1-\phi)} e^{-t\Delta'}(H_0 + Q(u)) d\tilde{t} \, \mathrm{dvol}(\tilde{p}) \right\|_{\gamma',2+\alpha} \\
&=: E_1 + E_2
\end{aligned}$$

To show that F is a contraction, we now determine B and T separately, by examining D_1 and E_1 near the tip and D_2 and E_2 near infinity. By the mapping properties all terms but the $e^{-t\Delta'}H_0$-terms are clearly well-defined.

Consequently we have to impose restrictions on the initial data. Indeed, to see that F maps $Z_{B,T}$ onto itself for suitable B, T, we will invoke Theorem 5.1 to estimate $\|e^{-t\Delta'}H_0\|_{(\gamma+2,2+\alpha),(\gamma',2+\alpha)}$. Hence we have to require $H_0 \in C^{1+\alpha}_{\gamma+\varepsilon} \oslash C^{1+\alpha}_{\gamma'+3}(M)$.

We can produce an initial manifold satisfying this condition by perturbing a minimal cone $\varphi_{\mathrm{ex}} : M \to \mathbb{R}^{n+1}$ by a function $f_{\mathrm{in}} \in C^{3+\alpha}_{\gamma+2+\varepsilon} \oslash C^{3+\alpha}_{\gamma'+1}(M)$. Plugging it into the linearization

$$H_0 = \Delta_{\mathrm{ex}} f_{\mathrm{in}} - 3\|A(\mathrm{ex})\|^2 f_{\mathrm{in}} + Q(f_{\mathrm{in}})$$

shows that then H_0 has the necessary regularity.

Hence we make the following assumption.

Assumption 6.1. We will assume that M is the perturbation of a minimal cone such that $H_0 \in C^{1+\alpha}_{\gamma+\varepsilon} \oslash C^{1+\alpha}_{\gamma'+3}(M)$.

We first carry out the argument near the tip and determine corresponding constants B_1 and T_1 there; then we make the necessary adjustments to the argument to carry it out near infinity and determine corresponding constants B_2 and T_2. Taking the respective minima we have then shown that F as a whole is a contraction. Thus it has a fixpoint, solving Eq. (MCF).

6.1 Contraction near the cone tip

6.1.1 Estimating D_1

To estimate D_1 we give a fairly general argument. Furthermore we notationally assume the case $\gamma = 0$; the general case follows immediately from replacing u by $x^\gamma u$. Observe that in this case we get even better x-behavior for $Q(u), Q(v)$, namely $x^{2\gamma+1}$. Recall that in establishing Proposition 2.7 we obtained the quadratic terms mostly as contractions with the inverse of the metric, which in turn was obtained via the von Neumann series. So we can think of Q as a series with Hölder coefficients of the form

$$Q(u)(p,t) = x^{-1} \sum_{(i,j,k)\in\mathcal{I}} a_{ijk}(p,t)(x^{-1}u)^i (\mathcal{V}_{\mathrm{b}} x^{-1}u)^j (\mathcal{V}_{\mathrm{b}}^2 x^{-1}u)^k,$$

where $\mathcal{I} = \big\{(i,j,k) \in \mathbb{N}^3 \mid i+j+k \geq 2\big\}$. We now iteratively separate out the terms which have at least one occurence of u, then those, who do not have any occurence of u, but at least one of $\mathcal{V}_{\mathrm{b}} u$ and finally those who consist only of second derivatives. For this let

$$\begin{aligned}
\mathcal{I}_1 &= \{(i,j,k) \in \mathcal{I} \mid i \geq 1\} \\
\mathcal{I}_2 &= \{(i,j,k) \in \mathcal{I} \setminus \mathcal{I}_1 \mid j \geq 1\} \\
\mathcal{I}_3 &= \mathcal{I} \setminus (\mathcal{I}_1 \cup \mathcal{I}_2),
\end{aligned}$$

all while pretending notationally that there is only one direction in which we would differentiate in, lest it would be required to write down terms for the m first derivatives and $m(m-1)/2$ second derivatives. Although not necessary for the remaining argument, we note that the index set $\mathcal{I}_3$ is empty, as all second derivatives in the linearization occur accompanied by terms of lower order. To see this, one has to go back to the arguments in Section 2.2 and Section 2.3: Any second order term stems from the difference $\Delta_t - \Delta_0$. A closer examination of Lemma 2.5 and Lemma 2.9 respectively shows that these terms come from two different places, the first being terms of the form $\mathcal{V}_{\mathrm{b}}\mathcal{O}_2(u, \mathcal{V}_{\mathrm{b}} u)$, so that by the Leibniz rule always some lower order term remains in the product. The second place are the second derivatives of the Laplacian itself, more specifically the terms $G^{ij}\partial_i\partial_j$ applied to f. But as $G \sim \mathcal{O}(u)$, we see that these also are coupled with lower order terms.

We now define

$$Q_1(u) = x^{-1} \sum_{(i,j,k)\in\mathcal{I}_1} a_{ijk}(p,t)(x^{-1}u)^i(\mathcal{V}_{\mathrm{b}}x^{-1}u)^j(\mathcal{V}_{\mathrm{b}}^2x^{-1}u)^k$$
$$Q_2(u) = x^{-1} \sum_{(i,j,k)\in\mathcal{I}_2} a_{ijk}(p,t)(x^{-1}u)^i(\mathcal{V}_{\mathrm{b}}x^{-1}u)^j(\mathcal{V}_{\mathrm{b}}^2x^{-1}u)^k,$$

so that by the previous consideration $Q = Q_1 + Q_2$. Assuming $\mathrm{supp}(a_{ijk}) \subset \mathrm{supp}(\phi)$, we can now estimate D_1 via

$$\begin{aligned} D_1 &= \left\| e^{-t\Delta'}\left((Q_1+Q_2)(u) - (Q_1+Q_2)(v)\right)\right\|_{2,2+\alpha} \\ &\leq \left\| e^{-t\Delta'}(Q_1(u) - Q_1(v))\right\|_{2,2+\alpha} \\ &+ \left\| e^{-t\Delta'}(Q_2(u) - Q_2(v))\right\|_{2,2+\alpha} \\ &=: \|d_1\|_{2,2+\alpha} + \|d_2\|_{2,2+\alpha} \end{aligned}$$

As $e^{-t\Delta'}$ is a bounded operator, establishing uniform $C_2^{2+\alpha}$-bounds on the d_i reduces to finding C_0^α-bounds for the differences $Q_i(u) - Q_i(v)$.

We begin the estimation by discussing d_1. As $Q_1(u)$ consists of those terms having at least one factor of u, we can factor it out and write

$$Q_1(u) = x^{-2}uq_1(u).$$

We now write the difference as

$$\begin{aligned} x^2(Q_1(u) - Q_1(v)) &= uq_1(u) - vq_1(v) \\ &= uq_1(u) - uq_1(v) + uq_1(v) - vq_1(v) \\ &= u(q_1(u) - q_1(v)) + q_1(v)(u-v). \end{aligned}$$

By the definition of the Hölder norms, we can easily estimate

$$\|q_1(v)\|_\alpha \leq C\|v\|_{2,2+\alpha},$$

where C only depends on the algebraic structure and the initial data. To estimate the difference of $q_1(u)$ and $q_1(v)$ observe that we are actually dealing with a function

$$q_1(u, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2u),$$

again pretending in notation there is only one dimension to differentiate in. Akin to the setup in the spatial Hölder estimates, we intersperse now some terms, in order to apply the mean value theorem:

$$\begin{aligned} q_1(u, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2u) - q_1(v, \mathcal{V}_{\mathrm{b}}v, \mathcal{V}_{\mathrm{b}}^2v) &= q_1(u, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2u) - q_1(v, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2u) \\ &+ q_1(v, \mathcal{V}_{\mathrm{b}}u, \mathcal{V}_{\mathrm{b}}^2u) - q_1(v, \mathcal{V}_{\mathrm{b}}v, \mathcal{V}_{\mathrm{b}}^2u) \\ &+ q_1(v, \mathcal{V}_{\mathrm{b}}v, \mathcal{V}_{\mathrm{b}}^2u) - q_1(v, \mathcal{V}_{\mathrm{b}}v, \mathcal{V}_{\mathrm{b}}^2v) \\ &=: e_1 + e_2 + e_3 \end{aligned}$$

Now the difference e_1 is a sum of terms of the form

$$a_{ijk}x^{-i}\left(u^i - v^i\right)(\mathcal{V}_{\mathrm{b}}x^{-1}u)^j(\mathcal{V}_{\mathrm{b}}\mathcal{V}_{\mathrm{b}}x^{-1}u)^k,$$

where we we can estimate the terms involving differentials by

$$\|\mathcal{V}_{\mathrm{b}}^{\{1,2\}}x^{-1}u\|_\alpha \leq C\|u\|_{2+\alpha}.$$

For the difference $x^{-i}u^i - x^{-i}v^i$ we obtain, using the mean value theorem for the function $s \mapsto s^i$, the pointwise estimate

$$|u^i - v^i|(p,t) \leq i\xi^{i-1}|u(p,t) - v(p,t)|$$

for some ξ between $u(p,t)$ and $v(p,t)$. Hence

$$\begin{aligned}
&\|a_{ijk}x^{-i}(u^i - v^i)(\mathcal{V}_{\mathrm{b}}x^{-1}u)^j(\mathcal{V}_{\mathrm{b}}\mathcal{V}_{\mathrm{b}}x^{-1}u)^k\|_\alpha \\
&\leq C\|x^{-i}\xi^{i-1}|u-v|\|_\alpha\|u\|_{2,2+\alpha}^{j+k} \\
&\leq C\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\}^{i-1}\|u-v\|_{2,2+\alpha}\|u\|_{2,2+\alpha}^{j+k}.
\end{aligned}$$

Consequently we may estimate

$$\|e_1\|_\alpha \leq C\|u-v\|_{2,2+\alpha}\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\}.$$

Similarly we obtain

$$\begin{aligned}
\|e_2\|_\alpha &\leq C|\mathcal{V}_{\mathrm{b}}x^{-1}u - \mathcal{V}_{\mathrm{b}}x^{-1}v|\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\} \\
&= C\|u-v\|_{2,2+\alpha}\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\}
\end{aligned}$$

and

$$\begin{aligned}
\|e_3\|_\alpha &\leq C|\mathcal{V}_{\mathrm{b}}^2x^{-1}u - \mathcal{V}_{\mathrm{b}}^2x^{-1}v|\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\} \\
&= C\|u-v\|_{2,2+\alpha}\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\},
\end{aligned}$$

where the C are again uniform constants, which may differ for each of the estimates, only depending on the initial data and algebraic structure of the equations.

Finally we can estimate the difference

$$\begin{aligned}
\|Q_1(u) - Q_1(v)\|_\alpha &= \|x^{-2}u(q_1(u) - q_1(v)) + x^{-2}q_1(v)(u-v)\|_\alpha \\
&\leq \|x^{-2}u(q_1(u) - q_1(v))\|_\alpha + \|q_1(v)x^{-2}(u-v)\|_\alpha \\
&\leq C\|u\|_{2,2+\alpha}\|u-v\|_{2,2+\alpha}\max\left\{\|u\|_{2,2+\alpha}, \|v\|_{2,2+\alpha}\right\} \\
&+ C'\|v\|_{2,2+\alpha}\|u-v\|_{2,2+\alpha} \\
&\leq \|u-v\|_{2,2+\alpha}BC.
\end{aligned}$$

After possibly enlarging the constant C again, we obtain $\|d_1\| \leq \|u-v\|BC$.

Estimation of the terms $\|d_2\|_{2,2+\alpha}$ now follows along the same lines, with the only difference being that $\mathcal{V}_\mathrm{b}u$ is factored out instead of u – the rest of the argument follows ad verbatim.

Hence we arrive, for $u, v \in Z_{B,T}$ at the estimate

$$D_1 \leq C_1 \|u - v\|_{2,2+\alpha} B,$$

where C_1 is a uniform constant only depending on the algebra and initial data.

6.1.2 Estimating E_1

We have to find that F actually maps $Z_{B,T}$ onto itself, for suitable choices of B and T. For this observe that

$$\begin{aligned} E_1 = \|F(u)\|_{\gamma+2,2+\alpha} &= \|e^{-t\Delta'} H_0 + e^{-t\Delta'} Q(u)\|_{\gamma+2,2+\alpha} \\ &\leq \|e^{-t\Delta'} H_0\|_{\gamma+2,2+\alpha} + \|e^{-t\Delta'} Q(u)\|_{\gamma+2,2+\alpha}. \end{aligned}$$

As Q consists of terms at least quadratic in $x^{-1}u, \mathcal{V}_\mathrm{b}x^{-1}u$ and $\mathcal{V}_\mathrm{b}^2 x^{-1}u$, we can bound $\|Q(u)\|_{\gamma,\alpha} \leq C\|u\|^2_{\gamma+2,2+\alpha}$, where C only depends on the algebraic structure of the equation and the initial data. As the heat kernel acts as a bounded operator, we also have $\|e^{-t\Delta'} Q(u)\|_{\gamma+2,2+\alpha} \leq C\|u\|^2_{\gamma+2,2+\alpha}$ (as usual the value of C may differ from line to line). By setting $B \leq 1/2C^{-1}$ (and positing $B \leq 1$) we can arrange that for any $u \in Z_{B,T}$

$$\|e^{-t\Delta'} u\|_{\gamma+2,2+\alpha} \leq C\|u\|^2 \leq \frac{1}{4}B.$$

This bound clearly also persists if we decrease T.

As $\|H_0\|_{\gamma,\alpha}$ is part of the initial data, we have by virtue of Theorem 5.1 that $\|e^{-t\Delta'} H_0\|_{\gamma+2,2+\alpha} \leq C\sqrt{T}^{\varepsilon}$. As C depends only on the initial data, by possibly decreasing T, we can arrange that $E_1 \leq \frac{1}{2}B$ independent of the choice of $u \in Z_{B,T}$. We denote these B and T by B_1 and T_1.

Furthermore note that also the condition that $F(u)(p,0) = 0$ is fulfilled due to the time scaling property.

6.2 Contraction near infinity

Establishing bounds on E_2 and D_2 follows along the same lines as the corresponding bounds on E_1 and D_1. The estimation of E_2 follows exactly as the one of E_1 in Section 6.1.2, noting that now y takes the role of x^{-1}, so that again we find constants B and T such that for $u \in Z_{B,T}$

$$E_1 \leq B,$$

where B and T are determined by the initial data.

The estimates for D_2 follow mutatis mutandis. As before we split up the quadratic term into Q_1 and Q_2, where Q_1 consists of those summands having at least one linear term in yf and Q_2 consists of those summands having at least one term $\mathcal{V}_b yf$. Again we wish to estimate D_2 using the triangle inequality, i.e. we proceed with

$$D_1 \leq \|d_1\|_{0,2+\alpha} + \|d_2\|_{0,2+\alpha}.$$

Since we established that $e^{-t\Delta'} : C^\alpha_{e,3} \to C^{2+\alpha}_{e,0}$ is a bounded operator, this now reduces to establishing C^α_3-bounds for the differences $Q_i(u) - Q_i(v)$. This in turn works the same as before, just with different norms involved, always noting that we can estimate the norms of derivatives of a function by the norm of the function without derivatives, i.e. we have by definition $\|\mathcal{V}_b u\|_{\gamma,\alpha} \leq \|u\|_{\gamma+1,1+\alpha}$ and so on.

So we again obtain B_2, T_2 such that we can conclude for $u, v \in Z_{B_2,T_2}$

$$\begin{aligned} D_2 &\leq C_2\|u - v\|_{0,2+\alpha} B_2 \\ E_2 &\leq \frac{1}{2} B_2. \end{aligned}$$

6.3 The fixpoint argument

By taking $T = \min\{T_1, T_2\}$ and $B \leq \min\{B_1, B_2\}$, we can, by possibly decreasing B and T even further, arrange that

$$B(C_1 + C_2) < 1,$$

so that actually $F : C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \to C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I)$ is a contraction. By the Banach fixpoint theorem thus F has a fixpoint f, and we observe that now

$$(\partial_t - A)f = (\partial_t - A) \circ F(f) = (\partial_t - A)H \circ Q(f) = Q(f),$$

so indeed f solves (MCF).

6.4 Assumptions and the main theorem

It is now time to discuss the compatibility of Assumption 3.1 and Assumption 6.1. We begin with the latter assumption, namely that we start with a perturbation of a minimal cone, as this is necessary for $H_0 \in C^{1+\alpha}_{\gamma+\varepsilon} \oslash C^{1+\alpha}_{\gamma'+3}(M)$.

The simplest minimal cone is a hyperplane, which is the cone over an equator of the sphere and we will call these *trivial.* A class of non-trivial examples is given by the Clifford or Simon's cones in $\mathbb{R}^n$ for $n \geq 8$. Their minimality was shown by Bombieri, De Giorgi and Guisti in [BDGG69].

These are cones over Clifford tori with specific scaling, e.g. the Simon's cones are those over

$$\mathbb{S}^n(1/\sqrt{2}) \times \mathbb{S}^n(1/\sqrt{2}) \subset \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2}.$$

Simons has shown in his classic paper [Sim68] that the only complete minimal hypersurfaces of $\mathbb{R}^n$ for $n < 8$ are hyperplanes, which can be seen as cones over an equator.

As a further result he obtained that for any immersed codimension-1 minimal submanifold N of $\mathbb{S}^n$ one has that either $\|A_N\|^2 \equiv 0$, i.e. N is an equator, or $\|A_N\|^2 \geq \dim N$.

Recall that Assumption 3.1 was that the minimal element of side face index sets μ is greater than 2. Writing $S_N = \|A_N(0)\|^2$, we have that the minimal eigenvalue of the shifted Laplacian $-\Delta + 3S_N$ is actually given by $3S_N$. Consequently, as elaborated in Section 1.2.3, μ can be obtained as

$$\mu = \min E = \sqrt{3S_N + \left(\frac{n-1}{2}\right)^2} - \frac{n-1}{2},$$

where $n = m - 1$ is the dimension of the cross section. It is immediately clear that if N is an equator that the condition on μ can not be fulfilled. If N is not an equator, the aforementioned result of Simon asserts that then $S_N \geq n$ and consequently

$$\begin{aligned}\mu \geq \min E &= \sqrt{3n + \left(\frac{n-1}{2}\right)^2} - \frac{n-1}{2} \\ &= \frac{1}{2}\sqrt{n^2 + 10n + 1} - \frac{n-1}{2}.\end{aligned}$$

As this situation can only arise in ambient dimension 8 or higher, we have $n \geq 6$, so that

$$\begin{aligned}\mu &\geq \frac{1}{2}\sqrt{n^2 + 8n + 13} - \frac{n-1}{2} \\ &> \frac{1}{2}\sqrt{(n-3)^2} - \frac{n-1}{2} \\ &= \frac{n+3}{2} - \frac{n-1}{2} \\ &= \frac{4}{2} = 2.\end{aligned}$$

This discussion so far only concerns *exact* cones. As minimal cones are stationary under mean curvature flow, we have to consider small perturbations of exact cones, i.e. $\varphi_0 = \varphi_{\text{ex}} + xu\nu_{\text{ex}}$. Combining Lemma 2.6, stating that $\|A(0)\|^2 = \langle \Delta_0 \nu_0, \nu_0 \rangle$ and Lemma 2.5, one quickly sees that for sufficiently small u we obtain bounds such that for small $\varepsilon > 0$

$$x^{-2}\left(\|A_N(\text{ex}) - \varepsilon\right) \leq \|A(0)\|^2 \leq x^{-2}\left(\|A_N(\text{ex}) + \varepsilon\right),$$

with the same result holding near infinity, invoking Lemma 2.9 instead.

Theorem 6.2 (Short-time existence)**.** Let $\varphi_0 : M \to \mathbb{R}^{m+1}$ be a perturbation of a minimal cone with unit normal vector ν_0, satisfying that

(i) the side face index sets of the Heat kernel near the conical singularity are bounded from below by 2, in particular, that the cross section is not a sphere, and

(ii) there exist $\gamma, \gamma' \geq 0$ and $\varepsilon > 0$ such that its mean curvature lies in $C^{1+\alpha}_{\gamma+\varepsilon} \oslash C^{1+\alpha}_{\gamma'+3}(M)$.

Then there exists for a small time T a function $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$ with $f(p,0) \equiv 0$ for all $p \in M$, such that

$$\varphi(p,t) = \varphi_0(p) + f(p,t)\nu_0(p)$$

is a solution to the mean curvature flow.

7 Curvatures along the flow

In this section we briefly discuss the behaviour of curvatures along the flow. In the first part we show how one can read off regularity and asymptotic behaviour quite easily. In the second part we prove the classical result of preservation of mean convexity, although possibly for a shorter time than the existence time of the solution.

7.1 Evolution of curvatures

Let $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$ be a solution to the mean curvature flow, as obtained in Theorem 6.2. For the mean curvature we see that the regularity of H_t is dictated by that of $\partial_t f \sim \Delta_0 f \sim x^{-2}\mathcal{V}_b^2$ (as well as $\Delta_0 \sim y^2 \mathcal{V}_b^2$ near infinity, see also Theorem 7.5), so that

$$H_t \in C^{\alpha}_{e,\gamma} \oslash C^{\alpha}_{e,\gamma'+2}(M \times [0,T]).$$

Recall that in the extrinsic setting we can express the Riemann tensor, the Ricci tensor and the scalar curvature by means of the second fundamental form, i.e.

$$\begin{aligned}
\mathrm{Riem}_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk} \\
\mathrm{Ric}_{ij} &= H h_{ij} - h_{il} g^{lk} h_{kj} \\
\mathrm{scal} &= H^2 - \|A\|^2,
\end{aligned}$$

so that regularity of the curvature reduces to regularity of the second fundamental form.

Using the results from Section 2 we can express the second fundamental form $h_{ij}(t)$ in terms of the initial embedding and the solution f

$$\begin{aligned}
h_{ij}(t) &= \langle \partial_i \partial_j \varphi_t, \nu_t \rangle \\
&= \langle \partial_i \partial_j (\varphi_0 + f\nu_0), \nu_0 - g^{kl} \partial_k f \partial_l (\varphi_0 + f\nu_0) \rangle.
\end{aligned}$$

Performing the tedious computation, one sees that the regularity is governed by that of $\partial_i \partial_j f$, while the asymptotic behaviour is governed by that of the initial second fundamental form, i.e. $h_{ij}(0)$. Recall that in local coordinates, we always imply that $y = x^{-1}$, so that $\partial_x f$ still makes sense for the solution $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$. In particular we have

$$\begin{aligned}
\partial_x \partial_x f &\in C^{\alpha}_{e,\gamma} \oslash C^{\alpha}_{e,\gamma'-2}(M \times [0,T]) \\
\partial_x \partial_z f &\in C^{\alpha}_{e,\gamma+1} \oslash C^{\alpha}_{e,\gamma'-1}(M \times [0,T]) \\
\partial_z \partial_z f &\in C^{\alpha}_{e,\gamma+2} \oslash C^{\alpha}_{e,\gamma'}(M \times [0,T]).
\end{aligned}$$

Recalling the asymptotic behaviour of $h_{ij}(0)$ from Section 2.1.1 and Section 2.1.2, we obtain

$$\begin{aligned} h_{xx} &\in C^{\alpha}_{e,0} \oslash C^{\alpha}_{e,-3}(M \times [0,T]) \\ h_{xz} &\in C^{\alpha}_{e,1} \oslash C^{\alpha}_{e,-2}(M \times [0,T]) \\ h_{zz} &\in C^{\alpha}_{e,1} \oslash C^{\alpha}_{e,-1}(M \times [0,T]). \end{aligned}$$

In particular we note that the sectional curvatures have Hölder regularity with asymptotic behaviour as in the exact case, i.e.

$$\begin{aligned} K_{xz} &\in C^{\alpha}_{e,-1} \oslash C^{\alpha}_{e,2}(M \times [0,T]) \\ K_{zz'} &\in C^{\alpha}_{e,-2} \oslash C^{\alpha}_{e,2}(M \times [0,T]). \end{aligned}$$

7.2 Preservation of mean convexity

A priori we can apply classical maximum principle techniques only to compact manifolds with isolated conical singularities, as the arguments require us to find minima and maxima. However, we found a solution f in the weighted Hölder space $C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$, where $M = (0,\infty) \times N$. Recall from Section 2.4 that f extends to a continuous function $\bar{f}$ on $\overline{M} = [0,\infty] \times N$. As a continuous function over a compact space, $\bar{f}$ attains maxima and minima. By construction of the Hölder spaces, even for $(\gamma,\gamma') = (0,0)$, we have that $f \in x^{\alpha}C^0 \oslash y^{\alpha}C^0(M \times [0,T])$. Consequently, for both $x \to 0$ and $x \to \infty$ (or equivalently, $y \to 0$), $f \to 0$. Thus we have, writing $p = (x,z)$, $\bar{f}(p,t) = 0$ for all p with $x = 0$ or $x = \infty$. It follows that either $f \equiv 0$ or f attains a non-zero maximum or minimum in the interior of M.

This now allows us to adapt the classical maximum principle to our situation.

Lemma 7.1 (Classical maximum principle)**.** Assume that for a function $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$ we have $\partial_t f \leq \Delta f$, $f(p,0) \leq 0$. Then the global maximum occurs on the parabolic boundary

$$\{x = 0\} \times (0,T) \cup \{x = \infty\} \times (0,T) \cup \tilde{M} \times (t = 0).$$

Proof. Suppose towards a contradiction that the maximum occurs at a positive time $t > 0$. By the previous reasoning it has to occur in the interior of the space, i.e. away from $\{x = 0\} \cup \{x = \infty\}$, or $f \equiv 0$, in which case we are done.

The proof now is reduced to the classical case. Consider first the case of a strict inequality, i.e. that we have $\partial_t f - \Delta f < 0$. However, at a maximum point p_0 we also have $\partial_t f(p_0) = 0$ and $\Delta f(p_0) \leq 0$, so that

$$0 \leq \partial_t f(p_0) - \Delta f(p_0) < 0,$$

which is a contradiction. In the case that only $\partial_t f - \Delta f \leq 0$, set for $\varepsilon > 0$ $f_\varepsilon = f - t\varepsilon$, so that

$$\partial_t f_\varepsilon = \partial_t f - \varepsilon \leq \Delta f_\varepsilon - \varepsilon < 0.$$

By the previous argument any maximum of f_ε occurs at the parabolic boundary and by letting $\varepsilon \to 0$ we also obtain the desired result for f. □

Corollary 7.2. Suppose $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$ satisfies

$$\partial_t f < \Delta f, f(p,0) \leq 0.$$

Then $f \leq 0$ on $M \times [0,T]$.

Proof. This follows directly from the previous lemma: The maximum of f occurs at the parabolic boundary, but both on $\{x = 0\} \cup \{x = \infty\}$ we have $f \equiv 0$ and on $\{t = 0\}$ we have $f \leq 0$ by assumption. □

As an immediate consequence we have:

Corollary 7.3. Let $\gamma' > 0$. Assume $c \in C^\alpha_e(M \times [0,T])$ is a non-positive function. Suppose $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$ satisfies

$$\partial_t f < \Delta f + cf, f(p,0) \leq 0.$$

Then $f \leq 0$ on $M \times [0,T]$.

We now wish to apply the maximum principle to the well-known evolution equation

$$\partial_t H = \|A\|^2 H + \Delta H,$$

but so far our solution $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0,T])$ only allows for one time derivative, which is already used up for $H = \partial_t f \langle \nu, \nu_0 \rangle$. However, we can deduce from the additional mapping properties from Section 5.2 that solutions actually lie in

$$\Lambda^{2+\alpha} = \left\{ f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \mid \partial_t f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \right\}$$

for some $I = [0, T'] \subset [0,T]$ and hence allow us to consider a second time derivative:

Lemma 7.4. Let $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I)$ be a solution to the conical mean curvature flow as in Section 6. Then there exists T', possibly $T' < T$ such that $f \in \Lambda^{2+\alpha}(M \times [0,T'])$.

Proof. By the results from Section 5.2 we can, via the fixpoint argument, obtain a solution $\tilde{f} \in \Lambda^{2+\alpha}(M \times [0, T']$. Write $I = [0, \min T, T']$. We have

$$\Lambda^{2+\alpha}(M \times I) \subset C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I)$$

and the corresponding inclusion of small balls in these spaces. By possibly restricting f or $\tilde{f}$ to the shorter time interval, we find that both $f \in Z_\mu$ and $\tilde{f} \in Z_{\tilde{\mu}}$ are fixpoints of F as in Section 6.3. Consequently, for $\mu' = \max\{\mu, \tilde{\mu}\}$ we have that both $f, \tilde{f} \in Z_{\mu'}$ are fixpoints of F. Thus they have to coincide by the Banach fixpoint theorem. □

We are now in a position to exploit aforementioned evolution equation and obtain the following, classical result.

Theorem 7.5 (Preservation of mean convexity)**.** Let $\gamma' > 0$ and let $f \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times [0, T])$ be a solution to Eq. (MCF) such that the initial mean curvature satisfies $H_0 \geq 0$, then we have $H \geq 0$ for a short-time. More specifically, if $f \in \Lambda^{2+\alpha}(M \times [0, T])$, this solution satisfies $H \geq 0$ on the whole of $[0, T]$.

Proof. We have to make sure that we have sufficient regularity to write down the evolution equation

$$\partial_t H = \|A\|^2 H + \Delta H, \qquad (\partial_t H)$$

as then the result follows immediately from Corollary 7.3. For this let $I = [0, \min\{T, T'\}]$ as in the previous Lemma 7.4. Recall that

$$\Lambda^\alpha = \left\{u \in C^\alpha_{e,\gamma} \oslash C^\alpha_{e,\gamma'+3}(M \times I) \mid t\partial_t \in C^\alpha_{e,\gamma} \oslash C^\alpha_{e,\gamma'+3}(M \times I)\right\}$$
$$\Lambda^{2+\alpha} = \left\{u \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \mid t\partial_t \in C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I)\right\}.$$

Write $\varphi_t = \varphi_0 + f\nu_0$ and indicate by a t in the index all time-dependent quantities. For the mean curvature we have

$$H_t \nu_t = \partial_t \varphi_t = \partial_t f \nu_0,$$

and consequentially $H_t = \partial_t f \langle \nu_0, \nu_t \rangle$. So we know that

$$H_t \in \frac{1}{t} C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \cap \Lambda^\alpha(M \times I).$$

In particular we may write down $\partial_t H_t$ and observe

$$\partial_t H_t \in \frac{1}{t^{3/2}} C^{2+\alpha}_{e,\gamma+2} \oslash C^{2+\alpha}_{e,\gamma'}(M \times I) \cap \frac{1}{t} C^\alpha_{e,\gamma} \oslash C^\alpha_{e,\gamma'+3}(M \times I).$$

For the sake of completeness we also observe

$$\|A\|^2 H, \Delta H \in C^\alpha_{e,\gamma} \oslash C^\alpha_{e,\gamma'+2}(M \times I).$$

This allows us to write down Eq. $(\partial_t H)$ and invoke Corollary 7.3, leading to the conclusion. □

8 A small survey of related problems

We conclude this thesis with a small survey of related problems. First we discuss the case of the cross section being a sphere and how it relates to other results concerning the mean curvature flow, in particular the classical results by Ecker and Huisken. Second, we relate our results to the works of Bahuaud and Vertman concerning the Yamabe flow and Ricci flow in presence of conical singularities. Third we discuss some results on the porous medium equation. Last we want to discuss cusp singularities, as the techniques used in this paper might yield results for these, too.

8.1 Other results for mean curvature flow

We had to impose the seemingly major restriction on the initial data that it was a perturbation close to a minimal cone and moreover that the cross section can not be a sphere. The first assumption is somewhat inherent to our Ansatz, as, for the fixpoint argument, one has to be able to apply the heat kernel to the terms occurring in the linearization and it is almost tautological to say that in the mean curvature flow one expects the mean curvature to occur in the linearization. The second assumption however was only needed to ensure that the heat kernel lifts to a polyhomogeneous distribution whose side face index sets only contain elements larger than two.

However, cones over spheres can be seen as Lipschitz graphs with linear growth and as such are treated in the works by Ecker and Huisken, most prominently their Annals paper [EH89]. They assert the existence of smooth solutions for all times, which in particular implies that the conical singularity gets smoothed out instantaneously.

However, the study of solutions near non-trivial cones also bears interest in itself. Appleby studied in his thesis [App10] smooth hypersurfaces asymptotic to Simon's cone in $\mathbb{R}^8$ and obtained short- and long-time results for this class of hypersurfaces. Using that Simon's cone is a member of a whole family of (otherwise smooth) minimal hypersurfaces foliating $\mathbb{R}^8$ (cf. [BDGG69]) and adapting Ecker and Huisken's non-compact maximum principle (cf. [EH91]), he showed that under initial gradient bounds and a prescribed sign of curvature, solutions stay between these minimal hypersurfaces and evolve towards Simon's cone.

A recent approach to dealing with singularities was developed by Sáez and Schnürer in [SS14]. There one considers the graphical mean curvature flow of a graph over a bounded open domain, such that it approaches infinity at the boundary of this domain and possibly on interior points. Then the projection, or as it is called by Maurer in [Mau16], its shadow, can be seen as a weak formulation of the mean curvature flow of hypersurface one dimension lower and one can flow the graph through singularities of its shadow.

8.2 Conical Yamabe and Ricci flow

The study of geometric evolution equations in the conical setting, using the tools of microlocal analysis was initiated by Bahuaud and Vertman. In [BV14] they established short-time existence of the Yamabe flow and this thesis is modeled after this approach. Furthermore, in [BV16] they established long-term results. As they employed slightly more general Hölder spaces, this required the development of a maximum principle adapted to the conical situation. Functions in these spaces do not necessarily decay to 0 as one approaches the conical singularity. It asserts that even for a maximum point of a function f on the edge (i.e. on $\{x = 0\}$), one still has $\Delta f \leq 0$.

In a subsequent paper [Ver16] Vertman established short-time results for the Ricci-de Turck flow, and, under a loss of regularity, for the Ricci flow. A major difficulty in trying to adapt the arguments from [BV14] to our situation in the mean curvature flow was the lack of stochastic completeness of the heat kernel for the shifted Laplacian. The same problem occurs also with the Lichnerowicz-Laplacian, leading to the consideration of weighted Hölder spaces. Lately, jointly with Kröncke in [KV18], long-term results were established, based on long-term heat kernel estimates. For cones near Ricci-flat cones a more recent version of [Ver16] immediately applies these estimates to obtain long-time existence.

As a first note, in both the Yamabe and the Ricci flow the results were also established for more general edge singularities, where one considers a fibration of cones along a singular strata. This does not seem to be a very natural setting for the mean curvature flow, but in general the mapping properties needed for the edge setting follow along the same lines and do not impose additional difficulties.

On a second note, in the intrinsic flows the results where obtained for compact manifolds with conical singularities, meaning that outside of the singular neighbourhood the manifold is compact. This again seems to be a rather unnatural setting for the mean curvature flow, which lead to the consideration of the asymptotics near infinity in this thesis. As the arguments are local in nature, our results also encompass such manifolds, as well as manifolds which do not possess a singularity, but are asymptotically conical. Another natural assumption would be to consider cones inside the unit sphere, with boundary conditions on the intersection of the cone with the sphere, but this of course leads to different problems unrelated to singularity analysis.

There are also some more technical similarities and differences worth pointing out. One is our assumption on the cross section N being close to a minimal submanifold which is not a sphere and subsequently imposing a lower bound on the norm of the cross sectional second fundamental form $\|A_N\|^2 \geq n$, where n is the dimension of the cross section. As discussed previously, in background of the results on graphical mean curvature flow by Ecker and Huisken, such a restriction bears some naturality. In the Yamabe flow one requires $\mathrm{scal}(N) = n(n-1)$, also restricting the topology of the cross section. Again this restriction is natural, as it is a necessary condition for solutions to the Yamabe problem in this setting (cf. [AB03]). A further condition, namely the minimal non-zero eigenvalue of the Laplacian being larger than the dimension of the cross section, is imposed. In the intrinsic setting one can always achieve this via rescaling the cross section. In our situation we can not apply a rescaling technique, as we require the cross section to be near a minimal surface of the sphere, which is not an equator. Not only would a rescaling in the extrinsic setting modify the mean curvature, but such a minimal surface can not lie completely in one hemisphere (cf. [Sim68]) and consequently there is no canonical way to rescale the cross section. In Vertman's work on the Ricci flow, conditions on the initial metric force the cross section to be Einstein. Further restrictions arise, as, again in order to obtain good behaviour of the heat kernel near the side faces, one has to posit lower bounds on the first non-zero eigenvalues of the Laplacian and the Lichnerowicz-Laplacian. In the subsequent paper [KV18] the authors give an extensive list of examples and non-examples of Einstein manifolds satisfying these spectral properties.

8.3 Porous medium equation on conical manifolds

Most existing results concern linear flows and naturally the question arises if these techniques can be applied to study non-linear flows as well. So far there already exist short-time results for the porous medium equation

$$\begin{cases} \partial_t u = \Delta u^m, \ m > 1 \\ u(0) = u_0. \end{cases}$$

It describes the dissipation of a gas in a porous medium, or, in case of surfaces, can be thought of a film, e.g. ink on a needle. Roidos and Schrohe showed in [RS16] short-time existence in certain L^p-spaces with different techniques.

One of the ingredients in the long-term existence result in [BV16] of the Yamabe flow is an adaption of a freezing-of-coefficient construction of parametrices to equations of the type $\partial_t - a(p,t)\Delta_t$, where a is a Hölder function of suitable regularity. Using similar, but much simpler calculations like those in Section 6.1.1, we can again linearize the equation and then use these parametrices to perform a fixpoint argument, yielding a short-time solution almost effortlessly. Application of the maximum principle immediately shows preservation of sign. So far we were unable to show uniqueness of the solution in the corresponding Hölder spaces.

8.4 Cusp singularities

A very interesting question would be the short-time existence of mean curvature flow in presence of cusp singularities. There the metric on $\mathbb{R}_+ \times N$ takes the form $g = dr^2 + e^{-2r} g_N$ or, after a change of coordinates, it reads as $g = x^{-2}dx^2 + x^2 g_N$. These singularities are of particular interest, as in mean curvature flow with surgery, the occurring singularities are either neck pinches, or degenerate neck pinches, with the latter being cusp-like formations. Furthermore the fibred cusp-singularity setting has been studied extensively in global analysis – heat kernel asymptotics as those in Theorem 1.4 are readily available, cf. the work of Vaillant [Vai01].

A Uniformly equivalent distance functions

Our estimates rely on uniform equivalences of the distance functions

$$d((x,z),(x',z')) \sim \sqrt{|x-x'|^2 + |x+x'|^2|z-z'|^2}$$

for $g = dx^2 + x^2 g_N(z)$ and

$$d((y,z),(y',z')) \sim \sqrt{|y-y'|^2 \left(\frac{y+y'}{yy'}\right)^4 + \left(\frac{y+y'}{yy'}\right)^2 |z-z'|^2}$$

for $g = y^{-4}dy^2 + y^{-2}g_N(z)$, where $\sim$ denotes uniform equivalence, i.e.

$$d \sim \tilde{d} \iff \exists C, C' : Cd(x,y) \leq \tilde{d}(x,y) \leq C'd(x,y).$$

While intriguing and often used, there seems to be no account of this fact.

The main step in the proof is calculating the first entries of the Taylor series and relies on the fact that the squared distance function from a point satisfies a Hamilton-Jacobi-equation.

Tayloring the squared distance function

Let $r(x) := \operatorname{dist}(x_0, x)$ be the distance function to a fixed point x_0 and let $\eta(x) = r(x)^2$. For $x \neq x_0$ we have the Eikonal equation $\|\nabla r\| \equiv 1$. While r is not differentiable in $x = x_0$, η is and satisfies the Hamilton-Jacobi-equation $4\eta = \|\nabla\eta\|^2$. Clearly this equation holds for $x = x_0$ as there η attains its minimum and $\nabla\eta$ vanishes, so assume $x \neq x_0$:

$$\|\nabla\eta\|^2 = g(\nabla\eta, \nabla\eta) = 4g(r\nabla r, r\nabla r) = 4r^2\|\nabla r\|^2 = 4r^2 = 4\eta$$

Now we can calculate the Taylor expansion in local coordinates:

$$\begin{aligned}\eta(x) = \eta(x_0) &+ \sum_i \partial_i\, \eta|_{x_0} (x-x_0)^i \\ &+ \sum_{i,j} \partial_i\partial_j\eta|_{x_0} (x-x_0)^i(x-x_0)^j \\ &+ \mathcal{O}(\|x-x_0\|^3)\end{aligned}$$

Clearly we have $\eta(x_0) = 0$ and $\partial_i\eta(x_0) = 0$ for x_0 is the minimum of η. It remains to see that $\partial_i\partial_j\, \eta|_{x_0} = g_{ij}|_{x_0}$. While it seems like there should be an elegant and intuitive argument, simply calculating quickly yields the desired result. Observe that in general

$$\begin{aligned}\partial_i\partial_j\eta &= \frac{1}{4}\partial_i\partial_j\|\nabla\eta\|^2 = \frac{1}{4}\partial_i\partial_j g(\nabla\eta, \nabla\eta) \\ &= \frac{1}{2}\partial_i g(\nabla_{\partial_i}\nabla\eta, \nabla\eta) \\ &= \frac{1}{2}\left(g(\nabla_{\partial_j}\nabla_{\partial_i}\nabla\eta, \nabla\eta) + g(\nabla_{\partial_i}\nabla\eta, \nabla_{\partial_j}\nabla\eta)\right).\end{aligned}$$

At x_0 the gradient of η vanishes, so the first summand is zero there, so we are left with

$$\partial_i\partial_j\eta\big|_{x_0} = \frac{1}{2}\, g(\nabla_{\partial_i}\nabla\eta, \nabla_{\partial_j}\nabla\eta)\Big|_{x_0}.$$

Observing that $\partial_k\eta = 0$ at x_0, we calculate the remaining term in local coordinates at x_0:

$$\begin{aligned}
\nabla_{\partial_i}\nabla\eta &= \nabla_{\partial_i} g^{kl}\partial_k\eta\partial_l \\
&= \partial_i(g^{kl}\partial_k\eta)\partial_l + g^{kl}\underbrace{\partial_k\eta}_{=0}\nabla_{\partial_i}\partial_j \\
&= g^{kl}\partial_i\partial_k\eta\partial_l + (\partial_i g^{kl})\underbrace{\partial_k\eta}_{=0}\partial_l
\end{aligned}$$

Consequently we have at x_0

$$\begin{aligned}
\partial_i\partial_j\eta &= \frac{1}{2}g(\nabla_{\partial_i}\nabla\eta, \nabla_{\partial_j}\nabla\eta) \\
&= \frac{1}{2}g(g^{kl}\partial_i\partial_k\eta\partial_l, g^{mn}\partial_j\partial_m\eta\partial_n) \\
&= \frac{1}{2}g^{kl}g_{ln}g^{nm}\partial_i\partial_k\partial_j\partial_m\eta \\
&= \frac{1}{2}g^{km}\partial_i\partial_k\eta\partial_j\partial_m\eta.
\end{aligned}$$

Let $H = (h_{ij}) = (\partial_i\partial_j\eta)\big|_{x_0}$. Algebraically we have shown

$$H = \frac{1}{2}H^t g^{-1}(x_0)H.$$

We know that H is invertible, as in geodesic coordinates H would be $2\,\mathrm{id}$ and coordinate changes corresponds to conjugating with orthogonal matrices. Hence

$$\mathrm{id} = HH^{-1} = \frac{1}{2}H^t g^{-1}(x_0)$$

and consequently $\frac{1}{2}H^t = g(x_0)$. As H is obviously symmetric it follows that $g(x_0) = \frac{1}{2}H$, proving

$$\eta(x) = g_{ij}(x_0)(x_0^i - x^i)(x_0^j - x^j) + \mathcal{O}(\|x - x_0\|^3).$$

Estimating the error term

This step is rather easy. Our manifold is topologically a cylinder $M = \mathbb{R}\times N$ with N compact. Hence, on N, one can find finite collection of charts $\{U_i\}$ in each of these $d(z_0, z)^2 \sim g_{ij}(z_0 - z)^i(z_0 - z)^j$, by choosing the charts small enough, and using as a constant say 2. To extend this result to the cone, extend $\{U_i\}$ to $\{(0, 1] \times U_i\}$ for both cases, i.e. both near the tips, as well as near infinity. This gives $\|p - p_0\| < 2$, so that again we have finitely many charts where $d(p_0, p)^2 \sim g_{ij}(p_0 - p)^i(p_0 - p)^j$ holds, establishing the uniform equivalence.

Deriving the explicit formulae

First we consider the case $(M, g) = ((0, 1]_x \times N, dx^2 + x^2 g_N)$ and give a proof for the formula which was already used, e.g. in [BV14]. We begin with using the results from the previous steps, once for $d(p, \cdot)^2$ and once for $d(p', \cdot)^2$, where $p = (x, z), p' = (x', z')$. Then

$$\begin{aligned} 2d(p, p')^2 &= (g_{ij}(p) + g_{ij}(p'))(p - p')^i (p - p')^j \\ &= 2|x - x'|^2 + (x^2 + x'^2)(z - z')^2. \end{aligned}$$

Now clearly $x^2 + x'^2 \leq (x + x')^2 = x^2 + 2xx' + x'^2 \leq 2(x^2 + x'^2)$, hence $(x^2 + x'^2) \sim 2(x + x')^2$ and we obtain

$$2d(p, p')^2 \sim 2|x - x'|^2 + 2(x + x')^2 |z - z'|^2$$

in each one of the finitely many charts covering M constructed before.

Remark A.1. While *a priori* these charts do not need to be compatible with the partition of unity we used in the Schauder estimates to restrict the heat kernel to the different regimes, the latter can clearly be subordinated to these charts.

Finally let us establish the corresponding formula in the asymptotically conical case, i.e. $(M, g) = ((0, 1]_y \times N, y^{-4} dy^2 + y^{-2} g_N)$. We begin as in the previous case, now considering points $p = (y, z), p' = (y', z')$. Then

$$2d(p, p')^2 \sim 2(y^{-4} + y'^{-4})(y - y')^2 + 2(y^{-2} + y'^{-2})(z - z')^2.$$

For the first term we iterate Young's inequality and obtain the desired result:

$$2d(p, p')^2 \sim 2(y^{-1} + y'^{-1})^4 (y - y') + 2(y^{-1} + y'^{-1})^2 (z - z')^2$$

Note that we will usually write

$$\frac{1}{y} + \frac{1}{y'} = \frac{y + y'}{yy'}.$$

References

[AB03] K. Akutagawa and B. Botvinnik. Yamabe metrics on cylindrical manifolds. *Geom. Funct. Anal.*, 13(2):259–333, 2003.

[Alb07] Pierre Albin. A renormalized index theorem for some complete asymptotically regular metrics: the Gauss-Bonnet theorem. *Adv. Math.*, 213(1):1–52, 2007.

[App10] Paul Appleby. *Mean Curvature flow of hypersurfaces asymptotic to minimal cones.* PhD thesis, Freie Universität Berlin, 2010.

[BDGG69] E. Bombieri, E. De Giorgi, and E. Giusti. Minimal cones and the Bernstein problem. *Invent. Math.*, 7:243–268, 1969.

[BV14] Eric Bahuaud and Boris Vertman. Yamabe flow on manifolds with edges. *Math. Nachr.*, 287(2-3):127–159, 2014.

[BV16] Eric Bahuaud and Boris Vertman. Long-time existence of the edge yamabe flow. arXiv:1605.03935, 2016.

[Che83] Jeff Cheeger. Spectral geometry of singular Riemannian spaces. *J. Differential Geom.*, 18(4):575–657 (1984), 1983.

[EH89] Klaus Ecker and Gerhard Huisken. Mean curvature evolution of entire graphs. *Ann. of Math. (2)*, 130(3):453–471, 1989.

[EH91] Klaus Ecker and Gerhard Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Invent. Math.*, 105(3):547–569, 1991.

[Joy12] Dominic Joyce. On manifolds with corners. In *Advances in geometric analysis*, volume 21 of *Adv. Lect. Math. (ALM)*, pages 225–258. Int. Press, Somerville, MA, 2012.

[KV18] Klaus Kröncke and Boris Vertman. Stability of Ricci de Turck flow on singular spaces. arXiv:1802.02908, 2018. preprint.

[Les97] Matthias Lesch. *Operators of Fuchs type, conical singularities, and asymptotic methods*, volume 136 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.

[Man11] Carlo Mantegazza. *Lecture notes on mean curvature flow*, volume 290 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2011.

[Mau16] Wolfgang Maurer. Shadows of graphical mean curvature flow. arXiv:1604.05238, 2016. preprint.

[Mel93] Richard B. Melrose. *The Atiyah-Patodi-Singer index theorem*, volume 4 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 1993.

[MV12] Rafe Mazzeo and Boris Vertman. Analytic torsion on manifolds with edges. *Adv. Math.*, 231(2):1000–1040, 2012.

[Per02] Oscar Perdomo. Stable minimal cones in $\mathbb{R}^8$ and $\mathbb{R}^9$ with constant scalar curvature. *Rev. Colombiana Mat.*, 36(2):97–106, 2002.

[RS16] Nikolaos Roidos and Elmar Schrohe. Existence and maximal L^p-regularity of solutions for the porous medium equation on manifolds with conical singularities. *Comm. Partial Differential Equations*, 41(9):1441–1471, 2016.

[See64] R. T. Seeley. Extension of C^∞ functions defined in a half space. *Proc. Amer. Math. Soc.*, 15:625–626, 1964.

[She13] David A. Sher. The heat kernel on an asymptotically conic manifold. *Anal. PDE*, 6(7):1755–1791, 2013.

[Sim68] James Simons. Minimal varieties in Riemannian manifolds. *Ann. of Math. (2)*, 88:62–105, 1968.

[SS14] Mariel Sáez and Oliver C. Schnürer. Mean curvature flow without singularities. *J. Differential Geom.*, 97(3):545–570, 2014.

[Vai01] Boris Vaillant. Index and spectral theory for manifolds with generalized fibred cusps. arXiv:math/0102072, 2001.

[Ver16] Boris Vertman. Ricci de Turck flow on singular manifolds. arXiv:1603.06545, 2016.