Münster J. of Math. **8** (2015), 211–228
DOI 10.17879/65219674085
urn:nbn:de:hbz:6-65219674337

Münster Journal of Mathematics

Conformally Kähler geometry and quasi-Einstein metrics

Wafaa Batat, Stuart J. Hall, Ali Jizany, and Thomas Murphy

(Communicated by Karsten Grove)

Abstract. We prove that the quasi-Einstein metrics found by Lü, Page and Pope on $\mathbb{CP}^1$-bundles over Fano Kähler–Einstein bases are conformally Kähler and that the Kähler class of the conformal metric is a multiple of the first Chern class. A detailed study of the lowest-dimensional example of such metrics on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ using the methods developed by Abreu and Guillemin for studying toric Kähler metrics is given. Our methods yield, in a unified framework, proofs of the existence of the Page, Koiso–Cao and Lü–Page–Pope metrics on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. Finally, we investigate the properties that similar quasi-Einstein metrics would have if they also exist on the toric surface $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$.

1. Introduction

A quasi-Einstein metric is a complete Riemannian manifold (M, g) satisfying

$$\mathrm{Ric}(g) + \nabla^2\phi - \frac{1}{m} d\phi \otimes d\phi = \lambda g, \tag{1}$$

for some function $\phi \in C^\infty(M)$ and constants λ, m with $m > 0$. Setting ϕ to be constant yields an Einstein metric, so solutions to equation (1) with nonconstant ϕ are referred to as *nontrivial* quasi-Einstein metrics. When m is a positive integer such metrics are important as the base manifolds for warped product constructions of Einstein metrics. As well as generalizing the Einstein condition, quasi-Einstein metrics can be thought of as deformations of gradient Ricci solitons, which are of central importance in the theory of Ricci flow. By formally taking $m \to \infty$ in equation (1), one recovers the equation defining a gradient Ricci soliton. Given their relationship with both types of canonical metric, a fundamental question is: in what way are quasi-Einstein metrics like Ricci solitons, and in what way are they like Einstein metrics?

The only known examples of compact quasi-Einstein metrics where m varies continuously are essentially due to a construction of Lü, Page and Pope [25] on $\mathbb{CP}^1$-bundles over Fano Kähler–Einstein manifolds, where the total space is denoted by W_q. This construction was generalized by the second author in [16]. On such spaces, non-Kähler Einstein metrics were known to exist due to a similar construction of Bérard-Bergery [3] (later generalized by Wang and Wang [28]). These spaces also admit shrinking gradient Ricci solitons due to Koiso [22], Cao [6] and Chave and Valent [8]. A more general construction of such solitons was later given by Dancer and Wang [11]. All the examples of Ricci solitons on these manifolds are Kähler. The Bérard-Bergery Einstein metrics turn out to be conformally Kähler. In the lowest-dimensional case, this was demonstrated by Derdzinski [12] who also showed that the Kähler metric must be extremal. However quasi-Einstein metrics are *never* Kähler, due to a foundational result of Case, Shu and Wei [7]. Nevertheless, we will show Kähler geometry plays a role in the theory of quasi-Einstein metrics on these spaces.

In Section 2 we show that the Lü–Page–Pope metrics are conformally Kähler and so are similar to the Bérard-Bergery Einstein metrics on these spaces. Maschler, in [26], suggested that this was likely to be the case, and it is probably known to experts. What is more surprising is that we are able to show that the Kähler metrics always lie in a multiple of the first Chern class. In this way the Lü–Page–Pope metrics are similar to the Dancer–Wang Ricci solitons, since any Kähler–Ricci soliton must lie in the first Chern class.

The next, and most significant, part of the article makes the link with Kähler geometry even more explicit for the lowest-dimensional case of the Lü–Page–Pope construction. Here the underlying manifold in this case is the nontrivial $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$, which can also be described as the one-point blow-up of the complex projective plane, $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. The Einstein metric given by this construction was originally discovered by Page [27], and the associated conformally Kähler metric is due to Calabi [5]. The Kähler–Ricci soliton on this manifold was originally discovered independently by Koiso [22] and Cao [6]. All of these Kähler metrics are toric, and therefore have a beautiful description due to Abreu [1, 2] and Guillemin [14]. In Section 3, we show that the Lü–Page–Pope metrics can also be explicitly described in this framework. This has a number of consequences; it leads to a greatly simplified proof of the existence of the quasi-Einstein metrics, and also gives a straightforward proof of the results of Section 2 in this special case. Moreover, our construction provides a unified framework for constructing the Page, Koiso–Cao, and Lü–Page–Pope metrics in one fell swoop.

In Section 4, the related toric surface $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$ is studied. This manifold is known to admit a conformally Kähler Einstein metric, analogous to the Page metric, due to Chen–LeBrun–Weber [9]. It also admits a Kähler–Ricci soliton, analogous to the Koiso–Cao metric, due to Wang and Zhu [29] (Donaldson gives an alternative proof of the existence theorem using the Abreu–Guillemin framework in [13]). The problem of constructing quasi-Einstein metrics analogous to the Lü–Page–Pope metrics on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$ is a natural open problem. If a family of quasi-Einstein metric with the same properties as the Lü–Page–Pope were to exist on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$, then our methods allow us to determine the explicit form of the potential function ϕ for metrics in a given cohomology class. Finally, in Section 5 we discuss some open problems and areas for future research.

2. The geometry of Lü–Page–Pope metrics

In this section we make precise the relationship between the Lü–Page–Pope quasi-Einstein metrics on certain $\mathbb{CP}^1$-bundles and the Kähler geometry of such manifolds.

2.1. **Construction of metrics on the manifolds W_q.** We begin by describing the construction of the manifolds W_q. Let (M, h, J) be a Fano Kähler–Einstein manifold of complex dimension n. Write the first Chern class of M as $c_1(M) = pa$, where $p \in \mathbb{N}$ and $a \in H^2(M, \mathbb{Z})$ is an indivisible class. For example, in the case $M = \mathbb{CP}^n$, one has $p = n + 1$ and $a = c_1(\mathcal{O}(1))$. The metric h is normalized so that

$$\mathrm{Ric}(h) = ph.$$

In other words, if $\eta(\cdot, \cdot) = h(J\cdot, \cdot)$ is the Kähler form of h, then $[\eta] = a$.

Denote by P_q the principal $U(1)$-bundle over M with Euler class $e = qa$ where $q \in \mathbb{Z}$. Let θ be the connection with curvature $\Omega = q\eta$. Finally, denote by W_q the projectivization $\mathbb{P}(L_q \oplus \mathcal{O})$ where L_q is the associated holomorphic line-bundle of P_q. It is useful to view the manifolds W_q as the compactification of $P_q \times (0, 4)$ obtained by collapsing a $U(1)$-fiber at 0 and 4. This gives rise to Riemannian metrics on W_q of the form

$$g = \alpha(s)^{-1} ds^2 + \alpha(s)\theta \otimes \theta + \beta(s)\pi^* h, \tag{2}$$

where s denotes the coordinate on $(0, 4)$, $\pi : W_q \to M$ is the projection and $\alpha, \beta \in C^\infty((0, 4))$. In order for the metrics of the form (2) to extend smoothly to the compactification W_q, the functions α and β satisfy

$$\alpha(0) = \alpha(4) = 0 \quad \text{and} \quad \alpha'(0) = -\alpha'(0) = 2.$$

The precise theorem that guarantees existence of nontrivial quasi-Einstein metrics on the manifolds W_q is [16, Thm. 3] (the case when m is integral was first proved in [25]). In the case where the base manifold is a single factor this can be restated as follows.

Theorem 2.2. *For $0 < |q| < p$, let W_q be as described above. Then, for all $m > 1$, there exists a nontrivial quasi-Einstein metric of the form (2) on W_q. Furthermore the function β is given by*

$$\beta(s) = A(s + s_0)^2 - \frac{q^2}{4A},$$

where s_0 and A are constants satisfying

$$s_0(s_0 + 4) = \frac{8Ap + q^2}{4A^2}. \tag{3}$$

Remark 2.3. Theorem 3 in [16] involves a constraint (the non-vanishing of a certain integral) which the Lü–Page–Pope examples automatically satisfy. The constraint is suggestive of a link between the quasi-Einstein metrics on W_q and Kähler geometry as the integral is essentially the Futaki invariant. A nontrivial Kähler–Ricci soliton must have non-vanishing Futaki invariant.

2.4. **The complex geometry of W_q.** The complex structure on W_q can be described in the (s,θ)-coordinates. One can lift the complex structure J on the base component and then define $J(\partial_s) = -(1/\sqrt{\alpha})\partial_\theta$. Hence the Kähler form of equation (2) is given by

$$\omega = \theta \wedge ds + \beta\pi^*\eta.$$

Lemma 2.5 (see [28, Cor. 7.3])**.** *Let g be a Lü–Page–Pope metric. If*

$$\sigma(s) = -\log\big(|2A(s+s_0) - q|\big),$$

then the conformally related metric $g_{\mathrm{K}} = e^{2\sigma}g$ is Kähler.

Proof. This follows from straightforward calculation. The Hermitian form of g_{K} is given by

$$\omega_{\mathrm{K}} = e^{2\sigma}(\theta \wedge ds + \beta\pi^*\eta)$$

and so

$$d\omega_{\mathrm{K}} = e^{2\sigma}\big(d\theta \wedge ds + (\beta'(s) + 2\sigma'(s)\beta(s))ds \wedge \pi^*\eta\big).$$

Using the fact that $d\theta = \pi^*\eta$, this vanishes if

$$\beta'(s) + 2\sigma'(s)\beta(s) + q = 0.$$

Hence as $\beta(s) = A(s+s_0)^2 - q^2/4A$ it follows that (up to a constant)

$$\sigma(s) = -\log\big(|2A(s+s_0) - q|\big). \qquad \square$$

In order to compute the first Chern class of W_q we revert to considering the manifold as the projectivization of a rank-two holomorphic vector bundle, namely $\mathbb{P}(L_q \oplus \mathcal{O})$. Some of the topology we need is presented in [28, §6]. Over each $\mathbb{CP}^1$-fiber there is the tautological line bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$. We denote the first Chern class of the dual of this line bundle over W_q by $F = c_1(\mathcal{O}_{\mathbb{CP}^1}(1))$. The Leray–Hirsch theorem states that

$$H^2(W_q;\mathbb{Z}) \cong H^2(M;\mathbb{Z}) \oplus \langle F\rangle.$$

In such a setting, we have the following lemma.

Lemma 2.6 ([28, Prop. 6.4])**.** *Let $W_q = \mathbb{P}(L_q \oplus \mathcal{O})$ and F be as described above. Then*

$$c_1(W_q) = (p+q)\pi^*a + 2F.$$

We can now compute the cohomology class of the Kähler metric g_{K} conformal to the Lü–Page–Pope metric.

Theorem 2.7. *Let (W_q, g) be a Lü–Page–Pope metric. Then*

(1) *the metric g is conformal to a Kähler metric g_{K}, and*
(2) *the cohomology class of the associated class ω_{K} is a scalar multiple of the first Chern class of W_q.*

Proof. The first part of the theorem follows immediately from Lemma 2.5. In order to compute the cohomology class of the metric some distinguished homology classes will be introduced. Let $\tau_1 \in H_2(M, \mathbb{R})$ be the class dual to $a \in H^2(M, \mathbb{R})$, in the sense that $\int_{\tau_1} a = 1$. Similarly, let $\tau_2 \in H_2(W_q, \mathbb{R})$ be the homology class dual to F, which means that $\int_{\tau_2} F = 1$ (here one represents τ_2 by a $\mathbb{CP}^1$-fiber divided by 2π). Denote by τ_1^0 the class of τ_1 in the copy of M glued in to W_q at $s = 0$ and by τ_1^4 the τ_1 in the copy of M glued in at $s = 4$. We claim that these classes satisfy the equation

$$\tau_1^0 - \tau_1^4 = q\tau_2. \tag{4}$$

Establishing the claim starts with the fact that $H_2(W_q, \mathbb{Z}) \cong H_2(M, \mathbb{Z}) \oplus \tau_2$. Here the first factor represents the pushforward of classes in the base M via a generic section of the vector bundle L_q, and τ_2 is the homology class of the $\mathbb{CP}^1$-fiber.

The copy of M at $s = 0$ represents the zero section of the line bundle L_q. The class τ_1^4 obviously does not intersect τ_1^0 in homology as the manifolds M at $s = 0$ and $s = 4$ do not intersect. If we restrict to the bundle defined over a representative cycle of the homology class τ_1^0 then, because the pullback of the first Chern class is the first Chern class of the pullback, we see that generic sections of the restriction of L_q intersect q times. Hence the homology of the subbundle is generated by τ_1^0 and τ_2 with

$$\tau_1^0 \cap \tau_1^0 = q, \quad \tau_2 \cap \tau_1^0 = 1 \quad \text{and} \quad \tau_2 \cap \tau_2 = 0,$$

where $\cap$ denotes oriented intersection. Hence if $\tau_1^0 \cap (a\tau_1^0 + b\tau_2) = 0$, it follows that $a = 1$ and $b = -q$, because the coefficients are elements of $\mathbb{Z}$. Hence $\tau_1^4 = \tau_0 - q\tau_2$. By construction, none of the homology classes τ_0^1, τ_4^1 or τ_2 vanish when they are embedded in W_q, as they were defined as the classes one gets in the image of this embedding. Therefore this identity must also hold in $H_2(W_q, \mathbb{R})$ and equation (4) is established.

Using equation (4), Lemma 2.6 can be restated as

$$\int_{\tau_1^0} c_1(W_q) = p + q \quad \text{and} \quad \int_{\tau_1^4} c_1(W_q) = p - q.$$

This implies that in order to prove the theorem one needs to evaluate the metric on the copies of τ_1 at $s = 0$ and at $s = 4$ and take the ratio. We compute

$$\begin{aligned}\frac{e^{2\sigma(4)}\beta(4)}{e^{2\sigma(0)}\beta(0)} &= \frac{(2As_0 - q)^2(4A^2(4 + s_0)^2 - q^2)}{(2A(4 + s_0) - q)^2(4A^2 s_0^2 - q^2)} \\ &= \frac{(2As_0 - q)(2A(4 + s_0) + q)}{(2A(4 + s_0) - q)(2As_0 + q)}.\end{aligned}$$

Using (3), we see

$$\frac{e^{2\sigma(4)}\beta(4)}{e^{2\sigma(0)}\beta(0)} = \frac{8Ap + q^2 - 8Aq - q^2}{8Ap + q^2 + 8Aq - q^2} = \frac{p - q}{p + q}.$$

The result now follows. □

Remark 2.8. In [26], conformally Kähler quasi-Einstein metrics were investigated by Maschler. He showed that in complex dimension 3 and greater, assuming the Kähler metric is not a local product, the square root of the conformal factor is a Killing potential, and the potential function and the conformal factor are functionally dependent, then the manifold is biholomorphic to one of the manifolds W_q.

3. Metrics on $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$ in the Abreu–Guillemin framework

In this section we study in more detail the lowest-dimensional example of the Lü–Page–Pope construction which occurs on the nontrivial $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$. As mentioned in the introduction, we shall switch perspectives and consider this manifold as the blow-up of the complex projective plane $\mathbb{CP}^2$ at one point, written $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. In this section we write the conformally Kähler Lü–Page–Pope metric (and Page's conformally Einstein metric and the Koiso–Cao soliton) explicitly in symplectic (also known as action-angle) coordinates. This has a number of nice features; it simplifies the existence theory in [25] and [16] and the results of Theorem 2.7 are almost immediate in this setting. It also suggests how the existence theory might run on the other toric Fano surface with non-vanishing Futaki invariant, $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$.

3.1. ***U*(2)-invariant Kähler Metrics on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$.** To begin with, we consider Kähler metrics that are invariant under a Hamiltonian action by the torus $\mathbb{T}^2$. The moment polytope (i.e. the image of the moment map) for the manifold $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ is the trapezium (trapezoid) T described by the linear inequalities

$$\begin{aligned} l_1(x) &= (1 + x_1), & l_2(x) &= (1 + x_2), \\ l_3(x) &= (1 - x_1 - x_2), & l_4(x) &= (a + x_1 + x_2). \end{aligned}$$

The parameter $a \in (-1, 2)$ determines the volume of the exceptional divisor and hence the cohomology class that the associated metric ω is in. The case where $a = 1$ corresponds to the case when $[\omega] = c_1$. We note in this case, the volume of the exceptional divisor is one third of the volume of the projective line at infinity. The Guillemin theory states that there is an open set in the manifold, diffeomorphic to $T^\circ \times \mathbb{T}^2$ where the metric takes the form

$$g = u_{ij}dx_i dx_j + u^{ij}d\theta_i d\theta_j, \tag{5}$$

where u is a function on T known as the *symplectic potential*. Here u_{ij} is the Hessian matrix in the Euclidean coordinates x_1, x_2 on the trapezium T, and u^{ij} is the inverse matrix. Furthermore, Guillemin showed that the symplectic potential u has the form

$$u(x) = \frac{1}{2}\bigg(\sum_{i=1}^{4} l_i(x)\log(l_i(x)) + f(x_1, x_2)\bigg), \tag{6}$$

where f is a smooth function with all derivatives continuous up to the boundary ∂T. The manifold $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$ inherits a $U(2)$ action from $\mathbb{CP}^2$ which fixes the point that is blown up. Hence the action lifts to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}}^2$. It is an example of a cohomogeneity-one action as the orbit of a generic point is a three-sphere $\mathbb{S}^3$. If we restrict to $U(2)$-invariant Kähler metrics then one can take $f(x_1, x_2) = f(x_1 + x_2)$ in (6). For the remainder of the article we will take $t = x_1 + x_2$. We can write the metric explicitly by noting that the Euclidean Hessian of u is given by

$$D^2 u = \frac{1}{2}\begin{pmatrix} \frac{1}{x_1+1} + P(t) & P(t) \\ P(t) & \frac{1}{x_2+1} + P(t) \end{pmatrix},$$

where

$$P(t) = \frac{1}{1-t} + \frac{1}{a+t} + f''(t).$$

It will also be useful to introduce, in terms of $t = x_1 + x_2$ the related functions

$$F(t) = 1 + (2+t)P(t) \quad \text{and} \quad z(t) = F^{-1}(t).$$

The function $z(t)$ satisfies the following conditions at the boundaries:

$$z(-a) = z(1) = 0 \quad \text{and} \quad z'(-a) = (2-a)^{-1}, \quad z'(1) = -1/3. \tag{7}$$

The determinant of the metric and the inverse of the matrix $D^2 u$ are given by

$$\det(D^2 u) = \frac{F(t)}{4(x_1+1)(x_2+1)}$$

and

$$(D^2 u)^{-1} = \frac{2(x_1+1)(x_2+1)}{F(t)}\begin{pmatrix} \frac{1}{x_2+1} + P(t) & -P(t) \\ -P(t) & \frac{1}{x_1+1} + P(t) \end{pmatrix}.$$

The x_i-components of the Ricci tensor in these coordinates are given by

$$\mathrm{Ric}_{ij} = \frac{1}{2}\left(\frac{\partial^2}{\partial x_i \partial x_j} - u^{kl}\frac{\partial u_{ij}}{\partial x_k}\frac{\partial}{\partial x_l}\right)\log\det(D^2 u).$$

The following quantity will be especially useful in our calculations:

$$\mathrm{Ric}_{11} - \mathrm{Ric}_{22} = \frac{1}{2}\frac{x_2 - x_1}{(x_1+1)(x_2+1)}\left(\frac{F'}{F^2} + \frac{2(F-1)}{F(2+t)}\right). \tag{8}$$

Functions on the manifold that are invariant under the $U(2)$ action can be expressed as functions $\phi(t) : [-a, 1] \to \mathbb{R}$. We will also need a similar expression to the one above for the Hessian (calculated with the metric (5)) of such functions:

$$\nabla^2\phi_{11} - \nabla^2\phi_{22} = \frac{1}{2}\frac{x_2 - x_1}{(x_1+1)(x_2+1)}\frac{\phi'}{F}. \tag{9}$$

Throughout this paper we will use the analyst's Laplacian $\Delta = \mathrm{tr}(\nabla^2)$. The Laplacian of a $U(2)$-invariant function $\phi(t)$ is given by

$$\Delta\phi = \left(-2(2+t)\frac{F'}{F^2} + \frac{4}{F}\right)\phi' + \frac{2(2+t)}{F}\phi''. \tag{10}$$

We will also need the formulae for how some of the above quantities transform under a conformal rescaling of the metric. If $\tilde{g} = e^{2\sigma} g$ for $\sigma \in C^{\infty}(M)$ then

$$\mathrm{Ric}(\tilde{g}) = \mathrm{Ric}(g) - 2(\nabla^2\sigma - d\sigma \otimes d\sigma) - (2|\nabla\sigma|^2 + \Delta\sigma)g, \tag{11}$$

where all the quantities on the right-hand side are computed with the metric g. The Hessian of a function ϕ transforms under conformal rescaling via

$$\widetilde{\nabla}^2\phi = \nabla^2\phi + e^{\sigma}\big[de^{-\sigma} \otimes d\phi + d\phi \otimes de^{-\sigma} - g(\nabla e^{-\sigma}, \nabla\phi)g\big], \tag{12}$$

and hence the Laplacian transforms via

$$\widetilde{\Delta}\phi = e^{-2\sigma}\big(\Delta\phi + 2g(\nabla\sigma, \nabla\phi)\big). \tag{13}$$

3.2. **Explicit metrics.** We now determine explicit representations for the function $z(t)$. One could then rearrange and perform the required integration in order to determine the function $f(t)$ in the symplectic potential (6).

3.2.1. *The Lü–Page–Pope metrics.* As before, we write $g_{\mathrm{LPP}} = e^{2\sigma} g_{\mathrm{K}}$, with g_{K} a Kähler metric. The Lü–Page–Pope metrics have J-invariant Ricci tensor and so the function $e^{-\sigma}$ is a Killing potential and thus $\sigma(t) = -\log(bt + c)$ for constants b and c. We can take the potential function to be invariant under the $U(2)$-action and so ϕ is also a function of t. We are hence in the setting considered by Maschler and so, by the discussion following [26, (2.2)], we have

$$\phi(t) = -m\log(e^{\sigma(t)} + d) = -m\log\left(\frac{dbt + dc + 1}{bt + c}\right),$$

for a constant d. We note that the constants a, b, c, d must satisfy

$$bt + c > 0 \quad \text{and} \quad dbt + dc + 1 > 0,$$

for all $t \in [-a, 1]$. Using equations (11), (12) and (13), for conformally related quantities, together with equations (8), (9) and (10), we can rewrite the equation

$$\mathrm{Ric}_{11}(g_{\mathrm{LPP}}) - \mathrm{Ric}_{22}(g_{\mathrm{LPP}}) + \nabla^2\phi_{11} - \nabla^2\phi_{22} = (g_{\mathrm{LPP}})_{11} - (g_{\mathrm{LPP}})_{22}$$

as a first-order ODE for $z(t)$:

$$\begin{aligned} \frac{dz}{dt} &+ \left(\frac{2}{2+t} - \frac{3b}{bt+c} - \frac{b}{bt+4b-c} - \frac{mb}{(dbt+dc+1)(bt+c)}\right)z \\ &+ \frac{2(bt+c)^2 - (2+t)}{(2+t)(bt+c)(bt+4b-c)} = 0. \end{aligned} \tag{14}$$

Quasi-Einstein metrics have a first integral due to Kim and Kim [21] coming from the contracted second Bianchi identity. For any solution to (1), there is a constant μ for which the quasi-Einstein potential ϕ satisfies

$$1 - \frac{1}{m}\big(\Delta\phi - |\nabla\phi|^2\big) = \mu e^{\frac{2\phi}{m}}. \tag{15}$$

We calculate with respect to the Kähler metric and obtain another ODE:

$$\frac{dz}{dt}+\left(\frac{2}{2+t}-\frac{3b}{bt+c}-\frac{b}{bt+c+d^{-1}}-\frac{mb}{(dbt+dc+1)(bt+c)}\right)z$$
$$+\left(\frac{\mu(bt+c)^2-(dbt+dc+1)^2}{2bd(2+t)(bt+c)(bt+c+d^{-1})}\right)=0.$$

In order to have consistency we must have

$$d=(2(2b-c))^{-1}\quad\text{and}\quad \mu=d^2+4bd.$$

Using the boundary conditions (7), we obtain

$$\frac{3-2(b+c)^2}{3(b+c)(5b-c)}=-\frac{1}{3}\quad\text{and}\quad\frac{(2-a)-2(c-ab)^2}{(2-a)(c-ab)((4-a)b-c)}=\frac{1}{(2-a)}.$$

Rearranging we see that

$$c^2=b^2+1\quad\text{and}\quad c^2=a(4-3a)b^2+4(a-1)bc+(2-a).$$

If $a\neq 1$ then $c=(2\pm\sqrt{3(2-a)})b$. In the case $c=(2+\sqrt{3(2-a)})b$, then $c>2b$ and so $d=(2(2b-c))^{-1}<0$. This means that

$$bt+c+d^{-1}=bt+4b-c<0$$

on $[-a,1]$. Hence $c>5b$; this is a contradiction as $a\in(-1,2)$. In the case where $c=(2-\sqrt{3(2-a)})b$ we have $b+c>0$ and so $(3-\sqrt{3(2-a)})b>0$. As $-1<a<2$ this means we must have $b>0$. We also have $c-ba>0$ hence $2-\sqrt{3(2-a)}>a$ and so $a<-1$. This is a contradiction. Hence $a=1$ and we have proved in a straightforward fashion the second part of Theorem 2.7.

The solution of (14) is given by

$$z(t)=Z(t)\int_{-1}^{t}\frac{(2+s)-2(bs+c)^2}{(bs+c)^{m+4}}(dbs+dc+1)^{m-2}(2+s)ds,$$

where

$$Z(t)=\frac{d(bt+c)^{m+3}}{(dbt+dc+1)^{m-1}(2+t)^2},\quad c=\sqrt{1+b^2},\quad d=\frac{1}{2(2b-c)}.$$

In order that we get a smooth metric we must be able to choose a compatible b so that $z(1)=0$, which is equivalent to

$$I(b):=\int_{-1}^{1}\frac{(2+s)-2(bs+c)^2}{(bs+c)^{m+4}}(dbs+dc+1)^{m-2}(2+s)ds=0.$$

We note that

$$I(0)=\left(\frac{1}{2}\right)^{m-2}\left(\frac{2}{3}\right)>0$$

and, for $m>1$,

$$I\left(\frac{1}{\sqrt{24}}\right)=-\frac{1}{12}\int_{-1}^{1}\frac{(s-1)^2}{(bs+c)^{m+4}}(dbs+dc+1)^{m-2}(2+s)ds<0.$$

Hence we see that there exists $b \in (0, 1/\sqrt{24})$ giving a smooth solution of (14), and hence of (1). For a fixed value of m it is very easy to find the approximate values of b (and hence c and d) numerically. When $m = 2$ we find

$$b \approx 0.076527, \quad c \approx 1.002924 \quad \text{and} \quad d \approx -0.588325.$$

For $m = 50$ we find

$$b \approx 0.005120, \quad c \approx 1.000013 \quad \text{and} \quad d \approx -0.505167.$$

Using the above values for $m = 50$, we compute

$$\frac{\phi(1) - \phi(-1)}{2} \approx 0.517374,$$

which suggests ϕ is approximately a linear function with gradient close to 0.52. If we compare the construction of the Koiso–Cao soliton $(g_{\mathrm{KC}}, \phi_{\mathrm{KC}})$ in Section 3.2.3, we see there is strong numerical evidence that, as $m \to \infty$,

$$\sigma \to 0, \quad \phi \to \phi_{\mathrm{KC}} \quad \text{and} \quad g_{\mathrm{LPP}} \to g_{\mathrm{KC}},$$

where one expects to get convergence in the C^∞ topology. A careful study of the above ansatz and the dependence of the values of b upon m would probably yield this result.

3.2.2. *Page's Einstein metric.* Toric constructions of the Kähler metric g_{K} conformal to Page's Einstein metric g_{P} were already given by Abreu [1] and Dammerman [10]. Both authors constructed Calabi's family of extremal metrics on the manifold which involves solving a fourth-order ODE. Our approach via the function $z(t)$ is slightly different as it uses only the Einstein equation and is therefore second-order. As with the Lü–Page–Pope metrics we write $g_{\mathrm{P}} = e^{2\sigma} g_{\mathrm{K}}$. As the Page metric must have J-invariant Ricci tensor where J is the complex structure, the gradient $\nabla_{\mathrm{K}} e^{-\sigma}$ must be a holomorphic vector field. As the function σ is also $U(2)$-invariant we find

$$\sigma(t) = -\log(bt + c),$$

where b, c are constants that satisfy $bt + c > 0$ for $t \in [-a, 1]$. We can always perform a homothetic rescaling to fix $\mathrm{Ric}(g_{\mathrm{P}}) = g_{\mathrm{P}}$ and, as in the Lü–Page–Pope case, rewrite the equation

$$\mathrm{Ric}_{11}(g_{\mathrm{P}}) - \mathrm{Ric}_{22}(g_{\mathrm{P}}) = (g_{\mathrm{P}})_{11} - (g_{\mathrm{P}})_{22}$$

as a first-order ODE for the function $z(t)$:

$$\frac{dz}{dt} + \left(\frac{2}{2+t} - \frac{3b}{bt+c} - \frac{b}{bt+4b-c}\right)z + \left(\frac{2(bt+c)^2 - (2+t)}{(2+t)(bt+c)(bt+4b-c)}\right) = 0.$$

This has a solution $z(t) = \frac{\mathcal{A}(t)}{\mathcal{B}(t)}$, with

$$\begin{aligned}\mathcal{A}(t) &= 6Kb^7t^4 + 24Kb^7t^3 + 12Kb^6ct^3 + 72Kcb^6t^2 + 72Kb^5c^2t - 12Kb^4c^3t \\ &\quad + 24Kb^4c^3 - 6Kb^3c^4 + 6b^3t^2 + 12b^2ct + 6bc^2 - 2bt - 2b - c, \\ \mathcal{B}(t) &= (6b^3(t+2)^2),\end{aligned}$$

and K a constant. In order to determine the constants we consider the boundary behavior of $z(t)$. Using the form of the solution and the boundary conditions, we let

$$z(t) = \frac{(t-1)(t+a)(At^2+Bt+C)}{(t+2)^2}.$$

Then using the conditions on the derivative, we have

$$A+B+C = \frac{-3}{(1+a)} \quad \text{and} \quad a^2A - aB + C = \frac{-(2-a)}{a+1}.$$

Using the form of $z(t)$, we obtain

$$A = Kb^4, B + (a-1)A = 4Kb^4 + 2Kb^3c$$

and

$$-aA + (a-1)B + C = 12Kb^3c + 1.$$

This yields

$$(30-7a)A + (a-7)B + C = 1.$$

Hence

$$\begin{pmatrix} 1 & 1 & 1 \\ (30-7a) & (a-7) & 1 \\ a^2 & -a & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} \frac{-3}{1+a} \\ 1 \\ \frac{-(2-a)}{(1+a)} \end{pmatrix}.$$

Solving this system yields

$$A = \frac{2(a-2)}{(1+a)(a^2-16a+37)}, \quad B = \frac{a^2+10a-33}{(1+a)(a^2-16a+37)}$$

and

$$C = \frac{-2(2a^2-18a+37)}{(1+a)(a^2-16a+37)}.$$

From this we can deduce

$$\frac{c}{b} = \frac{3a^2-4a-13}{4(a-2)}.$$

On the other hand, the conditions (7) yield

$$\frac{3-2(b+c)^2}{3(b+c)(5b-c)} = -\frac{1}{3} \quad \text{and} \quad \frac{(2-a)-2(c-ab)^2}{(2-a)(c-ab)((4-a)b-c)} = \frac{1}{(2-a)}.$$

Rearranging we see that

$$c^2 = b^2 + 1 \quad \text{and} \quad c^2 = a(4-3a)b^2 + 4(a-1)bc + (2-a).$$

Hence

$$(a-1)((1-3a)b^2 + 4bc - 1) = 0.$$

If $a = 1$ then $c = 7b/2$ and

$$A = -\frac{1}{22}, \quad B = -\frac{1}{2} \quad \text{and} \quad C = -\frac{21}{22}.$$

By examining the form of the solution $z(t) = \mathcal{A}(t)/\mathcal{B}(t)$ one can see that this choice leads to inconsistency. If $a \neq 1$ we have $c = (2 \pm \sqrt{3(2-a)})b$ and we deduce

$$3a^4 - 8a^3 - 42a^2 + 168a - 125 = 0$$

and so $a \approx 1.057769$. We note with this value of a, that the ratio of the volume of the exceptional divisor and the projective line that does not intersect it is

$$\frac{3}{2-a} \approx 3.183933$$

which agrees with values calculated by other methods in [4] and [23].

3.2.3. *The Koiso–Cao Kähler–Ricci soliton.* We now look for a $U(2)$-invariant solution to the equation

$$\mathrm{Ric}(g) + \nabla^2\phi = \lambda g.$$

If we assume the metric is Kähler, this forces the function ϕ to have holomorphic gradient and so we can assume $\phi = c(x_1 + x_2)$ for some constant c. We will also factor out homothety by setting $\lambda = 1$. Hence we obtain the following equation for $z(t)$:

$$\frac{dz}{dt} + \frac{2 - c(2+t)}{(2+t)} z + \frac{t}{2+t} = 0.$$

This yields

$$z(t) = \frac{de^{c(t+2)}}{(t+2)^2} + \frac{c^2 t(t+2) + 2c(t+1) + 2}{c^3(t+2)^2},$$

where d is a constant. Using the boundary conditions $z(1) = z(-1) = 0$, we get the equations

$$de^c + \frac{2-c^2}{c^3} = 0$$

and

$$\frac{de^{3c}}{9} + \frac{3c^2 + 4c + 2}{9c^3} = 0.$$

This means that c solves

$$e^{2c}(c^2 - 2) + 3c^2 + 4c + 2 = 0,$$

which yields $c \approx 0.5276$ and $d \approx -6.91561$. This agrees with the value found by other methods in [15].

As with the Lü–Page–Pope metric, one could also recover the relevant equations by considering a first integral due to Hamilton [19] and Ivey [20]; namely

$$\Delta_\phi \phi + 2\phi = 0,$$

where Δ_ϕ is the Bakry–Émery Laplacian

$$\Delta_\phi(\cdot) := \Delta(\cdot) - g(\nabla\phi, \nabla\cdot),$$

and ϕ is normalized so that $\int_M \phi e^{-\phi} dV_g = 0$. This yields

$$\left(-2(2+t)\frac{F'}{F^2} + \frac{4}{F}\right)c - \frac{2(2+t)}{F}c^2 + 2ct = 0.$$

Hence

$$\frac{dz}{dt} + \left(\frac{2}{2+t} - c\right)z + \frac{t}{2+t} = 0.$$

4. Metrics on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$

We now perform a very similar analysis on the toric surface $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$. In this case the moment polytope is the pentagon P given by $l_i(x) > 0$ for the linear functions:

$$\begin{aligned} &l_1(x) = 1 + x_1, \quad l_2(x) = 1 + x_2, \quad l_3(x) = a - 1 - x_1, \\ &l_4(x) = a - 1 - x_2, \quad l_5(x) = a - 1 - x_1 - x_2. \end{aligned}$$

Here we assume that the metric is also symmetric under the $\mathbb{Z}_2$ action that swaps x_1 and x_2. This is sensible as both the Wang–Zhu Kähler–Ricci soliton and the Chen–LeBrun–Weber metric have this symmetry. A $\mathbb{Z}_2$-invariant toric Kähler metric g on this manifold can be written in symplectic coordinates given by equation (5) with D^2u equal to

$$\begin{pmatrix} \frac{(a^2-ax_2-x_1^2-2x_1-1)}{2(x_1+1)(a-1-x_1)(a-1-x_1-x_2)} + f_{11} & \frac{1}{2(a-1-x_1-x_2)} + f_{12} \\ \frac{1}{2(a-1-x_1-x_2)} + f_{12} & \frac{(a^2-ax_1-x_2^2-2x_2-1)}{2(x_2+1)(a-1-x_2)(a-1-x_1-x_2)} + f_{22} \end{pmatrix},$$

where $f : P \to \mathbb{R}$ is a smooth function with $f(x_1, x_2) = f(x_2, x_1)$. One can show that the determinant of g is given by

$$\det(g) = \frac{\mathcal{D}}{4(x_1+1)(x_2+1)(a-1-x_1)(a-1-x_2)(a-1-x_1-x_2)},$$

where

$$\begin{aligned} \mathcal{D} = {} & a(a^2 + a - (x_1^2 + x_2^2) - 2(x_1 + x_2) - 2) \\ & + 2(x_1+1)(a-1-x_1)P_2 f_{11} \\ & + 2(x_2+1)(a-1-x_2)P_1 f_{22} \\ & - 4(x_1+1)(x_2+1)(a-1-x_1)(a-1-x_2) f_{12} \\ & + 4(x_1+1)(x_2+1)(a-1-x_1)(a-1-x_2)(a-1-x_1-x_2) \\ & \quad \cdot (f_{11}f_{22} - f_{12}^2), \end{aligned}$$

where

$$P_1 = (a^2 - ax_2 - x_1^2 - 2x_1 - 1), \quad P_2 = (a^2 - ax_1 - x_2^2 - 2x_2 - 1).$$

The inverse is thus given by

$$(D^2u)^{-1} = \begin{pmatrix} u^{11} & u^{12} \\ u^{21} & u^{22} \end{pmatrix} = \begin{pmatrix} \frac{A_{11}}{\mathcal{D}} & \frac{A_{12}}{\mathcal{D}} \\ \frac{A_{12}}{\mathcal{D}} & \frac{A_{22}}{\mathcal{D}} \end{pmatrix},$$

where

$$\begin{aligned} A_{11} = 2(x_1+1)(a-1-x_1)\big(& (a^2 - ax_1 - x_2^2 - 2x_2 - 1) \\ & + 2(x_2+1)(a-1-x_2)(a-1-x_1-x_2)f_{22}\big), \end{aligned}$$

$$\begin{aligned}A_{12} &= -2(x_1+1)(x_2+1)(a-1-x_1)(a-1-x_2)\\ &\quad\cdot\big(1+2(a-1-x_1-x_2)f_{12}\big),\\ A_{22} &= 2(x_2+1)(a-1-x_2)\big((a^2-ax_2-x_1^2-2x_1-1)\\ &\quad+2(x_1+1)(a-1-x_1)(a-1-x_1-x_2)f_{11}\big).\end{aligned}$$

Straightforward calculation yields the following lemma.

Lemma 4.1. *Let g be a toric Kähler metric on $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$ of the form given by equation* (5)*. Then the inverse u^{ij} satisfies*

$$u^{ij}(-1,-1)=u^{ij}(-1,a-1)=u^{ij}(0,a-1)=0$$

for $i,j\in\{1,2\}$. The derivatives satisfy

$$\begin{aligned}&u_1^{11}(-1-1)=2, && u_1^{12}(-1,-1)=u_2^{12}(-1,-1)=0,\\ &u_2^{22}(-1,-1)=2, && u_1^{11}(-1,a-1)=2,\\ &u_1^{12}(-1,a-1)=u_2^{12}(-1,a-1)=0, && u_2^{22}(-1,a-1)=-2,\\ &u_1^{11}(0,a-1)=-2, && u_1^{12}(0,a-1)=0,\\ &u_2^{12}(0,a-1)=2, && u_2^{22}(0,a-1)=-2.\end{aligned}$$

The quasi-Einstein metrics on $\mathbb{CP}^2\sharp\overline{\mathbb{CP}}^2$ are Hermitian and have J-invariant Ricci tensor. If we search for metrics on $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$ that are invariant under the $\mathbb{T}^2\times\mathbb{Z}_2$ action described above and which are also J-invariant, we have the following result.

Proposition 4.2. *Let (g,ϕ) be a Hermitian quasi-Einstein metric on the toric surface $\mathbb{CP}^2\sharp 2\overline{\mathbb{CP}}^2$. Suppose g is invariant under the action of $\mathbb{T}^2\times\mathbb{Z}_2$, has J-invariant Ricci tensor and $g=e^{2\sigma}g_{\mathrm{K}}$ for a Kähler metric g_{K}. Then*

$$\sigma=-\log(bt+c)\quad\textit{and}\quad \phi=-m\log\left(\frac{dbt+dc+1}{bt+c}\right).$$

Proof. The form of σ follows from the fact that the Ricci tensor of g is J-invariant. This forces the gradient with respect to g_{K}, $\nabla e^{-\sigma}$, to be a holomorphic vector field. Hence $e^{-\sigma}$ is an affine function in the polytope coordinates, invariant under the $\mathbb{Z}_2$-action. With respect to the metric g, $\nabla^2\phi-\frac{1}{m}d\phi\otimes d\phi$ is J-invariant. Calculation yields

$$\phi_{ij}-\left(\sigma'(\phi_i+\phi_j)+\frac{1}{m}\phi_i\phi_j\right)=0,$$

for $1\le i,j\le 2$. This equation can then be solved explicitly, yielding the result. □

The Kim–Kim first integral (15) and boundary behavior of the metric given in Lemma 4.1 give some constraints on the quantities b,c,d and μ.

Proposition 4.3. *Suppose* (g, ϕ) *is a quasi-Einstein metric on* $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$ *of the form given in Proposition 4.2. Then*

$$\frac{4b}{(c-2b)(dc+1-2db)} = \frac{1}{(c-2b)^2} - \frac{\mu}{(dc+1-2db)^2},$$
$$0 = \frac{1}{(c+(a-2)b)^2} - \frac{\mu}{(dc+1+(a-2)db)^2},$$

and

$$\frac{-2b}{(c+(a-1)b)(dc+1+(a-1)db)} = \frac{1}{(c+(a-1)b)^2} - \frac{\mu}{(dc+1+(a-1)db)^2}.$$

Moreover, we have

$$\int_{\mathrm{P}} (e^{-\phi} - \mu e^{(\frac{2}{m}-1)\phi}) e^{4\sigma} dx = 0.$$

Proof. All the equations above can be derived by examining the Kim–Kim first integral (15). Calculating quantities with respect to the Kähler metric g_{K}, this equation becomes

$$m(e^{2\sigma} - \mu e^{\frac{2\phi}{m}+2\sigma}) = \Delta\phi + 2g_{\mathrm{K}}(\nabla\sigma, \nabla\phi) - |\nabla\phi|^2. \tag{16}$$

The first three equations now follow from Lemma 4.1. For the integral constraint we note that the right-hand side of (16) can be written as $\Delta_{\mathcal{F}}\phi$ where

$$\Delta_{\mathcal{F}}(\cdot) := \Delta(\cdot) - g_{\mathrm{K}}(\nabla\mathcal{F}, \nabla\cdot) \quad \text{and} \quad \mathcal{F} = \phi - 2\sigma.$$

The result follows from noting that, for any $\mathcal{F}$,

$$\int_M \Delta_{\mathcal{F}}(\cdot) e^{-\mathcal{F}} dV_g = 0. \qquad \square$$

If one fixes the value of a (and so the particular Kähler class) and the value of m, then Proposition 4.3 yields four equations in the four unknowns b, c, d and μ. Given the result of Theorem 2.7 it is sensible to look for conformally Kähler quasi-Einstein metrics on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$ in the first Chern class c_1 which corresponds to setting $a = 2$. Using a numerical program to evaluate the integral, we find in the case $a = 2$ and $m = 2$ that the system of equations in Proposition 4.3 has the unique admissible solution (values are given to six significant figures)

$$b \approx -0.0744357, \quad c \approx 1.00482, \quad d \approx -0.463585 \quad \text{and} \quad \mu \approx 0.282617.$$

Another use of the proposition is to rule out certain limiting behaviors of hypothetical families of quasi-Einstein metrics. Suppose that a family of conformally Kähler quasi-Einstein metrics of the form given in Proposition 4.2 converges smoothly to a conformally Kähler gradient Ricci soliton. Then, as well as the Wang–Zhu soliton, one could in theory also converge to the Chen–LeBrun–Weber metric. By the calculations in [18] this would mean, as $m \to \infty$,

$$b \to -0.217907 \quad \text{and} \quad c \to 1.000632.$$

As ϕ converges to a constant, the integral constraint of Proposition 4.3 would mean $\mu \to 1$. One can then check that these values are not admissible as solutions of the constraints of Proposition 4.3 and so conclude that the Chen–LeBrun–Weber metric is not the smooth limit of such a hypothetical family.

5. Future work

In this section we list and comment on some future directions for research that our current work raises.

(1) Are there conformally Kähler analogues of the Lü–Page–Pope metric on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$? The second and fourth authors investigated this question numerically using an algorithm they developed for numerically approximating toric Kähler metrics on $\mathbb{CP}^2 \sharp 2\overline{\mathbb{CP}}^2$ in [17, 18].

(2) What is the significance of the conformally Kähler quasi-Einstein metrics? For example, in dimension 4, the Kähler metrics conformal to the Page and the Chen–LeBrun–Weber metrics are extremal Kähler metrics. An extremal Kähler metric is a critical point of the Calabi energy and such metrics are the subject of intense research activity at the time of writing. The Kähler classes of the extremal metrics conformal to the Page and Chen–LeBrun–Weber Einstein metrics are distinguished by minimizing the Calabi energy over all possible Kähler classes on these manifolds [24].

(3) The existence theorem for compact quasi-Einstein metrics in [16] can be paraphrased as: "whenever the manifold W_q admits a nontrivial Kähler–Ricci soliton, it admits at least one family of quasi-Einstein metrics". Is this true in general? Or is it at least true for other constructions of Kähler–Ricci solitons, such as that of Wang–Zhu [29]?

Acknowledgements. The majority of this work was undertaken whilst WB and TM paid research visits to SH in December 2014 and January 2015. The visit of WB was supported by a Scheme 5 grant from the London Mathematical Society. The visit of TM was funded by a Dennison research grant from the University of Buckingham and a grant from California State University Fullerton. The authors wish to warmly thank Gideon Maschler for useful comments on an early draft of this article and the anonymous referee for their comments.

References

[1] M. Abreu, Kähler geometry of toric varieties and extremal metrics, Internat. J. Math. **9** (1998), no. 6, 641–651. MR1644291

[2] M. Abreu, Kähler geometry of toric manifolds in symplectic coordinates, in *Symplectic and contact topology: Interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*, 1–24, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, 2003. MR1969265

[3] L. Bérard-Bergery, Sur de nouvelles variétés riemanniennes d'Einstein, in *Institut Élie Cartan, 6*, 1–60, Inst. Élie Cartan, 6, Univ. Nancy, Nancy, 1983. MR0727843

[4] A. L. Besse, *Einstein manifolds*, reprint of the 1987 edition, Classics in Mathematics, Springer, Berlin, 2008. MR2371700

[5] E. Calabi, Extremal Kähler metrics, in *Seminar on Differential Geometry*, 259–290, Ann. of Math. Stud., 102, Princeton Univ. Press, Princeton, NJ, 1982. MR0645743

[6] H.-D. Cao, Existence of gradient Kähler–Ricci solitons, in *Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994)*, 1–16, A K Peters, Wellesley, MA, 1996. MR1417944

[7] J. Case, Y.-J. Shu and G. Wei, Rigidity of quasi-Einstein metrics, Differential Geom. Appl. **29** (2011), no. 1, 93–100. MR2784291

[8] T. Chave and G. Valent, On a class of compact and non-compact quasi-Einstein metrics and their renormalizability properties, Nuclear Phys. B **478** (1996), no. 3, 758–778. MR1419065

[9] X. Chen, C. Lebrun and B. Weber, On conformally Kähler, Einstein manifolds, J. Amer. Math. Soc. **21** (2008), no. 4, 1137–1168. MR2425183

[10] B. Dammerman, *Metrics of special curvature with symmetry*, D. Phil. thesis, University of Oxford (2006), arXiv:math/0610738v1

[11] A. S. Dancer and M. Y. Wang, On Ricci solitons of cohomogeneity one, Ann. Global Anal. Geom. **39** (2011), no. 3, 259–292. MR2769300

[12] A. Derdziński, Self-dual Kähler manifolds and Einstein manifolds of dimension four, Compositio Math. **49** (1983), no. 3, 405–433. MR0707181

[13] S. K. Donaldson, Kähler geometry on toric manifolds, and some other manifolds with large symmetry, in *Handbook of geometric analysis. No. 1*, 29–75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008. MR2483362

[14] V. Guillemin, Kaehler structures on toric varieties, J. Differential Geom. **40** (1994), no. 2, 285–309. MR1293656

[15] S. J. Hall, Computing Perelman's ν-functional, Differential Geom. Appl. **29** (2011), no. 3, 426–432. MR2795848

[16] S. J. Hall, Quasi-Einstein metrics on hypersurface families, J. Geom. Phys. **64** (2013), 83–90. MR3004017

[17] S. J. Hall and T. Murphy, Approximating Ricci solitons and quasi-Einstein metrics on toric surfaces, New York J. Math. **22** (2016), 615–635.

[18] S. J. Hall and T. Murphy, Numerical approximations to extremal toric Kähler metrics with arbitrary Kähler class, to appear in Proc. Edinb. Math. Soc. (2).

[19] R. S. Hamilton, The formation of singularities in the Ricci flow, in *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, 7–136, Int. Press, Cambridge, MA, 1995. MR1375255

[20] T. Ivey, Ricci solitons on compact three-manifolds, Differential Geom. Appl. **3** (1993), no. 4, 301–307. MR1249376

[21] D.-S. Kim and Y. H. Kim, Compact Einstein warped product spaces with nonpositive scalar curvature, Proc. Amer. Math. Soc. **131** (2003), no. 8, 2573–2576 (electronic). MR1974657

[22] N. Koiso, On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics, in *Recent topics in differential and analytic geometry*, 327–337, Adv. Stud. Pure Math., 18-I, Academic Press, Boston, MA, 1990. MR1145263

[23] C. LeBrun, Einstein metrics on complex surfaces, in *Geometry and physics (Aarhus, 1995)*, 167–176, Lecture Notes in Pure and Appl. Math., 184, Dekker, New York, 1997. MR1423163

[24] C. LeBrun, On Einstein, Hermitian 4-manifolds, J. Differential Geom. **90** (2012), no. 2, 277–302. MR2899877

[25] H. Lü, D. N. Page and C. N. Pope, New inhomogeneous Einstein metrics on sphere bundles over Einstein–Kähler manifolds, Phys. Lett. B **593** (2004), no. 1-4, 218–226. MR2076718

[26] G. Maschler, Conformally Kähler base metrics for Einstein warped products, Differential Geom. Appl. **29** (2011), no. 1, 85–92. MR2784290

[27] D. Page, A compact rotating gravitational instanton, Phys. Lett. B **79** (1978), 235-238.

[28] J. Wang and M. Y. Wang, Einstein metrics on S^2-bundles, Math. Ann. **310** (1998), no. 3, 497–526. MR1612258

[29] X.-J. Wang and X. Zhu, Kähler–Ricci solitons on toric manifolds with positive first Chern class, Adv. Math. **188** (2004), no. 1, 87–103. MR2084775

Received March 2, 2015; accepted May 7, 2015

Wafaa Batat
Ecole Nationale Polytechnique d'Oran,
B.P 1523 El M'naouar, 31000 Oran, Algeria
E-mail: batatwafa@yahoo.fr

Stuart J. Hall
School of Computing, Mathematics and Digital Technology,
John Dalton Building, Manchester Metropolitan University
Chester Street, Manchester M1 5GD, UK
E-mail: stuart.hall@mmu.ac.uk

Ali Jizany
Department of Applied Computing, University of Buckingham,
Hunter St., Buckingham MK18 1G, UK
E-mail: ali.jizany@buckingham.ac.uk

Thomas Murphy
Department of Mathematics, California State University Fullerton,
800 N. State College Blvd., Fullerton, CA 92831, USA
E-mail: tmurphy@fullerton.edu