Westfälische
Wilhelms-Universität
Münster

On homological invariants of some twisted group C*-algebras

Sayan Chakraborty

Mathematik

On homological invariants of some twisted group C*-algebras

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Westfälischen Wilhelms-Universität Münster

vorgelegt von:
Sayan Chakraborty
aus Balurghat
2018

Dekan:	Prof. Dr. Xiaoyi Jiang
Erster Gutachter:	Prof. Dr. Siegfried Echterhoff
Zweiter Gutachter:	Prof. Dr. Joachim Cuntz
Tag der mündlichen Prüfung:	27.06.2018
Tag der Promotion:	....

Zusammenfassung

We study K-theory of the twisted group C*-algebras of discrete groups of the kind $\mathbb{Z}^n$ and $\mathbb{Z}^n \rtimes F$, where F is a finite cyclic group. Twisted group C*- algebras can be thought of as deformations of classical group C*-algebras where the deformation parameter is an element of the second group cohomology of the given group. For $\mathbb{Z}^n$ and $\mathbb{Z}^n \rtimes F$, we show how some K-theory elements (projective modules) behave under the deformation. Also we study homological invariants of smooth twisted algebras (holomorphically closed, dense sub-algebras of twisted group C*-algebras) of the groups $\mathbb{Z}^n$ and $\mathbb{Z}^n \rtimes F$.

Acknowledgements

I would like to thank my advisor Prof. Siegfried Echterhoff for constant encouragement and fruitful discussions. I am indebted to Jean Belissard, Joachim Cuntz, Franz Luef, Xiang Tang, Makoto Yamashita for helpful mathematical discussions and ideas. I thank my friends Christian, Paulo, Ingo who made my life easier during the years I wrote this thesis. Special thanks to my parents and my sister for making me the person what I am today: this thesis is dedicated to them. Last but not least, I want to thank SFB for funding my PhD studies.

Contents

1 Introduction

This thesis in a nutshell

In noncommutative geometry one studies noncommutative algebras which are motivated from geometry. Important example are noncommutative tori. Let us recall that, given a skew-symmetric real $n \times n$ matrix $\theta = (\theta_{ij})$, the n-dimensional noncommutative torus, A_θ, is defined as the universal C*-algebra generated by unitaries $U_1, \ldots, U_n$ subject to the relations

$$U_k U_j = e^{2\pi i \theta_{jk}} U_j U_k \quad \text{for} \quad j, k = 1, \cdots, n.$$

For 2-dimensional noncommutative tori, since θ is determined by only one real number, we will write the matrix θ just as a real number θ.

Noncommutative tori are model examples of so called "deformation quantisations" or twisted group C*-algebras. Twisted group C*-algebras are in some sense deformations of group C*-algebras while the deformation parameter is an element of the second group cohomology of the group. "Deformation quantisations" and twisted group C*-algebras give us a huge list of examples to study in noncommutative geometry. These can be thought of noncommutative versions of quantised Poisson manifolds where the deformation parameter is a Poisson bivector.

One studies vector bundles over classical manifolds. This comes naturally when one talks about notions of curvature or differential operators on manifolds. A well known theorem of Serre–Swan suggests that the natural analog of vector bundles over noncommutative algebras is projective modules.

Rieffel ([54]) constructed projective modules (which are known as Heisenberg modules) over all noncommutative tori. These constitute the framework to study geometry of noncommutative tori such as connections, curvature and Dirac operators on noncommutative tori. While the 2-dimensional noncommutative tori are analysed quite well, there remains quite a large number of open questions for higher dimensional noncommutative tori.

We call a skew-symmetric matrix totally irrational if the off-diagonal entries are rationally linearly independent and not rational.

While Rieffel constructed projective modules over n-dimensional noncommutative tori, Elliott in [24] computed the K-theory of these algebras. Elliott showed that the K-theory of n-dimensional noncommutative tori is independent of the parameter θ and he also computed the image of the canonical trace of A_θ. It follows from Elliott's computations that for totally irrational θ the canonical trace on A_θ is injective as a map from $\mathrm{K}_0(A_\theta)$ to $\mathbb{R}$. So using the description of the image of the trace and Rieffel's ([54]) computations of traces of projective modules, we can compute a basis of K_0 for A_θ in the case where θ is totally irrational.

Based on the results of Rieffel and Elliott, Echterhoff et al.([22]) constructed a continuous field of projective modules over the parameter space of 2-dimensional noncommutative tori. Along with other results, the authors gave a basis of $\mathrm{K}_0(A_\theta)$ using the range of the trace of 2-dimensional noncommutative tori. They also showed how this basis could be extended to provide elements of the basis of K-theory of some crossed products $A_\theta \rtimes F$, where F is a finite cyclic group which is compatible with θ.

We take a similar approach to [22] to provide bases for $\mathrm{K}_0(A_\theta)$ for any higher dimensional noncommutative tori.

Theorem 1.0.1. *Rieffel's projective modules over n-dimensional noncommutative tori can be chosen in a continuous way over the parameter space.*

Theorem 1.0.1 can be used to give explicit bases of K-theory of n-dimensional noncommutative tori, which we describe in Chapter 3.

One can also consider crossed product algebras for group actions on noncommutative tori. Crossed product algebras of finite group actions on noncommutative tori go back to the work of Bratteli, Elliott, Evans and Kishimoto. They considered ([10]) the flip action of $\mathbb{Z}_2$ on two dimensional noncommutative tori and the associated crossed products. Recall that the flip action of $\mathbb{Z}_2$ on a 2-dimensional noncommutative torus is given by mapping the generators U_i to U_i^{-1} for $i = 1, 2$. Walters ([59]) computed the K-theory of the crossed product $A_\theta \rtimes \mathbb{Z}_2$ for two dimensional noncommutative tori A_θ for some irrational θ. For these special θs, Walters also wrote down generators of K-theory of $A_\theta \rtimes \mathbb{Z}_2$ by showing that generators of the K_0 group of A_θ can be made "flip equivariant". (A result of Green and Julg shows that equivariant K-theory elements provide elements of the K-theory of the crossed product.) Later in [60] and [62], Walters considered $\mathbb{Z}_4$- and $\mathbb{Z}_6$- actions on two dimensional noncommutative tori. Recall that the following defines $\mathbb{Z}_4$- and $\mathbb{Z}_6$- actions on two dimensional noncommutative tori:

$$U_1 \to U_2,\ U_2 \to U_1^{-1} \quad (\text{for } \mathbb{Z}_4),$$

$$U_1 \to U_2,\ U_2 \to e^{-\pi i\theta} U_1^{-1} U_2 \quad (\text{for } \mathbb{Z}_6).$$

For these actions Walters showed that for irrational θ, the generators (as projective modules) of the K_0 group of A_θ are $\mathbb{Z}_2$- and $\mathbb{Z}_4$- equivariant to construct projective modules over $A_\theta \rtimes \mathbb{Z}_4$ and $A_\theta \rtimes \mathbb{Z}_6$. These constitute generators of the corresponding K_0 groups.

Later, in [22], Echterhoff, Lück, Phillips and Walters studied 2-dimensional A_θ with actions of finite cyclic subgroups (we denote these subgroups by F) of $SL_2(\mathbb{Z})$. Note that $SL_2(\mathbb{Z})$ has a canonical action on $\mathbb{Z}^2$, which can be lifted to actions on A_θ. The previous actions of $\mathbb{Z}_2$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are implemented by matrices in $SL_2(\mathbb{Z})$. It is demonstrated that the standard canonical projective module over A_θ, aka the Bott class (which is a completion of $\mathscr{S}(\mathbb{R})$, Schwartz functions on $\mathbb{R}$), can be made equivariant by the action of F yielding a projective module over the crossed product algebra $A_\theta \rtimes F$. It can be shown that the Bott class along with the identity element generate the K_0-group of the noncommutative torus.

Recently, actions of finite groups on higher-dimensional noncommutative tori have been considered in the article [32]. Let $W \in GL_n(\mathbb{Z})$ be the generator of the finite cyclic group F acting on $\mathbb{Z}^n$ such that $W^T\theta W = \theta$. Then the authors in [32] showed that there exists an action of F on the n-dimensional A_θ. Let us assume that n is an even number, $n = 2m$. To analyse projective modules over the corresponding crossed product algebras, we restrict our analysis to the class of Heisenberg modules which are some completions $\mathcal{E}$ of $\mathscr{S}(\mathbb{R}^m)$. We show that the classes of these projective modules (which may be thought of as higher dimensional versions of the Bott class) over the higher dimensional noncommutative tori can be made F-equivariant. The metaplectic action (action of the symplectic group on a suitable Hilbert space) is the key tool in this problem. This generalises the previous results of Walters and Echterhoff et al..

Theorem 1.0.2. *The metaplectic action of W on $\mathscr{S}(\mathbb{R}^m)$ extends to an action of $\mathcal{E}$ such that $\mathcal{E}$ becomes an F-equivariantly finitely generated projective A_θ module and thus a finitely generated projective module over $A_\theta \rtimes F$.*

Coming back to the flip case, note that this action can be defined for general n-dimensional tori A_θ. It is not hard to see that all the Heisenberg modules can be extended as modules over the crossed product $A_\theta \rtimes \mathbb{Z}_2$. Though, in [27], the authors have computed the K-theory of $A_\theta \rtimes \mathbb{Z}_2$, some computations in [27] are not clear to us. They used an exact sequence by Natsume ([45]) to compute the K-thoery of $A_\theta \rtimes \mathbb{Z}_2$. For crossed products like $A \rtimes (\mathbb{Z}_2 * \mathbb{Z}_2)$, Natsume's exact sequence looks like

$$\begin{array}{ccccc}
\mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A \rtimes \mathbb{Z}_2) \oplus \mathrm{K}_0(A \rtimes \mathbb{Z}_2) & \longrightarrow & \mathrm{K}_0(A \rtimes \mathbb{Z}_2 * \mathbb{Z}_2) \\
\uparrow & & & & \downarrow e_1 \\
\mathrm{K}_1(A \rtimes \mathbb{Z}_2 * \mathbb{Z}_2) & \longleftarrow & \mathrm{K}_1(A \rtimes \mathbb{Z}_2) \oplus \mathrm{K}_1(A \rtimes \mathbb{Z}_2) & \longleftarrow & \mathrm{K}_1(A).
\end{array}$$

In the final chapter we study this exact sequence, especially the connecting map e_1. We find its connection to the classical Pimsner-Voiculescu exact sequence. Recall that for crossed products like $A \rtimes \mathbb{Z}$, the Pimsner-Voiculescu sequence looks like

$$\begin{array}{ccccc}
\mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A \rtimes \mathbb{Z}) \\
\uparrow & & & & \downarrow e_2 \\
\mathrm{K}_1(A \rtimes \mathbb{Z}) & \longleftarrow & \mathrm{K}_1(A) & \longleftarrow & \mathrm{K}_1(A).
\end{array}$$

The result can be stated as follows:

Theorem 1.0.3. *For unital A, the connecting maps of the above two sequences commute in the following sense:*

$$\begin{array}{ccc}
\mathrm{K}_0(A \rtimes \mathbb{Z}_2 * \mathbb{Z}_2) & \xrightarrow{\;e_1\;} & \mathrm{K}_1(A) \\
& \searrow^{p} & \uparrow e_2 \\
& & \mathrm{K}_0(A \rtimes \mathbb{Z}),
\end{array}$$

where p is the map induced by the natural map from $A \rtimes \mathbb{Z}_2 * \mathbb{Z}_2 = (A \rtimes \mathbb{Z}) \rtimes \mathbb{Z}_2$ to $M_2(A \rtimes \mathbb{Z})$.

Using this result, we discuss the K-theory of crossed products of 3-dimensional noncommutative tori with respect to the flip action and describe the generators of K-theory (see Corollary 7.0.4). This explains the computations of [27] for the three dimensional case. Presumably this can be done for the whole K-theory computations for the n-dimensional case, that we plan to write down elsewhere.

Though, till now, mostly we talked about C*-algebras, we can also define smooth sub-algebras of noncommutative tori. For the noncommutative torus A_θ, we denote the smooth sub-algebra by A_θ^∞, which is a locally convex Fréchet algebra. Then it can be shown that the action of F on A_θ restricts to an action of A_θ^∞. As the algebras A_θ^∞ or $A_\theta^\infty \rtimes F$ can be thought of smooth functions on noncommutative spaces (in this case tori or quotients of tori by finite groups), it is also interesting to consider (co)-homology theory of these algebras. Indeed, one can define K-theory or periodic cyclic theory (HP) on the category of these locally convex algebras. HP can be thought of as a noncommutative version of de-Rham theory. Connes showed that for a locally convex algebra A, there is a morphism $\mathrm{ch} : \mathrm{K}(A) \to \mathrm{HP}(A)$, which resembles the classical Chern character.

Though there is a good number of tools to compute K-theory, computation of HP even for well known algebras is an active area of research. The computation of K-theory or HP for A_θ^∞ or $A_\theta^\infty \rtimes F$ is somewhat reasonable since these algebras are in a sense not too far from commutative algebras. HP for A_θ^∞ is computed by Nest using his version of the Pimsner-Voiculescu exact sequence for cyclic cohomology. Also one can use Connes' version of the Thom isomorphism (in K-theory or HP setting) to compute K-theory or HP of A_θ^∞.

Connes' version of the Thom isomorphism also holds in the category of C*-algebras, which we prove and show how it can be made F equivariant for suitable F action on $\mathbb{R}^n$ and A.

Let F be a compact group. Then the theorem is as follows.

Theorem 1.0.4. *Let $\mathbb{R}^n$ act on a C*-algebra A and F also acts on $\mathbb{R}^n$ and on A in a compatible way. Then*

$$\mathrm{K}_*^F(A \rtimes \mathbb{R}^n) = \mathrm{K}_{*+n}^F(A),$$

where K^F is the F-equivariant K*-theory on the category of C*-algebras.*

The algebras $A_\theta \rtimes F$ are examples of "F-equivariant version" of Rieffel's deformation quantisation. Using some ideas of Neshveyev ([46]) on deformation quantisation, the above theorem leads to the K-theory computations for $A_\theta \rtimes F$ (see Corollary 5.5.7). The methods implemented to prove the above theorem involve Kasparov's KK-theory and explicit construction of Thom and Dirac element. One of the merit of this method is that the proof can be modified to prove a similar theorem for smooth algebras also.

For the algebras $A_\theta^\infty \rtimes F$, we also describe some zeroth cyclic cocycles which are coming from the conjugacy classes of $\mathbb{Z}^n \rtimes F$ (see Chapter 6). This leads to some explicit K-theory computations of some $A_\theta \rtimes F$ (Corollary 7.0.4).

- Theorem 1.0.2 and Theorem 1.0.3 have been obtained in a joint work with Luef ([14]).

- The results regarding zeroth cyclic cocycles have been obtained in a joint work with Yamashita ([15]).
- I am indebted to Xiang Tang for some ideas regarding Theorem 1.0.4.

2 Preliminaries

We assume that the reader is familiar with basics of group C*-algebras, crossed products of C*-algebras by locally compact groups and KK-theory of graded C*-algebras. To fix notations we will give a brief overview of twisted group C*-algebras and equivariant KK-theory. Since we will work mostly with discrete or abelian groups, we assume that all our groups are unimodular.

In what follows, we use the following notations (unless otherwise specified):

1. $\mathcal{H}$: a separable Hilbert space or Hilbert module over some separable C*-algebra.
2. $\mathcal{B}(\mathcal{H})$: bounded operators on the separable Hilbert space $\mathcal{H}$.
3. $\mathbb{T}$: the circle group.
4. $\mathcal{U}(\mathcal{H})$: the group of unitary operators on the Hilbert space $\mathcal{H}$.
5. $\mathcal{K}$: the C*-algebra of compact operators.
6. $\mathcal{L}(\mathcal{H})$: C*-algebra of adjointable operators on a Hilbert module $\mathcal{H}$.
7. $C[0,1]$: the C*-algebra of complex valued continuous functions on the interval $[0,1]$.
8. $C([0,1],A)$: the C*-algebra of A-valued continuous functions on the interval $[0,1]$, for C*-algebra A.
9. $L^2(G)$: the Hilbert space of complex valued square integrable functions on a the locally compact group G.

2.1 Twisted group algebras

2.1.1 2-cocycles on a group and twisted group C*-algebras

Let G be a locally compact unimodular group. A Borel map $\omega : G \times G \to \mathbb{T}$ is called a *2-cocycle* (sometimes just cocycle) if

$$\omega(x,y)\omega(xy,z) = \omega(x,yz)\omega(y,z)$$

whenever $x, y, z \in G$, and if

$$\omega(x, 1) = 1 = \omega(1, x)$$

for any $x \in G$.

If ω is a 2-cocycle on G, we can define the twisted convolution algebra $L^1(G, \omega)$ as the vector space of all integrable complex functions on G with convolution and involution given by

$$(f *_\omega g)(x) = \int_G f(y)g(y^{-1}x)\omega(y,\, y^{-1}x)\, dy,$$

and

$$f^*(x) = \overline{\omega(x, x^{-1})f(x^{-1})}.$$

An *ω-representation* of G on a Hilbert space $\mathcal{H}$ is a Borel map $V\colon G \to \mathcal{U}(\mathcal{H})$, the unitary group of $\mathcal{H}$, such that

$$V(x)V(y) = \omega(x, y)V(xy)$$

far all $x, y \in G$. One example of such representation is the *regular ω-representation*, which is by definition the representation $L_\omega\colon G \to \mathcal{U}(L^2(G))$ given by

$$(L_\omega(x)\xi)(y) = \omega(x,\, x^{-1}y)\xi(x^{-1}y)$$

for $\xi \in L^2(G)$.

From an ω-representation $V\colon G \to \mathcal{U}(\mathcal{H})$ one defines a representation of $L^1(G, \omega)$ to $B(\mathcal{H})$ also denoted V, via the formula

$$V(f) = \int_G f(x)V(x)\, dx$$

for $f \in L^1(G, \omega)$. The reduced twisted group algebra $C^*(G, \omega)$ is defined to be the closure (under norm topology) of the image of $L^1(G, \omega)$ under the regular ω-representation L_ω.

Remark 2.1.1. *One can also define the full twisted group C*-algebra in the same sprit as the usual full group C*-algebra, but since the groups we will be working with are amenable, it coincides (like in the untwisted case) with the reduced twisted group C*-algebra.*

Remark 2.1.2. *When ω is trivial, obviously $C^*(G, \omega)$ is isomorphic to $C^*(G)$, the usual reduced group C*-algebra of G.*

2.1.2 The group $\mathrm{H}^2(G, \mathbb{T})$

Let us denote the set of all 2-cocycles of the group G by $Z^2(G, \mathbb{T})$.

Definition 2.1.3. *Two cocycles $\omega, \omega' \in Z^2(G, \mathbb{T})$ are called cohomologous if there exists a Borel function $u \colon G \to \mathbb{T}$ such that $\omega' = \partial u \cdot \omega$, where ∂u is defined as*

$$\partial u(x, y) = \overline{u(xy)}u(x)u(y).$$

Remark 2.1.4. *Let ω, ω' be cohomologous 2-cocycles. Then it is easy to check that the corresponding reduced twisted group C^*-algebras are isomorphic. The isomorphism is given by $f \mapsto u \cdot f$, from $L^1(G, \omega') \to L^1(G, \omega)$, which extends to an isomorphism of the corresponding C^*-algebras.*

Define $\mathrm{H}^2(G, \mathbb{T})$ to be the set $Z^2(G, \mathbb{T})$ modulo cohomologous 2-cocycles. It is easily checked to be an abelian group with obvious multiplication.

Remark 2.1.5. *If one is familiar with group (co)homology of discrete groups (see* [11] *for an account of that), then there is a universal coefficient theorem saying that*

$$\mathrm{H}^2(G, \mathbb{T}) = \mathrm{Hom}(\mathrm{H}_2(G, \mathbb{Z}), \mathbb{T}).$$

Remark 2.1.6. *Let G be finitely generated abelian. A map $\sigma : G \times G \to \mathbb{T}$ is said to be a bicharacter if $\sigma(g, \cdot)$ and $\sigma(\cdot, g)$ are homomorphisms from G to $\mathbb{T}$ for each $g \in G$. It is easy to see that every bicharacter is a 2-cocycle.*

Sometimes it is easy to determine the bicharacters. Also there is a theorem [36, Theorem 7.1] *saying that every 2-cocycle on a torsion free finitely generated abelian group is cohomologous to a bicharacter.*

2.1.3 Some basic examples

Example 2.1.7. *Let F be a finite cyclic group. Then any 2-cocycle is cohomologus to the trivial one. One might use Remark 2.1.5 to show this (using the fact that $H_2(F, \mathbb{Z})$ is trivial).*

Example 2.1.8. *Let G be the group $\mathbb{Z}^n$. For each $n \times n$ real antisymmetric matrix θ, we can construct a 2-cocycle on this group by defining $\omega_\theta(x, y) = e^{\pi i(\langle -\theta x, y\rangle)}$. The corresponding twisted group C^*-algebra $C^*(G, \omega_\theta)$ is called noncommutative torus (also denoted by A_θ). It is true that any 2-cocycle on $\mathbb{Z}^n$ is cohomologous to ω_θ for some $n \times n$ real antisymmetric matrix θ. By Remark 2.1.6, it is enough to look at bicharacters. It is not difficult to see that any bicharacter of $\mathbb{Z}^n$ can be written in the form $\sigma_{\theta'}(x, y) = e^{\pi i(\langle -\theta' x, y\rangle)}$, for some $n \times n$ real matrix θ'. Set $\theta = \theta' - (\theta')^T$. Then $\sigma_{\theta'}$ and ω_θ are cohomologous by the function $u(x) = e^{\pi i(\langle -\theta' x, x\rangle)}$.*

2.1.4 Group actions on $\mathcal{K}$

In this subsection we will see how twisted group C*-algebras of a group G are naturally associated to crossed product-C*-algebras of $\mathcal{K}$ by some action of G.

Let $\mathcal{P}(\mathcal{U}) = \mathcal{U}/\mathbb{T}1$ denote the projective unitary group. Note that $\mathrm{Aut}(\mathcal{K}) = \mathcal{P}(\mathcal{U})$. Any ω-representation $V\colon G \to \mathcal{U}$ gives rise to an action $\alpha\colon G \to \mathrm{Aut}(\mathcal{K})$ in the following way. Define $\alpha_x = \mathrm{Ad}(V(x))$ for $x \in G$ (one might take the regular ω-representation as V). Set $\overline{\omega}(x,y) = \overline{\omega(x,y)}$. Then the reduced crossed product $\mathcal{K} \rtimes_\alpha G$ is isomorphic to $C^*(G,\overline{\omega}) \otimes \mathcal{K}$

The isomorphism is given by the map $\phi\colon L^1(G,\overline{\omega}) \odot \mathrm{K} \to L^1(G,\mathcal{K})$

$$\phi(f \otimes L)(x) = f(x)LV(x)^* \tag{2.1.1}$$

for $f \in L^1(G,\overline{\omega})$, $L \in \mathcal{K}$.

On the other hand if G acts on $\mathcal{K}$ by α, one has the following diagram

$$\begin{array}{ccc} G & \xrightarrow{\ \alpha\ } & \mathrm{Aut}(\mathcal{K}) = \mathcal{P}(\mathcal{U}) \\ & \searrow^{V} & \downarrow s \\ & & \mathcal{U}, \end{array}$$

where s is a Borel section. It is easy to check that the Borel map V satisfies

$$V(x)V(y) = \omega_\alpha(x,y)V(xy)$$

for some 2-cocycle $\omega_\alpha \in Z^2(G,\mathbb{T})$ and also $\alpha_x = \mathrm{Ad}(V(x))$.

The connection between twisted group C*-algebras and crossed products as discussed above is a manifestation of so-called Packer-Raeburn stabilisation trick [49].

2.2 Equivariant KK-theory

We recall the basic definitions of Kasparov's KK group. For details we refer to the wonderful book of Blackadar [8]. KK-theory is a bivariant theory on the category of C*-algebras. This turns out to be a useful tool to compute K-theory of C*-algebras.

Let G be a locally compact group. Let A, B be separable $\mathbb{Z}_2$-graded (often we say just graded) G-C*-algebras. Define $\mathcal{E}^G(A,B) = \{(\mathcal{H},\psi,T)\}$, where $\mathcal{H}$ is a G-Hilbert B-module (countably generated and graded), $\psi : A \to \mathcal{L}(B)$ is a graded *-homomorphism with

$$\psi(ga) = g\psi(a), \; g \in G, a \in A,$$

and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned} (T - T^*)\psi(a) &\in \mathcal{K}(\mathcal{H}) \\ (gT - T)\psi(a) &\in \mathcal{K}(\mathcal{H}) \\ [\psi(a), T] &\in \mathcal{K}(\mathcal{H}) \\ \psi(a)(I - T^2) &\in \mathcal{K}(\mathcal{H}) \end{aligned}$$

for all $g \in G$, $a \in A$. Here $[x, y]$ denotes the graded commutator of $x, y \in A$ in the graded algebra A.

Definition 2.2.1.

- *Let $(\mathcal{H}, \psi, T) \in \mathcal{E}^G(A, B)$. Let C be another graded G-C^*-algebra. Also let $\phi : B \to C$ be a grading preserving morphism. Then there is a natural element $\phi_*(\mathcal{H}, \psi, T)$ in $\mathcal{E}^G(A, C)$ given by $(\phi_*(\mathcal{H}), \psi\otimes 1, T\otimes 1)$, $\phi_*(\mathcal{H}))$ denotes the tensor product of the graded Hilbert modules $\mathcal{H}$ (over B) and C (viewed as Hilbert module over B via the map ϕ).*
- *Let $(\mathcal{H}, \psi, T) \in \mathcal{E}^G(B, C)$. Let A be another graded G-C^*-algebra. Let $\phi : A \to B$ be a grading preserving morphism. Then we get an element $\phi^*(\mathcal{H}, \psi, T)$ in $\mathcal{E}^G(A, C)$ given by $(\mathcal{H}, \psi \circ \phi, T)$.*

We call $(\mathcal{H}_1, \psi_1, T_1) \cong (\mathcal{H}_2, \psi_2, T_2)$ if there exists a unitary element in $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ (adjointable operators between $\mathcal{H}_1$ and $\mathcal{H}_2$) which intertwines ψ_i and T_i for $i = 1, 2$. Now we define an equivalence relation $\sim_h$ on $\mathcal{E}^G(A, B)$. Let $(\mathcal{H}_1, \psi_1, T_1), (\mathcal{H}_2, \psi_2, T_2)$ be in $\mathcal{E}^G(A, B)$. We say $(\mathcal{H}_1, \psi_1, T_1) \sim_h (\mathcal{H}_2, \psi_2, T_2)$ if there exists $(\hat{\mathcal{H}}, \hat{\psi}, \hat{T})$ in $\mathcal{E}^G(A, C([0, 1], B))$ such that $\mathrm{ev}_{0,*}(\hat{\mathcal{H}}, \hat{\psi}, \hat{T}) \cong (\mathcal{H}_1, \psi_1, T_1)$ and $\mathrm{ev}_{1,*}(\hat{\mathcal{H}}, \hat{\psi}, \hat{T}) \cong (\mathcal{H}_2, \psi_2, T_2)$, where

$$\mathrm{ev}_x : C([0, 1], B) \to B, f \mapsto f(x).$$

Definition 2.2.2. *We define the* equivariant KK-theory *of A, B as*

$$\mathrm{KK}^G(A, B) := \mathcal{E}(A, B)/\sim_h$$

$\mathrm{KK}^G(A, B)$ *is an abelian group with addition and additive inverse given by*

$$\begin{aligned} (\mathcal{H}, \psi, T) + (\mathcal{H}', \psi', T') &= (\mathcal{H} \oplus \mathcal{H}', \psi \oplus \psi', T \oplus T') \\ -(\mathcal{H}, \psi, T) &= (-\mathcal{H}, -\psi, -T). \end{aligned}$$

$(-\mathcal{H}, -\psi, -T)$ *denotes* $(\mathcal{H}, \psi, T)$ *with opposite grading (see* [8, Chapter VIII] *for details).*

2.2.1 Some formal properties of KK-theory

- KK^G is a bivariant functor on the category of G-C*-algebras with $\mathrm{KK}^G(A, \mathbb{C})$ and $\mathrm{KK}^G(\mathbb{C}, A)$ are equivariant K-homology and equivariant K-theory, respectively.

- KK^G is homotopy invariant and stable in both variables.

- Write $\mathrm{KK}_0^G(A, B) = \mathrm{KK}^G(A, B)$. If one sets $\mathrm{KK}_1^G(A, B) = \mathrm{KK}^G(A, B \otimes \mathbb{C}_1)$, $\mathbb{C}_1$ being the one dimensional Clifford algebra (with canonical grading), one has the formal *Bott periodicity*: $\mathrm{KK}_1^G(A, B \otimes \mathbb{C}_1) \cong \mathrm{KK}^G(A, B)$.

2.2.2 Kasparov product

Theorem 2.2.3 (Kasparov)**.** *Let A, B, C be separable G-C*-algebras. Then there is a product*

$$\mathrm{KK}^G(A, B) \times \mathrm{KK}^G(B, C) \to \mathrm{KK}^G(A, C)$$

and a descent homomorphism

$$j_G : \mathrm{KK}^G(A, B) \to \mathrm{KK}(A \rtimes G, B \rtimes G).$$

Remark 2.2.4. *Though we talk about* KK*-theory of C*-algebras in general, one should really understand* $\mathrm{KK}(X, Y)$ *purely in terms of topology for nice topological spaces X and Y. We refer to the article* [9] *for an account of that.*

2.3 Baum-Connes conjecture for twisted group algebras

2.3.1 The Baum-Connes conjecture with coefficients

Let G be a locally compact group and let A be a G-C*-algebra. Then we can consider the reduced crossed product $A \rtimes G$. The Baum–Connes conjecture is about understanding the C*-algebra K-theory of $A \rtimes G$. The conjecture relates its K-theory to equivariant K-homology (with coefficient A) of some space.

Let $\underline{\mathcal{E}}G$ be the universal space for proper action of G. We refer to the article [6, section 1] for an overview of $\underline{\mathcal{E}}G$. The important thing is that the space always exists and unique upto G-homotopy.

Let $\mathrm{KK}_j^G(\underline{\mathcal{E}}G, A)$ denote the equivariant K-homology of $\underline{\mathcal{E}}G$ with G-compact supports and coefficients A, that is

$$\mathrm{KK}_j^G(\underline{\mathcal{E}}G, A) := \lim_{X \subset \underline{\mathcal{E}}G,\, X \text{is } G-\text{compact}} \mathrm{KK}_j^G(C_0(X), A) \;\; j = 0, 1.$$

$\mathrm{KK}_j^G(\underline{\mathcal{E}}G, A)$ is sometimes also called the topological K-theory of G with coefficient in A, we sometimes denote this by $\mathrm{K}_j^{\mathrm{Top}}(G; A)$.

Conjecture 2.3.1 ([5])**.** *Let G be a locally compact Hausdorff second countable topological group, and let A be any separable G-C^*-algebra. Then there is a group homomorphism*

$$\mu : \mathrm{KK}_j^G(\underline{\mathcal{E}}G, A) \to \mathrm{K}_j(A \rtimes G), \quad j = 0, 1$$

which is an isomorphism.

The above conjecture is the Baum-Connes conjecture with coefficients.

Remark 2.3.2. *The conjecture was not formulated originally (in* [5]*) in the form above. It was later modified in the form. We will see how the conjecture with coefficient is useful for some* K*-theory computations.*

Let us say a bit about the map μ. As we have seen in the definition of KK^G, an element of $\mathrm{KK}_G^j(C_0(X), A)$ consists of an adjointable operator on some Hilbert module over A. This adjointable operator has some good properties, specifically there is a $\mathrm{K}_j(A \rtimes G)$ valued “index”. When $A = \mathbb{C}$, G is trivial (so X is a point), this “index” is exactly the Fredholm index of some operator on a Hilbert space. And the map μ is assigning the operator to its index. Technically this is achieved by the Kasparov product and the descent map. We will see a geometric interpretation of this map in the next remark (subsection).

The conjecture in this form is not true in general. There is a counter example due to Higson, Lafforgue and Skandalis (see [29]). Although for a vast number of groups, the conjecture is shown to be true, for example amenable groups (see [30]).

2.3.2 A brief remark on topological K-theory of G

Let G be a (countable) discrete group. We describe a geometric picture for the above conjecture with coefficient $\mathbb{C}$. We refer to [7] for more details on this part.

Consider pairs (M, E) such that M is a manifold without boundary, with a given smooth proper co-compact action of G and a given G-equivariant Spin^c-structure, and E is a G-equivariant vector bundle on M. Two such pairs are equivalent via an equivalence relation $\sim$ generated by the following three relations (see [7] for details):

- Bordism,
- Direct sum - disjoint union,
- Vector bundle modification.

Then we define the *geometric K-theory of G* as

$$\mathrm{K}_j^{\mathrm{Geo}}(G) = \{(M, E)\}/\sim$$

for j=0,1, according to the following condition (respectively) on M: all of whose components have either even or odd dimension. Addition will be disjoint union

$$(M, E) + (M', E') = (M \cup M', E \cup E').$$

and inverse is given by reversing the Spin^c-structure of M. The main result of this section is:

Theorem 2.3.3 (Baum, Higson, Schick [7])**.**

$$\mathrm{K}_j^{\mathrm{Geo}}(G) \cong \mathrm{K}_j^{\mathrm{Top}}(G; \mathbb{C}),$$

via a natural isomorphism.

Remark 2.3.4. *If G is a discrete group and if one considers the Baum-Connes conjecture without coefficients, one gets the LHS of the conjecture as $\mathrm{K}_j^{\mathrm{Geo}}(G)$. In this picture μ sends (M, E) to the G-index of the Dirac operator on the* Spin^c *manifold M which is an operator on the Hilbert space $L^2(M, E)$, the space of square integrable sections of the vector bundle E.*

2.3.3 The conjecture for twisted group algebras

Let G be as before and let $\omega \in \mathrm{H}^2(G, \mathbb{T})$. Then, as we have seen we get an action of G on $\mathcal{K}$ corresponding to ω. Let us denote $\mathcal{K}$ to be $\mathcal{K}_\omega$ when $\mathcal{K}$ is equipped with the action of G corresponding to regular ω-representation as in Section 2.1.4.

Recall from Section 2.1.4 that $C^*(G, \omega) \otimes \mathcal{K} \cong \mathcal{K}_{\overline{\omega}} \rtimes G$. Since K-theory is stable we have $\mathrm{K}_*(\mathcal{K}_{\overline{\omega}} \rtimes G) \cong \mathrm{K}_*(C^*(G, \omega))$. So using 2.3, we have the following conjecture

Conjecture 2.3.5. *Let $[\omega] \in \mathrm{H}^2(G, \mathbb{T})$. Then we define ω-twisted topological* K*-theory $\mathrm{K}_j^{\mathrm{Top}}(G; \omega) := \mathrm{K}_j^{\mathrm{Top}}(G; \mathcal{K}_{\overline{\omega}})$. The twisted assembly map for G is then defined to be*

$$\mu_\omega \colon \mathrm{K}_j^{\mathrm{Top}}(G; \omega) \to \mathrm{K}_j(\mathcal{K}_\omega \rtimes G) \cong \mathrm{K}_j(C^*(G, \omega)).$$

It is obvious that the definition does not depend on the choice of the representative ω of the class $[\omega] \in \mathrm{H}^2(G, \mathbb{T})$. For more details of this map we refer to [22, Section 1].

If G satisfies the Baum-Connes conjecture with coefficients (for example amenable groups) as in 2.3.1, then obviously for G, μ_ω is an isomorphism for all $\omega \in \mathrm{H}^2(G, \mathbb{T})$. It might be possible that there are groups for which the Baum-Connes conjecture with coefficients is true for $\mathcal{K}$ as coefficient but not in general.

2.4 Some tools to compute K-theory

2.4.1 Crossed products by by $\mathbb{Z}$ and by $\mathbb{R}$

Let us talk about two particular cases of the Baum-Connes conjecture. We start with Connes' version of the Thom isomorphism.

Theorem 2.4.1. *(Connes) For a strongly continuous action $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, we have*

$$\mathrm{K}_j(A) \cong \mathrm{K}_{1+j}(A \rtimes_\alpha \mathbb{R}). \tag{2.4.1}$$

Though it is called the Connes-Thom isomorphism theorem, this is not a generalisation of the classical Thom isomorphism theorem which deals with vector bundles over manifolds and their K-theory. Though the Connes-Thom isomorphism and the classical Thom isomorphism are generalisations of Bott periodicity. In (2.4.1), when the action of $\mathbb{R}$ is trivial, we have $\mathrm{K}_j(A) \cong \mathrm{K}_{1+j}(A \otimes C_0(\mathbb{R}))$ which is exactly the Bott periodicity.

Theorem 2.4.2. *(Pimsner–Voiculescu) If $\alpha \in \mathrm{Aut}(A)$, one has the following exact sequence*

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{\mathrm{id}-\mathrm{K}_0(\alpha^{-1})} & \mathrm{K}_0(A) & \xrightarrow{i_0} & \mathrm{K}_0(A \rtimes_\alpha \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A \rtimes_\alpha \mathbb{Z}) & \xleftarrow[i_1]{} & \mathrm{K}_1(A) & \xleftarrow[\mathrm{id}-\mathrm{K}_1(\alpha^{-1})]{} & \mathrm{K}_1(A) \end{array} \tag{2.4.2}$$

where i_0, i_1 are induced from inclusions.

Remark 2.4.3. *The above two theorems can be derived from one another. See* [18, 10.2.2] *for the proof of the Connes-Thom isomorphism theorem from the Pimsner-Voiculescu. As mentioned, These two theorems can also be derived from the Baum-Connes conjecture for the groups $\mathbb{R}$ and $\mathbb{Z}$ (though the computations are not straight forward).*

2.4.2 The Künneth formula

This is a generalisation of the classical Künneth formula to the K-theory of C*-algebra setting. Classically this formula says that the cohomology of a product of two spaces is the tensor product of the cohomologies of the two factors. One special case of the C*-algebrais version of this, is the following:

Theorem 2.4.4. *(*[8, Section 23]*) Let A, B be C*-algebras. Assume $\mathrm{K}_j(B)$ is torsion-free and A is separable and type I (hence nuclear). Then*

$$\begin{aligned}\mathrm{K}_0(A\otimes B) &\cong (\mathrm{K}_0(A)\otimes \mathrm{K}_0(B))\oplus(\mathrm{K}_1(A)\otimes \mathrm{K}_1(B)),\\ \mathrm{K}_1(A\otimes B) &\cong (\mathrm{K}_0(A)\otimes \mathrm{K}_1(B))\oplus(\mathrm{K}_1(A)\otimes \mathrm{K}_0(B)).\end{aligned}$$

.

2.4.3 An easy computation

Let us recall that for an $n\times n$ real antisymmetric matrix θ, we defined the noncommutative torus A_θ as the twisted group C*-algebra $C^*(\mathbb{Z}^n,\omega_\theta)$. Let us give another picture of the noncommutative torus. A_θ is defined as the universal C*-algebra generated by unitaries $U_1,\dots,U_n$ subject to the relations

$$U_kU_j = e^{2\pi i\theta_{jk}}\,U_jU_k \quad \text{for} \quad j,k=1,\cdots,n,$$

where $\theta=(\theta_{ij})$ is a skew-symmetric real $n\times n$ matrix. Both pictures can be easily checked to be compatible. The isomorphism (of $C^*(\mathbb{Z}^n,\omega_\theta)$ and A_θ as defined just now) sends $\delta_{x_i}\in C^*(\mathbb{Z}^n,\omega_\theta)$ to U_i where $x_i=(0,\cdots,1,\cdots,0)$, 1 being in the i-th coordinate. Denote by θ' the $(n-1)\times(n-1)$ upper left corner of θ. So the generators of $A_{\theta'}$ are given by $U_1,\dots,U_{n-1}$.

Let φ be the automorphism on $A_{\theta'}$ given by $\varphi(U_j)=e^{2\pi i\theta_{j,n}}U_j$ for $j=1,\dots,(n-1)$ which is homotopic to the identity automorphism.

Proposition 2.4.5. $A_{\theta'}\rtimes_\varphi\mathbb{Z}\cong A_\theta$

Proof. Follows from generators and relations of both algebra. □

For the crossed product algebra $A_{\theta'}\rtimes_\varphi\mathbb{Z}$, the Pimsner–Voiculescu sequence is

$$\begin{array}{ccccc}\mathrm{K}_0(A_{\theta'}) & \xrightarrow{\mathrm{id}-\mathrm{K}_0(\varphi^{-1})} & \mathrm{K}_0(A_{\theta'}) & \xrightarrow{i_0} & \mathrm{K}_0(A_\theta)\\ \uparrow & & & & \downarrow\\ \mathrm{K}_1(A_\theta) & \xleftarrow[i_1]{} & \mathrm{K}_1(A_{\theta'}) & \xleftarrow[\mathrm{id}-\mathrm{K}_1(\varphi^{-1})]{} & \mathrm{K}_1(A_{\theta'}).\end{array}$$

Since φ is homotopic to the identity, $\mathrm{id} - \mathrm{K}_i(\alpha_\theta^{-1})$ is the zero map. Hence we have reduced the computation of $\mathrm{K}_0(A_\theta)$ to $\mathrm{K}_0(A_{\theta'})$, and $A_{\theta'}$ is an $n-1$ dimensional noncommutative torus. Since $\mathrm{K}_0(C(\mathbb{T})) = \mathrm{K}_1(C(\mathbb{T})) = \mathbb{Z}$, by induction on n we get

$$\mathrm{K}_0(A_\theta) \cong \mathbb{Z}^{2^{n-1}} \cong \mathrm{K}_1(A_\theta). \tag{2.4.3}$$

Remark 2.4.6. *Notice that* $\mathrm{K}_0(A_\theta)$ *is independent of the parameter* θ. *We will investigate this phenomenon more conceptually later.*

Remark 2.4.7. *When* $\theta = 0$, *one can also use the Künneth formula and the fact* $\mathrm{K}_0(C(\mathbb{T})) = \mathrm{K}_1(C(\mathbb{T})) = \mathbb{Z}$ *to compute the* K*-theory of* $A_\theta = A_0$ *which is just* $C(\mathbb{T}^n)$ *by Pontryagin duality.*

2.5 Some application of the twisted Baum-Connes conjecture

We have just seen that the K-theory of $A_\theta = C^*(\mathbb{Z}^n, \omega_\theta)$ is independent of θ. In other words, K-theory is rigid under deformation of the group $\mathbb{Z}^n$. As the Baum-Connes conjecture is an incarnation of Pimsner–Voiculescu exact sequence, this phenomenon can be explained for more general groups using the Baum-Connes conjecture. The main idea used in this explanation is homotopy invariance of KK^G.

Let us denote the set of all 2-cocycles of a group G with values in $\mathbb{R}$ (defined in the same way as if taking values in $\mathbb{T}$) by $Z^2(G, \mathbb{R})$. If we have an element $c \in Z^2(G, \mathbb{R})$, we can define the element $\omega_c \in Z^2(G, \mathbb{T})$ by $\omega_c(x, y) = e^{\pi i c(x,y)}$.

Definition 2.5.1. *An element* $[\omega] \in \mathrm{H}^2(G, \mathbb{T})$ *is called* real *if there exists a cocycle* $c \in Z^2(G, \mathbb{R})$ *such that* ω *is cohomologous to* ω_c.

The main theorem ([22, Corollary 1.13]) of this section is the following:

Theorem 2.5.2. *Let* G *be a group which satisfies the twisted version of the Baum-Connes conjecture for all 2-cocycles of* G, *and let* $\omega \in Z^2(G, \mathbb{T})$ *be a cocycle such that* $[\omega]$ *is real. Then* $\mathrm{K}_j(C^*(G, \omega)) \cong \mathrm{K}_j(C^*(G))$.

Idea of the proof. Using that $[\omega]$ is real, one defines a homotopy (a continuous path of 2-cocycles in $Z^2(G, \mathbb{T})$) between $[\omega]$ and the trivial cocycle. Since G satisfies the twisted version of the Baum-Connes conjecture for all 2-cocycles of G (in particular for the 2-cocycles lying at the continuous path), using the left hand side of the Baum–Connes conjecture and homotopy invariance one can prove the theorem. For details we refer to [22, Corollary 1.13].

□

Problem 2.5.3. *One can also define a twisted version of Geometric* K*-homology as in 2.3.2. The author in* [4] *describes how equivariant geometric* K*-homology of a discrete group can be twisted by a 2-cocycle. It would be interesting to see how the above theorem makes sense using the geometric LHS of the twisted Baum-Connes conjecture (*[4]*).*

Coming back to the example considered in 2.4.3, let θ be any skew symmetric real $n \times n$ matrix. Then we have defined $A_\theta = C^*(\mathbb{Z}^n, \omega_\theta)$.

Since $[\omega_\theta]$ is a real cocycle (by definition of the 2-cocycle), Theorem 2.5.2 immediately gives the computation of the K-theory of a higher dimensional noncommutative torus, as in 2.4.3. The K-theory in the untwisted case is computed using the Künneth formula.

We want to describe one more example, which shows the power of the above theorem.

Example 2.5.4. *Suppose $W \in GL_n(\mathbb{Z})$ is a matrix of finite order. Let $F := \langle W \rangle$ be the finite group (generated by W) acting on $\mathbb{Z}^n$ by matrix multiplication and let θ be an $n \times n$ real skew-symmetric matrix such that $W^T \theta W = \theta$. Let $G := \mathbb{Z}^n \rtimes_W F$. We call such group G a crystallographic group. Then we can define a 2-cocycle ω'_θ on G by*

$$\omega'_\theta((x, s), (y, t)) = \omega_\theta(x, s \cdot y). \tag{2.5.1}$$

It is easy to check that ω'_θ satisfies the 2-cocycle conditions. Similar to the previous case, $[\omega'_\theta]$ is also real. So the K*-theory of $C^*(G, \omega'_\theta)$ is same as the* K*-theory of $C^*(G)$.*

One might say that 2.5.1 is not the only way to define 2-cocycle on G. Indeed, there are other ways of defining cocycles on G. We will shortly see that K-theory changes when we define 2-cocycle on G in a slightly different way.

Let us consider the action of $\mathbb{Z}_2$ on $\mathbb{Z}^3$ by the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Also consider the cocycle ω_θ on $\mathbb{Z}^3$ where

$$\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta_1 \\ 0 & -\theta_1 & 0 \end{pmatrix}.$$

Let us construct the following cocycle ω on $G = \mathbb{Z}^3 \rtimes_A \mathbb{Z}_2$. Write $(x_1, x_2, x_3) = x \in \mathbb{Z}^3$ and $(y_1, y_2, y_3) = y \in \mathbb{Z}^3$. Define

$$\omega((x,s),(y,t)) = \begin{cases} -\omega'_\theta((x,s),(y,t)) & \text{when } s = -1, y_1 \text{ is odd}, \\ \omega'_\theta((x,s),(y,t)) & \text{otherwise}, \end{cases}$$

where ω'_θ is as before. It is checked that $C^*(G,\omega)$ is isomorphic to $A_\theta \rtimes_\omega \mathbb{Z}_2$ where the $\mathbb{Z}_2$ action on A_θ is given by

$$U_1 \to -U_1,\ U_2 \to U_2^{-1},\ U_3 \to U_3^{-1}.$$

One can also use [49, Theorem 4.1] to see the above. On the other hand $C^*(G, \omega'_\theta)$ is isomorphic to $A_\theta \rtimes_{\omega'_\theta} \mathbb{Z}_2$ where the $\mathbb{Z}_2$ action on A_θ is given by

$$U_1 \to U_1,\ U_2 \to U_2^{-1},\ U_3 \to U_3^{-1}.$$

And also $A_\theta \rtimes_{\omega'_\theta} \mathbb{Z}_2 \cong C(\mathbb{T}) \otimes (A_{\theta_1} \rtimes \mathbb{Z}_2)$ (A_{θ_1} being the two dimensional noncommutative torus which has the flip action of $\mathbb{Z}_2$). Now if we compute the K-theory of $C^*(G,\omega)$, one sees that there is a torsion in K_1 (see the computation [48, Proposition 3.9]). On the other hand using the Künneth formula, there is no torsion in the K-theory of $C(\mathbb{T}) \otimes (A_{\theta_1} \rtimes \mathbb{Z}_2)$, using the fact that $\mathrm{K}_0(A_{\theta_1} \rtimes \mathbb{Z}_2) = \mathbb{Z}^6, \mathrm{K}_1(A_{\theta_1} \rtimes \mathbb{Z}_2) = 0$ (see [27]).

If one assumes that the action of F on $\mathbb{Z}^n$ is "nice" and free away from the origin, the counterexamples like above do not work. We shall discuss this in detail.

Let us consider the matrix

$$W = \begin{pmatrix} a_{11} & a_{12} & \cdots & & \cdots & a_{1n} \\ a_{21} & \ddots & \ddots & & & a_{21} \\ \vdots & \ddots & & & & \\ & & & & & \\ & & & & \ddots & \vdots \\ a_{(n-1)1} & & & \ddots & \ddots & a_{(n-1)n} \\ a_{n1} & \cdots & & \cdots & a_{n(n-1)} & a_{nn} \end{pmatrix} \in GL_n(\mathbb{Z}).$$

Suppose there is a well defined action $\phi = (\alpha, \lambda, \eta)$ of $F = \langle W \rangle$ on the n-dimensional noncommutative torus A_θ by:

$$U_i \to \lambda_i \eta_i U_1^{a_{1i}} \cdots U_n^{a_{ni}}$$

for $1 \le i \le n$, $\lambda = (\lambda_i)_{1\le i\le n} \in \mathbb{T}$, $\eta = (\eta_i)_{1\le i\le n} \in \mathbb{T}$.

Also let there is an action (α, η) of F given by

$$U_i \to \eta_i U_1^{a_1 i} \cdots U_n^{a_n i}$$

for $1 \le i \le n$.

We have the following theorem.

Theorem 2.5.5. *If W acts on $\mathbb{Z}^n$ freely (away from the origin), then $A_\theta \rtimes_{\alpha,\lambda,\eta} F$ is isomorphic to $A_\theta \rtimes_{\alpha,\eta} F$.*

Proof. Let

$$\alpha = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & & \cdots & \alpha_{1n} \\ \alpha_{21} & \ddots & \ddots & & & \alpha_{21} \\ \vdots & \ddots & & & & \\ & & & & \ddots & \vdots \\ \alpha_{(n-1)1} & & & \ddots & \ddots & \alpha_{(n-1)n} \\ \alpha_{n1} & \cdots & & \cdots & \alpha_{n(n-1)} & \alpha_{nn} \end{pmatrix} \in M_n(\mathbb{Q}).$$

For each i, $1 \le i \le n$, let us define the following unitaries

$$U_i' = \lambda_1^{\alpha_{1i}} \lambda_2^{\alpha_{2i}} \cdots \lambda_n^{\alpha_{ni}} U_i.$$

Clearly $U_i', 1 \le i \le n$ generate A_θ.

Now

$$\phi(U_i') = \lambda_1^{\alpha_{1i}} \lambda_2^{\alpha_{2i}} \cdots \lambda_n^{\alpha_{ni}} \lambda_i \eta_i U_1^{a_{1i}} \cdots U_n^{a_{ni}}.$$

A simple computation shows that

$$\phi(U_i') = \mu_i \eta_i U_1'^{a_{1i}} \cdots U_n'^{a_{ni}}$$

where

$$\begin{aligned} \mu_i = {} & \lambda_1^{\alpha_{1i} - \alpha_{11} a_{1i} - \alpha_{12} a_{2i} \cdots - \alpha_{1n} a_{ni}} \\ & \lambda_2^{\alpha_{2i} - \alpha_{21} a_{1i} - \alpha_{22} a_{2i} \cdots - \alpha_{2n} a_{ni}} \cdots \\ & \lambda_i^{(\alpha_{ii}+1) - \alpha_{i1} a_{1i} - \alpha_{i2} a_{2i} \cdots - \alpha_{in} a_{ni}} \cdots \\ & \lambda_n^{\alpha_{ni} - \alpha_{n1} a_{1i} - \alpha_{n2} a_{2i} \cdots - \alpha_{nn} a_{ni}}. \end{aligned}$$

Now

$$\mu_i = \lambda_1^{-\alpha_{11}a_{1i}-\alpha_{12}a_{2i}\cdots+\alpha_{1i}(1-a_{ii})\cdots-\alpha_{1n}a_{ni}} \lambda_2^{-\alpha_{21}a_{1i}-\alpha_{22}a_{2i}\cdots+\alpha_{2i}(1-a_{ii})\cdots-\alpha_{2n}a_{ni}} \cdots \lambda_i^{1-\alpha_{i1}a_{1i}-\alpha_{i2}a_{2i}\cdots+\alpha_{ii}(1-a_{ii})\cdots-\alpha_{in}a_{ni}} \cdots \lambda_n^{-\alpha_{n1}a_{1i}-\alpha_{n2}a_{2i}\cdots+\alpha_{ni}(1-a_{ii})\cdots-\alpha_{nn}a_{ni}}.$$

If we want $\mu_i = 1$ for all i, we can put exponents of λ_m to be zero for $1 \leq m \leq n$, for each μ_i. This boils down to a system of equation given by the matrix equation $\alpha^t W' = \mathrm{Id}$, where

$$W' = \begin{pmatrix} a_{11}-1 & a_{12} & \cdots & & \cdots & a_{1n} \\ a_{21} & a_{22}-1 & \ddots & & & a_{21} \\ \vdots & \ddots & & & & \\ & & & & & \\ & & & & \ddots & \vdots \\ a_{(n-1)1} & & & \ddots & \ddots & a_{(n-1)n} \\ a_{n1} & \cdots & & \cdots & a_{n(n-1)} & a_{nn}-1 \end{pmatrix}.$$

By Cramer's rule, we have a solution for $\mu_i = 1$, for all i, of the variable (α_{ij}) under the condition

$$\det(W') \neq 0.$$

Note that, since the matrix W' has integer entries, and as we use elementary row operations to get a solution, we always get a rational solution. Since W acts freely (away from zero) on $\mathbb{Z}^n$, the above condition is satisfied. So for the variables U_i', the action ϕ corresponds to the action α. Hence $A_\theta \rtimes_{\alpha,\lambda,\eta} F$ is isomorphic to $A_\theta \rtimes_{\alpha,\eta} F$.

□

Let W be as above. For the following corollaries, we describe $\mathrm{H}^2(\mathbb{Z}^n \rtimes_W F, \mathbb{T})$ first. Borrowing notations from above, any cocycle in $\mathrm{H}^2(\mathbb{Z}^n \rtimes_W F, \mathbb{T})$ is cohomologus to a cocycle ω given by

$$\omega((x,s),(y,t)) = \omega_\theta(x, s\cdot y) f(s, s\cdot y),$$

where f is a function from $F \times \mathbb{Z}^n$ to $\mathbb{T}$ satisfying

$$f(st,x) = f(t,x) f(s, t^{-1}\cdot x),$$

$$\omega_\theta(s\cdot x, s\cdot y) = \omega_\theta(x,y) f(s, s\cdot(x+y))(f(s,s\cdot x)f(s,s\cdot y))^{-1}.$$

See [50, Page 715] for the proof of above fact. Since the above ω is dependent on θ and f, we denote it by $\omega_{\theta,f}$. Note that when $f\equiv 1$, then $\omega_{\theta,f}$ is exactly ω'_θ which was discussed earlier. Note that $f(s,0)=1$ and $f(1,x)=1$, which imply $\omega_{\theta,f}$ is ω_θ and trivial when restricted to $\mathbb{Z}^n$ and F, respectively.

For the following corollaries assume that $\omega_\theta(s\cdot x, s\cdot y) = \omega_\theta(x,y)$, i.e.

$$f(s, s\cdot(x+y))(f(s,s\cdot x)f(s,s\cdot y))^{-1} = 1.$$

Corollary 2.5.6. $C^*(\mathbb{Z}^n \rtimes_W F, \omega_{\theta,f}) \cong C^*(\mathbb{Z}^n \rtimes_W F, \omega'_\theta)$.

Proof. From [49, Theorem 4.1], $C^*(\mathbb{Z}^n \rtimes_W F, \omega'_\theta)$ is isomorphic to $A_\theta \rtimes_{\alpha,\eta} F$, and $C^*(\mathbb{Z}^n \rtimes_W F, \omega_{\theta,f})$ is isomorphic to $A_\theta \rtimes_{\alpha,\lambda,\eta} F$, for set of scalars $\lambda = (\lambda_i)_{1\le i\le n} \in \mathbb{T}$, $\eta = (\eta_i)_{1\le i\le n} \in \mathbb{T}$. By the above theorem they are isomorphic. □

Corollary 2.5.7. $\mathrm{K}_*(C^*(\mathbb{Z}^n \rtimes_W F, \omega_{\theta,f})) \cong \mathrm{K}_*(C^*(\mathbb{Z}^n \rtimes_W F))$.

Remark 2.5.8. *Though in the above corollaries we assumed that $\omega_\theta(s\cdot x, s\cdot y) = \omega_\theta(x,y)$, the actions in Example 2.5.4 satisfy this property with respect to the 2-cocycles of $\mathbb{Z}^n$ (Example 2.1.8).*

2.6 K-theory of twisted crystallographic group algebras

Let us continue the discussion on crystallographic group algebras. Recall from Example 2.5.4 that a crystallographic group G is of the form $\mathbb{Z}^n \rtimes F$, for some finite cyclic group group $F = \mathbb{Z}_m$ acting on $\mathbb{Z}^n$. So G fits the following exact sequence

$$1 \to \mathbb{Z}^n \to G \to F \to 1.$$

Let us assume that the action of F on $\mathbb{Z}^n$ is free away from the origin. Then Lück and Langer, in [37], computed the K-theory of group C*-algebras of these groups using the Baum–Connes conjecture. Let us recall their construction. One may take $\mathbb{R}^n$ as a model of $\underline{E}G$ (universal space of the proper G-actions), hence $G\backslash\mathbb{R}^n$ as a one for $\underline{B}G = G\backslash\underline{E}G$ (note that $\mathbb{R}^n$ has a canonical action of $\mathbb{Z}^n \rtimes F$). Let $\mathcal{M}$ denote the set of conjugacy classes of maximal finite subgroups of G. Let us write $C^*(P)$ as $\mathbb{C}[P]$ for $P\in\mathcal{M}$. In the following theorem, $\tilde{\mathrm{K}}_0(\mathbb{C}[P])$ denotes the reduced K_0-group, that is, the kernel of the map $\mathrm{K}_0(\mathbb{C}[P]) \to \mathrm{K}_0(\mathbb{C})$ induced by the trivial representation. We then have the following theorem

Theorem 2.6.1 ([37]). *In the even degree, there exists an exact sequence*

$$0 \longrightarrow \bigoplus_{P \in \mathcal{M}} \tilde{\mathrm{K}}_0(\mathbb{C}[P]) \longrightarrow \mathrm{K}_0(C^*(G)) \longrightarrow \mathrm{K}_0(\underline{B}G) \longrightarrow 0, \tag{2.6.1}$$

and in the odd degree, we have $\mathrm{K}_1(C^*(G)) \simeq \mathrm{K}_1(\underline{B}G)$.

Let us further assume that $p = m$ is prime. Then any non-trivial finite subgroup P of G must be isomorphic to $\mathbb{Z}_p$ via the restriction of the projection map $G \to \mathbb{Z}_p$. Thus any nontrivial finite subgroup of G represents an element of $\mathcal{M}$. We have the following precise description of the K-groups.

Theorem 2.6.2 ([37]). *The number* $n' = n/(p-1)$ *is an integer, and* $|\mathcal{M}| = p^{n'}$. *The rank of the* K*-homology groups of* $\underline{B}G$ *are given by*

$$\text{rk } \mathrm{K}_0(\underline{B}G) = \frac{2^n + p - 1}{2p} + \frac{(p-1)p^{n'-1}}{2},$$
$$\text{rk } \mathrm{K}_1(\underline{B}G) = \frac{2^n + p - 1}{2p} - \frac{(p-1)p^{n'-1}}{2}.$$

Theorem 2.6.3 ([37]). $\mathrm{K}_i(C^*(G))$ *is torsion free and*

$$\text{rk } \mathrm{K}_0(C^*(G)) = \frac{2^n + p - 1}{2p} + \frac{(p-1)p^{n'-1}}{2} + (p-1)p^{n'},$$
$$\text{rk } \mathrm{K}_1(C^*(G)) = \frac{2^n + p - 1}{2p} - \frac{(p-1)p^{n'-1}}{2}.$$

Thus we know the dimensions of the K-theory groups of the group algebras of $G = \mathbb{Z}^n \rtimes \mathbb{Z}_p$. As we have noticed in Example 2.5.4, real cocycles of G do not give different K-theory of twisted group algebra of G, we get the understanding of $\mathrm{K}_i(C^*(G,\omega))$, for exponential cocycle ω. Though it is very important to get the explicit description (description of the generators) of $\mathrm{K}_i(C^*(G,\omega))$, which is, in general, very difficult question. We will see, to which extent we can get an explicit description of $\mathrm{K}_i(C^*(G,\omega))$, in terms of explicit K-theory elements.

First we want to explain how projections in $\tilde{\mathrm{K}}_0(\mathbb{C}[P])$ contribute to $\mathrm{K}_0(C^*(G)$ and $K_0(C^*(G,\omega'_\theta))$. Since $\mathbb{C}[P]$ is isomorphic to the algebra $C(\hat{P}) \simeq \mathbb{C}^p$, $\mathrm{K}_0(\mathbb{C}[P])$ is the free abelian group of rank p. Now suppose g is a generator of P. The minimal projections of $\mathbb{C}[P]$ are given by

$$Q_{j,g} = \frac{1}{p} \sum_{k=0}^{p-1} \exp\left(i\frac{\pi}{p} jk\right) \lambda_{g^k}$$

for $j = 0, \cdots, p-1$, which also represent a basis of $\mathrm{K}_0(\mathbb{C}[P])$. Since the projections sum up to one in $\mathrm{K}_0(\mathbb{C}[P])$, a basis of $\tilde{\mathrm{K}}_0(\mathbb{C}[P])$ is given by $Q_{0,g}, \cdots, Q_{p-2,g}$.

By the above theorem, these projections are still linearly independent in $\mathrm{K}_0(C^*(G))$. To get elements of $\mathrm{K}_0(C^*(G, \omega'_\theta))$ (notations from 2.5.1), we need to modify this presentation a bit. Continuing to denote a generator of P by g, the unitary λ_g has order p in $C^*(G)$. But since we modified the product in $C^*(G, \omega'_\theta)$, the unitary $\lambda_g^{(\omega'_\theta)}$ need not be so. Still, since ω'_θ is cohomologically trivial on the finite group P and hence $C^*(P, \omega'_\theta) \simeq \mathbb{C}[P]$, we can always multiply suitable $z \in \mathbb{T}$ so that the order of the unitary $z\lambda_g^{(\omega'_\theta)}$ is p. Then a similar formula

$$Q_{j,g}^{(\theta)} = \frac{1}{p} \sum_{k=0}^{p-1} \exp\left(i\frac{\pi}{p}jk\right) z^k \lambda_{g^k}^{(\omega'_\theta)}$$

for $j = 1, \cdots, p-1$ will give projections which are elements of $\mathrm{K}_0(C^*(G, \omega'_\theta))$.

Now how $\mathrm{K}_0(\underline{B}G)$ contributes to the $\mathrm{K}_0(C^*(G, \omega'_\theta))$, is very hard to understand and is not known in general. Echterhoff et al in [22] started studying this question. For $n = 2$, they showed how Equation 6.4.3 can be understood in a C*-algebraic way. We shall try to understand this question in the later part of this thesis: as it is beyond the scope of this introductory chapter.

2.7 Locally convex algebras and m-algebras

In noncommutative geometry, as we also study geometry of noncommutative space, one must not stick to only continuous functions on noncommutative space but one should also study smooth functions. This gives much more geometric information. In this section we will introduce smooth group algebras which are in a way more nicely behaved compared to the twisted group C*-algebra. These smooth algebras are locally multiplicatively closed convex algebras, or m-algebras, which form a subclass in the category of locally convex algebras. We start by recalling the basic definitions.

2.7.1 m-algebras

Recall that a *locally convex algebra* is a complete locally convex space A endowed with a continuous multiplication $A \times A \to A$. A *locally multiplicatively convex algebra*, or an *m-algebra* A is a locally convex algebra whose topology is determined by a family of seminorms $\{q_\alpha\}_{\alpha \in I}$, such that each q_i is submultiplicative, i.e. :

$$q_\alpha(ab) \leqslant q_\alpha(a) q_\alpha(b) \quad \text{for all } a, b \in A,\ \alpha \in I.$$

The m-algebras form a full subcategory $\mathfrak{mAlg}$ in the category of locally convex algebras.

A sequence $0 \to A \xrightarrow{p} B \xrightarrow{\pi} C \to 0$ in $\mathfrak{mAlg}$ is *exact* if it is exact in the algebraic sense and the image $p(A)$ is closed in B. It is *weakly split* if there exists a continuous linear *splitting map* $\sigma\colon C \to B$ such that $\pi \circ \sigma = \mathrm{id}_C$ and *split exact* if there exists a splitting map which is a homomorphism.

The projective tensor product $A \hat{\otimes} B = A \hat{\otimes}_\pi B$ of m-algebras has a natural structure of an m-algebra, thus turning $\mathfrak{mAlg}$ into a monoidal category.

To spare the notation, we shall from now on denote the seminorms by $\|\cdot\|_\alpha$. Also, in this paper we often shall, for the sake of simplicity, restrict our attention to *Fréchet* m-algebras, i.e. those on which the topology is given by a countable system of seminorms. In this case we may always have the seminorms being indexed by $\mathbb{Z}_{>0}$ and assume $\|a\|_m \leq \|a\|_n$, for $m \leq n$, as we may iteratively replace them by $\|a\|'_{m+1} := \max\{\|a\|_m, \|a\|_{m+1}\}$.

The algebra $\mathrm{C}^\infty([0,1])$ of smooth functions on the unit interval is an m-algebra with respect to the system of (semi)norms

$$\|f\|_m = \|f\| + \|f'\| + \left\|\frac{f''}{2}\right\| + \cdots + \left\|\frac{f^{(m)}}{m!}\right\|.$$

The *cylinder algebra* is the subalgebra $\mathscr{Z}A \subseteq \mathrm{C}^\infty([0,1], A)$ consisting of A-valued functions whose derivatives of order ≥ 1 vanish at the endpoints.

Definition 2.7.1. *Two morphisms $\phi_0, \phi_1\colon A \to B$ of m algebras are called* diffeotopic *if there exists a morphism $\phi\colon A \to \mathscr{Z}B$ called* diffotopy, *such that* $\mathrm{ev}_0 \circ \phi = \phi_0$, $\mathrm{ev}_1 \circ \phi = \phi_1$.

We will now discuss some relevant examples of m-algebras.

2.7.2 Smooth compacts

Definition 2.7.2. *We define the algebra $\mathscr{K}^\infty$ of* smooth compacts *by taking the completion of the algebra $\mathbb{M}(\mathbb{Z}_{>0})$ (algebra of $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ matrices) in the system of seminorms given by:*

$$\|(a_{ij})\|_m := \sum_{i,j} |1+i|^m |1+j|^m |a_{ij}|, \quad m \in \mathbb{Z}_{\geq 0}.$$

It is straightforward to verify that the algebra $\mathscr{K}^\infty$ is an m-algebra.

2.7.3 Smooth Group Algebras

Definition 2.7.3. *A* length function, *or simply* length *on a discrete group G is a function $L\colon G \to \mathbb{R}_+$, such that for all $g, g' \in G$*

- $L(gg') \leq L(g) + L(g')$,
- $L(g^{-1}) = L(g)$,
- $L(e) = 0$ *where e is the identity of G.*

Two length functions $L, L' \colon G \to \mathbb{R}_+$ are *equivalent* if there exist positive constants c_1, c_2, C_1, C_2 such that for all $g \in G$

$$-c_1 + c_2 L(g) \leq L'(g) \leq C_1 + C_2 L(g).$$

It is convenient to assume length functions on groups to be integer-valued. Note that for any length function L there exists an equivalent integer-valued length L', e.g. $L'(g) = \lceil L(g) \rceil$.

Definition 2.7.4. *Let G be a discrete group and let L be a length function on G. We denote by $\mathcal{S}(G) = \mathcal{S}_L(G)$ the space of functions $\gamma = \sum_{g \in G} c_g \delta_g$ on G such that for all $m \in \mathbb{Z}_{>0}$ the following holds:*

$$\|\gamma\|_m = \|\gamma\|_{m,G,L} \coloneqq \sum_{g \in G} |c_g| (1 + L(g))^m < \infty.$$

Remark 2.7.5. *For equivalent length functions we get isomorphic algebras.*

Remark 2.7.6. *The algebra $\mathcal{S}(G)$ coincides with $H_L^{1,\infty}(G)$ introduced in* [33, Defn. 2.1]. *It is verified directly (*[33]*) that $\mathcal{S}(G)$ is an m-algebra with respect to the norms $\|\cdot\|_{m,L}$ and the convolution product.*

The algebra $\mathcal{S}(G)$ is a dense subalgebra of $\ell^1(G)$. Moreover, according to [33, Prp. 2.3, Cor. 2.4] *it is stable under holomorphic functional calculus on $\ell^1(G)$ and thus has the same* K*-theory. In* [33]*, Jolissaint also defined another smooth algebra $H_L^\infty(G)$, with respect to a length L on the group G, which is presently widely used in operator theory. A group G is said to have property (RD) if the reduced group C^*-algebra $C^*(G)$ contains $H_L^\infty(G)$. If the group has (RD), then $H_L^\infty(G)$ is dense and stable under holomorphic functional calculus in $C^*(G)$, which in some cases may simplify the calculations of* K*-theory for the latter algebra. However, in general there seems to be no natural m-algebra structure on $H_L^\infty(G)$, therefore we choose $H_L^{1,\infty}(G)$ as an example.*

The algebra $H_L^{1,\infty}(G)$ is a dense subalgebra of $H_L^\infty(G)$, yet not necessarily stable under holomorphic functional calculus in the latter. In the case when G is finitely generated and has polynomial growth, both algebras coincide.

2.7.4 Smooth (twisted) group algebras of $\mathbb{Z}^n$ and crystallographic groups

For $G = \mathbb{Z}^n$ or $G = \mathbb{Z}^n \rtimes F$ (for the choice of natural length function) we know that various classes of rapid decay functions coincide [33]. $\mathcal{S}(\mathbb{Z}^n)$ turns out to be isomorphic to $C^\infty(\mathbb{T}^n)$, the algebra of smooth functions on the torus.

Motivated by twisted group C*-algebras, one can also define a twisted version of $\mathcal{S}(G)$ which is denoted by $\mathcal{S}(G, \omega)$, $\omega \in \mathrm{H}^2(G, \mathbb{T})$. See [44] for an account of that. Also the author in [44] proved that for a nice class of groups (which includes $\mathbb{Z}^n, \mathbb{Z}^n \rtimes F$), $\mathcal{S}(G, \omega)$ is holomorphically closed in $C^*(G, \omega)$.

2.7.5 K-theory and cyclic (co-)homology of m-algebras

The K-theory of m-algebras was first defined by Phillips [51]. Cuntz in [17] defined a bivariant kk-theory on $\mathfrak{mAlg}$, which is a variant of Kasparov's KK-theory on the category of m-algebras. The K-theory satisfies the usual properties (smooth version) which are satisfied by K-theory in the category of C*-algebras :

1. Stability: $\mathrm{K}_i(A \otimes \mathscr{K}^\infty) \cong \mathrm{K}_i(A)$
2. Bott periodicity: $\mathrm{K}_{i+2}(A) \cong \mathrm{K}_i(A)$
3. Diffeotopy-invariance: if ϕ is a diffeotopy of two m-algebras, then $\mathrm{K}_i(\phi_0) = K_i(\phi_1)$
4. Excision: if I is a closed ideal of A then there exists a six term exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(I) & \longrightarrow & \mathrm{K}_0(A) & \longrightarrow & \mathrm{K}_0(A/I) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A/I) & \longleftarrow & \mathrm{K}_1(A) & \longleftarrow & \mathrm{K}_1(I) \end{array} \tag{2.7.1}$$

5. Morita stability: we will discuss this point later.

In the category $\mathfrak{mAlg}$, or more generally in the category of locally convex algebras one can define another functor HP, periodic cyclic cohomology, which has similar properties as K-theory (at least in $\mathfrak{mAlg}$). Periodic cyclic (co-)homology was first defined by Connes for locally convex algebras. He showed that it is a noncommutative generalisation of the classical de-Rham theory. The explicit theorem that connects HP to the de-Rham theory is so-called Hoschild-Konstant-Rosenberg theorem. Similar to the classical Chern character, Connes constructed the Chern character from HP to K-theory. Cuntz later defined a bivariant version of HP_* (which we denote by HP_* again) and proved that there exists a (unique in some sense) Chern character on the category of m-algebras

$$ch : \mathrm{kk}_* \to \mathrm{HP}_*$$

In $\mathfrak{mAlg}$, Cuntz also proved excision for bivariant HP, in both variables.

Theorem 2.7.7. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of m-algebras. Then excision holds for HP_*.*

$$\begin{CD} \mathrm{HP}_0(I) @>>> \mathrm{HP}_0(A) @>>> \mathrm{HP}_0(A/I) \\ @AAA @. @VVV \\ \mathrm{HP}_1(A/I) @<<< \mathrm{HP}_1(A) @<<< \mathrm{HP}_1(I), \end{CD} \tag{2.7.2}$$

where the maps (except vertical ones) are induced from the maps in the short exact sequence.

Now we recall the versions of the Connes–Thom isomorphism and the Pimsner–Voiculescu exact sequence for m-algebras (and also for locally convex algebras). By an automorphism of a locally convex algebra we mean an automorphism which is isometric in each seminorm. For $\mathbb{Z}$- and $\mathbb{R}$-action on m-algebras A, one can also define smooth versions of the crossed products $A \rtimes \mathbb{Z}$ and $A \rtimes \mathbb{R}$, which are again m-algebras.

Theorem 2.7.8. [23] *[Elliot-Natsume-Nest] For an action $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$, we have*

$$\mathrm{HP}_j(A) \cong \mathrm{HP}_{1+j}(A \rtimes_\alpha \mathbb{R}). \tag{2.7.3}$$

In 2.7.3, when the $\mathbb{R}$-action is trivial, we have $\mathrm{HP}_j(A) \cong \mathrm{HP}_{1+j}(A \otimes \mathcal{S}(\mathbb{R}))$, which is the Bott periodicity for HP.

Theorem 2.7.9. [47]*[Nest] If $\alpha \in \mathrm{Aut}(A)$, one has the following exact sequence (in the homology level)*

$$\begin{CD} \mathrm{HP}_0(A) @>{\mathrm{id}-\mathrm{HP}_0(\alpha^{-1})}>> \mathrm{HP}_0(A) @>{i_0}>> \mathrm{HP}_0(A \rtimes_\alpha \mathbb{Z}) \\ @AAA @. @VVV \\ \mathrm{HP}_1(A \rtimes_\alpha \mathbb{Z}) @<<{i_1}< \mathrm{HP}_1(A) @<<{\mathrm{id}-\mathrm{HP}_1(\alpha^{-1})}< \mathrm{HP}_1(A) \end{CD} \tag{2.7.4}$$

And in the cohomology, we have

$$\begin{array}{ccccc} \mathrm{HP}^0(A) & \xleftarrow[\mathrm{id}-\mathrm{HP}^0(\alpha)]{} & \mathrm{HP}^0(A) & \xleftarrow[i_0]{} & \mathrm{HP}^0(A \rtimes_\alpha \mathbb{Z}) \\ \downarrow & & & & \uparrow \\ \mathrm{HP}^1(A \rtimes_\alpha \mathbb{Z}) & \xrightarrow{i_1} & \mathrm{HP}^1(A) & \xrightarrow{\mathrm{id}-\mathrm{HP}^1(\alpha)} & \mathrm{HP}^1(A) \end{array} \tag{2.7.5}$$

where i_0, i_1 are induced from inclusions.

Example 2.7.10. *Using the above exact sequence one can easily compute the periodic cyclic (co)-homology groups for smooth noncommutative tori. Recall that a smooth noncommutative torus is the smooth group algebra $A_\theta^\infty := \mathcal{S}(\mathbb{Z}^{n+1}, \omega_\theta)$, which also can be written as the smooth crossed product $\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'}) \rtimes \mathbb{Z}$, where the action is smoothly homotopic to the trivial action (just like the C^*-case). Using the computations similar to* K*-theory computations of noncommutative tori, one gets the following two short exact sequences:*

$$0 \longrightarrow \mathrm{HP}_0(\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'})) \longrightarrow \mathrm{HP}_0(\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'}) \rtimes \mathbb{Z}) \longrightarrow \mathrm{HP}_1(\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'})) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{HP}_1(\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'})) \longrightarrow \mathrm{HP}_1(\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'}) \rtimes \mathbb{Z}) \longrightarrow \mathrm{HP}_0(\mathcal{S}(\mathbb{Z}^n, \omega_{\theta'})) \longrightarrow 0$$

Using induction on n, and knowing that

$$\mathrm{HP}_0(C^\infty(\mathbb{T})) = \mathbb{C}, \ \mathrm{HP}_1(C^\infty(\mathbb{T})) = \mathbb{C},$$

we get

$$\mathrm{HP}_0(A_\theta^\infty) = \mathbb{C}^{2^{(n-1)}}, \ \mathrm{HP}_1(A_\theta^\infty) = \mathbb{C}^{2^{(n-1)}}.$$

Remark 2.7.11. *Since the Chern character from $\mathrm{K}_*(C^\infty(\mathbb{T}^2)) \otimes \mathbb{C}$ to $\mathrm{HP}_*(C^\infty(\mathbb{T}^2))$ is an isomorphism (by a result of Baum and Connes, see* [58, Page 279-80, equation 11 and 13]*), we also know $\mathrm{HP}_*(C^\infty(\mathbb{T}^2))$ from the $\mathrm{K}_*(C^\infty(\mathbb{T}^2))$, which was discussed earlier. Using this one can compute the $\mathrm{HP}^*(C^\infty(\mathbb{T}^2)$, which results to the same result as in the above example for the case $\theta = 0$.*

3 Noncommutative tori and K-theory

This chapter is organised as follows:

In the first section of this chapter we recall some basics of groupoids, twisted groupoid C*-algebras and their K-theory. In the second section we construct the continuous field of projective modules and prove our main theorem. In the third section we show how our theorem could be used to give explicit generators of the K_0 groups of the noncommutative tori: we work out the three and four dimensional cases in detail.

Notation: $e(x)$ denotes the number $e^{2\pi i x}$.

3.1 Twisted groupoid algebras and their K-theory

We assume that the reader is familiar with basic notions of locally compact Hausdorff groupoids. We refer to the book of Renault [53] for a basic course on groupoids and representations of those. To introduce the notations we recall the definition of 2-cocycle on a groupoid.

Definition 3.1.1. *Let G be a locally compact Hausdorff groupoid. A continuous map $\omega : G^{(2)} \to \mathbb{T}$ is called a 2-cocycle if*

$$\omega(x, y)\omega(xy, z) = \omega(x, yz)\omega(y, z),$$

whenever $(x, y), (y, z) \in G^{(2)}$ and

$$\omega(x, d(x)) = 1 = \omega(t(x), x),$$

for any $x \in G$, where $G^{(2)}$ denotes the composable pairs of G and d, t denote the domain and the range map, respectively.

Definition 3.1.2. *The C^*-algebra $C^*(G, \omega)$ is defined to be the enveloping C^*-algebra of the ω-twisted left regular representation of the groupoid G (which is assumed to have a Haar system).*

Example 3.1.3. *Let G be the group (hence groupoid) $\mathbb{Z}^n$. Then for G, the above definition coincides with the definition of 2-cocycle of G. So for an antisymmetric matrix θ, we have noncommutative torus $C^*(\mathbb{Z}^n, \omega_\theta) = A_\theta$. It is well known that* $\mathrm{H}^2(\mathbb{Z}^n, \mathbb{T}) \cong \mathbb{T}^{\frac{n(n-1)}{2}}$.

Let $[a, b]$ be a closed interval. Let us consider the transformation groupoid $\mathbb{Z}^n \times [a, b]$ for trivial $\mathbb{Z}^n$ action on $[a, b]$. Suppose ω_r be a continuous family (with respect to $r \in [a, b]$ and in the topology of $\mathbb{T}^{\frac{n(n-1)}{2}}$) of 2-cocycles on the group $\mathbb{Z}^n$. We define the following 2-cocycle ω on the groupoid $\mathbb{Z}^n \times [a, b]$: $\omega(x, y, r) = \omega_r(x, y)$, when x, y belong to the r-fiber, and when x, y do not belong to same r-fiber, then $\omega(x, y, r)$ is defined to be zero. Then we have the following evaluation map

$$ev_r : C^*(\mathbb{Z}^n \times [a, b], \omega) \to C^*(\mathbb{Z}^n, \omega_r), \;\; r \in [a, b].$$

The following theorem is due to Echterhoff et al. [22].

Theorem 3.1.4. *Let $[p_1], [p_2], \cdots, [p_m] \in \mathrm{K}_0(C^*(\mathbb{Z}^n \times [a, b], \omega))$. Then the following are equivalent:*

1. $[p_1], [p_2], \cdots, [p_m]$ *form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n \times [a, b], \omega))$.
2. *For some $r \in [a, b]$, the evaluated classes $[ev_r(p_1)], [ev_r(p_2)], \cdots, [ev_r(p_m)]$ form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_r))$.
3. *For every $r \in [a, b]$, the evaluated classes $[ev_r(p_1)], [ev_r(p_2)], \cdots, [ev_r(p_m)]$ form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n, \omega_r))$.

Proof. See remark 2.3 of [22]. □

3.2 Projective modules over bundles of noncommutative tori

As the pfaffian of an even dimensional skew symmetric matrix will play a central role in the construction of our continuous field, we recall the definition of the pfaffian.

Definition 3.2.1. *The pfaffian of a $2p \times 2p$ skew symmetric matrix $A := (a_{ij})$ is a polynomial, denoted by $\mathrm{pf}(A)$, in the entries a_{ij} $(i < j)$ such that $\mathrm{pf}(A)^2 = \det A$ and $\mathrm{pf}(J_0'') = 1$, where J_0'' is the block diagonal matrix constructed from p identical 2×2 blocks of the form*

$$J_0' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So

$$J_0'' = \begin{pmatrix} J_0' & & & \\ & J_0' & & \\ & & \ddots & \\ & & & J_0' \end{pmatrix}.$$

It can be shown that pf(A) always exists and is unique. To give some examples,

$$\text{pf} \begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix} = \theta_{12},$$

$$\text{pf} \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & \theta_{14} \\ -\theta_{12} & 0 & \theta_{23} & \theta_{24} \\ -\theta_{13} & -\theta_{23} & 0 & \theta_{34} \\ -\theta_{14} & -\theta_{24} & -\theta_{34} & 0 \end{pmatrix} = \theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23}.$$

More generally, if $n = 2m$, for

$$\theta := \begin{pmatrix} 0 & \theta_{12} & \cdots & & \cdots & \theta_{1n} \\ -\theta_{12} & \ddots & \ddots & & & \theta_{2n} \\ \vdots & \ddots & & & & \\ & & & & & \\ & & & & \ddots & \vdots \\ -\theta_{1(n-1)} & & & \ddots & \ddots & \theta_{(n-1)n} \\ -\theta_{1n} & \cdots & & \cdots & -\theta_{(n-1)n} & 0 \end{pmatrix},$$

pfaffian of θ is given by $\sum_{\xi}(-1)^{|\xi|}\prod_{s=1}^{m}\theta_{\xi(2s-1)\xi(2s)}$, where the sum is taken over all elements ξ of the permutation group $\mathcal{S}_n$ such that $\xi(2s-1) < \xi(2s)$ for all $1 \leq s \leq m$ and $\xi(1) < \xi(3) < \cdots < \xi(2m-1)$. Define $SP(p, \mathbb{R}) := \{W \in GL_{2p}(\mathbb{R}) : W^t J_0'' W = J_0''\}$. Let us fix a number n. Let $n = 2p + q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{Z}_{>0}$. Recall that a skew-symmetric matrix is totally irrational if the off diagonal entries are rationally linearly independent and not rational. Let us fix any totally irrational $n \times n$ skew symmetric matrix $\psi := \begin{pmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{pmatrix}$ with the top upper $2p \times 2p$ left corner ψ_{11} having positive pfaffian. Also let $\theta := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ be any $n \times n$ skew-symmetric matrix such that it has similar properties as ψ, i.e θ_{11}, the left $2p \times 2p$ corner, has positive pfaffian.

Let $I := [0,1]$ and choose a path γ parametrised by I from ψ to θ in the set of $n \times n$ antisymmetric matrices, where ψ_{11} and θ_{11} are connected by a path γ_{11} in the space of $2p \times 2p$ antisymmetric matrices with positive pfaffian. Since the latter is path connected (being isomorphic to $GL^{+}_{2p}(\mathbb{R})/SP(p,\mathbb{R})$[1]), the choice is always possible. The matrices ψ_{12} and θ_{12}, ψ_{21} and θ_{21}, ψ_{22} and θ_{22} are connected by straight line homotopies, which will be denoted by γ_{12}, γ_{21} and γ_{22}, respectively.

For $r \in I$, we have

$$\gamma(r) = \begin{pmatrix} \gamma(r)_{11} & \gamma(r)_{12} \\ \gamma(r)_{21} & \gamma(r)_{22} \end{pmatrix}$$

(notice that $\gamma(r)_{11}$ is the $2p \times 2p$ block). Let $\mathbb{Z}^n \times I$ be the transformation groupoid with the action of $\mathbb{Z}^n$ on I being trivial. We will construct a 2-cocycle on this groupoid. Fibre-wise, the C*-algebra of this twisted groupoid algebra will be just the n-dimensional noncommutative torus with parameter $\gamma(r)$. Define $\Omega(x,y,r) = e((x \cdot \gamma(r)y)/2)$, when x, y are in the r-fibre ($r \in I$), and $\Omega(x,y,r) = 0$ if x, y belong to different fibres. We will use the same approach of Rieffel [54] to construct finitely generated projective $C^*(\mathbb{Z}^n \times I, \Omega)$-modules, which represent a suitable class $(\mathcal{E}_r)$ of projective modules over $C^*(\mathbb{Z}^n, \Omega_r) := C^*(\mathbb{Z}^n, \gamma(r))$, for each $r \in I$. To do this, we recall some constructions by Rieffel and Schwartz from [40]. Define a new cocycle Ω^{-1} (not to be confused with the inverse of Ω) on the groupoid by setting $\Omega^{-1}(x,y,r) = e((\gamma(r)'x \cdot y)/2)$, where

$$\gamma(r)' = \begin{pmatrix} \gamma(r)_{11}^{-1} & -\gamma(r)_{11}^{-1}\gamma(r)_{12} \\ \gamma(r)_{21}\gamma(r)_{11}^{-1} & \gamma(r)_{22} - \gamma(r)_{21}\gamma(r)_{11}^{-1}\gamma(r)_{12} \end{pmatrix},$$

when x, y are in the r-fibre, otherwise we define $\Omega^{-1}(x,y,r) = 0$. Set $\mathcal{A} = C^*(\mathbb{Z}^n \times I, \Omega)$ and $\mathcal{B} = C^*(\mathbb{Z}^n \times I, \Omega^{-1})$. Then the fibre $\mathcal{B}_r$ of $\mathcal{B}$, at $r \in I$, is the noncommutative torus $C^*(\mathbb{Z}^n, -\gamma(r)')$. Let M be the space $\mathbb{R}^p \times \mathbb{Z}^q$, $G := M \times \hat{M}$ and $\langle \cdot, \cdot \rangle$ the natural $\mathbb{T}$-valued pairing between M and $\hat{M}$, where $\hat{M}$ denotes the Pontryagin dual of M. Consider the space $\mathcal{E}^\infty := \mathscr{S}(M, I)$ consisting of all complex functions on $M \times I$ which are smooth and rapidly decreasing in the first variable and continuous in the second variable in each derivative of the first variable. Denote the set of rapidly decreasing $C(I)$-valued functions on $\mathbb{Z}^n$ by $\mathcal{A}^\infty = \mathscr{S}(\mathbb{Z}^n \times I, \Omega)$, viewed as a (dense) subalgebra of $C^*(\mathbb{Z}^n \times I, \Omega)$, and let $\mathcal{B}^\infty = \mathscr{S}(\mathbb{Z}^n \times I, \Omega^{-1})$, viewed as a (dense) subalgebra of $C^*(\mathbb{Z}^n \times P, \Omega^{-1})$, which is constructed similarly.

Following Li [40], we have the following theorem:

[1]For any $2p \times 2p$ antisymmetric matrix θ, there exists an invertible matrix T such that $TJ_0''T^t = \theta$. T is unique upto right multiplication by elements of $SP(p,\mathbb{R})$. Now $\mathrm{pf}(\theta) = \det(T)$ gives the isomorphism.

Theorem 3.2.2. *$\mathcal{E}^\infty$ may be given an $\mathcal{A}^\infty$-$\mathcal{B}^\infty$ Morita equivalence bimodule structure, which can be extended to a strong Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$.*

Proof. Following [40], let

$$T(r) = \begin{pmatrix} T(r)_{11} & 0 \\ 0 & I_q \\ T(r)_{31} & T(r)_{32} \end{pmatrix},$$

where $T(r)_{11}$ is a continuous family (with respect to r) of invertible matrices such that $T(r)_{11}^t J_0 T(r)_{11} = \gamma(r)_{11}$, $J_0 := \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$, $T(r)_{31} = \gamma(r)_{21}$ and $T(r)_{32}$ is the matrix obtained from $\gamma(r)_{22}$ by replacing the lower diagonal entries by zero.

We also define

$$S(r) = \begin{pmatrix} J_0(T(r)_{11}^t)^{-1} & -J_0(T(r)_{11}^t)^{-1}T(r)_{31}^t \\ 0 & I_q \\ 0 & T(r)_{32}^t \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{pmatrix}$$

and J' be the matrix obtained from J by replacing each negative entry of it by zero. Let us now denote the matrix $T(r)$ by T_r and $S(r)$ by S_r. Note that T_r can be thought as map from $\widehat{\mathbb{R}^n}$ to $\mathbb{R}^p \times \widehat{\mathbb{R}^p} \times \mathbb{R}^q \times \widehat{\mathbb{R}^q}$ and when restricted to $\mathbb{Z}^n$, it lands in $\mathbb{R}^p \times \widehat{\mathbb{R}^p} \times \mathbb{Z}^q \times \widehat{\mathbb{R}^q}$. So T_r maps $\mathbb{Z}^n$ to G. This is an example of an embedding map discussed in [40, 2.1]. Let P' and P'' be the canonical projections of G to M and $\hat{M}$ respectively and T_r' , T_r'' be the maps $P' \circ T_r$ and $P'' \circ T_r$, respectively. Similarly we define S_r' and S_r'' as $P' \circ S_r$ and $P'' \circ S_r$, respectively. Then the following formulas define an $\mathcal{A}^\infty$-$\mathcal{B}^\infty$ bimodule structure on $\mathcal{E}^\infty$:

$$(fU_l^\theta)(x,r) = e(-T_r(l) \cdot J'T_r(l)/2)\langle x, T_r''(l)\rangle f(x - T_r'(l), r), \tag{3.2.1}$$

$$\langle f,g\rangle_{\mathcal{A}^\infty}(l) = e(-T_r(l) \cdot J'T_r(l)/2) \int_G \langle x, -T_r''(l)\rangle g(x + T_r'(l), r)\bar{f}(x,r)dx, \tag{3.2.2}$$

$$(V_l^\theta f)(x,r) = e(-S_r(l)\cdot J'S_r(l)/2)\langle x, -S_r''(l)\rangle f(x+S_r'(l),r), \qquad (3.2.3)$$

$$_{\mathcal{B}^\infty}\langle f,g\rangle(l) = e(S_r(l)\cdot J'S_r(l)/2)\int_G \langle x, S_r''(l)\rangle \bar{g}(x+S_r'(l),r)f(x,r)dx, \quad (3.2.4)$$

where $l \in \mathbb{Z}^n$.

Using the proposition 2.2 of [40] and the continuity of the families T_r and S_r, the result follows. Completing the space $\mathcal{E}^\infty$ with respect to the defined inner products, we get an $\mathcal{A}$-$\mathcal{B}$ Morita equivalence bimodule. □

If we denote the completion of $\mathcal{E}^\infty$ with respect to the inner product by $\mathcal{E}$, the fibre-wise Morita equivalence $\mathcal{E}_r$ is just the Morita equivalence between $A_{\gamma(r)}$ and $A_{-\gamma(r)'}$ which Rieffel [54] had considered. Since both B and A are unital, $\mathcal{E}$ is a finitely generated projective $\mathcal{A}$-module with respect to the given action of $\mathcal{A}$ on $\mathcal{E}$ (see the argument before proposition 4.6 in [22]). The trace of this module $\mathcal{E}_r$, which was originally computed by Rieffel [54, proposition 4.3, page 289] , can be shown to be exactly the absolute value of the pfaffian of the upper left $2p \times 2p$ corner of the matrix $\gamma(r)$. Indeed, as [54, proposition 4.3, page 289] says that trace of $\mathcal{E}_r$ is $|\det\widetilde{T(r)}|$, where

$$\widetilde{T(r)} = \begin{pmatrix} T(r)_{11} & 0 \\ 0 & I_q \end{pmatrix},$$

the relation $T(r)_{11}^t J_0 T(r)_{11} = \gamma(r)_{11}$ and the fact $\det(J_0) = 1$ give the claim.

3.3 Generators of K_0 groups of noncommutative tori

From Elliott's computation of the image of the traces for noncommutative tori and the fact that the trace $\mathrm{Tr} : \mathrm{K}_0(A_\theta) \to \mathbb{R}$ is injective for totally irrational θ, we can use the main theorem and 3.1.4 to compute explicit generators of $\mathrm{K}_0(A_\theta)$ for all θ. We will explain the 3-dimensional and 4-dimensional cases in details and the $n \geq 5$ case will be just simple extrapolation of these two examples.

We recall the following facts which will play the key role. These facts are due to Elliott (taken from [25, beginning of page 836]):

Lemma 3.3.1. $\mathrm{Tr}(\mathrm{K}_0(A_\theta))$ *is the subgroup of* $\mathbb{R}$ *generated by* 1 *and the numbers* $\sum_\xi (-1)^{|\xi|} \prod_{s=1}^m \theta_{j_{\xi(2s-1)} j_{\xi(2s)}}$ *for* $1 \le j_1 < j_2 < \cdots < j_{2m} \le n$*, where the sum is taken over all elements* ξ *of the permutation group* $\mathcal{S}_{2m}$ *such that* $\xi(2s-1) < \xi(2s)$ *for all* $1 \le s \le m$ *and* $\xi(1) < \xi(3) < \cdots < \xi(2m-1)$.

Lemma 3.3.2. Tr *is injective for totally irrational* θ.

Before going to explicit computations, we shall say some words about the pfaffian of an $n \times n$ skew symmetric matrix $A := (a_{ij})$. Let $1 \le l \le [\frac{n}{2}]$.

Definition 3.3.3. *A* $2l$*-pfaffian minor (or just pfaffian minor)* M_{2l}^A *of a skew symmetric matrix* A *is the pfaffian of a submatrix of* A *consisting of rows and columns indexed by* $i_1, i_2, ..., i_{2l}$ *for some* $i_1 < i_2 < ... < i_{2l}$.

Note that the number of $2l$-pfaffian minors is $\binom{n}{2l}$ and the number of all pfaffian minors is $2^{n-1} - 1$.

Let us consider the $n \times n$ antisymmetric matrix Z whose entries above the diagonal are all 1:

$$Z = \begin{pmatrix} 0 & 1 & \cdots & & \cdots & 1 \\ -1 & \ddots & \ddots & & & \vdots \\ \vdots & \ddots & & & & \\ & & & & & \\ & & & & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 1 \\ -1 & \cdots & & \cdots & -1 & 0 \end{pmatrix}.$$

Proposition 3.3.4. *For any skew symmetric* $n \times n$ *matrix* $A := (a_{ij})$*, there exists some positive integer* t*, such that all pfaffian minors of* $A + tZ$ *are positive.*

Proof. For fixed l with $1 \le l \le [\frac{n}{2}]$, M_{2l}^{A+tZ} is a polynomial in t and

$$M_{2l}^{A+tZ} = t^l + t^{l-1} A_{l-1} + t^{l-2} A_{l-2} + \cdots + t^1 A_1 + A_0,$$

for polynomials $A_{l-1}, A_{l-2}, \cdots A_0$ in entries of $A := (a_{ij})$. Now we can choose such integer t such that t^l dominates the other entries of M_{2l}^{A+tZ}.

Since we have only a finite number of pfaffian minors, we can also choose such integer t such that $M_{2l}^{A+tZ} > 0$ for all l. □

With the above results in hand, we can describe the generators of the K_0-group of general n-dimensional non-commutative tori. Let us start with the general $n \times n$ matrix θ.

$$\theta = \begin{pmatrix} 0 & \theta_{12} & \cdots & & \cdots & \theta_{1n} \\ -\theta_{12} & \ddots & \ddots & & & \theta_{2n} \\ \vdots & \ddots & & & & \\ & & & & & \\ & & & & \ddots & \vdots \\ -\theta_{1(n-1)} & & & \ddots & \ddots & \theta_{(n-1)n} \\ -\theta_{1n} & \cdots & & \cdots & -\theta_{(n-1)n} & 0 \end{pmatrix}.$$

Using the above proposition, we can assume that all pfaffian minors of θ are positive (since in the above proposition, A and $A + tZ$ define the same noncommutative torus). Fix $1 \le l \le [\frac{n}{2}]$. For $i_1 < i_2 < ... < i_{2l}$, let us denote the corresponding pfaffian minor also by $M^{\theta}_{i_1,i_2,\dots,i_{2l}}$. Choose a permutation $\sigma \in \mathcal{S}_n$ such that $\sigma(1) = i_1, \sigma(2) = i_2, \cdots, \sigma(2l) = i_{2l}$. If $U_1, U_2, \cdots, U_n$ are generators of A_θ, there exists an $n \times n$ skew symmetric matrix, denoted by $\sigma(\theta)$, such that $U_{\sigma(1)}, U_{\sigma(2)}, \cdots, U_{\sigma(n)}$ are generators of $A_{\sigma(\theta)}$ and $A_{\sigma(\theta)} \cong A_\theta$. Note that since $i_1 < i_2 < ... < i_{2l}$, the upper left $2l \times 2l$ block has the following form

$$\sigma(\theta)|_{2l} := \begin{pmatrix} 0 & \theta_{i_1 i_2} & \cdots & & \cdots & \theta_{i_1 i_{2l}} \\ -\theta_{i_1 i_2} & \ddots & \ddots & & & \theta_{i_2 i_{2l}} \\ \vdots & \ddots & & & & \\ & & & & & \\ & & & & \ddots & \vdots \\ -\theta_{i_1 i_{(2l-1)}} & & & \ddots & \ddots & \theta_{i_{(2l-1)} i_{2l}} \\ -\theta_{i_1 i_{2l}} & \cdots & & \cdots & -\theta_{i_{(2l-1)} i_{2l}} & 0 \end{pmatrix}.$$

Now consider the projective module constructed as completion of $\mathscr{S}(\mathbb{R}^l \times \mathbb{Z}^{n-2l})$ over $A_{\sigma(\theta)}$ and denote it by $\mathcal{E}_{\sigma(\theta)|_{2l}}$. The trace of this module is the pfaffian of $\sigma(\theta)|_{2l}$, which is exactly $\sum_\xi (-1)^{|\xi|} \prod_{s=1}^m \theta_{i_{\xi(2s-1)} i_{\xi(2s)}}$ where the sum is taken over all elements ξ of the permutation group $\mathcal{S}_{2l}$ such that $\xi(2s-1) < \xi(2s)$ for all $1 \le s \le l$ and $\xi(1) < \xi(3) < \cdots < \xi(2l-1)$.

Varying l, we get $2^{n-1} - 1$ projective modules whose traces are given by exactly the numbers which appeared in Proposition 3.3.1. If θ is totally irrational (so that Tr is injective), these modules along with the trivial class [1] generate the K_0-group of A_θ. For general θ, since all pfaffian minors are positive, for any pfaffian minor we find a path between the corresponding $\sigma(\theta)$ to $\sigma(\theta')$ for a fixed totally irrational skew-symmetric matrix θ', whose pfaffian minors are all positive. Using our main theorem and Theorem 3.1.4, $\mathcal{E}_{\sigma(\theta')|_{2l}}$, for different l's, along with the trivial element generate K_0 of A'_θ. Hence $\mathcal{E}_{\sigma(\theta)|_{2l}}$, for different l's, along with trivial element generate K_0 of A_θ.

3.3.1 The 3-dimensional case

Let

$$\theta = \begin{pmatrix} 0 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 0 & \theta_{23} \\ -\theta_{13} & -\theta_{23} & 0 \end{pmatrix}.$$

Using the above proposition, assume that the 2-pfaffian minors of A_θ, $\mathrm{pf}(M^\theta_{ij})$, are positive (indeed, $A_{\theta+tZ}$ is isomorphic to A_θ for any integer t) , where

$$M^\theta_{ij} = \begin{pmatrix} 0 & \theta_{ij} \\ -\theta_{ij} & 0 \end{pmatrix}, \quad j > i \geq 1 \ .$$

From Lemma 3.3.1, one has

$$Tr(\mathrm{K}_0(A_\theta)) = \mathbb{Z} + \theta_{12}\mathbb{Z} + \theta_{13}\mathbb{Z} + \theta_{23}\mathbb{Z}.$$

When θ is totally irrational (so that the trace is injective), we consider the projective A_θ module $\mathcal{E}^\theta_{12} := \overline{\mathscr{S}(\mathbb{R}\times\mathbb{Z})_{12}}$ constructed as in the main theorem at $r = 0$ fibre for the choice of $M = \mathbb{R}\times\mathbb{Z}$. The trace of this module is θ_{12} (see the discussion at the end of section 3.2).

Consider the following matrices

$$\theta_1 = \begin{pmatrix} 0 & \theta_{23} & -\theta_{12} \\ -\theta_{23} & 0 & -\theta_{13} \\ \theta_{12} & \theta_{13} & 0 \end{pmatrix},$$

$$\theta_2 = \begin{pmatrix} 0 & \theta_{13} & \theta_{12} \\ -\theta_{13} & 0 & -\theta_{23} \\ -\theta_{12} & \theta_{23} & 0 \end{pmatrix}.$$

Note that A_θ, A_{θ_1} and A_{θ_2} are just "rotations" of each other and represent the same noncommutative tori. Let $\mathcal{E}^\theta_{13} := \overline{\mathscr{S}(\mathbb{R}\times\mathbb{Z})_{13}}$ and $\mathcal{E}^\theta_{23} := \overline{\mathscr{S}(\mathbb{R}\times\mathbb{Z})_{23}}$ be the projective modules over A_{θ_2} and A_{θ_1}, respectively, as discussed above. Now, similarly to the previous case, we see that $\mathrm{Tr}(\mathcal{E}^\theta_{13}) = \theta_{13}$ and $\mathrm{Tr}(\mathcal{E}^\theta_{23}) = \theta_{23}$. Since Tr is injective for θ (as it is totally irrational), we conclude that $\mathcal{E}^\theta_{12}, \mathcal{E}^\theta_{13}, \mathcal{E}^\theta_{23}$ along with the trivial element generate $\mathrm{K}_0(A_\theta)$. Using our description of the continuous field and 3.1.4, we conclude that $\mathcal{E}^\theta_{12}, \mathcal{E}^\theta_{13}, \mathcal{E}^\theta_{23}$ along with the trivial element generate $\mathrm{K}_0(A_\theta)$ for all θ.

3.3.2 The 4 dimensional case

Let

$$\theta = \begin{pmatrix} 0 & \theta_{12} & \theta_{13} & \theta_{14} \\ -\theta_{12} & 0 & \theta_{23} & \theta_{24} \\ -\theta_{13} & -\theta_{23} & 0 & \theta_{34} \\ -\theta_{14} & -\theta_{24} & -\theta_{34} & 0 \end{pmatrix} .$$

Without loss of generality (again using 3.3.4), we can assume that the pfaffians $\mathrm{pf}(\theta)$ and $\mathrm{pf}(M^{\theta}_{ij})$ are positive, where

$$M^{\theta}_{ij} = \begin{pmatrix} 0 & \theta_{ij} \\ -\theta_{ij} & 0 \end{pmatrix}, \ \ j > i \geq 1 \ .$$

Let θ be totally irrational again. Then, similarly to the previous case, we get the six modules $\mathcal{E}^{\theta}_{12}$, $\mathcal{E}^{\theta}_{13}$, $\mathcal{E}^{\theta}_{14}$, $\mathcal{E}^{\theta}_{23}$, $\mathcal{E}^{\theta}_{24}$, $\mathcal{E}^{\theta}_{34}$. These modules are completions of $\mathscr{S}(\mathbb{R} \times \mathbb{Z}^2)$ for different actions of A_θ. Since $\mathrm{K}_0(A_\theta) = \mathbb{Z}^8$, we need to find another projective module which has trace $\theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23}$ (according to Lemma 3.3.1). This module turns out to be the Bott class given by the completion of $\mathscr{S}(\mathbb{R}^2)$ (as in the main theorem for $M = \mathbb{R}^2$). Denote this module by $\mathcal{E}^{\theta}_{1234}$. Now the pfaffian of θ is $\mathrm{Tr}(\mathcal{E}^{\theta}_{1234})$ which is $\theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23}$. So again using Theorem 3.2.2 and 3.1.4 we conclude that $\mathcal{E}^{\theta}_{12}, \mathcal{E}^{\theta}_{13}, \mathcal{E}^{\theta}_{14}, \mathcal{E}^{\theta}_{23}, \mathcal{E}^{\theta}_{24}, \mathcal{E}^{\theta}_{34}$ and $\mathcal{E}^{\theta}_{1234}$ along with the trivial element generate $\mathrm{K}_0(A_\theta)$ for each θ.

4 Projective modules over some noncommutative orbifolds

4.1 Notation

Notation: $e(x)$ will always denote the number $e^{2\pi ix}$; the standard symplectic matrix on $\mathbb{R}^{2m}$ is defined by $J = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, where I_m is the $m \times m$ unit matrix, and $\mathscr{S}(\mathbb{R}^m)$ will denote the space of rapidly decreasing functions on $\mathbb{R}^m$.

4.2 A quick look into noncommutative orbifolds

Let $W := (a_{ij})$ be an $n \times n$ matrix of finite order with integer entries acting on $\mathbb{Z}^n$ and let F be the cyclic group generated by W. In addition, we assume that W is a θ-symplectic matrix i.e $W^T\theta W = \theta$. Hence F is a finite subgroup of $SP(n, \mathbb{Z}, \theta) := \{A \in GL(n, \mathbb{Z}) : A^T\theta A = \theta\}$. By Lemma 2.1 of [22] we have $C^*(\mathbb{Z}^n \rtimes F, \omega'_\theta) = A_\theta \rtimes_\alpha F$ with respect to the action ([32, Equation 2.6]) :

$$\alpha(U_i) = e(\sum_{k=2}^{n}\sum_{j=1}^{k-1} a_{ki}a_{ji}\theta_{jk})U_1^{a_{1i}} \cdots U_n^{a_{ni}}, \tag{4.2.1}$$

where $U_1, ..., U_n$ are the generators of A_θ.

Let us look into the case where $n = 2$. Note that $SP(2, \mathbb{Z}, \theta) = SL(2, \mathbb{Z})$. Finite cyclic subgroups of $SL(2, \mathbb{Z})$ are up to conjugacy generated by the following 4 matrices:

$$W_2 := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},\ W_3 := \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$W_4 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ W_6 := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

where the notation W_r indicates that it is a matrix of order r. The above fact can be derived using the fact that $SL(2,\mathbb{Z}) = \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$, i.e $SL(2,\mathbb{Z})$ can be written as amalgamated free product of $\mathbb{Z}_6$ and $\mathbb{Z}_4$, over $\mathbb{Z}_2$.

The actions of these matrices are considered already in [22], where the authors constructed projective modules over the corresponding crossed products and used these projective modules to prove some classification results for these crossed products.

For $n \geq 3$ finding such a matrix W is a non-trivial question. For $n = 3$, there is only one such matrix ($-I_3$) acting on all A_θ's (see [32, Theorem 5.2]). In [32] the authors found some W's and associated actions for $n \geq 4$ such that the crossed products are well defined.

4.3 Projective modules over noncommutative tori

In the last chapter we already discussed the construction of projective modules over noncommutative tori. Since, in this section, we use a simpler version of the construction, we recall it again (which will be suitable for this chapter). We fix $n = 2p + q$ for p, $q \in \mathbb{Z}_{>0}$. Let us choose $\theta := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$ any $n \times n$ skew-symmetric matrix partitioned into an invertible $2p \times 2p$ matrix θ_{11} and $q \times q$ matrix θ_{22}. We recall the approach of Rieffel [54] to the construction of finitely generated projective $C^*(\mathbb{Z}^n, \omega_\theta)$-modules and follow the presentation in [40]. Denote ω_θ by ω and define a new cocycle ω_1 on $\mathbb{Z}^n$

by $\omega_1(x, y) = e(\langle \theta' x, y\rangle/2)$, where

$$\theta' = \begin{pmatrix} \theta_{11}^{-1} & -\theta_{11}^{-1}\theta_{12} \\ \theta_{21}\theta_{11}^{-1} & \theta_{22} - \theta_{21}\theta_{11}^{-1}\theta_{12} \end{pmatrix}.$$

Set $\mathcal{A} = C^*(\mathbb{Z}^n, \omega)$ and $\mathcal{B} = C^*(\mathbb{Z}^n, \omega_1)$. Let M be the group $\mathbb{R}^p \times \mathbb{Z}^q$, $G := M \times \hat{M}$ and $\langle \cdot, \cdot \rangle$ be the natural pairing between M and its dual group $\hat{M}$ (we do not distinguish between the notations of pairing and usual inner product of linear spaces). Consider the Schwartz space $\mathcal{E}^\infty := \mathcal{S}(M)$ consisting of all complex functions on M which are smooth and rapidly decreasing. Denote by $\mathcal{A}^\infty = \mathcal{S}(\mathbb{Z}^n, \omega)$ and $\mathcal{B}^\infty = \mathcal{S}(\mathbb{Z}^n, \omega_1)$ the sub-algebras of $\mathcal{A}$ and $\mathcal{B}$, respectively. Let us consider the following $(2p + 2q) \times (2p + q)$ real valued matrix:

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & I_q \\ T_{31} & T_{32} \end{pmatrix},$$

where T_{11} is an invertible matrix such that $T_{11}^t J_0 T_{11} = \theta_{11}$, $J_0 := \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$,

$T_{31} = \theta_{21}$ and T_{32} is the matrix obtained from θ_{22} replacing the lower diagonal entries by zero.

We also define the following $(2p+2q) \times (2p+q)$ real valued matrix:

$$S = \begin{pmatrix} J_0(T_{11}^t)^{-1} & -J_0(T_{11}^t)^{-1}T_{31}^t \\ 0 & I_q \\ 0 & T_{32}^t \end{pmatrix}.$$

Let

$$J = \begin{pmatrix} J_0 & 0 & 0 \\ 0 & 0 & I_q \\ 0 & -I_q & 0 \end{pmatrix}$$

and J' be the matrix obtained from J by replacing the negative entries of it by 0. Note that T and S can be thought as maps $\mathbb{R}^p \times \mathbb{R}^{*p} \times \mathbb{Z}^q \to G$. Let P' and P'' be the canonical projections of G to M and $\hat{M}$ respectively and T' , T'' be the maps $P' \circ T$ and $P'' \circ T$ respectively. Similarly denote S' and S'' for the maps $P' \circ S$ and $P'' \circ S$ respectively. Then the following formulas define an $\mathcal{A}^\infty$-$\mathcal{B}^\infty$ bimodule structure on $\mathcal{E}^\infty$:

$$(fU_l^\theta)(x) = e(\langle -T(l), J'T(l)/2\rangle)\langle x, T''(l)\rangle f(x - T'(l)), \tag{4.3.1}$$

$$\langle f, g\rangle_{\mathcal{A}^\infty}(l) = e(\langle -T(l), J'T(l)/2\rangle) \int_G \langle x, -T''(l)\rangle g(x + T'(l))\bar{f}(x)dx, \tag{4.3.2}$$

$$(V_l^\theta f)(x) = e(\langle -S(l), J'S(l)/2\rangle)\langle x, -S''(l)\rangle f(x + S'(l)), \tag{4.3.3}$$

$${}_{\mathcal{B}^\infty}\langle f, g\rangle(l) = e(\langle S(l), J'S(l)/2\rangle) \int_G \langle x, S''(l)\rangle \bar{g}(x + S'(l)) f(x)dx, \tag{4.3.4}$$

where $l \in \mathbb{Z}^n$ and U_l^θ, V_l^θ denotes the canonical unitaries with respect to the group element $l \in \mathbb{Z}^n$ in $\mathcal{A}^\infty$ and $\mathcal{B}^\infty$, respectively.

See Proposition 2.2 in [40] for the following well-known result.

Theorem 4.3.1 (Rieffel)**.** *$\mathcal{E}^\infty$, with the above structures, is an $\mathcal{A}^\infty$-$\mathcal{B}^\infty$ Morita equivalence bimodule which can be extended to a strong Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$.*

Let $\mathcal{E}$ be the completion of $\mathcal{E}^\infty$ with respect to the inner products given above. Now $\mathcal{E}$ becomes a right projective A-module which is also finitely generated (see the discussion above proposition 4.6 of [22]). When $q = 0$, we call the corresponding projective module *Bott class.* Note that this class appears only for even dimensional tori.

4.4 Projective modules over noncommutative orbifolds

One natural question is how does one extend the projective modules over noncommutative tori to the aforementioned crossed product? Our main theorem addresses this question for the Bott classes.

In the following sections (except Section 7) we consider n to be an even number, $n = 2m$. Suppose $F = \langle W \rangle$ is a finite cyclic group acting on $\mathbb{Z}^n$. We want to build some projective modules over $C^*(\mathbb{Z}^n \rtimes F, \omega'_\theta)$. Note that W needs to be a θ-symplectic matrix, i.e $W^T \theta W = \theta$, as noted earlier.

In order to construct projective modules over $C^*(\mathbb{Z}^n \rtimes F, \omega'_\theta)$, we will use the so-called metaplectic representation of the symplectic matrix W. When θ is the standard skew-symmetric matrix J then W is also a standard symplectic matrix.

We denote the group of all J-symplectic matrices (also known as standard symplectic matrices) by $\mathcal{SP}(n)$, which is known as the symplectic group. We refer to chapter 2 of the book [20] for preliminaries on symplectic groups and their metaplectic representations. We recall the metaplectic action associated to the symplectic matrix W. Any symplectic matrix can be written as product of two free symplectic matrices (see page 38, [20]) which are by definition symplectic matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad A, B, C, D \in M_n(\mathbb{R})$$

such that $\det(B) \neq 0$. Let W be a free symplectic matrix. We now associate to W the *generating function*:

$$W(x, x') = \frac{1}{2}\langle DB^{-1}x, x\rangle - \langle B^{-1}x, x'\rangle + \frac{1}{2}\langle B^{-1}Ax', x'\rangle, \tag{4.4.1}$$

when $x, x' \in \mathbb{R}^m$.

Definition 4.4.1. *The metaplectic operator (metaplectic transformation) associated to W on $\mathscr{S}(\mathbb{R}^m)$ is given by*

$$\mathcal{F}_W f(x) = e(\frac{1}{4}(s - \frac{m}{2}))\sqrt{|\det(B^{-1})|}\int_{\mathbb{R}^m} e(W(x, x'))f(x')dx';$$

the integer s (sometimes called Maslov index) corresponds to a choice of the argument arg *of* $\det B^{-1}$*:*

$$s\pi \equiv \text{Arg}\ (\det B^{-1}) \quad mod\ 2\pi.$$

These operators can be extended to $L^2(\mathbb{R}^m)$ giving unitary operators on $L^2(\mathbb{R}^m)$ (see page 81, [20]). We denote by $\mathcal{MP}(n)$ the group of metaplectic operators which is a subgroup of the group of unitary operators of $L^2(\mathbb{R}^m)$.

Theorem 4.4.2. *There exists an exact sequence:*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathcal{MP}(n) \longrightarrow \mathcal{SP}(n) \longrightarrow 0,$$

where the map $\mathcal{MP}(n) \to \mathcal{SP}(n)$ is uniquely determined by the map $\mathcal{F}_W \to W$.

Proof. See page 84, [20]. □

One also defines the circle extension of $\mathcal{SP}(n)$, $\mathcal{MP}^c(n)$. This is defined to be the group $\mathcal{MP}(n) \times_{\mathbb{Z}_2} \mathbb{S}^1 : (\mathcal{MP}(n) \times \mathbb{S}^1)/\Delta(\mathbb{Z}_2)$, $\Delta(\mathbb{Z}_2)$ being the diagonal $\mathbb{Z}_2 \times \mathbb{Z}_2$ sitting inside $\mathcal{MP}(n) \times \mathbb{S}^1$. This gives rise to the exact sequence

$$0 \longrightarrow \mathbb{S}^1 \longrightarrow \mathcal{MP}^c(n) \longrightarrow \mathcal{SP}(n) \longrightarrow 0,$$

where $\mathbb{S}^1$ denotes the circle group.

In the following, we shall often write fW for $\mathcal{F}_W(f)$.

Following [20, Section 3.2.2] the following matrices generate all symplectic matrices:

$$J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \; M_L := \begin{pmatrix} L & 0 \\ 0 & (L^T)^{-1} \end{pmatrix}, \; V_P := \begin{pmatrix} I & 0 \\ P & I \end{pmatrix}, \qquad (4.4.2)$$

for a symmetric $m \times m$ matrix P and an invertible $m \times m$ matrix L.

Following [20, Section 7.1.2], we write down the metaplectic operators (up to some constant which will not matter in the proof) corresponding to J, M_L and V_P:

$$(f\,J)(x) = \int_{\mathbb{R}^m} e(\langle -x, x' \rangle) f(x') dx' \qquad (4.4.3)$$

$$(f\,M_L)(x) = \sqrt{|det(L)|} f(L(x)) \qquad (4.4.4)$$

$$(f\,V_P)(x) = e(\frac{1}{2}\langle Px, x \rangle) f(x), \qquad (4.4.5)$$

Hence it suffices to check statements of multiplicative type about metaplectic transformations for these transformations and the Schwartz space is invariant under metaplectic transformations, see [20, Corollary 63]. Now we are in the position to formulate our main theorem. For our result we always assume θ to be a non-degenerate matrix.

We recall the following proposition from [22, Proposition 4.5].

Proposition 4.4.3. *Suppose F is a finite group acting on a C^*-algebra A by the action α. Also suppose that $\mathcal{E}$ is a finitely generated projective (right) A-module with a right action $T : F \to \mathrm{Aut}(\mathcal{E})$, written $(\xi, g) \rightarrowtail \xi W_g$, such that $\xi(W_g)a = (\xi\alpha_g(a))W_g$ for all $\xi \in \mathcal{E}, a \in A$, and $g \in F$. Then $\mathcal{E}$ becomes a finitely generated projective $A \rtimes F$ module with action defined by*

$$\xi \cdot (\sum_{g \in F} a_g \delta_g) = \sum_{g \in F} (\xi a_g) W_g.$$

Also, if we restrict this $A \rtimes F$ module to A, we get the original A-module $\mathcal{E}$, with the action of F forgotten.

Proof. See the proof of Proposition 4.5 [22]. □

Theorem 4.4.4. *Let W be a θ-symplectic matrix of finite order and let $F = \langle W \rangle$ be the finite cyclic group generated by W. Let $\alpha : F \to \mathrm{Aut}(C^*(\mathbb{Z}^n, \omega_\theta))$ denote the corresponding action on $C^*(\mathbb{Z}^n, \omega_\theta)$. Then the metaplectic action of W on $\mathscr{S}(\mathbb{R}^m)$ extends to an action on $\mathcal{E}$ such that $\mathcal{E}$ becomes an F-equivariantly finitely generated projective $C^*(\mathbb{Z}^n, \omega_\theta)$ module and thus a finitely generated projective module over $C^*(\mathbb{Z}^n, \omega_\theta) \rtimes F$.*

Proof. We divide the proof in two parts.
First part: (the case $\theta = -J$): Recall from (4.3.1) that for the choice of $T := \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$ the action of $\mathscr{S}(\mathbb{Z}^n, \omega_{-J}) = C^\infty(\mathbb{T}^n)$ on $\mathscr{S}(\mathbb{R}^m)$ is given by the following:

$$f(U_i)^p(y_1, y_2, \ldots, y_m) = f(y_1, y_2, \ldots, y_i + p, \ldots, y_m), \text{ if } i \leq m, \tag{4.4.6}$$

$$f(U_i)^p(y_1, y_2, \ldots, y_m) = e(py_{i-m}) f(y_1, y_2, \ldots, y_m), \text{ if } i > m, \tag{4.4.7}$$

where the U_i's are the generators of the n-dimensional smooth torus $C^\infty(\mathbb{T}^n)$. [Note that $\theta = -J$ is chosen instead of $\theta = J$ to keep the formulas somewhat similar to [22]]. Let α_W denote the action of the matrix W on $\mathscr{S}(\mathbb{Z}^n)$: $\alpha_W(\phi)(x) = \phi(W^{-1}x)$. According to 4.4.3, we still have to check the following equation to complete the proof:

$$f(W)\phi = (f\alpha_W(\phi))W, \tag{4.4.8}$$

for all $f \in \mathscr{S}(\mathbb{R}^m)$ and $\phi \in \mathscr{S}(\mathbb{Z}^n, \omega_{-J})$, which will then imply that $\mathcal{E}$ becomes F-equivariant. Also, since $\mathscr{S}(\mathbb{Z}^n, \omega_{-J})$ is generated by $U_1, U_2, \ldots, U_n$, it is enough to check (4.4.8) for $\phi = U_1, U_2, \ldots, U_n$.

So we are left with checking the following equations:

$$fJU_i = (f\alpha_J(U_i))J, \tag{4.4.9}$$

$$fM_LU_i = (f\alpha_{M_L}(U_i))M_L, \tag{4.4.10}$$

$$fV_PU_i = (f\alpha_{V_P}(U_i))V_P, \tag{4.4.11}$$

for all $1 \leq i \leq n$, with J, M_L, V_P as in 4.4.2.

First we check the equations (4.4.9), (4.4.10) and (4.4.11) for $1 \leq i \leq m$. The left hand side (LHS) of (4.4.9) is

$$\begin{aligned}(fJU_i)(x_1, x_2, \ldots, x_m) &= (fJ)(x_1, x_2, \ldots, x_i + 1, \ldots, x_m) \\ &= \int_{\mathbb{R}^m} e(-\langle (x_1, x_2, \ldots, x_i + 1, \ldots, x_m), (x'_1, x'_2, \ldots, x'_m)\rangle) f(x')dx' \\ &= \int_{\mathbb{R}^m} e(-\langle (x_1, x_2, \ldots, x_m), (x'_1, x'_2, \ldots, x'_m)\rangle).e(-x'_i)f(x')dx';\end{aligned}$$

and the right hand side (RHS):

$$\begin{aligned}(f\alpha_J(U_i))J(x_1, x_2, \ldots, x_m) &= \int_{\mathbb{R}^m} e(-\langle (x_1, x_2, \ldots, x_m)(x'_1, x'_2, \ldots, x'_m)\rangle) f\alpha_J(U_i)(x')dx' \\ &= \int_{\mathbb{R}^m} e(-\langle (x_1, x_2, \ldots, x_m), (x'_1, x'_2, \ldots, x'_m)\rangle)(fU_{i+m}^{-1})(x')dx' \\ &= \int_{\mathbb{R}^m} e(-\langle (x_1, x_2, \ldots, x_m), (x'_1, x'_2, \ldots, x'_m)\rangle).e(-x'_i)f(x')dx'.\end{aligned}$$

Hence we proved that in (4.4.9), LHS = RHS. The LHS of (4.4.10) equals

$$\begin{aligned}(fM_LU_i)(x_1, x_2, \ldots, x_m) &= (fM_L)(x_1, x_2, \ldots, x_i + 1, \ldots, x_m) \\ &= \sqrt{det(L)} f(L(x_1, x_2, \ldots, x_i + 1, \ldots, x_n));\end{aligned}$$

and the RHS is

$$\begin{aligned}(f\alpha_{M_L}(U_i))M_L(x_1, x_2, \ldots, x_m) &= \sqrt{det(L)}(f\alpha_{M_L}(U_i))L(x_1, x_2, \ldots, x_m) \\ &= \sqrt{det(L)} f(L(x_1, x_2, \ldots, x_m) + L(x_i)) \\ &= \sqrt{det(L)} f(L(x_1, x_2, \ldots, x_i + 1, \ldots, x_m)).\end{aligned}$$

Hence we have demonstrated that in (4.4.10) the LHS = RHS. We have for the LHS of (4.4.11):

$$\begin{aligned}(fV_PU_i)(x_1,x_2,\dots,x_m) &= (fV_P)(x_1,x_2,\dots,x_i+1,\dots,x_m)\\ &= e(\frac{1}{2}\langle P(x_1,x_2,\dots,x_i+1,\dots,x_m),\cdots\\ &\quad (x_1,x_2,\dots,x_i+1,\dots,x_m)\rangle)\cdots\\ &\quad f(x_1,x_2,\dots,x_i+1,\dots,x_m);\end{aligned}$$

and the RHS is

$$\begin{aligned}(f\alpha_{V_P}(U_i))V_P(x_1,x_2,\dots,x_m) &= e(\frac{1}{2}(Px\cdot x))(f\alpha_{V_P}(U_i))(x)\\ &= e(\frac{1}{2}\langle P(x_1,x_2,\dots,x_i+1,\dots,x_m),\cdots\\ &\quad (x_1,x_2,\dots,x_i+1,\dots,x_m)\rangle)\cdots\\ &\quad f(x_1,x_2,\dots,x_i+1,\dots,x_m).\end{aligned}$$

Hence we have shown that LHS = RHS for (4.4.11).

Now, let $m < i \leq n$. We check the equations (4.4.9), (4.4.10) and (4.4.11) for these values of i .
For (4.4.9) the LHS is

$$\begin{aligned}(fJU_i)(x_1,x_2,\dots,x_m) &= e(ix_{i-m})(fJ)(x_1,x_2,\dots,x_m)\\ &= e(x_{i-m})\int_{\mathbb{R}^m} e(-\langle x,x'\rangle)f(x')dx';\end{aligned}$$

and the RHS

$$\begin{aligned}(f\alpha_J(U_i))J(x_1,x_2,\dots,x_m) &= \int_{\mathbb{R}^m} e(-\langle x,x'\rangle)(f\alpha_J(U_i))(x')dx'\\ &= \int_{\mathbb{R}^m} e(-\langle x,x'\rangle)(f(U_{i-m}))(x')dx'\\ &= \int_{\mathbb{R}^m} e(-\langle x,x'\rangle)f(x'_1,x'_2,\dots,x'_{i-m}+1,\dots,x'_m)dx'\\ &= e(x_{i-m})\int_{\mathbb{R}^m} e(-\langle x,x'\rangle)f(x'_1,x'_2,\dots,x'_{i-m},\dots,x'_m)dx'\\ &= e(x_{i-m})\int_{\mathbb{R}^m} e(-\langle x,x'\rangle)f(x')dx'.\end{aligned}$$

Hence we have proved that for (4.4.9), LHS = RHS.
For (4.4.10), the LHS

$$\begin{aligned}(fM_LU_i)(x_1,x_2,\dots,x_m) &= e(x_{i-m})(fM_L)(x_1,x_2,\dots,x_m)\\ &= \sqrt{det(L)}e(x_{i-m})f(L(x_1,x_2,\dots,x_m));\end{aligned}$$

and the RHS

$$\begin{aligned}(f\alpha_{M_L}(U_i))M_L(x_1,x_2,\ldots,x_m) &= \sqrt{det(L)}(f\alpha_{M_L}(U_i))(L(x_1,x_2,\ldots,x_m))\\ &= \sqrt{det(L)}e(\langle (L^{-1})^T(e_{i-m}),L(x_1,x_2,\ldots,x_m)\rangle)\cdots\\ &\quad f(L(x_1,x_2,\ldots,x_m))\\ &= \sqrt{det(L)}e(\langle e_{i-m},(L^{-1}L)(x_1,x_2,\ldots,x_m)\rangle)\cdots\\ &\quad f(L(x_1,x_2,\ldots,x_m))\\ &= \sqrt{det(L)}e(x_{i-m})f(L(x_1,x_2,\ldots,x_m)).\end{aligned}$$

Thus in (4.4.10) the LHS is equal to the RHS.
For (4.4.11), the LHS

$$\begin{aligned}(fV_PU_i)(x_1,x_2,\ldots,x_m) &= e(x_{i-m})(fV_P)(x_1,x_2,\ldots,x_m)\\ &= e(x_{i-m})e(\frac{1}{2}\langle Px,x\rangle)f(x);\end{aligned}$$

which equals to the RHS:

$$\begin{aligned}(f\alpha_{V_P}(U_i))V_P(x_1,x_2,\ldots,x_m) &= e(\frac{1}{2}\langle Px,x\rangle)(f\alpha(U_i)(x)\\ &= e(\frac{1}{2}\langle Px,x\rangle)(fU_i)(x)\\ &= e(x_{i-m})e(\frac{1}{2}\langle Px,x\rangle)f(x).\end{aligned}$$

Now we have the following diagram:

$$\begin{array}{ccccccccc} & & & & & & F & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathcal{MP}(n) & \longrightarrow & \mathcal{SP}(n) & \longrightarrow & 0.\end{array}$$

In the above diagram it is not assured that the inclusion $F \hookrightarrow \mathcal{SP}(n)$ lifts to an inclusion $F \hookrightarrow \mathcal{MP}(n)$. Since F is cyclic the following lift is always possible:

$$\begin{array}{ccccccccc} & & & & & & F & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & \mathcal{MP}^c(n) & \longrightarrow & \mathcal{SP}(n) & \longrightarrow & 0,\end{array}$$

where $\mathcal{MP}^c(n)$ is the circle extension of $\mathcal{SP}(n)$. Indeed, for the generator $W \in F$, we can choose $z \in \mathbb{T}$ to make sure that the order of the operator $z \cdot \mathcal{F}_W \in \mathcal{MP}^c(n)$ is same as the order of the element $W \in F$. So the inclusion $F \hookrightarrow \mathcal{MP}^c(n)$ gives the required action of W on $\mathscr{S}(\mathbb{R}^m)$.

Second part (the general case):

Let θ be a general non-degenerate anti-symmetric matrix. In this case $W_\theta^T \theta W_\theta = \theta$. We recall how the projective modules are constructed in this case. Since θ is non-degenerate, there exists an invertible matrix T such that $T^T J T = \theta$. Recall that the action of U_l^θ (for $l \in \mathbb{Z}^n$) on $\mathscr{S}(\mathbb{R}^m)$ is defined by

$$(fU_l^\theta)(x) = e((-T(l) \cdot J'T(l)/2))e(\langle x, T''(l)\rangle)f(x - T'(l)). \tag{4.4.12}$$

First note that $W := TW_\theta T^{-1}$ is a J-symplectic matrix (a matrix A is J-symplectic if $A^T J A = J$). So we can define $fW = f(TW_\theta T^{-1}) := \mathcal{F}_{TW_\theta T^{-1}}(f)$ for $f \in \mathscr{S}(\mathbb{R}^m)$ and fW_θ to be the function fW. So in this case we need to check the following equation:

$$(fW_\theta)U_l^\theta(x) = (f\alpha_{W_\theta}(U_l^\theta))W_\theta(x), \;\; x \in \mathbb{R}^m. \tag{4.4.13}$$

This follows from:

$$\begin{aligned}
(fW_\theta)U_l^\theta(x) &= e(\langle -T(l), J'T(l)/2\rangle)e(\langle x, T''(l)\rangle)(fW_\theta)(x - T'(l)) \\
&= e(\langle -T(l), J'T(l)/2\rangle)e(\langle x, T''(l)\rangle)(fW)(x - T'(l)) \\
&= e(\langle -T(l), J'\mathrm{Id}(Tl)/2\rangle)e(\langle x, \mathrm{Id}''(Tl)\rangle)(fW)(x - \mathrm{Id}'(Tl)) \\
&= (fW)U_{Tl}^J(x) \\
&= (f\alpha_W(U_{Tl}^J))W(x) \qquad \text{(using (4.4.11))} \\
&= c\int_{\mathbb{R}^m} e(W(x,x'))(f\alpha_W(U_{Tl}^J))(x')dx'; \qquad \text{(for a constant } c\text{)}
\end{aligned}$$

and the RHS

$$\begin{aligned}(f\alpha_{W_\theta}(U_l^\theta))W_\theta(x) &= (f\alpha_{W_\theta}(U_l^\theta))W(x)\\ &= c\int_{\mathbb{R}^m} e(W(x,x'))(f\alpha_{W_\theta}(U_l^\theta))(x')dx'\\ &= c\int_{\mathbb{R}^m} e(W(x,x'))(f(U^\theta_{W_\theta(l)}))(x')dx'\\ &= c\int_{\mathbb{R}^m} e(W(x,x'))(f(U^J_{W(Tl)}))(x')dx' \qquad \text{(using (4.4.14))}\\ &= c\int_{\mathbb{R}^m} e(W(x,x'))(f\alpha_W(U^J_{Tl}))(x')dx',\end{aligned}$$

where

$$(f(U^\theta_{W_\theta(l)}))(x') = (f(U^J_{W(Tl)}))(x'), \qquad x'\in\mathbb{R}^m \tag{4.4.14}$$

follows from:

$$(f(U^\theta_{W_\theta(l)}))(x') = e(\langle -T(W_\theta(l)), J'T(W_\theta(l))/2\rangle)e(\langle x', T''(W_\theta(l))\rangle)f(x'-T'(W_\theta(l))),$$

which is equal to

$$\begin{aligned} & e(\langle -T(W_\theta(T^{-1}Tl)), J'T(W_\theta(T^{-1}Tl))/2\rangle)e(\langle x', T''(W_\theta(T^{-1}Tl))\rangle)f(x'-T'(W_\theta(T^{-1}Tl)))\\ = \; & e(\langle -(TW_\theta T^{-1})(Tl), J'(TW_\theta T^{-1})(Tl)/2\rangle)e(\langle x', (TW_\theta T^{-1})''(Tl)\rangle)f(x'-(TW_\theta T^{-1})'(Tl))\\ = \; & e(\langle -W(Tl), J'W(Tl)/2\rangle)e(\langle x', \mathrm{Id}''(W(Tl))\rangle)f(x'-\mathrm{Id}'(W(Tl)))\\ = \; & (f(U^J_{W(Tl)}))(x').\end{aligned}$$

We finish the proof with the compatibility of the action with the inner product $\langle .,.\rangle_{\mathcal{A}^\infty}$ as defined in (4.3.2):

$$\langle fW_\theta, gW_\theta\rangle_{\mathcal{A}^\infty} = \alpha_{W_\theta^{-1}}(\langle f,g\rangle_{\mathcal{A}^\infty}).$$

Replacing f by fW_θ, it suffices to check:

$$\langle f, gW_\theta\rangle_{\mathcal{A}^\infty} = \alpha_{W_\theta^{-1}}(\langle fW_\theta^{-1}, g\rangle_{\mathcal{A}^\infty}). \tag{4.4.15}$$

The argument is based on some observations: (i) the explicit description of $\langle .,.\rangle_{\mathcal{A}^\infty}$ in terms of the right action of $\mathcal{A}^\infty$ on $\mathscr{S}(\mathbb{R}^m)$:

$$\langle f,g\rangle_{\mathcal{A}^\infty}(l) = \langle gU^\theta_{-l}, f\rangle_{L^2},$$

for $\langle f,g\rangle_{L^2} = \int_{\mathbb{R}^m} f(x)\overline{g(x)}dx$, and (ii) the relations:

$$\alpha_{W_\theta^{-1}}(\langle f,g\rangle_{\mathcal{A}^\infty})(l) = \langle g\alpha_{W_\theta}(U^\theta_{-l}), f\rangle_{L^2}.$$

The realization of $\langle .,. \rangle_{\mathcal{A}^\infty}$ in terms of the right action allows us to use equation (4.4.13):

$$(fW_\theta)U_l^\theta(x) = (f\alpha_{W_\theta}(U_l^\theta))W_\theta(x), \;\; x \in \mathbb{R}^m$$

in the proof of (4.4.15):

$$\begin{aligned}
\langle f, gW_\theta\rangle_{\mathcal{A}^\infty}(l) &= \langle (gW_\theta)U_{-l}^\theta, f\rangle_{L^2}, \\
&= \int_{\mathbb{R}^m} (g\alpha_{W_\theta}(U_{-l}^\theta)W_\theta(x))\overline{f(x)}dx, \\
&= \int_{\mathbb{R}^m} (g\alpha_{W_\theta}(U_{-l}^\theta))(x)\overline{(fW_\theta^{-1})(x))}dx, \\
&= \alpha_{W_\theta^{-1}}(\int_{\mathbb{R}^m} (gU_{-l}^\theta)(x)\overline{(fW_\theta^{-1})(x))}dx) \\
&= \alpha_{W_\theta^{-1}}(\langle fW_\theta^{-1}, g\rangle_{\mathcal{A}^\infty}),
\end{aligned}$$

which is the desired identity. □

4.5 The 2-dimensional case - revisited

The results for the 2-dimensional case [22] are revisited from the perspective of metaplectic transformations. As mentioned before, there are up to conjugation four matrices of finite order in $\mathrm{SL}_2(\mathbb{Z})$ generating $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6$. For the $\mathbb{Z}_2$ action on $\mathscr{S}(\mathbb{R})$, given by $f \to \tilde{f}$, where $\tilde{f}(x) = f(-x)$, the corresponding module, called the flip module, over $A_\theta \rtimes \mathbb{Z}_2$ is quite well studied by Walters [61]. In the next section we discuss flip modules in the higher-dimensional setting in detail. The $\mathbb{Z}_4$ action is given by the Fourier automorphism $f \to \tilde{f}$ where $\tilde{f}(x) = \int_{\mathbb{R}} e(\langle x, x'\rangle)f(x')dx'$. Walters has studied these modules extensively and among other things he computed the Chern character for the flip modules and Fourier modules. The $\mathbb{Z}_3$ and $\mathbb{Z}_6$ actions are similar so we only treat the $\mathbb{Z}_6$ action.

The cyclic group $\mathbb{Z}_6$ is generated by the matrix $W_6 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ that we denote by W. Note that this W_6 slightly differs from W_6 from section 4. We choose this W_6 to keep the final formula similar to the formula for $W_6 := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ in [22]. One should note that the action of the finite group on the module as in Proposition 4.4.3 is not unique.

The generating function associated to $W = W_6$ is given by

$$W(x, x') = xx' - \frac{1}{2}x'^2,$$

which follows from (4.4.1). The corresponding metaplectic transformation (for the choice $s = 1$) is

$$\mathcal{F}_W(f)(x) = e(\frac{1}{8}) \int_{\mathbb{R}} e(xx' - \frac{1}{2}x'^2) f(x') dx', \ f \in \mathscr{S}(\mathbb{R}).$$

The following proposition is due to Walters:

Proposition 4.5.1.

$$(\mathcal{F}_W)^6 = -\mathrm{Id}.$$

We modify the operator $\mathcal{F}_W$ to $e(\frac{1}{24})\mathcal{F}_W$, which amounts to including the Maslov index of the transformation. Then $\mathcal{F}_W^6 = \mathrm{Id}$. The corresponding module over $A_J \rtimes \mathbb{Z}_6$ is called the hexic module by Walters, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic form on $\mathbb{R}^2$. For a general A_θ, choosing $T := \begin{pmatrix} -\theta & 0 \\ 0 & 1 \end{pmatrix}$, we get from the main theorem:

$$\mathcal{F}_{W_\theta}(f)(x) = e(\frac{1}{24})\theta^{-\frac{1}{2}} \int_{\mathbb{R}} e(\frac{1}{2\theta}(2xx' - x'^2)) f(x') dx', f \in \mathscr{S}(\mathbb{R});$$

which is the exactly the formula for the $\mathbb{Z}_6$ action considered in [22].

5 Equivariant Connes-Thom isomorphism for C*-algebras

5.1 Introduction

This chapter is organised as follows. In the second section we recall some basics of equivariant KK-theory for compact group actions on C*-algebras. After introducing setups and notation, in the third section we introduce Connes' pseudo-differential calculus which will play an important role in the main theorem. In the fourth section we finally prove the main theorem of this chapter. In the last section we describe deformation quantization of C*-algebras due to Rieffel and give a computation of the K-theory of deformation quantization of C*-algebras with a compact group action, which leads to the computations of K-theory for twisted crystallographic group algebras.

5.2 Some basic definitions and notations

5.2.1 Equivariant KK-theory

In this subsection we recall equivariant KK-theory for compact group actions on C*-algebras.

Let G be a compact group. Let A, B be separable graded G-C*-algebras. Recall that $\mathcal{E}^G(A,B)$ consists of triples$\{(\mathcal{H},\psi,T)\}$, where $\mathcal{H}$ is a G-Hilbert B-module (countably generated and graded), $\psi : A \to \mathcal{L}(\mathcal{H})$ is a graded *-homomorphism with

$$\psi(ga) = g\psi(a),\ g \in G, a \in A,$$

and $T \in \mathcal{L}(\mathcal{H})$ is such that

$$\begin{aligned}(T - T^*)\psi(a) &\in \mathcal{K}(\mathcal{H}),\\ (gT - T)\psi(a) &\in \mathcal{K}(\mathcal{H}),\\ [\psi(a), T] &\in \mathcal{K}(\mathcal{H}),\\ \psi(a)(I - T^2) &\in \mathcal{K}(\mathcal{H}),\end{aligned}$$

for all $g \in G$, $a \in A$. The group $\mathrm{KK}^G(A, B)$ is defined to be $\mathcal{E}^G(A, B)$ modulo the equivalence relations which were introduced in 2.2. Since G is compact, we can actually choose T to be G-equivariant (since $\mathcal{H}$ is G-space (by action κ, say) and $T \in \mathcal{L}(\mathcal{H})$, we can replace T by $\int_G \mathrm{Ad}\kappa_g(T)dg$).

5.2.2 The setup

Let $\mathbb{R}^n$ act strongly continuously on a C*-algebra A by the action α. Also let G, a compact group, act on A by the action β. Define $GL_n(J)$ to be the group of invertible matrices g such that $g^t Jg = J$, where J is any real, skew-symmetric matrix, and $SL_n(\mathbb{R}, J) := SL_n(\mathbb{R}) \bigcap GL_n(J)$. Let $\rho : G \to SL_n(\mathbb{R}, J)$ be a group homomorphism such that

$$\beta_g \alpha_x = \alpha_{\rho_g(x)} \beta_g, \quad \text{for any } g \in G, x \in \mathbb{R}^n. \tag{5.2.1}$$

When ρ is a trivial group homomorphism, the actions α and β commute. The above condition really means that there is an action of the semi-direct product $\mathbb{R}^n \rtimes G$ on A. Here G acts on $\mathbb{R}^n$ by the map ρ, i.e $g \cdot x = \rho_g(x), g \in G, x \in \mathbb{R}^n$ and $\mathbb{R}^n \rtimes G$ acts on A by $(x, g) \cdot a = \alpha_x \beta_g(a), g \in G, x \in \mathbb{R}^n, a \in A$.

Let $\rho' : G \to GL_n(J)$ be defined by $\rho'(g) = ((\rho(g))^{-1})^t$. If we denote the dual group of $\mathbb{R}^n$ by $\mathbb{R}_n$, G also acts on $\mathbb{R}_n$ by $g \cdot x = \rho'_g(x), g \in G, x \in \mathbb{R}_n$. Note that the image of ρ' consists of invertible matrices g satisfying $gJg^t = J$.

As G is compact, there is always a G-invariant metric on $\mathbb{R}^n$. Without loss of generality, we will assume in what follows that G preserves the standard metric on $\mathbb{R}^n$.

5.2.3 The equivariant Takesaki-Takai duality theorem

There is a dual action $\hat{\alpha}$ of $\mathbb{R}_n$ on $A \rtimes_\alpha \mathbb{R}^n$ given by

$$\hat{\alpha}_x(f)(s) = e(\langle x, s \rangle) f(s)$$

The action of G on a compact operator (viewed as an operator on $L^2(\mathbb{R}^n)$) is $g(T')(f)(x) = T'(g^{-1} \cdot (f))(g^{-1}x)$. If we realise a compact operator by a kernel function $k(r, s)$ on $\mathbb{R}^n \times \mathbb{R}^n$ then the G action is the diagonal action i.e $g \cdot k(x, y) = k(g^{-1}x, g^{-1}y)$

Following [63, Page 190], we prove a G-equivariant version of the Takesaki-Takai duality theorem

Theorem 5.2.1. *$A \rtimes_\alpha \mathbb{R}^n \rtimes_{\hat{\alpha}} \mathbb{R}_n$ is isomorphic to $A \otimes \mathcal{K}(L^2(\mathbb{R}^n))$. And the isomorphism can be made G-equivariant.*

Proof. Let γ be the action of $\mathbb{R}^n$ on $C_0(\mathbb{R}^n, A)$ given by

$$(\gamma_t f)(s) = f(s-t),$$

From the proof of Theorem 7.1 ([63]), we have the isomorphism of $A \rtimes_\alpha \mathbb{R}^n \rtimes_{\hat{\alpha}} \mathbb{R}_n$ and $C_0(\mathbb{R}^n, A) \rtimes_\gamma \mathbb{R}^n$ given by Φ, where

$$\Phi(F)(s,r) = \int_{\mathbb{R}_n} \alpha_r^{-1}(F(t,s))e(\langle r-s, t\rangle)dt, \;\; F \in C_c(\mathbb{R}_n \times \mathbb{R}^n, A).$$

The isomorphism ([63, Lemma 7.6]) between $C_0(\mathbb{R}^n, A) \rtimes_\gamma \mathbb{R}^n$ and $A \otimes \mathcal{K}(L^2(\mathbb{R}^n))$ is easily checked to be G-equivariant. Let us now show that Φ is G-equivariant i.e $g \cdot \Phi(F) = \Phi(g \cdot F)$.

$$\begin{aligned}
\Phi(g\cdot F)(s,r) &= \int_{\mathbb{R}_n} \alpha_r^{-1}(g\cdot F(x,s))e(\langle r-s,x\rangle)dx \\
&= \int_{\mathbb{R}_n} \alpha_r^{-1}(\beta_g(F(g^t x, g^{-1}s)))e(\langle r-s,x\rangle)dx \\
\Phi(g\cdot F)(gs,r) &= \int_{\mathbb{R}_n} \alpha_r^{-1}(\beta_g(F(g^t x, s)))e(\langle r-gs,x\rangle)dx \\
&= \int_{\mathbb{R}_n} \alpha_r^{-1}(\beta_g(F(x, s)))e(\langle r-gs,(g^t)^{-1}x\rangle)dx \\
&= \int_{\mathbb{R}_n} \alpha_r^{-1}(\beta_g(F(x, s)))e(\langle r-gs,(g^t)^{-1}x\rangle)dx \\
&= \int_{\mathbb{R}_n} \alpha_r^{-1}(\beta_g(F(x, s)))e(\langle g^{-1}r-s,x\rangle)dx \\
\Phi(g\cdot F)(gs,gr) &= \int_{\mathbb{R}_n} \alpha_{gr}^{-1}(\beta_g(F(x, s)))e(\langle r-s,x\rangle)dx \\
&= \int_{\mathbb{R}_n} \beta_g(\alpha_r^{-1}(F(x, s)))e(\langle r-s,x\rangle)dx \\
\Phi(g\cdot F)(s,r) &= \int_{\mathbb{R}_n} \beta_g(\alpha_{g^{-1}r}^{-1}(F(x, g^{-1}s)))e(\langle g^{-1}r-g^{-1}s,x\rangle)dx \\
&= \beta_g(\Phi(F)(g^{-1}s, g^{-1}r)) \\
&= (g\cdot\Phi(F))(s,r).
\end{aligned}$$

□

5.3 Connes' pseudo-differential calculus

We assume that the reader is familiar with the definition of classical pseudo-differential calculus of $\mathbb{R}^n$ i.e Hörmander classes of symbols. We refer to [31] for a throughout discussion of that. Connes in [16] introduced an anisotropic version of Hörmander classes of symbols. Later it was studied in [2,3] in detail.

Suppose $\mathbb{R}^n$ acting (with an action α) on a unital (adjoining unit if necessary) C*-algebra A and A^∞ be the smooth algebra of A of smooth vectors for the action. It is well known that A^∞ is a locally convex algebra with respect to a system of seminorms (p_i), which is induced from the norm of A. For the C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$, let $x \mapsto V_x$ be the canonical representation of $\mathbb{R}^n$ in $M(A \rtimes_\alpha \mathbb{R}^n)$, the multiplier algebra, with $V_x a V_x^* = \alpha_x(a)$ $(a \in A)$. Let $\mathbb{R}_n$ be the Fourier dual of $\mathbb{R}^n$ as before. We shall say that ρ, a C^∞ map from $\mathbb{R}_n$ to A^∞, is a symbol of order m, $\rho \in S^m(\mathbb{R}_n, A^\infty)$ iff :

1. for all multi-indices i, j, there exists $C_{ij} < \infty$ such that
$$p_i\left(\left(\frac{\partial}{\partial \xi}\right)^j \rho(\xi)\right) \le C_{ij}(1 + |\xi|)^{m-|j|}\,;$$
2. there exists $s \in C^\infty(\mathbb{R}_n \setminus \{0\}, A^\infty)$ such that when $\lambda \to +\infty$ one has $\lambda^{-m}\,\rho(\lambda\,\xi) \to s(\xi)$ [for the topology of $C^\infty(\mathbb{R}_n \setminus \{0\}, A^\infty)$].

When A is $C_0(\mathbb{R}^n)$ and $\mathbb{R}^n$ is acting on A by the translation action, we may think ρ as a two variable function. In this case we get back the classical symbols ([38, Lemma 2.7]).

Then Connes proved that an order zero symbol gives rise to an element of the multiplier algebra of the crossed product $A \rtimes_\alpha \mathbb{R}^n$. Indeed, if ρ is a symbol of order zero then we can take the Fourier transform (in sense of distribution):
$$\widehat{\rho}(x) = \int_{\mathbb{R}_n} \rho(\xi) e(-\langle x, \xi\rangle) d\xi,$$
which is a well-defined distribution on $\mathbb{R}^n$ with values in A^∞ (it will be clear later what a distribution means). Following [16, Prop. 8], we define the multiplier $D_\rho \in M(B \rtimes \mathbb{R}^n)$ by
$$D_\rho := \int_{\mathbb{R}^n} \widehat{\rho}(x) V_x dx.$$
Following [2, Definition 3.1], D_ρ acts on the smooth sub-algebra $\mathcal{S}(\mathbb{R}^n, A^\infty)$ of $A \rtimes_\alpha \mathbb{R}^n$ by the oscillatory integral (see [1, Section 3.3])
$$D_\rho(u)(t) := \int_{\mathbb{R}^n} \int_{\mathbb{R}_n} \alpha_{-t}(\rho(\xi)) u(s) e(-\langle (t-s), \xi\rangle) ds d\xi.$$

To motivate the above equation, let us take $\rho \in \mathcal{S}(\mathbb{R}^n, A^\infty)$. Then

$$\begin{aligned} D_\rho(u)(t) &= (\int_{\mathbb{R}^n} \widehat{\rho}(s) V_s u ds)(t) \\ &= \int_{\mathbb{R}^n} \alpha_{-t}(\widehat{\rho}(s)) V_s(u(t)) ds \\ &= \int_{\mathbb{R}^n} \alpha_{-t}(\widehat{\rho}(s)) u(t-s) ds \\ &= \int_{\mathbb{R}^n} \alpha_{-t}(\widehat{\rho}(t-s)) u(s) ds \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}_n} \alpha_{-t}(\rho(\xi)) u(s) e(-\langle (t-s), \xi \rangle) ds d\xi. \end{aligned}$$

Note that the above integrals exist in the usual sense.

The set (norm closure) of all multipliers, which are coming from order zero symbols, we denote by $\mathcal{D}(A \rtimes \mathbb{R}^n)$. From [16, Prop. 8] and [2] there is an exact sequence

$$0 \longrightarrow A \rtimes_\alpha \mathbb{R}^n \xrightarrow{\varphi} \mathcal{D}(A \rtimes_\alpha \mathbb{R}^n) \xrightarrow{\psi} A \otimes C(S^{n-1}) \longrightarrow 0,$$

This exact sequence is often called the pseudo-differential extension. It is well known that there is a non-degenerate morphism $C^*(\mathbb{R}^n)$ to $M(A \rtimes_\alpha \mathbb{R}^n)$. So this morphism extends to the multiplier algebra of $C^*(\mathbb{R}^n)$ and in particular to the sub-algebra $\mathcal{D}(C^*(\mathbb{R}^n))$. So if we say $D \in \mathcal{D}(C^*(\mathbb{R}^n))$, we view D inside $M(A \rtimes_\alpha \mathbb{R}^n)$.

Theorem 5.3.1. *For $a \in A \hookrightarrow M(A \rtimes \mathbb{R}^n)$ and $D \in \mathcal{D}(C^*(\mathbb{R}^n))$, we have $[D, a] \in A \rtimes_\alpha \mathbb{R}^n$.*

Proof. See [2, Section 4], also [21, Proposition 4.3] for a more general version of this property. □

Let us recall the definition of asymptotic expansion of a symbol. For a decreasing divergent sequence $(m_j)_{j\in\{0,1,2,\cdots\}}$, and $\rho_j \in S^{m_j}(\mathbb{R}_n, A^\infty)$, we say $\rho \in S^{m_0}(\mathbb{R}_n, A^\infty)$ admits an asymptotic expansion $\sum \rho_j$ (written as $\rho \sim \sum \rho_j$), if for all integers $k \geq 1$,

$$\rho - \sum_{j\leq k} \rho_j \in S^{m_k}(\mathbb{R}_n, A^\infty).$$

For a multi-index k, and $a \in A^\infty$, let us denote the k-th derivative of a (with respect to the action of $\mathbb{R}^n$) by $\delta^k(a)$.

Theorem 5.3.2. *For $\rho_1 \in S^{m_1}(\mathbb{R}_n, A^\infty)$ and $\rho_2 \in S^{m_2}(\mathbb{R}_n, A^\infty)$, there exists a unique $\rho \in S^{m_1+m_2}(\mathbb{R}_n, A^\infty)$ such that $D_\rho = D_{\rho_1} D_{\rho_2}$. Also ρ admits an asymptotic expansion:*

$$\rho(\xi) \sim \sum_k \frac{i^{|k|}}{k!} \rho_1^{(k)}(\xi) \delta^k(\rho_2(\xi)).$$

Proof. See [2, Proposition 3.2]. Also [39, Theorem 2.2], for twisted dynamical systems. □

Theorem 5.3.3. *For $\rho \in S^0(\mathbb{R}_n, A^\infty)$ the the adjoint of D_ρ, $(D_\rho)^*$ exists and $(D_\rho)^* = D'_\rho$, where ρ' admits an asymptotic expansion:*

$$\rho'(\xi) \sim \sum_k \frac{i^{|k|}}{k!} \delta^k((\rho')^{(k)}(\xi)^*).$$

Proof. See [2, Proposition 3.3]. Also [39, Theorem 2.2], for twisted dynamical systems. □

Remark 5.3.4. *Since the unitisation of A, A', sits inside $M(A)$ non-degenerately, we get (*[21, Proposition 3.2]*) a non-degenerate morphism from $A' \rtimes \mathbb{R}^n$ to $M(A \rtimes \mathbb{R}^n)$ giving a morphism from $\mathcal{D}(A' \rtimes_\alpha \mathbb{R}^n)$ to $M(A \rtimes \mathbb{R}^n)$. Hence, though we adjoin a unit for the non-unital A, ultimately we end up getting an element in $M(A \rtimes \mathbb{R}^n)$.*

5.4 Equivariant Connes–Thom isomorphism for equivariant KK theory

Let $\mathbb{C}_n$ be the complexified Clifford algebra associated with $\mathbb{R}^n$. It is a graded C*-algebra with its natural grading. For the following theorem, assume $\mathbb{R}^n$ acts on $\mathbb{C}_n$ trivially and G has induced action on $\mathbb{C}_n$ coming from its action on $\mathbb{R}^n$.

Theorem 5.4.1. *Let $\mathbb{R}^n$ and a compact Lie group G act on an ungraded C*-algebra A, as previously. Let $\rho : G \to SL_n(\mathbb{R}, J)$ be a homomorphism. If the actions α and β satisfy equation (5.2.1), then*

$$\mathrm{K}^G\big((A \otimes \mathbb{C}_n) \rtimes_\alpha \mathbb{R}^n\big) \cong \mathrm{K}^G(A),$$

Let $B^\infty := A^\infty \otimes \mathbb{C}_n$, with an extended action of α by $\mathbb{R}^n$, which acts trivially on the second component.

We first construct Kasparov's Dirac and dual-Dirac elements for the G-equivariant $\mathbb{R}^n$ action on the C*-algebra A. We write the standard Hodge-de Rham operator as $d+d^*$ on $\mathbb{R}^n$ which is acting on $\mathbb{H} = L^2(\mathbb{R}^n, \wedge^\bullet \mathbb{C}^n)$. Let $\Delta = dd^* + d^*d$ be the Laplace-Beltrami operator, and Σ be the principal symbol of $(d+d^*)(1+\Delta)^{-1/2}$, then

$$\Sigma(x,\xi) = \frac{c(\xi)}{(1+\|\xi\|^2)^{\frac{1}{2}}}, \qquad x \in \mathbb{R}^n, \xi \in \mathbb{R}_n,$$

where $c(\xi)$ is the Clifford multiplication of ξ on $\wedge^\bullet \mathbb{C}^{n*}$. The key point is that the symbol is independent of x. Furthermore, we also observe that $(d+d^*)(1+\Delta)^{-1/2}$ is G-invariant, as G is assumed to preserve the metric. Let us point out that we are working with Clifford algebra valued symbols and we have the pseudo-differential extension

$$0 \longrightarrow \mathbb{C}_n \otimes \mathbb{R}^n \xrightarrow{\varphi} \mathcal{D}(\mathbb{C}_n \otimes \mathbb{R}^n) \xrightarrow{\psi} C(S^{n-1}, \mathbb{C}_n) \longrightarrow 0\cdot$$

Consider $B := A \otimes \mathbb{C}_n$ with an extended action α by $\mathbb{R}^n$, which acts trivially on the component $\mathbb{C}_n$. The smooth subalgebra B^∞ of α is identified with $A^\infty \otimes \mathbb{C}_n$. Let us define $\Sigma \in S^0(\mathbb{R}_n, B^\infty)$ as

$$\Sigma(\xi) = 1 \otimes \frac{\xi}{(1+\|\xi\|^2)^{\frac{1}{2}}}, \qquad \xi \in \mathbb{R}_n \subset \mathbb{C}_n$$

It is now a well-defined distribution on $\mathbb{R}^n$ with values in B^∞. We have $D_\alpha \in M(B \rtimes \mathbb{R}^n)$ by

$$D_\alpha := \int_{\mathbb{R}^n} \widehat{\Sigma}(x) V_x dx.$$

As $(d+d^*)(1+\Delta)^{-1/2}$ defines an element in $\mathrm{KK}^G(B, \mathbb{C})$, it is natural to expect that $(B \rtimes \mathbb{R}^n, \iota, D_\alpha)$ defines an element (also called the Thom element or Thom class) $y^G_{n,\alpha}$ in $\mathrm{KK}^G(A, B \rtimes_\alpha \mathbb{R}^n)$. Here ι denotes the inclusion of A in the multiplier algebra $M(B \rtimes \mathbb{R}^n)$. The following series of results prove that $(B \rtimes \mathbb{R}^n, \iota, D_\alpha)$ is indeed a Kasparov module.

Lemma 5.4.2. *The Fourier transform map which sends Σ to $\widehat{\Sigma}$ is G-equivariant, i.e $\widehat{g \cdot \Sigma} = g \cdot \widehat{\Sigma}$.*

Proof.

$$\begin{aligned}
\widehat{g\cdot\Sigma}(x) &= \int_{\mathbb{R}_n} (g\cdot\Sigma)(\xi)e(-\langle x,\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \beta_g(\Sigma(g^t\xi))e(-\langle x,\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \beta_g(\Sigma(\xi))e(-\langle x,(g^t)^{-1}\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \beta_g(\Sigma(\xi))e(-\langle g^{-1}x,\xi\rangle)d\xi \\
&= \beta_g(\widehat{\Sigma}(g^{-1}x)) \\
&= (g\cdot\widehat{\Sigma})(x).
\end{aligned}$$

□

Warning 5.4.3. *The integrands of the above give rise to divergent integrals: we took the Fourier transformation of order zero symbol (as tempered distribution). To regularise divergent oscillatory integrals, one does the following. Since we realise $\widehat{\Sigma}$ as a distribution, for $u\in\mathcal{S}(\mathbb{R}^n,A^\infty)$, $\langle\widehat{\Sigma},u\rangle$ is given by*

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^n}\int_{\mathbb{R}_n}\Sigma(\xi)u(x)e\left(-\frac{\epsilon\|\xi\|^2}{2}\right)e(-\langle x,\xi\rangle)d\xi dx.$$

Now since the expression $e\left(-\frac{\epsilon\|\xi\|^2}{2}\right)$ is G-invariant, the change of variable in the above proof makes sense. From now on we will use change of variables in oscillatory integrals for the action of G without any further explanation.

Lemma 5.4.4. *D_α is G equivariant.*

Proof. First note that the symbol is independent of A and the $\mathbb{R}^n$-action on

$\mathbb{C}_n$ is trivial. So $\alpha_{-t}(\Sigma(\xi)) = \Sigma(\xi)$. Then, for $u \in \mathcal{S}(\mathbb{R}^n, A^\infty)$,

$$\begin{aligned}
(g \cdot D_\alpha u)(t) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}_n} \beta_g(\alpha_{-t}(\Sigma(\xi)))\beta_g(u(s))e(-\langle (g^{-1}t - s), \xi\rangle) ds d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}_n} \beta_g(\Sigma(\xi))\beta_g(u(s))e(-\langle (g^{-1}t - s), \xi\rangle) ds d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}_n} \beta_g(\Sigma(\xi))\beta_g(u(g^{-1}s))e(-\langle (g^{-1}t - g^{-1}s), \xi\rangle) ds d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}_n} 1 \otimes \frac{(g^t)^{-1}\xi}{(1+\|\xi\|^2)^{\frac{1}{2}}}\beta_g(u(g^{-1}s))e(-\langle (t - s), (g^t)^{-1}\xi\rangle) ds d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}_n} 1 \otimes \frac{\xi}{(1+\|\xi\|^2)^{\frac{1}{2}}}\beta_g(u(g^{-1}s))e(-\langle (t - s), \xi\rangle) ds d\xi \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}_n} \Sigma(\xi)\beta_g(u(g^{-1}s))e(-\langle (t - s), \xi\rangle) ds d\xi \\
&= D_\alpha(g \cdot u)(t).
\end{aligned}$$

□

Lemma 5.4.5. *D_α is self adjoint.*

Proof. By definition $(D_\alpha)^* = (D_\Sigma)^*$. From 5.3.3, we have $(D_\Sigma)^* = D_\rho$ and from the asymptotic expansion of ρ, $\rho(\xi) = \Sigma(\xi)^*$ as all higher derivatives in the expression vanish (since the $\mathbb{R}^n$ action on the Clifford algebra is trivial). Now the result follows as Σ is self-adjoint being a real valued map.

□

Lemma 5.4.6. $[a, D_\alpha] \in B \rtimes_\alpha \mathbb{R}^n$ *for* $a \in A \hookrightarrow M(B \rtimes_\alpha \mathbb{R}^n)$.

Proof. First note that we have the pseudo-differential extension:

$$0 \longrightarrow B \rtimes_\alpha \mathbb{R}^n \xrightarrow{\varphi} \mathcal{D}(B \rtimes_\alpha \mathbb{R}^n) \xrightarrow{\psi} A \otimes C(S^{n-1}, \mathbb{C}_n) \longrightarrow 0.$$

Now for $a \in A^\infty$, let us compute the asymptotic expansions of aD_α and $D_\alpha a$. From 5.3.2, we have symbol of aD is $a\Sigma$ and symbol of Da is

$$\Sigma a + \text{negative order symbol}.$$

The above negative order symbol involvs higher order derivatives of Σ and higher order derivatives of a. Since ψ in the above exact sequence is principal symbol map, $\psi(aD_\alpha - D_\alpha a)$ vanishes which proves our claim, for $a \in A^\infty$. Since A^∞ is dense in A, the same is true for $a \in A$, too.

□

Lemma 5.4.7. *$a(1 - D_\alpha^2) \in B \rtimes_\alpha \mathbb{R}^n$ for $a \in A \hookrightarrow M(B \rtimes_\alpha \mathbb{R}^n)$*

Proof. Let us first compute the symbol of D_α^2. From 5.3.2, we have $D_\alpha^2 = F_\rho$, where $\rho(\xi) = (\Sigma(\xi))^2$ (as the $\mathbb{R}^n$ action on the Clifford algebra is trivial, the higher order derivatives in the asymptotic expansion of σ vanishes). Now

$$\begin{aligned}(1 - D_\alpha^2) &= \int_{\mathbb{R}^n} \widehat{1}(z) V_z dz - \int_{\mathbb{R}^n} \hat{\rho}(z) V_z dz \\ &= \int_{\mathbb{R}^n} \widehat{1}(z) V_z dz - \int_{\mathbb{R}^n} \widehat{\Sigma^2}(z) V_z dz \\ &= \int_{\mathbb{R}^n} (\widehat{1 - \Sigma^2}(z)) V_z dz.\end{aligned}$$

Now

$$1 - \Sigma^2 = 1 \otimes \frac{1}{(1 + \|\xi\|^2)} \qquad \xi \in \mathbb{R}_n$$

is a negative order symbol. So $\int_{\mathbb{R}^n} (\widehat{1 - \Sigma^2}(z)) V_z dz$ is in $B \rtimes_\alpha \mathbb{R}^n$ (follows from the pseudo-differential extension) which proves our claim. □

Lemma 5.4.8. *$g \cdot D_\alpha = D_\alpha$.*

Proof.

$$\begin{aligned}g \cdot D_\alpha &= \int_{\mathbb{R}^n} g \cdot \widehat{\Sigma}(x) V_{gx} dx \\ &= \int_{\mathbb{R}^n} \widehat{\Sigma}(gx) V_{gx} dx \\ &= \int_{\mathbb{R}^n} \widehat{\Sigma}(x) V_x dx \\ &= D_\alpha.\end{aligned}$$

Where we used that

$$\widehat{\Sigma}(gx) = g \cdot \widehat{\Sigma}(x),$$

which is in fact easy. Indeed

$$\begin{aligned}
\widehat{\Sigma}(gx) &= \int_{\mathbb{R}_n} \Sigma(\xi)e(-\langle gx,\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \Sigma(\xi)e(-\langle x,g^t\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \Sigma((g^t)^{-1}\xi)e(-\langle x,\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \Sigma(g\cdot\xi)e(-\langle x,\xi\rangle)d\xi \\
&= \int_{\mathbb{R}_n} \beta_g(\Sigma(\xi))e(-\langle x,\xi\rangle)d\xi \\
&= g\cdot\widehat{\Sigma}(x).
\end{aligned}$$

□

Now let $\hat{\alpha}$ be the dual action of $\mathbb{R}_n$ on $A\rtimes_\alpha\mathbb{R}^n$ and also on $B\rtimes_\alpha\mathbb{R}^n$, and $x^G_{n,\alpha}$ be the element defined by $D_{\hat{\alpha}}$ in $\mathrm{KK}^G(B\rtimes\mathbb{R}^n,(B\rtimes\mathbb{R}^n\otimes\mathbb{C}_n)\rtimes_{\hat{\alpha}}\mathbb{R}^n)$. Via the G-equivariant Morita equivalence $B\otimes\mathbb{C}_n = A\otimes\mathbb{C}_n\otimes\mathbb{C}_n\sim_M A$ and the G-equivariant Takai duality, we have $\mathrm{KK}^G(B\rtimes\mathbb{R}^n,(B\rtimes\mathbb{R}^n\otimes\mathbb{C}_n)\rtimes_{\hat{\alpha}}\mathbb{R}^n) = KK^G(B\rtimes\mathbb{R}^n, A)$.

Theorem 5.4.9 (Equivariant Connes-Thom)**.** *$x^G_{n,\alpha}$ and $y^G_{n,\alpha}$ are inverses to one another in* KK^G*-theory.*

Of course, this theorem will imply the Theorem 5.4.1.

5.4.1 Proof of equivariant Connes-Thom isomorphism

We prove the theorem showing that the elements $y^G_{n,\alpha}$ and $x^G_{n,\alpha}$ are inverses to each other (following closely the Fack–Skandalis proof).

When $A=\mathbb{C}$, α is also trivial, we call the elements $y^G_{n,\alpha}$ and $x^G_{n,\alpha}$ by y^G_n and x^G_n respectively. We claim that these are the usual G-equivariant Kasparov Dirac-dual dirac elements, which Kasparov constructed to prove Bott periodicity (see [35]). This is in fact easy: as when $A=\mathbb{C}$, y^G_n is clearly given by the module $(C_0(\mathbb{R}^n,\mathbb{C}_n),\iota,f)$, where f is the function

$$f(\xi)=\frac{\xi}{(1+\|\xi\|^2)^{\frac{1}{2}}},\qquad \xi\in\mathbb{R}_n=\mathbb{R}^n.$$

Also as we mentioned before that, in pseudo-differential calculus when one takes $A=C_0(\mathbb{R}^n)$ equipped with the standard $\mathbb{R}^n$-translation action $\hat{\alpha}$, one get the classical pseudo-differential calculus back. Since here we started with the symbol of the operator $(d+d^*)(1+\Delta)^{-1/2}$, taking the symbol with values in $C_0(\mathbb{R}^n)\otimes\mathbb{C}_n\otimes\mathbb{C}_n$, $x^G_{n,\alpha}$ is identified with the Dirac element of Kasparov in $\mathrm{KK}^G(C_0(\mathbb{R}^n)\otimes\mathbb{C}_n,\mathbb{C})$ which is nothing but $(d+d^*)(1+\Delta)^{-1/2}$ acting on $\mathbb{H}=L^2(\mathbb{R}^n,\wedge^\bullet\mathbb{C}^n)$. Note that we identified the class $\mathrm{KK}^G(C_0(\mathbb{R}^n)\otimes\mathbb{C}_n,\mathcal{K})$ with $\mathrm{KK}^G(C_0(\mathbb{R}^n)\otimes\mathbb{C}_n,\mathbb{C})$ by the Morita equivalence of $\mathcal{K}$ and $\mathbb{C}$. But this does not change the Dirac operator as the operator appears in the Morita class $\mathrm{KK}^G(\mathcal{K},\mathbb{C})$ is zero and the identification is made by taking Kasparov product with this Morita class. So in the case $A=\mathbb{C}$, $y^G_{n,\alpha}$ and $x^G_{n,\alpha}$ are inverses of each other (using Kasparov's Bott periodicity).

Suppose A and A' be such that $\mathbb{R}^n$ (by α and α') and G (by β and β') act on them like the way as discussed in 5.2.2. Let $\rho: A\to A'$ be an $\mathbb{R}^n\rtimes G$ equivariant homomorphism of C^* algebras. Also assume that $\rho(A)$ contains an approximate unit of A'. Then ρ induces a G-morphism $\hat\rho: A\otimes\mathbb{C}_n\rtimes_\alpha\mathbb{R}^n\to A'\otimes\mathbb{C}_n\rtimes_{\alpha'}\mathbb{R}^n$. Then naturality of the Thom class means

$$\hat\rho_*(x^G_{n,\alpha})=\rho^*(x^G_{n,\alpha'}).$$

Now $\rho^*(x^G_{n,\alpha'})$ by definition is $(A'\otimes\mathbb{C}_n\rtimes\mathbb{R}^n,\iota\circ\rho,D_{\alpha'})\in\mathrm{KK}^G(A,A'\otimes\mathbb{C}_n\rtimes\mathbb{R}^n)$. And $\hat\rho_*(x^G_{n,\alpha})$ is $(\rho_*(A\otimes\mathbb{C}_n\rtimes\mathbb{R}^n),\iota\otimes 1,D_\alpha\otimes 1)\in\mathrm{KK}^G(A,A'\otimes\mathbb{C}_n\rtimes\mathbb{R}^n)$. Now using [8, Example 13.5.2], $\rho_*(A\otimes\mathbb{C}_n\rtimes\mathbb{R}^n)$ becomes $A'\otimes\mathbb{C}_n\rtimes\mathbb{R}^n$ and $\iota\otimes 1$ becomes $\iota\circ\rho$. Since $D_\alpha\in\mathcal{D}(C^*(\mathbb{R}^n)\otimes\mathbb{C}_n)\hookrightarrow M(A\otimes\mathbb{C}_n\rtimes\mathbb{R}^n)$, it is natural in the sense that, under the map $\hat\rho: A\otimes\mathbb{C}_n\rtimes_\alpha\mathbb{R}^n\to A'\otimes\mathbb{C}_n\rtimes_{\alpha'}\mathbb{R}^n$, we get $D_\alpha\otimes 1$ is same as $D_{\alpha'}$ which proves the naturality of the Thom class.

Thus having these properties of the elements, the rest of the proof goes exactly like [26, Theorem 2]. The main idea is follows: define a deformation α^s of the action α for $s\in[0,1]$. Define $\alpha^s:\mathbb{R}^n\times A\to A$ by

$$\alpha^s_x(a):=\alpha_{sx}(a),\qquad a\in A, x\in\mathbb{R}^n.$$

So we get the elements $y_s:=y^G_{n,\alpha^s}$ and $x_s:=x^G_{n,\alpha^s}$.

Let us consider the algebra $D=C([0,1],A)$ and let $\mathbb{R}^n$ act on this algebra by $\gamma_t(f)(x)=\alpha_{tx}(f(x))$. Now the map $\mathrm{ev}_s: D\to A$ defined by $\mathrm{ev}_s(f)=f(s)$ is an $\mathbb{R}^n\rtimes G$ equivariant map, where the G action on $C([0,1],A)=C[0,1]\otimes A$ is given by the diagonal action (G-action on $C[0,1]$ being trivial). Now we get a G-map $\widehat{\mathrm{ev}_s}: D\rtimes\mathbb{R}^n\to A\rtimes\mathbb{R}^n$ which is also a $\mathbb{R}_n\rtimes G$ equivariant map with respect to the dual action. Now by the definition of the Kasparov product and naturality of Thom elements, we have the following elements:

$$[\mathrm{ev}_s]\in\mathrm{KK}^G(D,A)$$
$$[\widehat{\mathrm{ev}_s}]\in\mathrm{KK}^G(D\otimes\mathbb{C}_n\rtimes\mathbb{R}^n,A\otimes\mathbb{C}_n\rtimes\mathbb{R}^n)$$

$$[\widehat{\widehat{\mathrm{ev}_s}}] = [\mathrm{ev}_s] \in \mathrm{KK}^G(D, A) \quad \text{(using Takesaki-Takai)}$$
$$y^\gamma := y^G_{n,\gamma} \in \mathrm{KK}^G(D, D \otimes \mathbb{C}_n \rtimes_\gamma \mathbb{R}^n)$$
$$x^\gamma := x^G_{n,\gamma} \in \mathrm{KK}^G(D \otimes \mathbb{C}_n \rtimes_\gamma \mathbb{R}^n, D)$$

which satisfy (by naturality of the Thom class)

$$[\mathrm{ev}_s] \times y_s = y^\gamma \times [\widehat{\mathrm{ev}_s}]$$

and

$$x^\gamma \times [\mathrm{ev}_s] = [\widehat{\mathrm{ev}_s}] \times x_s.$$

Now

$$\begin{aligned}(y^\gamma \times x^\gamma) \times [\mathrm{ev}_s] &= y^\gamma \times (x^\gamma \times [\mathrm{ev}_s]) \\ &= y^\gamma \times [\widehat{\mathrm{ev}_s}] \times x_s \\ &= (y^\gamma \times [\widehat{\mathrm{ev}_s}]) \times x_s \\ &= [\mathrm{ev}_s] \times y_s \times x_s.\end{aligned}$$

But using homotopy invariance of KK^G-theory, $[\mathrm{ev}_s] = [\mathrm{ev}_0]$. And $[\mathrm{ev}_0]$ has homotopy inverse (class of the map f which sends a to $a \otimes 1$). So

$$y_s \times x_s = [f] \times (y^\gamma \times x^\gamma) \times [\mathrm{ev}_0]$$

shows that $y_s \times x_s$ is independent of s. Similarly starting with $(x^\gamma \times y^\gamma) \times [\widehat{\mathrm{ev}_s}]$, we conclude that $y_s \times x_s$ is independent of s. Since we know that $y_0 \times x_0 = 1$ (using Kasparov's Bott periodicity), we get $y_1 \times x_1 = 1$. Similarly $x_1 \times y_1 = 1$.

5.5 Application: K-theory of equivariant quantization

Recall that if α is a strongly continuous action of $\mathbb{R}^n$ on a C^*-algebra A, and J is a skew-symmetric form on $\mathbb{R}^n$, Rieffel [55] constructed a deformation quantization A_J of A via oscillatory integrals

$$a \times_J b := \int_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{Jx}(a)\alpha_y(b) e(x \cdot y) dx dy, \tag{5.5.1}$$

for $x, y \in \mathbb{R}^n$, and $a, b \in A^\infty$. The first copy of $\mathbb{R}^n$ in $\mathbb{R}^n \times \mathbb{R}^n$ is basically $\mathbb{R}_n$ after identification of $\mathbb{R}_n$ and $\mathbb{R}^n$ (see the discussion at Page 11 of [55].) $\mathbb{R}^n$ acts on A_J by the same action α ([55, Proposition 2.5]), and we denote the smooth vectors for this action to be A_J^∞.

We recall the following results about Rieffel deformation from [46]:

Theorem 5.5.1. *The map Θ_J from $A_J^\infty \rtimes \mathbb{R}^n$ to $A^\infty \rtimes \mathbb{R}^n$ defined by*

$$\Theta_J(f)(x) = \int_{\mathbb{R}^n} \alpha_{Jy}(\hat{f}(y))e(x \cdot y)dy$$

is an isomorphism, where $\hat{f}$ is the Fourier transformation of $f \in \mathcal{S}(\mathbb{R}^n, A)$ and $e(t) := e^{2\pi it}$.

Proof. See [46, Theorem 1.1]. □

Theorem 5.5.2. *The map Θ_J extends to isomorphism of $A_J \rtimes \mathbb{R}^n$ to $A \rtimes \mathbb{R}^n$.*

Proof. See [46, Theorem 2.1]. □

Now let G as in 5.2.2. That means G acts on A by β and we have

$$\beta_g\alpha_x = \alpha_{\rho_g(x)}\beta_g, \quad \text{for any } g \in G, x \in \mathbb{R}^n. \tag{5.5.2}$$

Now

$$\begin{aligned}
\beta_g(a) \times_J \beta_g(b) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \alpha_{Jx}(\beta_g(a))\alpha_y(\beta_g(b))e(x \cdot y)dxdy \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \alpha_{(g^t)^{-1}Jg^{-1}x}\beta_g(a)\alpha_{gy}\beta_g(b)e(x \cdot gy)dxdy, \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \alpha_{(g^t)^{-1}Jx}\beta_g(a)\alpha_{gy}\beta_g(b)e(gx \cdot gy)dxdy, \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} \beta_g\alpha_{Jx}(a)\beta_g\alpha_y(b)e(x \cdot y)dxdy, \\
&= \beta_g(a \times_J b)
\end{aligned}$$

shows that the action of G on A_J by β is well defined. So we get G action on $A_J^\infty \rtimes \mathbb{R}^n$ and $A^\infty \rtimes \mathbb{R}^n$. Abusively we call both actions by β again.

The following theorem ensures that we can make the isomorphism of 5.5.1 G-equivariant.

Theorem 5.5.3. *With the notations introduced in the beginning of the previous section,*

$$\beta_g(\theta_J(f)) = \theta_J(\beta_g(f))$$

Proof.

$$\begin{aligned}
\theta_J(\beta_g(f))(x) &= \int_{\mathbb{R}^n} \alpha_{Jy}(\widehat{\beta_g f}(y))e(x\cdot y)dy \\
&= \int_{\mathbb{R}^n} \alpha_{Jy}(\int_{\mathbb{R}^n} \beta_g f(t)e(-y\cdot t)dt))e(x\cdot y)dy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \alpha_{Jy}\beta_g(f(g^{-1}t))e(-y\cdot t)e(x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{g^{-1}Jy}(f(g^{-1}t))e(-y\cdot t)e(x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{g^{-1}Jy}(f(t))e(-y\cdot gt)e(x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{g^{-1}J(g^t)^{-1}y}(f(t))e(-y\cdot t)e(g^{-1}x\cdot y)dtdy \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \beta_g\alpha_{Jy}(f(t))e(-y\cdot t)e(g^{-1}x\cdot y)dtdy \\
&= \beta_g(\theta_J(f))(x).
\end{aligned}$$

Warning 5.5.4. *In the above, since $y \in \mathbb{R}_n$, abusively we have written gy for $(g^t)^{-1}y$ and we used the fact that $g^{-1}J(g^t)^{-1} = J$. In general one should be careful about $\mathbb{R}_n$ and $\mathbb{R}^n$.*

□

Example 5.5.5. *Recall that an n-dimensional noncommutative torus A_θ is the universal C*-algebra generated by unitaries U_1, U_2, U_3, $\cdots$, U_n subject to the relations*

$$U_kU_j = \exp(2\pi i\theta_{jk})U_jU_k$$

for $j,k = 1,2,3,\cdots,n$ and $\theta =: (\theta_{jk})$ being a skew symmetric real $n\times n$ matrix. If we look at the smooth holomorphically closed subalgebra A_θ^∞ of A_θ. The algebra A_θ^∞ can also be viewed as Rieffel deformation of $C^\infty(\mathbb{T}^n)$ by translation action of $\mathbb{R}^n$ and θ is the skew symmetric form.

Example 5.5.6. *Let G be $\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_4$ or $\mathbb{Z}_6$ as finite cyclic groups (can be viewed as matrices in $SL_2(\mathbb{Z})$) acting on $\mathbb{R}^2$. Since the action is $\mathbb{Z}^2$ preserving, the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ inherits an action of G from the G action on $\mathbb{R}^2$. Let A be the C^*-algebra of continuous functions on $\mathbb{T}^2$. The group $\mathbb{R}^2$ acts on $\mathbb{T}^2$ by translation. For $\theta \in \mathbb{R}$, we consider the symplectic form $\theta dx_1 \wedge dx_2$, also denoted by θ. So A_θ is just like the previous example of a 2 dimensional noncommutative torus. The action α (and β) of $\mathbb{R}^2$ (and G) on A satisfy Eq. (5.2.1). Now the G action on A_θ is well defined.*

Also recall that we considered the twisted group algebra (as defined in Chapter 2) $C^*(\mathbb{Z}^2 \rtimes G, \omega_\theta)$*, where* ω_θ *is a 2-cocycle of* $\mathbb{Z}^2$ ($\omega_\theta(x,y) := e^{2\pi i \langle \theta x, y\rangle}$, θ *being a real number) which was extended trivially to the semi-direct product. These actions are considered in* [22]. *Now it is not hard to see that* $C^*(\mathbb{Z}^2 \rtimes G, \omega_\theta) = A_\theta \rtimes G$*, where the latter is defined as in the previous paragraph. In general, with the above 2-cocycles, the twisted group algebras of groups like* $\mathbb{Z}^n \rtimes G$ *are basically coming from equivariant (in the sense discussed in the section 5.2.2) deformation quantization of* $\mathbb{R}^n$ *action on* $A = C(\mathbb{T}^n)$.

Corollary 5.5.7. $\mathrm{K}_*(A_\theta \rtimes G)$ *is independent of* θ *parameter.*

Proof. Here we have $A = C(\mathbb{T}^2)$ and $J = \theta$. From the above theorem and from Theorem 5.5.2 we get,

$$A_\theta \rtimes \mathbb{R}^2 \rtimes G \simeq A \rtimes \mathbb{R}^2 \rtimes G$$

.

Now applying the K functor on the both sides we get,

$$\mathrm{K}_*(A_\theta \rtimes \mathbb{R}^2 \rtimes G) = \mathrm{K}_*(A \rtimes \mathbb{R}^2 \rtimes G).$$

Now since in this particular case G is a finite cyclic group, hence *spin*^{c} preserving. Indeed, the diagram

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & Spin^c(n) & \longrightarrow & SO(n) & \longrightarrow & 0. \end{array}$$

determines a group 2-cocycle on $SO(n)$, and since the restriction of this cocycle to G is trivial (as G is cyclic), the lift

$$\begin{array}{ccccccccc} & & & & & & G & & \\ & & & & & \swarrow & \downarrow & & \\ 0 & \longrightarrow & \mathbb{S}^1 & \longrightarrow & Spin^c(n) & \longrightarrow & SO(n) & \longrightarrow & 0, \end{array}$$

is always possible. Hence

$$\begin{aligned}
\mathrm{K}_*(A_\theta \rtimes \mathbb{R}^2 \rtimes G) &= \mathrm{K}_*^G(A_\theta \rtimes \mathbb{R}^2), \\
&= \mathrm{K}_*^G(A_\theta \otimes \mathbb{C}_2 \rtimes \mathbb{R}^2) \\
&= \mathrm{K}_*^G(A_\theta) \\
&= \mathrm{K}_*(A_\theta \rtimes G)
\end{aligned}$$

and similarly $\mathrm{K}_*((A \rtimes \mathbb{R}^2) \rtimes G) = \mathrm{K}_*(A \rtimes G)$. So the claim follows from the second isomorphism.

□

6 Traces on the twisted group algebras of crystallographic groups

6.1 Introduction

Let us recall that we have introduced the smooth sub-algebras $A_\theta^\infty \rtimes F$ of the C*-algebras $A_\theta \rtimes F$ arising from actions of the finite group F on A_θ. In this chapter we look at some of the zeroth cyclic cocycles of the smooth algebras $A_\theta^\infty \rtimes F$, which will be helpful for computing explicit generators of $\mathrm{K}_0(A_\theta \rtimes F)$, at least in some cases. Like in the C*-algebra case, we shall view the algebras $A_\theta^\infty \rtimes F$ as twisted smooth group algebras of the groups like $\mathbb{Z}^n \rtimes F$. Zeroth cyclic cocycles are basically the traces of the algebra. In the untwisted case, each conjugacy class of the group $\mathbb{Z}^n \rtimes F$ gives a trace of the smooth algebra. But it is not clear that how these traces can be twisted to give traces of the twisted smooth algebra. We will see that an appropriate twisting of these traces is always possible for the group $\mathbb{Z}^n \rtimes F$. A general result for more general groups has been obtained in a joint work with Yamashita (see [15]). The results in this chapter involve careful study of group cocycles, Hochschild cocycles and their interaction.

6.2 Some basic definitions

6.2.1 Noncommutative calculus

Let us first review the fundamental concepts in Hochschild homology theory (see [19] for details).

Let A be a unital $\mathbb{C}$-algebra, and put $\mathcal{C}_n(A) = A^{\otimes n+1}$. Recall that the *Hochschild differential* $\mathcal{C}_n(A) \to \mathcal{C}_{n-1}(A)$ is given by

$$b(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \tag{6.2.1}$$

Then the Hochschild homology groups of A are given by

$$\mathrm{H}_*(A) := \mathrm{H}_*(\mathcal{C}(A), b)$$

Its dual theory we denote by H^*, and the corresponding complex by $\mathcal{C}(A)'$. It is easily checked that the zeroth Hochschild cohomology group $\mathrm{H}^0(A)$ consists of traces of A i.e.

$$\mathrm{H}^0(A) = \{f : A \to \mathbb{C} | f(ab) = f(ba), \forall a, b \in A\}.$$

We also define the zeroth cyclic cohomology group of A, $\mathrm{HC}^0(A)$, to be the group $\mathrm{H}^0(A)$. Since we won't use higher cyclic cohomology groups, we won't discuss much of that here. It is good to know that the the higher cyclic cohomology groups are different from the higher Hochschild cohomology groups.

6.2.2 Standard complex for group cohomology

Let G be a discrete group and M be a trivial G-module. The group cohomology $\mathrm{H}^*(G; M)$ is equal to the cohomology of the *standard complex* $(\mathcal{C}^*(G; M), d)$, where $\mathcal{C}^n(G; M) = \mathrm{Map}(G^n, M)$ and $d \colon \mathcal{C}^n(G; M) \to \mathcal{C}^{n+1}(G; M)$ is given by

$$\begin{aligned} d\phi(g_1, \ldots, g_{n+1}) &= \phi(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} \phi(g_1, \ldots, g_n). \quad (6.2.2) \end{aligned}$$

We formally put $\mathcal{C}^0(G; M) = \mathrm{Map}(*, M) = M$ and d to be the zero map from $\mathcal{C}^0(G; M)$ to $\mathcal{C}^1(G; M)$. Consequently, $\mathrm{H}^0(G; M) = M$ and the 2-cocycles are the 2-point functions satisfying

$$\phi(h, k) + \phi(g, hk) = \phi(g, h) + \phi(gh, k).$$

Obviously, $\mathcal{C}^*(G) = \mathcal{C}^*(G; \mathbb{C})$ is the dual complex of $\mathcal{C}_*(G) = (\mathbb{C}[G^n])_{n=0}^{\infty}$, endowed with differential $d \colon \mathcal{C}_{n+1}(G) \to \mathcal{C}_n(G)$ analogous to that of $\mathcal{C}^*(G)$.[1]

A group cochain $\phi \in \mathcal{C}^n(G; M)$ is *normalized* if

$$\phi(g_1, \ldots, g_i, e, g_{i+1}, \ldots, g_{n-1}) = 0 \quad (0 \leq i \leq n-1)$$

and $\phi(g_1, \ldots, g_n) = 0$ whenever $g_1 \cdots g_n = e$.

Proposition 6.2.1 ([34; 41, Section 4])**.** *When K is a field of characteristic zero and $n > 0$, any K-valued n-cocycle is cohomologous to a normalized one.*

[1] not to be confused with group C*-algebra.

6.2.3 Hochschild cocycles on group algebra

Let us now describe the Hochschild theory for group algebra $\mathbb{C}[G]$, which is essentially from [13]. For determining zeroth cyclic cocycles of twisted algebras of G, this general discussion will be useful as we shall see later.

For each $x \in G$, let us denote the centraliser of x in G by $C_G(x)$, and the conjugacy class of x by $\mathrm{Ad}_G(x)$. Let $\mathcal{C}_n^x(\mathbb{C}[G])$ be the subspace of $\mathcal{C}_n(\mathbb{C}[G]) = \mathbb{C}[G]^{\otimes n+1}$ spanned by those $g_0 \otimes \cdots \otimes g_n$ such that $g_0 \cdots g_n \in \mathrm{Ad}_G(x)$. Let $G\backslash_{\mathrm{Ad}} G$ denote the conjugacy classes of G. The spaces $(\mathcal{C}_n^x(\mathbb{C}[G]))_{n=0}^\infty$ give a subcomplex for b, which leads to the direct sum decomposition (see [13])

$$\mathrm{H}_*(\mathbb{C}[G]) = \bigoplus_{x \in G\backslash_{\mathrm{Ad}} G} \mathrm{H}_*(\mathcal{C}_*^x(\mathbb{C}[G]), b).$$

where $\mathrm{CC}(\mathcal{C}_*^x(\mathbb{C}[G]))$ is the direct summand of $\mathrm{CC}(\mathbb{C}[G])$ formed by the spaces $\mathcal{C}_n^x(\mathbb{C}[G])$. Dualizing this, we also obtain the direct product decomposition

$$\mathrm{H}^*(\mathbb{C}[G]) = \prod_{x \in G\backslash_{\mathrm{Ad}} G} \mathrm{H}^*(\mathcal{C}_*^x(\mathbb{C}[G])', b).$$

Let us denote the factors labeled by x by $\mathrm{H}^*(\mathbb{C}[G])_x$ and $\mathrm{HC}^*(\mathbb{C}[G])_x$. We want to describe them in terms of group cohomology.

From now on let us fix x, and also choose and fix $g^y = g_x^y \in G$ such that $(g^y)^{-1} x g^y = y$ for each element y in $\mathrm{Ad}_G(x)$.

Let $(g_0, \ldots, g_n) \in G^{n+1}$ be such that $g_0 \cdots g_n \in \mathrm{Ad}_G(x)$. We put

$$y_i = g_i \cdots g_n g_0 \cdots g_{i-1}.$$

Note that these elements are also in $\mathrm{Ad}_G(x)$.

Lemma 6.2.2. *The elements $g^{y_i} g_i (g^{y_{i+1}})^{-1}$ for $0 \le i < n$, and $g^{y_n} g_n (g^{y_0})^{-1}$ are in $C_G(x)$.*

Proof. This follows from direct calculation using $x g^y = g^y y$ and $y_i g_i = g_i y_{i+1}$. □

We next consider the map

$$\Xi \colon \mathcal{C}_n^x(\mathbb{C}[G]) \to \mathcal{C}_n(C_G(x)), \quad g_0 \otimes \cdots \otimes g_n \mapsto (g^{y_1} g_1 (g^{y_2})^{-1}, \ldots, g^{y_n} g_n (g^{y_0})^{-1}).$$

Proposition 6.2.3. *The map Ξ intertwines the Hochschild differential b on $(\mathcal{C}_n^x(\mathbb{C}[G]))_{n=0}^\infty$ and d on $\mathcal{C}_*(C_G(x))$.*

Proof. We can compare (6.2.1) and (6.2.2) directly term by term. □

Consequently, any cocycle on $C_G(x)$ induces a Hochschild cocycle on $\mathbb{C}[G]$ which is supported on the conjugacy class of x. For example, the trivial class represented by $1 \in \mathbb{C} = \mathcal{C}^0(C_G(x))$ corresponds to the trace

$$\tau^x(f) = \delta_{\mathrm{Ad}_G(x)}(f) = \sum_{g \in \mathrm{Ad}_G(x)} f(g) \tag{6.2.3}$$

on $\mathbb{C}[G]$.

The following proposition is well known.

Proposition 6.2.4. *The map* Ξ *induces an isomorphism* $\mathrm{H}^*(\mathbb{C}[G])_x \simeq \mathrm{H}^*(C_G(x); \mathbb{C})$.

Proof. This corresponds to the first half of [13, Theorem I']. More concretely, consider a map $\Upsilon\colon \mathcal{C}_*(C_G(x)) \to \mathcal{C}^x_*(\mathbb{C}[G])$ defined by

$$\Upsilon(g_1, \ldots, g_n) = (g_1 \cdots g_n)^{-1} x \otimes g_1 \otimes \cdots \otimes g_n.$$

Then, on the one hand, $\Xi\Upsilon$ is equal to the identity map on $\mathcal{C}_*(C_G(x))$. On the other, $\Upsilon\Xi$ can be computed as

$$g_0 \otimes \cdots \otimes g_n \mapsto g^{y_0} g_0 (g^{y_1})^{-1} \otimes g^{y_1} g_1 (g^{y_2})^{-1} \otimes \ldots \otimes g^{y_n} g_n (g^{y_0})^{-1}.$$

This map is homotopic to the identity on $\mathcal{C}^x_*(\mathbb{C}[G])$ as follows (cf. [28, Section III.2]). Put

$$\begin{aligned}
\theta_0(g_0 \otimes \cdots \otimes g_n) &= g^{y_0} g_0 \otimes g_1 \otimes \cdots \otimes g_n \otimes (g^{y_0})^{-1}, \\
\theta_1(g_0 \otimes \cdots \otimes g_n) &= g^{y_0} g_0 \otimes g_1 \otimes \cdots \otimes g_{n-1} \otimes (g^{y_n})^{-1} \otimes g^{y_n} g_n (g^{y_0})^{-1}, \ldots \\
\theta_n(g_0 \otimes \cdots \otimes g_n) &= g^{y_0} g_0 \otimes (g^{y_1})^{-1} \otimes g^{y_1} g_1 (g^{y_2})^{-1} \otimes \cdots \otimes g^{y_n} g_n (g^{y_0})^{-1}.
\end{aligned}$$

Then the alternating sum

$$h_n = \sum_{i=0}^{n} (-1)^{n+i+1} \theta_i \colon \mathcal{C}^x_n(\mathbb{C}[G]) \to \mathcal{C}^x_{n+1}(\mathbb{C}[G]) \tag{6.2.4}$$

satisfies $b h_n + h_{n-1} b = \mathrm{id} - \Upsilon\Xi$. Consequently, Ξ and Υ induce isomorphism of cohomology between the dual complexes. □

6.3 Inducing traces on twisted algebras

Let $\omega_0(g,h)$ be a $\mathbb{C}$-valued normalized 2-cocycle on G. We consider the twisted group algebras for the family of $\mathbb{C}^\times$-valued 2-cocycles $\omega^t(g,h) = e^{t\omega_0(g,h)}$, so that the new product is $g *_t h = \omega^t(g,h)gh$.

We want to 'deform' the trace $\tau = \delta_{\mathrm{Ad}_G(x)}$ (as in the equation 6.2.3) on $\mathbb{C}[G]$ to a one on $\mathbb{C}_{\omega^t}[G]$. Note that when $gh \in \mathrm{Ad}_G(x)$, we have

$$\delta_{\mathrm{Ad}_G(x)}(g *_t h) = \omega^t(g,h), \qquad \delta_{\mathrm{Ad}_G(x)}(h *_t g) = \omega^t(h,g),$$

but these factors do not need to be equal. We want to correct this situation by setting $\tau^x_{\omega^t}(g) = e^{t\xi(g)}\delta_{\mathrm{Ad}_G(x)}(g)$ for some function ξ on $\mathrm{Ad}_G(x)$. Then $\tau^x_{\omega^t}$ becomes a trace if we have

$$\omega_0(g,h) + \xi(gh) = \omega_0(h,g) + \xi(hg) \tag{6.3.1}$$

This means that the bilinear extension of $\omega_0(g,h) - \omega_0(h,g)$ is the Hochschild coboundary of the linear extension of ξ. Let us put $\omega_0^a(g,h) = \omega_0(g,h) - \omega_0(h,g)$.

Lemma 6.3.1. *The bilinear extension of ω_0^a represents a 1-cocycle in the complex $(\mathcal{C}^x_*(\mathbb{C}[G])', b)$.*

Proof. We need to show

$$\omega_0^a(gh,k) - \omega_0^a(g,hk) + \omega_0^a(kg,h) = 0$$

whenever $ghk \in \mathrm{Ad}_G(x)$. Expanding the definition of ω_0^a, this is the same as

$$\omega_0(gh,k) + \omega_0(hk,g) + \omega_0(kg,h) = \omega_0(k,gh) + \omega_0(g,hk) + \omega_0(h,kg).$$

Adding $\omega_0(g,h)+\omega_0(h,k)+\omega_0(k,g)$ to both sides and using the cocycle identity, we indeed obtain the equality. □

By Proposition 6.2.4, ω_0^a is a Hochschild coboundary if and only if its image in $\mathrm{H}^1(C_G(x);\mathbb{C})$ is trivial. Since $\mathrm{H}^1(C_G(x);\mathbb{C}) = Z^1(C_G(x);\mathbb{C})$, so ω_0^a is a coboundary if and only if its pullback by Υ vanishes.

Proposition 6.3.2. *Suppose x is of finite order. Then the function $\omega_0^a\Upsilon(g) = \omega_0^a(g^{-1}x, g)$ on $C_G(x)$ is trivial.*

Proof. Using $g^{-1}x = xg^{-1}$ and that ω_0 is a normalized 2-cocycle, we have $\omega_0^a(g^{-1}x,g) = -\omega_0^a(x,g^{-1})$. Note that x and g^{-1} play the same role in this expression, and $x \in C_G(g^{-1})$. Thus, $\omega_0^a(h,g^{-1})$ as a function in $h \in C_G(g^{-1})$ is in $Z^1(C_G(g^{-1});\mathbb{C}) = \mathrm{Hom}(C_G(g^{-1}),\mathbb{C})$. Since x is of finite order, it has to vanish on x. □

It follows that we have a solution for ξ in (6.3.1) when x is of finite order.

Example 6.3.3. *Suppose we have $G = \mathbb{Z}^2 \rtimes \mathbb{Z}_3$, where the generator 1 of $\mathbb{Z}_3$ acts on $\mathbb{Z}^2$ by the matrix*

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}.$$

In the following we denote elements of G by triples (a, b, i) for $a, b \in \mathbb{Z}$ and $i \in \mathbb{Z}_3$. This group has the following conjugacy classes that contain elements of finite order:

$$\{(0,0,0)\}, \{(a,b,1) \mid a \equiv b \bmod 3\}, \quad \{(a,b,1) \mid a \equiv b+1 \bmod 3\},$$
$$\{(a,b,1) \mid a \equiv b+2, \mathrm{mod} 3\}, \quad \{(a,b,2) \mid a \equiv b \bmod 3\},$$
$$\{(a,b,2) \mid a \equiv b+1, \mathrm{mod} 3\},$$
$$\{(a,b,2) \mid a \equiv b+2, \mathrm{mod} 3\}.$$

Let us consider the 2-cocycle ω_0 on $\mathbb{Z}^2$ given by $\omega_0((a,b),(c,d)) = ad - bc$, and extend it to G by

$$\omega_0((a,b,g),(c,d,h)) = \omega_0((a,b), g.(c,d)) \quad (a,b,c,d \in \mathbb{Z}, g,h \in \mathbb{Z}_3).$$

Writing t as θ, for $x = (0,0,0)$, $\tau^x_{\omega_\theta} = \tau^x_{\omega_0}$ is the usual trace on the twisted group algebra of G. Let us now consider the case $x = (0,0,1)$. Then we have

$$\omega_\theta((a,b,0),(c,d,1)) = \omega_\theta((a,b),(c,d)) = \exp\left(-\frac{i\theta}{2}(ad - bc)\right),$$
$$\omega_\theta((c,d,1),(a,b,0)) = \omega_\theta((c,d),(-a-b,a)) = \exp\left(-\frac{i\theta}{2}(ca + ad + bd)\right).$$

Thus, in the twisted group algebra $\mathbb{C}_{\omega_\theta}[G]$, we have

$$\lambda^{(\omega_\theta)}_{(a,b,0)}\lambda^{(\omega_\theta)}_{(c,d,1)} = \exp\left(-\frac{i\theta}{2}(ad - bc)\right)\lambda^{(\omega_\theta)}_{(a+c,b+d,1)}, \tag{6.3.2}$$
$$\lambda^{(\omega_\theta)}_{(c,d,1)}\lambda^{(\omega_\theta)}_{(a,b,0)} = \exp\left(-\frac{i\theta}{2}(ca + ad + bd)\right)\lambda^{(\omega_\theta)}_{(c-a-b,d+a,1)}. \tag{6.3.3}$$

In particular, $\delta_{\mathrm{Ad}_G(x)}$ is not a trace, but a correction like

$$\tau^x_{\omega_\theta}(\lambda_{(a,b,1)}) = \exp\left(-\frac{i\theta(a^2 + ab + b^2)}{6}\right)\delta_{\mathrm{Ad}_G(x)}(a,b,1) \tag{6.3.4}$$

is (see Section 6.4.1 for comparison with [12]). Let us relate the extra factor appearing here to the above general discussion. On the one hand, using Equation 6.2.4

$$bh_1 + h_0 b = \mathrm{id} - \Upsilon\Xi,$$

and $\omega_0^a b = 0$, $\omega_0^a \Upsilon = 0$ *implies that we can take*

$$\xi = \omega_0^a h_0 \tag{6.3.5}$$

On the other, for $y = (a, b, 1) \in \mathrm{Ad}_G(x)$*, the element* $g^y = (\frac{b-a}{3}, \frac{-a-2b}{3}, 0)$ *satisfies* $(g^y)^{-1} x g^y = y$*. Thus* ξ *is given by*

$$\xi(g_0) = \omega_0^a h_0(g_0) = \omega_0(g^{g_0} g_0, (g^{g_0})^{-1}) - \omega_0((g^{g_0})^{-1}, g^{g_0} g_0) = \frac{a^2 + ab + b^2}{3}$$

for $g_0 = (a, b, 1)$ *with* $a \equiv b \bmod 3$*, and* $\xi(g_0) = 0$ *otherwise. To check that the functional* (6.3.4) *agrees on* (6.3.2) *and* (6.3.3)*, one needs*

$$\begin{aligned} ad - bc + \frac{1}{3}\left((a+c)^2 + (a+c)(b+d) + (b+d)^2\right) = \\ ca + ad + bd + \frac{1}{3}\left((c-a-b)^2 + (c-a-b)(a+d) + (a+d)^2\right), \end{aligned}$$

which is indeed the case.

6.4 Zeroth cyclic cocycles of twisted crystallographic group algebras

In this section we only look at the example $G = \mathbb{Z}^n \rtimes F$, $F = \mathbb{Z}_m$. Since this is a finitely generated group, it carries a natural length function which will be called ℓ. Let us again recall the example of following 2-cocyle:

Example 6.4.1. *Let* F *be a finite subgroup of* $\mathrm{GL}_n(\mathbb{Z})$ *such that each* $W \in F$ *leaves* θ *invariant, i.e.,* $W^T \theta W = \theta$*. Then we can define a 2-cocycle* ω'_θ *on* $G = \mathbb{Z}^n \rtimes F$ *by* $\omega'_\theta((x, g), (y, h)) = \omega_\theta(x, g.y)$ *for* $x, y \in \mathbb{Z}^n$ *and* $g, h \in F$*. So by definition* $\omega'_\theta((x, g), (y, h)) = \omega_\theta(x, g.y) = e^{\pi i \langle -\theta x, g\cdot y\rangle}$*. Also define the two cocycles* $\omega^t_\theta((x, g), (y, h)) = e^{\pi i t\langle -\theta x, g\cdot y\rangle}$*, for* $t \in [0, 1]$*.*

In the context of cyclic cohomology, the natural algebra to consider is the smooth algebras associated to G and ω'_θ. Let us recall that *a Fréchet algebra* is given by a $\mathbb{C}$-algebra A and a sequence of (semi)norms $\|a\|_m$ $(a \in A)$ for $m = 1, 2, 3, \ldots$, such that A is complete with respect to the locally convex topology defined by the $\|a\|_m$, and that the product map $A \times A \to A$ is (jointly) continuous. Also if each $\|a\|_m$ is submultiplicative, that is $\|ab\|_m \leq \|a\|_m \|b\|_m$, A is then called an *m-algebra*.

Let A be a Fréchet m-algebra with seminorms $\|a\|_m$, and let $\alpha\colon G \curvearrowright A$ be an action of G which is *ℓ-tempered* [57]:

$$\forall m, \exists\, C, k, n\colon\ \|\alpha_g(x)\|_m \leq C(\ell(g)+1)^k \|x\|_n\,.$$

We then put

$$\mathcal{S}(G;A) = \left\{ f\colon G \to \mathcal{A} \;\middle|\; \forall k, m\colon \sum_g (\ell(g)+1)^k \|f(g)\|_m < \infty \right\}. \tag{6.4.1}$$

The seminorms

$$\|f\|_{d,m} = \sum_g (\ell(g)+1)^d \|f(g)\|_m$$

topologize $\mathcal{S}(G;A)$, and since $\|f\|_m$ is increasing in m, the seminorms $\|f\|'_m = \|f\|_{m,m}$ also topologize $\mathcal{S}(G;A)$, which is an m-algebra [56, Theorem 3.1.7]. $\mathcal{S}(G;A)$ is also sometimes written as $A \rtimes G$, a crossed product algebra. As a Fréchet space this is just the projective tensor product of $\mathcal{S}(G) = \mathcal{S}(G;\mathbb{C})$ and A. Let us denote the twistings of $\mathcal{S}(G)$ by $\mathcal{S}(G,\omega'_\theta)$, i.e we change the multiplication in $\mathcal{S}(G)$ to twisted convolution. Like in the case of C*-algebra, $\mathcal{S}(G)$ is naturally isomorphic to $\mathcal{S}(\mathbb{Z}^n,\omega_\theta) \rtimes F$, for some action of F on the Fréchet algebra $\mathcal{S}(\mathbb{Z}^n,\omega_\theta)$.

We also look at the algebra $\mathcal{S}(G;\mathcal{C}^\infty([0,1]))$, where G acting trivially on $\mathcal{C}^\infty([0,1])$. Let f and f' be elements of $\mathcal{S}(G;\mathcal{C}^\infty([0,1])) \simeq \mathcal{C}^\infty([0,1];\mathcal{S}(G))$. Their product is represented by the function

$$(t,g) \mapsto \sum_{h\in G} \omega^t_\theta(h,h^{-1}g) f_t(h) f'_t(h^{-1}g). \tag{6.4.2}$$

Also we have natural the evaluation maps $\mathrm{ev}_t, t \in [0,1]$, from $\mathcal{S}(G;\mathcal{C}^\infty([0,1])) \simeq \mathcal{C}^\infty([0,1];\mathcal{S}(G))$ to $\mathcal{S}(G,\omega^t_\theta)$.

For the following theorem, we use K-theory for Fréchet m-algebras (as in Chapter 2), and continuous version of Hochschild and cyclic (co-)cycles. Let $\langle , \rangle$ denote the Chern–Connes pairing between projections and cyclic cocycles. Though the following theorem can be more general, we only state it for our special G.

Theorem 6.4.2 (Theorem 4.8, [15])**.** *Let x be an element of G with finite centraliser. Then $\phi^{(t)} = \tau^x_{\omega^t_\theta}$ is well defined zeroth Hochschild (also cyclic) cocycle on $\mathcal{S}(G,\omega^t_\theta)$. When P is a projection inside $\mathrm{K}_0(\mathcal{S}(G;\mathcal{C}^\infty([0,1])))$, the pairing $\langle \phi^{(t)}, \mathrm{ev}_t(P)\rangle$ is independent of t.*

Proof. The expression $\langle -\theta x', g \cdot y' \rangle$ is the $\omega_0((x',g),(y',h)$, where the notation ω_0 is from previous section. Now the growth of $\omega_0((x',g),(y',h)$ is bounded by some polynomial in $\ell(x',g)$ and $\ell(y',h)$ and so the continuous version of Lemma 6.3.1 holds. Also the explicit formula of $\phi^{(t)}$ (6.2.3, 6.3.5), shows that it extends to the smooth algebra $\mathcal{S}(G,\omega_\theta^t)$. For the rest, see [15, Theorem 3.4, Theorem 4.8]. □

Remark 6.4.3. *The zeroth periodic cyclic cohomology group (*HP^0*), as introduced in Preliminaries (2), is defined to be certain direct limit of cyclic cohomology groups. As we have seen in Example 2.7.10 that* $\mathrm{HP}^0(\mathcal{S}(\mathbb{Z}^2)) = \mathbb{Z}^2$*, one might ask how this is connected to* $\mathrm{HC}^0(\mathcal{S}(\mathbb{Z}^2))$*. In fact* $\mathrm{HC}^0(\mathcal{S}(\mathbb{Z}^2))$ *is very big as it contains traces from infinitely many conjugacy classes. But It can be shown that they all (except the trace coming from the conjugacy class of the trivial element) disappear in* $\mathrm{HP}^0(\mathcal{S}(\mathbb{Z}^2))$*. The other generator of* $\mathrm{HP}^0(\mathcal{S}(\mathbb{Z}^2))$ *comes from a distinguished cyclic 2-cocycle which is an element of* $\mathrm{HC}^2(\mathcal{S}(\mathbb{Z}^2))$*.*

Let us now look at the K-theory of $\mathcal{S}(G,\omega'_\theta)$ and $C^*(G,\omega'_\theta)$. Assume $W \in F$ acts free away from origin. Again note that when $n = 2$, any finite subgroup of $\mathrm{SL}_2(\mathbb{Z})$ will work. Let $\mathcal{M}$ denote the set of conjugacy classes of maximal finite subgroups of G. In the following theorem, $\tilde{\mathrm{K}}_0(\mathbb{C}[P])$ denotes the reduced K_0-group, that is, the kernel of the map $\mathrm{K}_0(\mathbb{C}[P]) \to \mathrm{K}_0(\mathbb{C})$ induced by the trivial representation. Recall the following theorem from Chapter 2:

Theorem 6.4.4. *There exists an exact sequence*

$$0 \longrightarrow \bigoplus_{P \in \mathcal{M}} \tilde{\mathrm{K}}_0(\mathbb{C}[P]) \longrightarrow \mathrm{K}_0(C^*(G)) \longrightarrow \mathrm{K}_0(\underline{B}G) \longrightarrow 0. \tag{6.4.3}$$

Also, we have $\mathrm{K}_1(C^*(G)) \simeq \mathrm{K}_1(\underline{B}G)$*.*

Under the above setting, any nontrivial finite subgroup of G has a finite normaliser [42, Lemma 6.1]. In particular, any nontrivial torsion element has a finite centraliser.

Let us further assume that $p = m$ is prime. Then any non-trivial finite subgroup P of G must be isomorphic to $\mathbb{Z}_p$ via the restriction of the projection map $G \to \mathbb{Z}_p$. Thus any nontrivial finite subgroup of G represents an element of $\mathcal{M}$. In Chapter 2, we explained how projections in $\widetilde{\mathrm{K}}_0(\mathbb{C}[P])$ contribute to the $\mathrm{K}_0(C^*(G)$ and $\mathrm{K}_0(C^*(G,\omega'_\theta))$. Let us recall that quickly. Since $\mathbb{C}[P]$ is isomorphic to the algebra $C(\hat{P}) \simeq \mathbb{C}^p$, $\mathrm{K}_0(\mathbb{C}[P])$ is the free abelian group of rank p. Now let g be a generator of P. The minimal projections of $\mathbb{C}[P]$ are given by

$$Q_{j,g} = \frac{1}{p}\sum_{k=0}^{p-1} \exp\left(i\frac{\pi}{p}jk\right)\lambda_{g^k}$$

for $j = 0, \cdots, p-1$, which also represent a basis of $\mathrm{K}_0(\mathbb{C}[P])$. Since $Q_{0,g}$ represents the trivial representation, a basis of $\widetilde{\mathrm{K}}_0(\mathbb{C}[P])$ is given by $Q_{1,g}, \cdots, Q_{p-1,g}$.

For the generator of P, g, let us denote the canonical element $\lambda_g^{(\omega'_\theta)} \in \mathcal{C}^*(G, \omega'_\theta)$. Still, since ω'_θ is cohomologically trivial on P, we can always multiply a suitable $z \in \mathbb{T}$ so that order of the unitary $z\lambda_g^{(\omega'_\theta)}$ is p. Then the formula

$$Q_{j,g}^{(\theta)} = \frac{1}{p} \sum_{k=0}^{p-1} \exp\left(i\frac{\pi}{p} jk\right) z^k \lambda_{g^k}^{(\omega'_\theta)}$$

for $j = 1, \cdots, p-1$ will give projections which are elements of $\mathrm{K}_0(C^*(G, \omega'_\theta))$. Also note that the projections are in $\mathrm{K}_0(\mathcal{S}(G, \omega'_\theta))$, since they all basically come from $\mathrm{K}_0(\mathbb{C}[P])$. Varying t in $[0, 1]$, we can think $Q_{j,g}^{(t\theta)}$ as a continuous (smooth) field of projections and we denote the corresponding element in $\mathrm{K}_0(\mathcal{S}(G; \mathcal{C}^\infty([0, 1])))$ by $Q_{j,g}$. Of course, $\mathrm{ev}_t(Q_{j,g}) = Q_{j,g}^{(t\theta)}$.

6.4.1 The case $\mathbb{Z}^2 \rtimes \mathbb{Z}_3$

Let us come back to Example 6.3.3, and compare it to [12]. This group $\mathbb{Z}^2 \rtimes \mathbb{Z}_3$ can be presented as

$$G = \left\langle u, v, w \mid w^3 = e, uv = vu, wuw^{-1} = u^{-1}v, wvw^{-1} = u^{-1} \right\rangle.$$

Up to conjugation, the finite subgroups of G are generated by one of the elements w, uw or, u^2w all of which are of order 3. Since $\omega'_\theta(w, w) = 1$, the first element still has order 3 in the twisted group algebra, and the corresponding projections $Q_{1,w}^{(\theta)}$ and $Q_{2,w}^{(\theta)}$ give nontrivial classes $\mathrm{K}_0(C^*(G, \omega'_\theta))$. On the other hand,

$$\omega'_\theta(uw, uw)\omega'_\theta(uwuw, uw) = \omega_\theta(u, u^{-1}v)\omega_\theta(v, v^{-1}) = \exp(-i\frac{\theta}{2})$$

shows that $\lambda_{uw}^{(\omega'_\theta)}$ is not of order 3. But an adjustment like $\exp(i\frac{\theta}{6})\lambda_{uw}^{(\omega'_\theta)}$ gives the right order.

For simplicity, let us write $e(m)$ instead of $\exp(im\theta)$. Then as noticed above $e(\frac{1}{6})\lambda_{uw}^{(\omega'_\theta)}$ is an unitary of order 3. Similarly one can see that $e(\frac{2}{3})\lambda_{u^2w}^{(\omega'_\theta)}$ is also an unitary of order 3. Then we have the projections $Q_{1,w}^{(\theta)}$, $Q_{2,w}^{(\theta)}$, $Q_{1,uw}^{(\theta)}$, $Q_{2,uw}^{(\theta)}$, $Q_{1,u^2w}^{(\theta)}$, $Q_{2,u^2t}^{(\theta)}$ defining nontrivial classes in $\mathrm{K}_0(C^*(G, \omega'_\theta))$.

Recall from [12] that if α defines an action of $\mathbb{Z}_p$ on A, an α-invariant functional ϕ on A is said to be an *α-trace* if

$$\phi(xy) = \phi(\alpha(y)x)$$

holds for any $x, y \in A$. An α^s-trace ϕ gives rise to a trace T_ϕ on $A \rtimes \mathbb{Z}_p$ defined by

$$T_\phi(x_0 + x_1 w + \cdots + x_{p-1} w^{p-1}) = \phi(x_{p-s})$$

for $x_i \in A$ and w being the copy of $1 \in \mathbb{Z}_p$.

Coming back to the example, let us consider the induced action of $\mathbb{Z}_3$ on $S(\mathbb{Z}^2, \omega_\theta)$. From [12, Theorem 3.3], we have following α-traces ϕ_l^1 on $S(\mathbb{Z}^2, \omega_\theta)$:

$$\phi_l^1(\lambda_u^{(\omega_\theta)m} \lambda_v^{(\omega_\theta)n}) = e\left(\frac{1}{6}((m-n)^2 - l^2)\right) \delta_{3\mathbb{Z}}(m-n-l)$$

for $l = 0, 1, 2$. Similarly we have following α^2-traces ϕ_l^2 on $S(\mathbb{Z}^2, \omega_\theta)$:

$$\phi_l^2(\lambda_u^{(\omega_\theta)m} \lambda_v^{(\omega_\theta)n}) = e\left(-mn - \frac{1}{6}((m-n)^2 - l^2)\right) \delta_{3\mathbb{Z}}(m-n-l)$$

for $l = 0, 1, 2$. Each ϕ_l^i then gives the traces $T_l^i = T_{\phi_l^i}$ on $S(\mathbb{Z}^2, \omega_\theta) \rtimes \mathbb{Z}_3 = S(G, \omega'_\theta)$. Using $\lambda_u^{(\omega_\theta)m} \lambda_v^{(\omega_\theta)n} = e(-\frac{mn}{2}) \lambda_{u^m v^n}^{(\omega_\theta)}$, it is straightforward to check that the trace $\tau_{\omega'_\theta}^w$ of Example 6.3.3 is exactly equal to T_0^2.

The pairing of the above projections and traces are given by the following table (cf. [12, Theorem 1.2]), which is independent of θ as suggested by Theorem 6.4.2.

	$\tau_{\omega'_\theta}^{w^2} = T_0^1$	$\tau_{\omega'_\theta}^{uw^2} = T_1^1$	$\tau_{\omega'_\theta}^{u^2w^2} = T_2^1$	$\tau_{\omega'_\theta}^{w} = T_0^2$	$\tau_{\omega'_\theta}^{uw} = T_1^2$	$\tau_{\omega'_\theta}^{u^2w} = T_2^2$
$Q_{1,w}^{(\theta)}$	$\frac{1}{3}e^{\frac{4\pi}{3}i}$	0	0	$\frac{1}{3}e^{\frac{2\pi}{3}i}$	0	0
$Q_{2,w}^{(\theta)}$	$\frac{1}{3}e^{\frac{2\pi}{3}i}$	0	0	$\frac{1}{3}e^{\frac{4\pi}{3}i}$	0	0
$Q_{1,uw}^{(\theta)}$	0	0	$\frac{1}{3}e^{\frac{4\pi}{3}i}$	0	$\frac{1}{3}e^{\frac{2\pi}{3}i}$	0
$Q_{2,uw}^{(\theta)}$	0	0	$\frac{1}{3}e^{\frac{2\pi}{3}i}$	0	$\frac{1}{3}e^{\frac{4\pi}{3}i}$	0
$Q_{1,u^2w}^{(\theta)}$	0	$\frac{1}{3}e^{\frac{4\pi}{3}i}$	0	0	0	$\frac{1}{3}e^{\frac{2\pi}{3}i}$
$Q_{2,u^2w}^{(\theta)}$	0	$\frac{1}{3}e^{\frac{2\pi}{3}i}$	0	0	0	$\frac{1}{3}e^{\frac{4\pi}{3}i}$

6.4.2 The case $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$

Let $G = \mathbb{Z}^2 \rtimes \mathbb{Z}_2$, where the generator 1 of $\mathbb{Z}_2$ acts on $\mathbb{Z}^2$ by the matrix

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We sometimes call this action of $\mathbb{Z}_2$ a flip action. This group has following conjugacy classes that contain finite order elements:

$$\begin{aligned} X_0 &= \{(0,0,0)\}, \\ X_1 &= \{(a,b,1) \mid a \equiv 0 \bmod 2, b \equiv 0 \bmod 2\}, \\ X_2 &= \{(a,b,1) \mid a-1 \equiv 0 \bmod 2, b \equiv 0 \bmod 2\}, \\ X_3 &= \{(a,b,1) \mid a \equiv 0 \bmod 2, b-1 \equiv 0 \bmod 2\}, \\ X_4 &= \{(a,b,1) \mid a-1 \equiv 0 \bmod 2, b-1 \equiv 0 \bmod 2\}. \end{aligned}$$

X_0 does not contain element with finite centraliser. It contains the identity element of the group giving rise to the usual trace τ on $\mathcal{S}(G,\omega'_\theta)$. Now for the conjugacy classes X_i, for $1 \le i \le 4$, the function $\xi = 0$ (with the computations similar to Example 6.3.3).

Let us compare this computation of Walters [59]. This group $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$ has presentation

$$G = \left\langle u, v, w \mid w^2 = e, uv = vu, wuw^{-1} = u^{-1}, wvw^{-1} = v^{-1} \right\rangle.$$

Up to conjugation, the finite subgroups of G are generated elements w, uw, vw, or uvw all of which are of order 2. Let us denote the corresponding projections by $Q_w^{(\theta)}$, $Q_{uw}^{(\theta)}$, $Q_{vw}^{(\theta)}$, $Q_{uvw}^{(\theta)}$. These give nontrivial classes $\mathrm{K}_0(C^*(G,\omega'_\theta))$. Explicitly these projections are:

$$Q_w^{(\theta)} = \frac{1}{2}(1+w), Q_{uw}^{(\theta)} = \frac{1}{2}(1-uw), Q_{vw}^{(\theta)} = \frac{1}{2}(1-vw), Q_{uvw}^{(\theta)} = \frac{1}{2}(1-e(\frac{1}{2})uvw).$$

Now using Theorem 6.4.2, the value of $\tau^x_{\omega'_\theta}(Q_y^{(\theta)})$ remains constant, for $x \in X_i$, for $1 \le i \le 4$ and $y \in \{w, uw, vw, uvw\}$.

Consider the induced action of $\mathbb{Z}_2$ on $S(\mathbb{Z}^2, \omega_\theta)$. From [59, Page 592], we have following α-traces ϕ_{ij} on $S(\mathbb{Z}^2, \omega_\theta)$:

$$\phi_{ij}(\lambda_u^{(\omega_\theta)m}\lambda_v^{(\omega_\theta)n}) = e\left(-\frac{mn}{2}\right)\delta_{2\mathbb{Z}}(m-i)\delta_{2\mathbb{Z}}(n-j)$$

for $i,j = 0,1,2$. Each ϕ_{ij} then gives the traces $T_{ij} = T_{\phi_{ij}}$ on $S(\mathbb{Z}^2,\omega_\theta) \rtimes \mathbb{Z}_2 = S(G,\omega'_\theta)$. Using $\lambda_u^{(\omega_\theta)m}\lambda_v^{(\omega_\theta)n} = e(-\frac{mn}{2})\lambda^{(\omega_\theta)}_{u^m v^n}$, it is straightforward to check that the trace $\tau^w_{\omega'_\theta} = T_{00}$, $\tau^{uw}_{\omega'_\theta} = T_{10}$, $\tau^{vw}_{\omega'_\theta} = T_{01}$, $\tau^{uvw}_{\omega'_\theta} = T_{11}$.

The pairing of the above projections and traces are given by the following table (cf.[59, Page 592]), which is independent of θ as suggested by Theorem 6.4.2.

	$\tau^{w}_{\omega'_\theta} = T_{00}$	$\tau^{uw}_{\omega'_\theta} = T_{10}$	$\tau^{vw}_{\omega'_\theta} = T_{01}$	$\tau^{uvw}_{\omega'_\theta} = T_{11}$
$Q^{(\theta)}_{w}$	$\frac{1}{2}$	0	0	0
$Q^{(\theta)}_{uw}$	0	$\frac{1}{2}$	0	0
$Q^{(\theta)}_{vw}$	0	0	$\frac{1}{2}$	0
$Q^{(\theta)}_{uvw}$	0	0	0	$\frac{1}{2}$

For irrational θ, in [59, Lemma 2.1], Walters constructed a projection P_θ in $S(\mathbb{Z}^2, \omega_\theta) \rtimes \mathbb{Z}_2$ which is a variant of Power-Rieffel projection. Along with the other results, he proved the following theorem ([59, Lemma 2.3]:

Theorem 6.4.5. $\mathrm{K}_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_2, \omega'_\theta)$ *is isomorphic to* $\mathbb{Z}^6$*, which is generated by the* K*-theory classes of the elements: 1,* $Q^{(\theta)}_{w}, Q^{(\theta)}_{uw}, Q^{(\theta)}_{vw}, Q^{(\theta)}_{uvw}, P_\theta$.

Also we have the pairing of the traces with P_θ as follows ([59, Page 596]): for $0 < \theta < \frac{1}{2}$

	$\tau^{w}_{\omega'_\theta} = T_{00}$	$\tau^{uw}_{\omega'_\theta} = T_{10}$	$\tau^{vw}_{\omega'_\theta} = T_{01}$	$\tau^{uvw}_{\omega'_\theta} = T_{11}$
P_θ	1	-1	1	-1

and for $\frac{1}{2} < \theta < 1$

	$\tau^{w}_{\omega'_\theta} = T_{00}$	$\tau^{uw}_{\omega'_\theta} = T_{10}$	$\tau^{vw}_{\omega'_\theta} = T_{01}$	$\tau^{uvw}_{\omega'_\theta} = T_{11}$
P_θ	1	1	-1	-1

If we denote the tuple of traces as $T := (\tau, T_{00}, T_{10}, T_{01}, T_{11})$, T is a function on $\mathrm{K}_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_2, \omega'_\theta)$. Then Walters has the following result ([59, Proposition 3.2]:

Theorem 6.4.6. *T is injective on* $\mathrm{K}_0(C^*(\mathbb{Z}^2 \rtimes \mathbb{Z}_2, \omega'_\theta)$.

At the end of the next chapter, we shall use the above theorem to compute the generators of $\mathrm{K}_0(C^*(\mathbb{Z}^3 \rtimes \mathbb{Z}_2, \omega)$, for the flip action of $\mathbb{Z}_2$ on $\mathbb{Z}^3$ and for a quite general $\omega \in \mathrm{H}^2(\mathbb{Z}^3 \rtimes \mathbb{Z}_2, \mathbb{T})$.

7 Some non-commutative orbifolds: the flip-case

In this chapter we discuss the K-theory of the crossed products of noncommutative tori with flip action. We use the notations from Chapter 4.

We consider the $n \times n$ matrix $W = -I_n$ which generates the two element group. Suppose this group acts on a $n = 2p + q$-dimensional noncommutative torus with respect to the parameter θ with $\theta := \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}$, θ_{11} being the left $2p \times 2p$ corner, which amounts to the condition $W^T \theta W = \theta$ that holds in this case. We call this action the flip action.

We define the following operator on $\mathscr{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ with respect to W:

$$T_W(f)(x,t) := f(-x,-t). \tag{7.0.1}$$

$\mathscr{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ with respect to this action is a $A_\theta^\infty \rtimes \mathbb{Z}_2$ module which can be completed to an $A_\theta \rtimes \mathbb{Z}_2$ module. In Chapter 3, we saw that any generator of $\mathrm{K}_0(A_\theta)$ can be given by completions of modules of the type $\mathscr{S}(\mathbb{R}^p \times \mathbb{Z}^q)$. Now, it directly follows from the previous observation that all the generators of $\mathrm{K}_0(A_\theta)$ can be extended to provide classes in K_0 of $A_\theta \rtimes \mathbb{Z}_2$. We shall show that K-theory classes of these modules can be extended to a generating set of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ for 3-dimensional noncommutative tori. Our results will show that this should also be the case for the general n-dimensional case, but at this moment we are unable to compute the generators of $\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ for n-dimensional noncommutative tori A_θ. It should be noted that $\mathrm{K}_1(A_\theta \rtimes \mathbb{Z}_2)$ is trivial ([22]).

Let θ be a real antisymmetric $n \times n$ matrix and let θ' be the upper left $(n-1) \times (n-1)$ block of θ. In this case, A_θ can be written as a crossed product $A_{\theta'} \rtimes \mathbb{Z}$, where the action (γ) of $\mathbb{Z}$ on $A_{\theta'}$ given (by the generator of $\mathbb{Z}$) by $U_i \to e(\theta_{in})U_i$, for $i = 1, \cdots, n-1$. Now $A_\theta \rtimes \mathbb{Z}_2 = A_{\theta'} \rtimes \mathbb{Z} \rtimes \mathbb{Z}_2 = A_{\theta'} \rtimes \mathbb{Z}_2 * \mathbb{Z}_2$, since $\mathbb{Z}_2 * \mathbb{Z}_2$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}_2$ as groups. Note that one copy of $\mathbb{Z}_2$ acts on $A_{\theta'}$ by flip action (β) and the other by $\alpha = \gamma \circ \beta$. Our next step is to understand the K-theory of $A_{\theta'} \rtimes (\mathbb{Z}_2 * \mathbb{Z}_2)$.

For a general crossed product $A \rtimes_\beta \mathbb{Z}_2$, we first define a map p which goes from $A \rtimes \mathbb{Z}_2$ to $M_2(A)$ which sends the element $a + Wb$ to $\begin{pmatrix} a & b \\ WbW & WaW \end{pmatrix}$, where W is the unitary in $A \rtimes_\beta \mathbb{Z}_2$ implementing the action β. We recall the six term exact sequence by Natsume [45] which was used by Farsi-Watling [27] to compute the K-theory of $A_{\theta'} \rtimes (\mathbb{Z}_2 * \mathbb{Z}_2)$. For a free product $H_1 * H_2$ acting on a C*-algebra A, Natsume obtained the following exact sequence:

$$\begin{array}{ccccc} \mathrm{K}_0(A) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A \rtimes H_1) \oplus \mathrm{K}_0(A \rtimes H_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A \rtimes H_1 * H_2) \\ \uparrow & & & & \downarrow e_1 \\ \mathrm{K}_1(A \rtimes H_1 * H_2) & \xleftarrow[j_{1*}+j_{2*}]{} & \mathrm{K}_1(A \rtimes H_1) \oplus \mathrm{K}_1(A \rtimes H_2) & \xleftarrow[i_{1*}-i_{2*}]{} & \mathrm{K}_1(A), \end{array}$$

where i_1, i_2, j_1, j_2 are the natural inclusion maps. The right vertical map e_1, which we will describe in a while, is constructed in Natsume's paper. We call it exponential map since it is based on the exponential map in K-theory. We want to compare the above sequence with the six-term exact sequence obtained from the classical Toeplitz exact sequence (with coefficient in A) which is same as the Pimsner-Voiculescu exact sequence for actions of $\mathbb{Z}$ on the C*-algebra A.

From the definition of the crossed product, any crossed product algebra, $A \rtimes_\alpha G$, for a unital C*-algebra A and a discrete group G, has a natural representation (also called regular representation) ι on the Hilbert module $l^2(G, A)$ which is given by $\iota(a)(\xi)(g) = \alpha_{g^{-1}}(a)\xi(g)$ and $\iota(h)(\xi)(g) = \xi(h^{-1}g)$, for $a \in A$ and $g, h \in G$. Let $\mathbb{Z}$ act on a unital C*-algebra A by an action α. The classical Toeplitz algebra $\mathcal{T}^A$ with coefficients in A is defined as follows: we restrict the natural representation ι of A from $l^2(\mathbb{Z}, A)$ to $l^2(\mathbb{Z}_{\geq 0}, A)$ (note that the restriction is well defined). Call this restricted representation ι_1. When there is no confusion, we just call $\iota(a)$ and $\iota_1(a)$ by a. Take the right shift operator S on $l^2(\mathbb{Z}_{\geq 0}, A)$ which is given by $S(\xi)(n) = \xi(n-1)$, $\xi(-1) = 0$. Then $\mathcal{T}^A$ is generated by the elements $a \in l^2(\mathbb{Z}_{\geq 0}, A)$ and $S \in l^2(\mathbb{Z}_{\geq 0}, A)$. We have the following exact sequence:

$$0 \longrightarrow \mathcal{K}(l^2(\mathbb{Z}_{\geq 0}, A)) \xrightarrow{\varphi} \mathcal{T}^A \xrightarrow{\psi} A \rtimes \mathbb{Z} \longrightarrow 0$$

by defining $\psi(a) = a$ and $\psi(S) = U$, where U is the unitary in the crossed product $A \rtimes \mathbb{Z}$ coming from the generator of $\mathbb{Z}$. It can be easily checked that $\ker(\psi) = A \otimes \mathcal{K}$. This is the so-called Pimsner-Voiculescu exact sequence which gives rise to the Pimsner-Voiculescu six term exact sequence. Now we define the map e_2 to be the exponential map in K-theory for the above exact sequence. So we have

$$\begin{array}{ccccc}
\mathrm{K}_0(\mathcal{K}(l^2(\mathbb{Z}_{\geq 0}, A))) & \longrightarrow & \mathrm{K}_0(\mathcal{T}^A) & \longrightarrow & \mathrm{K}_0(A \rtimes \mathbb{Z}) \\
\uparrow & & & & \downarrow e_2 \\
\mathrm{K}_1(A \rtimes \mathbb{Z}) & \longleftarrow & \mathrm{K}_1(\mathcal{T}^A) & \longleftarrow & \mathrm{K}_1(\mathcal{K}(l^2(\mathbb{Z}_{\geq 0}, A))).
\end{array}$$

Pimsner-Voiculescu also proved that $\mathcal{T}^A$ is KK-equivalent to the algebra A.

The above (short exact) sequence also gives rise to the following exact sequence (tensoring with M_2)

$$0 \longrightarrow M_2(\mathcal{K}(l^2(\mathbb{Z}_{\geq 0}, A))) \xrightarrow{\varphi} M_2(\mathcal{T}^A) \xrightarrow{\psi} M_2(A \rtimes \mathbb{Z}) \longrightarrow 0.$$

We now describe the map e_1 using an exact sequence like the one above. Let the group $\mathbb{Z}_2 * \mathbb{Z}_2$ be generated by g and s, i.e. g and s generate the first and second copy of $\mathbb{Z}_2$ in $\mathbb{Z}_2 * \mathbb{Z}_2$, respectively. Natsume obtained the exact sequence

$$0 \longrightarrow \mathcal{K}(l^2(P)) \xrightarrow{\eta} \mathcal{T}_p \xrightarrow{\pi} C^*(\mathbb{Z}_2 * \mathbb{Z}_2) \longrightarrow 0,$$

with $P = P' \cup \{e\}$, where P' is the the set of all non-empty words in $\mathbb{Z}_2 * \mathbb{Z}_2$, which end in g and $\mathcal{T}_p$ is generated by $\mu(g)$ and $v(s)$, where $\mu(g)$ is the restriction of the left regular representation to $l^2(P)$ and $v(s) = \lambda(s)q(P)$, where $\lambda(s)$ is the restriction of the left regular representation to $l^2(P')$ and $q(P)$ is the projection onto the subspace generated by the inclusion $l^2(P') \subset l^2(P)$. When all these are defined, there is a map π sending $\mu(g)$ to $\lambda(g)$ and $v(s)$ to $\lambda(s)$. Denoting $\ker(\pi)$ to be $\mathcal{I}$, it can be shown that $\mathcal{I}$ is isomorphic to $\mathcal{K}(l^2(P))$. More details may be found in the paper by Natsume [45]. Denote e_1 to be map from $\mathrm{K}_0(C^*(\mathbb{Z}_2 * \mathbb{Z}_2))$ to $\mathrm{K}_1(\mathcal{K}(l^2(P))$ coming from the six-term exact sequence corresponding to the above exact sequence in K-theory.

The above construction can be easily extended to the case of crossed product. Let $\mathbb{Z}_2 * \mathbb{Z}_2$ acting on a unital A with the action α and β on A from $\langle s \rangle$ and $\langle g \rangle$, respectively. We denote the crossed product by $A \rtimes_{\alpha,\beta} \mathbb{Z}_2 * \mathbb{Z}_2$. $\mathcal{T}_p^A$ is constructed as follows. We have the natural representation ι' of $A \rtimes_{\alpha,\beta} \mathbb{Z}_2 * \mathbb{Z}_2$ on $l^2(\mathbb{Z}_2 * \mathbb{Z}_2, A)$ which we restrict to the Hilbert module $l^2(P, A)$ in the following sense: if we denote the restriction by ι_2, $a \in A, g, s$ act by the operators $\iota_2(a)(\xi)(x) = (\alpha, \beta)_{x^{-1}}(a)\xi(x)$, $\iota_2(g)(\xi)(x) = \xi(gx), \iota_2(s)(\xi)(x) = \xi(sx)$(by setting $\xi(s) = 0$). Then $\mathcal{T}_p^A$ is the C*-algebra generated by $\iota_2(a)(a \in A), \iota_2(g)$ and $\iota_2(s)$. Now we have the following exact sequence (see Lemma A.3 [45]):

$$0 \longrightarrow \mathcal{K}(l^2(P, A)) \xrightarrow{\eta} \mathcal{T}_p^A \xrightarrow{\pi} A \rtimes (\mathbb{Z} \rtimes \mathbb{Z}_2) \longrightarrow 0$$

by defining $\pi(\iota_2(a)) = \iota'(a)$, $\pi(\iota_2(g)) = \iota'(g)$ and $\pi(\iota_2(s)) = \iota'(s)$. Note that for the case $A = \mathbb{C}, \iota_2(g) = \mu(g)$(as a generator of $\mathcal{T}_p$), $\iota_2(s) = v(s), \iota'(g) = \mu(g), \iota'(s) = \mu(s)$. Denote the exponential map (of K-theory) of the above exact sequence still by e_1. The above exact sequence gives rise to Natsume's exact sequence in K-theory. So we have

$$\begin{array}{ccccc} \mathrm{K}_0(\mathcal{K}(l^2(P,A))) & \longrightarrow & \mathrm{K}_0(\mathcal{T}_p^A) & \longrightarrow & \mathrm{K}_0(A \rtimes (\mathbb{Z} \rtimes \mathbb{Z}_2)) \\ \uparrow & & & & \downarrow e_1 \\ \mathrm{K}_1(A \rtimes (\mathbb{Z} \rtimes \mathbb{Z}_2)) & \longleftarrow & \mathrm{K}_1(\mathcal{T}_p^A)) & \longleftarrow & \mathrm{K}_1(\mathcal{K}(l^2(P,A))). \end{array}$$

$\mathcal{T}_p^A$ can be shown to be KK-equivalent to $(A \rtimes_\alpha \mathbb{Z}_2) \oplus (A \rtimes_\beta \mathbb{Z}_2)$ (see [45], also [52]).

Let $S = \mathbb{Z} \rtimes \mathbb{Z}_2$ with generators a and b i.e a generates $\mathbb{Z}$ and b generates $\mathbb{Z}_2$. We saw that S is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2$, where the later group is generated by g and s and the isomorphism identifies a and b with sg and g, respectively. Now $l^2(P,A)$ could be identified with $l^2(P_1,A) \oplus l^2(P_2,A)$, where P_2 is the set $\{g, gsg, gsgsg, gsgsgsg, \ldots\}$ and P_1 is the set $\{e, sg, sgsg, sgsgsg, \ldots\}$. Counting the number of sg's, P_1 and P_2 have natural identifications with $\mathbb{Z}_{\geq 0}$. Under this identification $l^2(P,A)$ becomes $l^2(\mathbb{Z}_{\geq 0},A) \oplus l^2(\mathbb{Z}_{\geq 0},A)$, $\iota_2(s)$ becomes $\begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}$, $\iota_2(g)$ becomes $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\iota_2(a)$ becomes $\begin{pmatrix} \gamma^{-1}(a) & 0 \\ 0 & \gamma^{-1}\beta(a) \end{pmatrix}$, where γ is the action $\alpha \circ \beta$ on A and $\begin{pmatrix} \gamma^{-1}(a) & 0 \\ 0 & \gamma^{-1}\beta(a) \end{pmatrix}$ acts on $l^2(P_1,A) \oplus l^2(P_2,A)$. Now under the above identification $\mathcal{K}(l^2(P,A))$ becomes $\mathcal{K}(l^2(P_1,A) \oplus l^2(P_2,A)) = \mathcal{K}(l^2(\mathbb{Z}_{\geq 0},A) \oplus l^2(\mathbb{Z}_{\geq 0},A)) = M_2(\mathcal{K}((l^2(\mathbb{Z}_{\geq 0},A)))$.

Summarising we have the inclusions $P_{\mathcal{K}}^A$, $P_{\mathcal{T}}^A$, $P_{\mathbb{T}}^A$ as follows:

$$P_{\mathcal{K}}^A : \mathcal{K}(l^2(P,A)) \to M_2(\mathcal{K}((l^2(\mathbb{Z}_{\geq 0},A))),$$

$P_{\mathcal{T}}^A : \mathcal{T}_p^A \to M_2(\mathcal{T}^A)$, ($\mathcal{T}^A$ denotes the Toeplitz algebra of the $\mathbb{Z}$ action γ on A)

$P_{\mathbb{T}}^A : A \rtimes \mathbb{Z}_2 * \mathbb{Z}_2 = (A \rtimes_\gamma \mathbb{Z}) \rtimes \mathbb{Z}_2 \to M_2(A \rtimes_\gamma \mathbb{Z})$ are given by

$$P_{\mathcal{T}}^A(\iota_2(s)) = \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}, \quad P_{\mathcal{T}}^A(\iota_2(g)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{\mathcal{T}}^A(\iota_2(a)) = \begin{pmatrix} a & 0 \\ 0 & \beta(a) \end{pmatrix},$$

$P_{\mathcal{K}}^A$ is the identification map, $P_{\mathbb{T}}^A$ is the natural map p which was discussed before i.e

$$P_{\mathbb{T}}^A(U) = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}, \quad P_{\mathbb{T}}^A(W) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad P_{\mathbb{T}}^A(\iota_2(a)) = \begin{pmatrix} a & 0 \\ 0 & \beta(a) \end{pmatrix}.$$

So by construction we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2(\mathcal{K}(l^2(\mathbb{Z}_{\geq 0}),A)) & \xrightarrow{\varphi} & M_2(\mathcal{T}^A) & \xrightarrow{\psi} & M_2(A\rtimes\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow P^A_{\mathcal{K}} & & \uparrow P^A_{\mathcal{T}} & & \uparrow P^A_{\mathbb{T}} & & \\ 0 & \longrightarrow & \mathcal{K}(l^2(P),A) & \xrightarrow{\eta} & \mathcal{T}^A_p & \xrightarrow{\pi} & A\rtimes(\mathbb{Z}\rtimes\mathbb{Z}_2) & \longrightarrow & 0\end{array}$$

Theorem 7.0.1. *For unital A with the above notations, the connecting maps of the above both sequences commute in the following sense:*

$$\begin{array}{ccc} \mathrm{K}_0(A\rtimes\mathbb{Z}_2*\mathbb{Z}_2) & \xrightarrow{\quad e_1 \quad} & \mathrm{K}_1(A) \\ & \searrow^{p} & \uparrow e_2 \\ & & \mathrm{K}_0(A\rtimes_\gamma\mathbb{Z})) \end{array} \quad ,$$

where p is the natural map (viewed as a map of K*-groups).*

Proof. The result follows from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2(\mathcal{K}(l^2(\mathbb{Z}_{\geq 0}),A)) & \xrightarrow{\varphi} & M_2(\mathcal{T}^A) & \xrightarrow{\psi} & M_2(A\rtimes\mathbb{Z}) & \longrightarrow & 0 \\ & & \uparrow P^A_{\mathcal{K}} & & \uparrow P^A_{\mathcal{T}} & & \uparrow P^A_{\mathbb{T}} & & \\ 0 & \longrightarrow & \mathcal{K}(l^2(P),A) & \xrightarrow{\eta} & \mathcal{T}^A_p & \xrightarrow{\pi} & A\rtimes(\mathbb{Z}\rtimes\mathbb{Z}_2) & \longrightarrow & 0\end{array}$$

and the naturality of connecting maps. □

7.0.1 Continuous field of projective modules over $A_\theta \rtimes \mathbb{Z}_2$

We recall some results from Chapter 3. Let $[a, b]$ be a closed interval. We constructed the transformation groupoid $\mathbb{Z}^n \times [a, b]$ for trivial $\mathbb{Z}^n$ action on $[a, b]$. Let ω_r be a continuous family of 2-cocycles on the group $\mathbb{Z}^n$. We defined the following 2-cocycle ω on the groupoid $\mathbb{Z}^n \times [a, b]$: $\omega(x, y, r) = \omega_r(x, y)$. We can repeat the same construction for the group $\mathbb{Z}^n \rtimes \mathbb{Z}_2$. We define the following 2-cocycle ω' on the groupoid $\mathbb{Z}^n \rtimes \mathbb{Z}_2 \times [a, b]$: $\omega(x, y, r) = \omega'_r(x, y)$ (notations from previous chapters). We have the natural maps ev_r from $C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2 \times [a, b], \omega')$ to $C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2, \omega'_r)$ given by $f \to f'$, $f'(x, y) = f(x, y, r)$.

Theorem 7.0.2. *Let $[p_1], [p_2], \cdots, [p_m] \in \mathrm{K}_0(C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2 \times [a, b], \omega'))$. Then the following are equivalent:*

1. *$[p_1], [p_2], \cdots, [p_m]$ form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2 \times [a,b], \omega'))$.

2. *For some $r \in [a,b]$, the evaluated classes $[ev_r(p_1)], [ev_r(p_2)], \cdots, [ev_r(p_m)]$ form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2, \omega'_r))$.

3. *For every $r \in [a,b]$, the evaluated classes $[ev_r(p_1)], [ev_r(p_2)], \cdots, [ev_r(p_m)]$ form a basis of* $\mathrm{K}_0(C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2, \omega'_r))$.

Proof. See remark 2.3 of [22]. □

Now as in the beginning of the section we have seen that every projective module of A_θ has an action of $\mathbb{Z}_2$ making the projective module an $A_\theta \rtimes \mathbb{Z}_2 = C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2, \omega'_\theta)$ module. So we have the following immediate corollary of Theorem 3.2.2.

Corollary 7.0.3. *With the notations after Theorem 3.2.2, $\mathcal{E}$ is a finitely generated projective module over $C^*(\mathbb{Z}^n \rtimes \mathbb{Z}_2 \times [a,b], \omega')$*

7.0.2 Main computations with K-theory

Let us consider the 3×3 antisymmetric matrix:

$$\theta := \begin{pmatrix} 0 & \theta_{12} & \theta_{13} \\ -\theta_{12} & 0 & \theta_{23} \\ -\theta_{23} & -\theta_{23} & 0 \end{pmatrix},$$

where θ_{12} is an irrational number. Let A_θ be the corresponding 3-dimensional noncommutative torus generated by u_1, u_2 and u_3. Let $\mathbb{Z}_2 = \langle g \rangle$ acting on A_θ by flipping. We denote by $A_{\theta_{12}}$, the two dimensional noncommutative tori, which is generated by the matrix

$$\begin{pmatrix} 0 & \theta_{12} \\ -\theta_{12} & 0 \end{pmatrix}.$$

$\mathbb{Z}_2 = \langle g \rangle$ also action on $A_{\theta_{12}}$ by flip. Let us also define an action of another copy of $\mathbb{Z}_2 = \langle s \rangle$ on $A_{\theta_{12}}$ given by $u_1 \to e(\theta_{13})u_1^{-1}$ and $u_2 \to e(\theta_{23})u_2^{-1}$. So we have corresponding crossed products which we denote by $A_{\theta_{12}} \rtimes_g \mathbb{Z}_2$ and $A_{\theta_{12}} \rtimes_s \mathbb{Z}_2$, respectively. So we have, in fact, $\mathbb{Z}_2 * \mathbb{Z}_2 = \langle g, s \rangle$ acting on $A_{\theta_{12}}$. The corresponding crossed product we denote by $A_{\theta_{12}} \rtimes_\phi (\mathbb{Z}_2 * \mathbb{Z}_2)$. As we have seen in the beginning of this chapter that A_θ can be written as crossed product $A_{\theta_{12}} \rtimes \mathbb{Z}$ by the action $u_1 \to e(\theta_{13})u_1$ and $u_2 \to e(\theta_{23})u_2$. Now we have the isomorphism $A_\theta \rtimes \mathbb{Z}_2 \cong (A_{\theta_{12}} \rtimes \mathbb{Z}) \rtimes \mathbb{Z}_2 \cong A_{\theta_{12}} \rtimes_\phi (\mathbb{Z}_2 * \mathbb{Z}_2)$, $\lambda(sg)$ is identified with u_3 and $\lambda(g)$ is identified with w, where w is the unitary in $A_\theta \rtimes \mathbb{Z}_2$ coming from the unitary in $\mathbb{Z}_2$.

In this case, $A_{\theta_{12}} \rtimes_\phi (\mathbb{Z}_2 * \mathbb{Z}_2)$, Natsume's exact sequence looks like

$$\begin{array}{ccccc}
\mathrm{K}_0(A_{\theta_{12}}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta_{12}} \rtimes_s \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta_{12}} \rtimes_g \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta_{12}} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(A_{\theta_{12}} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) & \xleftarrow[j_{1*}+j_{2*}]{} & \mathrm{K}_1(A_{\theta_{12}} \rtimes_s \mathbb{Z}_2) \oplus \mathrm{K}_1(A_{\theta_{12}} \rtimes_g \mathbb{Z}_2) & \xleftarrow[i_{1*}-i_{2*}]{} & \mathrm{K}_1(A_{\theta_{12}}).
\end{array}$$

From Theorem 2.5.5, $A_{\theta_{12}} \rtimes_s \mathbb{Z}_2$ is isomorphic to the crossed product constructed from $\mathbb{Z}_2 = \langle s \rangle$ action on $A_{\theta_{12}}$ given by the flip. Let $S_g^{\theta_{12}}$, $S_s^{\theta_{12}}$ be the flip invariant Rieffel projections inside $A_{\theta_{12}} \rtimes_g \mathbb{Z}_2$ and $A_{\theta_{12}} \rtimes_s \mathbb{Z}_2$ respectively as constructed in Walters [59] in the same way as the classical Rieffel projection $S^{\theta_{12}} \in A_{\theta_{12}}$. Also there are projective modules S^1,S^2, S^3 over A_θ which are completions of $\mathscr{S}(\mathbb{R} \times \mathbb{Z})$ such that $[1], [S^1], [S^2], [S^3]$ generate K_0 of A_θ (as in Chapter 3, which are completions of the space of Schwartz functions on $\mathbb{R} \times \mathbb{Z}$). Let us now recall the Pimsner-Voiculescu exact sequence for A_θ.

$$\begin{array}{ccccc}
\mathrm{K}_0(A_{\theta_{12}}) & \xrightarrow{0} & \mathrm{K}_0(A_{\theta_{12}}) & \xrightarrow{i_*} & \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}) \\
\uparrow & & & & \downarrow \\
\mathrm{K}_1(A_{\theta_{12}} \rtimes \mathbb{Z}) & \xleftarrow[i*]{} & \mathrm{K}_1(A_{\theta_{12}}) & \xleftarrow[0]{} & \mathrm{K}_1(A_{\theta_{12}}).
\end{array}$$

This gives rise to the short exact sequence:

$$0 \longrightarrow \mathrm{K}_0(A_{\theta_{12}}) \xrightarrow{i_*} \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}) \xrightarrow{e_2} \mathrm{K}_1(A_{\theta_{12}}) \longrightarrow 0.$$

Among $[S^1], [S^2], [S^3]$, two of these, say S^2, S^3, map to generators of $\mathrm{K}_1(A_{\theta_{12}})$ via the map e_2.

As shown in the beginning of the section, the projective modules S^2, S^3 can be extended to projective modules over $A_{\theta_{12}} \rtimes \mathbb{Z}_2$ and by abuse of notations, we still denote the extended modules by S^2, S^3, as well.

Corollary 7.0.4. *$\mathrm{K}_0(A_\theta \rtimes \mathbb{Z}_2)$ is isomorphic to $\mathbb{Z}^{12}$, which is generated by the K-theory classes of the elements:*

1,
$P_1 = \frac{1}{2}(1 + w)$,
$P_2 = \frac{1}{2}(1 - u_1 w)$,
$P_3 = \frac{1}{2}(1 - u_2 w)$,
$P_4 = \frac{1}{2}(1 - e(\frac{1}{2}\theta_{12}) u_1 u_2 w)$,
$P_5 = \frac{1}{2}(1 + u_3 w)$,
$P_6 = \frac{1}{2}(1 - e(\frac{1}{2}\theta_{13}) u_1 u_3 w)$,
$P_7 = \frac{1}{2}(1 - e(\frac{1}{2}\theta_{23}) u_2 u_3 w)$,
$j_{1*}(S_g^{\theta_{12}})$, $j_{2*}(S_s^{\theta_{12}})$, S^2, S^3.

Proof. We have Natsume's exact sequence:

$$\begin{array}{ccccc} \mathrm{K}_0(A_{\theta_{12}}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta_{12}} \rtimes_s \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta_{12}} \rtimes_g \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta_{12}} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(A_{\theta_{12}} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2) & \xleftarrow[j_{1*}+j_{2*}]{} & \mathrm{K}_1(A_{\theta_{12}} \rtimes_s \mathbb{Z}_2) \oplus \mathrm{K}_1(A_{\theta_{12}} \rtimes_g \mathbb{Z}_2) & \xleftarrow[i_{1*}-i_{2*}]{} & \mathrm{K}_1(A_{\theta_{12}}). \end{array}$$

Since the K-theory groups of $A_\theta \rtimes \mathbb{Z}_2$ are already known as in the Introduction and in Chapter 5, we know that $\mathrm{K}_1(A_\theta \rtimes \mathbb{Z}_2) = 0$ and also $\mathrm{K}_1(A_{\theta_{12}} \rtimes_s \mathbb{Z}_2) \oplus \mathrm{K}_1(A_{\theta_{12}} \rtimes_g \mathbb{Z}_2) = 0$. So the upper left $i_{1*} - i_{2*}$ is injective. So we are left with

$$\begin{array}{ccccc} \mathrm{K}_0(A_{\theta_{12}}) & \xrightarrow{i_{1*}-i_{2*}} & \mathrm{K}_0(A_{\theta_{12}}) \rtimes_s \mathbb{Z}_2) \oplus \mathrm{K}_0(A_{\theta_{12}} \rtimes_g \mathbb{Z}_2) & \xrightarrow{j_{1*}+j_{2*}} & \mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z}_2 * \mathbb{Z}_2) \\ \uparrow & & & & \downarrow \\ 0 & \xleftarrow[0]{} & 0. & \xleftarrow[0]{} & \mathrm{K}_1(A_{\theta_{12}}). \end{array}$$

Using [59], the K-theory class of the elements P_i along with $P_8 = \frac{1}{2}(1 - e(\frac{1}{2}(\theta_{13} - \theta_{12} - \theta_{23}))u_1u_2u_3w)$ and $S_s^{\theta_{12}}$, $S_g^{\theta_{12}}$ form generators of $(\mathrm{K}_0(A_{\theta_{12}}) \rtimes_s \mathbb{Z}_2) \oplus (K_0(A_{\theta_{12}}) \rtimes_g \mathbb{Z}_2)$. Also it is well known that $\mathrm{K}_0(A_{\theta_{12}})$ is generated by 1 and the class of the Bott element $[S^{\theta_{12}}]$. Now for $0 < \theta_{12} < \frac{1}{2}$, $i_{1*}[S^{\theta_{12}}] = 2[S_g^{\theta_{12}}] + ([P_2] + [P_4]) - ([P_1] + [P_3])$. Indeed, these two elements have the same vector trace as in Theorem 6.4.6. Similarly, $i_{2*}[S^{\theta_{12}}] = 2[S_s^{\theta_{12}}] + ([P_6] + [P_8]) - ([P_5] + [P_7])$. Also for $\frac{1}{2} < \theta_{12} < 1$, the expression of $i_{1*}[S^{\theta_{12}}]$ is essentially the same (with some sign modification) and hence gives a similar result.

Now the elements S^2 and S^3 we lifted from $\mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z})$ to $\mathrm{K}_0(A_{\theta_{12}} \rtimes_\phi \mathbb{Z}_2 * \mathbb{Z}_2)$ via the p map. Since S^2 and S^3 are generators of $\mathrm{K}_0(A_{\theta_{12}} \rtimes \mathbb{Z})$ and hence they map to generators of $\mathrm{K}_1(A_{\theta_{12}})$ (in the Pimsner-Voiculescu sequence via the map e_2), by Theorem 7.0.1 the claim follows.

□

Remark 7.0.5. *So we have found generators of $A_\theta \rtimes \mathbb{Z}_2$ for quite general θ. Only assumption was that θ_{12} is irrational. Also for the assumption that one of the non-zero entries of θ is irrational, we also have found the generators. Indeed, if θ_{23} is irrational consider*

$$\theta_1 = \begin{pmatrix} 0 & \theta_{23} & -\theta_{12} \\ -\theta_{23} & 0 & -\theta_{13} \\ \theta_{12} & \theta_{13} & 0 \end{pmatrix}.$$

Since A_{θ_1} is isomorphic to A_θ, we continue working with A_{θ_1}. Similarly if θ_{13} is irrational, we work with A_{θ_2} where

$$\theta_2 = \begin{pmatrix} 0 & \theta_{13} & \theta_{12} \\ -\theta_{13} & 0 & -\theta_{23} \\ -\theta_{12} & \theta_{23} & 0 \end{pmatrix}.$$

If we want to drop this condition of irrationality of one of the entries, and use Theorem 7.0.2 (as in the Chapter 3), we have to compute the Chern character of the Projective module S^1 to identify this with $S_s^{\theta_{12}}$ and $S_g^{\theta_{12}}$. This is beyond the scope of this chapter and will appear in future work.

Remark 7.0.6. *I have an unpleasant obligation to warn the reader, that the corresponding result for* K*-theory of flipped crossed product in* [27] *contained a gap: it was claimed that (in the notations of the present paper) $i_{1*} - i_{2*}$ sends generators to generators. Fortunately, the arguments in the last corollary fill the gap for the three dimensional case which do not affect the final result of* [27] *at least for the three dimensional case.*

Question 7.0.7. *Our results suggest the following questions:*

- *Can one extend the Heisenberg modules of the form $\mathscr{S}(\mathbb{R}^p \times \mathbb{Z}^q)$ over A_θ to ones over general crossed products $A_\theta^\infty \rtimes F$, where the finite cyclic group F acts on A_θ^∞ as in section 4?*
- *What are the generators of $\mathrm{K}_0(A_\theta \rtimes F)$, where F acts on A_θ as in section 4?*
- *Heisenberg modules are also linked with results in signal analysis, concretely with Gabor frames, as shown in* [43]*. Thus one might wonder about the consequences of our results for the theory of Gabor frames.*

Bibliography

[1] Helmut Abels, *Pseudodifferential and singular integral operators*, De Gruyter Graduate Lectures, De Gruyter, Berlin, 2012. An introduction with applications. MR2884718

[2] Saad Baaj, *Calcul pseudo-différentiel et produits croisés de C^*-algèbres. I*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 11, 581–586. MR967366

[3] ———, *Calcul pseudo-différentiel et produits croisés de C^*-algèbres. II*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), no. 12, 663–666. MR967808

[4] Noe Barcenas, *Twisted geometric k-homology for proper actions of discrete groups*, 2015.

[5] Paul Baum and Alain Connes, *Geometric K-theory for Lie groups and foliations*, Enseign. Math. (2) **46** (2000), no. 1-2, 3–42. MR1769535

[6] Paul Baum, Alain Connes, and Nigel Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), 1994, pp. 240–291. MR1292018

[7] Paul Baum, Nigel Higson, and Thomas Schick, *On the equivalence of geometric and analytic K-homology*, Pure Appl. Math. Q. **3** (2007), no. 1, Special Issue: In honor of Robert D. MacPherson. Part 3, 1–24. MR2330153

[8] Bruce Blackadar, *K-theory for operator algebras*, Second, Mathematical Sciences Research Institute Publications, vol. 5, Cambridge University Press, Cambridge, 1998. MR1656031

[9] Jonathan Block and Nigel Higson, *Weyl character formula in KK-theory*, Noncommutative geometry and physics. 3, 2013, pp. 299–334. MR3098596

[10] Ola Bratteli, George A. Elliott, David E. Evans, and Akitaka Kishimoto, *Noncommutative spheres. I*, Internat. J. Math. **2** (1991), no. 2, 139–166. MR1094701

[11] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York-Berlin, 1982. MR672956

[12] J. Buck and S. Walters, *Connes-Chern characters of hexic and cubic modules*, J. Operator Theory **57** (2007), no. 1, 35–65. MR2301936

[13] Dan Burghelea, *The cyclic homology of the group rings*, Comment. Math. Helv. **60** (1985), no. 3, 354–365. MR814144

[14] Sayan Chakraborty and Franz Luef, *Metaplectic transformations and finite group actions on noncommutative tori*, 2017.

[15] Sayan Chakraborty and Makoto Yamashita, *Tracing cyclic homology pairings under twisting of graded algebras*, 2017.

[16] Alain Connes, *C^* algèbres et géométrie différentielle*, C. R. Acad. Sci. Paris Sér. A-B **290** (1980), no. 13, A599–A604. MR572645

[17] Joachim Cuntz, *Bivariante K-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter*, Doc. Math. **2** (1997), 139–182. MR1456322

[18] Joachim Cuntz, Ralf Meyer, and Jonathan M. Rosenberg, *Topological and bivariant K-theory*, Oberwolfach Seminars, vol. 36, Birkhäuser Verlag, Basel, 2007. MR2340673

[19] Joachim Cuntz, Georges Skandalis, and Boris Tsygan, *Cyclic homology in non-commutative geometry*, Encyclopaedia of Mathematical Sciences, vol. 121, Springer-Verlag, Berlin, 2004. Operator Algebras and Non-commutative Geometry, II. MR2052770

[20] Maurice A. de Gosson, *Symplectic methods in harmonic analysis and in mathematical physics*, Pseudo-Differential Operators. Theory and Applications, vol. 7, Birkhäuser/Springer Basel AG, Basel, 2011. MR2827662

[21] Claire Debord and Georges Skandalis, *Pseudodifferential extensions and adiabatic deformation of smooth groupoid actions*, Bull. Sci. Math. **139** (2015), no. 7, 750–776. MR3407514

[22] Siegfried Echterhoff, Wolfgang Lück, N. Christopher Phillips, and Samuel Walters, *The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$*, Journal für die Reine und Angewandte Mathematik **2** (2010), no. 639, 173–221.

[23] G. A. Elliott, T. Natsume, and R. Nest, *Cyclic cohomology for one-parameter smooth crossed products*, Acta Math. **160** (1988), no. 3-4, 285–305. MR945014

[24] George A. Elliott, *On the K-theory of the C*-algebra generated by a projective representation of a torsion-free discrete abelian group*, Operator Algebras and Group Representations, Vol. I (Neptun, 1980) **576** (1984), 157–184.

[25] George A. Elliott and Hanfeng Li, *Strong Morita equivalence of higher-dimensional noncommutative tori. II*, Mathematische Annalen **341** (2008), no. 4, 825–844.

[26] Thierry Fack and Georges Skandalis, *Connes' analogue of the Thom isomorphism for the Kasparov groups*, Invent. Math. **64** (1981), no. 1, 7–14. MR621767

[27] Carla Farsi and Neil Watling, *Symmetrized noncommutative tori*, Math. Ann. **296** (1993), no. 4, 739–741. MR1233495

[28] Paul G. Goerss and John F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999. MR1711612

[29] N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), no. 2, 330–354. MR1911663

[30] Nigel Higson and Gennadi Kasparov, *E-theory and KK-theory for groups which act properly and isometrically on Hilbert space*, Invent. Math. **144** (2001), no. 1, 23–74. MR1821144

[31] Lars Hörmander, *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517. MR0180740

[32] Ja A. Jeong and Jae Hyup Lee, *Finite groups acting on higher dimensional noncommutative tori*, J. Funct. Anal. **268** (2015), no. 2, 473–499. MR3283161

[33] Paul Jolissaint, *K-theory of reduced C^*-algebras and rapidly decreasing functions on groups*, K-Theory **2** (1989), no. 6, 723–735. MR1010979

[34] Max Karoubi, *Homologie cyclique des groupes et des algèbres*, C. R. Acad. Sci. Paris Sér. I Math. **297** (1983), no. 7, 381–384. MR732839

[35] G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **44** (1980), no. 3, 571–636, 719. MR582160

[36] Adam Kleppner, *Multipliers on abelian groups*, Math. Ann. **158** (1965), 11–34. MR0174656

[37] Martin Langer and Wolfgang Lück, *Topological K-theory of the group C^*-algebra of a semi-direct product $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ for a free conjugation action*, J. Topol. Anal. **4** (2012), no. 2, 121–172.

[38] Max Lein, Marius Măntoiu, and Serge Richard, *Magnetic pseudodifferential operators with coefficients in C^*-algebras*, Publ. Res. Inst. Math. Sci. **46** (2010), no. 4, 755–788. MR2791006

[39] Matthias Lesch and Henri Moscovici, *Modular curvature and Morita equivalence*, Geom. Funct. Anal. **26** (2016), no. 3, 818–873. MR3540454

[40] Hanfeng Li, *Strong morita equivalence of higher-dimensional noncommutative tori*, Journal für die Reine und Angewandte Mathematik **576** (2003), 167–180.

[41] Jean-Louis Loday and Daniel Quillen, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv. **59** (1984), no. 4, 569–591. MR780077

[42] Wolfgang Lück and Roland Stamm, *Computations of K- and L-theory of cocompact planar groups*, K-Theory **21** (2000), no. 3, 249–292. MR1803230

[43] Franz Luef, *Projective modules over noncommutative tori are multi-window Gabor frames for modulation spaces*, J. Funct. Anal. **257** (2009), no. 6, 1921–1946. MR2540994

[44] Varghese Mathai, *Heat kernels and the range of the trace on completions of twisted group algebras*, The ubiquitous heat kernel, 2006, pp. 321–345. With an appendix by Indira Chatterji. MR2218025

[45] Toshikazu Natsume, *On $K_*(C^*(\mathrm{SL}_2(\mathbb{Z})))$. Appendix to "$K$-theory for certain group C^*-algebras" [Acta Math.* **151** *(1983), no. 3-4, 209–230; MR0723010 (86f:46076)] by E. C. Lance*, J. Operator Theory **13** (1985), no. 1, 103–118. MR768305

[46] Sergey Neshveyev, *Smooth crossed products of Rieffel's deformations*, Lett. Math. Phys. **104** (2014), no. 3, 361–371. MR3164614

[47] Ryszard Nest, *Cyclic cohomology of crossed products with* **Z**, J. Funct. Anal. **80** (1988), no. 2, 235–283. MR961899

[48] P. Olczykowski and A. Sitarz, *K-theory of noncommutative Bieberbach manifolds*, Russ. J. Math. Phys. **22** (2015), no. 3, 389–399. MR3392526

[49] Judith A. Packer and Iain Raeburn, *Twisted crossed products of C^*-algebras*, Math. Proc. Cambridge Philos. Soc. **106** (1989), no. 2, 293–311. MR1002543

[50] ______, *On the structure of twisted group C^*-algebras*, Trans. Amer. Math. Soc. **334** (1992), no. 2, 685–718. MR1078249

[51] N. Christopher Phillips, *Representable K-theory for σ-C^*-algebras*, K-Theory **3** (1989), no. 5, 441–478. MR1050490

[52] M. Pimsner and D. Voiculescu, *K-groups of reduced crossed products by free groups*, J. Operator Theory **8** (1982), no. 1, 131–156. MR670181

[53] Jean Renault, *A groupoid approach to c^*-algebras*, Lecture notes in mathematics, Springer, 1980.

[54] Marc A. Rieffel, *Projective modules over higher-dimensional non-commutative tori*, Can. J. Math., 40(2), 257–338 **2** (1988), 257–338.

[55] ______, *Deformation quantization for actions of $\mathbf{R}^d$*, Mem. Amer. Math. Soc. **106** (1993), no. 506, x+93. MR1184061

[56] Larry B. Schweitzer, *Dense m-convex Fréchet subalgebras of operator algebra crossed products by Lie groups*, Internat. J. Math. **4** (1993), no. 4, 601–673. MR1232986

[57] ______, *Spectral invariance of dense subalgebras of operator algebras*, Internat. J. Math. **4** (1993), no. 2, 289–317. MR1217384

[58] Maarten Solleveld, *Some Fréchet algebras for which the Chern character is an isomorphism*, K-Theory **36** (2005), no. 3-4, 275–290 (2006). MR2275008

[59] Samuel G. Walters, *Projective modules over the non-commutative sphere*, J. London Math. Soc. (2) **51** (1995), no. 3, 589–602. MR1332894

[60] ______, *Chern characters of Fourier modules*, Canad. J. Math. **52** (2000), no. 3, 633–672. MR1758235

[61] ______, *K-theory of non-commutative spheres arising from the Fourier automorphism*, Canad. J. Math. **53** (2001), no. 3, 631–672. MR1827823

[62] ______, *Periodic integral transforms and C^*-algebras*, C. R. Math. Acad. Sci. Soc. R. Can. **26** (2004), no. 2, 55–61. MR2055227

[63] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007. MR2288954