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Generating derived categories of groups in Kropholler's hierarchy

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Abstract. We provide some proofs of generation of derived bounded and unbounded categories of chain complexes of groups in Kropholler's hierarchy in terms of the classes of modules induced up from subgroups that are at a lower level in the hierarchy compared to the big group. We formulate and use some generation results from the module category for this. The treatment is fairly straight-forward. We use these results to show that stable module categories for a large class of infinite groups, as defined by Mazza and Symonds in [14], are well-generated which is a generalization of the analogous result for finite groups whose stable module categories are compactly generated.

1. Introduction

The groups that this article will be mostly dealing with come from a hierarchy of groups that was first introduced by Peter Kropholler in the nineties in [12]. If we start with the class of all finite groups as our base class, we get an infinite family of infinite groups that satisfy many fascinating properties like, for example, admitting a finite-dimensional proper classifying space as long as they are of type FP_∞ (this highly nontrivial result was proved in [13]).

In this article, we will be looking at the derived unbounded, bounded above, bounded below, and bounded categories of chain complexes of modules of these groups over any commutative ring and prove some important generation properties pertaining to those derived categories in Section 4 with classes of chain complexes associated to modules induced up from subgroups belonging to a lower level of Kropholler's hierarchy. We shall then, in Section 6, comment on how those results can be handy in deriving some properties about the stable module categories of groups belonging to Kropholler's hierarchy whenever those stable module categories can be defined in the way of Mazza and Symonds [14].

To derive the generation results for all these derived and stable categories in Section 4 and Section 6, we have to first derive a number of useful analogous generation results in the module category, and we do that in Section 3, and before that, in Section 2, we set up a little bit of the abstract framework of generation in the context of module categories and prove some results which we then use in Section 4. Please note that, in [4], we elaborately study the abstract framework of generation that we introduce in Section 2 here. Regarding this abstract theory, in this article, we only develop those concepts and those results that we make use of to derive generation results in the derived and stable categories in the later sections.

We begin by providing a definition of Kropholler's hierarchy.

Definition 1.1 (see [12])**.** Let $\mathscr{X}$ be a class of groups. We define $H_0\mathscr{X} := \mathscr{X}$, and for any successor ordinal α, a group G is said to be in $H_\alpha\mathscr{X}$ if and only if there exists a finite-dimensional contractible CW-complex on which G acts by permuting the cells with cell stabilizers in $H_{\alpha-1}\mathscr{X}$. If α is a limit ordinal, then we define $H_\alpha\mathscr{X}$ as $\bigcup_{\beta<\alpha} H_\beta\mathscr{X}$. Further, G is said to be in $H\mathscr{X}$ if and only if G is in $H_\alpha\mathscr{X}$ for some ordinal α (note that α need not be a limit ordinal here).

The following result is easy to see from the above definition.

Lemma 1.2 ([12])**.** *Let $\mathscr{X}$ be a class of groups. Then $H_\alpha\mathscr{X} \subseteq H_\beta\mathscr{X}$, where α and β are any two ordinals such that $\alpha < \beta$.*

It is important to note that, in the above definition, if we start with the class of all finite groups, denoted $\mathscr{F}$, then the classes $H_0\mathscr{F}, H_1\mathscr{F}, \dots, H_n\mathscr{F}$, and so on are all distinct, by which we mean, for each positive integer n, there exists a group that is in $H_n\mathscr{F}$ but not in $H_{n-1}\mathscr{F}$. This is quite nontrivial and is due to a result by Januszkiewicz, Kropholler and Leary [11]. It is worth keeping in mind that the class $H\mathscr{F}$ is a much larger class than $\mathscr{F}$—just $H_1\mathscr{F}$ contains all groups of finite virtual cohomological dimension, all groups of finite Bredon cohomological dimension, etc. We give the following solid example of groups lying in one class of the hierarchy and not in the one immediately below up to $H_3\mathscr{F}$.

Theorem 1.3 ([8, Thm. 7.10])**.** *Let ω denote the first infinite ordinal, and let $\mathscr{F}$ denote the class of all finite groups.*
(a) *The free abelian group of rank t, where $1 \leq t < \aleph_0$, is in $H_1\mathscr{F}$ but not in $H_0\mathscr{F}$.*
(b) *The free abelian group of rank t, where $\aleph_0 \leq t < \aleph_\omega$, is in $H_2\mathscr{F}$ but not in $H_1\mathscr{F}$.*
(c) *The free abelian group of rank $\aleph_\omega$ is in $H_3\mathscr{F}$ but not in $H_2\mathscr{F}$.*

2. Some general results on generation in the module category

In this section, we define a notion of generation for modules that can be said to have been inspired from some notions of generation for triangulated categories which have been looked into by Raphael Rouquier [20], Jeremy Rickard [19], and others [1]. We prove a number of useful general results regarding generation of modules using our definition of generation, look into the significance of a module being generated in a finite number of steps by a class (in our definition), and investigate how that is related to the same module admitting finite resolutions by modules of that class. Some of the results discussed and proved in this discussion are interesting in their own right, and we will establish some new generation properties of modules of groups that lie in Kropholler's hierarchy. Those generation properties will be useful for us to prove our generation results in the derived categories later.

Definition 2.1. Let R be a ring, and let $\mathscr{T}$ be a class of R-modules. We define generation of modules from $\mathscr{T}$ inductively—we say an R-module is generated from $\mathscr{T}$ in n steps if and only if there exists a short exact sequence $0 \to M_1 \to M_2 \to M \to 0$, where M_1, M_2 are generated from $\mathscr{T}$ in a_1, a_2 steps respectively and $a_1 + a_2 \leq n-1$; to begin the induction, we say an R-module M is generated in 0 steps from $\mathscr{T}$ if and only if $M \in \mathscr{T}$. So if we are given a short exact sequence $0 \to M_1 \to M_2 \to M \to 0$ and we know that each M_i is generated from $\mathscr{T}$ in a_i steps, then M is generated from $\mathscr{T}$ in $a_1 + a_2 + 1$ steps.

We shall denote the class of all modules that can be generated in n steps from $\mathscr{T}$ by $\langle \mathscr{T} \rangle_n$ and the class of all modules that can be generated in finitely many steps from $\mathscr{T}$ by $\langle \mathscr{T} \rangle$.

For any R-module M, we define the $\mathscr{T}$-generation number, denoted $\alpha_{\mathscr{T}}(M)$, to be $\min\{n \in \mathbb{Z} \mid M \in \langle \mathscr{T} \rangle_n\}$. If $M \notin \langle \mathscr{T} \rangle_n$ for any finite n, we define $\alpha_{\mathscr{T}}(M)$ to be infinite.

For the rest of this section, we fix an arbitrary ring R. Note that saying an R-module M is generated in n steps from a class of R-modules $\mathscr{T}$ implies that M can be generated in m steps from $\mathscr{T}$ as well, where m is any integer greater than n. From this, the following lemma follows.

Lemma 2.2. *For any class of R-modules $\mathscr{T}$, $\mathscr{T} = \langle \mathscr{T} \rangle_0 \subseteq \langle \mathscr{T} \rangle_1 \subseteq \langle \mathscr{T} \rangle_2 \subseteq \cdots$.*

Remark 2.3. It is noteworthy that, in Definition 2.1 of generation of modules from a class of modules, we are not putting the new module in the middle of the short exact sequence that we are using to generate it. The main reason for this is that if we put the new module on the right as we are doing in Definition 2.1, then under suitable conditions, for a class of modules, our generation number of any module coincides with the dimension of that module over that class, i.e. the length of the shortest resolution of modules coming from that class admitted by that module. We prove this result in Lemma 2.8.

Also, putting the new module in the middle can give a different class. We prove this result below in Lemma 2.4. So putting the new module in the middle and putting it on the right are not equivalent for all classes of modules.

Lemma 2.4. *For any class of R-modules $\mathscr{T}$, let m-$\langle \mathscr{T} \rangle$ be the smallest class of R-modules containing $\mathscr{T}$ and satisfying the property that an R-module M is in m-$\langle \mathscr{T} \rangle$ if and only if there exists a short exact sequence $0 \to A \to M \to B \to 0$, where $A, B \in$ m-$\langle \mathscr{T} \rangle$. Then there exists a class of R-modules $\mathscr{U}$ such that m-$\langle \mathscr{U} \rangle \neq \langle \mathscr{U} \rangle$.*

Proof. Let $\mathscr{U}$ be the class of all simple R-modules. Then $\langle \mathscr{U} \rangle$ is the class of all simple modules and the zero modules. This is easy to see. Denote the class of all simple modules and the zero module by $\underline{\mathscr{U}}$. If $\alpha_{\mathscr{U}}(M) = 0$, then $M \in \underline{\mathscr{U}}$.

Assume if $\alpha_{\mathscr{U}}(M) \le n$, then $M \in \underline{\mathscr{U}}$—this is our induction hypothesis. If we take a module M such that $\alpha_{\mathscr{U}}(M) = n+1$, then we have a short exact sequence $0 \to A \to B \to M \to 0$, where $\alpha_{\mathscr{U}}(A), \alpha_{\mathscr{U}}(B) \le n$, and hence, by the induction hypothesis, $A, B \in \underline{\mathscr{U}}$, so M is a quotient of the zero module or a simple module, and therefore $M \in \underline{\mathscr{U}}$. Thus, we have shown that $\langle \mathscr{U} \rangle \subseteq \underline{\mathscr{U}}$. Clearly, $\underline{\mathscr{U}} \subseteq \langle \mathscr{U} \rangle$, as any simple module is in $\mathscr{U}$ by definition, and since we have an exact sequence $0 \to S \to S \to 0 \to 0$ for any simple S, the zero module is in $\langle \mathscr{U} \rangle$. Thus, $\langle \mathscr{U} \rangle = \underline{\mathscr{U}}$. Now note that, for any two simple modules S_1 and S_2, $S_1 \oplus S_2$ is not simple, but $S_1 \oplus S_2 \in m\text{-}\langle \mathscr{U} \rangle$ as we have an exact sequence $0 \to S_1 \to S_1 \oplus S_2 \to S_2 \to 0$. So $m\text{-}\langle \mathscr{U} \rangle \ne \langle \mathscr{U} \rangle$. □

We can see from the definition of a module being generated in n steps from a given class of modules that the module that we are trying to generate is occurring rightmost in a short exact sequence. In the following lemma, we look at what can be said about the number of steps required to generate a module if the module occurs rightmost in an exact sequence consisting of more than three modules.

Lemma 2.5. *Let $\mathscr{T}$ be a class of R-modules. If there exists an exact sequence $0 \to M_n \to \cdots \to M_1 \to M \to 0$ for some $n > 1$, where each M_i is generated in a_i steps from $\mathscr{T}$, then M can be generated from $\mathscr{T}$ in $n - 1 + \sum_{i=1}^{n} a_i$ steps.*

Proof. We will provide a proof by induction on n. Note that, when $n = 2$, this result holds true by definition of the number of steps of generation. Now let us assume that, for all $n \le k$, if there exists an exact sequence

$$0 \to M_n \to \cdots \to M_1 \to M \to 0,$$

where each M_i is generated in a_i steps from $\mathscr{T}$, then M can be generated from $\mathscr{T}$ in $n - 1 + \sum_{i=1}^{n} a_i$ steps—this is our induction hypothesis.

Now let $n = k + 1$. If we have an exact sequence

$$0 \to M_{k+1} \to M_k \to \cdots \to M_1 \to M \to 0,$$

we can split it into two exact sequences:
(S1) $0 \to M_{k+1} \to M_k \to \operatorname{Im}(M_k \to M_{k-1}) \to 0$,
(S2) $0 \to \operatorname{Im}(M_k \to M_{k-1}) \to M_{k-1} \to \cdots \to M_1 \to M \to 0$.
Since M_{k+1} is generated in a_{k+1} steps and M_k is generated in a_k steps from $\mathscr{T}$, looking at (S1), we can say that $\operatorname{Im}(M_k \to M_{k-1})$ can be generated from $\mathscr{T}$ in $a_{k+1} + a_k + 1$ steps. And since $\operatorname{Im}(M_k \to M_{k-1})$, $M_{k-1}, \ldots, M_2, M_1$ can be generated from $\mathscr{T}$ in $a_{k+1} + a_k + 1, a_{k-1}, \ldots, a_2, a_1$ steps respectively; looking at (S2), we can say using the induction hypothesis that M can be generated from $\mathscr{T}$ in

$$\begin{aligned}(k-1) + (a_{k+1} + a_k + 1) + a_{k-1} + \cdots + a_2 + a_1 \\ = k + \sum_{i=1}^{k+1} a_i = ((k+1) - 1) + \sum_{i=1}^{k+1} a_i\end{aligned}$$

steps. This completes our induction. □

Definition 2.6. For any class of R-modules $\mathscr{T}$, we define the $\mathscr{T}$-dimension of an R-module M, denoted $\mathscr{T}\text{-dim}(M)$, to be

$$\begin{aligned}\min\{i \in \mathbb{Z} \mid \text{ there exists an exact sequence} \\ 0 \to T_i \to T_{i-1} \to \cdots \to T_0 \to M \to 0, \\ \text{where each } T_i \in \mathscr{T}\}.\end{aligned}$$

If, for an R-module M, no such exact sequence exists for any i, we say that $\mathscr{T}\text{-dim}(M)$ is infinite.

- $[\mathscr{T}]_n$ is the class of all R-modules M such that $\mathscr{T}\text{-dim}(M) \leq n$.
- $[\mathscr{T}]_\infty$ is the class of all R-modules M such that there exists an exact sequence (of possibly infinite length) $T_* \twoheadrightarrow M$, where each $T_i \in \mathscr{T}$.
- $[\mathscr{T}]$ is the class of all R-modules with finite $\mathscr{T}$-dimension.

The following result is obvious.

Lemma 2.7. *For any class of R-modules $\mathscr{T}$,*
(a) $[\mathscr{T}]_0 \subseteq [\mathscr{T}]_1 \subseteq [\mathscr{T}]_2 \subseteq \cdots$,
(b) $[\mathscr{T}]_n \subseteq \langle \mathscr{T} \rangle_n$ *for any* $n \in \mathbb{Z}_{\geq 0}$,
(c) $[\mathscr{T}] \subseteq \langle \mathscr{T} \rangle$,
(d) *if $\mathscr{U}$ is a class of R-modules such that $\mathscr{T} \subseteq \mathscr{U}$, $\mathscr{U}\text{-dim}(M) \leq \mathscr{T}\text{-dim}(M)$ for all R-modules M.*

Proof. (a) follows from the definition of $[\mathscr{T}]_n$. (b) and (c) follow directly from Lemma 2.5. To prove (d), we can start with assuming that $\mathscr{T}\text{-dim}(M) = n < \infty$ and then note that if $0 \to T_n \to \cdots \to T_0 \to M \to 0$ is an exact sequence, where all the T_i's are in $\mathscr{T}$, then they are also in $\mathscr{U}$ as $\mathscr{T} \subseteq \mathscr{U}$, and therefore $\mathscr{U}\text{-dim}(M) \leq n$. □

The following result is important, in light of Remark 2.3, to show why putting the new module on the right of the generating exact sequence is useful, and it also shows how the generation number of a module over a given class can be a very useful invariant when the class satisfies some reasonable conditions.

Lemma 2.8. *Let $\mathscr{T}$ be a class of R-modules. If, for any short exact sequence of R-modules* $0 \to M_2 \to M_1 \to M \to 0$,

$$\mathscr{T}\text{-dim}(M) \leq 1 + \max\{\mathscr{T}\text{-dim}(M_1), \mathscr{T}\text{-dim}(M_2)\},$$

then $\mathscr{T}\text{-dim}(M) = \alpha_{\mathscr{T}}(M)$ for all R-modules M.

Proof. For any R-module M, it is clear from the definition of $\mathscr{T}\text{-dim}(M)$ and Lemma 2.5 that $\alpha_{\mathscr{T}}(M) \leq \mathscr{T}\text{-dim}(M)$. Assuming the conditions in the hypothesis of the statement of the lemma hold, we will prove by induction on $\alpha_{\mathscr{T}}(M)$ that $\mathscr{T}\text{-dim}(M) \leq \alpha_{\mathscr{T}}(M)$. If $\alpha_{\mathscr{T}}(M) = 0$, then $M \in \mathscr{T}$, and therefore $\mathscr{T}\text{-dim}(M) = 0$. Assume that, for all modules M such that $\alpha_{\mathscr{T}}(M) \leq n$, if the hypothesis of our lemma is satisfied, then $\mathscr{T}\text{-dim}(M) \leq \alpha_{\mathscr{T}}(M)$—this is our induction hypothesis. Now let $\alpha_{\mathscr{T}}(M) = n+1$, then by definition, we have an exact sequence $0 \to M_2 \to M_1 \to M \to 0$, where $\alpha_{\mathscr{T}}(M_1), \alpha_{\mathscr{T}}(M_2) \leq n$. By the induction hypothesis, it follows that $\mathscr{T}$-dim(M_1), $\mathscr{T}$-dim$(M_2) \leq n$, and therefore, from the hypothesis of the statement of the lemma, it follows that $\mathscr{T}$-dim$(M) \leq n+1 = \alpha_{\mathscr{T}}(M)$. This ends our induction. □

Remark 2.9. There are many examples of classes of R-modules that satisfy the hypothesis of Lemma 2.8. Classes of all projective R-modules, all Gorenstein projective R-modules, etc. satisfy it. In this remark, we provide a short proof for the case of projectives: if we take a short exact sequence $0 \to A \to B \to C \to 0$ of R-modules, where $\text{proj.dim}_R(A), \text{proj.dim}_R(B) \leq n$, then, in the long exact Ext-sequence

$$\cdots \to \text{Ext}_R^{n+1}(A,M) \to \text{Ext}_R^{n+2}(C,M) \\ \to \text{Ext}_R^{n+2}(B,M) \to \text{Ext}_R^{n+2}(A,M) \to \cdots$$

and since $\text{Ext}_R^{n+1}(A,M) = \text{Ext}_R^{n+2}(B,M) = 0$, it follows that $\text{Ext}_R^{n+2}(C,M) = 0$ for any R-module M, and therefore $\text{proj.dim}_R(C) \leq n+1$. We arrive at the same conclusion for the Gorenstein projective dimension of the rightmost module in a short exact sequence by noting that, for any R-module M, the Gorenstein projective dimension of M, denoted $\text{Gpd}_R(M)$, satisfies $\text{Gpd}_R(M) \leq n$ if and only if $\text{Ext}_R^k(M,L) = 0$ for all $k > n$ and all projective modules L (see [10, Thm. 2.20]).

In the next lemma, we see that, when we have a class where each module is generated in finitely many steps from another class, then every module generated in finitely many steps from the first class is also generated in finitely many steps from the second class, and a similar result is true when we have a bound on the number of steps required to generate every module in a similar situation.

Lemma 2.10. *Let $\mathscr{T}$ and $\mathscr{U}$ be two classes of R-modules.*

(a) *If $\mathscr{T} \subseteq \langle \mathscr{U} \rangle$, then $\langle \mathscr{T} \rangle \subseteq \langle \mathscr{U} \rangle$. In other words, any module that can be generated in finitely many steps from $\mathscr{T}$ can also be generated in finitely many steps from $\mathscr{U}$ if every module in $\mathscr{T}$ is generated in finitely many steps from $\mathscr{U}$.*

(b) *If $\mathscr{T} \subseteq \langle \mathscr{U} \rangle_m$, then $\langle \mathscr{T} \rangle_n \subseteq \langle \mathscr{U} \rangle_{mn+m+n}$.*

Proof. (a) We proceed by strong induction on the $\mathscr{T}$-generation number of modules. First, we check our base case. Note that our lemma holds true for modules in $\mathscr{T}$, i.e. for all modules whose $\mathscr{T}$-generation number is zero.

Now let us assume that all modules of $\mathscr{T}$-generation number $\leq n$ is in $\mathscr{U}$; this is our induction hypothesis. If $\alpha_{\mathscr{T}}(M) = n+1$, by definition, M admits a generation sequence $0 \to D_2 \to D_1 \to M \to 0$, where $D_1, D_2 \in \langle \mathscr{T} \rangle_n$. This means $\alpha_{\mathscr{T}}(D_1), \alpha_{\mathscr{T}}(D_2) \leq n$. It follows from our induction hypothesis that $D_1, D_2 \in \langle \mathscr{U} \rangle$. That means D_1 and D_2 can be generated from $\mathscr{U}$ in d_1 and d_2 steps respectively for some nonnegative integers d_1, d_2, and from that, it follows that M can be generated from $\mathscr{U}$ in $d_1 + d_2 + 1$ steps. Thus, $M \in \langle \mathscr{U} \rangle$, and that ends our induction.

(b) We proceed by strong induction on n. If $M \in \langle \mathscr{T} \rangle_0$,

$$M \in \mathscr{T} \subseteq \langle \mathscr{U} \rangle_m = \langle \mathscr{U} \rangle_{m.0+m+0}.$$

Let us assume that the result is true when $n \leq k$. If $M \in \langle \mathscr{T} \rangle_{k+1}$, there exists a short exact sequence $0 \to C_2 \to C_1 \to M \to 0$, where C_1, C_2 are generated from $\mathscr{T}$ in a_1, a_2 steps respectively, where $a_1 + a_2 \leq k$, so $a_1, a_2 \leq k$. By the induction hypothesis, C_i is generated from $\mathscr{U}$ in $ma_i + m + a_i$ steps for $i = 1, 2$. So M is generated from $\mathscr{U}$ in

$$\begin{aligned}(ma_1 + a_1 + m) + (ma_2 + a_2 + m) + 1 &= m(a_1 + a_2 + 2) + a_1 + a_2 + 1 \\ &\leq m(k+2) + k + 1 \\ &= m(k+1) + m + (k+1)\end{aligned}$$

steps. This ends our induction. □

The following result will come handy when we will handle generation of modules of groups in Kropholler's hierarchy. We are not including a proof because it is quite trivial.

Lemma 2.11. *For any class of R-modules, $\mathscr{T}$ and any nonnegative integer n, $[\mathscr{T}]_n$ is closed under arbitrary direct sums if $\mathscr{T}$ is closed under arbitrary direct sums.*

3. Generation of modules in Kropholler's hierarchy

For our dealings in this section, we fix an arbitrary commutative ring R.

Notation. For any class of R-modules $\mathscr{T}$, we denote by $\mathscr{T}^{\oplus}$ the smallest class containing $\mathscr{T}$ that is closed under arbitrary direct sums.

Lemma 3.1. *Let G be a group that acts cellularly on a G-CW-complex X with stabilizers in a class $\mathscr{L}$. Let $I(G, \mathscr{L})$ be a class of RG-modules consisting of all modules of the form $\operatorname{Ind}_H^G M$, where H is some subgroup of G that is in $\mathscr{L}$ and M is some RH-module. Then the number of steps needed to generate trivial module from $I(G, \mathscr{L})^{\oplus}$ is bounded by the dimension of X.*

Proof. We can assume that the maximal dimension of cells in X is finite because if it is not finite we have nothing to prove. Let this number be n. The augmented cell complex is of the form $0 \to A_n \to \cdots \to A_1 \to A_0 \to R \to 0$, where each A_i is an RG-permutation module that we get from the action of G as a group of permutations of the i-dimensional cells of X. Each A_i is a direct sum of the trivial module induced up to G from subgroups of G that are in $\mathscr{L}$. By Lemma 2.5, the trivial module can be generated from $I(G, \mathscr{L})^{\oplus}$ in $(n+1) - 1 + 0 = n$ steps. □

Definition 3.2. For any group G and a class of groups $\mathscr{X}$, we define $\Lambda_n(G, \mathscr{X})$ to be $\{\operatorname{Ind}_H^G M \mid M$ is some RH-module and H is some $H_n\mathscr{X}$-subgroup of $G\}$.

The following lemma will prove crucial in proving some important generation results later in Section 4. Before stating it, we recall that the successor of an ordinal number α is the smallest ordinal number greater than α. An ordinal number that is a successor is called a successor ordinal. If α is a successor ordinal, we define $\alpha - 1$ to be the ordinal number β whose successor is α (in ordinal addition notation, $\alpha = \beta + 1$).

Lemma 3.3. *For any group G, a class of groups $\mathscr{X}$, and any successor ordinal α, $\Lambda_\alpha(G, \mathscr{X}) \subseteq [\Lambda_{\alpha-1}(G, \mathscr{X})^{\oplus}]$.*

Proof. Let H be a $H_\alpha\mathscr{X}$-subgroup of G. Then, by definition, there exists a finite-dimensional contractible complex T on which H acts with stabilizers in $H_{\alpha-1}\mathscr{X}$. Its cellular chain complex is of the form

$$0 \to A_t \to \cdots \to A_1 \to A_0 \to R \to 0,$$

where each A_i is a permutation module that we get from the action of H as a group of permutations of the i-dimensional cells of T.

Let X be an arbitrary RH-module. If we tensor the above complex by X, we get the complex

$$0 \to A_t \otimes X \to \cdots \to A_1 \otimes X \to A_0 \otimes X \to X \to 0.$$

Now if we induce all these modules up to G, we get the complex

$$0 \to \operatorname{Ind}_H^G(A_t \otimes X) \to \cdots \to \operatorname{Ind}_H^G(A_0 \otimes X) \to \operatorname{Ind}_H^G X \to 0.$$

A_i can be written as a direct sum of the trivial module induced up to H from subgroups of H that are of the form H_σ, where H_σ denotes the stabilizer of the cell σ (note that $H_\sigma \in H_{\alpha-1}\mathscr{X}$ for all σ), with σ running over the set of H-orbit representatives for the i-dimensional cells (we can denote this set by Δ). Thus, $\operatorname{Ind}_H^G(A_i \otimes X) = \bigoplus_{\sigma\in\Delta} \operatorname{Ind}_{H_\sigma}^G X \in \Lambda_{\alpha-1}(G, \mathscr{X})^{\oplus}$. Thus, $\operatorname{Ind}_H^G X \in [\Lambda_{\alpha-1}(G, \mathscr{X})^{\oplus}]$. □

The following result follows straight-forwardly from Lemma 3.3.

Corollary 3.4. *For any class of groups $\mathscr{X}$, any group G, and any positive integer n,*

$$\Lambda_n(G, \mathscr{X}) \subseteq [\Lambda_{n-1}(G, \mathscr{X})^{\oplus}] \subseteq \cdots \subseteq \underbrace{[[\ldots[}_{n \text{ times}}\Lambda_0(G, \mathscr{X})\underbrace{^{\oplus}]^{\oplus}]\ldots^{\oplus}]}_{n \text{ times}}.$$

Theorem 3.5. *Let $\mathscr{X}$ be a class of groups and G a group. For any $n \geq 1$ and for any group J, let*

$$d_{n,\mathscr{X}}(J) := \inf\{\dim(X) \mid X \textit{ is a finite-dimensional contractible } \mathrm{CW}\textit{-complex on which } J \textit{ acts with stabilizers in } H_{n-1}\mathscr{X}\},$$

and let $t_{n,\mathscr{X}}(G) := \sup\{d_{n,\mathscr{X}}(H) \mid H \leq G, H \in H_n\mathscr{X}\}$; we write t_n for $t_{n,\mathscr{X}}(G)$ when there is no ambiguity over $\mathscr{X}$ and G. Then, for any fixed n, we have $\Lambda_n(G, \mathscr{X})^{\oplus} \subseteq \langle\Lambda_{n-m}(G, \mathscr{X})^{\oplus}\rangle_{\prod_{i=n-m+1}^{n}(1+t_i)-1}$ for any m such that $1 \leq m \leq n$.

Proof. We shall proceed by induction on m.

Let $\mathrm{Ind}_H^G(M) \in \Lambda_n(G, \mathscr{X})$, where H is some $H_n\mathscr{X}$-subgroup of G and M is some RH-module. From the proof of Lemma 3.3, $\mathrm{Ind}_H^G(M) \in [\Lambda_{n-1}(G, \mathscr{X})^{\oplus}]_{t_n}$, and by Lemma 2.11, any arbitrary direct sum of modules in $\Lambda_n(G, \mathscr{X})$ is in $[\Lambda_{n-1}(G, \mathscr{X})^{\oplus}]_{t_n}$. Thus, $\Lambda_n(G, \mathscr{X})^{\oplus} \subseteq [\Lambda_{n-1}(G, \mathscr{X})^{\oplus}]_{t_n} \subseteq \langle \Lambda_{n-1}(G, \mathscr{X})^{\oplus} \rangle_{t_n}$ (the last inclusion is by Lemma 2.7 (b)). This proves the theorem for $m = 1$.

Similarly, we get $\Lambda_a(G, \mathscr{X})^{\oplus} \subseteq [\Lambda_{a-1}(G, \mathscr{X})^{\oplus}]_{t_a} \subseteq \langle \Lambda_{a-1}(G, \mathscr{X})^{\oplus} \rangle_{t_a}$ for any a between 0 and n (this follows from the definition of t_a and Lemma 2.7 (b)).

We assume the statement of the theorem to be true for $m = d$. Now let $m = d + 1$. We have the following:
(a) $\Lambda_{n-d}(G, \mathscr{X})^{\oplus} \subseteq \langle \Lambda_{n-d-1}(G, \mathscr{X})^{\oplus} \rangle_{t_{n-d}}$;
(b) $\Lambda_n(G, \mathscr{X})^{\oplus} \subseteq \langle \Lambda_{n-d}(G, \mathscr{X})^{\oplus} \rangle_{\prod_{i=n-d+1}^{n}(1+t_i)-1}$ (induction hypothesis).

By Lemma 2.10, therefore, every module in $\Lambda_n(G, \mathscr{X})^{\oplus}$ is generated from $\Lambda_{n-d-1}(G, \mathscr{X})^{\oplus}$ in

$$\begin{aligned} t_{n-d} \prod_{i=n-d+1}^{n} (1 + t_i) - t_{n-d} + t_{n-d} + \Bigl(\prod_{i=n-d+1}^{n} (1 + t_i) - 1 \Bigr) \\ = (1 + t_{n-d}) \prod_{i=n-d+1}^{n} (1 + t_i) - 1 = \prod_{i=n-(d+1)+1}^{n} (1 + t_i) - 1 \end{aligned}$$

steps. This ends our induction. □

Corollary 3.6. *Let $\mathscr{X}$ be a class of groups and G an $H_n\mathscr{X}$ group. Using the notation of Theorem 3.5, every RG-module is generated from $\Lambda_0(G, \mathscr{X})^{\oplus}$ in $\prod_{i=1}^{n}(1 + t_i) - 1$ steps.*

Proof. We can assume that all the t_i's are finite because if any of them are not, we have nothing to prove. The corollary then follows by taking $m = n$ in the statement of Theorem 3.5 and by noting that, as $G \in H_n\mathscr{X}$, $\Lambda_n(G, \mathscr{X})$ is the class of all RG-modules. □

We end this section with the following remark.

Remark 3.7. For any class of RG-modules $\mathscr{U}$, define the finitistic $\mathscr{U}$-dimension of RG, denoted $\mathscr{F}\mathscr{U}$-dim(RG), to be

$$\sup\{\mathscr{U}\text{-dim}(M) \mid M \in \mathrm{Mod}(RG),\ \mathscr{U}\text{-dim}(M) < \infty\}.$$

It is obvious that if $\mathscr{T} \subseteq \mathscr{U}$, then $\mathscr{F}\mathscr{T}$-dim$(RG) \leq \mathscr{F}\mathscr{U}$-dim$(RG)$.

Note that one can just replace $t_{n,\mathscr{X}}(G)$ by the finitistic $\Lambda_{n-1}(G, \mathscr{X})^{\oplus}$-dimension of RG in the statement of Theorem 3.5. This way, it might be more algebraic for visualization purposes. Also, from the inequality mentioned in the previous paragraph, it follows that we can replace all the t_i's by t. So Corollary 3.6 can be restated as: for $G \in H_n\mathscr{X}$, every module can be generated from $\Lambda_m(G, \mathscr{X})^{\oplus}$, for any m satisfying $1 \leq m \leq n$, in $(1 + t)^{n-m} - 1$ steps. This shows very clearly that the number of levels we go down in the hierarchy to generate our class of modules gets reflected in the degree of the polynomial in t that we get as the number of steps. This number of steps need not be optimal, but this is very much in line with the spirit of Kropholler's hierarchy because, taking $\mathscr{X}$ to be the class of all finite groups for example, we see that we are generating all modules from the class of modules induced up from finite subgroups closed under direct sums.

4. Generation in the derived categories

For the rest of this article, for any ring R, we will use the following notations:

- Mod-R is the standard module category of R-modules,
- $\mathcal{D}^{b,+}(\text{Mod-}R)$ for the derived category of bounded above chain complexes of R-modules,
- $\mathcal{D}^{b,-}(\text{Mod-}R)$ for the derived category of bounded below chain complexes of R-modules,
- $\mathcal{D}^{b}(\text{Mod-}R)$ for the derived category of bounded chain complexes of R-modules,
- $\mathcal{D}(\text{Mod-}R)$ for the derived category of unbounded chain complexes of R-modules.

For any class of modules $\mathscr{C}$, when we write $\mathcal{D}^*(\mathscr{C})$, as we do in the statements of Theorem 4.3, we mean a class of all chain complexes in the relevant derived category where the modules in the chain complexes are from $\mathscr{C}$.

We begin straightaway with two very useful lemmas which are both standard knowledge (for Lemma 4.1, one can consult [5] for details and background).

Lemma 4.1. *Let R be a ring. Let $\mathcal{D}(\text{Mod-}R)$ be the derived unbounded category of chain complexes of R-modules, and let $\mathscr{U}$ be a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$. If $\mathscr{U}$ is closed under coproducts, then it is closed under direct limits of chain complexes, and if $\mathscr{U}$ is closed under products, then it is closed under inverse limits of chain complexes.*

Proof. Let $\mathscr{U}$ be closed under coproducts, and let $\{S_i\}_{i\geq 0}$ be a collection of chain complexes in $\mathscr{U}$ where we have chain complexes $f_i : S_i \to S_{i+1}$. The direct limit $\varinjlim_{i\geq 0} S_i$, from the definition of homotopy colimits, arises as a cokernel in the following short exact sequence:

$$0 \to \bigoplus_{i\geq 0} S_i \xrightarrow{\bigoplus_{i\geq 0}(id_{S_i} - f_i)} \bigoplus_{i\geq 0} S_i \to \varinjlim_{i\geq 0} S_i \to 0.$$

Here, the first two terms are in $\mathscr{U}$ as $\mathscr{U}$ is closed under coproducts, and therefore the third term is in $\mathscr{S}$ as well since $\mathscr{U}$ is a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$.

Similarly, now if we let $\mathscr{U}$ be closed under products where we have maps $f_i : S_i \to S_{i-1}$, then the inverse limit $\varprojlim_{i\geq 0} S_i$ arises as a kernel in the short exact sequence by the definition of homotopy limits

$$0 \to \varprojlim_{i\geq 0} S_i \to \prod_{i\geq 0} S_i \xrightarrow{\prod_{i\geq 1}(id_{S_i} - f_i)} \prod_{i\geq 0} S_i \to 0.$$

Here, the last two terms are in $\mathscr{U}$ as $\mathscr{U}$ is closed under products, and therefore the third term is in $\mathscr{U}$ since $\mathscr{U}$ is a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$. □

The following lemma is standard knowledge too. One can look up the proof of [18, Prop. 2.1 (f)] for an idea of the proof in the derived unbounded case; that same proof works for us.

Lemma 4.2. *Let R be a ring, and let $\mathscr{T}$ be a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$, $\mathcal{D}^{b,+}(\text{Mod-}R)$, $\mathcal{D}^{b,-}(\text{Mod-}R)$, or $\mathcal{D}^{b}(\text{Mod-}R)$. Then any chain complex X_* of the form $0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to 0$ is in $\mathscr{T}$ if each X_i, when considered as a chain complex concentrated in degree zero, is in $\mathscr{T}$.*

Proof. Let us assume that we are working in $\mathcal{D}(\text{Mod-}R)$. We will prove this by induction on the length of X_*. Of course, if X_* is of length 1, then it is in $\mathscr{T}$ by the hypothesis and since triangulated subcategories are closed under shifts. We now assume that if X_* is of length $\leq n$, then it is in $\mathscr{T}$—this is our induction hypothesis. Let $X_* : 0 \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0$ be a bounded complex of length $n+1$, where each X_i is in $\mathscr{T}$. We can fit this into a short exact sequence of bounded complexes as shown below.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \xrightarrow{0} & X_n & \xrightarrow{\mathrm{Id}} & X_n & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \xrightarrow{0} & X_2 & \xrightarrow{\mathrm{Id}} & X_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \xrightarrow{0} & X_1 & \xrightarrow{\mathrm{Id}} & X_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X_0 & \xrightarrow{\mathrm{Id}} & X_0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Here, the first chain complex is in $\mathscr{T}$ by the hypothesis of our lemma, and the third chain complex is in $\mathscr{T}$ by the induction hypothesis as each X_i is in S and it is a bounded complex of length $n-1$. So, since $\mathscr{T}$ is a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$, X_* is in $\mathscr{T}$, and we are done.

Note that the exact same proof works when $\mathscr{T}$ is a triangulated subcategory of $\mathcal{D}^{b,+}(\text{Mod-}R)$ or $\mathcal{D}^{b,+}(\text{Mod-}R)$ or $\mathcal{D}^{b}(\text{Mod-}R)$. ☐

We are now in a position to prove the following theorem about generation of the derived bounded above derived bounded and derived unbounded categories of chain complexes of modules with respect to classes of modules induced up from subgroups in Kropholler's hierarchy. In many of the upcoming statements, we will come across classes of modules, in most cases considered as classes of chain complexes concentrated in degree zero, with the superscript "$^{\oplus}$", which means closed under direct sums as explained earlier, and in some cases with the superscript "$^{\oplus,\Pi}$", which means we are taking the direct-sum closed class and closing it under products. Throughout this paper, for chain complexes, whenever we use the phrase "product", we mean "direct product of chain complexes".

Note that, although the statements of our next two results, Theorem 4.3 and Theorem 4.5, might seem a little technical, the proofs are actually quite straight-forward and easy to follow. The only reason why length-wise some of the proofs are quite long is we have chosen to elaborate in detail what limits we are using in what case and why.

Theorem 4.3. *Let G be a group, and let R be a commutative ring. We fix a class of groups, $\mathscr{X}$. For any triangulated category $\mathscr{T}$ and a class of objects in it denoted $\mathscr{U}$, we shall denote the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ by $\Delta_{\mathscr{T}}\mathscr{U}$.*

(a) *Let $\mathscr{T} = \mathcal{D}^{b,-}(\text{Mod-}RG)$. Then, for any $n \in \mathbb{N}$,*

$$\cdots = \Delta_{\mathscr{T}}\mathcal{D}^{b,-}(\Lambda_n(G,\mathscr{X})^{\oplus}) = \Delta_{\mathscr{T}}\mathcal{D}^{b,-}(\Lambda_0(G,\mathscr{X})^{\oplus}).$$

(b) *Let $\mathscr{T} = \mathcal{D}^{b,+}(\text{Mod-}RG)$. If $t_{n,\mathscr{X}}(G)$ or $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG)$ is finite for some $n \in \mathbb{N}$, then*

$$\begin{aligned}\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_n(G,\mathscr{X})^{\oplus,\Pi}) &= \Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})\\ &= \cdots = \Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_0(G,\mathscr{X})^{\oplus,\Pi}).\end{aligned}$$

(c) *Let $\mathscr{T} = \mathcal{D}(\text{Mod-}RG)$. In this case, for any class of objects $\mathscr{U}$ in $\mathscr{T}$, denote by $\mathscr{T}\text{-}\langle\mathscr{U}\rangle$ the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ closed under products and coproducts (direct sums). Then*

$$\begin{array}{ccccccc}\cdots = & \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_n(G,\mathscr{X}))\rangle & = & \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_{n-1}(G,\mathscr{X}))\rangle & = \cdots = & \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_0(G,\mathscr{X}))\rangle \\ & \| & & \| & & \| \\ \cdots = & \mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle & = & \mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle & = \cdots = & \mathscr{T}\text{-}\langle\Lambda_0(G,\mathscr{X})\rangle.\end{array}$$

(d) *Let $\mathscr{T} = \mathcal{D}^{b}(\text{Mod-}RG)$. Then, for any n,*

$$\Delta_{\mathscr{T}}\mathcal{D}^{b}(\Lambda_n(G,\mathscr{X})) \subseteq \Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}.$$

If additionally $t_{n,\mathscr{X}}(G)$ or alternatively $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG)$ is finite, then

$$\begin{array}{ccccc}\Delta_{\mathscr{T}}\mathcal{D}^{b}(\Lambda_n(G,\mathscr{X})^{\oplus}) & = & \Delta_{\mathscr{T}}\mathcal{D}^{b}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus}) & = \cdots = & \Delta_{\mathscr{T}}\mathcal{D}^{b}(\Lambda_0(G,\mathscr{X})^{\oplus}) \\ \| & & \| & & \| \\ \Delta_{\mathscr{T}}\Lambda_n(G,\mathscr{X})^{\oplus} & = & \Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus} & = \cdots = & \Delta_{\mathscr{T}}\Lambda_0(G,\mathscr{X})^{\oplus}.\end{array}$$

Proof. The techniques used in this proof for each of the subparts have some similarities.

(a) Note that, for any n, $\Lambda_{n-1}(G,\mathscr{X})^{\oplus} \subseteq \Lambda_n(G,\mathscr{X})^{\oplus}$. Thus,

$$\mathcal{D}^{b,-}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus}) \subseteq \mathcal{D}^{b,-}(\Lambda_n(G,\mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^{b,-}(\Lambda_n(G,\mathscr{X})^{\oplus}),$$

and therefore $\Delta_{\mathscr{T}}\mathcal{D}^{b,-}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^{b,-}(\Lambda_n(G,\mathscr{X})^{\oplus})$.

Take a bounded below chain complex $X_* = \cdots \to X_{m+k} \to \cdots \to X_{m+1} \to X_m \to 0$, where each X_i in $\Lambda_n(G, \mathscr{X})^{\oplus}$. We now look at the following truncations of X_*:

$$\begin{aligned} g_0(X_*) &= \cdots \to 0 \to 0 \to X_m \to 0 \\ g_1(X_*) &= \cdots \to 0 \to X_{m+1} \to X_m \to 0 \\ &\vdots \\ g_k(X_*) &= \cdots \to 0 \to X_{m+k} \to \cdots \to X_{m+1} \to X_m \to 0. \end{aligned}$$

In $g_i(X_*)$ as defined above, we have X_{m+j} in degree $m+j$ for all $j \in \{0, 1, \ldots, i\}$, and zero everywhere else. Note that each $X_i = \bigoplus_{\sigma \in \Sigma_i} X_{i,\sigma}$ for some indexing set Σ_i, where each $X_{i,\sigma} \in \Lambda_n(G, \mathscr{X})$. Now, as $\Lambda_n(G, \mathscr{X}) \subseteq [\Lambda_{n-1}(G, \mathscr{X})^{\oplus}]$ by Lemma 3.3, each $X_{i,\sigma}$ admits a finite length resolution $I^*_{i,\sigma} \twoheadrightarrow X_{i,\sigma}$ with modules from $\Lambda_{n-1}(G, \mathscr{X})^{\oplus}$. We thus get a resolution of $\bigoplus_{\sigma \in \Sigma_i} X_{i,\sigma}$ of possibly infinite length with modules from $\Lambda_{n-1}(G, \mathscr{X})^{\oplus} : \bigoplus_{\sigma \in \Sigma_i} I^*_{i,\sigma} \twoheadrightarrow \bigoplus_{\sigma \in \Sigma_i} X_{i,\sigma} = X_i$. Thus, a chain complex with X_i in degree zero and zero everywhere else is quasi-isomorphic to a bounded below complex

$$\cdots \to \bigoplus_{\sigma \in \Sigma_i} I^k_{i,\sigma} \to \cdots \to \bigoplus_{\sigma \in \Sigma_i} I^1_{i,\sigma} \to \bigoplus_{\sigma \in \Sigma_i} I^0_{i,\sigma} \to 0 \to 0 \to \cdots$$

with $\bigoplus_{\sigma \in \Sigma_i} I^k_{i,\sigma} \in \Lambda_{n-1}(G, \mathscr{X})^{\oplus}$ in degree k for $k \geq 0$ and zero in every other degree. So each X_i, considered as a complex concentrated in degree zero, is in $\Delta_{\mathscr{T}} \mathcal{D}^{b,-}(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$. Now note that each $g_i(X_*)$ is a bounded complex, where each module, when considered as a complex concentrated in degree zero, is in $\Delta_{\mathscr{T}} \mathcal{D}^{b,-}(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$. By Lemma 4.2, it follows that each $g_i(X_*)$ is in $\Delta_{\mathscr{T}} \mathcal{D}^{b,-}(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$. Note that $\bigoplus_{i \in \mathbb{N}} g_i(X_*)$ is in $\Delta_{\mathscr{T}} \mathcal{D}^{b,-}(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$ as each $g_i(X_*)$ is bounded below at degree m. We now apply the homotopy colimit construction artificially. We have a sequence of chain maps $g_0(X_*) \xrightarrow{\phi_0} g_1(X_*) \xrightarrow{\phi_1} g_2(X_*) \xrightarrow{\phi_2} \cdots$ between complexes, where $\phi_i : g_i(X_*) \to g_{i+1}(X_*)$ is given by the identity map at every degree between m and $m+i$ and the zero map at every other degree. In $\mathcal{D}(\text{Mod-}RG)$, the homotopy colimit of the $g_i(X_*)$'s is X_*, and it follows from the definition of homotopy colimits that, in $\mathcal{D}(\text{Mod-}RG)$, we have a short exact sequence (see Lemma 4.1)

$$0 \to \bigoplus_{i \in \mathbb{N}} g_i(X_*) \xrightarrow{\bigoplus_{i \geq 0}(id_{g_i(X_*)} - \phi_i)} \bigoplus_{i \in \mathbb{N}} g_i(X_*) \to X_* \to 0.$$

Now note that $\bigoplus_{i \in \mathbb{N}} g_i(X_*), X_* \in \mathcal{D}^{b,-}(\text{Mod-}RG)$ which is a triangulated subcategory of $\mathcal{D}(\text{Mod-}RG)$, and the above short exact sequence is a distinguished triangle in $\mathcal{D}^{b,-}(\text{Mod-}RG)$. We can see, in the short exact sequence above, the first two terms are in $\Delta_{\mathscr{T}} \mathcal{D}^{b,-}(\Lambda_{n-1}(G, \mathscr{X})^{\oplus})$ which is a triangulated subcategory of $\mathcal{D}^{b,-}(\text{Mod-}RG)$; it follows that

$$X_* \in \Delta_{\mathscr{T}} \mathcal{D}^{b,-}(\Lambda_{n-1}(G, \mathscr{X})^{\oplus}).$$

(b) Again, just like we saw in (a),

$$\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi}) \subseteq \Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n}(G,\mathscr{X})^{\oplus,\Pi}).$$

We start with an arbitrary bounded above chain complex

$$X_* := \cdots \to 0 \to X_m \to X_{m-1} \to X_{m-2} \to \cdots,$$

where each X_i is in $\Lambda_n(G,\mathscr{X})^{\oplus,\Pi}$, with X_{m-i} in degree $m-i$ for all $i \geq 0$ and zero in every other degree. Now look at the following truncations of X_*:

$$\begin{aligned} g_0(X_*) &= \cdots \to 0 \to X_m \to 0 \to 0 \to \cdots \\ g_1(X_*) &= \cdots \to 0 \to X_m \to X_{m-1} \to 0 \to \cdots \\ &\;\;\vdots \\ g_k(X_*) &= \cdots \to 0 \to X_m \to X_{m-1} \to \cdots \to X_{m-k} \to 0 \to 0 \to 0 \to \cdots. \end{aligned}$$

In $g_i(X_*)$, we have the module X_{m-j} in degree $m-j$ for all $j \in \{0,1,\ldots,i\}$ and zero in every other degree. The chain map $\phi_{k+1} : g_{k+1}(X_*) \to g_k(X_*)$ is given by the identity map in every degree between m and $m-k$ and the zero map in every other degree. As $t_n := t_{n,\mathscr{X}}(G) < \infty$ or $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$-dim$(RG) < \infty$, we denote either of these quantities by t, and we have

$$\Lambda_n(G,\mathscr{X}) \subseteq [\Lambda_{n-1}(G,\mathscr{X})^{\oplus}]_t,$$

and since, by Lemma 2.11, for any class of modules $\mathscr{T}$ that is closed under arbitrary direct sums, $[\mathscr{T}]_l$ is closed under arbitrary direct sums as well for any finite l, we have $\Lambda_n(G,\mathscr{X})^{\oplus} \subseteq [\Lambda_{n-1}(G,\mathscr{X})^{\oplus}]_t$. Now, for any i, $X_i = \prod_{j\in J} X_{i,j}$ for some indexing set J, where each $X_{i,j} \in \Lambda_n(G,\mathscr{X})^{\oplus}$, and we have a complex $\cdots \to 0 \to X_{i,j,t} \to X_{i,j,t-1} \to \cdots \to X_{i,j,0} \to 0 \to \cdots$ with $X_{i,j,k} \in \Lambda_{n-1}(G,\mathscr{X})^{\oplus}$ in degree k for all $k \in \{0,1,\ldots,t\}$ and zero in every other degree, quasi-isomorphic to the complex with $X_{i,j}$ in degree zero and zero in every other degree. Thus, each X_i, when considered as a complex concentrated in degree zero, is in $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})$—note that here the direct product of exact sequences, each of length t, is still an exact sequence of length t because we are in the module category. So, by Lemma 4.2, each

$$g_i(X_*) \in \Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi}).$$

Every $g_i(X_*)$ has just the zero module in every degree higher than m, so the bounded above chain complex $\prod_{i\in\mathbb{N}} g_i(X_*)$ is in $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})$. In $\mathcal{D}$(Mod-RG), the homotopy limit of the $g_i(X_*)$'s is X_*, and from the definition of homotopy limit, we get the short exact sequence (see Lemma 4.1)

$$0 \to X_* \to \prod_{i\in\mathbb{N}} g_i(X_*) \xrightarrow{\prod_{i\geq 1}(id_{g_i(X_*)}-\phi_i)} \prod_{i\in\mathbb{N}} g_i(X_*) \to 0.$$

All the terms here are in $\mathcal{D}^{b,+}(\text{Mod-}RG)$ which is a triangulated subcategory of $\mathcal{D}(\text{Mod-}RG)$, and the above short exact sequence is a distinguished triangle in $\mathcal{D}^{b,+}(\text{Mod-}RG)$. Note that the last two terms are in $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})$, which means X_* is in $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})$ as well since, by definition, $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})$ is a triangulated subcategory of $\mathcal{D}^{b,+}(\text{Mod-}RG)$.

Thus, we have $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n}(G,\mathscr{X})^{\oplus,\Pi}) = \Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi})$. Note that $\mathscr{F}\Lambda_{\alpha}(G,\mathscr{X})^{\oplus}\text{-dim}(RG) \leq \mathscr{F}\Lambda_{\beta}(G,\mathscr{X})^{\oplus}\text{-dim}(RG)$ whenever $\alpha < \beta$ (similarly, $t_{m,\mathscr{X}}(G) \leq t_{n,\mathscr{X}}(G)$ for all $m \leq n$, see Remark 3.7), which means

$$\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG) < \infty \implies \mathscr{F}\Lambda_{n-2}(G,\mathscr{X})^{\oplus}\text{-dim}(RG) < \infty,$$

and now we can show that

$$\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-1}(G,\mathscr{X})^{\oplus,\Pi}) = \Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{n-2}(G,\mathscr{X})^{\oplus,\Pi}).$$

We can go all the way down to $\Delta_{\mathscr{T}}\mathcal{D}^{b,+}(\Lambda_{0}(G,\mathscr{X})^{\oplus,\Pi})$ like this.

(c) We first show that, for any $n \geq 1$, $\mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle = \mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle$, which follows straight-forwardly from the fact that any complex in $\Lambda_n(G,\mathscr{X})$ is a module from the class of modules $\Lambda_n(G,\mathscr{X})$ concentrated in degree zero, and by Lemma 3.3, such a complex in $\Lambda_n(G,\mathscr{X})$ is quasi-isomorphic to a bounded complex of modules from $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$ and is therefore in $\mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$ by Lemma 4.2, and since

$$\mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle = \mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle$$

because $\mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle$ is closed under arbitrary direct sums, we have

$$\mathscr{T}\text{-}\langle\Lambda_{n}(G,\mathscr{X})\rangle \subseteq \mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle.$$

The other direction is obvious as $\Lambda_{n-1}(G,\mathscr{X}) \subseteq \Lambda_n(G,\mathscr{X}) \subseteq \mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle$ implies that $\mathscr{T}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle \subseteq \mathscr{T}\text{-}\langle\Lambda_{n}(G,\mathscr{X})\rangle$.

Now we show that, for any $n \geq 0$, $\mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_n(G,\mathscr{X}))\rangle = \mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle$. Again, it is clear that $\mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle \subseteq \mathscr{T}\text{-}\langle\mathcal{D}(\Lambda_n(G,\mathscr{X}))\rangle$. Now take an arbitrary unbounded chain complex (X_*, d_*), where each X_i is in $\Lambda_n(G,\mathscr{X})$. Over chain complexes, for any fixed m, it can be written as the inverse limit of its truncations given by

$$\begin{aligned}
g_{m,0}(X_*) &: \cdots \to X_m \to 0 \to 0 \to \cdots \\
g_{m,1}(X_*) &: \cdots \to X_m \to X_{m-1} \to 0 \to 0 \to \cdots \\
&\vdots \\
g_{m,k}(X_*) &: \cdots \to X_m \to X_{m-1} \to \cdots \to X_{m-k} \to 0 \to 0 \to \cdots .
\end{aligned}$$

In $g_{m,k}(X_*)$, we have X_{m-i} in degree $m-i$ for all $m \geq i \geq m-k$. The reason why we have an inverse limit here is because, like in the proof of (b) where we artificially used the short exact sequence used to define inverse limits in the derived unbounded category, our chain maps are from $g_{m,k+1}(X_*)$ to $g_{m,k}(X_*)$ for all k—the map from $g_{m,k+1}(X_*)$ to $g_{m,k}(X_*)$ is given by the identity map in all degrees strictly higher than $m-k$ and the zero map in every other degree.

Each $g_{m,k}(X_*)$ is a bounded below chain complex and, like in the proof of part (a), can be written as the direct limit of its truncations given by

$$\begin{aligned} j_0(g_{m,k}(X_*)) &: \cdots \to 0 \to X_{m-k} \to 0 \\ j_1(g_{m,k}(X_*)) &: \cdots \to X_{m-(k-1)} \to X_{m-k} \to 0 \\ &\vdots \\ j_t(g_{m,k}(X_*)) &: \cdots \to X_{m-(k-t)} \to \cdots \to X_{m-k} \to 0. \end{aligned}$$

Here, we have a direct limit because our chain maps go from $j_t(g_{m,k}(X_*))$ to $j_{t+1}(g_{m,k}(X_*))$ for all t—the map from $j_t(g_{m,k}(X_*))$ to $j_{t+1}(g_{m,k}(X_*))$ is given by the identity map in all degrees between $m-k$ and $m-k+t$ and the zero map in all other degrees. Thus, $X_* = \varprojlim_k \varinjlim_t j_t(g_{m,k}(X_*))$. Note that each $j_t(g_{m,k}(X_*))$ is a bounded complex of modules from $\Lambda_n(G,\mathscr{X})$, and so, by Lemma 4.2, each $j_t(g_{m,k}(X_*))$ is in $\mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle$, and now, since $\mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle$ is closed under both products and coproducts by definition, it is closed under both direct limits and inverse limits by Lemma 4.1, and therefore X_* is in $\mathscr{T}\text{-}\langle\Lambda_n(G,\mathscr{X})\rangle$.

(d) Take a bounded chain complex

$$X_* = 0 \to X_m \to X_{m-1} \to \cdots \to X_0 \to 0,$$

where each X_i is in $\Lambda_n(G,\mathscr{X})$. By Lemma 3.3, it follows that each X_i, considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded chain complex in $\mathcal{D}^b(\Lambda_{n-1}(G,\mathscr{X})^{\oplus})$, and by Lemma 4.2, that bounded chain complex is in $\Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$. This proves the first part.

Now note that, for any k,

$$\begin{aligned} \Lambda_{k-1}(G,\mathscr{X})^{\oplus} \subseteq \Lambda_k(G,\mathscr{X})^{\oplus} &\subseteq \mathcal{D}^b(\Lambda_k(G,\mathscr{X})^{\oplus}) \\ &\subseteq \Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_k(G,\mathscr{X})^{\oplus}), \end{aligned}$$

which implies that

$$\Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus} \subseteq \Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(G,\mathscr{X})^{\oplus}).$$

Denote $t_{n,\mathscr{X}}(G)$ (or $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG)$) by t. From Lemma 3.3, we have $\Lambda_n(G,\mathscr{X}) \subseteq [\Lambda_{n-1}(G,\mathscr{X})^{\oplus}]$, and when we have the additional assumption that $t<\infty$, we get $\Lambda_n(G,\mathscr{X}) \subseteq [\Lambda_{n-1}(G,\mathscr{X})^{\oplus}]_t$, which implies, courtesy of Lemma 2.11, that $\Lambda_n(G,\mathscr{X})^{\oplus} \subseteq [\Lambda_{n-1}(G,\mathscr{X})^{\oplus}]_t$. Using Lemma 4.2, we can say that all chain complexes in $\mathcal{D}^b(\Lambda_n(G,\mathscr{X})^{\oplus})$ are in $\Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$; therefore $\Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(G,\mathscr{X})^{\oplus}) \subseteq \Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$. Thus, we have

$$\Delta_{\mathscr{T}}\mathcal{D}^b(\Lambda_n(G,\mathscr{X})^{\oplus}) = \Delta_{\mathscr{T}}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}.$$

All the vertical equalities follow from Lemma 4.2.

Since $t_{n,\mathscr{X}}(G) \geq t_{m,\mathscr{X}}(G)$, and similarly,

$$\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG) \geq \mathscr{F}\Lambda_{m-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG),$$

for any $m \leq n$, we have $\Lambda_k(G, \mathscr{X})^{\oplus} \subseteq [\Lambda_{k-1}(G, \mathscr{X})^{\oplus}]_t$ for all $k \in \{1, 2, \ldots, n\}$, and again using Lemma 4.2 the way we used it above, we get that

$$\Delta_{\mathscr{T}}\Lambda_k(G, \mathscr{X})^{\oplus} \subseteq \Delta_{\mathscr{T}}\Lambda_{k-1}(G, \mathscr{X})^{\oplus}.$$

Since the inclusion in the other direction is obvious, this gives us our horizontal chain of equalities, and we are done. □

Definition 4.4. Let G be a group and R a commutative ring. We make the following definitions.

(a) Let $\mathscr{T} := \mathcal{D}(\text{Mod-}RG)$, and let $\mathscr{U}$ be a class of objects in $\mathscr{T}$. We denote by $\text{Loc}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$, called the localizing subcategory of $\mathscr{T}$ generated by $\mathscr{U}$, the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ and closed under coproducts (direct sums), and if this is all of $\mathscr{T}$, we say $\mathscr{U}$ generates $\mathscr{T}$.

We denote by $\text{Coloc}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$, called the colocalizing subcategory of $\mathscr{T}$ generated by $\mathscr{U}$, the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ and closed under products, and if this is all of $\mathscr{T}$, we say $\mathscr{U}$ cogenerates $\mathscr{T}$. We denote by $\mathscr{T}\text{-}\langle \mathscr{U} \rangle$, as in the statement of Theorem 4.3 (c), the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ closed under products and coproducts.

If $\text{Loc}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle = \mathscr{T}$ (resp. $\text{Coloc}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle = \mathscr{T}$), we say $\mathscr{U}$ generates (resp. cogenerates) $\mathscr{T}$. If $\mathscr{T}\text{-}\langle \mathscr{U} \rangle = \mathscr{T}$, we say $\mathscr{U}$ generates $\mathscr{T}$ with products and coproducts.

(b) Let $\mathscr{T} := \mathcal{D}^{b,+}(\text{Mod-}RG)$, and let $\mathscr{U}$ be a class of objects in $\mathscr{T}$. We denote by $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$ (resp. $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$) the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ satisfying the following property.

If $\{X_*^\lambda\}_{\lambda \in \Lambda}$ is a class of chain complexes in $\mathscr{U}$ such that there exists $n \in \mathbb{Z}$ such that, for every $\lambda \in \Lambda$, $\deg_i(X_*^\lambda) = 0$ for all $i > n$, then $\bigoplus_{\lambda \in \Lambda} X_*^\lambda$ (resp. $\prod_{\lambda \in \Lambda} X_*^\lambda$) is in $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$ (resp. $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$).

(c) Let $\mathscr{T} := \mathcal{D}^{b,-}(\text{Mod-}RG)$, and let $\mathscr{U}$ be a class of objects in $\mathscr{T}$. We denote by $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$ (resp. $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$) the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ satisfying the following property.

If $\{X_*^\lambda\}_{\lambda \in \Lambda}$ is a class of chain complexes in $\mathscr{U}$ such that there exists $n \in \mathbb{Z}$ such that, for every $\lambda \in \Lambda$, $\deg_i(X_*^\lambda) = 0$ for all $i < n$, then $\bigoplus_{\lambda \in \Lambda} X_*^\lambda$ (resp. $\prod_{\lambda \in \Lambda} X_*^\lambda$) is in $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$ (resp. $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle \mathscr{U} \rangle$).

(d) For any triangulated category $\mathscr{T}$, and any class of objects in it $\mathscr{U}$, we denote by $\langle \mathscr{U} \rangle$ the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$. Note that we used a different notation for this in the statement of Theorem 4.3, but this $\langle\rangle$ notation is more convenient in the context of generation.

With the aid of the above definitions, we get the following generation results for groups that are themselves in Kropholler's hierarchy (note that in the statement of Theorem 4.3, we did not require the big group G to be in Kropholler's hierarchy).

Theorem 4.5. *Let G be a group in $H_n\mathscr{X}$ for some class of groups $\mathscr{X}$. The following statements hold.*

(a) *If $\mathscr{T} = \mathcal{D}(\text{Mod-}RG)$, then*

$$\begin{array}{ccccc} \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G,\mathscr{X})\rangle & = & \cdots & = & \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{0}(G,\mathscr{X})\rangle \\ \| & & & & \\ \mathcal{D}(\text{Mod-}RG) = \mathscr{T}\text{-}\langle \Lambda_{n-1}(G,\mathscr{X})\rangle & = & \cdots & = & \mathscr{T}\text{-}\langle \Lambda_{0}(G,\mathscr{X})\rangle \\ \| & & & & \\ \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle. & & & & \end{array}$$

(b) *If $\mathscr{T} = \mathcal{D}^{b,+}(\text{Mod-}RG)$ and $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG)$ or alternatively $t_{n,\mathscr{X}}(G)$ is finite, then*

$$\begin{array}{l} \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G,\mathscr{X})\rangle \\ \qquad \| \\ \mathcal{D}^{b,+}(\text{Mod-}RG) = \underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle = \cdots = \underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle \Lambda_{0}(G,\mathscr{X})^{\oplus}\rangle. \end{array}$$

(c) *If $\mathscr{T} = \mathcal{D}^{b,-}(\text{Mod-}RG)$, then*

$$\begin{array}{l} \underline{\mathrm{Coloc}}_{\mathscr{T}}\text{-}\langle \Lambda^{\oplus}_{n-1}\rangle \\ \qquad \| \\ \mathcal{D}^{b,-}(\text{Mod-}RG) = \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}\rangle = \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-2}\rangle = \cdots = \underline{\mathrm{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{0}\rangle. \end{array}$$

(d) $\mathcal{D}^{b}(\text{Mod-}RG) = \langle \Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$. *If, additionally, $t_{n,\mathscr{X}}(G)$ or alternatively $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\text{-dim}(RG)$ is finite (see Remark 3.7), then*

$$\mathcal{D}^{b}(\text{Mod-}RG) = \langle \Lambda_{k}(G,\mathscr{X})^{\oplus}\rangle \quad \textit{for all } k \geq 0.$$

Proof. (a) To prove the first horizontal line of equalities, first note that, for any k, $\Lambda_k(G,\mathscr{X}) \subseteq [\Lambda_{k-1}(G,\mathscr{X})^{\oplus}]$, so

$$\mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{k}(G,\mathscr{X})\rangle \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{k-1}(G,\mathscr{X})^{\oplus}\rangle = \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{k-1}(G,\mathscr{X})\rangle$$

(the last equality follows from the fact that localizing subcategories are closed under arbitrary direct sums (coproducts) by definition). For the other direction, it is obvious that $\mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{k-1}(G,\mathscr{X})\rangle \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{k}(G,\mathscr{X})\rangle$ because $\Lambda_{k-1}(G,\mathscr{X}) \subseteq \Lambda_k(G,\mathscr{X})$. Note that, up to here, we have not used the fact that $G \in H_n\mathscr{X}$.

Note that, since $G \in H_n\mathscr{X}$, $\Lambda_n(G,\mathscr{X}) = \text{Mod-}RG$. We start by observing that the second horizontal line of equalities follow directly from the lower horizontal line of equalities in Theorem 4.3 (c). Take an arbitrary unbounded chain complex of RG-modules $(X_*, d_*) = \cdots \to X_k \to X_{k-1} \to \cdots$. We fix an m. We look at the following truncations:

$$\begin{aligned} j_0(X_*) &: \cdots \to X_{m+2} \to X_{m+1} \to \operatorname{Ker}(d_m) \to 0 \to 0 \to \cdots \\ j_1(X_*) &: \cdots \to X_{m+2} \to X_{m+1} \to X_m \to \operatorname{Ker}(d_{m-1}) \to 0 \to 0 \to \cdots \\ &\vdots \\ j_k(X_*) &: \cdots \to X_{m+2} \to \cdots \to X_{m-(k-2)} \to X_{m-(k-1)} \\ &\qquad \to \operatorname{Ker}(d_{m-k}) \to 0 \to \cdots . \end{aligned}$$

Here, in $j_k(X_*)$, there is X_{m-i} in degree $m-i$ for all $i \geq k-1$ and $\operatorname{Ker}(d_{m-k})$ in degree $m-k$ and zero in every other degree. The map f_k between $j_k(X_*)$ and $j_{k+1}(X_*)$ is given by the identity map in every degree bigger than or equal to $m-(k-1)$, the inclusion map in degree $m-k$ and the zero map in every other degree, and X_* can be written as the direct limit of these truncations, $\varinjlim_k j_k(X_*)$.

Each $j_k(X_*)$ is a bounded below chain complex and, as shown in the proof of part (a) of Theorem 4.3, can be written as the direct limit of their non-canonical truncations $g_i(j_k(X_*))$ (using the g_i notation from the proof of Theorem 4.3 (a)), each of which in turn are bounded chain complexes. So we have $X_* = \varinjlim_k \varinjlim_i g_i(j_k(X_*))$. Now note that each $g_i(j_k(X_*))$ is a bounded complex, where each module, when concentrated as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded complex with modules from $\Lambda_{n-1}(G, \mathscr{X})^{\oplus}$ by Lemma 3.3. So each of the modules in the bounded chain complex $g_i(j_k(X_*))$ for any i and k is in $\operatorname{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X})^{\oplus} \rangle$ by Lemma 4.2. Therefore, by Lemma 4.2 again, each

$$g_i(j_k(X_*)) \in \operatorname{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X})^{\oplus} \rangle = \operatorname{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X}) \rangle$$

(the last equality follows from the fact that localizing subcategories are closed under arbitrary direct sums by definition). Now localizing subcategories are closed under coproducts by definition, so they are closed under direct limits by Lemma 4.1, so $X_* \in \operatorname{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X}) \rangle$.

To prove the result about colocalizing subcategories, we start with the same chain complex X_* and the same fixed m and look at the following truncations (note that our g_i and j_i notations in the remaining part of the proof of (a) differ from the g_i and j_i notations used in the last paragraph):

$$\begin{aligned} g_0(X_*) &: \cdots \to X_m \to 0 \to 0 \to \cdots \\ g_1(X_*) &: \cdots \to X_m \to X_{m-1} \to 0 \to 0 \to \cdots \\ &\vdots \\ g_k(X_*) &: \cdots \to X_m \to X_{m-1} \to \cdots \to X_{m-k} \to 0 \to 0 \to \cdots . \end{aligned}$$

Here, in $g_k(X_*)$, we have the module X_{m-i} in degree $m-i$ for all $i \leq k$ and zero in every other degree. The map between $g_{k+1}(X_*)$ and $g_k(X_*)$ is given by the identity map in every degree bigger than or equal to $m-k$ and the zero map in every other degree, and X_* can be written as an inverse limit of these truncations, $\varprojlim_k g_k(X_*)$. Each $g_k(X_*)$ is a bounded below chain complex, and they can be written as the inverse limit of their canonical truncations in the following way.

Let (Y_*, δ_*) be a bounded below chain complex, where t is the smallest degree with a nonzero module. We define

$$j_k(Y_*) = \cdots \to 0 \to \operatorname{Ker}(\delta_{t+k}) \hookrightarrow Y_{t+k} \to Y_{t+k-1} \to \cdots \to Y_t \to 0,$$

where every degree i with $t \le i \le t+k$ has Y_i, degree $t+k+1$ has $\operatorname{Ker}(\delta_{t+k})$, and every other degree has zero. The map from $j_{k+1}(Y_*)$ and $j_k(Y_*)$ is given by δ_{t+k+1} in degree $t+k+1$, the identity map in every degree between t and $t+k$, and the zero map in every other degree. In this case, $Y_* = \varprojlim_k j_k(Y_*)$.

Using the above information, we can write each $g_k(X_*)$ as an inverse limit $\varprojlim_i j_i(g_k(X_*))$, and therefore we have

$$X_* = \varprojlim_k \varprojlim_i j_i(g_k(X_*)).$$

Again, each $j_i(g_k(X_*))$ is a bounded chain complex, where each module, when considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded chain complex with modules from $\Lambda_{n-1}(G, \mathscr{X})^{\oplus}$ by Lemma 3.3. So $j_i(g_k(X_*))$ is in $\operatorname{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X})^{\oplus} \rangle$ by Lemma 4.2 as colocalizing subcategories are triangulated subcategories. Now colocalizing subcategories are closed under products by definition, and so they are closed under inverse limits by Lemma 4.1; therefore X_* is in $\operatorname{Coloc}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X})^{\oplus} \rangle$.

(b) Here, the horizontal chain of equalities follows from Theorem 4.3 (b). To show that $\mathcal{D}^{b,+}(\text{Mod-}RG) = \underline{\operatorname{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X}) \rangle$, we start with an arbitrary bounded above chain complex (X_*, d_*) of RG-modules with m being the biggest degree without the zero module, and note that, like we saw in the proof of (a), over chain complexes, X_* can be realized as the direct limit of its truncations

$$\begin{aligned} j_{m,k}(X_*) : \cdots \to 0 \to X_m \to X_{m-1} \to \cdots \to X_{m-(k-1)} \\ \to \operatorname{Ker}(d_{m-k}) \to 0. \end{aligned}$$

Denote by f_k the chain map between $j_{m,k}(X_*)$ and $j_{m,k+1}(X_*)$ which is given by the identity map in every degree between m and $m-k+1$, the inclusion map in degree $m-k$, and the zero map in every other degree. Since all RG-modules are in $[\Lambda_{n-1}(G, \mathscr{X})^{\oplus}]$ by Lemma 3.3, each of the modules in $j_{m,k}(X_*)$, when considered as a chain complex concentrated in degree zero, is quasi-isomorphic to a bounded chain complex of modules from $\Lambda_{n-1}(G, \mathscr{X})^{\oplus}$ and, by Lemma 4.2, is therefore in $\underline{\operatorname{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X}) \rangle$ (note that we do not need the $^{\oplus}$ sign here since any module in $\Lambda_{n-1}(G, \mathscr{X})^{\oplus}$, as a chain complex concentrated in degree zero, can be written as a direct sum of chain complexes concentrated in degree zero with each of them having a module from $\Lambda_{n-1}(G, \mathscr{X})$ in degree zero, and is therefore in $\underline{\operatorname{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X}) \rangle$). And since each $j_{m,k}(X_*)$ is a bounded chain complex, we can say using Lemma 4.2 again that each $j_{m,k}(X_*)$ is in $\underline{\operatorname{Loc}}_{\mathscr{T}}\text{-}\langle \Lambda_{n-1}(G, \mathscr{X}) \rangle$. So, in $\mathcal{D}(\text{Mod-}RG)$, we have the short exact sequence (see Lemma 4.1)

$$0 \to \bigoplus_{k\in\mathbb{N}} j_{m,k}(X_*) \xrightarrow{\bigoplus_{i\geq 0}(id_{j_{m,i}(X_*)}-f_i)} \bigoplus_{k\in\mathbb{N}} j_{m,k}(X_*) \to X_* \to 0,$$

and again, we note, like in the proof of Theorem 4.3 (b), the above short exact sequence is a distinguished triangle in $\mathcal{D}^{b,+}(\text{Mod-}RG)$. Now the first two terms are in $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle$ by definition since each $j_{m,k}(X_*)$ is in $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle$ and is bounded above at degree m. Therefore, X_* is in $\underline{\text{Loc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle$ by definition of triangulated subcategories.

(c) Again, the horizontal chain of equalities follows from Theorem 4.3 (a). To show that $\mathcal{D}^{b,-}(\text{Mod-}RG) = \underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$, note that if $(X_i,d_i)_{i\geq m}$ is an arbitrary bounded below chain complex of RG-modules, over chain complexes, it is the inverse limit of its truncations

$$j_{m,k}(X_*) : \cdots \to 0 \to \text{Ker}(d_{m+k}) \hookrightarrow X_{m+k} \to \cdots \to X_m \to 0$$

which is bounded, and again, like in the proof of part (b), it follows that these truncations are in $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$. Here, the chain map f_k between $j_{m,k}(X_*)$ and $j_{m,k-1}(X_*)$ is given by the identity map in every degree between m and $m+k-1$, d_{m+k} in degree $m+k$, and the zero map in every other degree. In $\mathcal{D}(\text{Mod-}RG)$, we have the short exact sequence (see Lemma 4.1)

$$0 \to X_* \to \prod_{k\in\mathbb{N}} j_{m,k}(X_*) \xrightarrow{\prod_{i\geq 1}(id_{j_{m,i}(X_*)}-f_i)} \prod_{k\in\mathbb{N}} j_{m,k}(X_*) \to 0.$$

Again, we note that this is a distinguished triangle in $\mathcal{D}^{b,-}(\text{Mod-}RG)$, and since the last two terms are clearly in $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$ by definition, we have that X_* is in $\underline{\text{Coloc}}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$ as well by the definition of triangulated subcategories.

(d) This follows from Theorem 4.3 (d). □

The following corollary is obvious from the proof of Theorem 4.5.

Corollary 4.6. *Let G be a group, and let $\mathscr{X}$ be a class of groups. Assume that, for some n, $\Lambda_n(G,\mathscr{X})^{\oplus}$ is closed under kernels. Then*

$$\text{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})\rangle = \text{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle,$$

with $\mathscr{T}$ being the derived unbounded category $\mathcal{D}(\text{Mod-}RG)$.

Remark 4.7. It follows from the proof of Theorem 4.5 (a) that, for an $H_n\mathscr{X}$-group G, we can generate the whole derived unbounded category in three distinct ways from the class $\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$—with coproducts, with products, and with using both products and coproducts. We can of course also state this result by replacing n with a limit ordinal and having in place of $n-1$ some ordinal strictly smaller than α.

Comparing the class of localizing subcategories with the class of colocalizing subcategories arising from a given triangulated category with arbitrary products and coproducts is a question of classical interest. The following was proved by Amnon Neeman in [15, 17].

Theorem 4.8 ([15, 17]). *Let R be a commutative noetherian ring. For $\mathscr{U}$ a triangulated subcategory of $\mathcal{D}(\text{Mod-}R)$, write*

$$\phi(\mathscr{U}) := \{X \in \mathcal{D}(\text{Mod-}R) \mid \text{Hom}_{\mathcal{D}(\text{Mod-}R)}(U, X) = 0 \textit{ for all } U \in \mathscr{U}\},$$
$$\psi(\mathscr{U}) := \{X \in \mathcal{D}(\text{Mod-}R) \mid \text{Hom}_{\mathcal{D}(\text{Mod-}R)}(X, U) = 0 \textit{ for all } U \in \mathscr{U}\}.$$

(a) *If $\mathscr{U}$ is a localizing subcategory of $\mathcal{D}(\text{Mod-}R)$, then $\phi(\mathscr{U})$ is colocalizing, and if $\mathscr{U}$ is a colocalizing subcategory of $\mathcal{D}(R)$, then $\psi(\mathscr{U})$ is localizing. Also, if $\mathscr{U}$ is localizing, then $\psi(\phi(\mathscr{U})) = \mathscr{U}$.*
(b) *The assignment $\mathscr{U} \mapsto \phi(\mathscr{U})$ induces a bijection between the collection of localizing subcategories of $\mathcal{D}(\text{Mod-}R)$ and the collection of colocalizing subcategories of $\mathcal{D}(\text{Mod-}R)$.*

Remark 4.9. What Theorem 4.8 shows is that if the group ring is noetherian, then we do not get any "new" colocalizing subcategories other than the ones we get from the localizing subcategories. (To see an easy example, note that if we take a group not all of whose subgroups are finitely generated, then its group ring over any field is not noetherian. Theorem 1.3 tells us, for example, that the free abelian group of rank $\aleph_\omega$, where ω is the first infinite ordinal, is in $H_3\mathscr{F}$, with $\mathscr{F}$ being the class of all finite groups—so this group does not have a noetherian group ring over fields.) So it is nice to see in Theorem 4.5 (a) that, as far as the localizing and colocalizing subcategories generated by the smallest direct-sum closed class containing modules induced from subgroups in lower levels on the hierarchy are concerned, they coincide with each other. Can we find an example of a group G in $H_n\mathscr{X}$, for some n and $\mathscr{X}$, and a commutative ring R such that RG is not noetherian, where we get some colocalizing subcategories of $\mathcal{D}(\text{Mod-}RG)$ that do not come from localizing subcategories the way shown in Theorem 4.8?

We end this section with the following remark and a subsequent pair of questions.

Remark 4.10. For simplicity, in this remark, when there is no ambiguity over G, we shall denote the smallest direct sum closed class of modules induced up from $H_\alpha\mathscr{X}$-subgroups of G by Λ_α, where α is some ordinal.

Can we replace n in the statement of Theorem 4.5 by a limit ordinal α? If $G \in H_\alpha\mathscr{X}$ for some limit ordinal α, then it follows from Definition 1.1 that $G \in H_\beta\mathscr{X}$ for some successor ordinal $\beta < \alpha$. It follows from the arguments in the proof of Theorem 4.5 that, for any $H_\alpha\mathscr{X}$-group G, where $\mathscr{X}$ is a class of groups, we have the following equality of localizing subcategories and filtration of colocalizing subcategories of $\mathcal{D}(\text{Mod-}RG)$ for some fixed commutative ring R, where δ is the biggest limit ordinal strictly smaller than α:

$$\text{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_\delta\rangle = \text{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{\delta+1}\rangle = \cdots = \text{Loc}_{\mathscr{T}}\text{-}\langle\Lambda_{\beta-1}\rangle = \mathcal{D}(\text{Mod-}RG),$$
$$\text{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_\delta\rangle \subseteq \text{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_{\delta+1}\rangle \subseteq \cdots \subseteq \text{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_{\beta-1}\rangle = \mathcal{D}(\text{Mod-}RG),$$

where $\mathscr{T} = \mathcal{D}(\text{Mod-}RG)$ and each of the inclusion functors in the filtration is triangulated. The main reason, in short, as to why we do not get an equality for the generated colocalizing subcategories is because they need not be closed under coproducts.

Denoting ω to be the first infinite ordinal, if we now take

$$\alpha = \omega.n = \omega + \omega + \cdots + \omega \quad (n \text{ times})$$

and find a group $G \in H_\alpha \mathscr{X}$ for some $\mathscr{X}$ but not in $H_\beta \mathscr{X}$ for any $\beta < \alpha$ (note that we do not yet have examples of such groups—the best result that we have in the literature is that there are groups in $H_\alpha \mathscr{F} \setminus H_{<\alpha}\mathscr{F}$ for all ordinals α smaller than the first uncountable ordinal; this result is from [11]), then we do get a filtration of localizing subcategories that need not be a chain of equalities (this is because Lemma 3.3 need not hold when α is not a successor ordinal): for any integer k, let $\text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[k-1,k]}\rangle := \text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_\beta \rangle$ for any successor ordinal β satisfying $\omega.(k-1) < \beta < \omega.k$. This is well-defined because, for any two successor ordinals β_1 and β_2 between $\omega.(k-1)$ and $\omega.k$, $\text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{\beta_1}\rangle = \text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda_{\beta_2}\rangle$; this follows from the above chain of equalities. Now we have the following filtration of localizing subcategories:

$$\text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[0,1]}\rangle \subseteq \text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[1,2]}\rangle \subseteq \cdots \subseteq \text{Loc}_{\mathscr{T}}\text{-}\langle \Lambda^{[n-1,n]}\rangle = \mathcal{D}(\text{Mod-}RG)$$

We end this remark with a little comment on how the above filtration can be useful in studying the Krull dimension of the derived unbounded categories of groups in Kropholler's hierarchy which has not been studied at all before. Before providing the definition for the Krull dimension of triangulated categories, we recall that thick subcategories of a triangulated category are defined as triangulated subcategories closed under summands. For any two subcategories $\mathscr{I}_1, \mathscr{I}_2$ of $\mathscr{T}$, we define $\mathscr{I}_1 * \mathscr{I}_2$ to be the full subcategory of $\mathscr{T}$ consisting of objects M such that there is a distinguished triangle $M_1 \to M \to M_2 \rightsquigarrow$ with $M_i \in \mathscr{I}_i$. Rouqier [20] defines a thick subcategory $\mathscr{I}$ of $\mathscr{T}$ to be irreducible if given two thick subcategories $\mathscr{I}_1$ and $\mathscr{I}_2$ of $\mathscr{I}$ such that $\mathscr{I}$ is the smallest thick subcategory of $\mathscr{T}$ containing $\mathscr{I}_1 * \mathscr{I}_2$, then at least one of the $\mathscr{I}_i$'s is $\mathscr{I}$. The Krull dimension of $\mathscr{T}$ is the length of the maximal chain of thick irreducible subcategories $0 \neq \mathscr{I}_0 \subset \mathscr{I}_1 \subset \cdots \subset \mathscr{I}_n = \mathscr{T}$. Now, can we use the above filtration of localizing subcategories with the same G to comment on the Krull dimension of $\mathcal{D}(\text{Mod-}RG)$? Localizing subcategories are thick (this is standard knowledge; see Lemma 6.16), and making the inclusions strict can be possible with the choice of our group or possibly with making sure that, at every level of Kropholler's hierarchy below α, G has a subgroup which is not in any lower level. The most nontrivial part will be checking irreducibility of the localizing subcategories, but that can possibly be handled with looking at the irreducible components.

We end this section with the following question on groups that are beyond Kropholler's hierarchy.

Question 4.11. (a) *The two filtrations mentioned in Remark 4.10, except the last equality, where we have one of the subcategories in the filtration being equal to the whole derived unbounded category, hold for any arbitrary group. Now let G be Thompson's group given by $\langle x_0, x_1, x_2, \cdots : x_k^{-1} x_n x_k = x_{n+1}$ for $k < n\rangle$. We know this group is not in $H\mathscr{F}$ (see* [12]*), where $\mathscr{F}$ is the collection of all finite groups. Now, using the notation of Remark 4.10 with $\mathscr{X} = \mathscr{F}$, does any of the filtrations* $\mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda^{[0,1]}\rangle \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda^{[1,2]}\rangle \subseteq \cdots \subseteq \mathrm{Loc}_{\mathscr{T}}\text{-}\langle\Lambda^{[n-1,n]}\rangle \subseteq \cdots$ *and* $\mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_0\rangle \subseteq \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_1\rangle \subseteq \cdots \subseteq \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_{n-1}\rangle \subseteq \mathrm{Coloc}_{\mathscr{T}}\text{-}\langle\Lambda_n\rangle \subseteq \cdots$ *eventually stabilize, where $\mathscr{T} = \mathcal{D}(\text{Mod-}RG)$ and R is a fixed commutative ring?*

(b) *Again, using the notation from Remark 4.10, is there an example of a group G for which one of the filtrations, as mentioned in* (a)*, eventually stabilizes but the other one does not?*

(c) *In Theorem 4.5* (d)*, we see that if $G \in H_n\mathscr{X}$ for any $\mathscr{X}$ and R is some commutative ring, then $\mathcal{D}^b(\text{Mod-}RG) = \langle\Lambda_{n-1}(G,\mathscr{X})^{\oplus}\rangle$. Are there examples of groups not in $H_n\mathscr{X}$ for some $\mathscr{X}$, satisfying this result? Also, are there examples of groups $G \notin H\mathscr{X}$ for some $\mathscr{X}$ such that $\mathcal{D}^b(\text{Mod-}RG) = \langle\Lambda(G,\mathscr{X})^{\oplus}\rangle$ for any commutative ring R, where*

$$\Lambda(G,\mathscr{X}) := \{\mathrm{Ind}_H^G(M) \mid M \in \mathrm{Mod}(RH), H \in H\mathscr{X}\}?$$

5. Ending comments on generation of derived categories

In Theorem 4.3 and Theorem 4.5, we have seen that with derived unbounded and derived bounded below categories, when we look to generate them with the smallest direct-sum closed class of modules induced up from subgroups lower down the hierarchy, we can go all the way down to the zeroth level, whereas in the case of derived bounded above, we need additional conditions to get down even one level. So working with generating the derived category seems like the best bet. Below, we prove an easy result showing that if we introduce a similar definition for steps of generation like we did for modules in Section 2, we see that the number of steps taken to generate anything in the n-th hierarchy from the 0-th hierarchy is dependent exponentially on n but linearly on the length of the chain complex that we are generating.

Theorem 5.1. *Define generation of chain complexes in the following way, similar to the way we defined generation of modules: a chain complex X^* is generated from a class of chain complexes $\mathscr{T}$ in 0 steps if and only if $X^* \in \mathscr{T}$, and if we have a short exact sequence $0 \to X_1^* \to X_2^* \to X_3^* \to 0$, where the chain complex X_i^* for any two i, say $i = j, k \in \{1,2,3\}$, is generated from $\mathscr{T}$ in a_i steps, then the third chain complex is generated from $\mathscr{T}$ in $a_j + a_k + 1$ steps.*

Let G be a group and $\mathscr{X}$ a class of groups. Assume that $t_{n,\mathscr{X}}(G)$ or alternatively $\mathscr{F}\Lambda_{n-1}(G,\mathscr{X})^{\oplus}$-$\dim(RG)$ is finite, and denote any of these by t. Then a bounded chain complex, where each module is in $\Lambda_n(G,\mathscr{X})^{\oplus}$, of length m can be generated from $\Lambda_0(G,\mathscr{X})^{\oplus}$ in $m(t+1)^n - 1$ steps.

Proof. We proceed by induction on m. When $m = 1$, we have one module from the class $\Lambda_n(G, \mathscr{X})^\oplus$ in one degree and zero in every other degree. The result follows from Theorem 3.5 and Remark 3.7. Assume that, for all $k \leq m$, all bounded chain complexes, where each module is in $\Lambda_n(G, \mathscr{X})$, of length k are generated from $\Lambda_0(G, \mathscr{X})^\oplus$ (note that we are considering $\Lambda_0(G, \mathscr{X})^\oplus$ to be a class of complexes by considering all of the modules contained in it as chain complexes concentrated in degree zero) in $k(t + 1)^n - 1$ steps—this is our induction hypothesis. Now let X_* be a bounded chain complex of length $m + 1$. Like in the proof of Lemma 4.2, we can put X_* in the middle of a short exact sequence, where the other two terms are of lengths m and 1 respectively, so those other complexes are generated from $\Lambda_0(G, \mathscr{X})^\oplus$ in $m(t + 1)^n - 1$ and $(t + 1)^n - 1$ steps respectively. Thus, X_* can be generated from $\Lambda_0(G, \mathscr{X})^\oplus$ in $m(t + 1)^n - 1 + (t + 1)^n - 1 + 1 = (m + 1)(t + 1)^n - 1$ steps. This ends our induction. ☐

We end this section with a table on generation information in module and derived categories (Table 1). In it, generation results in the derived category have been gathered along with the analog results in the module category. The inclusion symbols on the first two columns on the right denote triangulated inclusion provided, for the terms those symbols are connecting, we consider the

Module category	Derived category
Anything in $\Lambda_n(G, \mathscr{X})^\oplus$ can be generated from $\Lambda_0(G, \mathscr{X})^\oplus$ in finite (given by $(1 + t_0)(1 + t_1) \dots (1 + t_n) - 1$) steps with $t_0, t_1, \dots, t_n < \infty$	$\mathcal{D}^{b,+}(\Lambda_n(G, \mathscr{X})^\oplus)$ $\veebar$ $\vdots \Uparrow$ (generates) $\veebar$ $\mathcal{D}^{b,+}(\Lambda_0(G, \mathscr{X})^\oplus)$ with $t_0, t_1, \dots, t_n < \infty$
$\Lambda_n(G, \mathscr{X})^\oplus \subseteq [\Lambda_{n-1}(G, \mathscr{X})^\oplus]_\infty$ $\subseteq [[\Lambda_{n-2}(G, \mathscr{X})^\oplus]^\oplus_\infty]_\infty$ $\subseteq \cdots$ $\subseteq \underbrace{[[\dots[}_{n\text{ times}}\Lambda_0(G, \mathscr{X})\underbrace{^\oplus]^\oplus_\infty]_\infty \dots {}^\oplus]_\infty}_{n\text{ times}}$	$\mathcal{D}^{b,-}(\Lambda_n(G, \mathscr{X})^\oplus)$ $\veebar$ $\vdots \Uparrow$ (generates) $\veebar$ $\mathcal{D}^{b,-}(\Lambda_0(G, \mathscr{X})^\oplus)$
$\Lambda_n(G, \mathscr{X}) \subseteq [\Lambda_{n-1}(G, \mathscr{X})^\oplus]$	$\Lambda_{n-1}(G, \mathscr{X})^\oplus$ generates $\mathcal{D}^b(\Lambda_n(G, \mathscr{X}))$
$\Lambda_n(G, \mathscr{X}) \subseteq \underbrace{[[\dots[}_{n\text{ times}}\Lambda_0(G, \mathscr{X})\underbrace{^\oplus]^\oplus] \dots {}^\oplus]}_{n\text{ times}}$	$\Lambda_0(G, \mathscr{X})$ generates $\mathcal{D}(\Lambda_n(G, \mathscr{X}))$ with products and coproducts

Table 1. Generation in module and derived categories

smallest triangulated subcategory of the relevant derived category containing those terms. Note that, for derived bounded above categories here, what we have is cogeneration because we cannot generate in that case without closing our generating classes under products. The diagram is informal, so we are a little loose with the word “generates” for every case. It is worth noting that, when we are talking about generating a derived category on $\Lambda_n(G, \mathscr{X})$, we are talking about generating the smallest triangulated subcategory of the relevant whole derived category.

6. Generation in stable module categories of infinite groups

Our results on the generation of the derived categories for groups in Kropholler's hierarchy (Theorem 4.3 (d), Theorem 4.5 (d)) can be used to comment on the generation of stable module categories for a large family of infinite groups. We need to provide some background material on this first, and we start with the stable module categories of finite groups.

6.1. Stable module categories of finite groups.

Definition 6.2. For a finite group G and a field k whose caracteristic divides the order of G, define the stable module category of G, denoted $\mathrm{StMod}(kG)$, as having the same objects as Mod-kG, and its morphisms are given by quotienting out those module homomorphisms that factor through some projective kG-module.

For finite groups, stable module categories are usually studied over fields of prime characteristic.

Theorem 6.3 (see [3, Thm. 2.31])**.** *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Then $\mathrm{StMod}(kG)$ is triangulated, with the suspension given by Ω^{-1}.*

Note that, since finite groups admit complete resolutions, we have the Ω^{-1} functor for finite groups, and also it follows from the definition of the stable module category that the Ω^{-1} functor is well-defined in the stable module category. It is now a standard fact that stable module categories of finite groups are compactly generated, i.e. generated by the compact objects in the stable module category which is a triangulated category. We provide a definition of compact objects of triangulated categories below.

Definition 6.4. Let $\mathscr{T}$ be a triangulated category with coproducts. An object $C \in \mathscr{T}$ is called compact if $\mathrm{Hom}_{\mathscr{T}}(C, ?)$ commutes with coproducts. We say $\mathscr{T}$ is compactly generated if the smallest localizing subcategory of $\mathscr{T}$ containing all the compact objects is the whole of $\mathscr{T}$.

Theorem 6.5 (see [3, Thm. 2.31])**.** *Let G be a finite group, and let k be a field whose characteristic divides the order of G. Then $\mathrm{StMod}(kG)$ is compactly generated, and the class of compact objects is precisely the class of finitely generated modules.*

The localizing subcategories of stable categories for finite groups have been classified by Benson, Iyengar and Krause in [2]. For infinite groups that admit complete resolutions, we can similarly define stable module categories.

6.6. **Stable module categories of infinite groups.** Now let G be a (not necessarily finite) group that admits complete resolutions over a commutative ring R of finite global dimension. Almost all of the material in this section up to Theorem 6.14 is from [21] which is connected to the paper [14].

Definition 6.7 (see [21]). Define a category $\mathrm{ModProj}(RG)$ in which the objects are the same as in $\mathrm{Mod}(RG)$ except all the projective RG-modules are identified with the zero module. For any two objects M, N in $\mathrm{ModProj}(RG)$, we define the Hom-sets of $\mathrm{ModProj}(RG)$ in the following way:

$$\mathrm{Hom}_{\mathrm{ModProj}(RG)}(M,N) = \mathrm{Hom}_{\mathrm{Mod}(RG)}(M,N)/P\,\mathrm{Hom}_{\mathrm{Mod}(RG)}(M,N),$$

where $P\,\mathrm{Hom}_{\mathrm{Mod}(RG)}(M,N)$ is the class of all morphisms $f : M \to N$ such that f is the composition of $g : M \to P$ and $h : P \to N$ for some projective RG-module P.

Remark 6.8. Note that, in comparison with Definition 6.2, $\mathrm{ModProj}(RG)$ as introduced in Definition 6.7 coincides with the stable module category of G when G is finite and R is a field whose characteristic divides the order of G.

In $\mathrm{ModProj}(RG)$, if we have a morphism $f : M \to N$, the syzygy functor Ω induces a map between $\Omega(M)$ and $\Omega(N)$. It is clear that, for any object M in $\mathrm{ModProj}(RG)$, $\Omega(M)$ is well-defined up to isomorphism. The following is clear now.

Lemma 6.9. *Ω is a functor from $\mathrm{ModProj}(RG)$ to itself.*

Definition 6.10. We define the stable module category of RG-modules, written $\mathrm{Stab}(RG)$ (to distinguish from the way we write stable module categories for finite groups), by stating it has the same objects as $\mathrm{Mod}(RG)$ and, for any two objects $M, N \in \mathrm{Stab}(RG)$,

$$\mathrm{Hom}_{\mathrm{Stab}(RG)}(M,N) = \varinjlim_{\Omega} \mathrm{Hom}_{\mathrm{ModProj}(RG)}(\Omega^n(M), \Omega^n(N)).$$

Recall that since G admits complete resolutions and since R is of finite global dimension, all RG-modules admit complete resolutions. In [14], the following was shown.

Theorem 6.11 ([14, Thm. 3.9]). *Any complex in $\mathcal{D}^b(\text{Mod-}RG)$ admits a complete resolution.*

We now need to expand a bit on $\Omega^0(M)$ for a given RG-module M.

Definition 6.12. Fix an RG-module M.

Take a complete resolution (F_*, d_*) of

$$M : \cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} F_{-1} \longrightarrow \cdots,$$

and denote $\Omega^t(M) := \mathrm{Ker}(d_{t-1})$ for any integer t. Note that t can be negative. It is easy to see that $\Omega^*(M)$ is well-defined in the stable category.

Remark 6.13. Note that, with the notation of Definition 6.12, $\Omega^0(M)$ need not be the same as M in the module category; however, $\Omega^0(M)$ and M are isomorphic in the stable category. We have a natural map $f : \Omega^0(M) \to M$ such that $\Omega^{\gg 0}(f) = \mathrm{id}$.

We now provide the following characterization of the stable module category of G in terms of other known triangulated categories. It was proved in [14] with G belonging to a large class of infinite groups called groups of type Φ over R (groups of type Φ were introduced in [22], and in the same paper, it was shown that they admit complete resolutions), but the proof works fine if it is just known that G admits complete resolutions over R.

Theorem 6.14 ([21, Thm. 3.7], also [14, Thm. 3.10])**.** *The following categories are equivalent as triangulated categories (here, $\mathcal{D}^b(\text{Proj-}RG)$ denotes the derived category of bounded complexes of projective RG-modules):*
(a) $\mathrm{Stab}(RG)$,
(b) $\mathcal{D}^b(\text{Mod-}RG)/\mathcal{D}^b(\text{Proj-}RG)$,
(c) *the category of acyclic complexes of projectives with the morphisms being given by chain homotopies,*
(d) *the category of Gorenstein projective RG-modules with the morphisms being given by* $\mathrm{ModProj}(RG)$.
Here, the (a) $\to$ (b) *map is given by considering modules as complexes concentrated in degree zero, the* (b) $\to$ (c) *map is given by complete resolutions (see Theorem 6.11), the* (c) $\to$ (d) *map is given by* Ω^0*, and the* (d) $\to$ (a) *map is given by inclusion, and the composition of these maps in this order is isomorphic to the identity map.*

We are now well settled to state the main result of this section. To do so, we need to first go through some technical definitions, all of which are from [16].

Definition 6.15. If we have a triangulated category $\mathscr{T}$ that admits coproducts of small sets of objects, for any regular cardinal α, we define α-localizing subcategories of $\mathscr{T}$ to be triangulated subcategories of $\mathscr{T}$ that are closed under taking fewer than α many coproducts. The α-localizing subcategory of $\mathscr{T}$ generated by $\mathscr{U}$, written $\langle \mathscr{U} \rangle^\alpha$, where $\mathscr{U}$ is a class of objects in $\mathscr{T}$, is the smallest triangulated subcategory of $\mathscr{T}$ closed under taking fewer than α many coproducts containing $\mathscr{U}$.

Of course, if $\alpha = \aleph_0$, then taking fewer than α many coproducts means taking finite coproducts. The following lemma, which provides a nice application of this terminology, is standard knowledge in localizing categories.

Lemma 6.16. *Let $\mathscr{T}$ be a triangulated category that admits arbitrary coproducts. Then α-localizing subcategories are thick if $\alpha > \aleph_0$.*

Proof. Let $\mathscr{U}$ be an α-localizing subcategory of $\mathscr{T}$, where $\alpha > \aleph_0$, and let $X \oplus Y \in \mathscr{U}$. Then, since $(X \oplus Y) \oplus (X \oplus Y) \oplus \cdots$ is isomorphic to

$$X \oplus (Y \oplus X) \oplus (Y \oplus X) \oplus \cdots ,$$

we have a triangle $X \to (X \oplus Y)^{(\mathbb{N})} \to (X \oplus Y)^{(\mathbb{N})} \to \Sigma X$, where Σ is the suspension. Here, $(X \oplus Y)^{(\mathbb{N})} \in \mathscr{U}$ because $\mathscr{U}$ is an α-localizing subcategory of $\mathscr{T}$, where $\alpha > \aleph_0$. Thus, X is in $\mathscr{U}$. □

For any regular cardinal α, there is a process of attaching to a triangulated category $\mathscr{T}$ admitting coproducts of small sets of objects a canonically defined α-localizing subcategory $\mathscr{T}^\alpha$ (see [16, Chap. 1] for details). We are not going to provide the definition of $\mathscr{T}^\alpha$ here. However, if $\mathscr{T}$ is generated by a class of objects $\mathscr{U}$ in our definition which is the strongest notion of generation (in our definition (see Definition 4.4 (d)), $\mathscr{U}$ generates $\mathscr{T}$ if the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$ is $\mathscr{T}$, and in general, we write $\langle \mathscr{U} \rangle$ for the smallest triangulated subcategory of $\mathscr{T}$ containing $\mathscr{U}$), then $\langle \mathscr{U} \rangle^\alpha$, meaning the canonically attached α-localizing subcategory of $\langle \mathscr{U} \rangle$, coincides with the α-localizing subcategory of $\mathscr{T}$ generated by $\mathscr{U}$ as in Definition 6.15 (see [16, Def. 1.12 and Lem. 1.13]). In [16], Neeman uses the generation sign $\langle\rangle$ to mean $\bigcup_\alpha \langle\rangle^\alpha$, but again, our notion of generation is stronger, so we need not worry.

Now assume that, in addition to admitting coproducts of small sets of objects, $\mathscr{T}$ also has small Hom-sets. If, for some regular cardinal α, $\mathscr{T}^\alpha$ is essentially small and $\mathscr{T}^\alpha$ generates $\mathscr{T}$ in Neeman's sense, then $\mathscr{T}$ is said to be *well-generated* (see [16, Def. 1.15]). Now if we take $\mathscr{T} = \mathcal{D}^b(\text{Mod-}RG)$, where G is in $H_n\mathscr{F}$ and admits complete resolutions, then $\mathscr{T}$ satisfies all these properties (due to Theorem 6.14) with $\alpha = \aleph_0$ and $\mathscr{T} = \langle \Lambda_{n-1}(G, \mathscr{F})^\oplus \rangle$ (we are using the symbol $\langle\rangle$ in our sense which is stronger than the sense in which it is used by Neeman, so we are fine). So $\mathscr{T}^\alpha = \langle \Lambda_{n-1}(G, \mathscr{F})^\oplus \rangle^\alpha = \mathscr{T}$ (the last equality follows from the fact that the α-localizing subcategory generated by $\Lambda_{n-1}(G, \mathscr{F})^\oplus$ is $\mathscr{T}$—see Theorem 4.5 (d)), and it follows that $\mathcal{D}^b(\text{Mod-}RG)$ is well-generated, and using [16, Rem. 1.16], we get that $\mathcal{D}^b(\text{Mod-}RG)/\mathcal{D}^b(\text{Proj}(RG))$ is well-generated. This gives us the following theorem.

Theorem 6.17. *Let G be an $H_n\mathscr{F}$-group that admits complete resolutions over a commutative ring R of finite global dimension. Then* $\text{Stab}(RG)$ *is well-generated.*

Remark 6.18. Well-generation is a generalized version of compact generation (see [16, Chap. 1]). So Theorem 6.17 generalizes the analogous result for finite groups, Theorem 6.5.

We seem to be seeing a combination of two classes of groups in the statement of Theorem 6.17. One being $H_n\mathscr{F}$ and the other being groups that admit complete resolutions over R with R being a commutative ring of finite global dimension. It is important to note that these two classes are not disconnected.

Remark 6.19. Combining [22, Conj. A] and [6, Conj. 43.1], we get the conjecture that a group G admits complete resolutions over $\mathbb{Z}$ if and only if it is in $H_1\mathscr{F}$.

One can state an algebraic version of this conjecture: for any commutative ring R of finite global dimension, a group admits complete resolutions over R if and only if the trivial module admits a finite-length resolution by direct sums of permutation modules with finite stabilizers. Conjecture A of [22] does not use the phrase "complete resolutions", but one of the conjectured equivalent statements there is that the Gorenstein projective dimension of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ is finite, which is equivalent to all $\mathbb{Z}G$-modules admitting complete resolutions (the same is true with any commutative ring of finite global dimension; see [9, Thm. 1.7]).

All known examples of groups that admit complete resolutions over the integers are in $H_1\mathscr{F}$. And for any $G \in H_1\mathscr{F}$, all RG-modules admit complete resolutions for any commutative ring R of finite global dimension—see [7, Thm. 1.5] and [9, Thm. 1.7]; alternatively, one can prove this by first proving it over $\mathbb{Z}$ by noting that permutation modules with finite stabilizers are Gorenstein projectives and using [9, Prop. 2.1 and Thm. 1.7].

We end with the following question. The case of finite groups was handled in [2].

Question 6.20. *For any fixed commutative ring R of finite global dimension, classify the localizing subcategories of* $\mathrm{Stab}(RG)$*, where G is an $H_n\mathscr{F}$ group, for some positive integer n, that admits complete resolutions over R.*

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