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Markov Random Walks Driven by General Markov Chains and Their Applications to Semi-Markov Queues

-2007-

Mathematik

Markov Random Walks Driven by General Markov Chains and Their Applications to Semi-Markov Queues

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von

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-2007-

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Zweiter Gutachter:	Prof. Dr. M. Löwe
Tag der mündlichen Prüfung:	21.11.2007
Tag der Promotion:	21.11.2007

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Introduction

Queueing theory is one of the important domains in applied probability. The basic idea has been borrowed from every-day experience of queues, for example, at the check-out counters in a supermarket, but a number of stochastic models may be formulated in queueing terms or are closely related. The great diversity of queueing problems gives rise to an enormous variety of queueing models.

The simplest and the most basic model in queueing theory is the single server queue, where customers arrive at one service station, are served one at a time on the first come first server basis and leave the system when service is completed. If arrival times form a renewal process and service times are distributed identically and independently, and if arrival times and service times are independent, then the queue is denoted by $GI/GI/1$, which is an old theme in the queueing theory. A specific feature of stable $GI/GI/1$ queues is the regeneration of the system, which means that the system reaches an empty state infinitely often and restarts from scratch at the empty state. The regeneration of a stable $GI/GI/1$ queue can be described in the framework of the theory of random walks. Let T_n be the interarrival time between customers $n-1$ and n, and U_n the service time of customer n. Denote by $(S_n)_{n\geq 0}$ the associated random walk given as

$$S_n := \sum_{k=0}^{n} X_k, \quad n \geq 0,$$

where $X_k = U_{k-1} - T_k, k \geq 1$ and $X_0 = 0$. Then the waiting time process forms the reflected random walk $(W_n)_{n\geq 0}$, i.e.,

$$W_0 = 0 \quad \text{and} \quad W_{n+1} = (W_n + X_{n+1})^+, \quad n \geq 0.$$

Moreover, the weak descending ladder epochs are regeneration epochs of the waiting time process.

$GI/GI/1$ queues have been extensively studied, because of their tractability. Yet the i.i.d. condition, on which a $GI/GI/1$ queue is based, is somewhat unnatural. In fact, almost everything in the world is occurred in a mutual interaction or under influence by some other things. Many efforts are made to generalize $GI/GI/1$ queues. A generalization can be obtained replacing the i.i.d. assumption by conditional independence given a temporally homogenous Markov chain M.

A semi-Markove queue denoted by $SM/SM/1$ is a generalization of $GI/GI/1$ queue, in which the sequence $(M_n, S_n)_{n\geq 0}$ forms a Markov random walk (MRW). A

MRW is a generalization of a random walk, in which the additive part is distributionally governed by a temporally homogenous Markov chain (see Chapter 2 for the precise definition). Markov modulation offers more flexibility in the modeling of the real world, but in general it is not easy to explicitly compute queueing quantities like stationary distributions of various queueing processes. In the case of finite modulation, some special types of queues known as $M/GI/1$ type and $GI/M/1$ type are extensively studied by various authors like Neuts, Ramaswami, etc. They have developed matrix-analytic methods for the computation of queueing characteristics such as stationary distributions, which becomes nowadays a popular tool in the applied probabilities. Some comprehensive treatments of matrix-analytic methods can be found in Neuts [43, 44] and Latouche and Ramaswami [34]. For a brief survey see Ramaswami [51]. On the contrary, the theory of queues with general modulation chains was not well developed to the same extent.
A study on semi-Markov queues with general modulation chains can be found in Nummelin [45]. He showed that, if the modulation chain M is positive Harris recurrent, then under the stability condition the waiting time process is one-dependent as well as wide-sense regenerative. Alsmeyer [4] showed that a unique stationary distribution for the waiting time process can be written as an occupation measure with respect to the weak descending ladder epoch.

This dissertation deals with MRW's driven by general Markov chains and semi-Markov queues. The first weak descending ladder epoch is one of the basic quantities in the theory of MRW's. In a semi-Markov queue it is interpreted as the index of the customers in the first busy cycle. Making use of some corollaries of Dynkin's formula (Corollary I.1.1, I.1.2 in Kalashnikov [33]), we first find moment conditions for the first weak descending ladder epochs of MRW's with negative drift. It should be pointed out that a similar method was used by Sharma [55] for the analysis of $R/R/1$ queues, in which interarrival times and service times form a classical sense regenerative process. In the same manner, we get moment conditions for regeneration epochs of reflected MRW's. These results can be directly applied to the queueing theory with the corresponding queueing interpretations. In particular, for a semi-Markov queue we find rates of convergence to the staionary distribution and conditions for the finiteness of moments of the stationary waiting time and workload processes.

This dissertation is organized as follows:

Chapter 1 contains basic definitions and some preliminary results from the theory of general Markov chains. Harris recurrence and ergodicity are reviewed briefly.

Chapter 2 deals with the basic theory of MRW's and reflected MRW's driven by general Markov chains. First we review some basic facts on the theory of MRW's. Most of the concepts and results can be found in Arjas [6, 7] and Arjas and Speed [8]. Next we are concerned with reflected MRW's and, following Nummelin [45] and Alsmeyer [4], we get a stationary distribution of a reflected MRW with negative drift as an occupation measure with respect to the first weak descending ladder epoch. The remainder of Chapter 2 deals with MRW's with lattice-type increments. In this case we obtain the joint stationary distributions of reflected MRW's in simpler forms.

Chapter 3 and Chapter 4 are the main parts of this dissertation. In chapter 3 we find moment conditions for the first weak descending ladder epochs of MRW's and for the regeneration epochs of reflected MRW's with negative drift. Moments of the first weak descending ladder epoch are of particular interest in the theory of MRW's. Moment conditions for the first weak descending ladder epoch of an ordinary random walk are known (see Theorem I.5.1 in Gut [30]). For a Markov random walk, some results on the finiteness of moments of the first weak descending ladder epoch can be found in Fuh and Lai [28] and Alsmeyer [5]. In particular, the results of Fuh and Lai can be regarded as special cases of our results.
In Chapter 4 we are concerned with semi-Markov queues. Throughout this chapter we assume that the stability conditions are satisfied. We first consider single server queues with general modulation chains and find rates of convergence to the stationary distribution and conditions for the finiteness of moments of stationary waiting time and workload processes. For $GI/GI/1$ queues, rates of convergence are available in Kalashnikov [33] (Chapter 5.3) and conditions for the finiteness of moments of stationary waiting time and workload processes in Asmussen [13] (Theorem X.2.1). Sharma [55] obtained the same results for a $R/R/1$ queue, which can be regarded as the special case of countable modulation. Finally, we point out that for a $G/G/1$ queue, in which interarrival times and service times form a stationary process, Daley, Foley and Rolski [26] obtained some conditions for finite moments of the stationary waiting time. In the remainder of this Chapter we examine multiserver semi-Markov queues. In particular, for 2-server queues with countable modulation chains, we show that under some additional conditions the workload process is regenerative.

Acknowledgements. I am grateful to my supervisor Prof. Dr. Gerold Alsmeyer for suggesting the topic of the thesis and for his many valuable comments during the work.

Chapter 1

Introduction to the theory of general Markov chains

A Markov process is one of successful stochastic processes. Its success is due to the relative simplicity of its theory and to the fact that Markov models can exhibit extremely varied and complex behavior. In this chapter we provide an introduction to the theory of Markov chains with general state space. The theory of general Markov chains forms a basis of this thesis. Although the analysis of general Markov chains requires more elaborate techniques than in the discrete case, nowadays the general theory has been developed to a matured state. There are plenty of literature on general Markov chains. For the comprehensive treatments see Meyn and Tweedie [37], Nummelin [46] and references therein. We are mainly interested in Harris recurrence and stationary distribution of general Markov chains. After introducing some fundamental notions on kernels and Markov chains, we deal with Harris recurrence. The analysis of Markov chains with countable state space is based on the recurrence of individual states. However, if the state space is uncountable, one can not expect the existence of such states in general. The Harris recurrence is an extension of the notion of recurrence from individual states to sets. A Harris chain possesses a regenerative scheme based on the splitting technique, which is suggested by Athreya and Ney. From the existence of regeneration epochs one can construct a stationary measure, which is unique up to constant multiples.

1.1 Definitions and elementary properties

Let $(E, \mathcal{E})$ be a measurable space. A function $\mathbf{K} : E \times \mathcal{E} \to [0, \infty)$ is called a *kernel* on $(E, \mathcal{E})$, if $\mathbf{K}(s, \cdot\,)$ is a measure on $(E, \mathcal{E})$ for all $s \in E$ and if $\mathbf{K}(\,\cdot, A)$ is a $\mathcal{E}$-measurable function for all $A \in \mathcal{E}$. If $\mathbf{K}(s, E) \leq 1$ for any $s \in E$, then the kernel $\mathbf{K}$ is called a *transition kernel*. It is known that such functions are well defined on Polish spaces[1]. Any kernel $\mathbf{K}$ can be interpreted as a nonnegative linear operator on the set of nonnegative measurable functions $\mathcal{F}_+(E)$ on E by defining

$$\mathbf{K}\,f(s) := \int_E f(s')\,\mathbf{K}(s,ds') = \langle \mathbf{K}(s,\cdot\,), f\rangle, \quad s \in E$$

for any $f \in \mathcal{F}_+(E)$. In particular, we have

$$\mathbf{K}(s,A) = \mathbf{K}\mathbf{1}_A(s), \quad s \in E, A \in \mathcal{E}.$$

By defining $\mathbf{K}f = \mathbf{K}f^+ - \mathbf{K}f^-$, we may extend this to every measurable function on $(E,\mathcal{E})$ such that $\mathbf{K}f^+$ and $\mathbf{K}f^-$ are not both infinite. Similarly $\mathbf{K}$ acts on the class of positive measures $\mathcal{M}_+(E)$ on E by

$$\lambda\mathbf{K}(\cdot) = \int_E \mathbf{K}(s,\cdot\,)\,\lambda(ds)$$

for any $\lambda \in \mathcal{M}_+(E)$. For any fixed $A \in \mathcal{E}$, one defines a kernel $\mathbf{I}_A$ by

$$\mathbf{I}_A(s,A') := \mathbf{1}_{A\cap A'}(s), \quad s \in E, A' \in \mathcal{E}.$$

If $\mathbf{K}_1$ and $\mathbf{K}_2$ are two kernels, their *composition* $\mathbf{K}_1\mathbf{K}_2$ is defined as

$$\mathbf{K}_1\mathbf{K}_2(s,A) = \int_E \mathbf{K}_2(s',A)\,\mathbf{K}_1(s,ds'), \quad s \in E, A \in \mathcal{E}.$$

The n-step iterates $\mathbf{K}^n$, $n \geq 0$, of a (transition) kernel $\mathbf{K}$ are defined iteratively,

$$\mathbf{K}^0 = \mathbf{I}_E \quad \text{and} \quad \mathbf{K}^n = \mathbf{K}\,\mathbf{K}^{n-1}, \quad n \geq 1.$$

Two kernels $\mathbf{K}$ and $\tilde{\mathbf{K}}$ on $(E,\mathcal{E})$ are said to be *adjoint* with respect to a positive σ-finite measure ν on E, if for any $f, g \in \mathcal{F}_+(E)$

$$\int_E (\mathbf{K}f)\,g\,d\nu = \int_E f\,(\tilde{\mathbf{K}}g)\,d\nu,$$

which will also be written as

$$\langle \mathbf{K}f, g\rangle_\nu = \langle f, \tilde{\mathbf{K}}g\rangle_\nu.$$

Assume that a measure space $(\Omega, \mathcal{S}, \mathbb{P})$ is given, which is called the *sample space*. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and denote by $\mathcal{F}_\infty = \sigma(\cup_{n=0}^\infty \mathcal{F}_n)$ the smallest σ-algebra

[1] A Polish space is a complete, separable metric space. Any locally compact space with a countable dense subset, any countable product of Polish spaces and function spaces with values in Polish space are examples of Polish spaces.
There are examples in probability theory, where non-Polish state spaces are required, but in general a state space is assumed to be Polish. There is a powerful and complete theory for probability on Polish spaces. For details see Appendix A1 in Asmussen [12] and references therein.

generated by $U_{n=0}^{\infty}\mathcal{F}_n$. A sequence $M = (M_n)_{n\geq 0}$ of $(E, \mathcal{E})$-valued random variables on $(\Omega, \mathcal{S}, \mathbb{P})$ is said to be $(\mathcal{F}_n)_{n\geq 0}$-*adapted*, if M_n is $\mathcal{F}_n$-measurable for any $n \geq 0$. Letting

$$\mathcal{F}_n^M = \sigma(M_k : k \leq n), \quad n \geq 0,$$

M is $(\mathcal{F}_n^M)_{n\geq 0}$ - adapted. The filtration $(\mathcal{F}_n^M)_{n\geq 0}$ is called the *canonical filtration of* M. An $(\mathcal{F}_n)_{n\geq 0}$-adapted chain $(M_n)_{n\geq 0}$ is called a *Markov chain* with respect to $(\mathcal{F}_n)_{n\geq 0}$, if for any $n \geq 0$

$$\mathbb{P}\left[M_{n+1} \in \cdot\, | \mathcal{F}_n\right] = \mathbb{P}\left[M_{n+1} \in \cdot\, | M_n\right] \quad \mathbb{P}\text{-a.s.}.$$

If, in addition, for a transition kernel $\mathbf{P} : E \times \mathcal{E} \to [0, 1]$

$$\mathbb{P}\left[M_{n+1} \in \cdot\, | \mathcal{F}_n\right] = \mathbf{P}(M_n, \cdot\,) \quad \mathbb{P}\text{-a.s.},$$

then M is called a *temporally homogeneous Markov chain with transition kernel* $\mathbf{P}$. The space $(E, \mathcal{E})$ or E is called the *state space* and the points of E are called *states*. Throughout this dissertation a state space E is assumed to be Polish, unless stated otherwise.

The distribution λ defined by

$$\lambda(\cdot) = \mathbb{P}(M_0 \in \cdot\,)$$

is called an *initial distribution*. For any initial distribution λ on $(E, \mathcal{E})$, we define a distribution $\mathbb{P}_\lambda$ by the requirements

$$\mathbb{P}_\lambda\left(M_0 \in \cdot\,\right) = \lambda(\cdot) \quad \text{and} \quad \mathbb{P}_\lambda\left[M_{n+1} \in \cdot\, | \mathcal{F}_n\right] = \mathbf{P}(M_n, \cdot\,), \quad n \geq 0.$$

Obviously

$$\mathbb{P}_\lambda\left(M_0 \in A_0, \cdots, M_n \in A_n\right) = \int_{A_0}\int_{A_1}\cdots\int_{A_n} \mathbf{P}(s_{n-1}, ds_n)\cdots\mathbf{P}(s_0, ds_1)\,\lambda(ds_0)$$

for any $n \in \mathbb{N}_0$ and $A_0, \cdots, A_n \in \mathcal{E}$. If M starts at a point $s \in E$, then we write $\mathbb{P}_s$ instead of $\mathbb{P}_{\delta_s}$. A σ-finite measure $\xi \neq 0$ is called a *stationary measure* or an *invariant measure* for $(M_n)_{n\geq 0}$ or $\mathbf{P}$, if for any $n \geq 1$

$$\xi\,\mathbf{P}^n(A) = \int_E \mathbf{P}^n(s, A)\,\xi(ds) = \xi(A), \quad A \in \mathcal{E}.$$

If ξ is a probability measure satisfying the above equality, then it is called a *stationary distribution* or an *invariant distribution*. If ξ is a stationary distribution, then by the Markov property we have

$$\mathbb{P}_\xi\left((M_n)_{n\geq m} \in \cdot\,\right) = \mathbb{P}_\xi\left((M_n)_{n\geq 0} \in \cdot\,\right)$$

for any $m \geq 0$. Two Markov chains M and $\tilde{M}$ are said to be *in duality* relative to ν, if their transition kernels are adjoint with respect to ν. One of the two chains is said to be the *dual* or *time-reversed chain* of the other one. If each of the two transition kernels $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{P}}'$ is adjoint to a transition kernel $\mathbf{P}$ with respect to ν, then there is an ν-null set N such that $\tilde{\mathbf{P}}(s,\cdot) = \tilde{\mathbf{P}}'(s,\cdot)$ for all $s \in N^c$. If E is countable, then the empty set is the only set of ν-measure zero and thus the duality condition is equivalent to the requirement

$$\tilde{\mathbf{P}} = \Delta_\nu^{-1}\mathbf{P}^T\Delta_\nu,$$

where $\mathbf{P}$ and $\tilde{\mathbf{P}}$ are the transition matrices of M and $\tilde{M}$, respectively, and Δ_ν is the diagonal matrix of ν.

A $\mathbb{N}_0 \cup \{\infty\}$-valued random variable τ is called a *stopping time* w.r.t. the filtration $(\mathcal{F}_n)_{n\geq 0}$, if

$$\{\tau = n\} \in \mathcal{F}_n \quad \text{for all} \quad n \geq 0.$$

If τ is a stopping time w.r.t. the canonical filtration $(\mathcal{F}_n^M)_{n\geq 0}$ of a Markov chain M, then it is called a *stopping time for the Markov chain* $(M_n)_{n\geq 0}$. Important examples of stopping times for the Markov chain $(M_n)_{n\geq 0}$ are the *first hitting time* $\kappa(A)$ and the *first return time* $\tau(A)$ to a set $A \in \mathcal{E}$ defined as

$$\kappa(A) := \inf\{n \geq 0 : M_n \in A\} \quad \text{and} \quad \tau(A) := \inf\{n \geq 1 : M_n \in A\}.$$

A random time τ is called a *randomized stopping time* for the Markov chain $(M_n)_{n\geq 0}$, if for every $n \geq 0$ the event $\{\tau = n\}$ and the post n-chain $(M_{n+1}, M_{n+2}, \cdots)$ are conditionally independent given the pre-n-chain $(M_0, \cdots, M_n)$, or equivalently,

$$\mathbb{P}\left[\tau = n | \mathcal{F}_\infty^M\right] = \mathbb{P}\left[\tau = n | \mathcal{F}_n^M\right] \quad \mathbb{P}\text{-a.s.}$$

If τ is a stopping time w.r.t. a filtration $(\mathcal{F}_n)_{n\geq 0}$ and if a Markov chain M is $(\mathcal{F}_n)_{n\geq 0}$-adapted, then τ is a randomized stopping time for M. Conversely, if τ is a randomized stopping time for $(M_n)_{n\geq 0}$, then $(M_n)_{n\geq 0}$ is adapted and τ is a stopping time w.r.t. the filtration $(\mathcal{F}_n^\tau)_{n\geq 0}$ defined as

$$\mathcal{F}_n^\tau := \sigma((M_k)_{k\leq n}, \{\tau = k : k \leq n\}), \quad n \geq 0.$$

By Pitman and Speed [47], the following are equivalent:

(i) τ is a randomized stopping time w.r.t. $(\mathcal{F}_n)_{n\geq 0}$;

(ii) for each $n \in \mathbb{N}_0$, $\mathcal{F}_n^\tau$ and $\mathcal{F}_\infty$ are conditionally independent given $\mathcal{F}_n$;

(iii) for each $n \in \mathbb{N}_0$, $\mathbb{E}[X|\mathcal{F}_n^\tau] = \mathbb{E}[X|\mathcal{F}_n]$ a.s. for each integrable $\mathcal{F}_\infty$-measurable random variable X.

Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$-adapted Markov chain and τ a randomized stopping time w.r.t. $(\mathcal{F}_n)_{n\geq 0}$. It is known that $(M_n)_{n\geq 0}$ possesses the *strong Markov property* w.r.t. a randomized stopping time τ, i.e., on $\{\tau < \infty\}$

$$\mathbb{P}\left[M_{\tau+n} \in \cdot\, | M_\tau\right] = \mathbf{P}^n(M_\tau, \cdot) \quad \mathbb{P}\text{-a.s.}$$

for any $n \geq 0$. We denote by $\mathcal{F}_\tau$ the σ-algebra of events which are observed up to time τ, i.e.,

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap \{\tau = n\} \in \mathcal{F}_n \text{ for all } n \geq 0\}.$$

1.2 Recurrence

A set $\Re \in \mathcal{E}$ is called a *recurrent set*, if for any $s \in E$

$$\mathbb{P}_s(M_n \in \Re \text{ i.o.}) = \mathbb{P}_s(\tau(\Re) < \infty) = 1.$$

Let $(\tau_n)_{n\geq 0}$ be the sequence of stopping times defined as

$$\tau_0 = \tau(\Re) \quad \text{and} \quad \tau_{n+1} = \inf\{k > \tau_n : M_k \in \Re\}, \quad n \geq 0.$$

If $\Re$ is a recurrent set, then by the strong Markov property the sequence $(M_n^\tau)_{n\geq 0} := (M_{\tau_n})_{n\geq 0}$ forms a temporally homogeneous Markov chain. The following assertions are well known (see Theorem I.2.3 in Borovkov [19] for example):

Proposition 1.1 *Suppose that a Markov chain M has a stationary distribution ξ and that there exists a recurrent set $\Re$ with $\xi(\Re) > 0$. Then the chain $(M_n^\tau)_{n\geq 0}$ has a stationary distribution $\xi^\Re$ defined as*

$$\xi^\Re(\cdot) = \frac{\xi(\,\cdot \cap \Re)}{\xi(\Re)}.$$

If, in addition, ξ is the unique stationary distribution for M, then

$$\begin{aligned} \xi(\cdot) &= \frac{1}{\mathbb{E}_{\xi^\Re}\tau(\Re)}\mathbb{E}_{\xi^\Re}\left(\sum_{n=0}^{\tau(\Re)-1} \mathbf{1}(M_n \in \cdot\,)\right) \\ &= \xi(\Re)\,\mathbb{E}_{\xi^\Re}\left(\sum_{n=0}^{\tau(\Re)-1} \mathbf{1}(M_n \in \cdot\,)\right) \\ &= \int_\Re \left(\sum_{n=0}^{\infty} \mathbb{P}_s(M_n \in \cdot\,, \tau(\Re) > n)\right)\xi(ds). \end{aligned}$$

For fixed $\Re \in \mathcal{E}, n \geq 1$, let ${}_\Re\mathbf{P}^n$ be the kernel defined as

$${}_\Re\mathbf{P}^n(s, A) = \mathbb{P}\left[M_n \in A, \tau(\Re) \geq n | M_0 = s\right], \quad s \in E, A \in \mathcal{E},$$

which is known as *n-step taboo kernel* with taboo set $\Re$. Obviously we have

$${}_\Re\mathbf{P}^n(s, \Re) = \mathbb{P}(\tau(\Re) = n), \quad n \geq 1, \quad \text{and} \quad \mathbb{E}_s\,\tau(\Re) = \sum_{n=1}^{\infty} n\,{}_\Re\mathbf{P}^n(s, \Re)$$

for any $s \in E$. Moreover, the transition kernel $\mathbf{P}_\Re$ for $(M_n^\tau)_{n\geq 0}$ is given as

$$\mathbf{P}_\Re(s, \Re \cap A) = \sum_{n=1}^{\infty} {}_\Re\mathbf{P}^n(s, \Re \cap A), \quad s \in \Re, A \in \mathcal{E}.$$

Note that for any $A \in \mathcal{E}$

$$\mathbb{E}_{\xi^{\Re}}\left(\sum_{n=0}^{\tau(\Re)-1} \mathbf{1}(M_n \in A)\right) = \mathbb{E}_{\xi^{\Re}}\left(\sum_{n=1}^{\tau(\Re)} \mathbf{1}(M_n \in A)\right) = \int_E \left(\sum_{n=1}^{\infty} {}_{\Re}\mathbf{P}^n(s,A)\right) \xi^{\Re}(ds).$$

Thus if ξ is the unique stationary distribution for M, then ξ can be also written as

$$\xi(A) = \xi(\Re) \int_E \left(\sum_{n=1}^{\infty} {}_{\Re}\mathbf{P}^n(s,A)\right) \xi^{\Re}(ds) = \int_{\Re} \left(\sum_{n=1}^{\infty} {}_{\Re}\mathbf{P}^n(s,A)\right) \xi(ds)$$

for any $A \in \mathcal{E}$.

Remark 1.2 In the situation of Proposition 1.1, the cycles

$$Z_n := \left(\tau_{n+1} - \tau_1, (M_k)_{\tau_n \le k < \tau_{n+1}}\right), \quad n \ge 0,$$

are stationary under $\mathbb{P}_{\xi^{\Re}}$. Let $(\tau'_n)_{n\ge 0}$ be a sequence of a.s. finite stopping times, such that there exists a distribution ζ with $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\tau'_n} \in \cdot\,)$ for any $n \ge 0$ and for any initial distribution λ. Then the cycles

$$Z'_n := \left(\tau'_{n+1} - \tau'_n, (M_k)_{\tau'_n \le k < \tau'_{n+1}}\right), \quad n \ge 0,$$

are also stationary under $\mathbb{P}_\zeta$. If, in addition, $\mathbb{E}_\zeta\, \tau'_1 < \infty$, then it can be shown that the measure defined as

$$\frac{1}{\mathbb{E}_\zeta\, \tau'_1} \mathbb{E}_\zeta \left(\sum_{n=0}^{\tau'_1 - 1} \mathbf{1}(M_n \in \cdot\,)\right)$$

is a stationary distribution for M. For details see Alsmeyer [3].

The following proposition is a consequence of Dynkin's formula and gives a criterion for a set to be positive recurrent.

Proposition 1.3 *For a set $\Re \in \mathcal{E}$, the expectation $\mathbb{E}_s\tau(\Re)$ is finite for any $s \in E$ if and only if there exists a nonnegative measurable function $V : E \to [0,\infty)$, and a constant $\Delta > 0$, such that*

(i) $\sup_{s\notin\Re} \int_E \left(V(s') - V(s)\right) \mathbb{P}_s(ds') \le -\Delta$ *;*

(ii) $\int_E \left(V(s') - V(s)\right) \mathbb{P}_s(ds') < \infty$ *for all* $s \in \Re$.

In this case, we have

$$\mathbb{E}_s\, \tau(\Re) \le \begin{cases} \frac{V(s)}{\Delta} & : \quad s \notin \Re \\ 1 + \frac{1}{\Delta}\left\{V(s) + \int_E \left(V(s') - V(s)\right) \mathbb{P}_s(ds')\right\} & : \quad s \in \Re. \end{cases}$$

Proof. See Corollary 5.2.1 in Kalashnikov [33]. QED

Moreover, we have the following criteria for the first return time to a set to have finite moments.

Proposition 1.4 *Let $\Re$ be a measurable set.*

(i) Suppose that there exist a nonnegative function $V : E \to [0, \infty)$, positive numbers Δ, b and $\alpha > 1$, and a random variable Λ defined on E, such that the following relations are fulfilled:

(a) $v_\Re := \sup_{s\in\Re} V(s) < \infty$;

(b) $\mathbb{P}\big(V(M_1) - V(s) \le \Lambda(s)\big) = 1$ *for all* $s \in E$;

(c) $\sup_{s\notin\Re} \mathbb{E}\,\Lambda(s) \le -\Delta$;

(d) $\sup_{s\in E} \mathbb{E}\,|\Lambda(s)|^\alpha \le b < \infty$.

Then

$$\mathbb{E}_s\,(\tau(\Re))^\alpha \le \begin{cases} \left(a(\Delta, b, \alpha) + \frac{2V(s)}{\delta}\right)^\alpha & : \quad s \notin \Re \\ c(\Delta, b, \alpha, v_\Re) & : \quad s \in \Re, \end{cases}$$

for some constants $a(\Delta, b, \alpha)$ and $c(\Delta, b, \alpha, v_\Re)$.

(ii) For some $\gamma > 0$, the expectation $\mathbb{E}_s \exp(\gamma\tau(\Re))$ is finite for any $s \in E$ if and only if there exists a nonnegative function $V : E \to [1, \infty)$, such that

(a) $\int_E \big(V(s') - V(s)\big)\,\mathbb{P}_s(ds') \le -\big(1 - \exp(-\gamma)\big)V(s)$ *for all* $s \notin \Re$;

(b) $\int_E \big(V(s') - V(s)\big)\,\mathbb{P}_s(ds') < \infty$ *for all* $s \in \Re$.

In this case, we have

$$\mathbb{E}_s \exp(\gamma\tau(\Re)) \le \begin{cases} V(s) & : \quad s \notin \Re \\ e^\gamma \cdot \left\{V(s) + \int_E \big(V(s') - V(s)\big)\,\mathbb{P}_s(ds')\right\} & : \quad s \in \Re. \end{cases}$$

Proof. See Theorem 5.2.2 and Corollary 5.2.2 in Kalashnikov [33]. QED

1.3 Harris recurrence

Let φ be a nontrivial σ-finite measure on $(E, \mathcal{E})$.

Definition 1.5 (i) A Markov chain $(M_n)_{n\ge0}$ with transition kernel $\mathbf{P}$ is called *φ-irreducible*, if for any $s \in E$ and $A \in \mathcal{E}$ with $\varphi(A) > 0$ there exists $n \ge 1$ with $\mathbb{P}_s\,(M_n \in A) > 0$.

(ii) A Markov chain M is called *(d-) periodic*, if there exists a finite sequence of sets $E_i \subset E, i = 1, \cdots, d$, such that

$$\mathbb{P}_s\,(M_1 \in E_{i+1}) = 1, \quad \text{if} \quad s \in E_i,$$

where we set $E_{d+1} = E_1$. If $d = 1$, then it is called *aperiodic.*

If a Markov chain is φ-irreducible, then the set $E \setminus \cup_{i=1}^{d} E_i$ is a φ-null set. It is known (see Theorem 3.11 in Asmussen [12]) that, if M is φ-irreducible and $\varphi(A) > 0$, then there exist a measurable set $\Re \subset A$, $r \geq 1$ and $p > 0$ such that

$$\mathbb{P}_s\left(M_r \in A'\right) \geq p\,\varphi(\Re \cap A')$$

for all $s \in \Re, A' \in \mathcal{E}$. If M is φ-irreducible, then φ is called an *irreducibility measure* for M. A φ-irreducible Markov chain has many different irreducibility measures. The measure ψ defined as

$$\psi = \sum_{n=1}^{\infty} 2^{-n}\,\varphi\,\mathbf{P}^n$$

is a maximal irreducibility measure, in the sense that all other irreducibility measures are absolutely continuous w.r.t. ψ. The maximal irreducibility measures are equivalent. It is known that, if a φ-irreducible Markov chain $(M_n)_{n\geq 0}$ has a stationary measure ξ, then it is unique up to constant multiples and is equivalent to maximal irreducibility measures.

Definition 1.6 A temporally homogeneous Markov chain $(M_n)_{n\geq 0}$ with transition kernel $\mathbf{P}$ is called *Harris recurrent* or *Harris chain*, if there exists a recurrent set $\Re$, such that for some $p \in (0, 1], r \geq 1$, and a distribution φ on E with $\varphi(\Re) = 1$

$$\mathbf{P}^r(s, A) \geq p\,\varphi(A) \quad \text{for any} \quad s \in \Re, A \in \mathcal{E}. \tag{1.1}$$

The set $\Re$ is called a *regeneration set*, and we say that $(M_n)_{n\geq 0}$ satisfies the minorization condition $\mathcal{M}(\Re, p, r, \varphi)$. If $r = 1$, then M is called *strong aperiodic.*

A Markov chain $(M_n)_{n\geq 0}$ is called *φ-recurrent*, if any $A \in \mathcal{E}$ with $\varphi(A) > 0$ is a recurrent set. Obviously a φ-recurrent Markov chain is φ-irreducible. Furthermore, a Markov chain is Harris recurrent if and only if it is φ-recurrent. In this case, any recurrent set of M contains a regeneration set of M.

Remark 1.7 A discrete Markov chain $(M_n)_{n\geq 0}$ is Harris recurrent if, and only if, it contains a communication class $\mathcal{K}$ of recurrent states such that $\mathbb{P}_i\left(\tau(\mathcal{K}) < \infty\right) = 1$ for all $i \in E$. Thus $(M_n)_{n\geq 0}$ can also possess transient states, from which $\mathcal{K}$ can be reached with probability 1. In this case, every set $\Re = \{j\}$ with $j \in \mathcal{K}$ is a regeneration set, since M satisfies the minorization condition $\mathcal{M}(\Re, p, r, \varphi)$ with $r \in \{n : p_{jj}^{(n)} > 0\}, p = p_{jj}^{(r)}$ and $\varphi = \delta_j$.

If M is a discrete, irreducible Markov chain, then the successive return times to a recurrent state form an identically and independently distributed (i.i.d.) sequence. At each time the chain enters the state, it starts a new tour with the same distribution, regardless of the preceding sample path. This leads to a decomposition of the chain into cycles with i.i.d. distribution. Unfortunately this is not true in general, if the state space is uncountable. But any Harris chain has or can be modified to have regeneration epochs in some generalized sense. We introduce some definitions, which are known for general stochastic processes.

Definition 1.8 Let $X = (X_t)_{t\in\Gamma}$ be a discrete- or continuous-time stochastic process with state space E ($\Gamma = \mathbb{N}_0$ or $\mathbb{R}_0^+$). Assume that there are random times $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \infty$ a.s. Consider cycles $Z_n := (\tau_{n+1} - \tau_n, (X_t)_{\tau_n \le t < \tau_{n+1}}), n \ge 0$.

(i) We call X or the pair (τ, X) *wide-sense regenerative*, if the cycles $Z_n, n \ge 1$, are identically distributed and the sequence $(Z_k)_{k\ge n}$ does not depend on $(\tau_0, \tau_1, \cdots, \tau_n)$ for $n \ge 1$:

(ii) A wide-sense regenerative process X or (τ, X) is called *classical-sense regenerative*, if the cycles $Z_n, n \ge 0$, are independent:

(iii) X or (τ, X) is called *l-dependent regenerative*, if the cycles $Z_n, n \ge 0$, are l-dependent and identically distributed for $n \ge 1$.

In each case of (i)-(iii), the random times $\tau_n, n \ge 0$, are called *regeneration epochs*. If further the cycles $Z_n, n \ge 0$, are identically distributed, then we say that X is *zero-delayed*. A (wide-sense or l-dependent) regenerative process X with regeneration epochs $\tau_n, n \ge 0$, is called *positive recurrent*, if $\mathbb{E}(\tau_2 - \tau_1) < \infty$, and *null recurrent*, otherwise.

Note that a l-dependent regenerative process X is always one-dependent regenerative, since to a given l-dependent cycles Z_n with regeneration epochs τ_n, $n \ge 0$, one can associate new cycles $\hat{Z}_n := (Z_k)_{\tau_{ln} \le k < \tau_{l(n+1)}}, n \ge 0$, which are one-dependent. If (τ, X) is wide-sense regenerative, then the sequence of regeneration epochs $(\tau_n)_{n\ge 0}$ forms a renewal process, which is called the *embedded renewal process*.

Remark 1.9 A wide-sense regenerative process with one-dependent cycles is often called *weak regenerative*.

Suppose that (τ, X) is positive recurrent, wide-sense regenerative and that there exists a distribution $\mathbb{P}_0$ such that

$$\mathbb{P}_0\Big(\big((\tau_n)_{n\ge 0}, (X_t)_{t\in\Gamma}\big) \in \cdot\,\Big) = \mathbb{P}\Big(\big((\tau_n - \tau_1)_{n\ge 0}, (X_{\tau_1+t})_{t\in\Gamma}\big) \in \cdot\,\Big).$$

Denote by $\theta_t, t \in \Gamma$, shift-operators defined as $\theta_t X = (X_{t+u})_{u\ge 0}$. Let $\mathbb{P}^*$ be the distribution defined as

$$\mathbb{P}^*(\cdot) = \frac{1}{\mathbb{E}_0\,\tau_1}\,\mathbb{E}_0\left(\int_0^{\tau_1} \mathbf{1}(\theta_t X \in \cdot\,)\,dt\right),$$

if $\Gamma = \mathbb{R}_0^+$, and

$$\mathbb{P}^*(\cdot) = \frac{1}{\mathbb{E}_0\,\tau_1}\,\mathbb{E}_0\left(\sum_{n=0}^{\tau_1 - 1} \mathbf{1}(\theta_n X \in \cdot\,)\right),$$

if $\Gamma = \mathbb{N}_0$, where $\mathbb{E}_0$ means the expectation under the distribution $\mathbb{P}_0$. It is known (see Kalashnikov [32] or Thorisson [63]) that, if $\Gamma = \mathbb{R}_0^+$ and the distribution $\mathbb{P}_0(\tau_1 \in \cdot\,)$ is spread out, then

$$\lim_{t\to\infty} \|\mathbb{P}(\theta_t X \in \cdot\,) - \mathbb{P}^*\| = \lim_{t\to\infty} \|\mathbb{P}_0(\theta_t X \in \cdot\,) - \mathbb{P}^*\| = 0.$$

If $\Gamma = \mathbb{N}_0$ and the span of τ_1 under $\mathbb{P}_0$ is 1, then

$$\lim_{n\to\infty} \|\mathbb{P}(\theta_n X \in \cdot\,) - \mathbb{P}^*\| = \lim_{n\to\infty} \|\mathbb{P}_0(\theta_n X \in \cdot\,) - \mathbb{P}^*\| = 0.$$

Regenerative processes play an important role in applied probability. There are plenty of literature on regenerative processes. A standard reference for regenerative processes is Thorisson [63]. Sigman and Wolf [60] give an expository survey including applications to the queueing theory.

We now return to Harris chains. The following proposition says that a Harris chain is wide-sense as well as one-dependent regenerative.

Proposition 1.10 (Regeneration lemma) *Given a Harris chain $(M_n)_{n\geq 0}$, there exist a filtration $(\mathcal{F}_n)_{n\geq 0}$ and a sequence $(\tau_n)_{n\geq 0}$ of random times, which have the following properties:*

(i) $0 = \tau_0 < \tau_1 < \tau_2 < \cdots < \infty$ a.s. under $\mathbb{P}_\lambda$ for any distribution λ on E;

(ii) $(M_n)_{n\geq 0}$ is Markov-adapted and each τ_k a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$;

(iii) under each $\mathbb{P}_s$, $s \in E$, the M_{τ_n} are independent for $n \geq 0$ and further identically distributed with common distribution $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\tau_1} \in \cdot)$ for any initial distribution λ and for $n \geq 1$;

(iv) for each $n \geq 0$ and $s \in E$

$$\mathbb{P}\left[(\tau_{n+j} - \tau_n, M_{\tau_{n+j}})_{j\geq 0} \in \cdot\,|\mathcal{F}_{\tau_n}\right] = \mathbb{P}_{M_{\tau_n}}((\tau_j, M_j)_{j\geq 0} \in \cdot\,) \quad \mathbb{P}_s\text{-}a.s.;$$

(v) $(\tau_{n+j} - \tau_n, M_{\tau_{n+j}})_{j\geq 0}$ is independent of $\tau_0, \cdots, \tau_n$ for each $n \geq 0$.

Proof. The sequence of random times $(\tau_n)_{n\geq 0}$ can be obtained by the splitting technique, which was suggested by Athreya and Ney [15]. The construction requires in general enlarging the probability space to support the new Bernoulli random variables. Suppose that M satisfies the minorization condition $\mathcal{M}(\Re, p, r, \varphi)$. Starting at any state, the chain M hits $\Re$ eventually. Conditional on doing so, the distribution of the transition r steps later can be written as

$$\mathbf{P}^r(s, \cdot\,) = p\varphi(\cdot) + (1-p)\bar{\mathbf{P}}^r(s, \cdot\,),$$

where

$$\bar{\mathbf{P}}^r(s, \cdot\,) = (1-p)^{-1}(\mathbf{P}^r(s, \cdot\,) - p\varphi(\cdot)).$$

Let $\eta_n, n \geq 0$, be i.i.d. $\{0, 1\}$-valued random variables with $\mathbb{P}_s(\eta_n = 1) = p$. Thus if $M_n \in \Re$, then M_{n+r} is generated according to φ if $\eta_n = 1$ and according to $\bar{\mathbf{P}}^r(s, \cdot\,)$, otherwise. The missing values of $M_{n+1}, \cdots, M_{n+r-1}$ are generated according to the conditional distribution given M_n and M_{n+r}, which exists on a Polish state space.
If $M_n \neq \Re$, then M_{n+1} is generated according to $\mathbf{P}(M_n, \cdot\,)$. Random times $\tau_n, n \geq 0$, are defined recursively:

$$\tau_0 = 0 \quad \text{and} \quad \tau_n := \inf\{k \geq \tau_{n-1} + r : M_{k-r} \in \Re, \eta_{k-r} = 1\}, \quad n \geq 1.$$

Then the properties (i)-(v) are fulfilled with $\zeta(\cdot) := \varphi(\cdot) = \mathbb{P}_\lambda(M_{\tau_1} \in \cdot\,)$ for any initial distribution λ. For details we refer to Alsmeyer [1], Kalashnikov [32] or Lindvall [35].

QED

We say that a sequence $(\tau_n)_{n\geq 0}$ forms a *sequence of regeneration epochs* for $(M_n)_{n\geq 0}$, if it satisfies properties (i) through (iv) in Proposition 1.10. In Alsmeyer [1] , it is shown that a Markov chain M is Harris recurrent, if (thus if and only if) it possesses a sequence of regeneration epochs. Note that a Harris chain M is d-periodic, if the span of τ_1 is d under $\mathbb{P}_\zeta$, where τ_1 is a regeneration epoch constructed by the splitting technique (see Proposition 3.10 in Asmussen [12]).

Remark 1.11 In the proof of Proposition 1.10, we have considered the bivariate Markov chain $\tilde{M} := (M_n, \eta_n)_{n\geq 0}$ with state space $(E \times \{0,1\}, \mathcal{E} \otimes \mathcal{P}(\{0,1\}))$ to get regeneration epochs. If M is strong aperiodic, then transition kernel $\tilde{\mathbf{P}}$ of $\tilde{M}$ can be given through

$$\begin{aligned}
\tilde{\mathbf{P}}((s,0), A \times \{\theta\}) &= \begin{cases} (p\theta + (1-p)(1-\theta))\mathbf{P}(s,A) & : \quad s \notin \Re \\ (p\theta + (1-p)(1-\theta))\bar{\mathbf{P}}(s,A) & : \quad s \in \Re \end{cases} \\
\tilde{\mathbf{P}}((s,1), A \times \{\theta\}) &= \begin{cases} (p\theta + (1-p)(1-\theta))\mathbf{P}(s,A) & : \quad s \notin \Re \\ (p\theta + (1-p)(1-\theta))\varphi(A) & : \quad s \in \Re, \end{cases}
\end{aligned}$$

for any $A \in \mathcal{E}, \theta \in \{0,1\}$, where $\bar{\mathbf{P}}$ is defined in Proposition 1.10. In this case M is classical-sense regenerative. For the construction of $\tilde{\mathbf{P}}$ in the general case, see Kalashnikov [32].

Remark 1.12 Borovkov introduced *renovative processes* (see Chapter 3 in Borovkov [19] or Foss and Kalashnikov [27]). Let $(Y_n)_{n\geq 0}$ be a sequence of random variables on E defined by the recursive relation

$$Y_{n+1} = f(Y_n, X_n), \quad n \geq 0,$$

where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. random variables taking values from a Polish space E' and the mapping $f : E \times E' \to E$ is supposed to be measurable. Note that the sequence $(Y_n)_{n\geq 0}$ forms a temporally homogeneous Markov chain on E. Denote by f_k the kth iteration of f: For any $y \in E, (x_0, \cdots, x_k) \in (E')^{k+1}$,

$$\begin{aligned}
f_1(y, x_0) &= f(y, x_0); \\
f_{k+1}(y, x_0, \cdots x_k) &= f(f_k(y, x_0, \cdots, x_{k-1}), x_k), \quad k \geq 1.
\end{aligned}$$

Suppose that there exist an integer $r > 0$ and measurable sets $B_r \subset E', C \in E$ such that for any $y, y' \in C$ and $(x_1, \cdots, x_r) \in B_r$

$$f_r(y, x_0, \cdots, x_{r-1}) = f_r(y', x_0, \cdots, x_{r-1}).$$

Define events

$$C_n = \{Y_n \in C\}, \quad B_n = \{(X_{n-r}, \cdots, X_{n-1}) \in B_r\}, \quad n \geq r, \quad A_n = C_{n-r} \cap B_n.$$

The events $A_n, n \geq 0$, are called *renovative* and their occurrence times the *renovation times*. Suppose that $\mathbb{P}(B_r) > 0$. Let further η_r be the common value of $f_r(y, x_0, \cdots, x_{r-1})$ for $y \in C$, and ζ a distribution defined as

$$\zeta(\cdot) = \mathbb{P}\,[\eta_r \in \cdot\,|B_r].$$

Then the chain $(Y_n)_{n\geq 0}$ satisfies the minorization condition

$$\mathbb{P}_s(Y_r \in A) \geq \mathbb{P}(B_r)\,\zeta(A) \quad \text{for all} \quad s \in C, A \in \mathcal{E}.$$

Thus if C is a recurrent set of $(Y_n)_{n\geq 0}$, then $(Y_n)_{n\geq 0}$ is Harris recurrent and the sequence of random variables $(\tau_n)_{n\geq 0}$ given as

$$\tau_0 = 0 \quad \text{and} \quad \tau_n = \inf\{k \geq \tau_{n-1} + r : \mathbf{1}(A_k) = 1\}, \quad n \geq 1,$$

forms a sequence of regeneration epochs.

From the existence of regeneration epochs for Harris chains one can construct a stationary measure, which is unique up to constant multiples.

Proposition 1.13 *With the same notations as in Proposition 1.10, the measure*

$$\xi(\cdot) := \mathbb{E}_\zeta \left(\sum_{n=0}^{\tau_1 - 1} \mathbf{1}(M_n \in \cdot) \right), \tag{1.2}$$

defines a stationary measure for $\mathbf{P}$*, which is unique up to constant multiples.*
If $\mathbb{E}_\zeta\, \tau_1 < \infty$*, then* $\xi^* := (\mathbb{E}_\zeta\, \tau_1)^{-1}\, \xi$ *is the unique stationary distribution for* $\mathbf{P}$.

Proof. See Satz 8.3.1, Satz 8.3.2 in Alsmeyer [1]. QED

A Harris chain $(M_n)_{n\geq 0}$ is called *positive Harris recurrent*, if M has a stationary distribution.

Remark 1.14 A continuous-time Markov process $(M_t)_{t\geq 0}$ is called a *Harris process*, if it is φ-recurrent for some σ-finite measure φ, i.e., for any $A \in \mathcal{E}$ with $\varphi(A) > 0$,

$$\mathbb{P}_s \left(\int_0^\infty \mathbf{1}(M_t \in A)\, dt = \infty \right) = 1, \quad s \in E.$$

Sigman [57] showed that, if a Markov process is one-dependent regenerative, it is a Harris process. It is also known that a Harris process has a unique (up to a multiplicative constant) stationary measure ξ. A Harris process with a finite stationary measure ξ is called *positive Harris recurrent.*

In Proposition 1.10 we have constructed regeneration epochs for a Harris chain from a minorization condition and the first return time to the regeneration set. It is thus reasonable, to expect some relations between moments of the regeneration epochs constructed by the splitting technique from a regeneration set and the first return time to the regeneration set. We need the following lemma.

Lemma 1.15 *Let $(X_n)_{n\geq 0}$ be a sequence of real valued random variables adapted to a filtration $(\mathcal{F}_n)_{n\geq 0}$ and τ an a.s. finite stopping time w.r.t. $(\mathcal{F}_n)_{n\geq 0}$.*

(i) Let $\alpha \geq 1$. If there exist $l_1 > 0$ and $l_2 > 0$ such that

$$\mathbb{E}[|X_n|^\alpha|\mathcal{F}_{n-1}] \leq l_1 < \infty \quad \text{and} \quad \mathbb{E}[\tau^\alpha|\mathcal{F}_0] \leq l_2 < \infty$$

for all $n \geq 1$, then

$$\mathbb{E}\Big[\Big(\sum_{n=1}^{\tau}|X_n|\Big)^\alpha|\mathcal{F}_0\Big] \leq c\, l_1 l_2 < \infty$$

for some constant c.

(ii) Let $\gamma > 0$. If there exists $l > 0$ such that

$$\mathbb{E}\Big[\exp(\gamma X_n)|\mathcal{F}_{n-1}\Big] \leq l < \infty$$

for all $n \geq 1$, then

$$\mathbb{E}\Big[\exp\Big(\frac{\gamma}{2}\sum_{n=1}^{\tau}X_n\Big)|\mathcal{F}_0\Big] \leq \big\{\mathbb{E}[l^\tau|\mathcal{F}_0]\big\}^{1/2}.$$

Proof. (i) See Theorem 1 in Borovkov and Utev [20].

(ii) The assertion can be deduced from Theorem 2 in Borovkov and Utev [20], but we give a simple proof. Let

$$R_n = l^{-n}\exp\Big(\gamma\sum_{k=1}^{n}X_k\Big).$$

Then the sequence $(R_n)_{n\geq 1}$ forms a positive supermartingale, and by the optional sampling theorem the sequence $(R_{n\wedge\tau})_{n\geq 1}$ also a positive supermartingale with $\mathbb{E}\left[R_n|\mathcal{F}_0\right] \leq 1$ for any $n \geq 1$, which implies $\mathbb{E}\left[R_{n\wedge\tau}|\mathcal{F}_0\right] \leq 1$ for any $n \geq 1$. Thus, by Fatou's lemma, $\mathbb{E}\left[R_\tau|\mathcal{F}_0\right] \leq 1$, since $\lim_{n\to\infty} R_{n\wedge\tau} = R_\tau$ a.s. Using Hölder's inequality, we have

$$\mathbb{E}\Big[\exp\Big(\frac{\gamma}{2}\sum_{n=1}^{\tau}X_n\Big)|\mathcal{F}_0\Big] = \mathbb{E}\left[R_\tau^{1/2}\, l^{\tau/2}|\mathcal{F}_0\right] \leq \left\{\mathbb{E}\left[l^\tau|\mathcal{F}_0\right]\right\}^{1/2}.$$

QED

The following assertions may be known, but we give full proofs, because we find no adequate proofs in literature.

Proposition 1.16 *Let M be a Harris chain satisfying the minorization condition $\mathcal{M}(\Re, p, r, \varphi)$ and $(\tau_n)_{n\geq 0}$ a sequence of regeneration epochs constructed by the splitting technique from the minorization condition.*

(i) Let $\alpha \geq 1$. *If* $\sup_{s\in\Re} \mathbb{E}_s (\tau(\Re))^\alpha < \infty$, *then* $\sup_{s\in\Re} \mathbb{E}_s \tau_1^\alpha < \infty$.

(ii) Let $\gamma > 0$. *If* $\sup_{s\in\Re} \mathbb{E}_s \exp(\gamma\tau(\Re)) < \infty$, *then* $\sup_{s\in\Re} \mathbb{E}_s \exp(\gamma'\tau_1) < \infty$ *for some* $\gamma' > 0$.

(iii) If $\Re = E$, *then* $\sup_{s\in E} \mathbb{E}_s \tau_1^\alpha < \infty$ *for any* $\alpha \geq 1$ *and* $\sup_{s\in E} \mathbb{E}_s \exp(\gamma\tau_1) < \infty$ *for some* $\gamma > 0$.

Proof. (i) A proof for the case $r = 1, \alpha = 1$ can be found in Borovkov [20], for example. To show the assertion for $r \geq 2$, $\alpha \geq 1$, let $\tilde{M} = (M_n, \eta_n)_{n\geq 0}$ be the Markov chain constructed in Proposition 1.10 (see also Remark 1.11). The transition kernel $\tilde{\mathbf{P}}$ of $\tilde{M}$ satisfies

$$\tilde{\mathbf{P}}^r\big((s,\theta), A \times \{0,1\}\big) = \begin{cases} (1-p)^{-1}\big(\mathbf{P}^r(s,A) - p\varphi(A)\big) & : \quad s \in \Re, \theta = 0 \\ \varphi(A) & : \quad s \in \Re, \theta = 1, \end{cases}$$

Furthermore, if $s \notin \Re$, then $\tilde{\mathbf{P}}\big((s,\theta), A \times \{0,1\}\big) = \mathbb{P}(s,A)$ for any $\theta \in \{0,1\}$. Let $\tilde{\tau}_n, n \geq 1$, be the random variables defined as

$$\tilde{\tau}_1 = \tau(\Re) \quad \text{and} \quad \tilde{\tau}_n \quad = \quad \inf\{k \geq r + \tilde{\tau}_{n-1} : \tilde{M}_k \in \Re \times \{0,1\}\}, \quad n \geq 2.$$

We set $\tilde{\tau}_0(\Re) = 0$. Let further ν be a random variable defined as

$$\nu := \inf\{k : \tilde{M}_{\tilde{\tau}_k} \in \Re \times \{1\}\}.$$

By the geometric trial argument

$$\mathbb{P}\,(\nu = k) = p(1-p)^{k-1}, \quad k \geq 1.$$

From the construction of τ_1, it is easy to see that for any $s \in \Re$

$$\begin{aligned} \mathbb{E}_s \tau_1^\alpha &\leq \sup_{(s,\theta)\in\Re\times\{0,1\}} \mathbb{E}_{(s,\theta)} \big(\tilde{\tau}_\nu + r\big)^\alpha \\ &= \sup_{s\in\Re} \mathbb{E}_{(s,0)} \big(\tilde{\tau}_\nu + r\big)^\alpha \\ &\leq \sup_{s\in\Re} \mathbb{E}_{(s,0)} \left(\sum_{n=1}^{\nu} \big(\tilde{\tau}_n - \tilde{\tau}_{n-1}\big) + r\right)^\alpha, \end{aligned}$$

since $\mathbb{E}_{(s,1)} \tau_1^\alpha = r^\alpha$ for any $s \in \Re$. Let $\mathcal{G}_n := \sigma\left(\tilde{M}_k : k \leq \tilde{\tau}_n\right)$ for $n \geq 1$. Then, for each $n \geq 1$,

$$\begin{aligned} &\mathbb{E}\left[\big(\tilde{\tau}_{n+1} - \tilde{\tau}_n\big)^\alpha \big| \mathcal{G}_n\right] \\ &\quad\leq \sup_{s\in\Re} \left\{ r^\alpha \tilde{\mathbf{P}}^r\big((s,0), \Re \times \{0,1\}\big) + \int_{\Re^c} \mathbb{E}_{s'}\big(r + \tau(\Re)\big)^\alpha \tilde{\mathbf{P}}^r\big((s,0), ds' \times \{0,1\}\big) \right\} \\ &\quad\leq \frac{1}{1-p} \sup_{s\in\Re} \left\{ r^\alpha \mathbf{P}^r(s,\Re) + \int_{\Re^c} \mathbb{E}_{s'}\big(r + \tau(\Re)\big)^\alpha \mathbf{P}^r(s,ds') \right\} \\ &\quad\leq \frac{1}{1-p} \sup_{s\in\Re} \mathbb{E}_s\big(r + \tau(\Re)\big)^\alpha < \infty. \end{aligned}$$

Since $\mathbb{E}\,\nu^\alpha < \infty$ for any $\alpha \geq 1$, we have $\sup_{s\in\Re} \mathbb{E}_s\,\tau_1^\alpha < \infty$ by Lemma 1.15 (i).

(ii) For any $s \in \Re$

$$\begin{aligned}\mathbb{E}_s \exp(\gamma\tau_1) &\leq \sup_{(s,\theta)\in\Re\times\{0,1\}} \mathbb{E}_{(s,\theta)} \exp\big(\gamma(r+\tilde{\tau}_\nu)\big)\\ &\leq \sup_{s\in\Re} \mathbb{E}_{(s,0)} \exp\left(\gamma\Big(r+\sum_{n=1}^{\nu}\big(\tilde{\tau}_n-\tilde{\tau}_{n-1}\big)\Big)\right).\end{aligned}$$

As in (i), for each $n \geq 1$

$$\mathbb{E}\left[\exp\Big(\gamma\big(\tilde{\tau}_{n+1}-\tilde{\tau}_n\big)\Big)\big|\mathcal{G}_n\right] \leq \frac{1}{1-p}\left\{\sup_{s\in\Re}\mathbb{E}_s\,\exp\big(\gamma(r+\tau)\big)-p\right\}.$$

Moreover, by assumption, there exists a $\gamma'', 0 < \gamma'' \leq \gamma$, such that

$$\sup_{s\in\Re}\mathbb{E}_s\,\exp\Big(\gamma''\big(2r+\tau(\Re)\big)\Big) < 1+\frac{p}{2}.$$

Letting

$$L := \frac{1}{1-p}\left\{\sup_{s\in\Re}\mathbb{E}_s\,\exp\Big(\gamma''\big(2r+\tau(\Re)\big)\Big)-p\right\},$$

for any $s \in \Re$

$$\mathbb{E}_s\,L^\nu \leq \sum_{n=1}^{\infty} L^n\,p(1-p)^{n-1} < \infty.$$

Put $\gamma' = \gamma''/2$. Then, by Lemma 1.15 (ii), for any $s \in \Re$

$$\begin{aligned}\mathbb{E}_s\,\exp(\gamma'\tau_1) &\leq \mathbb{E}_{(s,0)}\,\exp\left(\gamma'\Big(r+\sum_{n=1}^{\nu}\big(\tilde{\tau}_n-\tilde{\tau}_{n-1}\big)\Big)\right)\\ &\leq \left\{\sup_{s\in\Re}\mathbb{E}_s\Big(\frac{1}{1-p}\big(\sup_{s\in\Re}\mathbb{E}_s\,\exp\big(\gamma''(2r+\tau(\Re))\big)-p\big)\Big)^\nu\right\}^{1/2}\\ &\leq \left\{\sup_{s\in\Re}\mathbb{E}_s\,L^\nu\right\}^{1/2} < \infty.\end{aligned}$$

(iii) Clear from the proofs of (i) and (ii), since $\tau(\Re) = 1$. QED

The following proposition states the strong law of large numbers (SLLN) for real functions of Harris chains.

Proposition 1.17 *Let M be a positive Harris chain with a stationary distribution ξ. Consider a sequence of random variables $(Y_n)_{n\geq 0}$ with $Y_n := f(M_n), n \geq 0$, for a measurable, nonnegative real-valued function f. Then*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n} Y_k = \mathbb{E}_\xi Y_1 \quad \mathbb{P}_\lambda\text{-}a.s.$$

for any initial distribution λ on E.

Proof. See Theorem 4.3.6 in Revuz [53]. QED

1.4 Ergodicity

Definition 1.18 Let M be a Markov chain with transition kernel $\mathbf{P}$ and denote by M^λ a Markov chain with transition kernel $\mathbf{P}$ and initial distribution λ.

(i) We say that M admits *coupling*, if for any two initial distributions μ and λ there exist Markov chains M^μ and M^λ on a common probability space such that

$$M_n^\mu = M_n^\lambda, \quad n \geq T,$$

where T is a finite random time.

(ii) We say that M admits *shift-coupling*, if for any two initial distributions μ and λ there exist Markov chains M^μ and M^λ on a common probability space such that

$$M_{T+n}^\mu = M_{T'+n}^\lambda, \quad n \geq 0,$$

where T and T' are finite random times.

The following two propositions give characterizations of Harris chains.

Proposition 1.19 *Let M be a general Markov chain with a stationary distribution ξ. Then the following assertions are equivalent:*

(i) M admits shift-coupling;

(ii) for any initial distribution λ the distribution of M converges to ξ in Cesaro total variation, i.e.,

$$\lim_{n\to\infty} \left\| \frac{1}{n+1} \sum_{k=0}^{n} \mathbb{P}_\lambda (M_k \in \cdot\,) - \xi \right\| = 0;$$

(iii) M is positive Harris recurrent.

Proof. For the equivalence of (i) and (ii) see Theorem 5.5.4 in Thorisson [63]. The equivalence of (i) and (iii) follows from Theorem 10.4.6 in Thorisson [63]. QED

If M is positive Harris recurrent and aperiodic, then it is called *Harris ergodic.*

Proposition 1.20 *Let M be a general Markov chain with a stationary distribution ξ. Then the following assertions are equivalent:*

(i) M admits coupling;

(ii) for any initial distribution λ the distribution of M converges to ξ in total variation, i.e.,

$$\lim_{n\to\infty} \|\mathbb{P}_\lambda (M_n \in \cdot\,) - \xi\| = 0;$$

(iii) M is Harris ergodic.

Proof. For the equivalence of (i) and (ii) see Theorem 6.4.1 in Thorisson [63]. The equivalence of (i) and (iii) follows from Proposition VII 3.13 in Asmussen [12]. QED

It is known that rates of convergence of regenerative processes are closely related to moments of regeneration epoch. For details see Lindvall [35], Kalashnikov [32] and Thorisson [63].

The following assertions are known.

Proposition 1.21 *Let M be a Harris ergodic Markov chain with a stationary distribution ξ and τ_1 the first regeneration epoch constructed by the splitting technique. Let further φ be the distribution defined as $\varphi(\cdot) = \mathbb{P}_\lambda(M_{\tau_1} \in \cdot\,)$.*

(i) If $\mathbb{E}_\varphi\, \tau_1^{\alpha+1} < \infty$ for some $\alpha > 0$, then for some constant c

$$\|\mathbb{P}_\varphi(M_n \in \cdot\,) - \xi\| \le cn^{-\alpha}.$$

(ii) If $\mathbb{E}_\varphi\, \exp(\gamma\tau_1) < \infty$ for some $\gamma > 0$, then for some constants c and $\gamma' \in (0, \gamma]$

$$\|\mathbb{P}_\varphi(M_n \in \cdot\,) - \xi\| \le c\exp(-\gamma' n).$$

(iii) Let λ and μ be initial distributions on E. If for some $\alpha \ge 1$

$$\mathbb{E}_\lambda\, \tau_1^\alpha < \infty, \quad \mathbb{E}_\mu\, \tau_1^\alpha \quad \text{and} \quad \mathbb{E}_\varphi\, \tau_1^\alpha < \infty,$$

then

$$\lim_{n\to\infty} n^\alpha \|\mathbb{P}_\lambda(M_n \in \cdot\,) - \mathbb{P}_\mu(M_n \in \cdot\,)\| = 0.$$

Proof. See Corollary 5.1.1 in Kalashnikov [32] for the proof of (i) and (ii), and Theorem 10.7.5 in Thorisson [63] for (iii). QED

Corollary 1.22 *Let M be a Harris ergodic Markov chain with a stationary distribution ξ and $\Re$ a regeneration set.*

(i) Let $\alpha > 0$. If $\sup_{s\in\Re} \mathbb{E}_s(\tau(\Re))^{\alpha+1} < \infty$, then for some constant c

$$\|\mathbb{P}_\varphi\,(M_n \in \cdot\,) - \xi\| \le cn^{-\alpha}.$$

(ii) Let $\gamma > 0$. If $\sup_{s\in\Re} \mathbb{E}_s\, \exp(\gamma\tau(\Re)) < \infty$, then for some constants c and $\gamma' > 0$

$$\|\mathbb{P}_\varphi\,(M_n \in \cdot\,) - \xi\| \le c\exp(-\gamma' n).$$

Proof. All assertions follow directly from Proposition 1.16 and Proposition 1.21.

QED

A Markov chain M is called *uniformly φ-recurrent*, if it satisfies the condition

$$\sup_{s\in E} \mathbb{P}_s\left(\tau(A) > n\right) \to 0 \quad \text{as} \quad n \to \infty$$

for any $A \in \mathcal{E}$ with $\varphi(A) > 0$.

Proposition 1.23 *The following conditions are equivalent:*

(i) M is uniformly φ-recurrent and aperiodic;

(ii) M is aperiodic and E is a regeneration set, i.e., there exist an integer $n_0 \geq 1$, a constant $\alpha > 0$ and a distribution ψ such that

$$\sup_{s\in E} \mathbb{P}_s\left(M_{n_0} \in \cdot\,\right) \geq \alpha\psi(\cdot);$$

(iii) there exist positive constants $c < \infty$ and $\rho < 1$ such that

$$\left\|\mathbb{P}_s\left(M_n \in \cdot\,\right) - \xi\right\| < c\rho^n$$

for any $n \geq 0$ and $s \in E$.

Proof. See Theorem 6.15 in Nummelin [46]. QED

If one of the conditions (i) through (iii) of Proposition 1.23 holds true, M is called *uniformly (Harris) ergodic.*

Chapter 2

Markov random walks

A Markov random walk (MRW) is a bivariate sequence $(M_n, S_n)_{n\geq 0}$ consisting of a temporally homogeneous Markov chain $M = (M_n)_{n\geq 0}$ with arbitrary state space $(E, \mathcal{E})$ and a sequence $(S_n)_{n\geq 0}$ of real random variables, whose increments $X_0, X_1, \cdots$, say, are distributionally governed by M. The latter means that $X_0, X_1, \cdots$, are conditionally independent given M and that the conditional distribution of X_n given M depends only on M_{n-1} and M_n for $n \geq 1$ (on M_0 alone, if $n = 0$). The special case, where M is constant, leads back to ordinary random walks having i.i.d. increments. Since Markov modulation, as opposed to the i.i.d. case, offers greater flexibility in the modeling of fluctuations of additive random sequences without losing too much structural homogeneity, it is not surprising that MRW's have become a popular tool to provide more flexible and thus realistic models in areas like risk theory and queueing theory. The special case of finite modulation (E finite) has been extensively studied by various authors like Pyke, Cinlar and Arjas, and there is now a well developed theory for this case as to renewal and fluctuation theoretic aspects. Roughly speaking, if M has finite state space, then much of the theory can be obtained in an elegant manner via regenerative decomposition and subsequently resort to classical results for ordinary random walks. Unfortunately, this is not true to the same extent, when M has infinite, possibly uncountable state space, whence the theory in this case has not yet reached comparable maturity. This chapter deals with MRW's driven by general state space Markov chains including reflected MRW's. Throughout this chapter, the driving chain M is always assumed to be positive Harris recurrent with general state space $(E, \mathcal{E})$ and a unique stationary distribution ξ, unless stated otherwise.

2.1 Markov random walks

In this section we review the fundamental aspects of MRW's driven by general Markov chains. Our discussion is based on Arjas [6, 7], Arjas and Speed [8, 9].

2.1.1 Definitions

A mapping $\mathbf{K} : E \times (\mathcal{E} \times \mathcal{B}) \to [0,1]$ is called a *semi-Markov transition kernel*, if

(i) $s \mapsto \mathbf{K}(s, A \times B)$ is bounded, $\mathcal{E} \times \mathcal{B}$-measurable for any $A \times B \in \mathcal{E} \otimes \mathcal{B}$;

(ii) $A \times B \mapsto \mathbf{K}(s, A \times B)$ is a probability measure on $\mathcal{E} \times \mathcal{B}$ for any $s \in E$.

We define the composition $\mathbf{K}_1\mathbf{K}_2$ of semi-Markov transition kernels $\mathbf{K}_1$ and $\mathbf{K}_2$ as

$$(\mathbf{K}_1\mathbf{K}_2)(s, A\times B) := \int_E \int_{\mathbb{R}} \mathbf{K}_2(s', A\times(B-x))\, \mathbf{K}_1(s, ds'\times dx), \quad s \in E, A\times B \in \mathcal{E}\otimes\mathcal{B}.$$

The n-step iterates of a semi-Markov transition kernel $\mathbf{K}, n \geq 0$, are defined recursively

$$\mathbf{K}^0 = \mathbf{I} \quad \text{and} \quad \mathbf{K}^n = \mathbf{K}^{n-1}\mathbf{K}, \quad n \geq 1,$$

where the kernel $\mathbf{I}$ is defined as

$$\mathbf{I}(s, A \times B) = \delta_s(A)\, \delta_0(B), \quad s \in E, A \times B \in \mathcal{E} \otimes \mathcal{B}.$$

A *Markov random walk* (MRW) or *Markov additive process* (MAP)[1] is a bivariate Markov chain $(M_n, S_n)_{n\geq 0}$ with transition kernel $\mathbf{Q}$ of the form

$$\mathbf{Q}((s,x), A \times B) = \mathbf{K}(s, A \times (B - x)), \quad (s,x) \in E \times \mathbb{R}, A \times B \in \mathcal{E} \otimes \mathcal{B},$$

for some semi-Markov transition kernel $\mathbf{K}$.
For any $B \in \mathcal{B}$, define the operator $\mathbf{Q}(B)$ on the set of nonnegative measurable functions $\mathcal{F}_+(E)$ as

$$(\mathbf{Q}(B)f)(s) := \int_E f(s')\, \mathbf{K}(s, ds' \times B), \quad s \in E, f \in \mathcal{F}_+(E).$$

[1]A continuous-time Markov additive process can be defined in a similar manner. Let $\{\mathbf{K}_t : t \geq 0\}$ be a family of semi-transition kernels such that

$$\mathbf{K}_t(s, A \times B) = \int_{E\times\mathbb{R}} \mathbf{K}_{t-t'}(s', A \times (B - x))\, \mathbf{K}_{t'}(s, ds' \times dx)$$

for any $t' < t, t, t' \in \mathbb{R}_0^+, s, s' \in E, x \in \mathbb{R}, A \in \mathcal{E}, B \in \mathcal{B}$. A *Markov additive process* $(M_t, S_t)_{t\geq 0}$ is a bivariate Markov process with transition semigroup $(\mathbf{Q}_t)_{t\geq 0}$ defined as

$$\mathbf{Q}_t(s, x; A \times B) = \mathbf{K}_t(s, A \times (B - x)), \quad (s,x) \in E \times \mathbb{R}, A \times B \in \mathcal{E} \otimes \mathcal{B}, t \geq 0$$

for a family of semi-transition kernels $\{\mathbf{K}_t : t \geq 0\}$. It is clear that $(M_t)_{t\geq 0}$ is a Markov process with the transition semigroup $(\mathbf{Q}_t^M)_{t\geq 0}$ defined as

$$\mathbf{Q}_t^M(s, A) = \mathbf{Q}_t(s, A \times \mathbb{R}), \quad s \in E, A \in \mathcal{E}, t \geq 0.$$

Furthermore, it is known that, given the process $(M_t)_{t\geq 0}$, the process $(S_t)_{t\geq 0}$ has independent increments, that is,

$$\mathbb{E}\big[\Pi_{i=1}^n h_i(S_{t_i} - S_{t_{i-1}})|\mathcal{F}\big] = \Pi_{i=1}^n \mathbb{E}\big[h_i(S_{t_i} - S_{t_i-1})|\mathcal{F}\big],$$

for any $n \geq 1, 0 \leq t_0 < t_1 < \cdots < t_n$ and bounded measurable functions $h_1 \cdots, h_n$ on E, where $\mathcal{F}$ denotes the canonical filtration for $(M_t)_{t\geq 0}$.

In particular, we have for any $A \in \mathcal{E}$

$$(\mathbf{Q}(B)\mathbf{1}_A)(s) = \mathbf{K}(s, A \times B), \quad s \in E.$$

A MRW $(M_n, S_n)_{n\geq 0}$ is called a *Markov renewal process*, if the increments $S_{n+1}-S_n, n \geq 0$, of its additive part are a.s. positive, i.e., if $\mathbf{Q}((s,0), E \times (0,\infty)) = 1$ for all $s \in E$. The Markov chain $(M_n)_{n\geq 0}$ is called the *driving chain* or *underlying Markov chain.* Obviously a renewal process is equivalent to the special case of a Markov renewal process with a one-state driving chain. If $(N(t))_{t\geq 0}$ denotes the counting process for a Markov renewal process $(M_n, S_n)_{n\geq 0}$, i.e.,

$$N(t) := \sup\{n \geq 0 : S_n \leq t\}, \quad t \geq 0,$$

then the process $(M_t^S)_{t\geq 0}$ defined by $M_t^S = M_{N(t)}, t \geq 0$, is called a *semi-Markov process.*

The sequence of increments $X_n := S_n - S_{n-1}, n \geq 1$, of the additive part plays an important role in the theory of MRW's. Putting $X_0 = S_0$, the process $(M_n, X_n)_{n\geq 0}$ forms a temporally homogeneous Markov chain with transition kernel $\mathbf{P} : E\times(\mathcal{E}\otimes\mathcal{B}) \to [0,1]$ satisfying

$$\mathbf{P}(s, A \times B) = \mathbb{P}\left[M_1 \in A, X_1 \in B | M_0 = s\right], \quad s \in E, A \times B \in \mathcal{E} \otimes \mathcal{B}.$$

One can easily see that (M_{n+1}, X_{n+1}) depends on the past only through M_n for each $n \geq 0$ and that $(M_n)_{n\geq 0}$ forms a Markov chain with state space E and the transition kernel $\mathbf{P}_M$ defined as

$$\mathbf{P}_M(s, A) := \mathbf{P}(s, A \times \mathbb{R}), \quad s \in E, A \in \mathcal{E}.$$

Given $(M_n)_{n\geq 0}$, the $X_n, n \geq 0$, are conditionally independent with

$$\mathbb{P}\left[X_n \in B | (M_j)_{j\geq 0}\right] = \mathbf{F}(M_{n-1}, M_n, B) \quad \mathbb{P} - \text{a.s.},$$

for all $n \geq 1, B \in \mathcal{B}$ and a kernel $\mathbf{F} : E^2 \times \mathcal{B} \to [0,1]$. The process $(M_n, X_n)_{n\geq 0}$ is called a *Markov modulated sequence* with the *driving chain* $(M_n)_{n\geq 0}$.

Let throughout a canonical model be given with probability measures $\mathbb{P}_{s,x}, (s,x) \in E \times \mathbb{R}$, on $(\Omega, \mathcal{S})$ such that $\mathbb{P}_{s,x}(M_0 = s, X_0 = x) = 1$. For any distribution λ on $E \times \mathbb{R}$, define

$$\mathbb{P}_\lambda(\cdot) := \int_{E\times\mathbb{R}} \mathbb{P}_{s,x}(\cdot)\, \lambda(ds, dx),$$

in which case (M_0, X_0) has the initial distribution λ. The expectation under $\mathbb{P}_\lambda$ is denoted by $\mathbb{E}_\lambda$. For $s \in E$ and an initial distribution λ on E, we write $\mathbb{E}_s$ and $\mathbb{E}_\lambda$ instead of $\mathbb{E}_{s,0}$ and $\mathbb{E}_{\lambda\otimes\delta_0}$, respectively.

For any $C \in \mathcal{B}$, $\sigma(C)$ denotes the first return time of $(M_n, W_n)_{n\geq 0}$ to $E \times C$. For each fixed $C \in \mathcal{B}$ and $n \geq 1$, we define the probability distributions $\mathbf{H}_C^{(n)}(s, A; B)$ and $\mathbf{G}_C^{(n)}(s, A; B)$ as

$$\begin{aligned} \mathbf{H}_C^{(n)}(s, A; B) &= \mathbb{P}_s(M_n \in A, S_n \in B, \sigma(C) > n); \\ \mathbf{G}_C^{(n)}(s, A; B) &= \mathbb{P}_s(M_n \in A, S_n \in B, \sigma(C) = n), \quad s \in E, A \in \mathcal{E}, B \in \mathcal{B}. \end{aligned}$$

One can easily see that

$$\int_E \int_{\mathbb{R}} \mathbf{Q}(s', A \times (B - x'))\, \mathbf{H}_C^{(n)}(s, ds'; dx') = \mathbf{H}_C^{(n+1)}(s, A; B) + \mathbf{G}_C^{(n+1)}(s, A; B).$$

Define the corresponding transforms $\hat{\mathbf{H}}_C^{(\alpha,\beta)}$ and $\hat{\mathbf{G}}_C^{(\alpha,\beta)}$ as

$$\begin{aligned}
\hat{\mathbf{H}}_C^{(\alpha,\beta)}(s, A) &= \sum_{n=0}^{\infty} \alpha^n \int_0^{\infty} e^{\beta x}\, \mathbf{H}_C^{(n)}(s, A; dx) \\
&= \mathbb{E}_s \left(\sum_{n=0}^{\sigma(C)-1} \alpha^n\, e^{\beta S_n}; M_n \in A \right); \\
\hat{\mathbf{G}}_C^{(\alpha,\beta)}(s, A) &= \sum_{n=0}^{\infty} \alpha^n \int_0^{\infty} e^{\beta x}\, \mathbf{G}_C^{(n)}(s, A; dx) \\
&= \mathbb{E}_s \left(\alpha^{\sigma(C)} e^{\beta S_{\sigma(C)}}; M_{\sigma(C)} \in A, \sigma(C) < \infty \right)
\end{aligned}$$

for any $s \in E, A \in \mathcal{E}$. Further, define

$$\mathbf{H}_C(s, A; B) = \sum_{n=0}^{\infty} \mathbf{H}_C^{(n)}(s, A; B) \quad \text{and} \quad \mathbf{G}_C(s, A; B) = \sum_{n=0}^{\infty} \mathbf{G}_C^{(n)}(s, A; B).$$

In particular, if $B = \mathbb{R}$, then we write $\mathbf{H}_C(s, A)$ and $\mathbf{G}_C(s, A)$ instead of $\mathbf{H}_C(s, A; \mathbb{R})$ and $\mathbf{G}_C(s, A; \mathbb{R})$, respectively. Obviously, we have

$$\mathbf{H}_C(s, A) = \hat{\mathbf{H}}_C^{(1,0)}(s, A) \quad \text{and} \quad \mathbf{G}_C(s, A) = \hat{\mathbf{G}}_C^{(1,0)}(s, A), \quad s \in E, A \in \mathcal{E}.$$

2.1.2 The Harris recurrence of Markov modulated sequences

Nummelin [45] has shown that a Markov modulated sequence $(M_n, X_n)_{n \geq 0}$ is positive Harris recurrent and that the measure ν defined as

$$\nu(A \times B) := \int_E \mathbb{P}(s, A \times B)\, \xi(ds), \quad A \times B \in \mathcal{E} \otimes \mathcal{B},$$

is a unique stationary distribution for $(M_n, X_n)_{n \geq 0}$. Furthermore, a coupling argument shows that $(M_n, X_n)_{n \geq 0}$ is also Harris ergodic, provided that M is Harris ergodic.

Sometimes one needs to consider the sequence $(M_n, X_{n+1})_{n \geq 0}$. In this case, it turns out that regeneration epochs for M are also regeneration epochs for $(M_n, X_{n+1})_{n \geq 0}$.

Proposition 2.1 *Let $(\tau_n)_{n \geq 0}$ be a sequence of regeneration epochs for $(M_n)_{n \geq 0}$. Then the sequence $(M_n, X_{n+1})_{n \geq 0}$ is one-dependent as well as wide-sense regenerative with regeneration epochs $\tau_n, n \geq 0$, and for any initial distribution λ*

$$\mathbb{P}_\zeta \left((M_k, X_{k+1})_{k \geq 0} \in \cdot \right) = \mathbb{P}_\lambda \left((M_{\tau_n+k}, X_{\tau_n+k+1})_{k \geq 0} \in \cdot \right), \quad n \geq 1,$$

where $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\tau_1} \in \cdot\,)$.
If the state space E *is countable, then* $(M_n, X_{n+1})_{n\geq 0}$ *is classical-sense regenerative.*

Proof. Consider the cycles Z_n defined as

$$Z_n := \left(\tau_{n+1} - \tau_n, (M_k, X_{k+1})_{\tau_n \leq k < \tau_{n+1}}\right), \quad n \geq 0.$$

Obviously the sequence $(Z_k)_{k\geq n}$ does not depend on $(\tau_0, \tau_1, \cdots, \tau_n)$ for any $n \geq 1$. Moreover, by conditional independence of $(X_n)_{n\geq 0}$ given $(M_n)_{n\geq 0}$, one can easily see that for any $n \geq 1$ and any initial distribution λ

$$\mathbb{P}_\lambda(Z_n \in \cdot\,) = \int_E \mathbb{P}_s(Z_0 \in \cdot\,)\, \mathbb{P}_\lambda(M_{\tau_n} \in ds) = \mathbb{P}_\zeta(Z_0 \in \cdot),$$

thus $(M_n, X_{n+1})_{n\geq 0}$ is wide-sense regenerative. On the other hand, for any initial distribution λ and for any $n \geq 0$

$$\begin{aligned} \mathbb{P}_\lambda\left[Z_{n+2} \in \cdot\,|\mathcal{F}_{\tau_{n+1}}\right] &= \mathbb{P}\left[(\tau_{n+3} - \tau_{n+2}, (M_k, X_{k+1})_{\tau_{n+2}\leq k<\tau_{n+3}}) \in \cdot\,|M_{\tau_{n+1}}\right] \\ &= \mathbb{P}_\zeta(Z_0 \in \cdot\,), \end{aligned}$$

where $(\mathcal{F}_n)_{n\geq 0}$ is the canonical filtration for the sequence $(M_n, X_n)_{n\geq 0}$. Since Z_n is $\mathcal{F}_{\tau_{n+1}}$-measurable, the cycles $Z_n, n \geq 0$, are one-dependent. In particular, if M is discrete, then there exists a recurrent state i_0. Thus for any $i \in E$

$$\begin{aligned} \mathbb{P}_i\left[\mathbf{Z}_{n+1} \in \cdot\,|\mathcal{F}_{\tau_{n+1}}\right] &= \mathbb{P}\left[(\tau_{n+2} - \tau_{n+1}, (M_k, X_{k+1})_{\tau_{n+1}\leq k<\tau_{n+2}}) \in \cdot\,|M_{\tau_{n+1}}\right] \\ &= \mathbb{P}_{i_0}(Z_0 \in \cdot\,), \end{aligned}$$

which means that the cycles are independent. QED

Note that for any sequence of regeneration epochs $(\tau_n)_{n\geq 0}$ for $(M_n)_{n\geq 0}$

$$\mathbb{E}_\xi X_1 = \frac{1}{\mathbb{E}_\zeta\, \tau_1}\, \mathbb{E}_\zeta \left(\sum_{n=0}^{\tau_1 - 1} X_{n+1}\right) = \frac{1}{\mathbb{E}_\zeta\, \tau_1}\, \mathbb{E}_\zeta \left(\sum_{n=1}^{\tau_1} X_n\right),$$

where $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\tau_1} \in \cdot\,)$ for any initial distribution λ.

The SLLN for a MRW $(M_n, S_n)_{n\geq 0}$ is a direct consequence of the Harris recurrence of $(M_n, X_n)_{n\geq 0}$ and Proposition 1.16. But we give a full proof, in which the structure of one-dependence in MRW's is exploited.

Proposition 2.2 (SLLN for MRW's) *Given a Markov modulated sequence* $(M_n, X_n)_{n\geq 0}$, *it holds that*

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}_\xi\, X_1 \quad \mathbb{P}_\lambda - a.s.$$

for any initial distribution λ *on* $E \times \mathbb{R}$.

Proof. Let $(\tau_n)_{n\geq 0}$ be a sequence of regeneration epochs of $(M_n)_{n\geq 0}$ and let $\zeta(\cdot) = \mathbb{P}_\lambda\left(M_{\tau_1} \in \cdot\right)$. We note first that for any initial distribution λ

$$\mathbb{P}_\lambda\left(\lim_{n\to\infty}\frac{S_n}{n} = \mathbb{E}_\xi X_1\right) = \mathbb{P}_\lambda\left(\lim_{n\to\infty}\frac{S_{\tau_1+n}-S_{\tau_1}}{n} = \mathbb{E}_\xi X_1\right) = \mathbb{P}_\zeta\left(\lim_{n\to\infty}\frac{S_n}{n} = \mathbb{E}_\xi X_1\right).$$

Therefore it is sufficient to show the assertion only for $\mathbb{P}_\zeta$.
By assumption, the sequence $(S_n^*)_{n\geq 1} := (S_{\tau_n})_{n\geq 1}$ has stationary increments $X_n^* = \sum_{k=\tau_{n-1}+1}^{\tau_n} X_k, n \geq 1$, which is one-dependent under $\mathbb{P}_\zeta$ and in turn, by Birkhoff's ergodic theorem, we have as $n \to \infty$

$$\frac{S_n^*}{n} \to \mathbb{E}_\zeta X_1^* = \mathbb{E}_\zeta\left(\sum_{k=1}^{\tau_1} X_k\right) = \mathbb{E}_\zeta\,\tau_1 \cdot \mathbb{E}_\xi X_1 \quad \mathbb{P}_\zeta - \text{a.s.}$$

Let $T(n) := \inf\{k \geq 0 : \tau_k > n\}$. Then as $n \to \infty$

$$\frac{T(n)}{n} \to \frac{1}{\mathbb{E}_\zeta\,\tau_1} \quad \mathbb{P}_\zeta - \text{a.s.},$$

whence

$$\frac{S_{T(n)}^*}{n} = \frac{T(n)}{n}\cdot\frac{S_{T(n)}^*}{T(n)} \to \mathbb{E}_\xi\, X_1 \quad \text{and} \quad \frac{S_{T(n)-1}^*}{n} \to \mathbb{E}_\xi X_1 \quad \mathbb{P}_\zeta - \text{a.s.}$$

The assertion follows from the inequality

$$\frac{S_{T(n)-1}^*}{n} \leq \frac{S_n}{n} \leq \frac{S_{T(n)}^*}{n}.$$

QED

Let $\mu := \mathbb{E}_\xi X_1$, which is called *drift* of the MRW $(M_n, S_n)_{n\geq 0}$. As a direct consequence of Proposition 2.2, we get for any initial distribution λ

$$\mu < 0 \;\Rightarrow\; \lim_{n\to\infty} S_n = -\infty \quad \mathbb{P}_\lambda - \text{a.s.}; \tag{2.1}$$

$$\mu > 0 \;\Rightarrow\; \lim_{n\to\infty} S_n = +\infty \quad \mathbb{P}_\lambda - \text{a.s.} \tag{2.2}$$

2.1.3 Maximum of MRW's

Suppose that $S_0 = 0$ hereafter and put

$$\sigma^> := \sigma((0,\infty)), \quad \sigma^\geq := \sigma([0,\infty)), \quad \sigma^< := \sigma((-\infty,0)) \quad \text{and} \quad \sigma^\leq := \sigma((-\infty,0]),$$

which are called *the first strict ascending ladder epoch, the first weak ascending ladder epoch, the first strict descending ladder epoch* and *the first weak descending ladder epoch*, respectively. If σ^* is a.s. finite for $* \in \{\geq, >, \leq, <\}$, one can also define, in an obvious manner, the nth ladder epochs $\sigma_n^*, n \geq 1$, with $\sigma_1^* = \sigma^*$. Clearly $\sigma_n^*, n \geq 1$, are stopping times. For notational convenience, we write $\mathbf{H}_>^{(n)}$, $\mathbf{H}_\geq^{(n)}$, $\mathbf{H}_<^{(n)}$ and $\mathbf{H}_\leq^{(n)}$ instead of $\mathbf{H}_{(0,\infty)}^{(n)}$, $\mathbf{H}_{[0,\infty)}^{(n)}$, $\mathbf{H}_{(-\infty,0)}^{(n)}$ and $\mathbf{H}_{(-\infty,0]}^{(n)}$, respectively. The same notational conventions are used also for $\mathbf{G}_{(0,\infty)}^{(n)}$, $\mathbf{G}_{[0,\infty)}^{(n)}$, $\mathbf{G}_{(-\infty,0)}^{(n)}$ and $\mathbf{G}_{(-\infty,0]}^{(n)}$ and their transforms.

Consider the maximum of the partial sums

$$\overline{S}_n := \max_{0\leq k\leq n} S_k.$$

Noting that, for $0 \leq m \leq n$, S_m is maximal among the first n partial sums if and only if S_m is the last strict ascending ladder height before n, it can be easily seen that for all $|\alpha|, |\beta| < 1$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \alpha^n \int_E e^{\beta x} \mathbb{P}_s\left(M_n \in A, \overline{S}_n \in dx\right) \\
&= \sum_{n=0}^{\infty} \alpha^n \sum_{m=0}^{n} \int_E e^{\beta x}\, \mathbb{P}\left[M_n \in A, S_{m+1} - S_m \leq 0, \cdots, S_n - S_m \leq 0 | M_m = s'\right] \\
&\qquad \cdot \mathbb{P}_s\left(M_m \in ds', S_m \in dx, S_m \geq 0, S_m > S_1, \cdots, S_m > S_{m-1}\right).
\end{aligned}$$

Thus we get the following equality, which was obtained by Arjas [7]:

$$\sum_{n=0}^{\infty} \alpha^n\, \mathbb{E}_s\left(e^{\beta \overline{S}_n}; M_n \in A\right) \;=\; \sum_{n=0}^{\infty} \int_E \hat{\mathbf{H}}_>^{(\alpha,0)}(s', A) \left(\hat{\mathbf{G}}_>^{(\alpha,\beta)}\right)^n (s, ds'). \tag{2.3}$$

Now consider

$$\overline{S} := \sup_{n\geq 0} \overline{S}_n.$$

For any $s \in E, A \in \mathcal{E}$ and $x \in \mathbb{R}$, we define

$$\begin{aligned}
\mathbf{G}_>^0(s, A; (-\infty, x)) &= \delta_s(A)\delta_0((-\infty, x)); \\
\mathbf{G}_>^n(s, A; (-\infty, x)) &= \int_E \int_0^\infty \mathbf{G}_>(s', A; (-\infty, x - y))\, \mathbf{G}_>^{n-1}(s, ds'; dy), \quad n \geq 1.
\end{aligned}$$

Proposition 2.3 *If $\mu < 0$, then $\overline{S}$ is a.s. finite. Let $\tau := \inf\{n : S_n = \overline{S}\}$. Then for any $s \in E, A \in \mathcal{E}$ and $x \in \mathbb{R}$*

$$\mathbb{P}_s\left(M_\tau \in A, S_\tau < x\right) = \sum_{n=0}^{\infty} \int_A \left(1 - \mathbf{G}_>(s', E)\right)\, \mathbf{G}_>^n(s, ds'; (-\infty, x)) \tag{2.4}$$

and

$$\mathbb{E}_s\, e^{\beta \overline{S}} = \sum_{n=0}^{\infty} \int_E \left(1 - \mathbf{G}_>(s', E)\right) \left(\hat{\mathbf{G}}_>^{(1,\beta)}\right)^n (s, ds'). \tag{2.5}$$

Proof. Obviously $S_\tau = \sup\{S_{\sigma_n^>} : \sigma_n^> < \infty\}$. Furthermore, by (2.1), S_τ is $\mathbb{P}_{s,x}$-a.s. finite for any $(s,x) \in E \times \mathbb{R}$. Thus the probability that $\overline{S}$ is obtained in precisely n ladder steps and at state in A and does not exceed x is given by

$$\int_A \left(1 - \mathbf{G}_>(s', E)\right) \mathbf{G}_>^n(s, ds'; (-\infty, x)).$$

Summing over all n, we get the equality (2.4), and multiplying by β^n and summing over all n, the equality (2.5) follows. QED

2.1.4 Time-reversal

Definition 2.4 A Markov chain $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$ with transition kernel $\tilde{\mathbf{Q}}$ is called the *time-reversal* of (M_n, S_n), if, for any $B \in \mathcal{B}$, $\mathbf{Q}(B)$ and $\tilde{\mathbf{Q}}(B)$ are adjoint with respect to ξ, i.e., for any $f, g \in \mathcal{F}_+(E)$

$$\langle f, \mathbf{Q}(B) g\rangle_\xi = \langle \tilde{\mathbf{Q}}(B) f, g\rangle_\xi.$$

Obviously $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$ forms a MRW with driving chain $(\tilde{M}_n)_{n\geq 0}$.
Let $\hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}$ be the kernel defined as

$$\hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}(s, A) \quad = \quad \mathbb{E}\left[\alpha^{\tilde{\sigma}^>} e^{\beta \tilde{S}_{\tilde{\sigma}^>}}; \tilde{M}_{\tilde{\sigma}^>} \in A, \tilde{\sigma}^> < \infty | \tilde{M}_0 = s, \tilde{S}_0 = 0\right], \quad s \in E, A \in \mathcal{E},$$

where $\tilde{\sigma}^>$ is the first strict descending ladder epoch for the MRW $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$. The following assertion is due to Arjas and Speed [9].

Proposition 2.5 *The kernels* $\hat{\mathbf{H}}_{\leq}^{(\alpha,\beta)}$ *and* $\sum_{n=0}^{\infty} \left(\hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}\right)^n$ *are mutually adjoint with respect to* ξ, *i.e.,*

$$\langle f, \hat{\mathbf{H}}_{\leq}^{(\alpha,\beta)} g\rangle_\xi = \langle \sum_{n=0}^{\infty} \left(\hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}\right)^n f, g\rangle_\xi$$

for any $f, g \in \mathcal{F}_+(E)$.

Proof. Let

$$\tilde{\mathbf{G}}^{(n)}(s, A \times B) \quad = \quad \mathbb{P}\left[\tilde{M}_n \in A, \tilde{S}_n - \tilde{S}_k > 0, k < n, \tilde{S}_n \in B | \tilde{M}_0 = s, \tilde{S}_0 = 0\right].$$

Then for any $f, g \in \mathcal{F}_+(E)$

$$\begin{aligned} \langle f, \mathbf{H}_{\leq}^{(n)}(\cdot\,,\cdot; B) g\rangle_\xi &= \mathbb{E}_\xi \left(f(M_0) g(M_n) \mathbf{1}_B(S_n) \mathbf{1}_{\{S_1>0,\cdots,S_n>0\}}\right) \\ &= \mathbb{E}_\xi \left(f(\tilde{M}_n) g(\tilde{M}_0) \mathbf{1}_B(\tilde{S}_n) \mathbf{1}_{\{\tilde{S}_n - \tilde{S}_k>0,\cdots,\tilde{S}_n>0\}}\right) \\ &= \langle \tilde{\mathbf{G}}^{(n)}(\cdot\,,\cdot \times B) f, g\rangle_\xi, \end{aligned}$$

because for any $n \geq 0$

$$(M_k, S_k)_{0\leq k\leq n} \sim (\tilde{M}_{n-k}, \tilde{S}_n - \tilde{S}_{n-k})_{0\leq k\leq n}.$$

Note that for any $s \in E, A \in \mathcal{E}$

$$\begin{aligned}
&\sum_{n=0}^{\infty} \alpha^n \int e^{\beta x} \tilde{\mathbf{G}}^{(n)}(s, A \times dx) \\
&\quad = \sum_{n=0}^{\infty} \alpha^n \int e^{\beta x} \mathbb{P}\left[\tilde{M}_n \in A, \tilde{S}_n - \tilde{S}_k > 0, k < n, \tilde{S}_n \in B | \tilde{M}_0 = s, \tilde{S}_0 = 0\right] \\
&\quad = \sum_{n=0}^{\infty} \alpha^n \int_0^{\infty} e^{\beta x} \mathbb{P}\left[\tilde{M}_n \in A, n = \tilde{\sigma}_m^{>} \text{ for some } m \geq 1, \tilde{S}_n \in dx | \tilde{M}_0 = s, \tilde{S}_0 = 0\right] \\
&\quad = \sum_{n=0}^{\infty} \left(\hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}\right)^n (s, A),
\end{aligned}$$

where $\tilde{\sigma}_m^{>}$ denotes the mth strict ascending ladder epoch of the MRW $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$. The assertion follows from

$$\begin{aligned}
\langle f, \hat{\mathbf{H}}_{\leq}^{(\alpha,\beta)} g\rangle_\xi &= \langle f, \sum_{n=0}^{\infty} \alpha^n \int e^{\beta x} \mathbf{H}_{\leq}^{(n)}(\,\cdot\,, \cdot\,; dx) g\rangle_\xi \\
&= \sum_{n=0}^{\infty} \alpha^n \int e^{\beta x} \langle f, \mathbf{H}_{\leq}^{(n)}(\,\cdot\,, \cdot\,; dx) g\rangle_\xi \\
&= \sum_{n=0}^{\infty} \alpha^n \int e^{\beta x} \langle \tilde{\mathbf{G}}^{(n)}(\,\cdot\,, \cdot \times dx) f, g\rangle_\xi \\
&= \langle \sum_{n=0}^{\infty} \left(\hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}\right)^n f, g\rangle_\xi.
\end{aligned}$$

QED

If the state space E is countable, then the time-reversal $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$ is defined by the transition kernel matrix $\tilde{\mathbf{Q}} = (\tilde{q}_{ij})_{i,j\in E}$ with

$$\begin{aligned}
\tilde{q}_{ij}(B) &= \mathbb{P}\,[\tilde{M}_1 = j, \tilde{S}_1 \in B | \tilde{M}_0 = i, \tilde{S}_0 = 0] \\
&= \mathbb{P}\,[M_1 = i, S_1 \in B | M_0 = j, S_0 = 0], \quad B \in \mathcal{B}.
\end{aligned}$$

In this case, Proposition 2.5 can be written in the matrix form:

$$\left(I - \hat{\mathbf{G}}_{R(>)}^{(\alpha,\beta)}\right)^{-1} = \Delta_\xi^{-1} \hat{\mathbf{H}}_{\leq}^{(\alpha,\beta)} \Delta_\xi,$$

where I denotes the identity matrix.

2.2 Reflected Markov random walks

A reflected MRW appears naturally as the waiting time process in a Markov modulated queueing system. As known a reflected random walk $(W_n)_{n\geq 0}$ with negative drift is classical-sense regenerative, where the weak descending ladder epochs of the associated random walk are regeneration epochs. In that case many problems on reflected random walks can be attacked by the analysis of the weak descending ladder epoch. In the Markov modulated case, it is not true to the same extent in general. But Alsmeyer [4] showed that a reflected MRW with negative drift possesses a sequence of regeneration epochs, which are expressed in terms of the weak descending ladder epochs $\sigma_n, n \geq 0$, and regeneration epochs for $(M_{\sigma_n})_{n\geq 0}$.

In this section we review some elementary properties of reflected MRW's and examine their regeneration. Throughout this section we assume that a Markov modulated sequence $(M_n, X_n)_{n\geq 0}$ is given, and $(M_n, S_n)_{n\geq 0}$ denotes the associated MRW, where $S_n = X_0 + \cdots + X_n$ for each $n \geq 0$.

2.2.1 Reflected MRW's and their basic properties

A process $(M_n, W_n)_{n\geq 0}$ is called the *reflected Markov random walk* (reflected MRW) associated to $(M_n, X_n)_{n\geq 0}$, if it satisfies the recursive equation

$$W_0 = S_0^+ \quad \text{and} \quad W_{n+1} = (W_n + X_{n+1})^+, \quad n \geq 0,$$

where $X^+ = \max(X, 0)$. Obviously it is a Markov chain with transition kernel $\mathbf{\Phi}$ defined as

$$\mathbf{\Phi}((s,x), A \times B) = \mathbf{1}_B(0)\, \mathbb{P}_s(A \times (-\infty, -x)) + \mathbb{P}_s(A \times (B - x))$$

for $(s,x) \in E \times \mathbb{R}^+, A \times B \in \mathcal{E} \otimes \mathcal{B}^+$.

Let $\sigma_n, n \geq 0$, be the weak descending ladder epochs defined as

$$\sigma_0 := \inf\{n \geq 0 : S_n \leq 0\} \quad \text{and} \quad \sigma_n := \inf\{k > \sigma_{n-1} : S_k \leq S_{\sigma_{n-1}}\}, \quad n \geq 1.$$

The definition is slightly different from the definition in ordinary zero delayed random walks, in which σ_0 is defined to be 0. Obviously $\sigma_n, n \geq 0$, are a.s. finite stopping times, if the drift of the MRW $(M_n, S_n)_{n\geq 0}$ is negative. In the rest of this chapter we suppose that $\mu < 0$, unless stated otherwise.

Proposition 2.6 *Let $(M_n, W_n)_{n\geq 0}$ be the reflected MRW associated to $(M_n, X_n)_{n\geq 0}$. Then the following assertions hold true:*

(i) For any $n \geq 0$

$$W_n = \max_{k\leq n}(W_0 + S_n - S_0, S_n - S_k) = \max_{k\leq n-1}(W_0 - X_0 + S_n, S_n - S_k)^+, \quad n \geq 0.$$

(ii) $E \times \{0\}$ is a recurrent set of $(M_n, W_n)_{n\geq 0}$. Furthermore, it holds that

$$\sigma_0 = \inf\{k \geq 0 : W_k = 0\} \quad \textit{and} \quad \sigma_n = \inf\{k > \sigma_{n-1} : W_k = 0\}, \quad n \geq 1.$$

Proof. For (i) see Proposition III.6.3 in Asmussen [13]. (ii) is obvious. QED

By Proposition 2.6, for any $n \geq 0$

$$\begin{aligned}&\mathbb{P}_s\left(M_n \in A, W_n \leq x\right)\\ &= \sum_{l=0}^{n}\sum_{k=0}^{l}\int_E \mathbb{P}_s(M_l \in ds', \sigma_k = l)\mathbb{P}\left[M_n \in A, S_n - S_l \leq x, \sigma_{k+1} - \sigma_k > n - l | M_l = s'\right].\end{aligned}$$

Multiplying the above equation by α^n and summing over all n,

$$\sum_{n=0}^{\infty} \alpha^n\, \mathbb{E}_s\left(e^{\beta W_n}; M_n \in A\right) = \sum_{n=0}^{\infty}\int_E \hat{\mathbf{H}}_{\leq}^{(\alpha,\beta)}(s', A)\left(\hat{\mathbf{G}}_{\leq}^{(\alpha,0)}\right)^n(s, ds'). \tag{2.6}$$

for any $s \in E$.

Given a bivariate Markov chain $(M, T) = (M_n, T_n)_{n\geq 0}$ on $E \times [0, \infty)$, define the transform $\Lambda^{(\alpha,\beta)}_{(M,T)}$ as

$$\Lambda^{(\alpha,\beta)}_{(M,T)}(s, A) = \sum_{n=0}^{\infty} \alpha^n\, \mathbb{E}_s\left(e^{\beta T_n}; M_n \in A\right), \quad s \in E, A \in \mathcal{E}.$$

Two bivariate processes (M, T) and $(\tilde{M}, \tilde{T})$ on $E \times [0, \infty)$ are said to be mutually *adjoint*, if $\Lambda^{(\alpha,\beta)}_{(M,T)}$ and $\Lambda^{(\alpha,\beta)}_{(\tilde{M},\tilde{T})}$ are adjoint w.r.t. ξ. If two MRW's (M, S) and $(\tilde{M}, \tilde{S})$ are in time-reversal, then they are mutually adjoint.

The following assertions are due to Arjas and Speed [8, 9].

Corollary 2.7 *Let $(M_n, W_n)_{n\geq 0}$ be a reflected MRW associated to $(M_n, X_n)_{n\geq 0}$ with $W_0 = 0$. If $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$ is the time-reversal of $(M_n, S_n)_{n\geq 0}$, then the processes $(M_n, W_n)_{n\geq 0}$ and $(\tilde{M}_n, \max_{0\leq k\leq n} \tilde{S}_k)_{n\geq 0}$ are mutually adjoint.*

Proof. Let $\hat{\mathbf{H}}^{(\alpha,\beta)}_{R(>)}$ be a kernel defined as

$$\hat{\mathbf{H}}^{(\alpha,\beta)}_{R(>)}(s, A) = \mathbb{E}\left[\sum_{n=0}^{\tilde{\sigma}^{>}-1} \alpha^n\, e^{\beta \tilde{S}_n}; \tilde{M}_n \in A \middle| \tilde{M}_0 = s, \tilde{S}_0 = 0\right],$$

where $\tilde{\sigma}^{>}$ is the first strictly ascending ladder epoch of $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$. Then in a similar manner as in Proposition 2.5 one can show that the kernels $\sum_{n=0}^{\infty}\left(\hat{\mathbf{G}}_{\leq}^{(\alpha,0)}\right)^n$ and $\hat{\mathbf{H}}^{(\alpha,0)}_{R(>)}$ are mutually adjoint w.r.t. ξ. Thus the assertion follows from (2.3),(2.6) and Proposition 2.5. QED

The following proposition is due to Nummelin.

Proposition 2.8 *The reflected MRW $(M_n, W_n)_{n\geq 0}$ associated to $(M_n, X_n)_{n\geq 0}$ is positive Harris recurrent. It is further Harris ergodic, if the same holds true for M.*

Proof. See Nummelin [45]. For a simple proof using a coupling argument, see Asmussen [12]. Though the driving chain $(M_n)_{n\geq 0}$ is there assumed to be finite, the same arguments work also for the Markov chain with general state space. QED

Note that $E\times\{0\}$ is a recurrent set for $(M_n, W_n)_{n\geq 0}$ and thus contains a regeneration set $\Re \times \{0\}$ for a measurable set $\Re$. In this case the set $\Re$ is also a regeneration set of M.

Remark 2.9 If $\mu > 0$, then for any initial distribution λ

$$\lim_{n\to\infty} \frac{W_n}{n} = \mu \quad \mathbb{P}_\lambda - \text{a.s.},$$

since

$$\frac{S_n}{n} \to \mu > 0 \quad \mathbb{P}_\lambda - \text{a.s.} \quad \text{as} \quad n \to \infty$$

and thus there exists the smallest L such that $W_n > 0$ for $n > L$. Further,

$$\lim_{n\to\infty} \frac{W_n}{n} = \frac{S_n - S_L}{n} = \mu \quad \mathbb{P}_\lambda - \text{a.s.}$$

In particular, we have

$$\lim_{n\to\infty} W_n = \infty \quad \mathbb{P}_\lambda - \text{a.s.}$$

2.2.2 Regeneration

Let $(M_n, W_n)_{n\geq 0}$ be the reflected MRW associated to $(M_n, X_n)_{n\geq 0}$. Then $(M_n, W_n)_{n\geq 0}$ is positive Harris recurrent, and so there exists a sequence of regeneration epochs. Consider the subchain $M^\sigma := (M_n^\sigma)_{n\geq 0} := (M_{\sigma_n})_{n\geq 0}$, where $\sigma_n, n \geq 0$, are weak descending ladder epochs defined in 2.2.1. Obviously the sequence M^σ forms a temporally homogeneous Markov chain with transition kernel $\mathbf{G}_\leq$, where $\mathbf{G}_\leq$ is the kernel defined as

$$\mathbf{G}_\leq(s, A) = \hat{\mathbf{G}}_\leq^{(1,0)}(s, A), \quad s \in E, A \in \mathcal{E}.$$

Let $\Re \times \{0\}$ be a regeneration set of $(M_n, W_n)_{n\geq 0}$ and $(\hat{\tau}_n)_{n\geq 0}$ a sequence of regeneration epochs for it constructed by the splitting technique from $\Re \times \{0\}$. Then each $\hat{\tau}_n$ has the form $\sigma_{\bar{\tau}_n}$, where $\bar{\tau}_n$ is a stopping time w.r.t. the filtration $(\mathcal{F}_{\sigma_n})_{n\geq 0}$, where $(\mathcal{F}_n)_{n\geq 0}$ is a filtration such that $(M_n)_{n\geq 0}$ is Markov-adapted and each $\hat{\tau}_n$ a stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$. One can easily check that $(\bar{\tau}_n)_{n\geq 0}$ forms a sequence of regeneration epochs for the Markov chain $(M_n^\sigma)_{n\geq 0}$, which is thus Harris recurrent. Clearly, $\mathbb{E}_\zeta\, \bar{\tau}_1 \leq \mathbb{E}_\zeta\, \hat{\tau}_1 < \infty$, and thus $(M_n^\sigma)_{n\geq 0}$ is positive Harris recurrent. In fact, we have:

Proposition 2.10 *A sequence of random times $(\bar{\tau}_n)_{n\geq 0}$ is a sequence of regeneration epochs for M^σ if, and only if, $(\sigma_{\bar{\tau}_n})_{n\geq 0}$ is a sequence of regeneration epochs for $(M_n, W_n)_{n\geq 0}$.*

Furthermore, we get moment conditions for regeneration epoch of $(M_n, W_n)_{n\geq 0}$ in terms of moments of the weak descending ladder epochs and of the first return time to a regeneration set of M^σ:

Proposition 2.11 *Let $\bar{\Re} \times \{0\}$ be a regeneration set of $(M_n, W_n)_{n\geq 0}$. Let further $\bar{\tau}(\bar{\Re})$ denote the return time of $(M_n^\sigma)_{n\geq 0}$ to $\bar{\Re}$, i.e.,*

$$\bar{\tau}(\bar{\Re}) := \inf\left\{n > 0 : M_n^\sigma \in \bar{\Re}\right\}.$$

(i) Let $\alpha \geq 1$. Suppose that $\mathbb{E}[\sigma_1^\alpha | \mathcal{F}_{\sigma_0}]$ is bounded. If

$$\sup_{s\in\bar{\Re}} \mathbb{E}_s\, \sigma_1^\alpha < \infty \quad \textit{and} \quad \sup_{s\in\bar{\Re}} \mathbb{E}_s \left(\tau(\bar{\Re})\right)^\alpha < \infty,$$

then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\sup_{s\in\bar{\Re}} \mathbb{E}_s\, \hat{\tau}_1^\alpha < \infty$. If in addition M is Harris ergodic and $\alpha > 1$, then for some constant c

$$\|\mathbb{P}_\zeta\left((M_n, W_n) \in \cdot\,\right) - \pi\| \leq cn^{1-\alpha},$$

where π is a unique stationary distribution of $(M_n, W_n)_{n\geq 0}$ and $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\hat{\tau}_1} \in \cdot\,)$ for each λ on E.

(ii) Let $\gamma > 0$. Suppose that $\mathbb{E}\left[\exp(\gamma\sigma_1) | \mathcal{F}_{\sigma_0}\right]$ is bounded. If

$$\sup_{s\in\bar{\Re}} \mathbb{E}_s\, \exp(\gamma\sigma_1) < \infty \quad \textit{and} \quad \sup_{s\in\bar{\Re}} \mathbb{E}_s\, \exp(\gamma\bar{\tau}(\bar{\Re})) < \infty,$$

there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\sup_{s\in\bar{\Re}} \mathbb{E}_s\, \exp(\gamma'\hat{\tau}_1) < \infty$ for some $\gamma' > 0$. If in addition M is Harris ergodic, then for some constants c and $\gamma'' \in (0, \gamma']$

$$\|\mathbb{P}_\zeta\left((M_n, W_n) \in \cdot\,\right) - \pi\| \leq c\exp(-\gamma'' n).$$

Proof. Let $\hat{\tau}(\bar{\Re} \times \{0\})$ denote the return time of $(M_n, W_n)_{n\geq 0}$ to $\bar{\Re} \times \{0\}$. Then

$$\begin{aligned} \sup_{s\in\bar{\Re}} \mathbb{E}_s \left(\hat{\tau}(\bar{\Re} \times \{0\})\right)^\alpha &\leq \sup_{s\in\bar{\Re}} \mathbb{E}_s \left(\sum_{n=0}^{\bar{\tau}(\bar{\Re})-1} (\sigma_{n+1} - \sigma_n)\right)^\alpha; \\ \sup_{s\in\bar{\Re}} \mathbb{E}_s\, \exp\left(\gamma\hat{\tau}(\bar{\Re} \times \{0\})\right) &\leq \sup_{s\in\bar{\Re}} \mathbb{E}_s\, \exp\left(\gamma \sum_{n=0}^{\bar{\tau}(\bar{\Re})-1} (\sigma_{n+1} - \sigma_n)\right). \end{aligned}$$

Thus by Lemma 1.15 (i) $\sup_{s\in\bar{\Re}} \mathbb{E}_s \left(\hat{\tau}(\bar{\Re} \times \{0\})\right)^\alpha < \infty$ under the conditions of (i), and by Lemma 1.15 (ii) $\sup_{s\in\bar{\Re}} \mathbb{E}_s\, \exp\left(\gamma\hat{\tau}(\bar{\Re} \times \{0\})\right)$ under the conditions of (ii). All assertons follow from Proposition 1.16 and Corollary 1.22. QED

2.2.3 Stationary distribution

From the existence of a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ for $(M_n, W_n)_{n\geq 0}$ with $W_{\hat{\tau}_1} = 0$, we get a stationary distribution π of $(M_n, W_n)_{n\geq 0}$ given as

$$\pi(\cdot) = \frac{1}{\mathbb{E}_\zeta\, \hat{\tau}_1}\, \mathbb{E}_\zeta \left(\sum_{n=0}^{\hat{\tau}_1 - 1} \mathbf{1}\big((M_n, W_n) \in \cdot\,\big) \right),$$

where $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\hat{\tau}_1} \in \cdot\,)$ for any initial distribution λ. Moreover, since $E \times \{0\}$ is a recurrent set with $\pi(E \times \{0\}) > 0$, by Proposition 1.1 π can be also written as an occupation measure

$$\begin{aligned}
\pi(A \times B) &= \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1}\, \mathbb{E}_{\bar{\xi}} \left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}\big((M_n, S_n) \in A \times B\big) \right) \\
&= \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1} \int_E \mathbf{H}_{\leq}(s, A; B)\, \bar{\xi}(ds) \\
&= \int_E \mathbf{H}_{\leq}(s, A; B)\, \pi(ds \times \{0\}), \quad A \in \mathcal{E}, B \in \mathcal{B}^+,
\end{aligned}$$

where $\bar{\xi}$ is a stationary distribution for $\mathbf{G}_{\leq}$. Note that, for any $a \geq 0$, $E \times [0, a]$ is also a recurrent set with $\pi(E \times [0, a]) > 0$. Let

$$\tau_0(a) := \inf\{k \geq 0 : W_k \leq a\} \quad \text{and} \quad \tau_n(a) := \inf\{k > \tau_{n-1}(a) : W_k \leq a\}, \quad n \geq 1.$$

Then the random variables $\tau_n(a)$, $n \geq 0$, are a.s. finite stopping times and the chain $(M_n^{\tau(a)}, W_n^{\tau(a)})_{n\geq 0} = (M_{\tau_n(a)}, W_{\tau_n(a)})_{n\geq 0}$ forms a Harris chain with transition kernel $\mathbf{G}_{(-\infty,a]} = \hat{\mathbf{G}}^{(1,0)}_{(-\infty,a]}$. Denote by $\pi^{(a)}$ the stationary distribution.

Proposition 2.12 *Let $a \geq 0$. Then for any $A \in \mathcal{E}$ and $y \geq 0$*

$$\begin{aligned}
\pi(A \times [0, y]) &= \frac{1}{\mathbb{E}_{\pi^{(a)}}\, \sigma((-\infty, a])} \int_{[0,a]} \int_E \mathbf{H}_{(-\infty, a-x]}(s, A; [0, y - x])\, \pi^{(a)}(ds \times dx) \\
&= \pi(E \times [0, a]) \int_{[0,a]} \int_E \mathbf{H}_{(-\infty, a-x]}(s, A; [0, y - x])\, \pi^{(a)}(ds \times dx) \\
&= \int_{[0,a]} \int_E \mathbf{H}_{(-\infty, a-x]}(s, A; [0, y - x])\, \pi(ds \times dx).
\end{aligned}$$

Proof. By Proposition 1.1,

$$\begin{aligned}
\pi(A \times [0, y]) &= \frac{1}{\mathbb{E}_{\pi^{(a)}}\, \tau_1(a)} \mathbb{E}_{\pi^{(a)}} \left(\sum_{n=0}^{\tau_1(a) - 1} \mathbf{1}\big((M_n, S_n) \in A \times [0, y]\big) \right) \\
&= \pi(E \times [0, a]) \cdot \mathbb{E}_{\pi^{(a)}} \left(\sum_{n=0}^{\tau_1(a) - 1} \mathbf{1}\big((M_n, S_n) \in A \times [0, y]\big) \right).
\end{aligned}$$

Thus the assertions follow from

$$\begin{aligned}
&\mathbb{E}_{\pi^{(a)}}\left(\sum_{n=0}^{\tau_1(a)-1}\mathbf{1}\big((M_n,S_n)\in A\times[0,y]\big)\right)\\
&=\sum_{n=0}^{\infty}\mathbb{P}_{\pi^{(a)}}\big((M_n,S_n)\in A\times[0,y],\tau_1(a)>n\big)\\
&=\sum_{n=0}^{\infty}\int_E\int_0^a\mathbb{P}_{s,x}\big((M_n,S_n)\in A\times[0,y],S_k>a,k\le n\big)\,\pi^{(a)}(ds\times dx)\\
&=\sum_{n=0}^{\infty}\int_E\int_0^a\mathbb{P}_{s}\big((M_n,S_n)\in A\times[0,y-x],S_k>a-x,k\le n\big)\,\pi^{(a)}(ds\times dx)\\
&=\int_E\int_0^a\mathbf{H}_{(-\infty,a-x]}(s,A;y-x)\,\pi^{(a)}(ds\times dx).
\end{aligned}$$

QED

As a corollary we get the following assertion, which is a generalization of a result of Arjas [7]:

Corollary 2.13 *For any* $A\in\mathcal{E}$

$$\mathbb{E}_\pi\left(e^{\beta W_1};M_1\in A\right)=\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\int_E\hat{\mathbf{H}}_{\le}^{(1,\beta)}(s,A)\,\bar\xi(ds)=\int_E\hat{\mathbf{H}}_{\le}^{(1,\beta)}(s,A)\,\pi(ds\times\{0\}).$$

Proof. The assertion follows from

$$\begin{aligned}
\mathbb{E}_\pi\left(e^{\beta W_1};M_1\in A\right) &= \int_0^\infty e^{\beta x}\,\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\sum_{n=0}^{\infty}\mathbb{P}_{\bar\xi}\left(M_n\in A,S_n\in dx,\sigma_1>n\right)\\
&= \frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\int_E\left(\int_0^\infty\sum_{n=0}^{\infty}e^{\beta x}\,\mathbb{P}_s\left(M_n\in A,S_n\in dx,\sigma_1>n\right)\right)\bar\xi(ds)\\
&= \frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\int_E\hat{\mathbf{H}}_{\le}^{(1,\beta)}(s,A)\,\bar\xi(ds)\\
&= \int_E\hat{\mathbf{H}}_{\le}^{(1,\beta)}(s,A)\,\pi(ds\times\{0\}).
\end{aligned}$$

QED

2.3 MRW's with lattice-type increments

MRW's with lattice-type increments are of particular interest in the applied probability. In particular, reflected MRW's driven by finite Markov chains and with upward or downward skip-free increments have received a considerable attention in the queueing theory, and were extensively studied by Neuts and his school. They have developed matrix-analytic methods to algorithmically compute characteristics like stationary distribution. A comprehensive treatment can be found in Neuts [43, 44] and Latouche and Ramaswami [34] and some generalizations in Miyazawa [38], Sengupta [54] and Tweedie [65]. This section examines MRW's with lattice-type increments driven by general Markov chains. In this case we can not expect to find computational algorithms, but get stationary distributions in simpler forms.

2.3.1 MRW's with lattice-type increments

Consider a Markov modulated sequence $(M_n, X_n)_{n\geq 0}$ defined on $E \times \mathbb{Z}$ with transition kernel $\mathbf{P}$. Denote by $(M_n, S_n)_{n\geq 0}$ the associated MRW with transition kernel $\mathbf{Q}$ and by $(M_n, W_n)_{n\geq 0}$ the associated reflected MRW.

For $B \subset \mathbb{Z}$, $\sigma(B)$ denotes the first return time to $E \times B$. For each $l \in \mathbb{Z}$ we define the kernels $\hat{\mathbf{H}}_B^{[\alpha,l]}$ and $\hat{\mathbf{G}}_B^{[\alpha,l]}$ as

$$\begin{aligned}\hat{\mathbf{H}}_B^{[\alpha,l]}(s,A) &= \sum_{n=0}^{\infty} \alpha^n \mathbb{P}_s\left(M_n \in A, S_n = l, \sigma(B) > n\right);\\ \hat{\mathbf{G}}_B^{[\alpha,l]}(s,A) &= \sum_{n=0}^{\infty} \alpha^n \mathbb{P}_s\left(M_n \in A, S_n = l, \sigma(B) = n\right).\end{aligned}$$

If $\alpha = 1$, then we write $\mathbf{H}_B^{[l]}$ and $\mathbf{G}_B^{[l]}$ instead of $\hat{\mathbf{H}}_B^{[1,l]}$ and $\hat{\mathbf{G}}_B^{[1,l]}$, respectively. Furthermore, we put

$$\mathbf{H}_B = \sum_{l\in\mathbb{Z}} \mathbf{H}_B^{[l]} \quad \text{and} \quad \mathbf{G}_B = \sum_{l\in\mathbb{Z}} \mathbf{G}_B^{[l]}.$$

If $\sigma(B) = \sigma^*$ for some $* \in \{\geq, >, \leq, <\}$, we write $\hat{\mathbf{H}}_>^{[\alpha,l]}$, $\hat{\mathbf{H}}_\geq^{[\alpha,l]}$, $\hat{\mathbf{H}}_<^{[\alpha,l]}$ and $\hat{\mathbf{H}}_\leq^{[\alpha,l]}$ instead of $\hat{\mathbf{H}}_{[1,\infty)}^{[\alpha,l]}$, $\hat{\mathbf{H}}_{[0,\infty)}^{[\alpha,l]}$, $\hat{\mathbf{H}}_{(-\infty,-1]}^{[\alpha,l]}$ and $\hat{\mathbf{H}}_{(-\infty,0]}^{[\alpha,l]}$, respectively. The same notational conventions are used also for $\hat{\mathbf{G}}_{[1,\infty)}^{[\alpha,l]}$, $\hat{\mathbf{G}}_{[0,\infty)}^{[\alpha,l]}$, $\hat{\mathbf{G}}_{(-\infty,-1]}^{[\alpha,l]}$ and $\hat{\mathbf{G}}_{(-\infty,0]}^{[\alpha,l]}$. Note that

$$\sum_{n=0}^{\infty} \alpha^n\, \mathbb{P}_s\left(M_n \in A, W_n = l\right) = \sum_{n=0}^{\infty} \int_E \hat{\mathbf{H}}_\leq^{[\alpha,l-k]}(s',A)\, \left(\hat{\mathbf{G}}_\leq^{[\alpha,k]}\right)^n (s,ds').$$

Denote by $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$ the time-reversal of $(M_n, S_n)_{n\geq 0}$ and for each $l \in \mathbb{Z}$ define the kernel $\hat{\mathbf{G}}_{R(>)}^{[\alpha,l]}$ as

$$\hat{\mathbf{G}}_{R(>)}^{[\alpha,l]}(s,A) = \sum_{n=0}^{\infty} \alpha^n \mathbb{P}\left[\tilde{M}_n \in A, \tilde{S}_n = l, \tilde{\sigma}^> = n \middle| \tilde{M}_0 = s, \tilde{S}_0 = 0\right],$$

where $\tilde{\sigma}^<$ is the first return time of $(\tilde{M}_n, \tilde{S}_n)_{n\geq 0}$ to $E \times (-\infty, -1]$.

In the similar manner as in Proposition 2.5, we get the following assertions:

Proposition 2.14 *For each $l \geq 1$ the kernels*

$$\hat{\mathbf{H}}_{\leq}^{[\alpha,l]} \quad \text{and} \quad \sum_{n=1}^{l} \left(\hat{\mathbf{G}}_{R(>)}^{[\alpha,l]}\right)^n$$

are mutually adjoint with respect to ξ. In particular, $\hat{\mathbf{H}}_{\leq}^{[\alpha,1]}$ and $\hat{\mathbf{G}}_{R(>)}^{[\alpha,1]}$ are mutually adjoint with respect to ξ.

Proof. Note that

$$\begin{aligned}
&\mathbb{P}\left[\tilde{M}_n \in A, \tilde{S}_n = l, \tilde{S}_n - \tilde{S}_k > 0, k < n \middle| \tilde{M} = s, \tilde{S}_0 = 0\right] \\
&= \mathbb{P}_s\left[\tilde{M}_n \in A, \tilde{S}_n = l, n = \tilde{\sigma}_m^{>} \text{ for some } m, 1 \leq m \leq l \middle| \tilde{M}_0 = s, \tilde{S}_0 = 0\right].
\end{aligned}$$

The first assertion follows in the same manner as in the proof of Proposition 2.5. The second assertion is clear. QED

For each $k \geq 0$, $\pi^{(k)}$ denotes the stationary distribution for $\mathbf{G}_{(-\infty,k]}$. The following assertion is a direct consequence of Proposition 2.12.

Proposition 2.15 *For any $k \geq 0$, a unique stationary distribution π can be written as*

$$\begin{aligned}
\pi(A \times \{l\}) &= \frac{1}{\mathbb{E}_{\pi^{(k)}}\, \sigma((-\infty,k])} \sum_{m=0}^{k} \int_E \mathbf{H}_{(-\infty,k-m]}^{[l-m]}(s,A)\, \pi^{(k)}(ds \times \{l\}) \\
&= \pi(E \times \{0,1,\cdots,k\}) \sum_{m=0}^{k} \int_E \mathbf{H}_{(-\infty,k-m]}^{[l-m]}(s,A)\, \pi^{(k)}(ds \times \{m\}) \\
&= \sum_{m=0}^{k} \int_E \mathbf{H}_{(-\infty,k-m]}^{[l-m]}(s,A)\, \pi(ds \times \{m\}), \quad A \in \mathcal{E}, l \geq 0.
\end{aligned}$$

In particular, for any $A \in \mathcal{E}$ and $l \geq 0$

$$\pi(A \times \{l\}) = \frac{1}{\mathbb{E}_{\pi^{(0)}}\, \sigma_1} \int_E \mathbf{H}_{\leq}^{[l]}(s,A)\, \pi^{(0)}(ds) = \int_E \mathbf{H}_{\leq}^{[l]}(s,A)\, \pi(ds \times \{0\}).$$

Proof. Denote by $\underline{\mathbf{H}}_{(-\infty,k]}(s,m;A,l)$ the expectation that starting from (s,m), the process visits $A \times \{l\}$ avoiding levels $0,\cdots,k$. Since the process is independent of level, we have

$$\underline{\mathbf{H}}_{(-\infty,k]}(s,m;A,l) = \mathbf{H}_{(-\infty,k-m]}^{[l-m]}(s,A)$$

for any $m, 0 \leq m \leq k$. Noting that

$$\mathbb{E}_{\pi^{(k)}}\, \sigma((-\infty,k]) = \pi(E \times \{0,1,\cdots,k\}),$$

the assertions follow from

$$\pi(A \times \{l\}) = \frac{1}{\mathbb{E}_{\pi^{(k)}}\,\sigma((-\infty,k])} \sum_{m=0}^{k} \int_E \underline{\mathbf{H}}_{(-\infty,k]}(s,m;A,l)\pi^{(k)}(ds \times \{m\}).$$

QED

2.3.2 MRW's with upward skip-free increments

Consider a Markov modulated sequence $(M_n, X_n)_{n\geq 0}$ defined on $E \times \{\cdots,-2,-1,0,1\}$ with transition kernel $\mathbf{P}$ and let

$$\mathbf{A}_l(s,A) := \mathbf{P}(s, A \times \{-l+1\}), \quad s \in E, A \in E, l \geq 0.$$

The associated MRW $(M_n, S_n)_{n\geq 0}$ possesses upward skip-free increments and the transition kernel $\mathbf{Q}$ can be written as

$$\mathbf{Q}((s,k), A \times \{l\}) = \begin{cases} \mathbf{A}_{k-l+1}(s,A) & : \ l \geq k+1 \\ 0 & : \ \text{otherwise} \end{cases}$$

for any $s \in E, A \in \mathcal{E}$.

The following proposition is originally due to Neuts for the case of finite driving chain, and was extended by Tweedie [65] to the general case.

Proposition 2.16 *For all $l \geq 1$ it holds that*

$$\hat{\mathbf{H}}_{\leq}^{[\alpha,l]}(s,A) = (\hat{\mathbf{H}}_{\leq}^{[\alpha,1]})^l(s,A), \quad s \in E, A \in \mathcal{E}.$$

Moreover, $\hat{\mathbf{H}}_{\leq}^{[\alpha,1]}$ satisfies the nonlinear operator equation

$$\hat{\mathbf{H}}_{\leq}^{[\alpha,1]}(s,A) = \alpha \sum_{n=0}^{\infty} \int_E \mathbf{A}_n(s',A)\,(\hat{\mathbf{H}}_{\leq}^{[\alpha,1]})^n(s,ds') \quad \textit{for all} \quad |\alpha| \leq 1. \tag{2.7}$$

Proof. Decomposing over the time of the last entrance to level l, we have

$$\mathbf{H}_{\leq}^{(n)}(s,A;l+1) = \sum_{r=1}^{n-1} \int_E \mathbf{H}_{\leq}^{(n-r)}(s',A;1)\,\mathbf{H}_{\leq}^{(r)}(s,ds';l),$$

where $\mathbf{H}_{\leq}^{(n)}(s,A;l) := \mathbf{H}_{\leq}^{(n)}(s,A;\{l\})$ (see 2.1.1 for the definition). Multiplying the above equation by α^n and summing over all n, we get easily

$$\hat{\mathbf{H}}_{\leq}^{[\alpha,l]}(s,A) = \int_E \hat{\mathbf{H}}_{\leq}^{[\alpha,1]}(s',A)\,\hat{\mathbf{H}}_{\leq}^{[\alpha,l-1]}(s,ds'),$$

which yields the first assertion.
For the second assertion note that $\mathbf{H}_{\leq}^{(1)}(\,\cdot,\cdot\,;1) = \mathbf{A}_0(\,\cdot,\cdot\,)$. By conditioning on the state entered in step $n-1$, we get

$$\mathbf{H}_{\leq}^{(n)}(s,A;1) = \sum_{k=1}^{n-1} \int_E \mathbf{A}_k(s',A)\,\mathbf{H}_{\leq}^{(n-1)}(s,ds';k), \quad n \geq 2,$$

from which we obtain the equation (2.7). QED

Remark 2.17 Tweedie [65] showed that $\mathbf{H}_{\leq}^{[1]}$ is the minimal solution of nonlinear operator equation

$$\mathbf{H}_{\leq}^{[1]}(s,A) = \sum_{n=0}^{\infty} \int_E \mathbf{A}_n(s',A)\,(\mathbf{H}_{\leq}^{[1]})^n(s,ds'), \quad s \in E, A \in \mathcal{E},$$

in the sense that, if $\mathbf{R}$ is another kernel satisfying above operator equation, then

$$\mathbf{H}_{\leq}^{[1]}(s,A) \leq \mathbf{R}(s,A), \quad s \in E, A \in \mathcal{E}.$$

Now consider the reflected MRW $(M_n, W_n)_{n\geq 0}$ associated to $(M_n, X_n)_{n\geq 0}$. For $k, l \geq 0$, let

$$\mathbf{\Phi}_{kl}(s,A) = \mathbb{P}\,[M_1 \in A, W_1 = l | M_0 = s, W_0 = k], \quad s \in E, A \in \mathcal{E}.$$

Then one can easily see that the transition matrix kernel $\mathbf{\Phi} = (\mathbf{\Phi}_{kl})_{k,l\geq 0}$ is given as

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{B}_0(s,A) & \mathbf{A}_0(s,A) & 0 & 0 & 0 & \cdots \\ \mathbf{B}_1(s,A) & \mathbf{A}_1(s,A) & \mathbf{A}_0(s,A) & 0 & 0 & \cdots \\ \mathbf{B}_2(s,A) & \mathbf{A}_2(s,A) & \mathbf{A}_1(s,A) & \mathbf{A}_0(s,A) & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \end{pmatrix}, \tag{2.8}$$

where $\mathbf{B}_l, l \geq 0$, are kernels defined as

$$\mathbf{B}_l(s,A) \;=\; \sum_{k=l+1}^{\infty} \mathbf{A}_k(s,A), \quad l \geq 0.$$

If E is finite, then the Markov chain with this transition kernel $\mathbf{\Phi}$ is referred to as $M/GI/1$ *type*. A specific feature of $M/GI/1$-type processes is the existence of matrix-geometric stationary distributions (see Neuts [42, 43]).

The following proposition is due to Tweedie [65], which is a generalization of a result of Neuts [42].

Proposition 2.18 *For any $l \geq 0$, the stationary distribution π of $\mathbf{\Phi}$ satisfies the relation*

$$\pi(A \times \{l\}) = \frac{1}{\mathbb{E}_{\pi^{(0)}}\, \sigma_1} \int_E (\mathbf{H}_{\leq}^{[1]})^l\,(s, A)\,\pi^{(0)}(ds) = \int_E (\mathbf{H}_{\leq}^{[1]})^l\,(s, A)\,\pi(ds \times \{0\}) \quad (2.9)$$

for any $a \in \mathcal{E}$ and the recursive equation

$$\pi(A \times \{l+1\}) \;=\; c_l \cdot \int_E \mathbf{H}_{\leq}^{[1]}(s, A)\,\pi^{(l)}(ds \times \{l\}) = \int_E \mathbf{H}_{\leq}^{[1]}(s, A)\,\pi(ds \times \{l\}),$$

where

$$c_l = \pi(E \times \{0, 1, \cdots, l\}).$$

Proof. The first equality in (2.9) follows from

$$\pi(A \times \{l\}) = c_0 \cdot \int_E \mathbf{H}_{\leq}(s, A; l)\,\pi^{(0)}(ds), \quad A \in \mathcal{E},$$

and

$$\mathbf{H}_{\leq}(s, A; l) = \mathbf{H}_{\leq}^{[l]}(s, A) = \left(\mathbf{H}_{\leq}^{[1]}\right)^l (s, A)$$

for any $s, \in E, A \in \mathcal{E}, l \geq 0$. The second equality in (2.9) is the assertion of Proposition 1.1. For the last assertion, note that for any $l, 0 \leq l - 1$,

$$\mathbf{H}_{(-\infty, l-k]}(s, A; l + 1 - k) = 0,$$

since the process is upward skip-free. Therefore, from Proposition 2.15, for any $A \in \mathcal{E}$

$$\begin{aligned}
\pi(A \times \{l+1\}) &= c_l \cdot \sum_{k=0}^{l} \int_E \mathbf{H}_{(-\infty, l-k]}(s, A; l + 1 - k)\,\pi^{(l)}(ds \times \{k\}) \\
&= c_l \cdot \int_E \mathbf{H}_{\leq}(s, A; 1)\,\pi^{(l)}(ds \times \{l\}) \\
&= \int_E \mathbf{H}_{\leq}^{[1]}(s, A)\,\pi(ds \times \{l\}).
\end{aligned}$$

QED

2.3.3 MRW's with downward skip-free increments

Next consider a Markov modulated sequence $(M_n, X_n)_{n \geq 0}$ defined on $E \times \{-1, 0, \cdots\}$ with transition kernel $\mathbf{P}$ and let

$$\mathbf{D}_l(s, A) = \mathbf{P}(s, A \times \{l - 1\}), \quad s \in E, A \in \mathcal{E},$$

for each $l \geq 0$. The associated MRW $(M_n, S_n)_{n\geq 0}$ possesses downward skip-free increments and the transition kernel $\mathbf{Q}$ can be written as

$$\mathbf{Q}((s,k), A \times \{l\}) = \begin{cases} \mathbf{D}_{l-k+1}(s,A) & : \quad l \geq k-1 \\ 0 & : \quad \text{otherwise.} \end{cases}$$

The kernel $\hat{\mathbf{G}}_{<}^{[\alpha,-1]}$ can be represented in terms of the minimal solution satisfying a nonlinear operator equation.

Proposition 2.19 *The kernel* $\hat{\mathbf{G}}_{<}^{[\alpha,-1]}$ *satisfies the nonlinear operator equation*

$$\hat{\mathbf{G}}_{<}^{[\alpha,-1]}(s,A) = \alpha \sum_{l=0}^{\infty} \int_E \mathbf{D}_l(s',A)\,(\hat{\mathbf{G}}_{<}^{[\alpha,-1]})^l(s,ds'), \quad s \in E, A \in \mathcal{E},$$

for all $|\alpha| \leq 1$. *Furthermore, for* $0 \leq \alpha \leq 1$, $\hat{\mathbf{G}}_{<}^{[\alpha,-1]}$ *is the minimal nonnegative solution of the nonlinear operator equation.*

Proof. A proof for the finite modulation case can be found in Theorem 2.2.1 and Theorem 2.2.2 in Neuts [44]. The general case can be proved in the same manner.

QED

Consider the reflected MRW $(M_n, W_n)_{n\geq 0}$ associated to $(M_n, X_n)_{n\geq 0}$.
For $k, l \geq 0$, let

$$\mathbf{\Phi}_{kl}(s,A) = \mathbb{P}\,[M_1 \in A, W_1 = l | M_0 = s, W_0 = k], \quad s \in E, A \in \mathcal{E}.$$

Then the transition matrix kernel $\mathbf{\Phi} = (\mathbf{\Phi}_{kl})_{k,l\geq 0}$ is given as

$$\mathbf{\Phi} = \begin{pmatrix} \mathbf{C}_0(s,A) & \mathbf{C}_1(s,A) & \mathbf{C}_2(s,A) & \mathbf{C}_3(s,A) & \cdots \\ \mathbf{D}_0(s,A) & \mathbf{D}_1(s,A) & \mathbf{D}_2(s,A) & \mathbf{D}_3(s,A) & \cdots \\ 0 & \mathbf{D}_0(s,A) & \mathbf{D}_1(s,A) & \mathbf{D}_2(s,A) & \cdots \\ 0 & 0 & \mathbf{D}_0(s,A) & \mathbf{D}_1(s,A) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \tag{2.10}$$

where

$$\mathbf{C}_0(s,A) = \mathbf{D}_0(s,A) + \mathbf{D}_1(s,A), \quad \mathbf{C}_l(s,A) = \mathbf{D}_{l+1}(s,A), \quad l \geq 1.$$

If E is finite, then the Markov chain with this transition kernel $\mathbf{\Phi}$ is referred to as $M/GI/1$ *type*. A specific feature of $M/GI/1$-type precesses is the existence of stationary distributions satisfying a certain recursive equation, which is obtained by Ramaswami [49].

A stationary distribution π of $(M_n, W_n)_{n\geq 0}$ satisfies the relation

$$\pi(A\times\{l\}) = \int_E \mathbf{C}_l(s,A)\,\pi(ds\times\{0\}) + \sum_{k=0}^{l}\int_E \mathbf{D}_{l-k+1}(s,A)\pi(ds\times\{k\}). \quad (2.11)$$

For any $l\geq 1$ define kernels $\mathbf{\Phi}_l^{\mathbf{D}}$ and $\mathbf{\Phi}_l^{\mathbf{C}}$ as

$$\begin{aligned}\mathbf{\Phi}_l^{\mathbf{D}}(s,A) &= \sum_{k=l}^{\infty}\int_E \mathbf{D}_k(s',A)\,(\mathbf{G}_<^{[-1]})^{(k-l)}(s,ds');\\ \mathbf{\Phi}_l^{\mathbf{C}}(s,A) &= \sum_{k=l}^{\infty}\int_E \mathbf{C}_k(s',A)\,(\mathbf{G}_<^{[-1]})^{(k-l)}(s,ds').\end{aligned}$$

Note that $\pi(\cdot\times\{0\})$ satisfies the equation

$$\pi(A\times\{0\}) = \int_E \mathbf{\Phi}_0^{\mathbf{C}}(s,A)\,\pi(ds\times\{0\}),$$

since $\mathbf{\Phi}_0^{\mathbf{C}} = \mathbf{G}_{\leq}$.

The following assertion is a generalization of a result of Ramaswami [49].

Proposition 2.20 *For any $l\geq 1$ the stationary distribution π satisfies the relation*

$$\pi(A\times\{l\}) = \int_E \mathbf{\Phi}_l^{\mathbf{C}}(s,A)\,\pi(ds\times\{0\}) + \sum_{k=1}^{l}\int_E \mathbf{\Phi}_{l+1-k}^{\mathbf{D}}(s,A)\,\pi(ds\times\{k\}) \quad (2.12)$$

for any $l\geq 0, A\in\mathcal{E}$, or recursively,

$$\pi(A\times\{l+1\}) = \int_E \mathbf{K}_0(s,A)\,\pi(ds\times\{0\}) + \sum_{k=1}^{l}\int_E \mathbf{K}_k(s,A)\,\pi(ds\times\{k\}), \quad (2.13)$$

where kernels $\mathbf{K}_k, k\geq 0$, given as

$$\begin{aligned}\mathbf{K}_0(s,A) &= \sum_{m=0}^{\infty}\int_E (\mathbf{\Phi}_1^{\mathbf{D}})^m(s',A)\,\mathbf{\Phi}_{l+1}^{\mathbf{C}}(s,ds');\\ \mathbf{K}_k(s,A) &= \sum_{m=0}^{\infty}\int_E (\mathbf{\Phi}_1^{\mathbf{D}})^m(s',A)\,\mathbf{\Phi}_{l+2-k}^{\mathbf{D}}(s,ds'), \quad k\geq 1.\end{aligned}$$

Conversely, if a distribution π' satisfies the equation (2.12), then it is the stationary distribution.

Proof. For a fixed $k\geq 1$, consider the Markov renewal process at the epochs of visits to the set $E\times\{0,\cdots,k\}$. The Markov chain $(M_n^{(k)}, W_n^{(k)})_{n\geq 0} = (M_{\sigma_n(-\infty,k]}, W_{\sigma_n(-\infty,k]})_{n\geq 0}$ is positive Harris recurrent. Moreover, the transition kernel $\mathbf{P}^{(k)}$ of $(M_n^{(k)}, W_n^{(k)})_{n\geq 0}$ is given by the matrix form

$$\mathbf{P}^{(k)} = \begin{pmatrix} \mathbf{C}_0(s,A) & \mathbf{C}_1(s,A) & \cdots & \mathbf{C}_{k-1}(s,A) & \Phi_k^{\mathbf{C}}(s,A) \\ \mathbf{D}_0(s,A) & \mathbf{D}_1(s,A) & \cdots & \mathbf{D}_{k-1}(s,A) & \Phi_k^{\mathbf{D}}(s,A) \\ 0 & \mathbf{D}_0(s,A) & \cdots & \mathbf{D}_{k-2}(s,A) & \Phi_{k-1}^{\mathbf{D}}(s,A) \\ 0 & 0 & \cdots & \mathbf{D}_{k-3}(s,A) & \Phi_{k-2}^{\mathbf{D}}(s,A) \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & \mathbf{D}_0(s,A) & \Phi_1^{\mathbf{D}}(s,A) \end{pmatrix}.$$

Thus we obtain

$$\pi^{(k)}(A\times\{k\}) = \int_E \mathbf{\Phi}_k^{\mathbf{C}}(s,A)\,\pi^{(k)}(ds\times\{0\}) + \sum_{l=1}^{k}\int_E \mathbf{\Phi}_{k+1-l}^{\mathbf{D}}(s,A)\,\pi^{(k)}(ds\times\{l\}), \quad k\geq 0,$$

from which (2.12) follows.
On the other hand, by Proposition 2.15, we have

$$\pi(A\times\{k+1\}) = c_k\cdot\sum_{m=0}^{k}\int_E \mathbf{H}_{(-\infty,k-m]}(s,A;k+1-m)\,\pi^{(k)}(ds\times\{m\}), \quad A\in\mathcal{E},$$

where $c_k = \pi(E\times\{0,1,2,\cdots,k\})$. However, $\mathbf{H}_{(-\infty,m]}(s,A;m+1)$ is the expectation that, starting in $(s,0)$, the process visits $A\times\{m+1\}$ avoiding levels $0,1,\cdots,n$. Since $\mathbf{\Phi}_1^{\mathbf{D}}(s,A) = \mathbf{G}_{\leq}(s,A)$, we obtain the relations

$$\begin{aligned} \mathbf{H}_{(-\infty,k]}(s,A;k+1) &= \sum_{m=0}^{\infty}\int_E (\mathbf{\Phi}_1^{\mathbf{D}})^m(s',A)\,\mathbf{\Phi}_{k+1}^{\mathbf{C}}(s,ds') \quad \text{and} \\ \mathbf{H}_{(-\infty,k-l]}(s,A;k+1-l) &= \sum_{m=0}^{\infty}\int_E (\mathbf{\Phi}_1^{\mathbf{D}})^m(s',A)\,\mathbf{\Phi}_{k+2-l}^{\mathbf{D}}(s,ds') \quad \text{for} \quad l\leq k-1. \end{aligned}$$

Letting $\mathbf{K}_0(s,A) = \mathbf{H}_{(-\infty,k]}(s,A;k+1)$ and $\mathbf{K}_l(s,A) = \mathbf{H}_{(-\infty,k-l]}(s,A;k+1-l)$, (2.13) follows.
The converse is stated in Miyazawa [38] for the case of countable driving chain. The general case can be proved in the same manner. QED

Remark 2.21 Ramaswami [49] showed that, if $(M_n)_{n\geq 0}$ is a finite Markov chain, the stationary distribution π of $(M_n,W_n)_{n\geq 0}$ is given by the recursive formula

$$\pi_l = \left(\pi_0\mathbf{\Phi}_l^{\mathbf{C}} + \sum_{k=1}^{l-1}\pi_k\mathbf{\Phi}_{l+1-k}^{\mathbf{D}}\right)(I-\mathbf{\Phi}_1^{\mathbf{D}})^{-1}, \quad l\geq 1, \tag{2.14}$$

where $\pi_l, l\geq 0$, are measures on E defined as

$$\pi_l(\{i\}) = \pi(\{(i,l)\}), \quad i\in E.$$

If the state space has only one element, then the stochastic matrix $\mathbf{G}_{<}^{[-1]}$ reduces to the scalar 1, and (2.14) yields

$$\pi_l = \left(\pi_0 \mathbf{C}_l + \sum_{k=1}^{l-1} \pi_k \mathbf{D}_{l+1-k} \right) (1 - \mathbf{D}_1)^{-1}, \quad l \geq 1$$

for $M/GI/1$ queue.

Remark 2.22 For fixed $s \in E$ and $A \in \mathcal{E}$, $\mathbf{G}_{<}^{[-1]}(s, A)$ can be interpreted as the probability that starting in $(s, 1)$, the MRW $(M_n, S_n)_{n \geq 0}$ makes the first passage into level 0 and the state in A.
For $l \geq 1$ let $\mathbf{G}^{(n,l)}$ be the probability that starting in (s, l), the MRW $(M_n, S_n)_{n \geq 0}$ makes the first passage into level 0 and the state in A at step n. Define the corresponding transform $\hat{\mathbf{G}}^{[\alpha,l]}$ as

$$\hat{\mathbf{G}}^{[\alpha,l]}(s, A) = \sum_{n=0}^{\infty} \alpha^n \mathbf{G}^{(n,l)}(s, A), \quad s \in E, A \in \mathcal{E}.$$

Then it can be easily shown that for any $l \geq 1$

$$\hat{\mathbf{G}}^{[\alpha,l]} = \left(\hat{\mathbf{G}}_{<}^{[\alpha,-1]} \right)^l.$$

2.3.4 A duality

A MRW $(M'_n, S'_n)_{n \geq 0}$ with transition kernel $\mathbf{Q}'$ is called the *dual* of $(M_n, S_n)_{n \geq 0}$ with transition kernel $\mathbf{Q}$, if $\mathbf{Q}(\{k\})$ and $\mathbf{Q}'(\{-k\})$ are adjoint w.r.t. ξ for each $k \in \mathbb{Z}$. Consider two MRW's $(M_n, S_n)_{n \geq 0}$ and $(M'_n, S'_n)_{n \geq 0}$ governed by

$$\mathbb{P}\left[M_{n+1} \in A, S_{n+1} = l - k + 1 | M_n = s, S_n = l\right] = \mathbf{A}_k(s, A)$$

and

$$\mathbb{P}\left[M'_{n+1} \in A, S'_{n+1} = l + k - 1 | M'_n = s, S'_n = l\right] = \mathbf{D}_k(s, A),$$

where $\mathbf{A}_k, \mathbf{D}_k, k \in \mathbb{Z}$, are kernels on $E \times \mathcal{E}$ with $\mathbf{A}_k = \mathbf{D}_k = 0$ for $k \leq -1$. It is clear that $(M_n, S_n)_{n \geq 0}$ and $(M'_n, S'_n)_{n \geq 0}$ are in duality if, and only if, for each $k \geq 0$, $\mathbf{A}_k$ and $\mathbf{D}_k$ are mutually adjoint w.r.t. ξ.

Now we get a generalization of the duality theorem, which is obtained by Asmussen and Ramaswami [13] for the case of finite modulation.

Proposition 2.23 *If for each $k \geq 0$ the kernels $\mathbf{A}_k$ and $\mathbf{D}_k$ are mutually adjoint with respect to ξ, then for each $l \geq 1$ the kernels $\hat{\mathbf{H}}_{\leq}^{[\alpha,l]}$ and $(\hat{\mathbf{G}}_{D(<)}^{[\alpha,-1]})^l$ are mutually adjoint with respect to ξ, where*

$$\begin{aligned}
\hat{\mathbf{H}}_{\leq}^{[\alpha,l]}(s, A) &= \sum_{n=0}^{\infty} \alpha^n \mathbb{P}_s\left(M_n \in A, S_n = l, S_k \geq 0, k \leq n\right), \quad l \geq 1; \\
\hat{\mathbf{G}}_{D(<)}^{[\alpha,-1]}(s, A) &= \sum_{n=0}^{\infty} \alpha^n \mathbb{P}_s\left(M'_n \in A, S'_n = -1, S'_k \geq 0, k < n\right).
\end{aligned}$$

Proof. Noting that for any $n \geq 0$

$$(M_k, S_k)_{0 \leq k \leq n} \sim (M'_{n-k}, S'_{n-k} - S'_n)_{0 \leq k \leq n},$$

as in Proposition 2.14 one can show that the kernels $\hat{\mathbf{H}}_{\leq}^{[\alpha,1]}$ and $\hat{\mathbf{G}}_{D(<)}^{[\alpha,-1]}$ are mutually adjoint with respect to ξ. Therefore, the assertion follows from Proposition 2.16. QED

If the state space E is finite, then Proposition 2.23 can be written as a matrix form

$$\hat{\mathbf{H}}_{\leq}^{[\alpha,1]} = \Delta_{\xi}^{-1} \left(\hat{\mathbf{G}}_{<}^{[\alpha,-1]} \right)^{T} \Delta_{\xi},$$

where Δ_{ξ} denotes the diagonal matrix of stationary distribution ξ.

Chapter 3

Moment conditions

The first weak descending ladder epoch is a basic quantity in the analysis of MRW's, but also plays an important role in the study of semi-Markov queues. The first weak descending ladder epoch of a MRW is interpreted as the index of customers served in the first busy period of the corresponding semi-Markov queue. This chapter investigates moments of the first weak descending ladder epochs of MRW's and regeneration epochs of reflected MRW's. Throughout the driving chain M is assumed to be positive Harris recurrent with a stationary distribution ξ and to satisfy the minorization condition $\mathcal{M}(\Re, p, r, \varphi)$. We denote by $(\tau_n)_{n\geq 0}$ a sequence of regeneration epochs for M constructed by the splitting technique from the minorization condition. Throughout this chapter $(M_n, X_n)_{n\geq 0}$ denotes a Markov modulated sequence with transition kernel $\mathbf{P}$, $(M_n, S_n)_{n\geq 0}$ the associated MRW and $(M_n, W_n)_{n\geq 0}$ the associated reflected MRW. We assume that $-\infty < \mu = \mathbb{E}_\xi X_1 < 0$. Further, we denote by $\sigma_n, n \geq 0$, the weak descending ladder epochs defined in 2.1.3, i.e.,

$$\sigma_1 = \inf\{n \geq 1 : S_n \leq 0\} \quad \text{and} \quad \sigma_{n+1} = \inf\{k > \sigma_n : S_k \leq S_{\sigma_n}\}, \quad n \geq 1.$$

Let throughout a canonical model be given with probability measures $\mathbb{P}_{s,x}, s \in E, x \in \mathbb{R}$ on $(\Omega, \mathcal{S})$ such that $\mathbb{P}_{s,x}(M_n = s, X_0 = x) = 1$.

3.1 Moments of the first weak descending ladder epoch

In this section we find moment conditions for the first weak descending ladder epoch of a MRW $(M_n, S_n)_{n\geq 0}$. The main idea is simple. We first show that under some adequate conditions the associated reflected MRW $(M_n, W_n)_{n\geq 0}$ possesses a recurrent set of the form $\Re \times [0, x_0]$ for some $x_0 \geq 0$. Once $(M_n, W_n)_{n\geq 0}$ visits $\Re \times [0, x_0]$, by the SLLN for MRW's (see Proposition 2.2) $(W_n)_{n\geq 0}$ is reduced to 0 in some steps with positive probability. This procedure is repeated infinitely often, in order to obtain a random time ν, say, which is stochastically greater than the first weak descending ladder epoch of $(M_n, S_n)_{n\geq 0}$. Finally we try to find moment conditions for the random time ν. It turns out that moments of ν is connected with the regeneration structure of the sequence $(M_n, X_{n+1})_{n\geq 0}$. A similar method can be found in Sharma [55].

The following lemma is due to Alsmeyer and plays a central role in this chapter.

Lemma 3.1 *There exists $c \geq 0$ such that for any $\epsilon > 0$ there are measurable $C_\epsilon, D_\epsilon \subset \Re$ satisfying*

$$\begin{aligned} q &:= \mathbb{P}_\varphi\left((M_{\tau_1 - r}, M_{\tau_1}) \in C_\epsilon \times D_\epsilon\right) > 0; \\ q' &:= \inf_{(s,s')\in C_\epsilon \times D_\epsilon} \mathbb{P}\left[(X_1, \cdots, X_r) \in I_\epsilon(\mathbf{c}) | M_0 = s, M_r = s'\right] > 0, \end{aligned}$$

where we put

$$\mathbf{c} = (c_1, \cdots, c_r) \in \mathbb{R}^r \quad \textit{and} \quad I_\epsilon(\mathbf{c}) = [c_1 - \epsilon, c_1 + \epsilon] \times \cdots \times [c_r - \epsilon, c_r + \epsilon].$$

Proof. See Lemma 3.1 in Alsmeyer [2]. QED

From Lemma 3.1 one can develop another regeneration scheme for M. Let $(\eta'_n)_{n\geq 0}$ be a sequence of i.i.d. Bernoulli variables with success probability q', which is also independent of $(M_n, \tau_n)_{n\geq 0}$. Each time τ_j when $(M_{\tau_j - r}, M_{\tau_j}) \in C_\epsilon \times D_\epsilon$ we generate $(X_{\tau_j - r+1}, \cdots, X_{\tau_j})$ according to

$$\mathbb{P}\left[(X_{\tau_j - r+1}, \cdots, X_{\tau_j}) \in I_\epsilon(\mathbf{c}) \cap \cdot | M_{\tau_j - r}, M_{\tau_j}\right] / \mathbb{P}\left[(X_{\tau_j - r+1}, \cdots, X_{\tau_j}) \in I_\epsilon(\mathbf{c}) | (M_{\tau_j - r}, M_{\tau_j}\right]$$

if $\eta'_j = 1$, and according to

$$\mathbb{P}\left[(X_{\tau_j - r+1}, \cdots, X_{\tau_j}) \in \cdot \, | M_{\tau_j - r}, M_{\tau_j}\right]$$

otherwise. Next discard the old values of $M_{\tau_j - r+1}, \cdots, M_{\tau_j - 1}$ and regenerate according to

$$\mathbb{P}[(M_{\tau_j - r+1}, \cdots, M_{\tau_j - 1}) \in \cdot \, | M_{\tau_j - r}, M_{\tau_j}, X_{\tau_j - r+1}, \cdots, X_{\tau_j}].$$

At all remaining time points n we regenerate X_n according to $\mathbb{P}[X_n \in \cdot \, | M_{n-1}, M_n]$. It is easily verified that the new chain $(M_n, X_n)_{n\geq 0}$ is indeed a Markov chain with transition kernel $\mathbf{P}$. Let

$$\tau'_0 = 0 \quad \text{and} \quad \tau'_n := \inf\{\tau_j \geq \tau'_{n-1} + r : (M_{\tau_j - r}, M_{\tau_j}, \eta'_j) \in C_\epsilon \times D_\epsilon \times \{1\}\}, \quad n \geq 1.$$

Then $(\tau'_n)_{n\geq 0}$ is a sequence of regeneration epochs for M with

$$\zeta(\cdot) = \mathbb{P}_\lambda(M_{\tau'_1} \in \cdot) = \varphi(\cdot \cap D_\epsilon)/\varphi(D_\epsilon).$$

From the construction of the regeneration epochs one can show that

(i) $c_j - \epsilon \leq X_{\tau'_n - r+j} \leq c_j + \epsilon$ for each $1 \leq j \leq r$ and $n \geq 1$.

(ii) $(M_{\tau'_n+j}, X_{\tau'_n+j+1})_{j\geq 0}$ and $(M_j, X_j)_{0\leq j\leq \tau'_n-r}$ are independent for any $n\geq 1$ (see Lemma 3.2 in Alsmeyer [2]).

As direct consequences of (i) and (ii) we have: For any $m\geq 1$

$$\left|\mathbb{E}\left[\sum_{n=\tau'_m+1}^{\tau'_{m+1}} X_n | M_{\tau'_m-r}=s\right] - \mathbb{E}_\zeta\left(\sum_{n=1}^{\tau'_1} X_n\right)\right| \leq r\epsilon$$

and for any $\gamma>0$ such that $\mathbb{E}_\zeta \exp\left(\gamma\sum_{n=1}^{\tau'_1-r} X_n\right)<\infty$

$$\exp(-\gamma r\epsilon) \leq \mathbb{E}_\zeta \exp\left(\gamma\sum_{n=1}^{\tau'_m} X_n\right)\left\{\prod_{j=1}^{r}\exp(\gamma c_j)\mathbb{E}_\zeta \exp\left(\gamma\sum_{n=1}^{\tau'_1-r} X_n\right)\right\}^{-m} \leq \exp(\gamma r\epsilon).$$

In particular, if for some $\gamma>0$

$$\mathbb{E}_\zeta \exp\left(\gamma\sum_{n=1}^{\tau'_1} X_n\right)<1,$$

then one can choose $\epsilon>0$ so that

$$\prod_{j=1}^{r}\exp(\gamma c_j)\mathbb{E}_\zeta \exp\left(\gamma\sum_{n=1}^{\tau'_1-r} X_n\right)<1.$$

Let ν be the random variable defined as

$$\nu := \inf\{n : (M_{\tau_n-r}, M_{\tau_n}, \eta'_n) \in C_\epsilon \times D_\epsilon \times \{1\}\}.$$

Then it can be easily seen that

$$\mathbb{P}_\varphi(\nu>k) < (1-qq')^k, \quad k\geq 1,$$

from which

$$\begin{aligned}\mathbb{P}_\zeta(\nu>k) &= \frac{1}{\varphi(D_\epsilon)}\int_{D_\epsilon}\mathbb{P}_s(\nu>k)\,\varphi(ds)\\ &\leq \frac{1}{\varphi(D_\epsilon)}\mathbb{P}_\varphi(\nu>k)\\ &\leq \frac{1}{\varphi(D_\epsilon)}(1-qq')^k.\end{aligned}$$

In particular, for any $\alpha\geq 1$ and for some $\gamma>0$

$$\mathbb{E}_\zeta\,\nu^\alpha<\infty \quad\text{and}\quad \mathbb{E}_\zeta \exp(\gamma\nu)<\infty.$$

Note that for any $\alpha \geq 1$

$$E_\zeta\,(\tau_1')^\alpha \leq \mathbb{E}_\zeta \left(\sum_{n=1}^{\nu} (\tau_n - \tau_{n-1}) \right)^\alpha .$$

Since for each $n \geq 1$ and any $s \in \Re$

$$\mathbb{E}[(\tau_n - \tau_{n-1})^\alpha | M_{\tau_{n-1}-r} = s] = \mathbb{E}_\varphi\, \tau_1^\alpha$$

and by the construction of ζ

$$\mathbb{E}_\varphi\, \tau_1^\alpha < \infty \quad \Rightarrow \quad \mathbb{E}_\zeta\, \tau_1^\alpha < \infty,$$

by Lemma 1.15 (i)

$$\mathbb{E}_\varphi\, \tau_1^\alpha < \infty \;\; \Rightarrow \;\; \mathbb{E}_\zeta\, (\tau_1')^\alpha < \infty;$$

$$\mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty \;\; \Rightarrow \;\; \mathbb{E}_\zeta \left(\sum_{n=1}^{\tau_1'} X_n^+ \right)^\alpha < \infty.$$

In the same manner, by Lemma 1.15 (ii), for some $\gamma > 0$

$$\mathbb{E}_\varphi\, \exp(\gamma \tau_1) < \infty \;\; \Rightarrow \;\; \mathbb{E}_\zeta\, \exp(\gamma' \tau_1') < \infty \quad \text{for some} \quad \gamma' > 0;$$

$$\mathbb{E}_\varphi\, \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \;\; \Rightarrow \;\; \mathbb{E}_\zeta\, \exp\left(\gamma' \sum_{n=1}^{\tau_1'} X_n^+ \right) < \infty \text{ for some } \gamma' > 0.$$

In the sequel we assume that $|X_{\tau_n - r + j}| \leq c$ for some $c \geq 0$ and any $n \geq 1$, $j = 1, \cdots, r$.

Lemma 3.2 *Let $\alpha \geq 1$. If*

$$\mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty,$$

then there exist a nonnegative real number x_0 and $m \geq 1$ such that

$$\mathbb{E}\; [(\tilde{\tau}(x_0))^\alpha | M_{\tau_m - r} = s, W_{\tau_m - r} = w] \leq \begin{cases} (a + b\,w)^\alpha & : \;\; (s, w) \in \Re \times (x_0, \infty) \\ d & : \;\; (s, w) \in \Re \times [0, x_0] \end{cases}$$

for some constants a, b and d, where the random time $\tilde{\tau}(x_0)$ is defined as

$$\tilde{\tau}(x_0) = \inf\{n > 0 : W_{\tau_{mn} - r} \leq x_0\}.$$

Proof. Note first that $(M_{\tau_{mn}-r}, W_{\tau_{mn}-r})_{n\geq 1}$ forms a temporally homogeneous Markov chain. Consider the test function $V : \Re \times [0,\infty) \to [0,\infty)$ defined as $V(s,w) = w$. Then for any $(s,w) \in \Re \times [0,\infty)$

$$\begin{aligned} \Lambda(s,w) &:= \mathbb{E}\left[(W_{\tau_{mn}-r} - W_{\tau_{m(n-1)}-r})^{\alpha} | M_{\tau_{m(n-1)}-r} = s, W_{\tau_{m(n-1)}-r} = w\right] \\ &\leq \mathbb{E}_{\varphi}\left(rc + \sum_{k=1}^{\tau_m - r} X_k^+\right)^{\alpha} \\ &< \infty, \end{aligned}$$

because for any $s \in \Re$ and $k \geq 1$

$$\mathbb{E}\left[\left(\sum_{n=\tau_k - r+1}^{\tau_{k+1}-r} X_n^+\right)^{\alpha} \Big| M_{\tau_k - r} = s\right] \leq \mathbb{E}_{\varphi}\left(2rc + \sum_{n=1}^{\tau_1} X_n^+\right)^{\alpha} < \infty.$$

On the other hand, by Proposition 2.2 there exists $m \geq 1$ such that

$$\mathbb{E}_{\varphi}\left(\sum_{n=1}^{\tau_m} X_n\right) < -2rc - 2\epsilon$$

for some $\epsilon > 0$. For fixed $x \geq 0$ let $\tau_n(x), n \geq 0$, be random times defined as

$$\tau_n(x) := \sup\left\{k : \sum_{l=\tau_{m(n-1)}+1}^{k} X_l^- < x\right\}, \quad n \geq 1.$$

Then for each $n \geq 1$

$$\mathbb{E}_{\varphi}\left(\sum_{k=\tau_{m(n-1)}+1}^{\tau_n(x)\wedge\tau_{mn}} X_k^+\right) \uparrow \mathbb{E}_{\varphi}\left(\sum_{k=\tau_{m(n-1)}+1}^{\tau_{mn}} X_k^+\right) \quad \text{as} \quad x \uparrow \infty$$

and thus there exists a real number $x_0' \geq 0$ such that for any $n \geq 1$

$$\begin{aligned} \mathbb{E}_{\varphi}\left(\sum_{k=\tau_{m(n-1)}+1}^{\tau_{mn}} X_k^+ - \sum_{k=\tau_{m(n-1)}+1}^{\tau_n(x_0')\wedge\tau_{mn}} X_k^-\right) &< -2rc - \epsilon; \\ \mathbb{E}_{\varphi}\left(\sum_{k=\tau_n(x_0')\wedge\tau_{mn}+1}^{\tau_{mn}} X_k^+\right) &< \frac{\epsilon}{2}. \end{aligned}$$

In turn, for any $s \in \Re$ and $w > x_0 := 2rc + x_0'$

$$\begin{aligned} \mathbb{E}\Lambda(s,w) &= \mathbb{E}\left[W_{\tau_{nm}-r} - W_{\tau_{m(n-1)}-r} \Big| M_{\tau_{m(n-1)}} = s, W_{\tau_{m(n-1)}-r} = w\right] \\ &\leq \mathbb{E}\left[\sum_{k=\tau_{m(n-1)}+1}^{\tau_n(x_0')\wedge\tau_{mn}} X_k + \sum_{k=\tau_n(x_0')\wedge\tau_{mn}+1}^{\tau_{mn}-r} X_k^+ \Big| M_{\tau_{m(n-1)}} = s\right] + 2rc + \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned} &\le \mathbb{E}_\varphi \left(\sum_{k=1}^{\tau_1(x_0')\wedge\tau_m} X_k + \sum_{k=\tau_1(x_0')\wedge\tau_m+1}^{\tau_m} X_k^+ \right) + 2rc + \frac{\epsilon}{2} \\ &< -2rc - \epsilon + 2rc + \frac{\epsilon}{2} < 0. \end{aligned}$$

Thus the assertion follows from Proposition 1.4 (i). QED

Theorem 3.3 *Let $\alpha \ge 1$. Suppose that*

$$\mathbb{E}_\varphi \, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty,$$

then $\mathbb{E}_\lambda \, \sigma_1^\alpha < \infty$. Moreover, if

$$\sup_{s\in E} \mathbb{E}_s \, \tau_1^\alpha < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \, (X_1^+)^\alpha < \infty,$$

then $\sup_{s\in E} \mathbb{E}_{s,w} \, \sigma_1^\alpha < \infty$ for any $w \ge 0$.

Proof. We keep the notations of the proof of Lemma 3.2.

Note first that for any $w \le x_0$

$$\mathbb{E}\left[(\tilde\tau(x_0))^\alpha | M_{\tau_m - r} = s, W_{\tau_m - r} = w\right] \le \mathbb{E}\left[(\tilde\tau(x_0))^\alpha | M_{\tau_m - r} = s, W_{\tau_m - r} = x_0\right] < \infty.$$

W.l.o.g. we may assume $m = r = 1$. By Proposition 2.2, there exist $m' \ge 1$ and $q > 0$ such that

$$\mathbb{P}_\varphi \left(\sum_{k=1}^{\tau_{m'}} X_k < -x_0 - 2c \right) > q.$$

Thus for any $(s, w) \in \Re \times [0, x_0]$

$$\begin{aligned} &\mathbb{P}\left[W_k = 0 \text{ for some } k, \tau_n \le k \le \tau_{n+m'} - 1 \Big| M_{\tau_n - 1} = s, W_{\tau_n - 1} = w \right] \\ &\ge \mathbb{P}\left[\sum_{k=\tau_n+1}^{\tau_{n+m'}-1} X_n < -x_0 - c \Big| M_{\tau_n - 1} = s \right] \\ &\ge \mathbb{P}_\varphi \left(\sum_{k=1}^{\tau_{m'}} X_n < -x_0 - 2c \right) \\ &> q. \end{aligned}$$

Let $\eta_n, n \geq 1$, be random times defined as

$$\eta_0 = 0 \quad \text{and} \quad \eta_n = \inf\{k \geq \eta_{n-1} + m' : W_{\tau_k - 1} \leq x_0\}, \quad n \geq 1.$$

Let further ν be a random time defined as

$$\nu := \inf\{n : W_k = 0 \text{ for some } \tau_{\eta_n} \leq k \leq \tau_{\eta_n + m'} - 1\}.$$

Then by the definition of ν

$$\mathbb{P}(\nu > n) \leq (1-q)^n, \quad n \geq 1.$$

Moreover, for any $w \leq x_0$

$$\mathbb{E}\left[\sigma_1^\alpha | M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = w\right] \leq \mathbb{E}\Big[\Big(\sum_{n=2}^{\eta_\nu + m'} (\tau_n - \tau_{n-1})\Big)^\alpha \Big| M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0\Big].$$

Note that for any $n \geq 1$,

$$\mathbb{E}\big[(\eta_{n+1} - \eta_n)^\alpha | \mathcal{G}_n\big] \leq \sup_{s \in \Re} \mathbb{E}\big[(\tilde{\tau}(x_0) + m')^\alpha | M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0\big] < \infty,$$

where $\mathcal{G}_n$ is the σ-algebra generated by $\{(M_{\tau_k - 1}, W_{\tau_k - 1}) | k \leq \eta_n\}$. Since further

$$\sup_{s \in \Re} \mathbb{E}\left[\nu^\alpha | M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0\right] \leq \sum_{n=0}^{\infty} (n+1)^\alpha (1-q)^n < \infty,$$

by Lemma 1.14 (i) we have

$$\begin{aligned} \mathbb{E}\big[(\eta_\nu + m')^\alpha | \mathcal{G}_1\big] &\leq \sup_{s \in \Re} \mathbb{E}\Big[\Big(\sum_{n=1}^{\nu} (\eta_n - \eta_{n-1}) + m'\Big)^\alpha | M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0\Big] \\ &< l_1 < \infty \end{aligned}$$

for some constant l_1. Note that for any $n \geq 2$ and $(s, w) \in \Re \times [0, \infty)$

$$\mathbb{E}\left[(\tau_n - \tau_{n-1})^\alpha | M_{\tau_{n-1} - 1} = s, W_{\tau_{n-1} - 1} = w\right] = \mathbb{E}_\varphi \tau_1^\alpha.$$

Thus for any initial distribution λ on E

$$\begin{aligned} \mathbb{E}_\lambda \left(\sum_{n=\tilde{\tau}(x_0)}^{\eta_\nu + m'} (\tau_n - \tau_{n-1}) \right)^\alpha &\leq \sup_{s \in \Re} \mathbb{E}\left[\Big(\sum_{n=2}^{\eta_\nu + m'} (\tau_n - \tau_{n-1})\Big)^\alpha \Big| M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0\right] \\ &\leq c' \left(\mathbb{E}_\varphi \tau_1^\alpha\right) \sup_{s \in \Re} \mathbb{E}\big[(\eta_\nu + m')^\alpha \big| M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0\big] \\ &\leq c' \, l_1 \, \mathbb{E}_\varphi \tau_1^\alpha < \infty \end{aligned}$$

for some constant c'. Now let $w > x_0$. Then, from Lemma 3.2 and Lemma 1.14 (i),

$$\begin{aligned}\mathbb{E}&\left[\Big(\sum_{n=2}^{\tilde{\tau}(x_0)}(\tau_n-\tau_{n-1})\Big)^{\alpha}\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right]\\ &\le c''\,(\mathbb{E}_{\varphi}\,\tau_1^{\alpha})\,\mathbb{E}\big[\big(\tilde{\tau}(x_0)\big)^{\alpha}\big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\big]\\ &\le c''\,(\mathbb{E}_{\varphi}\,\tau_1^{\alpha})\,(a+bw)^{\alpha}\end{aligned}$$

for some constant c''. In turn,

$$\begin{aligned}&\mathbb{E}_{\lambda}\left(\sum_{n=2}^{\tilde{\tau}(x_0)}(\tau_n-\tau_{n-1})\right)^{\alpha}\\ &=\int_{\Re}\int_0^{\infty}\mathbb{E}\left[\Big(\sum_{n=2}^{\tilde{\tau}(x_0)}(\tau_n-\tau_{n-1})\Big)^{\alpha}\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right]\mathbb{P}_{\lambda}^{(M_{\tau_1-1},W_{\tau_1-1})}(ds,dw)\\ &\le c''\{\mathbb{E}_{\varphi}\,\tau_1^{\alpha}\}\int_0^{\infty}\big(a+b\,w+d\big)^{\alpha}\,\mathbb{P}_{\lambda}\,(W_{\tau_1-1}\in dw)\\ &\le c''\{\mathbb{E}_{\varphi}\,\tau_1^{\alpha}\}\,\mathbb{E}_{\lambda}\left(a+b\sum_{n=1}^{\tau_1}X_n^{+}+d\right)^{\alpha}\\ &<\infty.\end{aligned}$$

The first assertion follows from the inequality

$$\mathbb{E}_{\lambda}\,\sigma_1^{\alpha}\ \le\ \mathbb{E}_{\lambda}\left(\tau_1+\sum_{n=2}^{\tilde{\tau}(x_0)}(\tau_n-\tau_{n-1})+\sum_{n=\tilde{\tau}(x_0)}^{\eta_{\nu}+m'}(\tau_n-\tau_{n-1})\right)^{\alpha}.$$

For the second assertion it suffices to note that for any $w \ge 0$

$$\begin{aligned}\sup_{s\in E}\mathbb{E}_{s,w}\left(\sum_{n=2}^{\tilde{\tau}(x_0)}(\tau_n-\tau_{n-1})\right)^{\alpha}\ &\le\ c''\{\mathbb{E}_{\varphi}\,\tau_1^{\alpha}\}\left\{\sup_{s\in E}\mathbb{E}_s\left(a+bw+b\sum_{n=1}^{\tau_1}X_n^{+}+d\right)^{\alpha}\right\};\\ \sup_{s\in E}\mathbb{E}_{s,w}\left(\sum_{n=\tilde{\tau}(x_0)}^{\eta_{\nu}+m'}(\tau_n-\tau_{n-1})\right)^{\alpha}\ &\le\ c'\,l_1\,\mathbb{E}_{\varphi}\,\tau_1^{\alpha}.\end{aligned}$$

QED

From the proof of Theorem 3.3, one should notice that if

$$\sup_{s\in\Re}\mathbb{E}_s\,\tau_1^{\alpha}<\infty\quad\text{and}\quad\sup_{s\in\Re}\mathbb{E}_s\left(\sum_{n=1}^{\tau_1}X_n^{+}\right)^{\alpha}<\infty,$$

then $\sup_{s\in\Re} \mathbb{E}_{s,w}\, \sigma_1^\alpha < \infty$ for any $w \geq 0$. Moreover, if for a measurable subset A of E with $\Re \subset A$

$$\sup_{s\in A} \mathbb{E}_s\, \tau_1^\alpha < \infty \quad \text{and} \quad \sup_{s\in A} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty,$$

then $\sup_{s\in A} \mathbb{E}_{s,w}\, \sigma_1^\alpha < \infty$ for any $w \geq 0$.

Remark 3.4 Suppose that the condition of Lemma 3.2 holds true. Then, from Lemma 3.2, it is clear that for any $(s, w) \in \Re \times [0, \infty)$

$$\mathbb{P}\left[(M_n, W_n)_{n\geq 0} \in \Re \times [0, x_0] \quad \text{i.o.}\, | M_{\tau_m - r} = s, W_{\tau_m - r} = w\right] = 1.$$

Thus for any $(s, w) \in E \times [0, \infty)$

$$\begin{aligned}
&\mathbb{P}_{s,w}\left((M_n, W_n)_{n\geq 0} \in \Re \times [0, x_0] \quad \text{i.o.}\right) \\
&\quad = \int_{\{\tau_m < \infty\}} \mathbb{P}\left[(M_n, W_n)_{n\geq 0} \in \Re \times [0, x_0] \quad \text{i.o.}\, | M_{\tau_m - r}, W_{\tau_m - r}\right] d\mathbb{P}_{s,w} \\
&\quad = \mathbb{P}_{s,w}(\tau_m < \infty) \\
&\quad = 1,
\end{aligned}$$

which means that $\Re \times [0, x_0]$ is a recurrent set of $(M_n, W_n)_{n\geq 0}$.

As a consequence of Theorem 3.3 we obtain moment conditions for $|S_{\sigma_1}|^\alpha$.

Corollary 3.5 *(i) Let $\alpha \geq 1$. Suppose that*

$$\mathbb{E}_\xi\, X_1^+ < \infty \quad \textit{and} \quad \mathbb{E}_\xi\, (X_1^-)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} (X_n^-)^\alpha \right) < \infty,$$

then $\mathbb{E}_\lambda\, |S_{\sigma_1}|^\alpha < \infty$. Moreover, if

$$\sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} (X_n^-)^\alpha \right) < \infty,$$

then $\sup_{s\in E} \mathbb{E}_s\, |S_{\sigma_1}|^\alpha < \infty$.

(ii) Let $\gamma > 0$. Suppose that

$$\mathbb{E}_\xi\, X_1^+ < \infty \quad \textit{and} \quad \mathbb{E}_\xi\, \exp(\gamma X_n^-) < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} \exp(\gamma X_n^-) \right) < \infty,$$

then $\mathbb{E}_\lambda \exp(\gamma|S_{\sigma_1}|) < \infty$. *Moreover, if*

$$\sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} \exp(\gamma X_n^-) \right) < \infty,$$

then $\sup_{s\in E} \mathbb{E}_s \exp(\gamma|S_{\sigma_1}|) < \infty$.

Proof. (i) We keep the notations in the proof of Theorem 3.3.

Note first that

$$\mathbb{E}_\xi X_1^+ < \infty \Rightarrow \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \text{and} \quad \mathbb{E}_\xi (X_1^-)^\alpha < \infty \Rightarrow \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} (X_n^-)^\alpha \right) < \infty.$$

Note further that for any $n \geq 1$

$$\mathbb{E} \left[\sum_{k=\tau_n}^{\tau_{n+1}-1} (X_k^-)^\alpha \Big| M_{\tau_n - 1} = s \right] \leq 2c^\alpha + \mathbb{E}_\varphi \left(\sum_{k=1}^{\tau_1} (X_k^-)^\alpha \right) =: l'.$$

Thus, for any initial distribution λ on E,

$$\begin{aligned}
&\mathbb{E}_\lambda \left(\sum_{n=\tilde{\tau}(x_0)}^{\eta_\nu + m'} \sum_{k=\tau_{n-1}}^{\tau_n - 1} (X_k^-)^\alpha \right) \\
&\leq \mathbb{E} \left[\sum_{n=\tilde{\tau}(x_0)}^{\eta_\nu + m'} \sum_{k=\tau_{n-1}}^{\tau_n - 1} (X_k^-)^\alpha \Big| M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0 \right] \\
&\leq c_1 \mathbb{E} \left[\eta_\nu + m' | M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = x_0 \right] \cdot \sup_{s \in \Re} \mathbb{E} \left[\sum_{k=\tau_n}^{\tau_{n+1}-1} (X_k^-)^\alpha \Big| M_{\tau_n - 1} = s \right] \\
&\leq c_1 l_1 l',
\end{aligned}$$

where c_1 is a constant, and from the proof of Theorem 3.3

$$\begin{aligned}
&\mathbb{E}_\lambda \left(\sum_{n=2}^{\tilde{\tau}(x_0)} \sum_{k=\tau_{n-1}}^{\tau_n} (X_k^-)^\alpha \right) \\
&= \int_\Re \int_0^\infty \mathbb{E} \left[\sum_{n=2}^{\tilde{\tau}(x_0)} \sum_{k=\tau_{n-1}}^{\tau_n} (X_k^-)^\alpha \Big| M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = w \right] \mathbb{P}_\lambda^{(M_{\tau_1 - 1}, W_{\tau_1 - 1})} (ds, dw) \\
&\leq l'c' \{\mathbb{E}_\varphi \tau_1\} \int_0^\infty \big(a + b\, w + d\big) \, \mathbb{P}_\lambda (W_{\tau_1 - 1} \in dw) \\
&\leq l'c' \{\mathbb{E}_\varphi \tau_1\} \left\{ \mathbb{E}_\lambda \left(a + b \sum_{n=1}^{\tau_1} X_n^+ \right) + d \right\}.
\end{aligned}$$

Hence the first assertion follows from

$$\begin{aligned}
\mathbb{E}_\lambda |S_{\sigma_1}|^\alpha &\le \mathbb{E}_\lambda \left(\sum_{n=1}^{\sigma_1} (X_n^-)^\alpha \right) \\
&\le \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} (X_n^-)^\alpha \right) + \mathbb{E}_\lambda \left(\sum_{n=2}^{\tilde\tau(x_0)} \sum_{k=\tau_{n-1}}^{\tau_n} (X_k^-)^\alpha \right) + \mathbb{E}_\lambda \left(\sum_{n=\tilde\tau(x_0)}^{\eta_\nu+m'} \sum_{k=\tau_{n-1}}^{\tau_n} (X_k^-)^\alpha \right) \\
&\le \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} (X_n^-)^\alpha \right) + l'c' \{\mathbb{E}_\varphi \tau_1\} \left\{ \mathbb{E}_\lambda \left(a + b \sum_{n=1}^{\tau_1} X_n^+ \right) + d \right\} + c_1 l_1 l'.
\end{aligned}$$

The second assertion is obvious.

(ii) Noting that

$$\exp(\gamma |S_{\sigma_1}|) \le \sum_{n=1}^{\sigma_1} \exp(\gamma X_n^-),$$

one gets all assertions, in the same manner. QED

From the proof of Corollary 3.5 it is clear that

$$\begin{aligned}
\mathbb{E}_\xi X_1^+ < \infty \quad \text{and} \quad \mathbb{E}_\xi (X_1^-)^\alpha < \infty \;&\Rightarrow\; \mathbb{E}_\varphi |S_{\sigma_1}|^\alpha < \infty; \\
\mathbb{E}_\xi X_1^+ < \infty \quad \text{and} \quad \mathbb{E}_\xi \exp(\gamma X_n^-) < \infty \;&\Rightarrow\; \mathbb{E}_\varphi \exp(\gamma |S_{\sigma_1}|) < \infty.
\end{aligned}$$

If M is uniformly Harris ergodic, then we get the following assertions, which are obtained in Fuh and Lai [28].

Corollary 3.6 *Let $\alpha \ge 1$ and suppose that M is uniformly Harris ergodic.*

(i) If $\sup_{s\in E} \mathbb{E}_s (X_1^+)^\alpha < \infty$, then $\sup_{s\in E} \mathbb{E}_s \sigma_1^\alpha < \infty$.

(ii) If $\sup_{s\in E} \mathbb{E}_s X_1^+ < \infty$ and $\sup_{s\in E} \mathbb{E}_s (X_1^-)^\alpha < \infty$, then $\sup_{s\in E} \mathbb{E}_s |S_{\sigma_1}|^\alpha < \infty$.

Proof. If M is uniformly Harris ergodic, then by Proposition 1.15 (iii) $\sup_{s\in E} \mathbb{E}_s \tau_1^\alpha < \infty$ for any $\alpha \ge 1$, whence

$$\sup_{s\in E} \mathbb{E}_s (X_1^+)^\alpha < \infty \;\Rightarrow\; \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty.$$

Thus (i) follows directly from Theorem 3.3. For (ii) it suffices to note that

$$\mathbb{E}_s |S_{\sigma_1}|^\alpha \le \mathbb{E}_s \left(\sum_{n=1}^{\sigma_1} (X_n^-)^\alpha \right) \le c_1 \left\{ \sup_{s'\in E} \mathbb{E}_{s'} (X_1^-)^\alpha \right\} \left\{ \sup_{s'\in E} \mathbb{E}_{s'} \sigma_1 \right\}, \quad s \in E,$$

where c_1 is a suitable constant. QED

In particular, if E is a one-element set, then we have:

$$\begin{array}{rl} \mathbb{E}\,(X_1^+)^\alpha < \infty \quad \text{for some} \quad \alpha \geq 1 & \Rightarrow \quad \mathbb{E}\,\sigma_1^\alpha < \infty; \\ \mathbb{E}\,X_1^+ < \infty \quad \text{and} \quad \mathbb{E}\,(X_1^-)^\alpha < \infty \quad \text{for some} \quad \alpha \geq 1 & \Rightarrow \quad \mathbb{E}\,|S_{\sigma_1}|^\alpha < \infty, \end{array}$$

which are classical results in Gut [30].

As another consequence of Theorem 3.3 we get conditions for the uniform convergence of σ_1^α, in the sense that

$$\sup_{s\in A} \mathbb{E}_s\big(\sigma_1^\alpha\,\mathbf{1}(\sigma_1 > x)\big) \to 0 \quad \text{as} \quad x \to \infty$$

for some $A \in \mathcal{E}$. Remember that for any nonnegative real-valued random variable Z and $A \in \mathcal{E}$

$$\sup_{s\in A} \mathbb{E}_s\big(Z\,\mathbf{1}(Z > x)\big) = 0 \quad \text{as} \quad x \to \infty \quad \Leftrightarrow \quad \sup_{s\in A} \mathbb{E}_s\,G(Z) < \infty \quad \text{for some} \quad G \in \Theta_c,$$

where Θ_c is a set of real functions $G : [0,\infty) \to [0,\infty)$ having concave derivatives g with $\lim_{x\to\infty} g(x) = \infty$.

Corollary 3.7 *Let $\alpha \geq 1$ and $A \in \mathcal{E}$ with $\Re \subset A$. If for some $\epsilon > 0$*

$$\sup_{s\in A} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^{\alpha(1+\epsilon)} < \infty \quad \text{and} \quad \sup_{s\in A} \mathbb{E}_s\,\tau_1^{\alpha(1+\epsilon)} < \infty,$$

then

$$\sup_{s\in A} \mathbb{E}_s\big(\sigma_1^\alpha \mathbf{1}(\sigma_1 > x)\big) \to 0 \quad \text{as} \quad x \to \infty.$$

Proof. Since $x^{1+\epsilon} \in \Theta_c$ for $0 < \epsilon < 1$, the assertion follows from the proof of Theorem 3.3 and

$$\begin{aligned} &\sup_{s\in A} \mathbb{E}_s\,\sigma_1^{\alpha(1+\epsilon)} < \infty \\ &\qquad \Rightarrow \sup_{s\in A} \mathbb{E}_s\,\big(\sigma_1^\alpha \mathbf{1}(\sigma_1 > x)\big) \leq \sup_{s\in A} \mathbb{E}_s\,\big(\sigma_1^\alpha \mathbf{1}(\sigma_1^\alpha > x)\big) \to 0 \quad \text{as} \quad x \to \infty. \end{aligned}$$

QED

In particular, if M is uniformly Harris ergodic, then

$$\sup_{s\in E} \mathbb{E}_s\,(X_1^+)^{\alpha(1+\epsilon)} \quad \Rightarrow \quad \sup_{s\in E} \mathbb{E}_s\big(\sigma_1^\alpha \mathbf{1}(\sigma_1 > x)\big) \to 0 \quad \text{as} \quad x \to \infty.$$

Remark 3.8 Let $(X_n)_{n\geq 0}$ be a sequence of nonnegative real-valued random variables adapted to a filtration $(\mathcal{F}_n)_{n\geq 0}$ and τ an a.s. finite stopping time with respect to $(\mathcal{F}_n)_{n\geq 0}$. Suppose that for some $\alpha \geq 1$

$$\begin{aligned} \mathbb{E}\big[\tau^\alpha \mathbf{1}(\tau > x)|\mathcal{F}_0\big] \quad &< \quad \epsilon_1(x) \to 0; \\ \mathbb{E}\big[X_n^\alpha \mathbf{1}(X_n > x)|\mathcal{F}_{n-1}\big] \quad &< \quad \epsilon_2(x) \to 0, \quad n \geq 1, \end{aligned}$$

as $x \to \infty$. Borovkov and Utev [20] showed that

$$\mathbb{E}\left[\Big(\sum_{n=1}^{\tau} X_n\Big)^{\alpha}\mathbf{1}\Big(\sum_{n=1}^{\tau} X_n > x\Big)\big|\mathcal{F}_0\right] < \epsilon_3(x) \to 0 \quad \text{as} \quad x \to \infty.$$

Remark 3.9 It is known (see Kalashnikov [32, 34]) that if for a real function $G \in \Theta_c$ with derivative g there exist a nonnegative function $V : E \to [0, \infty)$, positive numbers Δ, b and a random variable Λ defined on E satisfying Proposition 1.4 (i) through (iii) and

$$\sup_{s \in E} \mathbb{E}\, G(|\Delta(s)|) < \infty,$$

then $\sup_{s\in\Re} \mathbb{E}_s\, G(\tau(\Re)) < \infty$. If $s \notin \Re$, then

$$\mathbb{E}_s\, G(\tau(\Re)) \le G\left(a_s + \frac{2V(s)}{\Delta}\right),$$

where

$$a_s = g^{-1}\left(\frac{1}{\Delta}\mathbb{E}_s\, G\Big(1 + \frac{2|\Delta(s)|}{\Delta}\Big)\right).$$

In combination with a result of Borovkov and Utev (see Remark 3.8), one can easily show that if for some $G \in \Theta_c$

$$\sup_{s\in\Re} \mathbb{E}_s\, G\left(\sum_{n=1}^{\tau_1} X_n^+\right) < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s\, G(\tau_1) < \infty,$$

then $\sup_{s\in\Re} \mathbb{E}_s\, G'(\sigma_1) < \infty$ for some $G' \in \Theta_c$.

Next we find conditions for the finiteness of exponential moments of the first weak descending ladder epoch.

Lemma 3.10 *Let $\gamma > 0$. If*

$$\mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1} X_n^+\Big) < \infty,$$

then for a suitable $m \ge 1$ there exist real numbers $y_0 \ge 0$ and $\gamma' > 0$ such that

$$\mathbb{E}[\exp(\gamma'\tilde{\tau}(y_0))|M_{\tau_m - r} = s, W_{\tau_m - r} = w] \le \begin{cases} \exp(\gamma' w) & : \ (s,w) \in \Re \times [y_0, \infty) \\ e^{\gamma'} \cdot \Big\{\exp(\gamma' y_0) + \mathbb{E}_\varphi \exp\Big(\gamma' \sum_{n=1}^{\tau_m} X_n^+\Big)\Big\} & : \ (s,w) \in \Re \times [0, y_0] \end{cases}$$

for some $\gamma' > 0$, where the random time $\tilde{\tau}(y_0)$ is defined as

$$\tilde{\tau}(y_0) = \inf\{n > 0 : W_{\tau_{mn} - r} \le y_0\}.$$

Proof. Note that under the condition $\mu < 0$

$$\mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1} X_n^+\Big) < \infty \quad \Rightarrow \quad \mathbb{E}_\varphi \exp\Big(\gamma' \sum_{n=1}^{\tau_1} X_n\Big) < 1$$

for some $\gamma' > 0$. For notational convenience we write γ instead of γ'. W.l.o.g. we may even assume that

$$\prod_{j=1}^{r} e^{\gamma c_j}\, \mathbb{E}_\zeta \exp\Big(\gamma \sum_{n=1}^{\tau_1 - r} X_n\Big) < 1.$$

and that $|X_{\tau_n - r + j}| \le c_j$ for any $n \ge 1$, $j = 1, \cdots, r$ and put $rc' = c_1 + \cdots + c_r$.

Consider the test function $V : [0,\infty) \to [1,\infty)$ defined as $V(w) = \exp(\gamma w)$. Then for any $w \ge 0$

$$\begin{aligned}
&\mathbb{E}\big[(V(W_{\tau_{mn}-r}) - V(W_{\tau_{m(n-1)}-r})|M_{\tau_{m(n-1)}-r} = s, W_{\tau_{m(n-1)}-r} = w\big] \\
&\le e^{\gamma(w-rc')}\, \mathbb{E}\Big[\exp\Big(2\gamma rc' + \gamma \sum_{k=\tau_{m(n-1)}+1}^{\tau_{mn}} X_k^+\Big) - 1\big|M_{\tau_{m(n-1)}-r} = s\Big] \\
&\le e^{\gamma(w-rc')}\, \mathbb{E}_\varphi \exp\Big(2\gamma rc' + \gamma \sum_{k=1}^{\tau_m} X_k^+\Big) - e^{\gamma(w-rc')} \\
&< \infty.
\end{aligned}$$

On the other hand, there exists $m \ge 1$ such that

$$\mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_m} X_n\Big) \quad < \quad e^{-\gamma(2rc'+1)} - 2\epsilon$$

for some $\epsilon > 0$, since

$$\mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_m} X_n\Big) \le \Big\{\prod_{j=1}^{r} e^{\gamma c_j} \mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1 - r} X_n\Big)\Big\}^m \to 0 \quad \text{as} \quad m \to \infty.$$

Furthermore, there exists a positive real number y_0 such that for any $n \ge 1$

$$\begin{aligned}
\mathbb{E}_\varphi \exp\Big[\gamma\Big(\sum_{k=1}^{\tau_m} X_k^+ - \sum_{k=1}^{\tau_1(y_0)\wedge\tau_m} X_k^-\Big)\Big] \quad &< \quad e^{-\gamma(2rc'+1)} - \epsilon; \\
\mathbb{E}_\varphi \exp\Big(\gamma \sum_{k=\tau_1(y_0)\wedge\tau_m+1}^{\tau_m - r} X_k^+\Big) \quad &< \quad 1 + \epsilon,
\end{aligned}$$

where $\tau_n(y_0), n \ge 1$, are random variables defined in the proof of Lemma 3.2. Thus for any $w \ge y_0 + rc'$

$$\mathbb{E}\Big[V(W_{\tau_{mn}-r}) - V(W_{\tau_{m(n-1)}-r})\big|M_{\tau_{m(n-1)}-r} = s, W_{\tau_{m(n-1)}-r} = w\Big]$$

$$
\begin{aligned}
&= \mathbb{E}\Big[\exp(\gamma W_{\tau_{mn}-r}) - \exp(\gamma W_{\tau_{m(n-1)}-r})\big| M_{\tau_{m(n-1)}-r} = s, W_{\tau_{m(n-1)}-r} = w\Big] \\
&\le e^{\gamma(w-rc')}\,\mathbb{E}\Big[\exp(\gamma(W_{\tau_{mn}-r} - w + rc')) - 1\big| M_{\tau_{m(n-1)}-r} = s\Big] \\
&\le e^{\gamma w}\,\mathbb{E}\Big[\exp\Big(\gamma \sum_{k=\tau_{m(n-1)}+1}^{\tau_n(y_0)\wedge\tau_{mn}} X_k + \gamma \sum_{k=\tau_n(y_0)\wedge\tau_{mn}+1}^{\tau_{mn}} X_k^+ + 2\gamma r c'\Big) - 1\big| M_{\tau_{m(n-1)}-r} = s\Big] \\
&\le e^{\gamma w}\left\{\mathbb{E}_\varphi \exp\Big(\gamma \sum_{k=1}^{\tau_1(y_0)\wedge\tau_m} X_k + \gamma \sum_{k=\tau_1(y_0)\wedge\tau_m+1}^{\tau_m} X_k^+ + 2\gamma r c'\Big) - 1\right\} \\
&\le e^{\gamma w}\big(e^{-\gamma} - 1\big).
\end{aligned}
$$

Therefore, by Proposition 1.4 (ii), we get for any $w \ge 0$

$$
\mathbb{E}[\exp(\gamma\tilde{\tau}(y_0))|M_{\tau_m-r} = s, W_{\tau_m-r} = w] \le \begin{cases} \exp(\gamma w) & : \quad (s,w) \in \Re \times [y_0,\infty) \\ l_{y_0} & : \quad (s,w) \in \Re \times [0,y_0], \end{cases}
$$

where

$$
\begin{aligned}
l_{y_0} &= \sup_{(s,w)\in\Re\times[0,y_0]} \Big\{ \exp\gamma\Big(\exp(\gamma w) + \int_E \big(\exp(\gamma w') - \exp(\gamma w)\big) \\
&\qquad \cdot \mathbb{P}\big[W_{\tau_{mn}-r} \in dw' | (M_{\tau_{m(n-1)}-r}, W_{\tau_{m(n-1)}-r}) = (s,w)\big]\Big)\Big\} \\
&\le e^{\gamma}\Big\{\exp(\gamma w) + \int_0^\infty \big(\exp(\gamma w') - \exp(\gamma y_0)\big)\,\mathbb{P}_\varphi(W_{\tau_{mn}-r} \in dw')\Big\} \\
&\le e^{\gamma}\Big\{\exp(\gamma y_0) + \mathbb{E}_\varphi \exp\big(\gamma \sum_{n=1}^{\tau_m} X_n^+\big)\Big\}.
\end{aligned}
$$

QED

Theorem 3.11 *Let $\gamma > 0$. Suppose that*

$$
\mathbb{E}_\varphi \exp(\gamma\tau_1) < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1} X_n^+\Big) < \infty.
$$

If for an initial distribution λ on E

$$
\mathbb{E}_\lambda \exp(\gamma\tau_1) < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \exp\Big(\gamma \sum_{n=1}^{\tau_1} X_n^+\Big) < \infty,
$$

then $\mathbb{E}_\lambda \exp(\gamma'\sigma_1) < \infty$ for some $\gamma' > 0$. Moreover, if

$$
\sup_{s\in E} \mathbb{E}_s \exp(\gamma\tau_1) < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \exp(\gamma X_1^+) < \infty,
$$

then then for any $w \ge 0$ there exists $\gamma' > 0$ such that $\sup_{s\in E} \mathbb{E}_{s,w} \exp(\gamma'\sigma_1) < \infty$.

Proof. By Lemma 3.10 there exist positive real numbers y_0 and $\gamma_1 > 0$ such that

$$\mathbb{E}[\exp(\gamma_1 \tilde{\tau}(y_0))|M_{\tau_m - r} = s, W_{\tau_m - r} = w] \leq l_{y_0} \exp(\gamma_1 w),$$

where l_{y_0} is given in Lemma 3.10. W.l.o.g. we may assume $m = r = 1$. Note that there exist $m' \geq 1$ and $q > 0$ such that

$$\mathbb{P}_\varphi \left(\sum_{k=\tau_n+1}^{\tau_{n+m'}} X_n < -y_0 - 2c \right) > q$$

for any $n \geq 1$. Let $\eta_n, n \geq 0$, be random variables defined as

$$\eta_0 = 0 \quad \text{and} \quad \eta_n = \inf\{k \geq \eta_{n-1} + m' : W_{\tau_k - 1} \leq y_0\}, \quad n \geq 1,$$

and ν be a random variable defined as

$$\nu := \inf\{n : W_k = 0 \text{ for some } \tau_{\eta_n} \leq k \leq \tau_{\eta_n + m'} - 1\}.$$

Then for any $n \geq 1$

$$\mathbb{P}(\nu > n) \leq (1-q)^n, \quad n \geq 1.$$

Since

$$\sup_{s \in \Re} \mathbb{E}[\exp(\gamma_1(\tilde{\tau}(y_0) + m'))|M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = y_0] < \infty,$$

there exists $\gamma_2, 0 < \gamma_2 \leq \gamma_1$, such that for any $w \leq y_0$ and $n \geq 1$

$$\begin{aligned} \mathbb{E}\big[\exp\big(\gamma_2(\eta_{n+1} - \eta_n)\big)|\mathcal{G}_n\big] &\leq \sup_{s \in \Re} \mathbb{E}\big[\exp\big(\gamma_2(\tilde{\tau}(y_0) + 2m')\big)|M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = y_0\big] \\ &< \min\Big(\frac{2}{2-q}, \exp \gamma_1\Big), \end{aligned}$$

where $\mathcal{G}_n$ is the σ-algebra generated by $\{(M_{\tau_k - 1}, W_{\tau_k - 1})|k \leq \eta_n\}$. Choose $\gamma_3 > 0$ such that

$$\mathbb{E}_\varphi \exp(2\gamma_3 \tau_1) < \exp \gamma_2.$$

Then,

$$\begin{aligned} \mathbb{E}\big[\exp(\frac{\gamma_3}{2}\sigma_1)|\mathcal{G}_1\big] &\leq \mathbb{E}\left[\exp\Big(\frac{\gamma_3}{2} \sum_{n=2}^{\eta_\nu + m'} (\tau_n - \tau_{n-1})\Big)\Big|\mathcal{G}_1\right] \\ &\leq \mathbb{E}\left[\left(\sup_{s \in \Re} \mathbb{E}\Big[\exp\Big(\gamma_3 \sum_{n=2}^{\eta_1 + m'} (\tau_n - \tau_{n-1})\Big)\Big|M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = y_0\Big]\right)^\nu \Big|\mathcal{G}_1\right] \\ &\leq \mathbb{E}\left[\left(\sup_{s \in \Re} \mathbb{E}\Big[\exp(\gamma_2(\tilde{\tau}(y_0) + 2m'))\Big|M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = y_0\Big]\right)^\nu \Big|\mathcal{G}_1\right] \\ &\leq \sup_{s \in \Re} \mathbb{E}\left[\Big(\frac{2}{2-q}\Big)^\nu \Big|M_{\tau_1 - 1} = s, W_{\tau_1 - 1} = y_0\right]. \end{aligned}$$

Note that for any $s \in \Re$

$$\mathbb{E}\left[\left(\frac{2}{2-q}\right)^{\nu}\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=y_0\right] \le \sum_{n=0}^{\infty}\left(\frac{2}{2-q}\right)^{n+1}(1-q)^n =: l' < \infty.$$

Thus for any $(s,w) \in \Re \times [0,\infty)$

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{\gamma_3}{2}\sum_{n=2}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right)\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right] \\
&\le \mathbb{E}\left[\exp\left(\frac{\gamma_3}{2}\,(\sum_{n=2}^{\tilde{\tau}(y_0)}(\tau_n-\tau_{n-1})+\sum_{n=\tilde{\tau}(y_0)}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right)\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right] \\
&\le \mathbb{E}\left[\exp\left(\gamma_3\sum_{n=2}^{\tilde{\tau}(y_0)}(\tau_n-\tau_{n-1})\right)\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right] \\
&\quad \cdot\mathbb{E}\left[\exp\left(\gamma_3\sum_{n=2}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right)\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=x_0\right] \\
&\le l'\,\mathbb{E}\left[\exp(\gamma_2\tilde{\tau}(y_0))|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right] \\
&\le l' l_{y_0}\,\exp(\gamma_1 w)
\end{aligned}$$

and in turn,

$$\begin{aligned}
&\mathbb{E}_\lambda\,\exp\left(\frac{\gamma_3}{2}\sum_{n=2}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right) \\
&= \int\int \mathbb{E}\left[\exp\left(\frac{\gamma_3}{2}\sum_{n=2}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right)\Big|M_{\tau_1-1}=s, W_{\tau_1-1}=w\right]\,\mathbb{P}_\lambda^{(M_{\tau_1-1},W_{\tau_1-1})}(ds,dw) \\
&\le \int l' l_{y_0}\,\exp(\gamma_1 w)\,\mathbb{P}_\lambda\,(W_{\tau_1-1}\in dw) \\
&\le l' l_{y_0}\,\mathbb{E}_\lambda\,\exp\left(\gamma_1\sum_{n=1}^{\tau_1}X_n^+\right).
\end{aligned}$$

Letting $\gamma' := \gamma_3/4$,

$$\begin{aligned}
\mathbb{E}_\lambda\,\exp(\gamma'\sigma_1) &\le \mathbb{E}_\lambda\,\exp\left(\gamma'\tau_1+\gamma'\sum_{n=2}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right) \\
&\le \left\{\mathbb{E}_\lambda\,\exp(\frac{\gamma_3}{2}\tau_1)\right\}\left\{\mathbb{E}_\lambda\,\exp\left(\frac{\gamma_3}{2}\sum_{n=2}^{\eta_\nu+m'}(\tau_n-\tau_{n-1})\right)\right\} \\
&\le \left\{\mathbb{E}_\lambda\,\exp(\gamma\tau_1)\right\}l' l_{y_0}\left\{\mathbb{E}_\lambda\,\exp\left(\gamma\sum_{n=1}^{\tau_1}X_n^+\right)\right\} \\
&< \infty,
\end{aligned}$$

which proves the first assertion. The second assertion is obvious, since for any $s \in E$

$$\begin{aligned}\mathbb{E}_s \exp(\gamma'\sigma_1) &\leq \left\{\sup_{s'\in E} \mathbb{E}_{s'} \exp(\gamma\tau_1)\right\} l' l_{y_0} \left\{\sup_{s'\in E} \mathbb{E}_{s'} \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+\right)\right\} \\ &< \infty.\end{aligned}$$

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From the proof of Theorem 3.11, it is clear that if for some $\gamma > 0$

$$\sup_{s\in\Re} \mathbb{E}_s \exp(\gamma\tau_1) < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+\right) < \infty,$$

then for any $w \geq 0$ there exists $\gamma' > 0$ such that $\sup_{s\in\Re} \mathbb{E}_{s,w} \exp(\gamma'\sigma_1) < \infty$. Moreover, if for a measurable subset A of E with $\Re \subset A$

$$\sup_{s\in A} \mathbb{E}_s \exp(\gamma\tau_1) < \infty \quad \text{and} \quad \sup_{s\in A} \mathbb{E}_s \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+\right) < \infty,$$

then for any $w \geq 0$ there exists $\gamma' > 0$ such that $\sup_{s\in A} \mathbb{E}_{s,w} \exp(\gamma'\sigma_1) < \infty$.

If M is uniformly ergodic, then by Proposition 1.16 (iii) $\sup_{s\in E} \mathbb{E}_s \exp(\gamma'\tau_1) < \infty$ for some $\gamma' > 0$, and so by Lemma 1.15 (ii)

$$\begin{aligned}&\sup_{s\in E} \mathbb{E}_s \exp(\gamma X_1) < \infty \quad \text{for some} \quad \gamma > 0 \\ &\qquad \Rightarrow \sup_{s\in E} \mathbb{E}_s \exp\left(\gamma' \sum_{n=1}^{\tau_1} X_n^+\right) < \infty \quad \text{for some} \quad \gamma' > 0.\end{aligned}$$

Thus, from Theorem 3.11, we get:

Corollary 3.12 *Let $\gamma > 0$. Suppose that M is uniformly ergodic.*

If $\sup_{s\in E} \mathbb{E}_s \exp(\gamma X_1^+) < \infty$, then $\sup_{s\in E} \mathbb{E}_s \exp(\gamma'\sigma_1) < \infty$ for some $\gamma' < 0$.

In particular, if E is a one-element set, then we have:

$$\mathbb{E} \exp(\gamma X_1^+) < \infty \quad \text{for some} \quad \gamma > 0 \;\; \Rightarrow \;\; \mathbb{E} \exp(\gamma'\sigma_1) < \infty \quad \text{for some} \quad \gamma' > 0.$$

3.2 Rates of convergence

In this section we find rates of convergence of reflected MRW's. Throughout this section we assume that M is Harris ergodic. A unique stationary distribution of a reflected MRW $(M_n, W_n)_{n\geq 0}$ is denoted by π. From Proposition 1.20 it is clear that for any initial distribution λ on E

$$\lim_{n\to\infty} \|\mathbb{P}_\lambda((M_n, W_n) \in \cdot\,) - \pi\| = 0.$$

Theorem 3.13 *Suppose that* $\Re \times \{0\}$ *is a recurrent set of* $(M_n, W_n)_{n\geq 0}$.

(i) Let $\alpha > 1$. *Suppose that*

$$\sup_{s\in\Re} \mathbb{E}_s\, \tau_1^\alpha < \infty \quad \textit{and} \quad \sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+\right)^\alpha < \infty.$$

If for an initial distribution λ *on* E

$$\mathbb{E}_\lambda\, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} X_n^+\right)^\alpha < \infty,$$

then there exists a sequence of regeneration epochs $(\hat\tau_n)_{n\geq 0}$ *of* $(M_n, W_n)_{n\geq 0}$ *such that* $\mathbb{E}_\zeta\, \hat\tau_1^\alpha < \infty$ *and* $\mathbb{E}_\lambda\, \hat\tau_1^\alpha < \infty$, *where* $\zeta(\cdot) = \mathbb{P}_{\lambda'}(M_{\hat\tau_1} \in \cdot\,)$ *for each initial distribution* λ' *on* E. *Moreover, for some constant* c

$$\|\mathbb{P}_\lambda\left((M_n, W_n) \in \cdot\,\right) - \pi\| \leq c n^{1-\alpha}.$$

(ii) Let $\gamma > 0$. *Suppose that*

$$\sup_{s\in\Re} \mathbb{E}_s\, \exp(\gamma\tau_1) < \infty \quad \textit{and} \quad \sup_{s\in\Re} \mathbb{E}_s\, \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+\right).$$

Then there exists a sequence of regeneration epochs $(\hat\tau_n)_{n\geq 0}$ *such that* $\mathbb{E}_\zeta\, \exp(\gamma'\hat\tau_1) < \infty$ *for some* $\gamma' > 0$. *Moreover, for some constants* c *and* $\gamma'' \in (0, \gamma']$

$$\|\mathbb{P}_\zeta\left((M_n, W_n) \in \cdot\,\right) - \pi\| \leq c \exp(-\gamma'' n).$$

Proof. (i) By Lemma 3.2 there exist $m \geq 1$ and $x_0 \geq 0$ such that for any $w \leq x_0$

$$\mathbb{E}\left[(\tilde\tau(x_0))^\alpha | M_{\tau_m - r} = s, W_{\tau_m - r} = w\right] \leq \mathbb{E}\left[(\tilde\tau(x_0))^\alpha | M_{\tau_m - r} = s, W_{\tau_m - r} = x_0\right] < \infty.$$

W.l.o.g. we may assume $m = r = 1$. By assumption there exist $r' \geq 1$, $p' > 0$, a regeneration set $\Re_0 \subset \Re$ of M and a distribution ϕ on $\Re_0$ such that

$$\mathbb{P}_s\left((M_{r'}, W_{r'}) \in \cdot\,\right) > p'(\phi \otimes \delta_0)(\cdot)$$

for any $s \in \Re_0$. We show that there exists a regeneration epoch $\hat\tau_1$ for $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\phi\, \hat\tau_1^\alpha < \infty$ and $\mathbb{E}_\lambda\, \hat\tau_1^\alpha < \infty$, from which the assertion follows (see Corollary 1.20).

By ergodicity of $(M_n, W_n)_{n\geq 0}$

$$\lim_{n\to\infty} \mathbb{P}_{\varphi\otimes\delta_{x_0+c}}((M_n, W_n) \in \Re_0 \times \{0\}) = \pi(\Re_0 \times \{0\}) > 0,$$

whence there exists $n_0 \geq 1$ such that

$$q := \mathbb{P}_{\varphi\otimes\delta_{x_0+c}}((M_{n_0}, W_{n_0}) \in \Re_0 \times \{0\}) > 0.$$

In turn, for any $(s, w) \in \Re \times [0, x_0]$ and $n \geq 1$

$$\begin{aligned}
&\mathbb{P}[(M_{\tau_n+n_0}, W_{\tau_n+n_0}) \in \Re_0 \times \{0\} | M_{\tau_n-1} = s, W_{\tau_n-1} = w] \\
&\quad \geq \mathbb{P}[(M_{\tau_n+n_0}, W_{\tau_n+n_0}) \in \Re_0 \times \{0\} | M_{\tau_n-1} = s, W_{\tau_n-1} = x_0] \\
&\quad \geq \mathbb{P}_{\varphi\otimes\delta_{x_0+c}}((M_{n_0}, W_{n_0}) \in \Re_0 \times \{0\}) \\
&\quad = q.
\end{aligned}$$

Let $\eta_n, n \geq 1$, be random times defined as

$$\eta_0 = 0 \quad \text{and} \quad \eta_n = \inf\{k > \eta_{n-1} : W_{\tau_k-1} \leq x_0,\, \tau_k \geq \tau_{\eta_{n-1}} + n_0\}, \quad n \geq 1.$$

Let further ν be a random time defined as

$$\nu := \inf\left\{n : (M_{\tau_{\eta_n}+n_0}, W_{\tau_{\eta_n}+n_0}) \in \Re_0 \times \{0\}\right\}.$$

Then for any $n \geq 1$

$$\mathbb{P}(\nu > n) \leq (1 - q)^n$$

and thus for any $w \leq x_0$

$$\mathbb{E}\left[(\tau(\Re_0 \times \{0\}))^\alpha | M_{\tau_1-1} = s, W_{\tau_1-1} = w\right] \leq \mathbb{E}\Big[\Big(\sum_{n=1}^{\eta_\nu+n_0} (\tau_n - \tau_{n-1})\Big)^\alpha \Big| M_{\tau_1-1} = s, W_{\tau_1-1} = x_0\Big],$$

where $\tau(\Re_0 \times \{0\})$ is the return time of $(M_n, W_n)_{n\geq 0}$ to $\Re_0 \times \{0\}$. As in the proof of Theorem 3.3, one can show, on the one hand, that

$$\mathbb{E}\big[(\eta_\nu + n_0)^\alpha | M_{\tau_1-1} = s, W_{\tau_1-1} = x_0\big] < \infty,$$

from which

$$\sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=\tilde{\tau}(x_0)}^{\eta_\nu+n_0} (\tau_n - \tau_{n-1})\right)^\alpha < \infty,$$

on the other hand, that for any $(s, w) \in \Re \times [0, \infty)$

$$\mathbb{E}\left[\Big(\sum_{n=2}^{\tilde{\tau}(x_0)} (\tau_n - \tau_{n-1})\Big)^\alpha \Big| M_{\tau_1-1} = s, W_{\tau_1-1} = w\right] \leq c''(a + bw)^\alpha\, \mathbb{E}_\varphi\, \tau_1^\alpha,$$

where c'', a and b are suitable constants and thus for any $s \in \Re$

$$\begin{aligned}
\mathbb{E}_s\left(\sum_{n=2}^{\tilde{\tau}(x_0)} (\tau_n - \tau_{n-1})\Big)^\alpha\right) &\leq c''\,\{\mathbb{E}_\varphi\, \tau_1^\alpha\} \int c''(a + bw)^\alpha\, \mathbb{P}_s(W_{\tau_1-1} \in dw) \\
&\leq c''\,\{\mathbb{E}_\varphi\, \tau_1^\alpha\} \left\{\sup_{s\in\Re} \mathbb{E}_s \left(a + b\sum_{n=1}^{\tau_1} X_n^+\right)^\alpha\right\} \\
&< \infty.
\end{aligned}$$

Thus as in the proof of Theorem 3.3

$$\begin{aligned}\sup_{s\in\Re}\mathbb{E}_s\,(\tau(\Re_0\times\{0\}))^\alpha &\le \sup_{s\in\Re}\mathbb{E}_s\left(\tau_1+\sum_{n=2}^{\tilde\tau(x_0)}(\tau_n-\tau_{n-1})+\sum_{n=\tilde\tau(x_0)}^{\eta_\nu+n_0}(\tau_n-\tau_{n-1})\right)^\alpha\\ &< \infty.\end{aligned}$$

Letting $\hat\tau_1$ be the first regeneration epoch of $(M_n, W_n)_{n\ge0}$ constructed by the splitting technique from $\Re_0\times\{0\}$, by Proposition 1.16 (i)

$$\sup_{s\in\Re}\mathbb{E}_s\,\hat\tau_1^\alpha<\infty$$

and, in particular, $\mathbb{E}_\zeta\,\hat\tau_1^\alpha<\infty$ with $\zeta:=\phi$. Note that $\zeta(\cdot)=\mathbb{P}_{\lambda'}\left(M_{\hat\tau_1}\in\cdot\right)$ for each initial distribution λ' on E. Furthermore,

$$\mathbb{E}_\lambda\,(\tau(\Re_0\times\{0\}))^\alpha \;<\; \infty,$$

since

$$\mathbb{E}_\lambda\left(\sum_{n=2}^{\tilde\tau(x_0)}(\tau_n-\tau_{n-1})\right)^\alpha\le c''\{\mathbb{E}_\varphi\,\tau_1^\alpha\}\,\mathbb{E}_\lambda\left(a+b\sum_{n=1}^{\tau_1}X_n^+\right)^\alpha<\infty.$$

Noting that

$$\begin{aligned}\mathbb{E}_\lambda(\hat\tau_1-\tau(\Re_0\times\{0\}))^\alpha &\le \sup_{s\in\Re_0}\mathbb{E}\left[\hat\tau_1^\alpha|M_{\tau(\Re_0\times\{0\})}=s\right]\\ &\le \sup_{s\in\Re}\mathbb{E}_s\,\hat\tau_1^\alpha\\ &< \infty,\end{aligned}$$

we have

$$\mathbb{E}_\lambda\,\hat\tau_1^\alpha\le\mathbb{E}_\lambda\Big(\tau(\Re_0\times\{0\})+(\hat\tau_1-\tau(\Re_0\times\{0\}))\Big)^\alpha<\infty.$$

The second assertion follows from Proposition 1.21 (i) and (iii), since

$$\|\mathbb{P}_\lambda((M_n,W_n)\in\cdot)-\pi\|\le\mathbb{P}_\lambda((M_n,W_n)\in\cdot)-\mathbb{P}_\zeta((M_n,W_n)\in\cdot)\|+\|\mathbb{P}_\zeta((M_n,W_n)\in\cdot)-\pi\|.$$

Applying Proposition 1.21 (ii), (ii) can be proved in the same manner.

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In particular, it is clear that

$$\begin{aligned}\sup_{s\in\Re}\mathbb{E}_s\,\tau_1^\alpha<\infty \quad\text{and}\quad \sup_{s\in\Re}\mathbb{E}_s\left(\sum_{n=1}^{\tau_1}X_n^+\right)^\alpha<\infty \quad\text{for some}\quad \alpha\ge1\\ \Rightarrow \|\mathbb{P}_\varphi\left((M_n,W_n)\in\cdot\right)-\pi\|\le cn^{1-\alpha}.\end{aligned}$$

If M is uniformly ergodic, then E is a regeneration set of M. In this case we have:

Corollary 3.14 *Suppose that M is uniformly ergodic.*

(i) Let $\alpha > 1$. If $\sup_{s\in E} \mathbb{E}_s \left(X_1^+\right)^\alpha < \infty$, then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ such that $\mathbb{E}_\lambda \hat{\tau}_1^\alpha < \infty$ for any initial distribution λ on E. Moreover, for any initial distribution λ on E there exists a constant c such that

$$\|\mathbb{P}_\lambda ((M_n, W_n) \in \cdot\,) - \pi\| \leq cn^{1-\alpha}.$$

(ii) Let $\gamma > 0$. If $\sup_{s\in E} \mathbb{E}_s \exp\left(\gamma X_1^+\right) < \infty$, then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ such that $\mathbb{E}_\zeta \exp(\gamma'\hat{\tau}_1) < \infty$ for some $\gamma' > 0$, where $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\hat{\tau}_1} \in \cdot\,)$. Moreover, for some constants c and $\gamma'' \in (0, \gamma']$

$$\|\mathbb{P}_\zeta ((M_n, W_n) \in \cdot\,) - \pi\| \leq c\exp(-\gamma'' n).$$

Proof. Note first that $E \times \{0\}$ is a recurrent set of $(M_n, W_n)_{n\geq 0}$. Since $\sup_{s\in E} \mathbb{E}_s \left(X_1^+\right)^\alpha < \infty$ implies

$$\sup_{s\in E} \mathbb{E}_s \tau_1^\alpha < \infty \quad \text{and} \quad \sup_{s\in \Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} X_n^+\right)^\alpha < \infty,$$

(i) follows from Theorem 3.13 (i). (ii) can be proved in the same manner. QED

In particular, if E is a one-element set, then $(\sigma_n)_{n\geq 0}$ forms a sequence of regeneration epochs for $(W_n)_{n\geq 0}$, and

$$\begin{aligned} &\mathbb{E}\left(X_1^+\right)^\alpha < \infty \quad \text{for some} \quad \alpha > 1 \\ &\qquad \Rightarrow \|\mathbb{P}\,(W_n \in \cdot\,) - \pi\| \leq cn^{1-\alpha} \quad \text{for some constant} \quad c; \\ &\mathbb{E}\exp\left(\gamma X_1^+\right) < \infty \quad \text{for some} \quad \gamma > 0 \\ &\qquad \Rightarrow \|\mathbb{P}\,(W_n \in \cdot\,) - \pi\| \leq c'\exp(-\gamma' n) \quad \text{for some constants} \quad c \quad \text{and} \quad \gamma' > 0. \end{aligned}$$

It should be noticed that in the proof of Theorem 3.13 the moments of $\hat{\tau}_1$ depend only on the moments of η_1 and τ_1. Suppose that $(M_n, W_n)_{n\geq 0}$ satisfies the minorization condition $\mathcal{M}(\Re, p', r', \varphi \otimes \delta_0)$. Let η_n', $n \geq 1$, be random times defined as

$$\eta_0' = 0, \quad \eta_n' = \inf\{k > \eta_{n-1}' + n_0 + r' : W_{\tau_k - 1} \leq x_0,\, \tau_k \geq \tau_{\eta_{n-1}'} + n_0 + r'\}, \quad n \geq 1.$$

Note that for any $(s, w) \in \Re \times [0, x_0]$

$$\begin{aligned} &\mathbb{P}\left[\left(M_{\tau_{\eta_n'} + n_0 + r'}, W_{\tau_{\eta_n'} + n_0 + r'}\right) \in \cdot \,| M_{\tau_{\eta_n'} - 1} = s, W_{\tau_{\eta_n'} - 1} = w\right] \\ &\geq \int_\Re \mathbb{P}_{s'}\left((M_{r'}, W_{r'}) \in \cdot\,\right) \mathbb{P}\left[M_{\tau_{\eta_n'} + n_0} \in ds', W_{\tau_{\eta_n'} + n_0} = 0 | M_{\tau_{\eta_n'} - 1} = s, W_{\tau_{\eta_n'} - 1} = x_0\right] \\ &\geq p'\,(\phi \otimes \delta_0)(\cdot) \int_\Re \mathbb{P}_{\varphi \otimes \delta_{x_0 + c}}\left(M_{n_0} \in ds', W_{n_0} = 0\right) \\ &> p'q\,(\phi \otimes \delta_0)(\cdot). \end{aligned}$$

Furthermore, as in the proof of Theorem 3.3, one can easily show that for any $n \geq 1$

$$\mathbb{E}_\varphi\, \tau_1^\alpha < \infty \quad \text{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty \;\Rightarrow\; \mathbb{E}_\varphi \left(\tau_{\eta'_n} - \tau_{\eta'_{n-1}} \right)^\alpha = \mathbb{E}_\varphi\, \tau_{\eta'_1}^\alpha < \infty,$$

Therefore, by the splitting technique, one can construct a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ for $(M_n, W_n)_{n\geq 0}$ such that

$$\mathbb{E}_\varphi\, \tau_1^\alpha < \infty \quad \text{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty \;\Rightarrow\; \mathbb{E}_\varphi\, \hat{\tau}_1^\alpha < \infty.$$

If in addition for an initial distribution λ on E

$$\mathbb{E}_\lambda\, \tau_1^\alpha < \infty \quad \text{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty,$$

then $\mathbb{E}_\lambda\, \hat{\tau}_1^\alpha < \infty$. In the same manner, one can show that

$$\mathbb{E}_\varphi\, \exp(\gamma\tau_1) < \infty \quad \text{and} \quad \mathbb{E}_\varphi\, \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \text{for some} \quad \gamma > 0$$
$$\Rightarrow \mathbb{E}_\varphi\, \exp(\gamma'\hat{\tau}_1) < \infty \quad \text{for some} \quad \gamma' > 0.$$

If in addition for an initial distribution λ on E

$$\mathbb{E}_\lambda\, \exp(\gamma\tau_1) < \infty \quad \text{and} \quad \mathbb{E}_\lambda\, \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+ \right) < \infty \quad \text{for some} \quad \gamma > 0,$$

then $\mathbb{E}_\lambda\, \exp(\gamma'\hat{\tau}_1) < \infty$ for some $\gamma' > 0$.

Corollary 3.15 *Suppose that there exist $u > 0$ and $q > 0$ such that*

$$\mathbb{P}_s\, (M_1 \in \cdot\,, X_1 \leq -u) > q\varphi(\cdot), \quad s \in \Re.$$

(i) Let $\alpha > 1$. Suppose that

$$\mathbb{E}_\varphi\, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda\, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} X_n^+ \right)^\alpha < \infty,$$

then for some constant c

$$\|\mathbb{P}_\lambda\, ((M_n, W_n) \in \cdot\,) - \pi\| \leq c n^{1-\alpha}.$$

(ii) Let $\gamma > 0$. Suppose that

$$\mathbb{E}_\varphi \exp(\gamma\tau_1) < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+\right).$$

Then for some constants c and $\gamma' > 0$

$$\|\mathbb{P}_\varphi\left((M_n, W_n) \in \cdot\,\right) - \pi\| \le c\exp(-\gamma' n).$$

Proof. (i) By Remark 3.4 $\Re \times [0, x_0]$ is a recurrent set of $(M_n, W_n)_{n\ge0}$ for some $x_0 \ge 0$. Put

$$r' = \left[\frac{x_0}{u}\right].$$

Then for any $(s, w) \in \Re \times [0, x_0]$

$$\mathbb{P}_{s,w}\left(M_{r'} \in \Re, W_{r'} = 0\right) \ge \mathbb{P}_{s,x_0}\left(M_{r'} \in \Re, W_{r'} = 0\right) > q^{r'}.$$

By the geometric trial argument one can easily see that $\Re \times \{0\}$ is also a recurrent set of $(M_n, W_n)_{n\ge0}$ and thus a regeneration set, since by assumption

$$\mathbb{P}_s\left(M_1 \in \cdot\,, W_1 = 0\right) > q\varphi(\cdot), \quad s \in \Re.$$

Thus $(M_n, W_n)_{n\ge0}$ satisfies the minorization condition $\mathcal{M}(\Re\times\{0\}, q, 1, \varphi\otimes\delta_0)$ and there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\ge0}$ such that $\mathbb{E}_\varphi\,\hat{\tau}_1^\alpha < \infty$. Furthermore, under the conditions we have $\mathbb{E}_\lambda\,\hat{\tau}_1^\alpha < \infty$, from which the assertion follows.

(ii) can be proved in the same manner. QED

In particular, if E is a countable space, then there exists a recurrent state i_0 of M. Denote by τ_1 the return time of M to a recurrent state i_0. Suppose that

$$\mathbb{P}_{i_0}\left(\tau_1 = 1, X_1 < 0\right) = \mathbb{P}_{i_0}(M_1 = i_0, X_1 < 0) > 0.$$

If for some $\alpha > 1$

$$\mathbb{E}_{i_0}\,\tau_1^\alpha < \infty \quad \text{and} \quad \mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau_1} X_n^+\right)^\alpha < \infty,$$

then for some constant c

$$\|\mathbb{P}_{i_0}\left((M_n, W_n) \in \cdot\,\right) - \pi\| \le cn^{1-\alpha}.$$

If for some $\gamma > 0$

$$\mathbb{E}_{i_0} \exp(\gamma\tau_1) < \infty \quad \text{and} \quad \mathbb{E}_{i_0} \exp\left(\gamma \sum_{n=1}^{\tau_1} X_n^+\right) < \infty,$$

then for some constants c and $\gamma' > 0$

$$\|\mathbb{P}_{i_0}\left((M_n, W_n) \in \cdot\,\right) - \pi\| \le c\exp(-\gamma' n).$$

Chapter 4

Semi-Markov queues

Semi-Markov queues are generalizations of classical queues, which are based on the i.i.d. assumption of interarrival and service times. In a semi-Markov queue interarrival times and service times of customers are governed by a Markov chain, which is called the *modulation chain*. Semi-Markov queues with finite modulation chains are extensively studied by Neuts and his school, and there are plenty of literature. However, the theory of queues with general modulation chains are not well developed to the same extent.

This final chapter studies semi-Markov queues with general modulation chains. As applications of previous chapters we obtain conditions for the finiteness of moments for the stationary waiting time and workload processes, and rates of convergence to the steady state distributions. Throughout a Markov chain M is assumed to be Harris ergodic with a unique stationary distribution ξ and to satisfy a minorization condition $\mathcal{M}(\Re, p, r, \varphi)$. Further, we denote by $(\tau_n)_{n\geq 0}$ a sequence of regeneration epochs constructed by the splitting technique from the minorization condition.

4.1 Single server queues

A single server queue is the simplest and the most basic model in queueing theory, where customers arrive at one service station, are served one at a time, and leave the system when the service is completed.

We number the customers $0, 1, 2, \cdots$. Denote by T_n the interarrival time between customers $n-1$ and n, and by U_n the service time of customer n. Let T_0 and U_{-1} be arbitrary random variables with values in $\mathbb{R}_0^+$. $T_n, n \geq 1$, and $U_n, n \geq 0$, are assumed to be positive. In our model the process $(M_n, T_n, U_{n-1})_{n\geq 0}$, which is called the *input process*, is assumed to be a Markov modulated sequence with driving chain M and transition kernel

$$\mathbf{P} : E \times (\mathcal{E} \otimes (\mathcal{B}|_{[0,\infty)})^2) \to [0, 1].$$

Let $\overline{T}_n = T_0 + T_1 + \cdots + T_n, n \geq 0$, and $\overline{U}_n = U_{-1} + U_0 + \cdots + U_n$, $n \geq -1$. If $T_0 = U_{-1} = 0$, which means that customer 0 arrives at time 0, then $\overline{T}_n$ is the arrival time of the customer n in the system and $\overline{U}_n$ the total workload up to the nth customer.

The queue discipline is assumed to be FIFO, i.e., the customers are served in the order of arrival. We say that the system is *stable*, if there exists a finite stationary distribution for the waiting time process. If the mean interarrival time $\mathbb{E}_\xi T_1$ and the mean service time $\mathbb{E}_\xi U_0$ are finite, we define the *traffic intensity* ρ as

$$\rho := \frac{\mathbb{E}_\xi U_0}{\mathbb{E}_\xi T_1}. \tag{4.1}$$

The condition $\rho < 1$, or equivalently $\mu = \mathbb{E}_\xi(U_0 - T_1) < 0$, is called the *stability condition* for the single server queue. Throughout this section we assume the stability condition.

4.1.1 The actual waiting time

We denote by W_n the actual waiting time of the customer n, i.e., the time from arrival to the system until service starts. Put $X_n := U_{n-1} - T_n$ for $n \geq 0$. Then $(M_n, X_n)_{n\geq 0}$ forms a Markov modulated sequence with the driving chain $(M_n)_{n\geq 0}$. Denote by $(M_n, S_n)_{n\geq 0}$ the associated MRW. One can easily see that the actual waiting time process $(M_n, W_n)_{n\geq 0}$ is the reflected MRW associated to $(M_n, X_n)_{n\geq 0}$ with $W_0 = S_0$. Consequently, the actual waiting time process $(M_n, W_n)_{n\geq 0}$ is Harris ergodic, whence there exists a unique stationary distribution π_W of $(M_n, W_n)_{n\geq 0}$, which by Proposition 2.14 satisfies the relation

$$\pi_W(A \times [0, y]) = \int_E \mathbf{H}_{\leq}(s, A; [0, y])\, \pi_W(ds \times \{0\}), \quad A \in \mathcal{E}, y \geq 0.$$

Moreover, by Corollary 2.15

$$\mathbb{E}_{\pi_W}\left(e^{\beta W_1}; M_1 \in A\right) = \int_E \hat{\mathbf{H}}_{\leq}^{(1,\beta)}(s, A)\, \pi_W(ds \times \{0\})$$

for any $A \in \mathcal{E}$.

Note that the process $(M_n, W_n, T_n, U_{n-1})_{n\geq 0}$ forms a temporally homogeneous Markov chin. Throughout this chapter we assume that a canonical model is given with probability measures $\mathbb{P}_{(s,w,x,y)}, (s, w, x, y) \in E \times [0, \infty)^3$ on $(\Omega, \mathcal{S})$ such that

$$\mathbb{P}_{(s,w,x,y)}(M_0 = s, W_0 = w, T_0 = x, U_{-1} = y) = 1.$$

For each $s \in E, w \geq 0$ and initial distribution λ on E, we write $\mathbb{E}_{(s,w)}, \mathbb{E}_s$ and $\mathbb{E}_\lambda$ instead of $\mathbb{E}_{(s,w,0,0)}, \mathbb{E}_{(s,0,0,0)}$ and $\mathbb{E}_{\lambda\otimes\delta_{(0,0,0)}}$, respectively.

Theorem 4.1 *(i) Suppose that* $\Re \times \{0\}$ *is a recurrent set of* $(M_n, W_n)_{n\geq 0}$.

(a) Let $\alpha > 1$. Suppose that

$$\sup_{s\in\Re} \mathbb{E}_s\, \tau_1^\alpha < \infty \quad \textit{and} \quad \sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda\, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty,$$

there exists a sequence of regeneration epochs $(\hat\tau_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\zeta\, \hat\tau_1^\alpha < \infty$ and $\mathbb{E}_\lambda\, \hat\tau_1^\alpha < \infty$, where $\zeta(\cdot) = \mathbb{P}_{\lambda'}(M_{\hat\tau_1} \in \cdot)$ for each initial distribution λ' on E. Moreover, for some constant c

$$\|\mathbb{P}_\lambda\left((M_n, W_n) \in \cdot\,\right) - \pi_W\| \leq c n^{1-\alpha}.$$

(b) Let $\gamma > 0$. Suppose that

$$\sup_{s\in\Re} \mathbb{E}_s \exp\left(\gamma\tau_1\right) < \infty \quad \textit{and} \quad \sup_{s\in\Re} \mathbb{E}_s \exp\left(\gamma \sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty.$$

Then there exists a sequence of regeneration epochs $(\hat\tau_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\zeta\, \exp(\gamma'\hat\tau_1) < \infty$ for some $\gamma' > 0$. Moreover, for some constants c and $\gamma'' \in (0, \gamma']$

$$\|\mathbb{P}_\zeta\left((M_n, W_n) \in \cdot\,\right) - \pi_W\| \leq c \exp(-\gamma'' n).$$

(ii) Suppose that for some $q > 0$

$$\mathbb{P}_s\left(M_1 \in \cdot\,, U_0 - T_1 < 0\right) > q\varphi(\cdot), \quad s \in \Re.$$

(a) Let $\alpha > 1$. Suppose that

$$\mathbb{E}_\varphi\, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda\, \tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty,$$

then there exists a sequence of regeneration epochs $(\hat\tau_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\varphi\, \hat\tau_1^\alpha < \infty$ and $\mathbb{E}_\lambda\, \hat\tau_1^\alpha < \infty$. Moreover, for some constant c

$$\|\mathbb{P}_\varphi\left((M_n, W_n) \in \cdot\,\right) - \pi_W\| \leq c n^{1-\alpha}.$$

(b) Let $\gamma > 0$. *Suppose that*

$$\mathbb{E}_\varphi \exp(\gamma\tau_1) < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\Big) < \infty.$$

Then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ *of* $(M_n, W_n)_{n\geq 0}$ *such that* $\mathbb{E}_\varphi \exp(\gamma'\hat{\tau}_1) < \infty$ *for some* $\gamma' > 0$. *Moreover, for some constants* c *and* $\gamma'' \in (0, \gamma']$

$$\|\mathbb{P}_\varphi((M_n, W_n) \in \cdot\,) - \pi_W\| \leq c\exp(-\gamma'' n).$$

Proof. All assertions follow directly from Theorem 3.13 and Corollary 3.15.

QED

If E is countable and if the condition

$$\mathbb{P}_{i_0}(M_1 = i_0, U_0 - T_1 < 0) > 0$$

is satisfied for a recurrent state $i_0 \in E$, then

$$\begin{aligned}
&\mathbb{E}_{i_0}(\tau(i_0))^\alpha < \infty \quad \text{and} \quad \mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(i_0)} U_{n-1}\right)^\alpha < \infty \quad \text{for some} \quad \alpha > 1\\
&\quad \Rightarrow \|\mathbb{P}_{i_0}((M_n, W_n) \in \cdot\,) - \pi_W\| \leq cn^{1-\alpha} \quad \text{for some constant} \quad c;\\
&\mathbb{E}_{i_0} \exp(\gamma\tau(i_0)) < \infty \quad \text{and} \quad \mathbb{E}_{i_0} \exp\Big(\gamma \sum_{n=1}^{\tau(i_0)} U_{n-1}\Big) < \infty \quad \text{for some} \quad \gamma > 0\\
&\quad \Rightarrow \|\mathbb{P}_{i_0}((M_n, W_n) \in \cdot\,) - \pi_W\| \leq c\exp(-\gamma' n) \text{ for some constants} \quad \gamma' > 0 \text{ and } c,
\end{aligned}$$

which are obtained in Sharma [55].

If M is uniformly ergodic, then by Corollary 3.14 for any initial distribution λ on E

$$\begin{aligned}
&\sup_{s\in E} \mathbb{E}_s U_0^\alpha < \infty \quad \text{for some} \quad \alpha > 1\\
&\qquad \Rightarrow \|\mathbb{P}_\lambda((M_n, W_n) \in \cdot\,) - \pi_W\| \leq cn^{1-\alpha} \quad \text{for some constant} \quad c.
\end{aligned}$$

If E is a one-element set, then

$$\begin{aligned}
&\mathbb{E}\, U_0^\alpha < \infty \quad \text{for some} \quad \alpha > 1\\
&\qquad \Rightarrow \|\mathbb{P}(W_n \in \cdot\,) - \pi_W\| \leq cn^{1-\alpha} \quad \text{for some} \quad c;\\
&\mathbb{E}\exp(\gamma U_0) < \infty \quad \text{for some} \quad \gamma > 0\\
&\qquad \Rightarrow \|\mathbb{P}(W_n \in \cdot\,) - \pi_W\| \leq c\exp(-\gamma' n) \quad \text{for some constants} \quad c \quad \text{and} \quad \gamma' > 0,
\end{aligned}$$

which is obtained in Kalashnikov [33].

Let W be a random variable having the steady state distribution of W_n, i.e.,

$$\mathbb{P}(W \le x) = \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1} \mathbb{E}_{\bar{\xi}} \left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}(W_n \le x) \right).$$

Theorem 4.2 *(i) Let $\alpha \ge 1$. If*

$$\mathbb{E}_{\bar{\xi}}\, \sigma_1^{\alpha+1} < \infty \quad \textit{and} \quad \mathbb{E}_{\bar{\xi}} \left(\sum_{n=1}^{\sigma_1} U_{n-1} \right)^{\alpha+1} < \infty,$$

then $\mathbb{E}\, W^\alpha < \infty$.

(ii) Let $\gamma > 0$. If for some $\epsilon > 0$

$$\mathbb{E}_{\bar{\xi}}\, \sigma_1^{1+\epsilon} < \infty \quad \textit{and} \quad \mathbb{E}_{\bar{\xi}} \exp\left(\gamma \sum_{n=1}^{\sigma_1} U_{n-1} \right) < \infty,$$

then $\mathbb{E} \exp(\gamma' W) < \infty$ for some $\gamma' > 0$.

Proof. (i) Since $W_k \le \sum_{n=1}^{\sigma_1} U_{n-1}$ for all $k \le \sigma_1$, it holds that

$$\begin{aligned} \mathbb{E}\, W^\alpha = \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1} \mathbb{E}_{\bar{\xi}} \left(\sum_{n=1}^{\sigma_1} W_n^\alpha \right) &\le \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1} \mathbb{E}_{\bar{\xi}} \left[\sigma_1 \left(\sum_{n=1}^{\sigma_1} U_{n-1} \right)^\alpha \right] \\ &\le \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1} \left\{ \mathbb{E}_{\bar{\xi}}\, \sigma_1^p \right\}^{1/p} \left\{ \mathbb{E}_{\bar{\xi}} \left(\sum_{n=1}^{\sigma_1} U_{n-1} \right)^{\alpha q} \right\}^{1/q}, \end{aligned}$$

where $p > 1$ and $1/p + 1/q = 1$. Taking $p = \alpha + 1$, the assertion follows.

(ii) In the similar manner as in (i), we have

$$\mathbb{E} \exp(\gamma' W) \;\le\; \frac{1}{\mathbb{E}_{\bar{\xi}}\, \sigma_1} \left\{ \mathbb{E}_{\bar{\xi}}\, \sigma_1^p \right\}^{1/p} \left\{ \mathbb{E}_{\bar{\xi}} \exp\left(\gamma' q \sum_{n=1}^{\sigma_1} U_{n-1} \right) \right\}^{1/q}.$$

Taking $\gamma < \gamma' < \gamma(1 - 1/\alpha)$ and $q = \gamma/\gamma'$, the assertion follows. QED

Corollary 4.3 *Suppose that M is uniformly Harris ergodic. Then the following assertions hold true:*

(i) Let $\alpha \ge 1$. If $\sup_{s \in E} \mathbb{E}_s\, U_0^{\alpha+1} < \infty$, then $\mathbb{E}\, W^\alpha < \infty$.

(ii) Let $\gamma > 0$. If $\sup_{s \in E} \mathbb{E}_s \exp(\gamma U_0) < \infty$, then $\mathbb{E} \exp(\gamma' W)$ for some $\gamma' > 0$.

Proof. (i) By Lemma 1.14 (i) and Theorem 3.3

$$\begin{aligned}
\sup_{s\in E} \mathbb{E}_s\, U_0^{\alpha+1} < \infty \;&\Rightarrow\; \sup_{s\in E} \mathbb{E}_s\, \tau_1^{\alpha+1} < \infty \quad \text{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1}\right)^{\alpha+1} < \infty \\
&\Rightarrow\; \sup_{s\in E} \mathbb{E}_s\, \sigma_1^{\alpha+1} < \infty \quad \text{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\sigma_1} U_{n-1}\right)^{\alpha+1} < \infty \\
&\Rightarrow\; \mathbb{E}_{\bar{\xi}}\, \sigma_1^{\alpha+1} < \infty \quad \text{and} \quad \mathbb{E}_{\bar{\xi}} \left(\sum_{n=1}^{\sigma_1} U_{n-1}\right)^{\alpha+1} < \infty.
\end{aligned}$$

Thus the assertion follows from Theorem 4.3 (i).

(ii) Using Lemma 1.14 (ii) and Theorem 3.11, the assertion follows in the same manner. QED

In particular, if E is a one-element space, then we obtain the classical result by Kiefer and Wolfowitz (see Theorem X.2.1. in Asmussen [12])

$$\mathbb{E}\, U^{1+\alpha} < \infty \quad \text{for some} \quad \alpha \geq 1 \;\Rightarrow\; \mathbb{E}\, W^{\alpha} < \infty.$$

Moreover, we get

$$\mathbb{E}\, \exp(\gamma U) < \infty \quad \text{for some} \quad \gamma > 0 \;\Rightarrow\; \mathbb{E}\, \exp(\gamma' W) < \infty \quad \text{for some} \quad \gamma' > 0.$$

Remark 4.4 Suppose that $\Re \times \{0\}$ is a recurrent set of $(M_n, W_n)_{n\geq 0}$. Then from Theorem 3.13 (i)

$$\begin{aligned}
&\sup_{s\in\Re} \mathbb{E}_s\, \tau_1^{\alpha+1} < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1}\right)^{\alpha+1} < \infty \\
&\quad\Rightarrow \mathbb{E}_{\varphi}\, \hat{\tau}_1^{\alpha+1} < \infty \quad \text{and} \quad \mathbb{E}_{\varphi} \left(\sum_{n=1}^{\hat{\tau}_1} U_{n-1}\right)^{\alpha+1} < \infty \\
&\quad\Rightarrow \mathbb{E}\, W^{\alpha} = \frac{1}{\mathbb{E}_{\varphi}\, \hat{\tau}_1} \mathbb{E}_{\varphi} \left(\sum_{n=0}^{\hat{\tau}_1 - 1} W_n^{\alpha}\right) < \infty.
\end{aligned}$$

In the same manner,

$$\begin{aligned}
&\sup_{s\in\Re} \mathbb{E}_s\, \exp(\gamma\tau_1) < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s\, \exp\left(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\right) < \infty \\
&\quad\Rightarrow \mathbb{E}\, \exp(\gamma' W) < \infty \quad \text{for some} \quad \gamma' > 0.
\end{aligned}$$

The same results are obtained in Sharma [55] for the countable modulation case.

Remark 4.5 Let $(Y_n)_{n\geq 0}$ be a sequence of random variables on the probability space $(\Omega, \mathcal{S}, \mathbb{P})$. Let $\mathcal{F}_n^m := \sigma(Y_k : n \leq k \leq m)$. Define

$$\beta(m) := \sup |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|,$$

where the supremum is taken over all $A \in \mathcal{F}_0^n$ and $B \in \mathcal{F}_{n+m}^\infty$. $(Y_n)_{n\geq 0}$ is called *strongly mixing*, if $\beta(m)$ tends to 0 as m increases to ∞. Daley, Foley and Rolski [26] studied moment conditions for the waiting time of $G/G/1$ queue, in which the input process forms a stationary process. They have shown that if the sequence of interarrival times $(T_n)_{n\geq 0}$ is strongly mixing with mixing coefficients $(\beta(n))_{n\geq 1}$ satisfying

$$\sum_{n=1}^{\infty} n^{\alpha-1}\beta(n) < \infty \tag{4.2}$$

and $\mathbb{E}\, U_0^{\alpha+1} < \infty$, then $\mathbb{E}\, W^{\alpha-1} < \infty$. Their result can be also applied to $SM/G/1$ queue. It is known (see Athreya and Pantula [16]) that a Harris ergodic Markov chain is strongly mixing for any initial distribution with $\beta(m) \leq 2\sup_{s\in E} \mathbb{E}\, K_{m-1}(s)$, where

$$K_m(s) := \|\mathbb{P}_s\,(M_m \in \cdot\,) - \xi\|, \quad m \geq 1.$$

If M is uniformly Harris ergodic, then the Markov modulated sequence $(M_n, T_n)_{n\geq 0}$ is also uniformly Harris ergodic, and consequently the mixing coefficients of $(M_n, T_n)_{n\geq 0}$ satisfy the condition (4.2). Thus in the $SM/G/1$ queue with uniformly Harris ergodic modulation chain M we obtain

$$\mathbb{E}\, U_0^{\alpha+1} < \infty \quad \text{for some} \quad \alpha \geq 1 \;\; \Rightarrow \;\; \mathbb{E}\, W^{\alpha} < \infty.$$

4.1.2 The busy cycle

Denote by I_n^B and I_n^L the nth busy period and the nth idle period, respectively. Then the nth *busy cycle* can be written as $I_n = I_n^B + I_n^L$. Let η_n^B and η_n^L be the beginning time of the nth busy period and the beginning time of the nth idle period, respectively. Obviously, for any $n \geq 1$,

$$\eta_n^B = \overline{T}_{\sigma_{n-1}} \quad \text{and} \quad \eta_n^L = \overline{T}_{\sigma_{n-1}} + \sum_{k=\sigma_{n-1}+1}^{\sigma_n} U_{k-1}.$$

Denoting

$$T_{\sigma_{n-1}}^{(\sigma_n)} := \sum_{k=\sigma_{n-1}+1}^{\sigma_n} T_k \quad \text{and} \quad U_{\sigma_{n-1}}^{(\sigma_n)} := \sum_{k=\sigma_{n-1}+1}^{\sigma_n} U_{k-1}, \quad n \geq 1,$$

we have

$$I_n^B = \eta_n^L - \eta_n^B = U_{\sigma_{n-1}}^{(\sigma_n)}, \quad I_n^L \;=\; \eta_{n+1}^B - \eta_n^L = T_{\sigma_{n-1}}^{(\sigma)} - U_{\sigma_{n-1}}^{(\sigma_n)} \quad \text{and} \quad I_n = T_{\sigma_{n-1}}^{(\sigma_n)}.$$

The following assertions are direct consequences of Corollary 3.5.

Proposition 4.6 *(i) Let $\alpha \geq 1$. Suppose that*

$$\mathbb{E}_\xi\, U_0^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\xi\, T_1^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} T_n^\alpha \right) < \infty,$$

then $\mathbb{E}_\lambda\, (I_1^L)^\alpha < \infty$. Moreover, if

$$\sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} T_n^\alpha \right) < \infty,$$

then $\sup_{s\in E} \mathbb{E}_s\, (I_1^L)^\alpha < \infty$.

(ii) Let $\gamma > 0$. Suppose that

$$\mathbb{E}_\xi\, U_0 < \infty \quad \textit{and} \quad \mathbb{E}_\xi\, \exp(\gamma T_1) < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} \exp(\gamma T_n) \right) < \infty,$$

then $\mathbb{E}_\lambda\, \exp(\gamma I_1^L) < \infty$. Moreover, if

$$\sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} \exp(\gamma T_n) \right) < \infty,$$

then $\sup_{s\in E} \mathbb{E}_s\, \exp(\gamma I_1^L) < \infty$.

Proof. All assertions are direct consequences of Corollary 3.5, since $I_1^L = |S_{\sigma_1}|$.

QED

Moreover, we obtain moments of the busy cycle as in the proof of Theorem 3.3 and Theorem 3.11.

Proposition 4.7 *(i) Let $\alpha > 0$. Suppose that*

$$\mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty,$$

then $\mathbb{E}_\lambda\, I_1^\alpha < \infty$. *Moreover, if*

$$\sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1}\right)^\alpha < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} T_n\right)^\alpha < \infty$$

then $\sup_{s\in E} \mathbb{E}_s\, (I_1^B)^\alpha < \infty$.

(ii) Let $\gamma > 0$. *Suppose that*

$$\mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\Big) < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \exp\Big(\gamma \sum_{n=1}^{\tau_1} T_n\Big) < \infty.$$

If for an initial distribution λ *on* E

$$\mathbb{E}_\lambda \exp\Big(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\Big) < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \exp\Big(\gamma \sum_{n=1}^{\tau_1} T_n\Big) < \infty,$$

then $\mathbb{E}_\lambda \exp(\gamma' I_1) < \infty$ *for some* $\gamma' > 0$. *Moreover, if*

$$\sup_{s\in E} \mathbb{E}_s \exp\Big(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\Big) < \infty \quad \textit{and} \quad \sup_{s\in E} \mathbb{E}_s \exp\Big(\gamma \sum_{n=1}^{\tau_1} T_n\Big) < \infty$$

then $\sup_{s\in E} \mathbb{E}_s \exp(\gamma' I_1) < \infty$ *for some* $\gamma' > 0$.

Proof. (i) Noting that from the proof of Theorem 3.3

$$\mathbb{E}_\lambda I_1^\alpha = \mathbb{E}_\lambda \left(\sum_{k=1}^{\sigma_1} T_k\right)^\alpha \le \mathbb{E}_\lambda \left(\sum_{k=1}^{\tau_1} T_k + \sum_{n=2}^{\tilde\tau(x_0)} \sum_{k=\tau_{n-1}}^{\tau_n} T_k + \sum_{n=\tilde\tau(x_0)}^{\eta_\nu+m'} \sum_{k=\tau_{n-1}}^{\tau_n} T_k\right)^\alpha,$$

all assertions follows in the same manner as in the proof of Theorem 3.3.

(ii) All assertions follow in the same manner as in the proof of Theorem 3.11. QED

From the inequality

$$I_1^B = \sum_{k=1}^{\sigma_1} U_{k-1} \le \sum_{k=1}^{\tau_1} U_{k-1} + \sum_{n=2}^{\tilde\tau(x_0)} \sum_{k=\tau_{n-1}}^{\tau_n} U_{k-1} + \sum_{n=\tilde\tau(x_0)}^{\eta_\nu+m'} \sum_{k=\tau_{n-1}}^{\tau_n} U_{k-1},$$

one can also obtain the corresponding assertions for I_1^B.

Remark 4.8 Denote by $\bar\xi$ the stationary distribution of M^σ. Using the strong Markov property, it can be easily seen that

$$\mathbb{E}_{\bar\xi}\, I_1 = \mathbb{E}_\xi\, T_1\, \mathbb{E}_{\bar\xi}\, \sigma_1, \quad \mathbb{E}_{\bar\xi}\, I_1^B = \mathbb{E}_\xi\, U_0\, \mathbb{E}_{\bar\xi}\, \sigma_1 \quad \text{and} \quad \mathbb{E}_{\bar\xi}\, I_1^L = -\mathbb{E}_{\bar\xi}\, S_{\sigma_1} = -\mu \mathbb{E}_{\bar\xi}\, \sigma_1.$$

4.1.3 Continuous-time processes

The *workload* V_t at time t is the total time the server has to work to clear the system at time t. Under FIFO discipline, it is the same as the waiting time a customer would have if he arrived at time t. Thus the workload in a single server queue with FIFO discipline is also called the *virtual waiting time*. It can be easily seen that

$$V_t = \sum_{n\geq 0} (\overline{T}_n + W_n + U_n - t)^+ \cdot \mathbf{1}([\overline{T}_n, \overline{T}_{n+1}))(t), \quad t \geq 0.$$

Obviously

$$\lim_{t\uparrow \overline{T}_n} V_t = \lim_{t\uparrow \overline{T}_n} (\overline{T}_{n-1} + W_{n-1} + U_{n-1} - t)^+ = (W_{n-1} + U_{n-1} - T_n)^+ = W_n, \quad n \geq 0.$$

The *queue length* Q_t at time t is the number in system at time t and can be written as

$$Q_t = \sum_{n\geq 0} \mathbf{1}(\overline{T}_n \leq t, \overline{T}_n + W_n + U_n > t), \quad t \geq 0.$$

By definitions it is clear that

$$\lim_{t\uparrow \overline{T}_{\sigma n}} V_t = \lim_{t\uparrow \overline{T}_{\sigma n}} Q_t = 0, \quad n \geq 0.$$

Proposition 4.9 *Let $(\hat{\tau}_n)_{n\geq 0}$ be a sequence of regeneration epochs for $(M_n, W_n)_{n\geq 0}$ with $W_{\hat{\tau}_n} = 0$ for $n \geq 0$. Then the process $(V_t, Q_t)_{t\geq 0}$ is one-dependent, positive recurrent regenerative under each $\mathbb{P}_\lambda$, i.e., the cycles $\tilde{Z}_n$ defined as*

$$\tilde{Z}_n = \left(\overline{T}_{\hat{\tau}_{n+1}} - \overline{T}_{\hat{\tau}_n}, (V_t, Q_t)_{\overline{T}_{\hat{\tau}_n} \leq t < \overline{T}_{\hat{\tau}_{n+1}}}, (M_k)_{\hat{\tau}_n \leq k < \hat{\tau}_{n+1}}\right), \quad n \geq 0,$$

are one-dependent for $n \geq 0$ and identically distributed for $n \geq 1$ with common distribution $\mathbb{P}_\lambda(\tilde{Z}_n \in \cdot\,)$ under each $\mathbb{P}_\lambda$, where $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\hat{\tau}_1} \in \cdot\,)$.
If E is countable, then the process $(V_t, Q_t)_{t\geq 0}$ is classical-sense regenerative.

Proof. Note first that the cycles Z_n defined as

$$Z_n := \left(\hat{\tau}_{n+1} - \hat{\tau}_n, (M_k, W_k, T_{k+1}, U_k)_{\hat{\tau}_n \leq k < \hat{\tau}_{n+1}}\right), \quad n \geq 0$$

are one-dependent for $n \geq 0$ and identically distributed for $n \geq 1$ with common distribution $\mathbb{P}_\zeta^{(Z_n)_{n\geq 0}} = \mathbb{P}_\lambda^{(Z_n)_{n\geq 1}}$, where $\zeta(\cdot) = \mathbb{P}_\lambda(M_{\hat{\tau}_1} \in \cdot\,)$ for any initial distribution λ. From the definition of the workload process,

$$\begin{aligned}
V_{\overline{T}_{\hat{\tau}_n + t}} &= \sum_{k=0}^{\infty} (T_k + W_k + U_k - \overline{T}_{\hat{\tau}_n} - t)^+ \cdot \mathbf{1}[\overline{T}_k, \overline{T}_{k+1})(\overline{T}_{\hat{\tau}_n} + t) \\
&= \sum_{k=\hat{\tau}_n}^{\infty} (T_k + W_k + U_k - \overline{T}_{\hat{\tau}_n} - t)^+ \cdot \mathbf{1}[\overline{T}_k, \overline{T}_{k+1})(\overline{T}_{\hat{\tau}_n} + t) \\
&= \sum_{k=0}^{\infty} \mathbf{1}[\overline{T}_{\hat{\tau}_n + k} - \overline{T}_{\hat{\tau}_n}, \overline{T}_{\hat{\tau}_n + k+1} - \overline{T}_{\hat{\tau}_n})(t) \cdot F\big((T_{m+1}, U_m)_{m\geq \hat{\tau}_n}\big)
\end{aligned}$$

for any $n \geq 0, t \geq 0$, and some measurable function F. Similarly we get

$$\begin{aligned} Q_{\overline{T}_{\hat{\tau}_n}+t} &= \sum_{k=0}^{\infty} \mathbf{1}(\overline{T}_k \leq \overline{T}_{\hat{\tau}_n} + t, \overline{T}_k + W_k + U_k > \overline{T}_{\hat{\tau}_n} + t) \\ &= \sum_{k=0}^{\infty} \mathbf{1}(\overline{T}_{\hat{\tau}_n+k} \leq \overline{T}_{\hat{\tau}_n} + t, \overline{T}_{\hat{\tau}_n+k} + W_{\hat{\tau}_n+k} + U_{\hat{\tau}_n+k} > \overline{T}_{\hat{\tau}_n} + t) \\ &= \sum_{k=0}^{\infty} \mathbf{1}(\sum_{m=0}^{k} T_{\hat{\tau}_n+m} \leq t, \sum_{m=0}^{k} T_{\hat{\tau}_n+m} + W_{\hat{\tau}_n+k} + U_{\hat{\tau}_n+k} > t). \end{aligned}$$

Thus the cycles $\tilde{Z}_n, n \geq 0$, can be written as images of Z_n under some measurable function. In particular, the cycles $\tilde{Z}_n$ are one-dependent for $n > 0$ and identically distributed for $n \geq 1$ with common distribution $\mathbb{P}_\zeta(\tilde{Z}_0 \in \cdot\,)$.
If E is countable, then the cycles $\tilde{Z}_n, n \geq 0$, are independent, since the cycles $Z_n, n \geq 0$, are independent. QED

Let us point out that the process $(V_t, Q_t)_{t\geq 0}$ forms a semi-regenerative process, which means that we can find a Markov renewal process $(M_n^\sigma, \overline{T}_{\sigma_n})_{n\geq 0}$ such that

$$\begin{aligned} &\mathbb{P}\left[(V_{t+\overline{T}_{\sigma_{n-1}}}, Q_{t+\overline{T}_{\sigma_{n-1}}})_{t\geq 0} \in \cdot\,|\overline{T}_{\sigma_0}, \cdots, \overline{T}_{\sigma_{n-1}}, M_0, \cdots, M_{n-1}, M_n = s\right] \\ &\qquad = \mathbb{P}_s\left((V_t, Q_t)_{t\geq 0} \in \cdot\,\right) \end{aligned}$$

for any $n \geq 1, s \in E$. Consequently, a steady state distribution π_{VQ} of $(V_t, Q_t)_{t\geq 0}$ is given as

$$\begin{aligned} \pi_{VQ}(\cdot) &= \frac{1}{\mathbb{E}_\zeta \overline{T}_{\hat{\tau}_1}} \mathbb{E}_\zeta\left(\int_0^{\overline{T}_{\hat{\tau}_1}} \mathbf{1}((V_t, Q_t) \in \cdot\,) dt\right) \\ &= \frac{1}{\mathbb{E}_{\bar{\xi}}\, I_1} \mathbb{E}_{\bar{\xi}}\left(\int_0^{I_1} \mathbf{1}((V_t, Q_t) \in \cdot\,) dt\right) \\ &= \frac{1}{\mathbb{E}_\xi\, T_1} \int_E \mathbb{E}_s\left(\int_0^{I_1} \mathbf{1}((V_t, Q_t) \in \cdot\,) dt\right) \pi_W(ds \times \{0\}), \end{aligned}$$

where $\bar{\xi}$ is the stationary distribution for $M^\sigma = (M_{\sigma_n})_{n\geq 0}$.

In particular, denoting by π_V a steady state distribution of $(V_t)_{t\geq 0}$,

$$\begin{aligned} \mathbb{P}_{\pi_V}(V_0 = 0) &= \frac{1}{\mathbb{E}_{\bar{\xi}}\, I_1} \mathbb{E}_{\bar{\xi}}\left(\int_0^{I_1} \mathbf{1}(V_t = 0)\, dt\right) \\ &= \frac{\mathbb{E}_{\bar{\xi}}\, \sigma_1\, (\mathbb{E}_\xi\, T_1 - \mathbb{E}_\xi\, U_0)}{\mathbb{E}_{\bar{\xi}}\, \sigma_1\, \mathbb{E}_\xi\, T_1} = 1 - \frac{\mathbb{E}_\xi\, U_0}{\mathbb{E}_\xi\, T_1}. \end{aligned}$$

Remark 4.10 A steady state distribution of the workload process can be written by more general relations. Let $(\tau_n(a))_{n\geq 0}$ ba a sequence of random variables defined as

$$\tau_0(a) := \inf\{k \geq 0 : W_k \leq a\} \quad \text{and} \quad \tau_n(a) := \inf\{k > \tau_{n-1}(a) : W_k \leq a\}, \quad n \geq 1$$

and denote by $\pi_W^{(a)}$ the stationary distribution of the positive Harris chain $(M_{\tau_n(a)}, W_{\tau_n(a)})_{n\geq 0}$. Then, for any fixed $a \geq 0$, a steady state distribution π_V of $(V_t)_{t\geq 0}$ given as

$$\begin{aligned}\pi_V(\cdot) &= \frac{1}{\mathbb{E}_{\pi_W^{(a)}} \overline{T}_{\tau_1(a)}} \mathbb{E}_{\pi_W^{(a)}} \left(\int_0^{\overline{T}_{\tau_1(a)}} \mathbf{1}(V_t \in \cdot\,) dt \right) \\ &= \frac{1}{\mathbb{E}_\xi T_1} \int_{E\times[0,a]} \mathbb{E}_{s,w} \left(\int_0^{I(a)} \mathbf{1}(V_t \in \cdot) dt \right) \pi_W(ds \times dw),\end{aligned}$$

where $I(a) = \sum_{n=1}^{\tau(a)} T_n$.

As in the actual waiting time process we get moments of regeneration epochs:

Proposition 4.11 *(i) Suppose that $\Re \times \{0\}$ is a recurrent set of $(M_n, W_n)_{n\geq 0}$.*

(a) Let $\alpha > 1$. Suppose that

$$\sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \text{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty,$$

then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\zeta \overline{T}_{\hat{\tau}_1}^\alpha < \infty$ and $\mathbb{E}_\lambda \overline{T}_{\hat{\tau}_1}^\alpha < \infty$, where $\zeta(\cdot) = \mathbb{P}_{\lambda'}(M_{\hat{\tau}_1} \in \cdot\,)$ for each initial distribution λ' on E.

(b) Let $\gamma > 0$. Suppose that

$$\sup_{s\in\Re} \mathbb{E}_s \exp\left(\gamma \sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \exp\left(\gamma \sum_{n=1}^{\tau_1} T_n \right) < \infty.$$

Then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\zeta \exp(\gamma' \overline{T}_{\hat{\tau}_1}) < \infty$ for some $\gamma' > 0$.

(ii) Suppose that there exists $q > 0$ such that

$$\mathbb{P}_s(M_1 \in \cdot\,, U_0 - T_1 < 0) > q\varphi(\cdot).$$

(a) Let $\alpha > 1$. Suppose that

$$\mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \text{and} \quad \mathbb{E}_\varphi \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty.$$

If for an initial distribution λ on E

$$\mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_\lambda \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty,$$

then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\varphi \overline{T}^\alpha_{\hat{\tau}_1} < \infty$ and $\mathbb{E}_\lambda \overline{T}^\alpha_{\hat{\tau}_1} < \infty$.

(b) Let $\gamma > 0$. Suppose that

$$\mathbb{E}_\varphi \exp\left(\gamma \sum_{n=1}^{\tau_1} U_{n-1} \right) < \infty \quad \textit{and} \quad \mathbb{E}_\varphi \exp\left(\gamma \sum_{n=1}^{\tau_1} T_n \right) < \infty.$$

Then there exists a sequence of regeneration epochs $(\hat{\tau}_n)_{n\geq 0}$ of $(M_n, W_n)_{n\geq 0}$ such that $\mathbb{E}_\varphi \exp(\gamma' \overline{T}_{\hat{\tau}_1}) < \infty$ for some $\gamma' > 0$.

Proof. All assertions are obvious from Theorem 4.1 and Proposition 4.9. QED

One should note that the process $(V_t, Q_t)_{t\geq 0}$ is not wide-sense regenerative in general, but one-dependent regenerative, and thus from the theory of point processes (see Sigman [59]) we get

$$\lim_{t\to\infty} \| \frac{1}{t} \int_0^t \mathbb{P}_\lambda((V_u, Q_u) \in \cdot)\, du - \pi_{VQ} \| = 0$$

for any initial distribution λ on E.

However, if E is countable, then $(V_t, Q_t)_{t\geq 0}$ is classical-sense regenerative and we get rates of convergence, which are obtained in Sharma [55].

Corollary 4.12 *Suppose that E is countable and that for a recurrent state i_0 of M*

$$\mathbb{P}_{i_0}(M_1 = i_0, U_0 - T_1 < 0) > 0.$$

Suppose further that the distribution $\mathbb{P}_{i_0}(\overline{T}_{\tau_1} \in \cdot)$ is spread out, where τ_1 is the first return time to i_0 of M.

(i) Let $\alpha > 1$. Suppose that

$$\mathbb{E}_{i_0} \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_{i_0} \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty.$$

If for an initial state $i \in E$

$$\mathbb{E}_i \left(\sum_{n=1}^{\tau_1} U_{n-1} \right)^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_i \left(\sum_{n=1}^{\tau_1} T_n \right)^\alpha < \infty,$$

then for some constant c

$$\|\mathbb{P}_i\left((V_t, Q_t) \in \cdot\,\right) - \pi_{VQ}\| \le ct^{1-\alpha}.$$

(ii) Let $\gamma > 0$. Suppose that

$$\mathbb{E}_{i_0} \exp\Big(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\Big) < \infty \quad \text{and} \quad \mathbb{E}_{i_0} \exp\Big(\gamma \sum_{n=1}^{\tau_1} T_n\Big) < \infty.$$

Then for some constants c and $\gamma'' \in (0, \gamma']$

$$\|\mathbb{P}_{i_0}\left((V_t, Q_t) \in \cdot\,\right) - \pi_{VQ}\| \le c\exp(-\gamma'' t).$$

Proof. As in the proof of Corollary 3.15, one can easily see that there exists a sequence of a.s. finite random times $(\nu_n)_{n\ge0}$ such that $(\hat\tau_n)_{n\ge0} = (\tau_{\nu_n})_{n\ge0}$ forms a sequence of regeneration epochs of $(M_n, W_n)_{n\ge0}$. Furthermore, one can easily see that $\mathbb{P}_{i_0}(\overline{T}_{\hat\tau_1} \in \cdot\,)$ is spread out if and only if $\mathbb{P}_{i_0}(\overline{T}_{\tau_1} \in \cdot\,)$ is so (cf. Proposition X.3.2 in Asmussen [12]). Thus all assertions follow from Corollary of Theorem 3.4.2 in Kalashnikov [32] and Theorem 10.7.5 in Thorisson [63]. QED

Denote by V a random variable having the steady state distribution of $(V_t)_{t\ge0}$, i.e.,

$$\mathbb{P}(V \le x) = \frac{1}{\mathbb{E}_{\bar\xi}\, I_1}\mathbb{E}_{\bar\xi}\left(\int_0^{I_1} \mathbf{1}(V_t \le x)\, dt\right).$$

Theorem 4.13 *(i) Let $\alpha \ge 1$. If $\mathbb{E}_{\bar\xi}\, I_1^{\alpha+1} < \infty$, then $\mathbb{E}\, V^\alpha < \infty$.*

(ii) Let $\gamma > 0$. If $\mathbb{E}_{\bar\xi} \exp(\gamma I_1) < \infty$, then $\mathbb{E} \exp(\gamma' V) < \infty$ for some $\gamma' > 0$.

Proof. (i) Since $V_t \le \sum_{n=1}^{\sigma_1} U_{n-1} \le I_1$ for all $k \le \sigma_1$, it holds that

$$\mathbb{E}\, V^\alpha = \frac{1}{\mathbb{E}_{\bar\xi}\, I_1}\mathbb{E}_{\bar\xi}\left(\int_0^{I_1} V_t^\alpha\, dt\right) \le \frac{1}{\mathbb{E}_{\bar\xi}\, \sigma_1}\mathbb{E}_{\bar\xi}\, I_1^{\alpha+1} < \infty,$$

which proves (i). (ii) can be proved in the same manner. QED

If M is uniformly Harris ergodic, then from Corollary 4.3

$$\sup_{s\in E} \mathbb{E}_s\, T_1^{\alpha+1} < \infty \quad \text{and} \quad \sup_{s\in E} \mathbb{E}_s\, U_0^{\alpha+1} < \infty \quad \text{for some} \quad \alpha > 0 \ \Rightarrow\ \mathbb{E}\, V^\alpha < \infty.$$

Moreover,

$$\sup_{s\in E} \mathbb{E}_s \exp(\gamma T_1) < \infty \quad \text{and} \quad \sup_{s\in E} \mathbb{E}_s \exp(\gamma U_0) < \infty \quad \text{for some} \quad \gamma > 0$$
$$\Rightarrow \mathbb{E} \exp(\gamma' V) < \infty \quad \text{for some} \quad \gamma' > 0.$$

Remark 4.14 Suppose that $\Re \times \{0\}$ is a recurrent set of $(M_n, W_n)_{n\geq 0}$. Then as in Remark 4.4

$$\sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} T_n\right)^{\alpha+1} < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\tau_1} U_{n-1}\right)^{\alpha+1} < \infty$$

$$\Rightarrow \sup_{s\in\Re} \mathbb{E}_s\, \overline{T}_{\hat{\tau}_1}^{\alpha+1} < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \left(\sum_{n=1}^{\hat{\tau}_1} U_{n-1}\right)^{\alpha+1} < \infty$$

$$\Rightarrow \mathbb{E}_\zeta\, \overline{T}_{\hat{\tau}_1}^{\alpha+1} < \infty \quad \text{and} \quad \mathbb{E}_\zeta \left(\sum_{n=1}^{\hat{\tau}_1} U_{n-1}\right)^{\alpha+1} < \infty$$

$$\Rightarrow \mathbb{E}\, V^\alpha = \frac{1}{\mathbb{E}_\zeta\, \overline{T}_{\hat{\tau}_1}} \mathbb{E}_\zeta \left(\int_0^{\overline{T}_{\hat{\tau}_1}} V_t^\alpha\right) < \infty.$$

Similarly,

$$\sup_{s\in\Re} \mathbb{E}_s \exp\Big(\gamma \sum_{n=1}^{\tau_1} T_n\Big) < \infty \quad \text{and} \quad \sup_{s\in\Re} \mathbb{E}_s \exp\Big(\gamma \sum_{n=1}^{\tau_1} U_{n-1}\Big) < \infty \quad \text{for some} \quad \gamma > 0$$
$$\Rightarrow \mathbb{E} \exp(\gamma' V) < \infty \quad \text{for some} \quad \gamma' > 0.$$

The same results are obtained in Sharma [55] for the countable modulation case.

Sometimes it is of interest to look at the queue length at certain random times. Denote by Q_n^A and Q_n^D the queue length just prior to the nth arrival and just after the nth departure, respectively.

Proposition 4.15 *Let $(\hat{\tau}_n)_{n\geq 0}$ be a sequence of regeneration epochs for $(M_n, W_n)_{n\geq 0}$ with $W_{\hat{\tau}_n} = 0$ for $n \geq 0$. Consider the cycles $\hat{Z}_n$ defined as*

$$\hat{Z}_n = \big(\hat{\tau}_{n+1} - \hat{\tau}_n, (Q_k^A, Q_k^D)_{\hat{\tau}_n \leq k < \hat{\tau}_{n+1}}\big), \quad n \geq 0.$$

Then $\hat{Z}_n$ are one-dependent for $n \geq 0$ and identically distributed for $n \geq 1$ with common distribution $\mathbb{P}_\lambda^{(\hat{Z}_n)_{n\geq 1}} = \mathbb{P}_\zeta^{(\hat{Z}_n)_{n\geq 0}}$, where $\zeta = \mathbb{P}_\lambda (M_{\hat{\tau}_1} \in \cdot\,)$ for any initial distribution λ. Furthermore, $(\hat{Z}_n)_{n\geq k}$ is independent of $(\hat{\tau}_0, \cdots, \hat{\tau}_k)$.
If the state space is countable, then the cycles are independent.

Proof. Note first that for any $k, n \geq 0$

$$\begin{aligned}
Q_{\hat{\tau}_n+k}^A &= \lim_{t\uparrow \overline{T}_{\hat{\tau}_n+k}} Q_t = \lim_{t\uparrow \overline{T}_{\hat{\tau}_n+k}} \sum_{r=0}^\infty \mathbf{1}(\overline{T}_r \leq t, \overline{T}_r + W_r + U_r > t) \\
&= \lim_{t\uparrow \overline{T}_{\hat{\tau}_n+k}} \sum_{r=0}^\infty \mathbf{1}(\overline{T}_{\hat{\tau}_n+r} \leq t, \overline{T}_{\hat{\tau}_n+r} + W_{\tau_n+r} + U_{\hat{\tau}_n+r} > t) \\
&= \lim_{t\uparrow T_{\hat{\tau}_n}^{(k)}} \sum_{r=0}^\infty \mathbf{1}(T_{\hat{\tau}_n}^{(r)} \leq t, T_{\hat{\tau}_n}^{(r)} + W_{\hat{\tau}_n+r} + U_{\hat{\tau}_n+r} > t) \\
&= f_k\big((M_r, W_r, A_{r+1}, U_r)_{r\geq \hat{\tau}_n}\big),
\end{aligned}$$

where f_k is a measurable function. Similarly one can show that for any $k, n \geq 0$

$$Q^D_{\hat{\tau}_n+k} = \lim_{t \downarrow \overline{T}_{\hat{\tau}_n+k} + U_{\hat{\tau}_n+k}} Q_t = g_k\big((M_r, W_r, T_{r+1}, U_r)_{r \geq \hat{\tau}_n}\big)$$

for some measurable function g_k. QED

In view of the previous result there exists a steady state distribution π' of $(Q^A_n, Q^D_n)_{n\geq 0}$, which is given as

$$\begin{aligned}
\pi'(\cdot) &= \frac{1}{\mathbb{E}_{\bar{\xi}}\,\sigma_1} \mathbb{E}_{\bar{\xi}} \left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}\big((Q^A_n, Q^D_n) \in \cdot\,\big) \right) \\
&= \int_E \mathbb{E}_s \left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}((Q^A_n, Q^D_n) \in \cdot\,) \right) \pi_W\,(ds \times \{(0,0)\}).
\end{aligned}$$

The rates of convergence of the sequence $(Q^A_n, Q^D_n)_{n\geq 0}$ to the stationary distribution π' are available from Theorem 4.1.

If the service times U_n are exponentially distributed with intensity β and independent of $(T_n)_{n\geq 0}$, then the sequence $(M_n, Q^A_n)_{n\geq 0}$ forms a Markov chain. Let $\mathbf{\Phi} = (\mathbf{\Phi}_{ij})_{i,j\geq 0}$, where

$$\mathbf{\Phi}_{ij}(s, A) = \mathbb{P}[M_{n+1} \in A, Q^A_n = j | M_n = s, Q^A_n = i].$$

Then $\mathbf{\Phi}$ can be written as the matrix form (2.8), where

$$\begin{aligned}
\mathbf{A}_l(s, A) &= \int_0^\infty e^{-\beta t} \frac{(\beta t)^l}{l!} d\mathbb{P}_s\,(M_1 \in A, U_0 \in dt), \quad l \geq 0, s \in E, A \in \mathcal{E}; \\
\mathbf{B}_l(s, A) &= \mathbb{P}_s\,(M_1 \in A) - \sum_{n=0}^{l} \mathbf{A}_n(s, A), \quad l \geq 0, s \in E, A \in \mathcal{E}.
\end{aligned}$$

If the arrival times $\overline{T}_n$ form a homogeneous Poisson process of rate β and independent of $(U_{n-1})_{n\geq 0}$, then the sequence $(M_n, Q^D_{n-1}, D_{n-1})_{n\geq 0}$ forms a Markov chain, where D_n is the time between the nth and $(n+1)$st departures. Let $\mathbf{\Phi} = (\mathbf{\Phi}_{ij})_{i,j\geq 0}$ with

$$\mathbf{\Phi}_{ij}(s, A; x) = \mathbb{P}[M_{n+1} \in A, Q^D_n = j, D_n \leq x | M_n = s, Q^D_{n-1} = i].$$

Then

$$\mathbf{\Phi} = \begin{pmatrix}
\mathbf{C}_0(s, A; x) & \mathbf{C}_1(s, A; x) & \mathbf{C}_2(s, A; x) & \mathbf{C}_3(s, A; x) & \cdots \\
\mathbf{D}_0(s, A; x) & \mathbf{D}_1(s, A; x) & \mathbf{D}_2(s, A; x) & \mathbf{D}_3(s, A; x) & \cdots \\
0 & \mathbf{D}_0(s, A; x) & \mathbf{D}_1(s, A; x) & \mathbf{D}_2(s, A; x) & \cdots \\
0 & 0 & \mathbf{D}_0(s, A; x) & \mathbf{D}_1(s, A; x) & \cdots \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot &
\end{pmatrix}$$

for $s \in E, A \in \mathcal{E}, x \geq 0$, where

$$\begin{aligned} \mathbf{D}_l(s, A; x) &= \int_0^x e^{-\beta t} \frac{(\beta t)^l}{l!} d\mathbb{P}_s(M_1 \in A, U_0 \in dt); \\ \mathbf{C}_l(s, A; x) &= \int_0^x \beta e^{-\beta t} \, \mathbf{D}_l(s, A; x-t) \, dt. \end{aligned}$$

Such queueing systems with finite modulation chains are extensively studied by various authors. For the comprehensive treatment the readers are referred to Neuts [43, 44].

4.1.4 Identities between steady state distributions

Now we find some relations between steady state distributions of the actual waiting time process, the workload process and the queue length process. For this purpose, we introduce random variables T_n^* and U_n^*, $n \geq 0$, defined as

$$\begin{aligned} \mathbb{P}[T_n^* \leq x | M_{n-1} = s, M_n = s'] &:= \frac{1}{\mathbb{E}_\xi T_1} \int_0^x \mathbb{P}[T_n > y | M_{n-1} = s, M_n = s'] \, dy; \\ \mathbb{P}[U_{n-1}^* \leq x | M_{n-1} = s, M_n = s'] &:= \frac{1}{\mathbb{E}_\xi U_0} \int_0^x \mathbb{P}[U_{n-1} > y | M_{n-1} = s, M_n = s'] \, dy \end{aligned}$$

for any $s_0, s_1 \in E$ and $x \geq 0$. Obviously the sequences $(M_n, T_n^*)_{n \geq 0}$ and $(M_n, U_{n-1}^*)_{n \geq 0}$ are Markov modulated chains with driving chain M.

Lemma 4.16 *For any $x \geq 0$, we have*

$$\begin{aligned} \frac{1}{\mathbb{E}_{\bar{\xi}} \sigma_1} \mathbb{E}_{\bar{\xi}} \left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}(T_{n+1}^* \leq x) \right) &= \frac{1}{\mathbb{E}_\xi T_1} \int_0^x \mathbb{P}_\xi(T_1 > y) \, dy; \\ \frac{1}{\mathbb{E}_{\bar{\xi}} \sigma_1} \mathbb{E}_{\bar{\xi}} \left(\sum_{n=0}^{\sigma_1 - 1} \mathbf{1}(U_n^* \leq x) \right) &= \frac{1}{\mathbb{E}_\xi U_0} \int_0^x \mathbb{P}_\xi(U_0 > y) \, dy. \end{aligned}$$

Proof. Letting $\mathcal{G}_n := \sigma\big((M_k, T_k, U_{k-1})_{k \leq n}\big), n \geq 1$, we have

$$\mathbb{P}\big[T_{n+1} | \{\sigma_1 > n\} \cup \mathcal{G}_n\big] = \mathbb{P}[T_{n+1} | \mathcal{G}_n] = \mathbb{P}[T_{n+1} | M_n].$$

Thus the events $\{T_{n+1} > y\}$ and $\{\sigma_1 > n\}$ are independent given M_n. In a similar manner one can check that $\{T_{n+1}^* \leq y\}$ and $\{\sigma_1 > n\}$ are independent given M_n. Hence we have

$$\begin{aligned} &\frac{1}{\mathbb{E}_\xi T_1} \int_0^x \mathbb{P}[T_{n+1} > y, \sigma_1 > n | M_n = s_0, M_{n+1} = s_1] \, dy \\ &= \frac{1}{\mathbb{E}_\xi T_1} \mathbb{P}[\sigma_1 > n | M_n = s_0, M_{n+1} = s_1) \int_0^x \mathbb{P}[T_{n+1} > y | M_n = s_0, M_{n+1} = s_1] \, dy, \\ &= \mathbb{P}[\sigma_1 > n | M_n = s_0, M_{n+1} = s_1] \, \mathbb{P}[T_{n+1}^* \leq x | M_n = s_0, M_{n+1} = s_1] \\ &= \mathbb{P}[T_{n+1}^* \leq x, \sigma_1 > n | M_n = s_0, M_{n+1} = s_1]. \end{aligned}$$

Consequently, by making use of the Markov property and Fubini Theorem,

$$\begin{aligned}\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\,\mathbb{E}_{\bar\xi}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}(T^*_{n+1}\le x)\right) &= \frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\sum_{n=0}^{\infty}\mathbb{P}_{\bar\xi}\,(T^*_{n+1}\le x,\sigma_1>n)\\ &= \frac{1}{\mathbb{E}_\xi T_1}\int_0^x\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\sum_{n=0}^{\infty}\mathbb{P}_{\bar\xi}\,(T_{n+1}>y,\sigma_1>n)\,dy\\ &= \frac{1}{\mathbb{E}_\xi\,T_1}\int_0^x\mathbb{P}_\xi(T_1>y)\,dy.\end{aligned}$$

In the same manner, one can prove the second equality. QED

For any fixed $k\ge 0$, we set

$$T_n^{(k)} := \overline{T}_{n+k}-\overline{T}_n,\quad k\ge 0\quad\text{and}\quad T_{-1}^{(k)} := T_0.$$

Then, for any $k\ge 0$, the sequence $(M_n, W_n, T^*_{n-1}, U^*_{n-1}, U_{n-1}, T^{(k)}_{n-1})_{n\ge 0}$ forms a Markov modulated chain with the driving chain $(M_n, W_n)_{n\ge 0}$ and thus is positive Harris recurrent. As a consequence, there exists a stationary distribution for the chain given as

$$\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\,\mathbb{E}_{\bar\xi}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}((M_n,W_n,T^*_n,U^*_n,U_n,T^{(k)}_n)\in\cdot\,)\right)$$

for any $k\ge 0$, where $\bar\xi$ is a stationary distribution of M^σ. Let W, T^*, U^*, U and $T^{(k)}$ be random variables given by

$$\mathbb{P}^{(W,T^*,U^*,U,T^{(k)})}(\cdot) := \frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\,\mathbb{E}_{\bar\xi}\left(\sum_{n=0}^{\sigma_1-1}\mathbf{1}((W_n,T^*_n,U^*_n,U_n,T^{(k)}_n)\in\cdot\,)\right).$$

The existence of such variables follows from Kolmogorov's consistency theorem. Further, we introduce a random variable χ with $\chi\sim\mathcal{B}(1,\rho)$, which is independent on all the other random variables and sequences.

Proposition 4.17 *It holds that*

$$V\sim(1-\chi)+\chi(W+U^*)\sim(W+U-T^*)^+.$$

Proof. The relations are known for $GI/GI/1$ queues (see Satz 11.3.2 in Alsmeyer [1]) and can be proved in the same manner also for $SM/SM/1$ queues.

Since $W_{n+1}=W_n+U_n-T_{n+1}=W_n+X_{n+1}$ for $0\le n<\sigma_1$ and $\{W_{n+1}-y>x\}$ can be exchanged by $\{W_n-y>x\}$,

$$P(V>x) = \frac{1}{\mathbb{E}_{\bar\xi}\,\overline{T}_{\sigma_1}}\,\mathbb{E}_{\bar\xi}\left(\sum_{n=0}^{\sigma_1-1}\int_{\overline{T}_n}^{\overline{T}_{n+1}}\mathbf{1}(W_n+U_n+\overline{T}_n-y>x)\,dy\right)$$

$$
\begin{aligned}
&= \frac{1}{\mathbb{E}_{\bar\xi}\,\overline{T}_{\sigma_1}}\,\mathbb{E}_{\bar\xi}\left(\sum_{n=0}^{\sigma_1-1}\int_0^{T_{n+1}} \mathbf{1}(W_n+U_n-y>x)\,dy\right)\\
&= \frac{1}{\mathbb{E}_{\bar\xi}\,\overline{T}_{\sigma_1}}\sum_{n=0}^{\infty}\int_0^{\infty} \mathbb{P}_{\bar\xi}\,(W_n>x-y, U_n>y, \sigma_1>n)\,dy\\
&= \rho\cdot\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1\mathbb{E}_{\xi}\,U_0}\sum_{n=0}^{\infty}\int_0^{\infty} \mathbb{P}_{\bar\xi}\,(W_n>x-y, U_n>y, \sigma_1>n)\,dy.
\end{aligned}
$$

However, the events $\{W_n > x-y, \sigma_1 > n\}$ and $\{U_n > y\}$ are conditionally independent given M_n and M_{n+1}, whence

$$
\begin{aligned}
&\mathbb{P}_{\bar\xi}\,(W_n>x-y, U_n>y, \sigma_1>n)\\
&= \int_{E^2} \mathbb{P}[W_n>x-y, U_n>y, \sigma_1>n|M_n, M_{n+1}]\,d\mathbb{P}_{\bar\xi}^{M_n,M_{n+1}}\\
&= \int_{E^2} \mathbb{P}[W_n-x>-y, \sigma_1>n|M_n, M_{n+1}]\,\mathbb{P}[U_n>y|M_n, M_{n+1}]\,d\mathbb{P}_{\bar\xi}^{M_n,M_{n+1}}.
\end{aligned}
$$

Therefore for any $x \geq 0$

$$
\begin{aligned}
\mathbb{P}(V>x) &= \rho\cdot\frac{1}{\mathbb{E}_{\bar\xi}\sigma_1\mathbb{E}_{\xi}U_0}\sum_{n=0}^{\infty}\int_{E^2}\int_0^{\infty}\mathbb{P}[W_n-x>-y, \sigma_1>n|M_n, M_{n+1}]\\
&\qquad\qquad \cdot\mathbb{P}[U_n>y|M_n, M_{n+1}]\,dy\,d\mathbb{P}_{\bar\xi}^{M_n,M_{n+1}}\\
&= \rho\cdot\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\sum_{n=0}^{\infty}\int_{E^2}\mathbb{P}[W_n+U_n^*>x, \sigma_1>n|M_n, M_{n+1}]\,d\mathbb{P}_{\bar\xi}^{M_n,M_{n+1}}\\
&= \rho\cdot\frac{1}{\mathbb{E}_{\bar\xi}\,\sigma_1}\sum_{n=0}^{\infty}\mathbb{P}_{\bar\xi}\,(W_n+U_n^*>x, \sigma_1>n)\\
&= \rho\mathbb{P}(W+U^*>x),
\end{aligned}
$$

from which the first relation follows. Moreover, the events $\{W_n + U_n > x + y, \sigma_1 > n\}$ and $\{T_{n+1} > y\}$ are conditionally independent given M_n, M_{n+1}. Thus the second relation follows from the equality

$$
\int_0^{T_{n+1}} \mathbf{1}(W_n+U_n-y>x)\,dy = \int_0^{\infty}\mathbf{1}(W_n+U_n-y>x, T_{n+1}>y)\,dy.
$$

QED

Next we find relations between steady state distributions of the actual waiting time and queue length process. Let Q be a random variable with distribution

$$
\mathbb{P}(Q\in\cdot\,) := \frac{1}{\mathbb{E}_{\bar\xi}\,I_1}\,\mathbb{E}_{\bar\xi}\left(\int_0^{I_1}\mathbf{1}(Q_t\in\cdot\,)\,dt\right).
$$

The following assertions are known for $GI/GI/1$ queues.

Proposition 4.18 *The following equalities hold true:*

(i)

$$\mathbb{E}Q = \frac{1}{\mathbb{E}T} \cdot \mathbb{E}(W+U) \quad \textit{(Little's Formula)};$$

(ii)

$$\mathbb{P}(Q > l) = \rho\, \mathbb{P}(W + U^* > T^{(l-1)}).$$

Proof. (i) Note first that each of the customers $n = 0, 1, \cdots, \sigma_1 - 1$, provides a contribution $W_n + U_n$ to $\int_{[0,\sigma_1)} Q_t\, dt$. Thus we have

$$\begin{aligned}
\mathbb{E}Q &= \frac{1}{\mathbb{E}_{\bar\xi}\, \overline{T}_{\sigma_1}}\, \mathbb{E}_{\bar\xi} \int_0^{\overline{T}_{\sigma_1}} Q_t\, dt \\
&= \frac{1}{\mathbb{E}_{\xi}\, T_1} \cdot \frac{1}{\mathbb{E}_{\bar\xi}\, \sigma_1}\, \mathbb{E}_{\bar\xi} \left(\sum_{n=0}^{\sigma_1 - 1} (W_n + U_n) \right) = \frac{1}{\mathbb{E}T} \cdot \mathbb{E}(W+U).
\end{aligned}$$

(ii) The assertion is known for $GI/GI/1$ queues (see Satz 11.4.2 in Alsmeyer [1]) and can be proved in the same manner also for $SM/SM/1$ queues.

For any $0 \le n < \sigma_1$ and $t \in [\overline{T}_n, \overline{T}_{n+1})$ let

$$\sum_{k=0}^{n} \mathbf{1}(U'_k > t) := \sum_{k=0}^{n} \mathbf{1}(\overline{T}_k + W_k + U_k > t) = Q_t.$$

Since $U'_0 < U'_1 < \cdots$, we have

$$\{Q_t \ge n\} = \{U'_{n-l+1} > t\} \quad \text{for all} \quad l \ge 1,$$

where we set $U'_l := 0$ for $l \le 0$. From

$$U'_{n-l+1} - \overline{T}_{n+1} \le U'_{n-l+1} - \overline{T}_n \le 0 \quad \text{for} \quad n \in \{0, \cdots, l-2, \sigma_1, \cdots, \sigma_1 + l - 2\}$$

and

$$\mathbb{P}_{\bar\xi} \left((M_0, W_0, U_0, T_0^{(l)}) \in \cdot \right) = \mathbb{P}_{\bar\xi} \left((M_{\sigma_1}, W_{\sigma_1}, U_{\sigma_1}, T_{\sigma_1}^{(l)}) \in \cdot \right),$$

we get for any $l \ge 0$

$$\begin{aligned}
\mathbb{P}(Q \ge l) &= \frac{1}{\mathbb{E}_{\bar\xi}\, \overline{T}_{\sigma_1}} \mathbb{E}_{\bar\xi} \left(\sum_{n=0}^{\sigma_1 - 1} \int_{\overline{T}_n}^{\overline{T}_{n+1}} \mathbf{1}(U'_{n-l+1} > t)\, dt \right) \\
&= \frac{1}{\mathbb{E}_{\bar\xi}\, \overline{T}_{\sigma_1}} \mathbb{E}_{\bar\xi} \left(\sum_{n=l-1}^{\sigma_1 + l - 2} \int_0^{\infty} \Big(\mathbf{1}(U'_{n-l+1} - \overline{T}_n > t) - \mathbf{1}(U'_{n-l+1} - \overline{T}_{n+1} > t) \Big)\, dt \right) \\
&= \frac{1}{\mathbb{E}_{\bar\xi}\, \overline{T}_{\sigma_1}} \mathbb{E}_{\bar\xi} \left(\sum_{n=0}^{\sigma_1 - 1} \int_0^{\infty} \Big(\mathbf{1}(W_n + U_n - T_n^{(l-1)} > t) - \mathbf{1}(W_n - T_n^{(l-1)} > t) \Big)\, dt \right).
\end{aligned}$$

Since the events $\{W_n - T_n^{(l-1)}\}$ and $\{U_n\}$ are conditionally independent given M_n, M_{n+1}, by the same computation as for $\mathbb{P}(V \in \cdot)$, we have

$$\mathbb{P}(Q > l) = \rho \mathbb{P}(W + U^* > T^{(l-1)}).$$

QED

Notice that Little's formula holds true for more general queues. It is known (see Glynn and Whitt [29]), if the limits

$$\frac{1}{\lambda} := \lim_{n\to\infty} \frac{\overline{T}_n}{n} \quad \text{and} \quad \overline{W} := \lim_{n\to\infty} \frac{1}{n}(W_n + U_n)$$

exist and are finite, then the limit $L := \lim_{t\to\infty} \frac{1}{t}\int_0^t Q_u du$ exists, and Little's formula is written as $L = \lambda \overline{W}$.

4.2 Multiserver queues

In this section we consider semi-Markov queues with N servers. As in the case of single server queues we denote by T_n the interarrival time between customers $n-1$ and n, and by U_n the service time of customer n. Let T_0 and U_{-1} be arbitrary random variables with values in $\mathbb{R}_0^+$. $T_n, n \geq 1$, and $U_n, n \geq 0$, are assumed to be positive. The input process $(M_n, T_n, U_{n-1})_{n\geq 0}$ is assumed to be a Markov modulated chain with driving chain M and transition kernel

$$\mathbf{P} : E \times (\mathcal{E} \otimes (\mathcal{B}|_{[0,\infty)})^2) \to [0,1].$$

Let $\overline{T}_n = T_0 + T_1 + \cdots + T_n, n \geq 0$, and $\overline{U}_n = U_{-1} + U_0 + \cdots + U_n$, $n \geq -1$. If $T_0 = U_{-1} = 0$, which means that customer 0 arrives at time 0, then $\overline{T}_n$ is the arrival time of the customer n in the system and $\overline{U}_n$ the total workload up to the nth customer.

The queue discipline is assumed to be FCFS, which means that the customers join service in the order they arrive. In the single server case, it is the same as the FIFO discipline but not in general for $N > 1$. We say that the system is *stable*, if there exists a finite stationary distribution for the discrete-time workload process. If the mean interarrival time $\mathbb{E}_\xi T_1$ and the mean service time $\mathbb{E}_\xi U_0$ are finite, we define the *traffic intensity* ρ_N as

$$\rho_N := \frac{\mathbb{E}_\xi U_0}{N \mathbb{E}_\xi T_1}. \tag{4.3}$$

The condition $\rho_N < 1$, or equivalently $\mathbb{E}_\xi U_0 < N\, \mathbb{E}_\xi T_1$, is called the *stability condition* for multiserver queue with N servers. Throughout this section we assume the stability condition.

4.2.1 Existence of a stationary version

We think of each server as having its own waiting line and the arriving customer joining the line that has the least residual work. We order the residual work in the various lines at time t and thus obtain a vector $V_t = (V_t(1) \cdots V_t(N))$ satisfying

$$V_t(1) \leq V_t(2) \leq \cdots \leq V_t(N), \quad t \geq 0.$$

It is of particular interest to observe V_t just before the arrival instants $\overline{T}_n$ and we write $W_n = (W_n(1), \cdots, W_n(N)) = V_{\overline{T}_n-}$. Thus $W_n(1)$ is the waiting time of the nth customer, before he is served. The process $(W_n)_{n\geq 0}$ of ordered vectors is called the *discrete-time workload process* and satisfies the *Kiefer-Wolfowitz recurrence relation*

$$W_{n+1} = \mathcal{R}(W_n + U_n\mathbf{e} - T_{n+1}\mathbf{I})^+, \quad n \geq 0,$$

where $\mathbf{e} = (1, 0, \cdots, 0), \mathbf{I} = (1, \cdots, 1)$, $(x_1, \cdots, x_N)^+ = (x_1^+, \cdots, x_N^+)$ and $\mathcal{R}$ is the operator arranging vectors of $\mathbb{R}^N$ in the increasing order. Obviously the process $(M_n, W_n)_{n\geq 0}$ forms a temporally homogeneous Markov chain. The following proposition says that under the stability condition the queueing system is stable.

Proposition 4.19 *There exists a stationary version* $(M_n^*, W_n^*)_{n\geq 0}$.

Proof. Let $(M_n^*, T_n^*, U_{n-1}^*)_{n\in\mathbb{Z}}$ be a stationary doubly infinite version of $(M_n, T_n, U_{n-1})_{n\geq 0}$ and consider a new queue with the input process $(M_n^*, T_n^*, U_{n-1}^*)_{n\geq 0}$. Let $W_{k,n}^*$ be the ordered workload vector in the new system found by the customer n if the customer $-k$ finds an empty system. For fixed $n \geq 0$, the sequence $(W_{k,n}^*)_{k\geq 0}$ increases coordinately. Now define

$$W_n^* := \lim_{k\to\infty} W_{k,n}^* = \sup\{W_{k,n}^* : -\infty < k \leq n\}.$$

It holds that

$$W_{n+1}^* = \lim_{k\to\infty} W_{k,n+1}^* = \lim_{k\to\infty} \mathcal{R}(W_{k,n}^* + U_{n-1}^*\mathbf{e} - T_n^*\mathbf{I})^+ = \mathcal{R}(W_n^* + U_n^*\mathbf{e} - T_{n+1}^*\mathbf{I})^+.$$

For the finiteness of W_n^*, see Theorem 2.3.1. in Baccelli-Bremaud [17]. Therefore, $(M_n^*, W_n^*)_{n\geq 0}$ is a stationary version of $(M_n, W_n)_{n\geq 0}$. QED

4.2.2 Regeneration of the discrete time workload process

In the rest of this chapter we suppose that E is countable and $N = 2$. We find some conditions that the discrete-time workload process $(M_n, W_n)_{n\geq 0}$ is Harris ergodic. i_0 denotes a recurrent state of M and τ_1 the first return time of M to the state i_0. Consider a test function V_β defined as

$$V_\beta(w(1), w(2)) = v_\beta(w(1)) + v_\beta(w(2)),$$

where

$$v_\beta(x) = x - \frac{1}{\beta}\big(1 - \exp(-\beta x)\big), \quad \beta > 0.$$

Let T and U be positive random variables satisfying

$$\mathbb{E}\big(2(u \wedge T) - U\big) > 0 \quad \text{and} \quad \mathbb{E}\big(U\mathbf{1}(U > u)\big) < \epsilon$$

for some positive real numbers ϵ and u. Then it can be shown (see pp.133 of Kalashnikov [33])that for $x \le x_1 := u + \frac{u^2}{2e\epsilon}$

$$\begin{aligned}\mathbb{E}\left(v_\beta((x+U-T)^+) - v_\beta(x)\right) &\le \mathbb{E}\left(v_\beta((x+U)^+) - v_\beta(x)\right)\\ &\le (1-\exp(-\beta(x_1+u)))\mathbb{E}U + \epsilon.\end{aligned}$$

and that for $x > x_1$

$$\begin{aligned}\mathbb{E}\left(v_\beta((x+U-T)^+) - v_\beta(x)\right) &\le (1-\exp(-\beta x))\mathbb{E}\,(U - u\wedge T);\\ \mathbb{E}\left(v_\beta((x-T)^+) - v_\beta(x)\right) &\le -(1-\exp(-\beta x))\mathbb{E}\,(u\wedge T) + \epsilon.\end{aligned}$$

Proposition 4.20 *There exists a compact subset $\mathcal{K}$ of $[0,\infty)^2$ such that $\{i_0\}\times\mathcal{K}$ is a recurrent set of $(M_n, W_n)_{n\ge 0}$.*

Proof. Consider the Markov chain $(W_{\tau_n})_{n\ge0}$. We will show that there exists a compact measurable subset $\mathcal{K}$ of $[0,\infty)^2$ such that $\mathbb{E}_{(i_0,w)}\,\tilde\tau(\mathcal{K}) < \infty$ for any $w \in [0,\infty)^2$, where

$$\tilde\tau(\mathcal{K}) := \inf\{k > 0 : W_{\tau_k} \in \mathcal{K}\},$$

from which the assertion follows (see Remark 3.4).

For fixed $x \ge 0$ denote by $\tau(x)$ the random time defined as

$$\tau(x) := \sup\left\{n : \sum_{k=1}^{n} T_k < x\right\}.$$

Then there exist $x_0 \ge 0$ and $\Delta > 0$ such that

$$\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau_1} U_{n-1} - 2\sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\right) < -\Delta \quad \text{and} \quad \mathbb{E}_{i_0}\left(\sum_{n=\tau(x_0)\wedge\tau_1+1}^{\tau_1} U_{n-1}\right) < \epsilon.$$

Choose $\epsilon < \min(1, \frac{\Delta}{16})$ and let the positive real numbers u, β and x_1 be given as

$$\begin{aligned}u &= 2x_0\\ \beta &= \min\left\{-\frac{1}{u}\ln\big(1 - \frac{\Delta}{7\Delta+6l}\big), \left(2u + \frac{u^2}{2e\epsilon}\right)^{-1}\ln\frac{4}{3}\right\},\\ x_1 &= \frac{1}{\beta}\ln\frac{4}{3} - u,\end{aligned}$$

where

$$l = \mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau_1} U_{n-1}\right) < \infty.$$

Note that

$$x_1 \geq u + \frac{u^2}{2e\epsilon}, \quad 1 - \exp(-\beta u) \leq \frac{\Delta}{7\Delta + 6l} \quad \text{and} \quad 1 - \exp\big(-\beta(x_1 + u)\big) = \frac{1}{4}.$$

Let x_2 be a positive real number satisfying the equation

$$1 - \exp(-\beta x_2) = \frac{1}{2}$$

and let

$$\mathcal{K} := \{(w(1), w(2)) : w(1) \leq w(2) \leq x_2\}.$$

Then for any $w \geq 0$

$$\mathbb{E}_{(i_0,w)}\big(V(W_{\tau_1}) - V(w)\big) \leq \mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau_1} U_{n-1}\right) + \frac{1}{\beta} < \infty.$$

Thus it suffices to show that

$$\sup_{w \notin \mathcal{K}} \mathbb{E}_{(i_0,w)}\big(V(W_{\tau_1}) - V(w)\big) < 0.$$

We consider three cases:

(i) $w_0(1) \leq x_1$:
Since $w_0(2) > x_2 \geq x_1 + u > 2x_0$,

$$w_{\tau(x_0)\wedge\tau_1}(2) = w_0(2) - \sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n.$$

Thus

$$\begin{aligned}
&\mathbb{E}_{i_0}\big(V_\beta(W_{\tau_1}) - V_\beta(W_0)\big) \\
&\quad = \sum_{k=1}^{2} \mathbb{E}_{i_0}\big(v_\beta(w_{\tau_1}(k)) - v_\beta(w_0(k))\big) \\
&\quad \leq \mathbb{E}_{i_0}\left[\left(v_\beta\Big(w_0(1) + \sum_{n=1}^{\tau(x_0)\wedge\tau_1} U_{n-1}\Big) - v_\beta(w_0(1))\right)\right] \\
&\qquad + \mathbb{E}_{i_0}\left[\left(v_\beta\Big(w_0(2) - \sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\Big) - v_\beta(w_0(2))\right)\right] + 2\epsilon \\
&\quad \leq \big(1 - \exp(-\beta(x_1 + u))\big)l - (1 - \exp(-\beta u))\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\right) + 2\epsilon
\end{aligned}$$

$$\le \frac{1}{4}\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau_1} U_{n-1}\right) - \frac{1}{2}\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\right) + 2\epsilon$$
$$\le -\frac{\Delta}{4} + 2\epsilon < 0.$$

(ii) $w_0(1) > x_1$ and $w_0(2) - w_0(1) < u$:
For $1 \le k, l \le 2$ denote by C_{kl} the set of events starting at $w_0(k)$ and ending at $w_{\tau_1}(l)$. Obviously $C_{11} = C_{22}$ and $C_{12} = C_{21}$. Define measurable functions $f_{kl}^{\tau_1} : (\mathbb{R}_0^+)^\infty \times (\mathbb{R}_0^+)^\infty \to \mathbb{R}, k, l = 1, 2$, as

$$f_{kl}^{\tau_1} = \big(w_{\tau_1}(l) - w_0(k)\big)\mathbf{1}(C_{kl}).$$

Note that

$$\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbb{E}_{i_0} f_{kl}^{\tau(x_0)\wedge\tau_1} = \mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1}(U_{n-1} - 2T_n)\right) < -\Delta.$$

For each $k, l \le 2$, denote $f_{kl} = f_{kl}^{\tau(x_0)\wedge\tau_1}$. Then

$$\begin{aligned}
\mathbb{E}_{i_0}\big(V_\beta(W_{\tau_1}) - V_\beta(W_0)\big) &= \sum_{k=1}^{2}\sum_{l=1}^{2}\mathbb{E}_{i_0}\big\{\big(v_\beta(w_{\tau_1}(l)) - v_\beta(w_0(k))\big)\mathbf{1}(C_{kl})\big\} \\
&\le \sum_{k=1}^{2}\sum_{l=1}^{2}\big\{1 - \exp(-\beta w_0(k))\big\}\mathbb{E}_{i_0}\left(f_{kl}\mathbf{1}\big(C_{kl}\big)\right) + 2\epsilon \\
&\le -\Delta - \sum_{k=1}^{2}\sum_{l=1}^{2}\exp(-\beta w_0(k))\mathbb{E}_{i_0}\left(f_{kl}\mathbf{1}\big(C_{kl}\big)\right) + 2\epsilon.
\end{aligned}$$

For $1 \le k, l \le 2$, let $C_{kl}^+ := \{f_{kl} \ge 0\} \cap C_{kl}$ and $C_{kl}^- := \{f_{kl} < 0\} \cap C_{kl}$. Then

$$\begin{aligned}
&\mathbb{E}_{i_0}\big(V_\beta(W_{\tau_1}) - V_\beta(W_0)\big) \\
&\le -\Delta - \sum_{k=1}^{2}\sum_{l=1}^{2}\mathbb{E}_{i_0}\Big\{\exp(-\beta x_1) f_{kl}\mathbf{1}\big(C_{kl}^-\big) + \exp\big(-\beta(x_1+u)\big) f_{kl}\mathbf{1}\big(C_{kl}^+\big)\Big\} + 2\epsilon \\
&= -\Delta - \exp(-\beta x_1)\sum_{k=1}^{2}\sum_{l=1}^{2}\mathbb{E}_{i_0}\Big\{f_{kl}\mathbf{1}(C_{kl}^-) + \exp(-\beta u) f_{kl}\mathbf{1}(C_{kl}^+)\Big\} + 2\epsilon \\
&= -\Delta - \exp(-\beta x_1)\sum_{k=1}^{2}\sum_{l=1}^{2}\Big\{\mathbb{E}_{i_0}\big(f_{kl}\mathbf{1}(C_{kl})\big) - (1 - \exp(-\beta u))\mathbb{E}_{i_0}\big(f_{kl}\mathbf{1}(C_{kl}^+)\big)\Big\} + 2\epsilon \\
&\le -\Delta + \exp(-\beta x_1)\Big\{\Delta + \big(1 - \exp(-\beta u)\big)l\Big\} + 2\epsilon \\
&\le -\Delta + \frac{3\Delta}{4}\left(1 - \frac{\Delta}{7\Delta + 6l}\right)^{-1}\left(1 + \frac{l}{7\Delta + 6l}\right) + \frac{\Delta}{8} \\
&< 0,
\end{aligned}$$

since

$$\exp(-\beta x_1) = \frac{3}{4}\exp(\beta u) \le \frac{3}{4}\left(1 - \frac{\Delta}{7\Delta + 6l}\right)^{-1}.$$

(iii) $w_0(1) > x_1$ and $w_0(2) - w_0(1) \ge u$:
In this case

$$\begin{aligned}
&\mathbb{E}_{i_0}\big(V_\beta(W_{\tau_1}) - V_\beta(W_0)\big) \\
&= \mathbb{E}_{i_0}\Big\{v_\beta\Big(w_0(1) + \sum_{n=1}^{\tau(x_0)\wedge\tau_1}(U_{n-1} - T_n)\Big) - v_\beta(w_0(1)) \\
&\quad + v_\beta\Big(w_0(2) - \sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\Big) - v_\beta(w_0(2))\Big\} + 2\epsilon \\
&\le \big\{1 - \exp(-\beta w(1))\big\}\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1} U_{n-1} - T_n\right) \\
&\quad - \big\{1 - \exp(-\beta w(2))\big\}\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\right) + 2\epsilon \\
&\le \big\{1 - \exp(-\beta w(1))\big\}\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1} U_{n-1} - 2T_n\right) \\
&\quad - \big\{\exp(-\beta w(1)) - \exp(-\beta w(2))\big\}\mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau(x_0)\wedge\tau_1} T_n\right) + 2\epsilon \\
&< 0.
\end{aligned}$$

QED

Suppose that the system reaches a state in $\{i_0\} \times \mathcal{K}$ for a compact subset $\mathcal{K} \subset [0,\infty) \times [0,\infty)$. If the workload can be successively reduced, then in a finite step the system reaches a state with a positive probability, which is independent of the starting point. The following theorem gives a condition that $(M_n, W_n)_{n\ge 0}$ is Harris recurrent. The same argument can be found in Asmussen [12] (Lemma XII.2.3) and Morozov [39].

Theorem 4.21 *If there exist $\eta, \epsilon, q > 0$, such that*

$$\mathbb{P}_{i_0}(U_0 < 2\eta - \epsilon, T_1 > \eta, \tau_1 = 1) > q, \tag{4.4}$$

then $(M_n, W_n)_{n\ge 0}$ is Harris ergodic.
If the more stronger condition

$$\mathbb{P}_{i_0}(U_0 < \eta - \epsilon, T_1 > \eta, \tau_1 = 1) > q \tag{4.5}$$

is satisfied, then $(M_n, W_n)_{n\ge 0}$ forms a classical-sense regenerative process.

Proof. By Proposition 4.20, there exists a bounded set $\mathcal{K} = \{(w_1, w_2) \in [0, \infty) \times [0, \infty) : w_k \le x_2, k = 1, 2\}$ for some positive real number x_2 such that $\{i_0\} \times \mathcal{K}$ forms a recurrent set of $(M_n, W_n)_{n \ge 0}$. Let $F_n := \{U_{n-1} < 2\eta - \epsilon, T_n > \eta\}$ and $r > 2x_2/\epsilon$. Note that each occurrence of F_n decreases residual work. Let $(M_0, W_0) = (i_0, w) \in \{i_0\} \times \mathcal{K}$. Then the customer n finds an empty server provided that $r - 4 \le n \le r$ and that $\cap_{n=0}^{r} F_n$ occurs. Hence the queue length at $r - 2$ is at most 1. This means that the customers $r-2, r-1$ enter service immediately and W_r is independent of initial value W_0. Furthermore, customers $r - N, \cdots, r-1$ enter service immediately. Thus with $\varphi(A) = \mathbb{P}_{i_0}[W_2 \in A | F_1 \cap F_2]$, we have

$$\mathbb{P}_{(i_0,w)}(W_r \in A) \ge q^r \varphi(A), \quad w \in \mathcal{K}.$$

To show the positivity, note that $\{i_0\} \times \mathcal{K}$ is a regeneration set of $(M_n, W_n)_{n \ge 0}$. Denoting by $\hat{\tau}(\{i_0\} \times \mathcal{K})$ the first return time of $(M_n, W_n)_{n \ge 0}$ to $\{i_0\} \times \mathcal{K}$, we have

$$\mathbb{E}_{i_0} \hat{\tau}(\{i_0\} \times \mathcal{K}) = C \cdot \mathbb{E}_{i_0} \tau_{\mathcal{K}} \cdot \mathbb{E}_{i_0} \tau_1 < \infty.$$

The aperiodicity follows, since (4.5) holds true for all sufficiently large r.
The second assertion is obvious, since then the customer n finds an empty system provided that $r - 2 \le n \le r$ and that $\cap_{n=0}^{r} F_n$ occurs. In this case, we have

$$\mathbb{P}_{(i_0,w)}(W_r \in A) \ge q^r \delta_{(i_0,0)}, \quad w \in \mathcal{K}.$$

QED

By Theorem 4.21 there exists a unique stationary distribution $\pi_{\mathbf{W}}$ of $(M_n, W_n)_{n \ge 0}$. Moreover, by Proposition 1.20 for any initial state $i \in E$

$$\lim_{n \to \infty} \|\mathbb{P}_i((M_n, W_n) \in \cdot\,) - \pi_{\mathbf{W}}\| = 0.$$

In the rest of this section, we assume that (4.5) holds true. Thus the random times $\sigma_n^*, n \ge 0$, defined as

$$\sigma_0^* := \inf\{k \ge 0 : W_k(2) = 0\} \quad \text{and} \quad \sigma_{n+1}^* := \inf\{k > \sigma_n^* : W_k(2) = 0\}, \quad n \ge 0$$

are a.s. finite. Obviously the chain $(M_{\sigma_n^*})_{n \ge 0}$ is positive Harris recurrent. Denote by ξ^* a unique stationary distribution of $(M_{\sigma_n^*})_{n \ge 0}$.

Corollary 4.22 *There exists a unique stationary distribution $\pi_{\mathbf{W}}$ for $(M_n, W_n)_{n \ge 0}$, which is given by*

$$\begin{aligned}
\pi_{\mathbf{W}}(\cdot) &= \frac{1}{\mathbb{E}_{\xi^*} \sigma_1^*} \mathbb{E}_{\xi^*} \left(\sum_{n=0}^{\sigma_1^*-1} \mathbf{1}((M_n, W_n) \in \cdot\,) \right) \\
&= \sum_{i \in E} \mathbb{E}_i \left(\sum_{n=0}^{\sigma_1^*-1} \mathbf{1}((M_n, W_n) \in \cdot\,) \right) \pi_{\mathbf{W}}(\{(i, 0, 0)\}).
\end{aligned}$$

Let further $\alpha > 1$ *and suppose that*

$$\mathbb{E}_{i_0}\,\tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_{i_0}\left(\sum_{n=1}^{\tau_1} U_{n-1}\right)^\alpha < \infty.$$

If for an initial state $j_0 \in E$

$$\mathbb{E}_{j_0}\,\tau_1^\alpha < \infty \quad \textit{and} \quad \mathbb{E}_{j_0}\left(\sum_{n=1}^{\tau_1} U_{n-1}\right)^\alpha < \infty,$$

then for some constant c

$$\|\mathbb{P}_{j_0}\left((M_n, W_n) \in \cdot\,\right) - \pi_{\mathbf{W}}\| \le c n^{1-\alpha}.$$

Proof. By assumption

$$\mathbb{E}_{(i_0,w)}\big(V_\beta(W_{\tau_1}) - V_\beta(w)\big)^\alpha \le \mathbb{E}_{i_0}\left[V_\beta\Big(\sum_{n=1}^{\tau_1} U_{n-1} + w\Big) - V_\beta(w)\right]^\alpha < \infty$$

for any $w \ge 0$. Thus, as in Theorem 3.3, one can show that there exists a sequence of regeneration epochs $(\hat\tau_n)_{n\ge 0}$ such that $\mathbb{E}_{i_0}\,\hat\tau_1^\alpha < \infty$ and $\mathbb{E}_{j_0}\,\hat\tau_1^\alpha < \infty$. Thus the assertion follows from Proposition 1.21. QED

Remark 4.23 Theorem 4.21 is obtained in Morozov [39] for a $R/GI/N$ queue, in which interarrival times form a regenerative process.

The corresponding assertions for the continuous-time workload process can be obtained in the same manner and are therefore omitted.

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