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Abstract

Let $r \geq 0$ be real number let $\mathbf{K}$ be a complete non-Archimedeanly non-trivially valued field. In the first chapter, we give the definition of a function $f : X \to \mathbf{E}$ on a domain $X \subseteq \mathbf{K}^d$ with values in a $\mathbf{K}$-Banach space $\mathbf{E}$ to be *r-times differentiable* or $\mathcal{C}^r$ at a point $a \in X$. Then we endow the $\mathbf{K}$-vector space of all such $\mathcal{C}^r$-functions with a locally convex topology and examine properties of theirs such as completeness, density of (locally) polynomial functions, closure under composition and, for the dual, under convolution.
For functions on open domains in one variable, we show this definition to equal a handier description through the convergence speed $o(1/|h|^r)$ of the rest term of the Taylor-polynomial at $x + h$ expanded around x up to degree $\lfloor r \rfloor$. Moreover on the special domain $X = \mathbb{Z}_p^d$ we show the $\mathcal{C}^r$-functions $f : \mathbb{Z}_p^d \to \mathbf{K}$ to be characterized by its Mahler coefficients $(a_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d}$ obeying $|a_{\boldsymbol{n}}||\boldsymbol{n}|^r \to 0$ as $|\boldsymbol{n}| \to \infty$, where we put $|\boldsymbol{n}| := n_1 + \cdots + n_d$. Then as a corollary, a characterization of $\mathcal{C}^r$-functions $f : X \to \mathbf{K}$ on open $X \subseteq \mathbb{Q}_p^d$ by partial Taylor-polynomials is obtained.

We turn to the second chapter: Let G be a connected reductive group over a local field $\mathbf{F}$ and P a minimal parabolic subgroup. Let $\mathbf{K}$ be a complete non-Archimedeanly non-trivally valued field of characteristic 0 with valuation ring $\mathbf{o}$. Let $\theta : P \to \mathbf{K}^*$ be an unramified character and denote by $I(\theta) = \operatorname{Ind}_P^G \theta$ the smooth principal series. Let U be an algebraic representation of G (and if U is nontrivial, also assume $\mathbf{K} \supseteq \mathbf{F}$ and G to split). Then $V = I(\theta) \otimes_{\mathbf{K}} U$ is a locally algebraic G-representation, and we let $\widehat{V}$ be the universal $\mathbf{K}$-Banach space representation with a G-invariant norm whereinto V maps continuously with respect to its finest locally convex topology. We will then show that the universal unitary lattice $\mathfrak{L} \subseteq V$, given by the preimage of the unit ball in $\widehat{V}$, is of the form $\mathfrak{L} = \sum_{w \in W} \mathfrak{L}_w$ with W denoting the Weyl group of G and each $\mathfrak{L}_w$ being a cyclic $\mathbf{o}[\bar{P}]$-module which is free as an $\mathbf{o}$-module.

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Introduction

Start with two normed finite dimensional vector spaces V and W over a valued field $\mathbf{K}$. Let $f : U \to W$ some map defined on an open subset $U \subseteq V$. Then f is called *differentiable* or $\mathcal{C}^1$ in the point $a \in U$ if there exists a linear map $D_a : V \to W$ such that for every $\varepsilon > 0$ there is a neighborhood $U_\varepsilon \ni a$ in U with

$$\|f(x+h) - f(x) - D_a \cdot h\| \leq \varepsilon \|h\| \quad \text{for all } x+h, x \in U_\varepsilon.$$

To iterate this differentiability notion, we need a choice of coordinates on the function's domain. We therefore assume $V = \mathbf{K}^d$ and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ be its canonical basis vectors. Then given any two points $x+h, x \in U$ with $h \in \mathbf{K}^{*d}$, we can define $A := f^{]1[}(x+h, h) \in \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V, W)$ by the partial difference quotients

$$A(h_k \cdot \boldsymbol{e}_k) = f(x + h_1 \cdot \boldsymbol{e}_1 + \cdots + h_k \cdot \boldsymbol{e}_k) - f(x + h_1 \cdot \boldsymbol{e}_1 + \cdots + h_{k-1} \cdot \boldsymbol{e}_{k-1}) \quad \text{for } k = 1, \ldots, d.$$

Then this map $f^{]1[} : U^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V, W)$ extends to a continuous function $f^{[1]} : U^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V, W)$ with $U^{[1]} = U \times U$ if and only if f is $\mathcal{C}^1$ at every point of a. (See Remark 1.35.) This function's domain lies again in the $\mathbf{K}$-vector space $V \times V$ inheriting a natural choice of coordinates, its range is in a natural way again a $\mathbf{K}$-vector space, and so we can define f to be *twice continuously differentiable* if

$$f^{]2[} = (f^{[1]})^{]1[} : (X^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, W), W)$$

extends to a continuous function $f^{[2]}$ on all of $X^{[2]} = X^{[1]} \times X^{[1]}$, and so on. This construction can also be carried out to yield a notion of pointwise differentiability.

As our goal is a definition of r-fold differentiability for $r \in \mathbb{R}_{\geq 0}$, we introduce the notion of a $\mathcal{C}^\rho$-point for $\rho \in [0, 1[$ as follows: The mapping f is $\mathcal{C}^\rho$ in the point $a \in U$ if for every $\varepsilon > 0$ there is a neighborhood $U_\varepsilon \ni a$ in U with

$$\|f(x+h) - f(x)\| \leq \varepsilon \|h\|^\rho \quad \text{for all } x+h, x \in U_\varepsilon.$$

Now write $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu \in \mathbb{N}$ and $\rho \in [0, 1[$. Then for f to be a $\mathcal{C}^r$*-function*, we demand its ν-th iterated difference quotient not merely to extend continuously, but $\mathcal{C}^\rho$-wise at all critical limit points.

Then to arrive at our Definition 3.1 of a $\mathcal{C}^r$-point, we notice that a mapping symmetric in two coordinates is partially differentiable in both coordinates if and if only if it is so in one of them. E.g. if $V = \mathbf{K}$ is one-dimensional, we can alternatively write $f^{[1]}(x,y) = [f(x) - f(y)]/(x-y)$ for its first difference quotient. This is a symmetric mapping in both coordinates. If we define a mapping to be twice differentiable if firstly $f^{[1]}$ exists on $U \times U$ and then is again differentiable, we are hence brought down to checking partial differentiability in $f^{[1]}$'s first coordinate, reducing an exponential growth of parameters to a linear one. This observation underlies the definition of iterated differentiability in the sense of [Schikhof, 1984], which we also employ here for our iterated partial difference quotients.

We first show this definition to satisfy a number of properties naively to be expected:

- Given a locally cartesian subset $X \subseteq \mathbf{K}^d$ (see Section 3.1) and a $\mathbf{K}$-Banach space $\mathbf{E}$, the $\mathbf{K}$-vector space of all such $\mathcal{C}^r$-functions $\mathcal{C}^r(X, \mathbf{E})$ can naturally be endowed with a locally convex topology, which is then complete and also a locally convex $\mathbf{K}$-algebra if the range $\mathbf{E}$ is so.

- As a large class of explicit examples, we also find all locally analytic functions to be r-times differentiable for any $r \geq 0$. Then we show all locally polynomial functions of total degree at most ν and consequently all polynomial functions to constitute dense subspaces on a compact domain X. By this density, we can view $\mathcal{D}(X,\mathbf{K}) = \varinjlim \mathcal{D}^r(X,\mathbf{K})$ as the filtered $\mathbf{K}$-vector space of all $\mathbf{K}$-linear forms defined on all arbitrarily often differentiable functions $\mathcal{C}^\infty(X,\mathbf{E})$ extending continuously onto $\mathcal{C}^r(X,\mathbf{E})$ for some $r \geq 0$. When X is moreover a group with $\mathcal{C}^\infty$-multiplication, we can endow $\mathcal{D}(X,\mathbf{K})$ with a convolution product and prove it to be filtered $\mathbf{K}$-algebra.

- Informed by the above interpretation of the ν-th difference quotient as a map with values in $\mathbf{K}$-linear homomorphisms, we will also see that $\mathcal{C}^r$-functions are closed under composition if $r \geq 1$. We note that thereby and since a $\mathcal{C}^r$-function is defined pointwise, it is a local notion, so that put together we arrive at a reasonable notion of a $\mathcal{C}^r$-manifold.

In its most naive way presented above, the notion of a $\mathcal{C}^r$-function is hardly handy, and the first reduction by the symmetry of these difference quotients can be taken further. We want to give a guideline on the order in which we obtain these simplification results:

- On domains $X \subseteq \mathbf{K}$ in one variable with locally sufficiently many points - such as open ones - the symmetry properties of these iterated difference quotients are strong enough to reduce the question of iterated differentiability to a more convenient Taylor polynomial criterion in which r-fold differentiability can be checked by only one additional variable.

- Let $\mathbf{K} \supseteq \mathbb{Q}_p$ as a valued field with valuation ring $\mathbf{o}$. There is a distinguished (see Subsection 2.3) orthogonal basis of the continuous $\mathbf{K}$-valued functions $\mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$ relating to the domain's topological group structure, the so called *Mahler polynomials* $\{\binom{*}{\boldsymbol{i}} : \boldsymbol{i} \in \mathbb{N}^d\}$. Denoting by $c_0(\mathbb{N},\mathbf{K})$ all zero sequences in $\mathbf{K}$, this means that we have an isomorphism of $\mathbf{K}$-Banach spaces $c_0(\mathbb{N},\mathbf{K}) \xrightarrow{\sim} \mathcal{C}^0(\mathbb{Z}_p,\mathbf{K})$ with $f \in \mathcal{C}^0(\mathbb{Z}_p^d,\mathbf{K})$ corresponding to its *Mahler coefficients* $(a_{\boldsymbol{n}})_{\boldsymbol{n}\in\mathbb{N}^d}$. We want to describe the topological $\mathbf{K}$-vector subspace $\mathcal{C}^r(\mathbb{Z}_p^d,\mathbf{K}) \hookrightarrow$

$\mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$ under this isomorphism:
We will firstly prove a function $f : \mathbb{Z}_p \to \mathbf{K}$ to be $\mathcal{C}^\rho$ if and only if $|a_n|n^\rho \to 0$ as $n \to \infty$ and infer a mapping $f : \mathbb{Z}_p^d \to \mathbf{K}$ to be a $\mathcal{C}^\rho$-function if and only if $|a_{\boldsymbol{n}}||\boldsymbol{n}|^\rho \to 0$ as $|\boldsymbol{n}| \to \infty$ with $|\boldsymbol{n}| := n_1 + \cdots + n_d$. Applying this to the ν-th difference quotient of f, we see f to be a $\mathcal{C}^r$-function if and only if $|a_n|n^r \to 0$ as $n \to \infty$. Then we will describe $\mathcal{C}^r$-functions $f : \mathbb{Z}_p^d \to \mathbf{K}$ in several variables as intersection of $\mathcal{C}^{\boldsymbol{r}}$-function spaces for $\boldsymbol{r} \in \mathbb{R}_{\geq 0}^d$, resembling tensor products of $\mathcal{C}^r$-functions in one variable. This shows the orthogonality of the multivariate Mahler polynomials. By the density of the subspace of all polynomial functions inside $\mathcal{C}^r(X, \mathbf{K})$ on a compact domain $X \subseteq \mathbf{K}^d$, we see that the $\mathbf{K}$-linear span of the multivariate Mahler polynomials in many variables is dense in $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. Put together these form an orthogonal base, and so we can describe $\mathcal{C}^r$-functions $f : \mathbb{Z}_p^d \to \mathbf{K}$ by its Mahler coefficients obeying $|a_{\boldsymbol{n}}||\boldsymbol{n}|^r \to 0$ as $|\boldsymbol{n}| \to \infty$.

- This last condition is equivalent to $|a_{\boldsymbol{n}}|n_k^r$ as $|\boldsymbol{n}| \to \infty$ for $k = 1, \ldots, d$, each condition describing the topological tensor product of $\mathcal{C}^r$-functions in the k-th variable with continuous functions in the other ones. Then by the Taylor polynomial description in one variable above, we infer an equivalent description of $\mathcal{C}^r$-functions $f : X \to \mathbf{K}$ on open $X \subseteq \mathbb{Q}_p^d$ through partial Taylor polynomials.

In the context of the existing literature, the notion of a $\mathcal{C}^\nu$-function in many variables for an integer $\nu \geq 0$ was also defined exemplarily in [Schikhof, 1984, Section 84] and studied more generally by Dr Stany de Smedt in his thesis. Also more recently another notion of $\mathcal{C}^\nu$-function by Bertram, Glöckner and Neeb was given, which was shown to coincide with the one by Schikhof and de Smedt in [Glöckner, 2007]. Our notion of a $\mathcal{C}^r$-function therefore differs from the one so far discussed in the literature in being defined pointwise and by allowing for a real number $r \geq 0$.
To give an idea of the cited work's already achieved results, we remark that the results on the characterization by Mahler coefficients and Taylor polynomials presented here generalize those already obtained in in the classic book [Schikhof, 1984] for one variable and an integral order of differentiability $\nu \geq 0$. We also note that it was already shown in [Bertram et al., 2004] that, using their equivalent notion of $\mathcal{C}^\nu$-function, these functions are directly seen to be closed under composition, and moreover these to contain all locally analytic functions.

We finally remark that this work was inspired by the aim of generalizing the definition of a $\mathcal{C}^r$-function on the domain $\mathbb{Z}_p$ as given in [Berger and Breuil, 2010, Section 4] by its Mahler coefficients (see also Example 3.67) to the one of an r-times differentiable function living on a finite dimensional manifold over a complete non-Archimedeanly non-trivially valued field $\mathbf{K}$.

0 Prerequisites

Throughout this paper $\mathbf{K}$ will denote a complete non-Archimedeanly valued field whose valuation v is nontrivial. If we fix a positive real constant $c_v < 1$, we obtain a norm $|x| := c_v^{v(x)}$. Define $\mathbf{o}_{<\lambda} = \{x \in \mathbf{K} : |x| < \lambda\}$ respectively $\mathbf{o}_{\leq\lambda} = \{x \in \mathbf{K} : |x| \leq \lambda\}$ for $\lambda \in \mathbb{R}_{\geq 0}$; put $\mathbf{o} = \mathbf{o}_{\leq 1}$ and $\mathbf{k} = \mathbf{o}/\mathbf{o}_{<1}$. If the residue field $\mathbf{k}$ of $\mathbf{K}$ has positive characteristic p, we will always put $c_v = p^{-1}$. Then $v(p) > 0$ and if this value is finite, we will assume $v(p) = 1$.

Cartesian products

Let $X = X_1 \times \cdots \times X_d$ be a finite cartesian product of sets. Then we will call a subset $A \subseteq X$ **cartesian** if $A = A_1 \times \cdots \times A_d$ with $A_1 \subseteq X_1, \ldots, A_d \subseteq X_d$.

Notation. Let I be an index set and let X_i for all $i \in I$ be a set.

1. Let $k \in I$. We denote the projection onto the k-th component by $\mathrm{p}_k : X \to X_k$.
2. Let I be finite and assume that $X_i \ni 0, 1$ for all $i \in I$. Then for $k \in I$, we let $\boldsymbol{e}_k = (0, \ldots, 1, \ldots, 0) \in \prod_{i \in I} X_i$ denote the tuple whose only nonzero entry is a 1 at the k-th place.

Let $A \subseteq X^I$ for a set X and an index set I. We will denote by $\triangle A$ the diagonal subset

$$\triangle A = \{(x, \ldots, x) \in A : x \in X\}$$

and by $\triangledown A$ its subset of tuples with pairwise distinct coordinates

$$\triangledown A = \{(x_i)_{i \in I} \in A : x_{i'} \neq x_{i''} \text{ if } i', i'' \in I \text{ distinct } \}.$$

If $d = 1$, then $\triangle A = \triangledown A = A$.

Metric and normed spaces

We will throughout assume all seminorms to be non-Archimedean. All normed respectively metric spaces are implicitly assumed to be endowed with a norm $\|\cdot\|$ respectively metric d, through whose arguments it will be clear whereon it is defined. Every normed space gives rise to a metric $\mathrm{d}(x, y) := \|x - y\|$.
Let the set $X = X_1 \times \cdots \times X_d$ be the cartesian product of normed respectively metric spaces $X_1, \ldots, X_d$ with correspondingly indexed norms respectively metrics. Then we endow X with the structure of a normed respectively metric space through the norm

$$\|x\| = \max\{\|x_1\|_1, \ldots, \|x_d\|_d\}$$

respectively metric

$$\mathrm{d}(x, y) = \max\{\mathrm{d}_1(x_1, y_1), \ldots, \mathrm{d}_d(x_d, y_d)\}.$$

We will then call X a **cartesian** normed respectively metric space.

We extend the addition on $\mathbb{R}_{\geq 0}$ to $\mathbb{R}_{\geq 0} \cup \{\infty\}$ by defining $x + \infty = \infty + x = \infty$ for all $x \in \mathbb{R}_{\geq 0} \cup \{\infty\}$, the multiplication on $\mathbb{R}_{>0}$ by $c \cdot \infty = \infty \cdot c = \infty$ for all $c \in \mathbb{R}_{>0} \cup \{\infty\}$ and the total ordering on $\mathbb{R}_{\geq 0}$ by setting $\infty \geq x$ for all $x \in \mathbb{R}_{\geq 0} \cup \{\infty\}$.
If X is an arbitrary set and Y a normed space, we define a *quasi-norm* $\|\cdot\|_{\sup}$ (a map with image in $\mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfying all axioms of a norm) on the mappings $f : X \to Y$ by

$$\|f\|_{\sup} = \begin{cases} \sup_{x \in X} \|f(x)\|, & \text{if this supremum exists,} \\ \infty, & \text{otherwise.} \end{cases}$$

For a subset $A \subseteq X$, we define $\|f\|_A := \|f_{|A}\|_{\sup}$. If moreover X respectively A is a compact topological space, $\|\cdot\|_{\sup}$ respectively $\|\cdot\|_A$ define a norm respectively seminorm on all continuous functions $f : X \to Y$.

A **K-Banach algebra** will be a **K**-Banach space which is a **K**-algebra whose multiplication is continuous.

Notation. Let X be a metric space.

- If S is any non-empty bounded subset of X, then we will denote by $\delta\, S$ its **diameter** defined by $\delta\, S = \sup_{x,y \in S} \mathrm{d}(x, y)$.
- For $\varepsilon > 0$ and $x \in X$, we denote by $\mathrm{B}_{\leq\varepsilon}(x) = \{y \in X : \mathrm{d}(x, y) \leq \varepsilon\}$ the **"closed" ball around x of radius ε in X**.

Notational conventions

Notation. We will adopt the following conventions:

- We denote by $\mathbb{N} = \{0, 1, 2, \dots\}$ the set of nonnegative integers. Then small Latin letters i, j, k, l, m and n will usually denote nonnegative integers if not explicitly mentioned otherwise.
- We define the **integral part** $\lfloor r \rfloor$ of a nonnegative real number r by $\lfloor r \rfloor := \max\{n \in \mathbb{N} : n \leq r\} \in \mathbb{N}$ and its **fractional** one by $\{r\} := r - \lfloor r \rfloor \in [0, 1[$.
- We might abbreviate $\min\{a, b\}$ respectively $\max\{a, b\}$ for two real numbers a and b by the associative logical conjunction respectively disjunction operator $a \wedge b$ respectively $a \vee b$.

We will also adopt the principle of any sum respectively product running over an empty index set being equal to 0 respectively 1.

1 Apparatus

1.1 Locally convex $\mathbb{K}$-vector spaces

Let V be a $\mathbf{K}$-vector space together with a family of seminorms $\{q_i\}$ with the index i running through an arbitrary index set I. The **locally convex topology** on V is then defined as the coarsest translationally closed one making all these seminorms q_i continuous. To make this family directed with respect to their natural partial order of pointwise comparison, we can replace this family through all the seminorms $q_F := \max_{i \in F} q_i$ for the finite subsets $F \subseteq I$. Since a pointwise greater seminorm induces a finer topology, this family $\{q_F\}$ induces the same topology as $\{q_i\}$ and we will in the following always tacitly assume the defining family of seminorms of a locally convex topology to be directed. We call a topological $\mathbf{K}$-vector space whose topology is locally convex a **locally convex $\mathbf{K}$-vector space**. A **locally convex $\mathbf{K}$-algebra** is a locally convex $\mathbf{K}$-vector space which is a $\mathbf{K}$-algebra whose multiplication is continuous.

A net (f_λ) will be called a **Cauchy net** if for every $\varepsilon > 0$ and any seminorm q_i there is an index λ_0 such that $q_i(f_\mu - f_\nu) \leq \varepsilon$ for any $\mu, \nu \geq \lambda_0$. A net (f_λ) will be said to **converge to** f if for any $\varepsilon > 0$ and seminorm q_i there is an index λ_0 such that $q_i(f_\lambda - f) \leq \varepsilon$ for all $\lambda \geq \lambda_0$. Then V will be said to be **complete** if every Cauchy net in V converges.

If V is a locally convex $\mathbf{K}$-vector space whose topology is defined by a family of seminorms $\{q_i\}$, the induced family of semimetrics d_i will be a *gauge* of the associated *uniform space*, its base of *entourages* given by $d_i^{-1}[0, \varepsilon[$ for $i \in I$ and some $\varepsilon > 0$. We note that any closed subset of a complete subset is also complete. Furthermore, let $V := \prod_{h \in H} V_h$ be the product of a family of locally convex $\mathbf{K}$-vector spaces with seminorms $\{q_{h,i}\}$. Its locally convex topology is induced by the family of seminorms $\{q_{h,i} \circ \pi_h\}$ for the projections $\pi_h : V \twoheadrightarrow V_h$. We point out that V is complete if and only if all its components V_h are so. For a proof of these facts, we confer the reader to [Kelley, 1975, Chapter 6].

Let X be a Hausdorff topological space, $\mathfrak{c}$ a family of subsets in X and Y a complete metric space. Denote by $\mathcal{C}^0_{|\mathfrak{c}}(X, Y)$ the set of all functions $f : X \to Y$ which are continuous on each $C \in \mathfrak{c}$. We equip $\mathcal{C}^0_{|\mathfrak{c}}(X, Y)$ with **the uniformity of $\mathfrak{c}$-convergence**: Its base is given by the entourages $d_C^{-1}[0, \varepsilon[$ with $d_C(f, g) := \max_{x \in C} \mathrm{d}(f(x), g(x))$ for every $C \in \mathfrak{c}$ and $\varepsilon > 0$. By [Kelley, 1975, Theorem 7.10(d)], the uniform space $\mathcal{C}^0_{|\mathfrak{c}}(X, Y)$ is complete. Then X will be called **$\mathfrak{c}$-generated** if a subset A of X is closed as soon as $A \cap C$ is closed in C for all subsets $C \in \mathfrak{c}$.

Proposition 1.1. *Let X be a Hausdorff topological space and Y a complete metric space. Let $\mathfrak{c}$ be a family of compact subsets in X. If X is $\mathfrak{c}$-generated, then the set $\mathcal{C}^0(X, Y) := \mathcal{C}^0_{|\{X\}}(X, Y)$ will be complete with respect to the uniformity of $\mathfrak{c}$-convergence.*

Proof. It suffices to prove $\mathcal{C}^0(X, Y) = \mathcal{C}^0_{|\mathfrak{c}}(X, Y)$ as uniform spaces: Note that a topological space is $\mathfrak{c}$-generated if and only if it has the final topology with respect to all inclusions $C \hookrightarrow X$ of subspaces $C \in \mathfrak{c}$. So a mapping $f : X \to Y$ is continuous if and only if all its restrictions

$f_{|C}$ to the subspaces $C \in \mathfrak{c}$ are so. Hence the space $\mathcal{C}^0_{|\mathfrak{c}}(X,Y)$ coincides with $\mathcal{C}^0(X,Y)$ as sets. As their families of semimetrics coincide, they are also equal as uniform spaces and hence $\mathcal{C}^0(X,Y)$ is complete. ■

Let X be a topological space and Y a normed $\mathbf{K}$-vector-space. For any compact subset C in X, we define **the uniform seminorm of the restriction to** C on the space of continuous functions by $\|\cdot\|_C : \mathcal{C}^0(X,Y) \to \mathbb{R}_{\geq 0}$. The induced locally convex topology is called the **topology of uniform convergence on compact sets**, for short the **topology of compact convergence**.

A function on a metric space is continuous if and only if it is sequentially continuous. In particular continuity can be tested on all compact subsets, as seen next.

Lemma 1.2. *A Hausdorff topological space X will be called **compactly generated** if X is $\mathfrak{c}$-generated for $\mathfrak{c} = \{$compacta in $X\}$. We will say that X is **sequential** if a set A in X is closed as soon as the limits of all convergent sequences in A remain therein. Then every sequential space is compactly generated.*

Proof. Let X be a sequential space and $B \subseteq X$ a subset which is not closed. We have to show that its intersection with some compact subset $C \subseteq X$ cannot be so, either. Now X being sequential, there is a convergent sequence (c_n) in B such that its limit c does not lie therein. Then $C := \{c_n\} \cup \{c\}$ is compact as any open neighborhood of c contains all but finitely many c_n. But its intersection with B lacks c and therefore cannot be closed, q.e.d. ■

Corollary 1.3. *If X is a metric space and Y a $\mathbf{K}$-Banach space, then the $\mathbf{K}$-vector space $\mathcal{C}^0(X,Y)$ will be complete with respect to the topology of compact convergence.*

Proof. Since X is metric, it is in particular sequential. By Lemma 1.2 it is compactly generated. The uniformity of $\mathcal{C}^0(X,Y)$ is the uniformity of uniform convergence on $\mathfrak{c}$-subsets with $\mathfrak{c}$ being the family of compact subsets. By Proposition 1.1, the uniform space $\mathcal{C}^0(X,Y)$ is complete. ■

1.2 $\mathcal{C}^\rho$-functions for $\rho \in [0,1[$

Assumption. Throughout this subsection, we will fix a real number $\rho \in [0,1[$.

Definition of $\mathcal{C}^\rho$-functions

We begin generally. For a point a in a metric space X, we define what it means for a function f to *oscillate negligibly versus the distance's ρ-th power at a*. This is when a will be called a **$\mathcal{C}^\rho$-point of** f.

Definition. Let X be a metric space, Y a complete metric space, $f : A \to Y$ a mapping defined on a subset $A \subseteq X$ and a some point in X; we will say that f is $\mathcal{C}^\rho$ **at** a, if for every $\varepsilon > 0$ there exists a neighborhood $U \ni a$ in X such that

$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x,y)^\rho \quad \text{for all } x, y \in U \cap A.$$

Then f will be a **$\mathcal{C}^\rho$-function** if f is $\mathcal{C}^\rho$ at all points $a \in A$, where we note that this notion is independent of the ambient space X. We will denote the set of all $\mathcal{C}^\rho$-functions $f : A \to Y$ by $\mathcal{C}^\rho(A, Y)$.

We emphasize that we also defined what it means for a point $a \in X$ not in the function's domain A to be $\mathcal{C}^\rho$. If there is a neighborhood of a disjoint to A, then this condition will be void. The interesting case occurs whenever a is a boundary point of A in X.

Remark 1.4. Keeping the notations above, let us assume that $a \in X$ is a boundary point in $\partial A = \bar{A} - A \subseteq X$. Then by completeness of Y, a function f is $\mathcal{C}^0$ at a if and only if there is a *unique* limit $f(a) \in Y$ such that for every $\varepsilon > 0$, there exists a neighborhood $U \ni a$ in X such that

$$\mathrm{d}(f(x), f(a)) \leq \varepsilon \quad \text{for all } x \in U \cap A.$$

If even $a \in A$, then a function $f : A \to Y$ will be $\mathcal{C}^0$ at a if and only if it will be continuous at a.

The next Proposition 1.6 tells us that we can at least assume all functions to be defined on their set of $\mathcal{C}^\rho$-points in the boundary of A in X. Note that in case $\rho = 0$, the above definition is also meaningful whenever X is merely a topological space.

Lemma 1.5. *Let X be a topological space, (Y, d) a complete metric space and $f : A \to Y$ a continuous mapping defined on a subset $A \subseteq X$. Let $A \subseteq B \subseteq \bar{A} \subseteq X$ denote the $\mathcal{C}^0$-points of f. Then f extends uniquely to a continuous mapping $F : B \to Y$.*

Proof. This is a well-known fact in general topology: For every $x \in B$ denote by $\mathcal{B}(x)$ the system of neighborhoods of x in X and consider the family $\{\overline{f(A \cap U)} : U \in \mathcal{B}(x)\}$ of closed subsets of Y. Because $\overline{f(A \cap U)} \supseteq f(\overline{A \cap U})$ by continuity and $\overline{A \cap U} \ni x$ as $U \ni x$ is open, the latter family's finite intersections are non-void. If x is a $\mathcal{C}^0$-point of f, this family will contain for every $\varepsilon > 0$ a set of diameter $\delta \leq \varepsilon$, so that - by [Kelley, 1975, Theorem 6.23] - the intersection $\cap_{U \in \mathcal{B}(x)} \overline{f(A \cap U)}$ is non-empty. Since its diameter equals zero, the intersection consists of a single point $F(x)$. Surely $F(x) = f(x)$ whenever $x \in A$, as all $f(A \cap U) \ni f(x)$.
Extending f by sending $x \in B$ to the point $F(x)$, we define a mapping F of B to Y; it remains to prove that F is continuous. Fix $\varepsilon > 0$ and some $b \in B$. Since b is a $\mathcal{C}^0$-point, there exists a $U \in \mathcal{B}(b)$ such that by continuity of the distance even $\delta\, \overline{f(A \cap U)} \leq \varepsilon$. For every $x' \in B \cap U$, we have $U \in \mathcal{B}(x')$ and thus $F(x') \in \overline{f(A \cap U)}$; since $F(b) \in \overline{f(A \cap U)}$ as well, we have $\mathrm{d}(F(b), F(x')) \leq \delta\, \overline{f(A \cap U)} \leq \varepsilon$, which proves that F is continuous at b. ∎

Proposition 1.6. *Let X be a metric space, Y a complete metric space and $f : A \to Y$ a $\mathcal{C}^\rho$-function defined on subset $A \subseteq X$. Let $A \subseteq B \subseteq \bar{A} \subseteq X$ denote the $\mathcal{C}^\rho$-points of f. Then f extends uniquely to a $\mathcal{C}^\rho$-function $F : B \to Y$.*

Proof. Through the foregoing Lemma 1.5, we know that f extends to a continuous function $F : B \to Y$. We want to show that F is even $\mathcal{C}^\rho$ there. For this, choose $a \in B$ and fix $\varepsilon > 0$. As f is $\mathcal{C}^\rho$ at a, we find a neighborhood $U \ni a$ in X such that

$$\mathrm{d}(F(x), F(y)) \leq \tilde{\varepsilon} \cdot \mathrm{d}(x, y)^\rho \quad \text{for all } x, y \in A \cap U, \qquad (*)$$

with $\tilde{\varepsilon} := \varepsilon/C^2$ and $C := 1 + 2^\rho \geq 1$. It remains to show that this inequality also holds in case x or y in $(B - A) \cap U$ with $\tilde{\varepsilon}$ replaced by ε. We firstly assume that $x \notin A$, but y so. Then x lies in the boundary of A and we can find $x' \in A$ so close to x that $\mathrm{d}(F(x), F(x')) \leq \tilde{\varepsilon}_{x,y}$ with $\tilde{\varepsilon}_{x,y} := \tilde{\varepsilon} \cdot \mathrm{d}(x,y)^\rho$ by the continuity of F. Convergency towards x does not harm, so we may as well assume that $x' \in U$ and $\mathrm{d}(x', x) \leq \mathrm{d}(x,y)$.

It follows

$$\begin{aligned}
\mathrm{d}(F(x), F(y)) &\leq \mathrm{d}(F(x), F(x')) + \mathrm{d}(F(x'), F(y)) \\
&\leq \tilde{\varepsilon}_{x,y} + \tilde{\varepsilon} \cdot \mathrm{d}(x', y)^\rho \\
&\leq \tilde{\varepsilon}_{x,y} + \tilde{\varepsilon} \cdot (\mathrm{d}(x', x) + \mathrm{d}(x,y))^\rho \\
&\leq C\, \tilde{\varepsilon} \cdot \mathrm{d}(x,y)^\rho \leq \varepsilon \cdot d(x,y)^\rho.
\end{aligned}$$

By symmetry, this shows that Inequality $(*)$ likewise holds in case that not both of x and y lie in $(B - A) \cap U$.

If x and y lie in $(B - A) \cap U$, we will reduce to the first case by inserting an element $z \in A \cap U$ in between: Since x is in the boundary of A, we find z such that $\mathrm{d}(x,z) \leq \delta_{x,y} := \mathrm{d}(x,y)$. Thence by the cases already considered,

$$\begin{aligned}
\mathrm{d}(F(x), F(y)) &\leq \mathrm{d}(F(x), F(z)) + \mathrm{d}(F(z), F(y)) \\
&\leq C\tilde{\varepsilon} \cdot \mathrm{d}(x,z)^\rho + C\tilde{\varepsilon} \cdot \mathrm{d}(y,z)^\rho \\
&\leq C\tilde{\varepsilon} \cdot \mathrm{d}(x,z)^\rho + C\tilde{\varepsilon} \cdot (\mathrm{d}(x,z) + \mathrm{d}(x,y))^\rho \\
&\leq C^2\, \tilde{\varepsilon} \cdot \mathrm{d}(x,y)^\rho \leq \varepsilon \cdot \mathrm{d}(x,y)^\rho.
\end{aligned}$$

This completes the proof of the remaining case, so Inequality $(*)$ holds for all $x, y \in B \cap U$ which was left to show. ■

Properties of the space of $\mathcal{C}^\rho$-functions

Assumption. We will from now let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space.

Definition. Let X and Y be metric spaces, $f : X \to Y$ a mapping on X and a some point in X; we will say that f **is** $\mathcal{C}^{\mathrm{lip}}$ or is **locally Lipschitzian** at a if there exists a constant $C > 0$ and a neighborhood $U \ni a$ such that

$$\mathrm{d}(f(x), f(y)) \leq C \cdot \mathrm{d}(x,y) \text{ for all } x, y \in U.$$

Then f will be a $\mathcal{C}^{\mathrm{lip}}$**-function** or **a locally Lipschitzian function** if f is $\mathcal{C}^{\mathrm{lip}}$ at all points $a \in X$. We will denote the set of all $\mathcal{C}^{\mathrm{lip}}$-functions $f : X \to Y$ by $\mathcal{C}^{\mathrm{lip}}(X, Y)$.

Definition. Let X and Y be metric spaces and let $f : X \to Y$ be a mapping on X. We define the function $|f^{]1[}| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{]1[}|(x,y) = \mathrm{d}(f(x), f(y))/\mathrm{d}(x,y).$$

Then for every function $f \in \mathcal{C}^{\text{lip}}(X, \mathbf{E})$ the mapping $|f^{]1[}|$ is locally bounded and hence also bounded on compacta. In addition f is also continuous. We can therefore establish:

Definition. For every compact subset $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathcal{C}^{\text{lip}},C}$ on $\mathcal{C}^{\text{lip}}(X, \mathbf{E})$ by

$$\|f\|_{\mathcal{C}^{\text{lip}},C} = \|f_{|C}\|_{\sup} \vee \||(f_{|C})^{]1[}|\|_{\sup}.$$

We equip $\mathcal{C}^{\text{lip}}(X, \mathbf{E})$ with the locally convex topology given by the family of seminorms $\{\|\cdot\|_{\mathcal{C}^{\text{lip}},C} : C \subseteq X \text{ compact}\}$.

Proposition 1.7. *(i) Let X, Y and Z be metric spaces. Then the $\mathcal{C}^\rho$-functions are closed under composition with locally Lipschitzian functions, i.e. if $g : X \to Y$ and $f : Y \to Z$, then if one of these functions will be $\mathcal{C}^\rho$ and the other one $\mathcal{C}^{\text{lip}}$, then $f \circ g \in \mathcal{C}^\rho(X, Z)$.*

(ii) If X is a metric space and $\mathbf{E}$ a $\mathbf{K}$-Banach algebra, then $\mathcal{C}^\rho(X, \mathbf{E})$ will be a $\mathbf{K}$-algebra.

(iii) The space of $\mathcal{C}^\rho$-functions is closed under direct sums and tensor products: Let X, Y be metric spaces and $\mathbf{E}$ a normed $\mathbf{K}$-algebra. If $f \in \mathcal{C}^\rho(X, \mathbf{E})$ and $g \in \mathcal{C}^\rho(Y, \mathbf{E})$, then $f \oplus g(x, y) := f(x) + g(y)$ and $f \odot g(x, y) := f(x) \cdot g(y)$ will lie in $\mathcal{C}^\rho(X \times Y \to \mathbf{E})$.

Proof. Ad (i): Fix a point $a \in X$ and $\varepsilon > 0$. Let $g : X \to Y$ and $f : Y \to Z$ be functions such that one is $\mathcal{C}^\rho$ and the other $\mathcal{C}^{\text{lip}}$ for a constant $C > 0$ in a neighborhood of a respectively $g(a)$. We see that

$$\mathrm{d}(f \circ g(x), f \circ g(y))$$

is at most either

$$C \cdot \mathrm{d}(g(x), g(y)) \leq C\varepsilon \cdot \mathrm{d}(x, y)^\rho \quad \text{or} \quad \varepsilon \cdot \mathrm{d}(g(x), g(y))^\rho \leq \varepsilon C \cdot \mathrm{d}(x, y)^\rho$$

for x, y in an appropriate neighborhood U of a. In other words $f \circ g$ is $\mathcal{C}^\rho$ at a.
Ad (ii): Only the fact that $\mathcal{C}^\rho(X, \mathbf{E})$ is closed under products requires attention: So let $f, g \in \mathcal{C}^\rho(X, \mathbf{E})$. Fix $\varepsilon > 0$. Because the multiplication in $\mathbf{E}$ is continuous, there is a uniform constant $M \geq 1$ such that $\|xy\| \leq M\|x\|\|y\|$. We compute

$$\begin{aligned}\|fg(x) - fg(y)\| \leq &\|fg(x) - f(x)g(y)\| \vee \|f(x)g(y) - fg(y)\| \\ \leq &M \cdot \|f(x)\|\|g(x) - g(y)\| \vee M \cdot \|g(y)\|\|f(x) - f(y)\| \\ \leq &C \cdot (\|g(x) - g(y)\| \vee \|f(x) - f(y)\|) \\ \leq &\varepsilon \cdot \mathrm{d}(x, y)^\rho\end{aligned}$$

for x, y in a sufficiently small neighborhood $U \ni a$, as f and g are in particular continuous by Lemma 1.5 and therefore bounded in a neighborhood of the point a.
Ad (iii): This follows from the foregoing statements (i) and (ii) as

$$f \circledast g = (f \circ \mathrm{p}_x) * (g \circ \mathrm{p}_y),$$

where p_x respectively p_y denotes the projection of $X \times Y$ onto X respectively Y and '$*$' denotes either '$+$' or '$\cdot$'. ■

More generally we can prove $f \circ g$ to be a $\mathcal{C}^{\rho\eta}$-function if f is $\mathcal{C}^\rho$ and g is $\mathcal{C}^\eta$, but nothing more: The following example shows that the above Proposition 1.7(i) fails in case that $f, g \in \mathcal{C}^\rho(X, Y)$, but neither one is locally Lipschitzian:

Example. Denote by $\overline{\mathbb{Q}}_p$ the algebraic closure of $\mathbb{Q}_p$ with normalized valuation v, i.e. $v(p) = 1$. Let $0 < \rho < 1$ and choose $\eta > 0$ such that $\rho < \eta$, but $\eta^2 < \rho$. Let $\alpha, \beta \in \overline{\mathbb{Q}}_p$ such that $v(\alpha) = \eta$ and $v(\beta) = \eta^2$. Let $\chi_\alpha : p^{\mathbb{Z}} \to \alpha^{\mathbb{Z}}$ respectively $\chi_\beta : \alpha^{\mathbb{Z}} \to \beta^{\mathbb{Z}}$ be the group homomorphisms defined by sending p to α respectively α to β.
We endow $X := p^{\mathbb{Z}} \cup \{0\}, Y := \alpha^{\mathbb{Z}} \cup \{0\}$ and $Z := \beta^{\mathbb{Z}} \cup \{0\}$ with the subspace metric of $\overline{\mathbb{Q}}_p$. We extend χ_α to a mapping $f : X \to Y$ by putting $f(0) = 0$ and χ_β to a mapping $g : Y \to Z$ by putting $g(0) = 0$. Then we show f and g to be $\mathcal{C}^\rho$-functions, but $g \circ f$ not so.

Proof. Firstly, observe that $f(x) = f(y)$ if $v(x - y) > v(x)$ implies $v(f(x) - f(y)) \geq \eta v(x - y)$ for any $x, y \in \mathbb{Q}_p$. We quickly check that $f \in \mathcal{C}^\rho(X, Y)$: As $f_{|p^{\mathbb{Z}}}$ is locally constant, it suffices to check this at 0. Let $\theta := \eta - \rho > 0$. Fix $C \geq 0$ and let $U := \{x \in X : v(x) \geq \delta\} \ni 0$ open in X with $\delta \geq C/\theta$. For $x, y \in U$, we have

$$v(f(x) - f(y)) \geq \eta v(x - y) = \rho(v(x - y)) + \theta(v(x - y)) \geq \rho(v(x - y)) + C.$$

The proof of $g \in \mathcal{C}^\rho(Y, Z)$ is analogous.

To show that $g \circ f$ is not a $\mathcal{C}^\rho$-function, note that $v(f(x)) = \eta v(x)$ on X and $v(g(x)) = \eta v(x)$ on Y. If therefore $x, y \in X$ with $v(x) < v(y)$, then $v(g \circ f(x)) = \eta^2 v(x) < \eta^2 v(y) = v(g \circ f(y)))$. Thus

$$v(g \circ f(x) - g \circ f(y)) = \eta^2 v(x).$$

Now let $C := 0$ and U be a neighborhood of 0. Let $\theta := \eta^2 - \rho < 0$. Then there exists a point $x \in U$ such that $\theta v(x) < C$ and another one, y say, such that $v(x) < v(y)$. Then

$$v(g \circ f(x) - g \circ f(y)) = \eta^2 v(x) = (\rho + \theta) v(x) = \rho v(x - y) + \theta v(x) < \rho v(x - y) + C.$$

I.e. $g \circ f$ is not $\mathcal{C}^\rho$ at 0. ■

The locally convex topology on $\mathcal{C}^\rho$-functions

Definition. Let X be a metric space and $f : X \to \mathbf{E}$ a mapping thereon. We define $|f^{]\rho[}| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\rho[}|(x, y) = \frac{\|f(x) - f(y)\|}{\mathrm{d}(x, y)^\rho}.$$

Then the mapping $f : X \to E$ is $\mathcal{C}^\rho$ if and only if the function $|f^{]\rho[}|$ extends to a continuous function $|f^{[\rho]}| : X \times X \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle X \times X$. Therefore the following definition is meaningful.

Definition. For every compact $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathcal{C}^\rho, C}$ on $\mathcal{C}^\rho(X, \mathbf{E})$ by

$$\|f\|_{\mathcal{C}^\rho, C} = \|f_{|C}\|_{\sup} \vee \||(f_{|C})^{[\rho]}|\|_{\sup}.$$

We equip the $\mathbf{K}$-vector space $\mathcal{C}^\rho(X, \mathbf{E})$ with the locally convex topology given by the set of seminorms $\{\|\cdot\|_{\mathcal{C}^\rho,C} : C \subseteq X \text{ compact}\}$.
If X itself is compact, then we will turn $\mathcal{C}^\rho(X, \mathbf{E})$ into a normed $\mathbf{K}$-vector space by endowing it with the norm $\|\cdot\|_{\mathcal{C}^\rho} := \|\cdot\|_{\mathcal{C}^\rho,X}$.

Remark 1.8. The locally convex $\mathbf{K}$-vector space $\mathcal{C}^\rho(X, \mathbf{E})$ is the initial locally convex $\mathbf{K}$-vector space with respect to all restriction mappings

$$\begin{aligned} \mathcal{C}^\rho(X, \mathbf{E}) &\to \mathcal{C}^\rho(C, \mathbf{E}), \\ f &\mapsto f_{|C}, \end{aligned}$$

with C running through the family of all compact $C \subseteq X$.

Proof. Define $|f^{[\rho]}| : X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{[\rho]}|(x, y) = \begin{cases} |f^{]\rho[}|(x, y), & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

By (the comment before) Lemma 1.2, the function $|f^{[\rho]}| : X \times X \to \mathbb{R}_{\geq 0}$ is continuous as soon as its restrictions to all compacta $\tilde{C} \subseteq X \times X$ are so. Given such compact $\tilde{C} \subseteq X \times X$, we find compact $C \subseteq X$ with $\tilde{C} \subseteq C \times C$, namely $C := \mathrm{p}_1 \tilde{C} \cup \mathrm{p}_2 \tilde{C}$. Therefore $|f^{[\rho]}| : X \times X \to \mathbb{R}_{\geq 0}$ is continuous as soon as its restrictions to all $C \times C \subseteq X$ for compact $C \subseteq X$ are continuous. That is, if and only if for all compact $C \subseteq X$ the mapping $f_{|C} : C \to \mathbf{E}$ is a $\mathcal{C}^\rho$-function. Thence $\mathcal{C}^\rho(X, \mathbf{E})$ is the initial $\mathbf{K}$-vector space and by definition then moreover the initial locally convex $\mathbf{K}$-vector space with respect to the restriction mappings

$$\begin{aligned} \mathcal{C}^\rho(X, \mathbf{E}) &\to \mathcal{C}^\rho(C, \mathbf{E}), \\ f &\mapsto f_{|C} \end{aligned}$$

for $C \subseteq X$ compact. ■

Proposition 1.9. *Let X be a metric space. Then the locally convex $\mathbf{K}$-vector space $\mathcal{C}^\rho(X, \mathbf{E})$ endowed with the family of seminorms $\{\|\cdot\|_{\mathcal{C}^\rho,C}\}$ running through all compact subsets $C \subseteq X$ is complete.*

Proof. By Remark 1.8, we find $\mathcal{C}^\rho(X, \mathbf{E})$ to be canonically isomorphic to the locally convex $\mathbf{K}$-vector space A, defined as the subspace

$$\begin{aligned} &\{(f_C) \in \prod_{C \subseteq X \text{ compact}} \mathcal{C}^\rho(C, \mathbf{E}) : f_{C|D\cap C} = f_{D|C\cap D} \text{ for all } C, D \subseteq X \text{ compact}\} \\ \subseteq &\prod_{C \subseteq X \text{ compact}} \mathcal{C}^\rho(C, \mathbf{E}) := P. \end{aligned}$$

Then A is closed in P, as convergence in $\mathcal{C}^\rho(C, \mathbf{E})$ implies in particular pointwise convergence. In more detail: If $\boldsymbol{f} \notin A$, then $\|f_{C|D\cap C} - f_{D|C\cap D}\|_{\sup} = \varepsilon > 0$ for two compacta $C, D \subseteq X$. Therefore $U := \prod_{K \neq C,D \text{ compact in } X} \mathcal{C}^\rho(K, \mathbf{E}) \times \mathrm{B}_{<\varepsilon}(f_C) \times \mathrm{B}_{<\varepsilon}(f_D) \ni \boldsymbol{f}$ is an open

neighborhood in the complement of A. As P is complete if and only if each factor is complete, we are reduced to the case X a compact metric space.
Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}^\rho(X,\mathbf{E})$. By completeness of $\mathcal{C}^0(X,\mathbf{E})$ with respect to $\|\cdot\|_{\sup}$, we find $f_n \to f$ with respect to $\|\cdot\|_{\sup}$ for the pointwise limit $f \in \mathcal{C}^0(X,\mathbf{E})$. It remains to prove $f \in \mathcal{C}^\rho(X,\mathbf{E})$ and $f_n \to f$ with respect to $\|\cdot\|_{\mathcal{C}^\rho}$. Let $x, y \in X$ be distinct. For $n, m \in \mathbb{N}$, we have

$$\begin{aligned}|(f-f_n)^{]\rho[}|(x,y) &\le |(f-f_m)^{]\rho[}|(x,y) \vee |(f_m-f_n)^{]\rho[}|(x,y)\\ &\le |(f-f_m)^{]\rho[}|(x,y) \vee \||(f_m-f_n)^{]\rho[}|\|_{\triangledown X^2}\\ &\le |(f-f_m)^{]\rho[}|(x,y) \vee \|f_m-f_n\|_{\mathcal{C}^\rho,C}.\end{aligned}$$

Fixing such $n \in \mathbb{N}$, put $c_n = \limsup_{m\ge 0}\|f_m - f_n\|_{\mathcal{C}^\rho,C}$. Since $f_n \to f$ pointwise, fixing such $x, y \in X$, we find $\limsup_{m\ge 0}|(f-f_m)^{]\rho[}|(x,y) = 0$. Hence $|(f-f_n)^{]\rho[}|(x,y) \le c_n$ and thus $\||(f-f_n)^{]\rho[}|\|_{\triangledown X^2} \le c_n$ as this bound does not depend on the chosen points $x, y \in X$. As (f_n) is a Cauchy-sequence with respect to $\|\cdot\|_{\mathcal{C}^\rho}$, we find $c_n \to 0$ and thus $\||(f-f_n)^{]\rho[}|\|_{\triangledown X^2} \to 0$. If we can prove $f \in \mathcal{C}^\rho(X,\mathbf{E})$, this will show $f_n \to f$ with respect to $\|\cdot\|_{\mathcal{C}^\rho}$ and we are done. For this, define $|f^{[\rho]}| : X^2 \to \mathbb{R}_{\ge 0}$ by

$$|f^{[\rho]}|(x,y) = \begin{cases}|f^{]\rho[}|(x,y), & \text{if } x \ne y,\\ 0, & \text{otherwise.}\end{cases}$$

Then $|f^{]\rho[}|$ is continuous on $\triangledown X^2$ and $f \in \mathcal{C}^\rho(X,\mathbf{E})$ if and only if $|f^{[\rho]}|$ is continuous on the diagonal $\triangle X^2$.
Fix $\varepsilon > 0$ and $a \in X$. We find $n_0 \in \mathbb{N}$ such that $\||(f-f_n)^{]\rho[}|\|_{\triangledown X^2} \le \varepsilon$ for all $n \ge n_0$. Since $f_{n_0} \in \mathcal{C}^\rho(X,\mathbf{E})$, there exists a neighborhood $V \ni a$ such that $|f_{n_0}^{]\rho[}|(x,y) \le \varepsilon$ for all distinct $x, y \in V$. Hence for distinct $x, y \in W := V^2 \subseteq X^2$ open, we find

$$|f^{]\rho[}|(x,y) \le |f_{n_0}^{]\rho[}|(x,y) \vee \||(f-f_{n_0})^{]\rho[}|\|_{\triangledown X^2} \le \varepsilon.$$

I.e. $|f^{[\rho]}|$ is continuous on $\triangle X^2$. ■

Lemma 1.10. *Let X be a compact metric space. If $\mathbf{E}$ is a $\mathbf{K}$-Banach algebra, then $\mathcal{C}^\rho(X,\mathbf{E})$ will be a $\mathbf{K}$-Banach algebra.*

Proof. We assume $\|xy\| \le M\|x\|\|y\|$ for all $x, y \in \mathbf{E}$ for a constant $M \ge 1$. We want to prove $\|fg\|_{\mathcal{C}^\rho} \le M \cdot \|f\|_{\mathcal{C}^\rho}\|g\|_{\mathcal{C}^\rho}$ for $f, g \in \mathcal{C}^\rho(X,\mathbf{E})$. Surely $\|fg\|_{\sup} \le M \cdot \|f\|_{\sup}\|g\|_{\sup}$. For distinct $x, y \in X$, we compute

$$\begin{aligned}\|fg(x) - fg(y)\| \le& \|fg(x) - f(x)g(y)\| \vee \|f(x)g(y) - fg(y)\|\\ =& M \cdot (\|f(x)\|\|g(x) - g(y)\| \vee \|g(y)\|\|f(x) - f(y)\|).\end{aligned}$$

It follows $\||fg|^{[\rho]}\|_{\sup} \le M \cdot (\|f\|_{\sup}\||g^{[\rho]}|\|_{\sup} \vee \|g\|_{\sup}\||f^{[\rho]}|\|_{\sup}) \le M \cdot \|f\|_{\mathcal{C}^\rho}\|g\|_{\mathcal{C}^\rho}$. ■

Definition. Let X be a metric spaces and Y a set; a mapping $g : X \to Y$ will be called **δ-constant** if $\mathrm{d}(x,y) \le \delta$ implies $g(x) = g(y)$.

Lemma 1.11. *Let X be a metric space and $f : X \to \mathbf{E}$ a mapping such that for fixed $\varepsilon > 0$, there exists $0 < \delta \leq 1$ such that $\mathrm{d}(x,y) \leq \delta$ implies $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x,y)^\rho$ for all $x, y \in X$. Then there exists a δ-constant function $g : X \to \mathbf{E}$ with $\|f - g\|_{\mathcal{C}^\rho, C} \leq \varepsilon$ for all $C \subseteq X$ compact.*

Proof. Because $\mathbf{E}$ is non-Archimedean, we can partition X into finitely many equivalence classes U_i by declaring

$$x \sim y \quad \text{if} \ \ \|f(x) - f(y)\| \leq \varepsilon\delta^\rho.$$

By assumption on f, two points x and y will be equivalent if $\mathrm{d}(x,y) \leq \delta$. In particular every U_i is open.

We now choose an element a_i from each U_i and define δ-constant $g : X \to \mathbf{E}$ by

$$g(x) := f(a_i) \quad \text{if} \ \ x \in U_i.$$

Then $\|f - g\|_{\sup} \leq \varepsilon\delta^\rho \leq \varepsilon$ and

$$\begin{aligned} & \| |(f-g)^{[\rho]}| \|_{\sup} \\ = & \| |(f-g)^{[\rho]}| \|_{\{(x,y) \in X^2 : \mathrm{d}(x,y) \leq \delta\}} \vee \| |(f-g)^{[\rho]}| \|_{\{(x,y) \in X^2 : \mathrm{d}(x,y) > \delta\}} \\ \leq & \quad \| |f^{[\rho]}| \|_{\{(x,y) \in X^2 : \mathrm{d}(x,y) \leq \delta\}} \vee \| |g^{[\rho]}| \|_{\{(x,y) \in X^2 : \mathrm{d}(x,y) \leq \delta\}} \\ & \vee \max_{x,y \in X : \mathrm{d}(x,y) > \delta} \left(\frac{\|f(x) - g(x)\|}{\mathrm{d}(x,y)^\rho} \vee \frac{\|f(y) - g(y)\|}{\mathrm{d}(x,y)^\rho} \right) \\ \leq & \varepsilon \vee 0 \vee \varepsilon\delta^\rho / \delta^\rho = \varepsilon. \end{aligned}$$

■

Corollary 1.12. *Let X be a compact metric space. Then the locally constant functions $g : X \to \mathbf{E}$ are dense in $\mathcal{C}^\rho(X, \mathbf{E})$.*

Proof. Fix $\varepsilon > 0$ and let $f \in \mathcal{C}^\rho(X, \mathbf{E})$. Then $|f^{[\rho]}| : X^2 \to \mathbb{R}_{\geq 0}$ is by compactness of X^2 a uniformly continuous function vanishing on $\triangle X^2$. Hence we find a $0 < \delta \leq 1$ such that in particular for all $(a,a) \in X^2$, it holds

$$||f^{[\rho]}|(x,y) - |f^{[\rho]}|(a,a)| = |f^{[\rho]}|(x,y) \leq \varepsilon \quad \text{for all } x, y \in X \text{ with } \mathrm{d}((x,y),(a,a)) \leq \delta.$$

By the triangle inequality, we will have $\delta(\{(x,y)\} \cup \triangle X^2) \leq \delta$ if $\mathrm{d}(x,y) \leq \delta$ for any $x, y \in X$. Thus

$$\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x,y)^\rho \quad \text{for all } x, y \in X \text{ with } \mathrm{d}(x,y) \leq \delta.$$

By Lemma 1.11, we find δ-constant g with $\|f - g\|_{\mathcal{C}^\rho} \leq \varepsilon$. In particular the locally constant functions are dense in $\mathcal{C}^\rho(X, \mathbf{E})$. ■

Componentwise criteria for being $\mathcal{C}^\rho$

Definition 1.13. We will call a subset $A \subseteq X$ of a cartesian metric space $X = X_1 \times \cdots \times X_d$ **telescopic**, if there exists a subset $B \subseteq A \times A$ such that:

(i) For all $(x, y) \in B$, also $(x_1, \ldots, x_k, y_{k+1}, \ldots, y_d) \in A$ for $k = 1, \ldots, d-1$;

(ii) For all distinct $x, y \in A$, there exists $z \in A$ with $(x, z), (z, y) \in B$ and $\mathrm{d}(x, z) < \mathrm{d}(x, y)$.

Remark 1.14. If $A \subseteq X$ is telescopic with cartesian ultrametric X, then every ball $U = \mathrm{B}_{\leq\delta}(a) \subseteq A$ of A will be telescopic: Assume that B proves A to be telescopic. We claim that $\tilde{B} := B \cap U \times U$ proves U to be telescopic.
Ad (i): Let $(x, y) \in \tilde{B}$. Then $(x_1, \ldots, x_k, y_{k+1}, \ldots, y_d) \in A$ for $k = 1, \ldots, d$. Since

$$\mathrm{d}((x_1, \ldots, x_k, y_{k+1}, \ldots, y_d), a) \leq \mathrm{d}(x, a) \vee \mathrm{d}(y, a) \leq \delta,$$

we find $(x_1, \ldots, x_k, y_{k+1}, \ldots, y_d) \in U$.
Ad (ii): Let $x, y \in U$ be distinct. Then there exists $z \in A$ with $(x, z), (z, y) \in B$ and $\mathrm{d}(x, z) < \mathrm{d}(x, y)$. Since U is an ultrametric ball with $x \in U$ as its center, $y \in U$ and $\mathrm{d}(x, z) < \mathrm{d}(x, y)$ implies $z \in U$. Hence $(x, z), (z, y) \in U \times U \cap B = \tilde{B}$.

Example 1.15. (0) Cartesian subsets $A \subseteq X$ without isolated points are surely telescopic for $B := A \times A$.

(i) Let X be a metric space without isolated points and $A := \nabla X^n \subseteq X^n =: \boldsymbol{X}$. Then the subset

$$B := \nabla X^{2n} \subseteq A \times A$$

shows $A \subseteq \boldsymbol{X}$ to be telescopic: By definition, we find that (i) is satisfied. To see (ii), let $x, y \in A$ be distinct and $\delta := \mathrm{d}(x, y) > 0$. Since X has no isolated points, we can find $z_1 \in B_{<\delta}(x_1) \subseteq X$ distinct from $x_1, \ldots, x_n, y_1, \ldots, y_n$. Then we can find $z_2 \in B_{<\delta}(x_2) \subseteq X$ distinct from $x_1, \ldots, x_n, y_1, \ldots, y_n$ and z_1. In this way, we construct $z \in A$ with $\mathrm{d}(x, z) < \mathrm{d}(x, y)$ such that in particular $(x, z), (z, y) \in \nabla X^{2n} = B$.

(ii) More generally, let $X_1, \ldots, X_d$ be metric spaces without isolated points and $A := A_1 \times \cdots \times A_d \subseteq X_1 \times \cdots \times X_d =: X$ with $A_k := \nabla X_k^{n_k} \subseteq X_k^{n_k} =: \boldsymbol{X}_k$ for natural numbers $n_1, \ldots, n_d$. Then the subset

$$B := \{(x, y) \in A \times A : (x_k, y_k) \in \nabla X_k^{2n_k} \text{ for } k = 1, \ldots, d\} \subseteq A \times A,$$

shows A to be telescopic: By definition, we find that (i) is satisfied. To see (ii), let $x, y \in A$ be distinct and $\delta := \mathrm{d}(x, y) > 0$. For $k = 1, \ldots, d$, we can find by the above Example 1.15(i) elements $z_k \in \boldsymbol{X}_k$ with $\mathrm{d}(x_k, z_k) < \delta$ such that in particular $(x_k, z_k), (z_k, y_k) \in \nabla X_k^{2n_k}$. Therefore $z = (z_1, \ldots, z_d) \in A$ satisfies $\mathrm{d}(x, z) < \mathrm{d}(x, y)$ and proves $(x, z), (z, y) \in B$.

Lemma 1.16. *We assume $\varrho \in [0,1]$. Let $f : U \to \mathbf{E}$ be a mapping defined on a telescopic subset $U \subseteq \mathbf{K}^d$. Fix $\varepsilon > 0$. Then*

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\varrho \quad \text{for all } x, y \in U$$

if and only if for $k = 1, \ldots, d$ holds

$$\|f(x + t \cdot \boldsymbol{e}_k) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_k, x \in U \text{ with } t \in \mathbf{K}.$$

Proof. Foremost, note that the second statement is the special case $y = x + t \cdot \boldsymbol{e}_k$ of the first one. We prove that the second statement also implies the first one.
Let $B \subseteq U \times U$ prove U to be telescopic. Let $y = x + t_1\boldsymbol{e}_1 + \cdots + t_d\boldsymbol{e}_d, x \in U$. Then

$$\begin{aligned}
&f(y) - f(x) \\
=&f(t_1\boldsymbol{e}_1 + \cdots + t_d\boldsymbol{e}_d + x) - f(x) \\
=&(f(t_1\boldsymbol{e}_1 + \cdots + t_d\boldsymbol{e}_d + x) - f(t_1\boldsymbol{e}_1 + \cdots + t_{d-1}\boldsymbol{e}_{d-1} + x)) + \cdots + (f(t_1\boldsymbol{e}_1 + x) - f(x)).
\end{aligned}$$

Thus, if $(x, y) \in B \cap U \times U$, it will follow

$$\begin{aligned}
&\|f(y) - f(x)\| \\
\leq &\max_{k=1,\ldots,d} \|f(t_1\boldsymbol{e}_1 + \cdots + t_k\boldsymbol{e}_k + x) - f(t_1\boldsymbol{e}_1 + \cdots + t_{k-1}\boldsymbol{e}_{k-1} + x)\| \\
\leq &\varepsilon \cdot \max(|t_1|^\varrho, \ldots, |t_d|^\varrho) \\
= &\varepsilon \cdot \|t_1\boldsymbol{e}_1 + \cdots + t_d\boldsymbol{e}_d\|^\varrho = \varepsilon \cdot \|y - x\|^\varrho;
\end{aligned}$$

here the second inequality by Property (i) of B in Definition 1.13.

We claim that this suffices, i.e. if

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\varrho \quad \text{for all } (x, y) \in B \cap U \times U,$$

then

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\varrho \quad \text{for all } x, y \in U.$$

To see this, let $x, y \in U$, which we may assume to be distinct. By Definition 1.13(ii), we find $z \in U$ with $(x, z), (z, y) \in B$ and $\|x - z\| < \|x - y\|$. As by the non-Archimedean triangle inequality $\|z - y\| = \|x - y\|$, we find

$$\|f(x) - f(y)\| \leq \|f(x) - f(z)\| \vee \|f(z) - f(y)\| \leq \varepsilon \|x - z\|^\varrho \vee \varepsilon \|z - y\|^\varrho = \varepsilon \|x - y\|^\varrho.$$

■

Symmetry properties

Definition 1.17. Let $A_1, \ldots, A_d$ be sets and put $A = A_1 \times \cdots \times A_d$. Denote by $\sigma : A \to A$ the mapping swapping the k-th and l-th coordinate. Then we will call:

(i) A point $a \in A$ **symmetric in its k-th and l-th coordinate** if $\sigma a = a$.

(ii) A subset $U \subseteq A$ **symmetric in its k-th and l-th coordinate** if $\sigma U = U$.

(iii) A function $f : U \to \mathbf{E}$ on a subset $U \subseteq A$ **symmetric in its k-th and l-th coordinate** if U is symmetric in its k-th and l-th coordinate and $f \circ \sigma = f$.

Lemma 1.18. *We assume $\varrho \in [0,1]$. Let $U \subseteq \mathbf{K}^d$ be a subset and $f : U \to \mathbf{E}$ a mapping symmetric in its k-th and l-th coordinate. Fix $\varepsilon > 0$. Then*

$$\|f(x + t \cdot \boldsymbol{e}_k) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_k, x \in U$$

if and only if

$$\|f(x + t \cdot \boldsymbol{e}_l) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_l, x \in U.$$

Proof. By symmetry it suffices to prove one direction, e.g. if

$$\|f(x + t \cdot \boldsymbol{e}_l) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_l, x \in U,$$

then

$$\|f(x + t \cdot \boldsymbol{e}_k) - f(x)\| \leq \varepsilon |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_k, x \in U.$$

Denote by $\sigma : K^d \to K^d$ the map swapping the k-th and l-th coordinate. By assumption, U is left stable under σ, i.e. if $y, x \in U$, then $y^\sigma, x^\sigma \in U$. Now let $y = x + t \cdot \boldsymbol{e}_k, x \in U$. By symmetry of f in its k-th and l-th coordinate, we find

$$\|f(x+t\cdot\boldsymbol{e}_k)-f(x)\| = \|f(y)-f(x)\| = \|f(y^\sigma)-f(x^\sigma)\| = \|f(x^\sigma+t\cdot\boldsymbol{e}_l)-f(x^\sigma)\| \leq \varepsilon|t|^\varrho,$$

the last inequality as $y^\sigma, x^\sigma \in U$.

■

Definition (1.17'). Let $A_1, \ldots, A_d$ be sets and put $A = A_1 \times \cdots \times A_d$. Let $I \subseteq \{1, \ldots, d\}$ be a subset of indices. Then we will call:

(i) A point $a \in A$ **symmetric in its coordinates indexed by** I if a is symmetric in its k-th and l-th coordinates for all $k, l \in I$.

(ii) A subset $U \subseteq A$ **symmetric in its coordinates indexed by** I if U is symmetric in its k-th and l-th coordinate for all $k, l \in I$.

(iii) A function $f : U \to \mathbf{E}$ on a subset $U \subseteq A$ **symmetric in its coordinates indexed by** I if f is symmetric in its k-th and l-th coordinate for all $k, l \in I$.

Lemma 1.19. *We assume $\varrho \in [0,1]$. Let $U \subseteq \mathbf{K}^d$ be a telescopic subset. Let $\{1, \ldots, d\} = I_1 \dot\cup \ldots \dot\cup I_e$ with representatives $i_1, \ldots, i_e$ and $f : U \to \mathbf{E}$ a mapping symmetric in its coordinates indexed by $I_1, \ldots, I_e$. Fix $\varepsilon > 0$. Then it holds*

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\varrho \quad \text{for all } x, y \in U$$

if and only if for $j = 1, \ldots, e$ holds

$$\|f(x + t \cdot \boldsymbol{e}_{i_j}) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_{i_j}, x \in U.$$

Proof. Let $j \in \{1, \ldots, e\}$ and put $I := I_j$. By Lemma 1.18, we find by symmetry of f in its coordinates $i, i' \in I$ that

$$\|f(x + t \cdot \boldsymbol{e}_i) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_i, x \in U$$

if and only if

$$\|f(x + t \cdot \boldsymbol{e}_{i'}) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_{i'}, x \in U.$$

In particular we can choose $i = i_j$. Since $\{1, \ldots, d\} = I_1 \dot\cup \ldots \dot\cup I_e$, we find that for $j = 1, \ldots, e$ holds

$$\|f(x + t \cdot \boldsymbol{e}_{i_j}) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_{i_j}, x \in U$$

if and only if for $k = 1, \ldots, d$ holds

$$\|f(x + t \cdot \boldsymbol{e}_k) - f(x)\| \leq \varepsilon \cdot |t|^\varrho \quad \text{for all } x + t \cdot \boldsymbol{e}_k, x \in U.$$

By Lemma 1.16, this is equivalent to

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\varrho \quad \text{for all } x, y \in U.$$

■

Corollary 1.20. *Let $X = X_1 \times \cdots \times X_d \subseteq \mathbf{K}^d$ and $A \subseteq X$ a telescopic subset. Let $\{1, \ldots, d\} = I_1 \dot\cup \ldots \dot\cup I_e$ with representatives $i_1, \ldots, i_d$. Let $f : A \to \mathbf{E}$ be a mapping and a some point in X, both symmetric in its coordinates indexed by $I_1, \ldots, I_d$. Then for f to be $\mathcal{C}^\rho$ at a, the following convergence condition suffices: For every $\varepsilon > 0$, there exists a ball $U \ni a$ in X such that for $j = 1, \ldots, e$ holds*

$$\|f(x + t \cdot \boldsymbol{e}_{i_j}) - f(x)\| \leq \varepsilon \cdot |t|^\rho \quad \text{for all } x + t \cdot \boldsymbol{e}_{i_j}, x \in U \cap A.$$

Proof. Fix $\varepsilon > 0$. Since a is symmetric in the coordinates indexed by $I_1, \ldots, I_d$, so is every ball $U \ni a$ in X. Since A is symmetric in the coordinates indexed by $I_1, \ldots, I_d$, so is the ball $U \cap A$ in A. With A, so is by Remark 1.14 the ball $U \cap A$ again telescopic. By Lemma 1.19 applied to $f_{|U \cap A}$, we find for $j = 1, \ldots, e$ that

$$\|f(x + t \cdot \boldsymbol{e}_{i_j}) - f(x)\| \leq \varepsilon \cdot |t|^\rho \quad \text{for all } x + t \cdot \boldsymbol{e}_{i_j}, x \in U \cap A$$

if and only if

$$\|f(x) - f(y)\| \leq \varepsilon \|x - y\|^\rho \quad \text{for all } x, y \in U \cap A.$$

Since the balls $U \ni a$ in X constitute a basis of neighborhoods of a, this is equivalent to f being $\mathcal{C}^\rho$ at a. ■

1.3 $\mathcal{C}^{\boldsymbol{\rho}}$-functions for $\boldsymbol{\rho} \in [0, 1[^d$

Assumption. Throughout this subsection, we will fix a tuple of real numbers $\boldsymbol{\rho} \in [0, 1[^d$.

Definition of $\mathcal{C}^{\boldsymbol{\rho}}$-functions

Definition 1.21. Let $f : X \to Y$ be a mapping on the metric spaces $X = X_1 \times \cdots \times X_d$ and Y. We put $\mathrm{d}(x,y)^{\boldsymbol{\rho}} := \mathrm{d}_1(x_1,y_1)^{\rho_1} \vee \ldots \vee \mathrm{d}_d(x_d,y_d)^{\rho_d}$ with the convention $0^0 = 0$ (See the following Remark).

(i) Let a be some point in X. We will say that f is $\mathcal{C}^{\boldsymbol{\rho}}$ **at** a if for every $\varepsilon > 0$, there exists a neighborhood $U \ni a$ such that

$$\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x,y)^{\boldsymbol{\rho}} \quad \text{for all } x, y \in U.$$

We will say that f is a $\mathcal{C}^{\boldsymbol{\rho}}$**-function** if f is $\mathcal{C}^{\boldsymbol{\rho}}$ at all points $a \in X$. The set of all $\mathcal{C}^{\boldsymbol{\rho}}$-functions $f : X \to Y$ will be denoted by $\mathcal{C}^{\boldsymbol{\rho}}(X,Y)$.

(ii) We define $|f^{]\boldsymbol{\rho}[}| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{]\boldsymbol{\rho}[}|(x,y) := \frac{\|f(x) - f(y)\|}{\mathrm{d}(x,y)^{\boldsymbol{\rho}}}.$$

Remark. (i) In case $\rho_k = 0$ for some $k \in \{1,\ldots,d\}$, the scurrilous convention $0^0 = 0$ ensures that in a neighborhood of the point $a \in X$, the condition $\mathrm{d}(f(x), f(y)) \leq \varepsilon \cdot \mathrm{d}(x,y)^{\boldsymbol{\rho}}$ is still stronger than the mere continuity condition $\mathrm{d}(f(x), f(y)) \leq \varepsilon$, which were in place if we had adopted the common convention $0^0 = 1$.

(ii) Keeping the above notations, we see that if $f : X \to Y$ is $\mathcal{C}^{\boldsymbol{\rho}}$ at $a \in X$, then it will be $\mathcal{C}^{\tilde{\boldsymbol{\rho}}}$ thereat for any $\tilde{\boldsymbol{\rho}} \leq \boldsymbol{\rho}$ componentwise. In particular if f is $\mathcal{C}^{\boldsymbol{\rho}}$ at $a \in X$, then it will be $\mathcal{C}^{\rho}$ thereat with $\rho := \rho_1 \wedge \ldots \wedge \rho_d \in [0,1[$.

Then the mapping $f : X \to \mathbf{E}$ is $\mathcal{C}^{\boldsymbol{\rho}}$ if and only if the function $|f^{]\boldsymbol{\rho}[}|$ extends to a continuous function $|f^{[\boldsymbol{\rho}]}| : X \times X \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle X \times X$. We moreover saw above that every $\mathcal{C}^{\boldsymbol{\rho}}$-function is in particular continuous. We can therefore establish:

Definition. For every compact $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}},C}$ on $\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$ by

$$\|f\|_{\mathcal{C}^{\boldsymbol{\rho}},C} = \|f_{|C}\|_{\sup} \vee \||(f_{|C})^{[\boldsymbol{\rho}]}|\|_{\sup}.$$

We equip the $\mathbf{K}$-vector space $\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$ with the locally convex topology given by the family of seminorms $\{\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}},C} : C \subseteq X \text{ compact}\}$.
If X itself is compact, then we will turn $\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$ into a normed $\mathbf{K}$-vector space by endowing it with the norm $\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}}} := \|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}},X}$.

Remark 1.22. We have an equality of locally convex $\mathbf{K}$-vector space $\mathcal{C}^{\vec{\rho}}(X,\mathbf{K}) = \mathcal{C}^{\rho}(X,\mathbf{K})$ with $\vec{\rho} = (\rho,\ldots,\rho)$. It holds $\|\cdot\|_{\mathcal{C}^{\vec{\rho}},C} = \|\cdot\|_{\mathcal{C}^{\rho},C}$ for any $C \subseteq X$ compact.

Remark 1.23. The locally convex $\mathbf{K}$-vector space of $\mathcal{C}^{\boldsymbol{\rho}}$-functions $\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$ is the initial locally convex $\mathbf{K}$-vector space with respect to all restriction mappings

$$\begin{aligned} \mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E}) &\to \mathcal{C}^{\boldsymbol{\rho}}(C,\mathbf{E}), \\ f &\mapsto f_{|C} \end{aligned}$$

with C running through the family of all compact subsets $C \subseteq X$.

Proof. Define $|f^{[\boldsymbol{\rho}]}| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{[\boldsymbol{\rho}]}|(x,y) = \begin{cases} |f^{]\boldsymbol{\rho}[}|(x,y), & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

By (the comment before) Lemma 1.2, the function $|f^{[\boldsymbol{\rho}]}| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ is continuous as soon as its restrictions to all compacta $C \subseteq \nabla X \times X$ are so. Given such compact $\tilde{C} \subseteq X \times X$, we find compact $C \subseteq X$ with $\tilde{C} \subseteq C \times C$, namely $C := \mathrm{p}_1 \tilde{C} \cup \mathrm{p}_2 \tilde{C}$. Therefore $|f^{[\boldsymbol{\rho}]}| : X \times X \to \mathbb{R}_{\geq 0}$ is continuous as soon as its restrictions to all $C \times C \subseteq X \times X$ for compact $C \subseteq X$ are continuous. That is, if and only if for all compact $C \subseteq X$ the mapping $f_{|C} : C \to \mathbf{E}$ is a $\mathcal{C}^{\boldsymbol{\rho}}$-function.
Thence $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ is the initial $\mathbf{K}$-vector space and by definition then moreover the initial locally convex $\mathbf{K}$-vector space with respect to the restriction mappings

$$\begin{aligned} \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E}) &\to \mathcal{C}^{\boldsymbol{\rho}}(C, \mathbf{E}), \\ f &\mapsto f_{|C} \end{aligned}$$

for $C \subseteq X$ compact. ∎

Properties of the space of $\mathcal{C}^{\boldsymbol{\rho}}$-functions

Proposition 1.24. *Let $X = X_1 \times \cdots \times X_d$ be a cartesian metric space and $\mathbf{E}$ a $\mathbf{K}$-Banach algebra.*

(i) The space $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ is a locally convex $\mathbf{K}$-algebra.

(ii) The tensor product of a $\mathcal{C}^{\boldsymbol{\rho}'}$-function with a $\mathcal{C}^{\boldsymbol{\rho}''}$-function is a $\mathcal{C}^{\boldsymbol{\rho}}$-function for $\boldsymbol{\rho} := (\boldsymbol{\rho}', \boldsymbol{\rho}'')$: Let $X' = X'_1 \times \cdots \times X'_d$ and $X'' = X''_1 \times \cdots \times X''_e$ be cartesian metric spaces. If $f \in \mathcal{C}^{\boldsymbol{\rho}'}(X', \mathbf{E})$ and $g \in \mathcal{C}^{\boldsymbol{\rho}''}(X'', \mathbf{E})$, then $f \odot g(x,y) := f(x)g(y)$ will lie in $\mathcal{C}^{\boldsymbol{\rho}}(X' \times X'', \mathbf{E})$ and for every compact subset $C \subseteq X$ holds $\|f \odot g\|_{\mathcal{C}^{\boldsymbol{\rho}}, C} \leq \|f\|_{\mathcal{C}^{\boldsymbol{\rho}'}, C} \|g\|_{\mathcal{C}^{\boldsymbol{\rho}''}, C}$ with equality if $\boldsymbol{\rho}$ has at most one nonzero entry.

Proof. Ad (i): Firstly, to see that that $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ is a $\mathbf{K}$-algebra, only its closure under products requires attention. Let $M \geq 1$ be the operator norm of the multiplication on $\mathbf{E}$. Let $f, g \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$, $a \in X$ and fix $\varepsilon > 0$. We compute

$$\begin{aligned} \|fg(y) - fg(x)\| \leq& \|fg(y) - f(y)g(x)\| \vee \|f(y)g(x) - fg(x)\| \\ =& M \cdot \|f(y)\| \|g(y) - g(x)\| \vee M \cdot \|g(x)\| \|f(y) - f(x)\| \\ \leq& C \cdot (\|g(y) - g(x)\| \vee \|f(y) - f(x)\|) \leq \varepsilon \cdot \mathrm{d}(y,x)^{\boldsymbol{\rho}} \end{aligned}$$

for x, y in a sufficiently small neighborhood $U \ni a$; the second inequality as f and g are in particular continuous and therefore bounded in a neighborhood of the point a.
Secondly, the above computation shows $\|fg\|_{\mathcal{C}^{\boldsymbol{\rho}}, C} \leq M \cdot \|f\|_{\mathcal{C}^{\boldsymbol{\rho}}} \|g\|_{\mathcal{C}^{\boldsymbol{\rho}}}$ for all $f, g \in \mathcal{C}^{\boldsymbol{\rho}}$ and $C \subseteq X$ compact, i.e. the continuity of multiplication of $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ with operator norm M.
Ad (ii): Firstly, $f \odot g \in \mathcal{C}^0(X' \times X'', \mathbf{E})$ by Proposition 1.7(iii). Fix $\varepsilon > 0$ and $a = (a', a'') \in$

$X' \times X''$. Since f is $\mathcal{C}^{\boldsymbol{\rho}'}$ at a', we find a $\delta' > 0$ such that $\|f(y') - f(x')\| \leq \varepsilon \cdot \mathrm{d}(y', x')^{\boldsymbol{\rho}'}$ for all $y', x' \in \mathrm{B}_{\leq \delta'}(a')$ and since g is continuous at a'', it is bounded in a neighborhood $V \ni a''$ by a constant $C'' > 0$. We compute

$$\|f \odot g(y) - f \odot g(x)\| \leq \|f(y') - f(x')\| \|g(x'')\| \leq \varepsilon \cdot \mathrm{d}(y', x')^{\boldsymbol{\rho}'} \cdot C \quad \text{for } y, x \in \mathrm{B}_{\leq \delta'}(a') \times V.$$

Similarly we find some $\delta'' > 0$, a neighborhood $U \ni a'$ and a constant $C' > 0$ such that $\|f \odot g(y) - f \odot g(x)\| \leq \varepsilon \cdot C \cdot \mathrm{d}(y'', x'')^{\boldsymbol{\rho}''}$ for $y, x \in U \times \mathrm{B}_{\leq \delta''}(a'')$. Hence for a ball $\mathrm{B}_{\leq \delta} \ni a$ in the intersection of these two product neighborhoods, we find

$$\|f \odot g(y) - f \odot g(x)\| \leq \varepsilon \cdot (\mathrm{d}(y', x')^{\boldsymbol{\rho}'} \vee \mathrm{d}(y'', x'')^{\boldsymbol{\rho}''}) = \varepsilon \cdot \mathrm{d}(y, x)^{\boldsymbol{\rho}} \quad \text{for } y, x \in \mathrm{B}_{\leq \delta}(a).$$

Secondly, we have foremost $\|f \odot g\|_{\sup} = \|f\|_{\sup} \|g\|_{\sup}$. Now for all $(y, x) \in X \times X$ with $X := X' \times X''$ compact and $y' \neq x'$ and $y'' \neq x''$ holds

$$\begin{aligned}
\frac{\|f \odot g(y) - f \odot g(x)\|}{\mathrm{d}(x, y)^{\boldsymbol{\rho}}} &\leq \frac{\|f(y') - f(x')\|}{\mathrm{d}(x, y)^{\boldsymbol{\rho}}} \cdot \|g(y'')\| \vee \|f(x')\| \cdot \frac{\|g(y'') - g(x'')\|}{\mathrm{d}(x, y)^{\boldsymbol{\rho}}} \\
&\leq \frac{\|f(y') - f(x')\|}{\mathrm{d}(x', y')^{\boldsymbol{\rho}'}} \cdot \|g(y'')\| \vee \|f(x')\| \cdot \frac{\|g(y'') - g(x'')\|}{\mathrm{d}(x'', y'')^{\boldsymbol{\rho}''}} \\
&\leq \| |f^{[\boldsymbol{\rho}']}| \|_{\sup} \|g\|_{\sup} \vee \|f\|_{\sup} \| |g^{[\boldsymbol{\rho}'']}| \|_{\sup} \leq \|f\|_{\mathcal{C}^{\boldsymbol{\rho}'}} \cdot \|g\|_{\mathcal{C}^{\boldsymbol{\rho}''}}.
\end{aligned}$$

We saw $f \odot g \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ and therefore $|f \odot g^{[\boldsymbol{\rho}]}|(y, x) = 0$ if $x' = y'$ or $x'' = y''$. We conclude $\|f \odot g\|_{\mathcal{C}^{\boldsymbol{\rho}}} \leq \|f\|_{\mathcal{C}^{\boldsymbol{\rho}'}} \|g\|_{\mathcal{C}^{\boldsymbol{\rho}''}}$.
In case $\boldsymbol{\rho}$ has at most one nonzero entry, e.g. among those of $\boldsymbol{\rho}'$, then we find $\mathrm{d}(x, y)^{\boldsymbol{\rho}} = 1$ as soon as $x'' \neq y''$ for any $x, y \in X$ and therefore

$$\|f \odot g\|_{\mathcal{C}^{\boldsymbol{\rho}}} = \|f \odot g\|_{\sup} \vee \| |f \odot g^{[\boldsymbol{\rho}]}| \|_{\{x, y \in X \times X : x'' = y''\}}.$$

If $x'' = y''$, we obtain

$$\|f \odot g(y) - f \odot g(x)\| / \mathrm{d}(x, y)^{\boldsymbol{\rho}} = \|f(y') - f(x')\| / \mathrm{d}(x', y')^{\boldsymbol{\rho}'} \cdot \|g(x'')\| = |f^{[\boldsymbol{\rho}']}|(y', x') \|g(x'')\|.$$

We conclude

$$\begin{aligned}
\|f \odot g\|_{\mathcal{C}^{\boldsymbol{\rho}}} &= \|f \odot g\|_{\sup} \vee \| |f \odot g^{[\boldsymbol{\rho}]}| \|_{\{x, y \in X \times X : x'' = y''\}} \\
&= \|f\|_{\sup} \|g\|_{\sup} \vee \| |f^{[\boldsymbol{\rho}']}| \|_{\sup} \|g\|_{\sup} = \|f\|_{\mathcal{C}^{\boldsymbol{\rho}'}} \|g\|_{\mathcal{C}^{\boldsymbol{\rho}''}}.
\end{aligned}$$

■

Proposition 1.25. *Let $X = X_1 \times \cdots \times X_d$ be a cartesian metric space. Then the locally convex $\mathbf{K}$-vector space $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ endowed with the family of seminorms $\{\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}}, C}\}$ running through all compact subsets $C \subseteq X$ is complete.*

Proof. By Remark 1.23, we find $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ to be canonically isomorphic to the locally convex $\mathbf{K}$-vector space A, defined as the subspace

$$\begin{aligned}
&\{(f_C) \in \prod_{C \subseteq X \text{ compact}} \mathcal{C}^{\boldsymbol{\rho}}(C, \mathbf{E}) : f_{C|D \cap C} = f_{D|C \cap D} \text{ for all } C, D \subseteq X \text{ compact}\} \\
&\subseteq \prod_{C \subseteq X \text{ compact}} \mathcal{C}^{\boldsymbol{\rho}}(C, \mathbf{E}) := P.
\end{aligned}$$

Then A is closed in P, as convergence in $\mathcal{C}^{\boldsymbol{\rho}}(C, \mathbf{E})$ implies in particular pointwise convergence. In more detail: Let $\boldsymbol{f}$ be in the boundary of A and fix two compacta $C, D \subseteq X$. Consider for any $\varepsilon > 0$ the open neighborhood $U := \mathrm{B}_{\leq\varepsilon}(f_C) \times \mathrm{B}_{\leq\varepsilon}(f_D) \times \prod'_{K \subseteq X \text{ compact}} \mathcal{C}^{\boldsymbol{\rho}}(K, \mathbf{E}) \ni \boldsymbol{f}$; here the prime indicating the exclusion of $C, D \subseteq X$ in the index set. Since $\boldsymbol{f} \in \partial A$, we find $U \cap A \neq \emptyset$ and thus $\|f_{C|D\cap C} - f_{D|C\cap D}\|_{\sup} \leq \varepsilon$ for every $\varepsilon > 0$, i.e. $\boldsymbol{f} \in A$. As P is complete if and only if each factor is complete, we are reduced to the case X a compact metric space.
Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$. By completeness of $\mathcal{C}^0(X, \mathbf{E})$ with respect to $\|\cdot\|_{\sup}$, we find $f_n \to f$ with respect to $\|\cdot\|_{\sup}$ for the pointwise limit $f \in \mathcal{C}^0(X, \mathbf{E})$. It remains to prove $f \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ and $f_n \to f$ with respect to $\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}}}$. Let $(x, y) \in \triangledown X \times X$. For $n, m \in \mathbb{N}$, we have

$$\begin{aligned} |(f - f_n)^{]\boldsymbol{\rho}[}|(x, y) &\leq |(f - f_m)^{]\boldsymbol{\rho}[}|(x, y) \vee |(f_m - f_n)^{]\boldsymbol{\rho}[}|(x, y) \\ &\leq |(f - f_m)^{]\boldsymbol{\rho}[}|(x, y) \vee \||(f_m - f_n)^{]\boldsymbol{\rho}[}|\|_{\triangledown X\times X} \\ &\leq |(f - f_m)^{]\boldsymbol{\rho}[}|(x, y) \vee \|f_m - f_n\|_{\mathcal{C}^{\boldsymbol{\rho}}}. \end{aligned}$$

Fixing such $n \in \mathbb{N}$, put $c_n = \limsup_{m\geq 0}\|f_m - f_n\|_{\mathcal{C}^{\boldsymbol{\rho}}}$. Since $f_n \to f$ pointwise, fixing such $x, y \in X$, we find $\limsup_{m\geq 0}|(f - f_m)^{]\boldsymbol{\rho}[}|(x, y) = 0$. Hence $|(f - f_n)^{]\boldsymbol{\rho}[}|(x, y) \leq c_n$ and thus $\||(f - f_n)^{]\boldsymbol{\rho}[}|\|_{\triangledown X\times X} \leq c_n$ as this bound does not depend on the chosen points $x, y \in X$. As (f_n) is a Cauchy-sequence with respect to $\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}}}$, we find $c_n \to 0$ and thus $\||(f - f_n)^{]\boldsymbol{\rho}[}|\|_{\triangledown X\times X} \to 0$. If we can prove $f \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$, this will show $f_n \to f$ with respect to $\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}}}$ and we are done. For this, define $|f^{[\boldsymbol{\rho}]}| : X \times X \to \mathbb{R}_{\geq 0}$ by

$$|f^{[\boldsymbol{\rho}]}|(x, y) = \begin{cases} |f^{]\boldsymbol{\rho}[}|(x, y), & \text{if } x \neq y, \\ 0, & \text{otherwise.} \end{cases}$$

Then $|f^{]\boldsymbol{\rho}[}|$ is continuous on $\triangledown X \times X = \triangledown X \times X$ and $f \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ if and only if $|f^{[\boldsymbol{\rho}]}|$ is continuous on the diagonal $\triangle X \times X$.
Fix $\varepsilon > 0$ and $a \in X$. We find $n_0 \in \mathbb{N}$ such that $\||(f - f_n)^{]\boldsymbol{\rho}[}|\|_{\triangledown X\times X} \leq \varepsilon$ for all $n \geq n_0$. Since $f_{n_0} \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$, there exists a neighborhood $U \ni a$ such that $|f_{n_0}^{]\boldsymbol{\rho}[}|(x, y) \leq \varepsilon$ for all $(x, y) \in U \times U \cap \triangledown X \times X$. Hence for distinct $x, y \in V := U \times U \cap X \times X \subseteq X \times X$ open, we find

$$|f^{]\boldsymbol{\rho}[}|(x, y) \leq |f_{n_0}|^{]\boldsymbol{\rho}[}(x, y) \vee \||(f - f_{n_0})^{]\boldsymbol{\rho}[}|\|_{\triangledown X\times X} \leq \varepsilon.$$

I.e. $|f^{[\boldsymbol{\rho}]}| : X \times X \to \mathbb{R}_{\geq 0}$ is continuous on $\triangle X \times X$. ■

Notation. Let $X = X_1 \times \cdots \times X_d$ be a cartesian metric spaces and put

$$\mathbf{d}(x, y) := (\mathrm{d}_1(x_1, y_1), \ldots, \mathrm{d}_d(x_d, y_d)) \quad \text{for } x, y \in X.$$

Let $\boldsymbol{\delta} \in \mathbb{R}^d_{\geq 0}$. Then we write

$$\mathbf{d}(x, y) \leq \boldsymbol{\delta} \quad \text{if} \quad \mathrm{d}_1(x_1, y_1) \leq \delta_1, \ldots, \mathrm{d}_d(x_d, y_d) \leq \delta_d.$$

Given $a \in X$, we denote $\mathrm{B}_{\leq\boldsymbol{\delta}}(a) := \{x \in X : \mathbf{d}(x, a) \leq \boldsymbol{\delta}\}$.

Definition 1.26. Let $X = X_1 \times \cdots \times X_d$ and Y be metric spaces. A mapping $g : X \to Y$ will be called **locally $\boldsymbol{\delta}$-constant** for $\boldsymbol{\delta} \in \mathbb{R}^d_{\geq 0}$ if it is locally constant and $\mathbf{d}(x, y) \leq \boldsymbol{\delta}$ implies $g(x) = g(y)$.

In case $\boldsymbol{\delta} \in \mathbb{R}^d_{>0}$, we find a locally $\boldsymbol{\delta}$-constant function to be δ-constant with $\delta = \delta_1 \wedge \ldots \wedge \delta_d > 0$. But we will also be interested in the case where there is only one positive $\delta_k > 0$ and where we do not know particular positive lower bounds for the other entries of $\boldsymbol{\delta}$ - even though they might exist, e.g. by compactness of X.

Lemma 1.27. *Let $f \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ with $X = X_1 \times \cdots \times X_d$ a compact metric space. Then*

$$\|f\|_{\mathcal{C}^{\boldsymbol{\rho}}} = \|f\|_{\sup} \vee \||f^{[\boldsymbol{\rho}]}|\|_{X^{[\boldsymbol{\rho}]}}$$

with $X^{[\boldsymbol{\rho}]} := \{(x, y) \in X \times X : x_k = y_k \text{ if } \rho_k = 0\} \subseteq X \times X.$

Proof. This reduces by definition of $\|\cdot\|_{\mathcal{C}^{\boldsymbol{\rho}}}$ to the assertion that $\||f^{[\boldsymbol{\rho}]}|\|_{X^{[\boldsymbol{\rho}]}} \geq \||f^{[\boldsymbol{\rho}]}|\|_{\sup}$. We have by definition $|f^{[\boldsymbol{\rho}]}|(x, y) \geq \|f\|_{\sup}$ only if $\mathrm{d}(x, y) \leq 1$. But if $\rho_k = 0$ and $x_k \neq y_k$, then $\mathrm{d}^{\boldsymbol{\rho}}(x, y) \geq \mathrm{d}_k(x_k, y_k)^0 = 1$. The assertion follows. ■

Lemma 1.28. *Let $\boldsymbol{\rho} \in [0, 1[^d$. Let $\boldsymbol{\delta} \in [0, 1]^d$ such that:*

1. *For any $k = 1, \ldots, d$, we have $\delta_k = 0$ only if $\rho_k = 0$.*
2. *Put $D = \{\delta_k^{\rho_k} : k = 1, \ldots, d \text{ and } \delta_k > 0\}$. Then we have $\max D = \min D$ and $\gamma := \max D > 0$.*

Let $X = X_1 \times \cdots \times X_d$ be a compact cartesian metric space and $f \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$ such that for fixed $\varepsilon > 0$, we find $\mathbf{d}(x, y) \leq \boldsymbol{\delta}$ to imply $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x, y)^{\boldsymbol{\rho}}$ for all $x, y \in X$. Then there exists locally $\boldsymbol{\delta}$-constant $g : X \to \mathbf{E}$ with $\|f - g\|_{\sup} \leq \varepsilon\gamma$ and $\|f - g\|_{\mathcal{C}^{\boldsymbol{\rho}}} \leq \varepsilon$.

Proof. Fix such $f : X \to \mathbf{E}$ and $\varepsilon > 0$. Because $f \in \mathcal{C}^{\boldsymbol{\rho}}(X, \mathbf{E})$, there exists a tuple $\tilde{\boldsymbol{\delta}} \in]0, 1]^d$ - and for which we may by premiss on f assume $\tilde{\delta}_k = \delta_k$ for all $k = 1, \ldots, d$ with $\delta_k > 0$ - such that $\mathbf{d}(x, y) \leq \tilde{\boldsymbol{\delta}}$ implies $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}^{\boldsymbol{\rho}}(x, y)$. Because $\mathbf{E}$ is non-Archimedean, we can partition X into equivalence classes $U_i \subseteq X$ by declaring

$$x \sim y \quad \text{if} \;\; \|f(x) - f(y)\| \leq \varepsilon\gamma.$$

Since f is in particular continuous, every U_i is open. We now choose an element a_i from each U_i and define locally constant $g : X \to \mathbf{E}$ by

$$g(x) := f(a_i) \quad \text{if} \;\; x \in U_i.$$

We note that two points x and y will be equivalent if $\mathbf{d}(x, y) \leq \tilde{\boldsymbol{\delta}}$. Because $\boldsymbol{\delta} \leq \tilde{\boldsymbol{\delta}}$, we find thus g to be in particular locally $\boldsymbol{\delta}$-constant.

By construction $\|f-g\|_{\sup} \leq \varepsilon\gamma \leq \varepsilon$ and

$$\begin{aligned}
|||(f-g)^{[\boldsymbol{\rho}]}|||_{\sup} &= |||(f-g)^{[\boldsymbol{\rho}]}|||_{X^{[\boldsymbol{\rho}]}} \\
&= |||(f-g)^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in X^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\leq\tilde{\boldsymbol{\delta}}\}} \vee |||(f-g)^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in X^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\not\leq\tilde{\boldsymbol{\delta}}\}} \\
&\leq |||f^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in X^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\leq\tilde{\boldsymbol{\delta}}\}} \vee |||g^{[\boldsymbol{\rho}]}|||_{\{(x,y)\in X^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\leq\tilde{\boldsymbol{\delta}}\}} \\
&\quad \vee \max_{(x,y)\in X^{[\boldsymbol{\rho}]}:\mathbf{d}(x,y)\not\leq\tilde{\boldsymbol{\delta}}} \left(\frac{\|f(x)-g(x)\|}{\mathrm{d}(x,y)^{\boldsymbol{\rho}}} \vee \frac{\|f(y)-g(y)\|}{\mathrm{d}(x,y)^{\boldsymbol{\rho}}}\right) \\
&\leq \varepsilon \vee 0 \vee \varepsilon\gamma/\gamma = \varepsilon;
\end{aligned}$$

the first equality by the preceding Lemma 1.27. Regarding the last inequality, we note that for $(x,y)\in X^{[\boldsymbol{\rho}]}$, we have $\mathbf{d}(x,y)\not\leq\tilde{\boldsymbol{\delta}}$ if and only if there is $k\in\{1,\dots,d\}$ with $\rho_k>0$ such that $\mathrm{d}_k^{\rho_k}(x_k,y_k) > \delta_k^{\rho_k} = \gamma$, and so $\mathrm{d}(x,y)^{\boldsymbol{\rho}} > \gamma$. ■

Corollary 1.29. *Let $X = X_1\times\cdots\times X_d$ be a compact cartesian metric space. For $\boldsymbol{\rho}\in[0,1[^d$, the locally constant functions $g : X\to\mathbf{E}$ are dense in $\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$.*

Proof. Fix $\varepsilon>0$ and $f\in\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$. Then $|f^{[\boldsymbol{\rho}]}| : X\times X\to\mathbb{R}_{\geq 0}$ is by compactness of $X\times X$ a uniformly continuous function vanishing on $\triangle(X\times X)$. Hence we find $\boldsymbol{\delta}\in]0,1]^d$ such that in particular for all $\vec{a}=(a,a)\in X\times X$, it holds

$$||f^{[\boldsymbol{\rho}]}|(x,y) - |f^{[\boldsymbol{\rho}]}|(\vec{a})| = |f^{[\boldsymbol{\rho}]}|(x,y) \leq \varepsilon \quad \text{for all } (x,y)\in X\times X\cap \mathrm{B}_{\leq(\boldsymbol{\delta},\boldsymbol{\delta})}(\vec{a}).$$

By possibly shrinking $\boldsymbol{\delta}\in]0,1]^d$ coordinatewise, we can moreover assume $\delta_k^{\rho_k}=\gamma$ for all $k\in L := \{l\in\{1,\dots,d\} : \rho_l>0\}$ with $\gamma := \min\{\delta_l^{\rho_l} : l\in L\} > 0$. Then $\boldsymbol{\delta}$ fulfills the conditions of the preceding Lemma 1.28.
By the triangle inequality, if we have $\mathbf{d}(x,y)\leq\boldsymbol{\delta}$ for $(x,y)\in X\times X$, then $\mathbf{d}((x,y),(a,a))\leq(\boldsymbol{\delta},\boldsymbol{\delta})$ for some $a\in X$. Thus for all $(x,y)\in X\times X$ holds

$$\|f(x)-f(y)\| \leq \varepsilon\cdot\mathrm{d}(x,y)^{\boldsymbol{\rho}} \quad \text{if } \mathbf{d}(x,y)\leq\boldsymbol{\delta}.$$

By Lemma 1.28 we find therefore locally $\boldsymbol{\delta}$-constant $g : X\to\mathbf{E}$ with $\|f-g\|_{\mathcal{C}^{\boldsymbol{\rho}}}\leq\varepsilon$. In particular the locally constant functions are dense in $\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$. ■

Another characterization

Lemma 1.30. *Let $X = X_1\times\cdots\times X_d$ be a compact cartesian metric space and $f\in\mathcal{C}^{\boldsymbol{\rho}}(X,\mathbf{E})$. Then*

$$|||f^{[\boldsymbol{\rho}]}|||_{\sup} = |||f^{[\rho_1\cdot e_1]}|||_{\sup} \vee \ldots \vee |||f^{[\rho_d\cdot e_d]}|||_{\sup}.$$

Proof. Firstly, we find by definition $|||f^{[\boldsymbol{\rho}]}|||_{\sup} \geq |||f^{[\rho_k\cdot e_k]}|||_{\sup}$ for $k=1,\dots,d$. Contrariwise, let $x, y \in X$ be distinct. We compute

$$\begin{aligned}
&|f^{[\rho]}|(x,y) \\
=&\frac{\|f(y)-f(x)\|}{\mathrm{d}(x,y)^{\rho}} \\
=&\frac{\|(f(y_1,\dots,y_{d-1},y_d)-f(y_1,\dots,y_{d-1},x_d))+\dots+(f(y_1,x_2,\dots,x_d)-f(x))\|}{\mathrm{d}(x,y)^{\rho}} \\
\leq&\frac{\max_{k=1,\dots,d}\|f(y_1,\dots,y_{k-1},y_k,x_{k-1},\dots,x_d)-f(y_1,\dots,y_{k-1},x_k,x_{k+1},\dots,x_d)\|}{\max_{k=1,\dots,d}\mathrm{d}(y_k,x_k)^{\rho_k}} \\
\leq&\max_{k=1,\dots,d \text{ with } x_k\neq y_k}\frac{\|f(y_1,\dots,y_{k-1},y_k,x_{k-1},\dots,x_d)-f(y_1,\dots,y_{k-1},x_k,x_{k+1},\dots,x_d)\|}{\mathrm{d}(y_k,x_k)^{\rho_k}} \\
=&\max_{k=1,\dots,d \text{ with } x_k\neq y_k}|f^{[\rho_k\cdot e_k]}|((y_1,\dots,y_{k-1},y_k,x_{k+1},\dots,x_d),(y_1,\dots,y_{k-1},x_k,x_{k+1},\dots,x_d)) \\
\leq&\max_{k=1,\dots,d}\||f^{[\rho_k\cdot e_k]}|\|_{\sup}.
\end{aligned}$$

■

Corollary 1.31. *Let $X = X_1 \times \dots \times X_d$ be a cartesian metric space. The locally convex $\mathbf{K}$-vector space $\mathcal{C}^{\rho}(X,\mathbf{E})$ is the initial locally convex $\mathbf{K}$-vector space with respect to the inclusion mappings*

$$\begin{array}{ccc}
 & & \mathcal{C}^{\rho_1\cdot e_1}(X,\mathbf{E}) \\
 & \overset{\text{incl.}}{\nearrow} & \\
\mathcal{C}^{\rho}(X,\mathbf{E}) & & \vdots \\
 & \underset{\text{incl.}}{\searrow} & \\
 & & \mathcal{C}^{\rho_d\cdot e_d}(X,\mathbf{E}).
\end{array}$$

It holds $\|\cdot\|_{\mathcal{C}^{\rho},C} = \|\cdot\|_{\mathcal{C}^{\rho_1\cdot e_d},C} \vee \dots \vee \|\cdot\|_{\mathcal{C}^{\rho_d\cdot e_d},C}$ for any compact $C \subseteq X$.

Proof. We assume firstly X to be compact. By definition, the function $f : X \to \mathbf{E}$ is $\mathcal{C}^{\rho}$ at a if and only if it is $\mathcal{C}^{\rho_k\cdot e_k}$ at a for $k = 1,\dots,d$ and by Lemma 1.30, we have $\|f^{[\rho]}\|_{\sup} = \|f^{[\rho_1\cdot e_1]}\|_{\sup} \vee \dots \vee \|f^{[\rho_d\cdot e_d]}\|_{\sup}$ for any compact $C \subseteq X$.
By the first step, we find $\mathcal{C}^{\rho}(C,\mathbf{E})$ to be initial locally convex $\mathbf{K}$-vector space with respect to the inclusions of $\mathcal{C}^{\rho_1\cdot e_1}(C,\mathbf{E}),\dots,\mathcal{C}^{\rho_d\cdot e_d}(C,\mathbf{E})$. We conclude by applying Remark 1.23 to $\mathcal{C}^{\rho}(C,\mathbf{E})$ as well as $\mathcal{C}^{\rho_1\cdot e_1}(C,\mathbf{E}),\dots,\mathcal{C}^{\rho_d\cdot e_d}(C,\mathbf{E})$. ■

Symmetry properties

Lemma 1.32. *Let $X = X_1 \times \dots \times X_d$ be a compact cartesian metric space and $f : X \to \mathbf{E}$ be a mapping symmetric in its k-th and l-th coordinate. Then $f \in \mathcal{C}^{\rho\cdot e_k}(C,\mathbf{E})$ if and only if $f \in \mathcal{C}^{\rho\cdot e_l}(C,\mathbf{E})$ and it holds $\|f\|_{\mathcal{C}^{\rho\cdot e_k}} = \|f\|_{\mathcal{C}^{\rho\cdot e_l}}$.*

Proof. Since X is compact, we find $f \in \mathcal{C}^{\rho\cdot e_k}(X,\mathbf{E})$ for some $k \in \{1,\dots,d\}$ if and only if for every $\varepsilon > 0$ exists $\delta > 0$ such that $\|f(x) - f(y)\| \leq \varepsilon \cdot \mathrm{d}(x,y)^{\rho\cdot e_k}$ for all $x, y \in X$ with $\mathrm{d}(x,y) \leq \delta$. Since f is symmetric in its k-th and l-th coordinate, we see by Lemma 1.18 that

$f \in \mathcal{C}^{\rho\cdot\mathbf{e}_k}(X,\mathbf{E})$ if and only if $f \in \mathcal{C}^{\rho\cdot\mathbf{e}_l}(X,\mathbf{E})$.

Moreover, denote by σ the permutation map on X swapping the k-th and l-th coordinate. Then (σ,σ) acts on $X \times X \supseteq X^{[\rho\cdot\mathbf{e}_k]}$ and

$$\||f^{[\rho\cdot\mathbf{e}_k]}|\|_{X^{[\rho\cdot\mathbf{e}_k]}} = \||f^{[\rho\cdot\mathbf{e}_l]}| \circ (\sigma,\sigma)\|_{X^{[\rho\cdot\mathbf{e}_k]}} = \||f^{[\rho\cdot\mathbf{e}_l]}|\|_{(\sigma,\sigma)X^{[\rho\cdot\mathbf{e}_k]}} = \||f^{[\rho\cdot\mathbf{e}_l]}|\|_{X^{[\rho\cdot\mathbf{e}_l]}}.$$

■

Corollary 1.33. *Let $X = X_1 \times \cdots \times X_d$ be a compact cartesian metric space. Let $\{1,\dots,d\} = I_1 \dot\cup \dots \dot\cup I_e$ with representatives $i_1,\dots,i_d$ and $f : X \to \mathbf{E}$ a mapping symmetric in its coordinates indexed by $I_1,\dots,I_e$. Then*

$$f \in \mathcal{C}^{\rho}(X,E) \quad \text{if and only if} \quad f \in \mathcal{C}^{\rho\cdot\mathbf{e}_{i_1}}(X,\mathbf{E}) \cap \dots \cap \mathcal{C}^{\rho\cdot\mathbf{e}_{i_e}}(X,\mathbf{E})$$

and it holds

$$\|f\|_{\mathcal{C}^\rho} = \|f\|_{\mathcal{C}^{\rho\cdot\mathbf{e}_{i_1}}} \vee \dots \vee \|f\|_{\mathcal{C}^{\rho\cdot\mathbf{e}_{i_e}}}.$$

Proof. We have

$$\mathcal{C}^{\rho}(X,\mathbf{E}) = \mathcal{C}^{\vec\rho}(X,\mathbf{E}) = \mathcal{C}^{\rho\cdot\mathbf{e}_1}(X,\mathbf{E}) \cap \dots \cap \mathcal{C}^{\rho\cdot\mathbf{e}_d}(X,\mathbf{E})$$

and

$$\|\cdot\|_{\mathcal{C}^\rho} = \|\cdot\|_{\mathcal{C}^{\vec\rho}} = \|\cdot\|_{\mathcal{C}^{\rho\cdot\mathbf{e}_1}} \vee \dots \vee \|\cdot\|_{\mathcal{C}^{\rho\cdot\mathbf{e}_d}},$$

the first equality by Remark 1.22 for $\vec\rho = (\rho,\dots,\rho)$ and the second one by Corollary 1.31. We can then conclude by applying Lemma 1.32. ■

1.4 $\mathcal{C}^{1+\rho}$-functions

Assumption. Throughout this subsection, we will make the following assumptions:

- We will fix a real number $\rho \in [0,1[$.
- We let $X = X_1 \times \cdots \times X_d \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point.

A quick remark on this assumption's origin: Define the - e.g. first - partial difference quotient of a function $f : X \to \mathbf{E}$ by

$$f^{]1,0,\dots,0[}(x,t) = \frac{f(x + t\cdot \boldsymbol{e}_1) - f(x)}{t} \quad \text{for } x \in X, t \in \mathbf{K}^* \text{ with } x + t\cdot\boldsymbol{e}_1 \in X.$$

Then f is defined to be once partially differentiable in its first coordinate at $a \in X$ if and only if this function is $\mathcal{C}^0$ at $(a,0)$. But this function $f^{]1,0,\dots,0[}$ has a **unique** extension onto $(a,0)$ with value $D_{1,0,\dots,0}f(a) := \lim_{t\to 0} f^{]1,0,\dots,0[}(a,t)$ if and only if a_1 is an accumulation point of X_1.

Definition. Let $f : X \to \mathbf{E}$ be a mapping.

(i) Put $\quad X^{]e_k[} = X_1 \times \cdots \times X_{k-1} \times \triangledown X_k^2 \times X_{k+1} \times \cdots \times X_d$.

We define $f^{]e_k[} : X^{]e_k[} \to \mathbf{E}$ by

$$f^{]e_k[}(-; y_k, x_k; -) = \frac{f(x + t \cdot \boldsymbol{e}_k) - f(x)}{t};$$

here $x := (x_1, \ldots, x_d)$ and $t := y_k - x_k \neq 0$ - the hyphenations to the left and right of the semicolons representing the omitted coordinate entries $x_1, \ldots, x_{k-1}$ and $x_{k+1}, \ldots, x_d$.

(ii) Put $\quad X^{[e_k]} = X_1 \times \cdots \times X_{k-1} \times X_k^2 \times X_{k+1} \times \cdots \times X_d$.

Then f will be a $\mathcal{C}^{(1+\rho)\cdot e_k}$**-function** if f is continuous and $f^{]e_k[} : X^{]e_k[} \to \mathbf{E}$ extends (uniquely) to a $\mathcal{C}^\rho$-function $f^{[e_k]} : X^{[e_k]} \to \mathbf{E}$. We denote the set of all $\mathcal{C}^{(1+\rho)\cdot e_k}$-functions $f : X \to \mathbf{E}$ by $\mathcal{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E})$. For compact cartesian $C \subseteq X$ we define the seminorm $\|\cdot\|_{\mathcal{C}^{(1+\rho)\cdot e_k}, C}$ on $\mathcal{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E})$ by

$$\|f\|_{\mathcal{C}^{(1+\rho)\cdot e_k}, C} = \|f\|_C \vee \|f^{[e_k]}\|_{\mathcal{C}^\rho, C^{[e_k]}}.$$

(iii) We define $\mathcal{C}^{1+\rho}(X, \mathbf{E})$ as the initial locally convex $\mathbf{K}$-vector space with respect to the inclusion mappings $\mathcal{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E}) \hookrightarrow \mathcal{C}^0(X, \mathbf{E})$ for $k = 1, \ldots, d$, that is we put $\mathcal{C}^{1+\rho}(X, \mathbf{E}) = \mathcal{C}^{(1+\rho)\cdot e_1}(X, \mathbf{E}) \cap \ldots \cap \mathcal{C}^{(1+\rho)\cdot e_d}(X, \mathbf{E}) \subseteq \mathcal{C}^0(X, \mathbf{E})$ and for compact cartesian $C \subseteq X$, we define the seminorm $\|\cdot\|_{\mathcal{C}^{1+\rho}, C}$ on $\mathcal{C}^{1+\rho}(X, \mathbf{E})$ by

$$\|f\|_{\mathcal{C}^{1+\rho}, C} = \|f\|_{\mathcal{C}^{(1+\rho)\cdot e_1}, C} \vee \ldots \vee \|f\|_{\mathcal{C}^{(1+\rho)\cdot e_d}, C}.$$

Proposition 1.34. *Let $f : X \to \mathbf{E}$ be a mapping. Consider the following*

Definition. (i) For all $y, x \in X$ with $y = x + t_1\boldsymbol{e}_1 + \cdots + t_d\boldsymbol{e}_d$ and $t_1, \ldots, t_d \in \mathbf{K}^*$, there is a unique $\mathbf{K}$-linear map $A =: f^{]1[}(y, x) : \mathbf{K}^d \to \mathbf{E}$ defined through

$$A \cdot t_k\boldsymbol{e}_k := f(x + t_1 \cdot \boldsymbol{e}_1 + \cdots + t_k \cdot \boldsymbol{e}_k) - f(x + t_1 \cdot \boldsymbol{e}_1 + \cdots + t_{k-1} \cdot \boldsymbol{e}_{k-1}) \quad \text{for } k = 1, \ldots, d.$$

(ii) Define $\quad X^{]1[} := \{(y, x) \in X \times X : y = x + t_1\boldsymbol{e}_1 + \cdots + t_d\boldsymbol{e}_d \text{ with } t_1, \ldots, t_d \in \mathbf{K}^*\}$ and $\quad X^{[1]} := X \times X$.

We will say that $f : X \to \mathbf{E}$ is a $\tilde{\mathcal{C}}^{1+\rho}$**-function** if $f^{]1[} : X^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E})$ extends (uniquely) to a $\mathcal{C}^\rho$-function $f^{[1]} : X^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E})$ with respect to the operator norm on its range. We will denote the set of all $\tilde{\mathcal{C}}^{1+\rho}$-functions $f : X \to \mathbf{E}$ by $\tilde{\mathcal{C}}^{1+\rho}(X, \mathbf{E})$. For every compact cartesian $C \subseteq X$ we define the seminorm $\|\cdot\|_{\tilde{\mathcal{C}}^{1+\rho}, C}$ on $\tilde{\mathcal{C}}^{1+\rho}(X, \mathbf{E})$ by

$$\|f\|_{\tilde{\mathcal{C}}^{1+\rho}, C} = \|f\|_C \vee \|(f^{[1]}\|_{\mathcal{C}^\rho, C^{[1]}}.$$

Then $\tilde{\mathcal{C}}^{1+\rho}(X, \mathbf{E}) = \mathcal{C}^{1+\rho}(X, \mathbf{E})$ and $\|\cdot\|_{\tilde{\mathcal{C}}^{1+\rho}, C} = \|\cdot\|_{\mathcal{C}^{1+\rho}, C}$ for $C \subseteq X$ compact cartesian.

Proof. We have $A \cdot \boldsymbol{e}_1 = f^{]e_1[}(y_1, x_1; x_2; \ldots; x_d), \ldots, A \cdot \boldsymbol{e}_d = f^{]e_d[}(y_1; \ldots; y_{d-1}; y_d, x_d)$ for $y, x \in X$ with $y = x + t_1 \cdot \boldsymbol{e}_1 + \cdots + t_d \cdot \boldsymbol{e}_d$ and $t_1, \ldots, t_d \in \mathbf{K}^*$. Hence under the isometric isomorphism of $\mathbf{K}$-Banach spaces

$$\begin{aligned}\mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E}) &\to \mathbf{E}^d \\ A &\mapsto (A \cdot \boldsymbol{e}_1, \ldots, A \cdot \boldsymbol{e}_d),\end{aligned}$$

we obtain $f^{]1[}(y, x) = (f^{]e_1[}(y_1, x_1; x_2; \ldots; x_d), \ldots, f^{]e_d[}(y_1; \ldots; y_{d-1}; y_d, x_d))$. Hence $f^{]1[} : X^{]1[} \to \mathbf{E}^d$ extends to a $\mathcal{C}^\rho$-function on all $x, y \in X$ if and only if $f^{]e_k[} : X^{]e_k[} \to \mathbf{E}$ extends to a $\mathcal{C}^\rho$-function $f^{[e_k]} : X^{]e_k[} \to \mathbf{E}$ for $k = 1, \ldots, d$. That is,

$$\mathcal{C}^{(1+\rho)\cdot e_1}(X, \mathbf{E}) \cap \ldots \cap \mathcal{C}^{(1+\rho)\cdot e_d}(X, \mathbf{E}) = \mathcal{C}^{1+\rho}(X, \mathbf{E}).$$

For $y, x \in X$ with $y = x + t_1 \boldsymbol{e}_1 + \cdots + t_d \boldsymbol{e}_d$ and $t_1, \ldots, t_d \in \mathbf{K}$, we find

$$\begin{aligned}\|f^{[1]}(y, x)\| =& \|A\| = \|A \cdot \boldsymbol{e}_1\| \vee \ldots \vee \|A \cdot \boldsymbol{e}_d\| \\ =& \|f^{[e_1]}(y_1, x_1; x_2; \ldots; x_d)\| \vee \ldots \vee \|f^{[e_d]}(y_1; \ldots; y_{d-1}; y_d, x_d)\|.\end{aligned}$$

If we let x, y run through $C \subseteq X$ compact cartesian, we find

$$\|f^{[1]}\|_{\mathcal{C}^\rho, C^{[1]}} = \|f^{[e_1]}\|_{\mathcal{C}^\rho, C^{[e_1]}} \vee \ldots \vee \|f^{[e_d]}\|_{\mathcal{C}^\rho, C^{[e_d]}}.$$

Therefore $\|f\|_{\tilde{\mathcal{C}}^{1+\rho}, C} = \|f\|_{\mathcal{C}^{1+\rho}, C}$. ■

Remark 1.35. Let $f \in \mathcal{C}^1(X, \mathbf{E})$ and $a \in X$. Consider the $\mathbf{K}$-linear mapping $D_a f := f^{[1]}(a, a) \in \mathrm{Hom}_{\mathbf{K}\text{-vctsp}}(\mathbf{K}^d, \mathbf{E})$. Then for every $\varepsilon > 0$, there exists a neighborhood $U \ni a$ in X such that $\|f^{[1]}(x + h, x) - D_a f\| \leq \varepsilon$ for all $x + h, x \in U$. In particular

$$\begin{aligned}\|f(x + h) - f(x) - D_a f \cdot h\| =& \|f^{[1]}(x + h, x) \cdot h - D_a f \cdot h\| \\ \leq& \|f^{[1]}(x + h, x) - D_a f\| \|h\| \leq \varepsilon \|h\| \quad \text{for all } x + h, x \in U.\end{aligned}$$

This is usually called *strict differentiability*. Therefore if a function is $\mathcal{C}^1$ at a point $a \in X$, then it is strictly differentiable at a. In the other direction, given $\varepsilon > 0$ we find a neighborhood $U \ni a$ in X such that in particular for all $y = x + t \cdot \boldsymbol{e}_k, x \in U$ with $k = 1, \ldots, d$ holds

$$\|f^{]e_k[}(-; x_k + t, x_k; -) - D_a f \cdot \boldsymbol{e}_k\| = \|1/t \cdot (f(x + t \cdot \boldsymbol{e}_k) - f(x) - D_a f)\| \leq \varepsilon \cdot |t| |t \boldsymbol{e}_k| = \varepsilon.$$

Therefore $f^{]e_k[}$ is $\mathcal{C}^0$ at $\vec{a} = (-; a_k, a_k; -) \in X^{[e_k]}$ for $k = 1, \ldots, d$ or, by the preceding Proposition 1.34, equivalently the function $f^{]1[}$ is $\mathcal{C}^0$ at (a, a).

Remark 1.36 (About (in)equalities of continuous functions on dense subsets). Let X be a topological space, Y a normed space and $f, g : X \to Y$ two continuous functions thereon. Let $A \subseteq X$ be a dense subset. Then $f(a) = g(a)$ respectively $\|f(a)\| \leq \|g(a)\|$ for all $a \in A$ implies $f(x) = g(x)$ respectively $\|f(x)\| \leq \|g(x)\|$ for all $x \in X$.

Proof. Let $h := f - g : X \times X \to Y$. We know that $F(\overline{Z}) \subseteq \overline{F(Z)}$ for any continuous function F and any subset Z in the domain of F. Putting $Z = A \times A$ and $F = h$, we have $\overline{A \times A} = \overline{A} \times \overline{A} = X \times X$ and therefore $h_{|A} = 0$ respectively $\|h\|_{|A} \leq 0$ pointwise implies $h = 0$ respectively $\|h\| \leq 0$ pointwise; the former since $\{0\} \subseteq Y$ is closed, the latter since $\|\cdot\| : Y \to \mathbb{R}_{\geq 0}$ is continuous (and $\mathbb{R}_{\geq 0} \subseteq \mathbb{R}$ closed). This means $f(x) = g(x)$ respectively $\|f(x)\| \leq \|g(x)\|$ for all $x \in X$. ∎

Assumption. Let X be a topological space, Y a normed space and $F : X \to Y$ a continuous function. Whenever we will in the following refer to the *continuous extension* of a proposition claiming a certain property only for the continuous function $f := F_{|A}$ defined on a dense subset $A \subseteq X$, we mean to invoke the proposition together with the above Remark's 1.36 observation to infer the claimed property for F itself.

The following two observations will mainly be used later on, but due to their basic character it seemed appropriate to state them here.

Lemma 1.37. *Let $X \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point. Then we have norm-nonincreasing inclusions $\mathcal{C}^1(X, \mathbf{E}) \subseteq \mathcal{C}^{\text{lip}}(X, \mathbf{E}) \subseteq \mathcal{C}^\rho(X, \mathbf{E})$.*

Proof. Firstly, let $f \in \mathcal{C}^1(X, \mathbf{E})$. Fix $\varepsilon > 0$ and $a \in X$. By definition $f(y) - f(x) = f^{]1[}(y, x) \cdot (y - x)$ and hence $\|f(x) - f(y)\| \leq \|f^{]1[}(x, y)\| \|x - y\|$ for all $(y, x) \in X^{]1[}$. Since $f \in \mathcal{C}^1(X, \mathbf{E})$, the function $f^{]1[} : X^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E})$ extends to a continuous function $f^{[1]} : X^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E})$. As moreover $X^{]1[} \subseteq X^{[1]} = X \times X$ densely, it follows by Remark 1.36 that $\|f(x) - f(y)\| \leq \|f^{[1]}(x, y)\| \|x - y\|$ for all $(x, y) \in X^{[1]} = X \times X$. By continuity, there exists a neighborhood $U \ni a$ such that $\|f^{[1]}(x, y)\| \leq M$ for all $x, y \in U$ and a constant $M > 0$. I.e. $f \in \mathcal{C}^{\text{lip}}(X, \mathbf{E})$.
Secondly, let $f \in \mathcal{C}^{\text{lip}}(X, \mathbf{E})$. Then

$$\|f(x) - f(y)\| \leq |f^{]1[}|(x, y) \|x - y\| \leq M \|x - y\| \leq \varepsilon \|x - y\|^\rho$$

for all (distinct) x, y in a neighborhood $V \subseteq U$ of a with $\|x - y\|^{1-\rho} \leq \varepsilon / M$, i.e. $f \in \mathcal{C}^\rho(X, \mathbf{E})$. This proves $\mathcal{C}^\rho(X, \mathbf{E}) \supseteq \mathcal{C}^{\text{lip}}(X, \mathbf{E}) \supseteq \mathcal{C}^1(X, \mathbf{E})$.

Regarding the norms, we first off remark that these inclusions to be norm-nonincreasing means that $\|\cdot\|_{\mathcal{C}^\rho, C} \leq \|\cdot\|_{\mathcal{C}^{\text{lip}}, C}$ on $\mathcal{C}^{\text{lip}}(X, \mathbf{E})$ for $C \subseteq X$ compact and $\|\cdot\|_{\mathcal{C}^{\text{lip}}, C} \leq \|\cdot\|_{\mathcal{C}^1, C}$ on $\mathcal{C}^1(X, \mathbf{E})$ for $C \subseteq X$ compact cartesian.
Let $C \subseteq X$ be compact. We firstly show $\||f|^{[\rho]}\|_{\sup} \leq \||f^{]1[}|\|_{\sup} \vee \|f\|_{\sup}$ for $f \in \mathcal{C}^{\text{lip}}(C, \mathbf{E})$. Let $x, y \in C$. If $x = y$, then surely $|f^{[\rho]}|(x, y) = 0 \leq \||f^{]1[}|\|_{\sup}$. Let them be distinct. We distinguish two cases.
Case 1: $\|x - y\| \leq 1$. By the above estimate, we find

$$|f^{[\rho]}|(x, y) = \|x - y\|^{\rho - 1} \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq \|x - y\|^{\rho - 1} |f^{]1[}|(x, y) \leq |f^{]1[}|(x, y) \leq \||f^{]1[}|\|_{\sup}.$$

Case 2: $\|x - y\| > 1$. Then

$$|f^{[\rho]}|(x, y) = \|x - y\|^{-\rho} \|f(x) - f(y)\| \leq \|f(x) - f(y)\| \leq \|f(x)\| \vee \|f(y)\| \leq \|f\|_{\sup}.$$

Secondly, let $C \subseteq X$ be compact cartesian and $f \in \mathcal{C}^1(C, \mathbf{E})$. By the proof of Proposition 1.34 for $\rho = 1$, we find $\|f^{[e_1]}\|_{\sup} \vee \ldots \vee \|f^{[e_d]}\|_{\sup} = \|f^{[1]}\|_{\sup}$. Because $f^{]1[}(x, y) \cdot (x - y) = f(x) - f(y)$ for $(x, y) \in X^{]1[}$, we have indeed $\||f^{]1[}|\|_{C^{]1[}} \leq \|f^{[1]}\|_{C^{]1[}}$, and so by continuity $\|f\|_{\mathcal{C}^{\text{lip}},C} = \|f\|_{\mathcal{C}^1,C}$. ■

Lemma 1.38. *Let X and Y be compact metric spaces, $g \in \mathcal{C}^\rho(Y, \mathbf{E})$ and $f \in \mathcal{C}^{\text{lip}}(X, Y)$. Then $\|g \circ f\|_{\mathcal{C}^\rho} \leq (1 \vee \||f^{]1[}|\|^\rho_{\sup}) \cdot \|g\|_{\mathcal{C}^\rho} \leq (1 \vee \|f\|^\rho_{\mathcal{C}^{\text{lip}}}) \cdot \|g\|_{\mathcal{C}^\rho}$.*

Proof. We find $g \circ f \in \mathcal{C}^\rho(X, \mathbf{E})$ by Proposition 1.7(i). For $x, y \in X$ with $f(x), f(y) \in Y$ distinct holds

$$\begin{aligned}|g \circ f^{]\rho[}|(x, y) =& \|g(f(x)) - g(f(y))\| / \|x - y\|^\rho \\ =& \|g(f(x)) - g(f(y))\| / \|f(x) - f(y)\|^\rho \cdot \|f(x) - f(y)\|^\rho / \|x - y\|^\rho \\ =& |g^{]\rho[}|(f(x), f(y)) \cdot |f^{]1[}|(x, y)^\rho.\end{aligned}$$

This equality extends by zero to all other distinct $x, y \in X$. Therefore

$$\|g \circ f\|_{\mathcal{C}^\rho} = \|g \circ f\|_{\sup} \vee \||g \circ f^{]\rho[}|\|_{\sup} \leq \|g\|_{\sup} \vee \|g\|_{\mathcal{C}^\rho} \||f^{]1[}|\|^\rho_{\sup} \leq (1 \vee \|f\|^\rho_{\mathcal{C}^{\text{lip}}}) \cdot \|g\|_{\mathcal{C}^\rho}.$$

■

Symmetry properties

Lemma 1.39. *Let $f : X \to \mathbf{E}$ be a mapping symmetric in its coordinates $k, l \in \{1, \ldots, d\}$. Then*

$$f \in \mathcal{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E}) \quad \text{if and only if} \quad f \in \mathcal{C}^{(1+\rho)\cdot e_l}(X, \mathbf{E})$$

and for every compact cartesian $C \subseteq X$ symmetric in its k-th and l-th coordinate holds

$$\|f\|_{\mathcal{C}^{(1+\rho)\cdot e_k},C} = \|f\|_{\mathcal{C}^{(1+\rho)\cdot e_l},C}.$$

Proof. By symmetry, it will suffice to prove one direction, e.g. if $f \in \mathcal{C}^{(1+\rho)\cdot e_l}(X, \mathbf{E})$, then $f \in \mathcal{C}^{(1+\rho)\cdot e_k}(X, \mathbf{E})$. Since f is symmetric, we find

$$\begin{aligned}& f^{]e_l[}(-; x_k; -; x_l, y_l; -) \\ =& \frac{f(\ldots, x_k, \ldots, x_l, \ldots) - f(\ldots, x_k, \ldots, y_l, \ldots)}{x_l - y_l} \\ =& \frac{f(\ldots, \overbrace{x_l}^{k\text{-th place}}, \ldots, \overbrace{x_k}^{l\text{-th place}}, \ldots) - f(\ldots, \overbrace{y_l}^{k\text{-th place}}, \ldots, \overbrace{x_k}^{l\text{-th place}}, \ldots)}{x_l - y_l} \\ =& f^{]e_k[}(-; \underbrace{x_l, y_l}_{k\text{-th post}}; -; \underbrace{x_k}_{l\text{-th post}}; -).\end{aligned}$$

Denote by $\sigma \in \mathcal{C}^{\text{lip}}(X^{[e_k]}, X^{[e_l]})$ the permutation map defined by

$$\sigma(-; x_k; -; x_l, y_l; -) = (-; \underbrace{x_l, y_l}_{k\text{-th post}}; -; \underbrace{x_k}_{l\text{-th post}}; -).$$

Then we just saw $f^{]e_k[} = f^{]e_l[} \circ \sigma_{|X^{]e_k[}}$. Hence $f^{]e_k[} : X^{]e_k[} \to \mathbf{E}$ extends to the $\mathcal{C}^\rho$-function $f^{[e_k]} : X^{[e_k]} \to \mathbf{E}$ if and only if $f^{]e_l[} \circ \sigma_{|X^{]e_k[}} : X^{]e_k[} \to \mathbf{E}$ extends to the $\mathcal{C}^\rho$-function $f^{[e_l]} \circ \sigma : X^{[e_k]} \to \mathbf{E}$. Here we used $f^{[e_k]} = f^{[e_l]} \circ \sigma$, as these are two continuous functions coinciding on the dense subset $X^{]e_k[} \subseteq X^{[e_k]}$. Since we assume $f^{[e_l]} \in \mathcal{C}^\rho(X^{[e_l]}, \mathbf{E})$, by Proposition 1.7(i) holds $f^{[e_k]} = f^{[e_l]} \circ \sigma \in \mathcal{C}^\rho(X^{[e_k]}, \mathbf{E})$. Therefore $f^{]e_k[} : X^{]e_k[} \to \mathbf{E}$ extends to the $\mathcal{C}^\rho$-function $f^{[e_k]} : X \to \mathbf{E}$.

Moreover, for every compact cartesian $C \subseteq X$ symmetric in its k-th and l-th coordinate holds $\sigma C^{[e_k]} = C^{[e_l]}$. Thus

$$\|f^{[e_k]}\|_{\mathcal{C}^\rho, C^{[e_k]}} = \|f^{[e_l]} \circ \sigma\|_{\mathcal{C}^\rho, C^{[e_k]}} = \|f^{[e_l]}\|_{\mathcal{C}^\rho, C^{[e_l]}};$$

here we used $\|\|(\sigma^{\pm 1}_{|C})^{]1[}\|\|_{\sup} = 1$ and Lemma 1.38 for the latter equality. We conclude $\|f\|_{\mathcal{C}^{(1+\rho)\cdot e_k}, C} = \|f\|_{\mathcal{C}^{(1+\rho)\cdot e_l}, C}$. ∎

Corollary 1.40. *Let $\{1, \ldots, d\} = I_1 \dot\cup \ldots \dot\cup I_e$ with representatives $i_1, \ldots, i_d$ and $f : X \to \mathbf{E}$ a mapping symmetric in its coordinates indexed by $I_1, \ldots, I_e$. Then*

$$f \in \mathcal{C}^{1+\rho}(X, E) \quad \text{if and only if} \quad f \in \mathcal{C}^{(1+\rho)\cdot e_{i_1}}(X, \mathbf{E}) \cap \ldots \cap \mathcal{C}^{(1+\rho)\cdot e_{i_e}}(X, \mathbf{E})$$

and for every compact cartesian $C \subseteq X$ symmetric in its coordinates indexed by $I_1, \ldots, I_d$ holds

$$\|f\|_{\mathcal{C}^{1+\rho}, C} = \|f\|_{\mathcal{C}^{(1+\rho)\cdot e_{i_1}}, C} \vee \ldots \vee \|f\|_{\mathcal{C}^{(1+\rho)\cdot e_{i_e}}, C}.$$

Proof. By Lemma 1.39. ∎

2 Fractional differentiability in one variable

Assumption. Throughout this section, we will fix a real number $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu = \lfloor r \rfloor \in \mathbb{N}$ and $\rho = \{r\} \in [0, 1[$.

2.1 $\mathcal{C}^r$-functions for $r \in \mathbb{R}_{\geq 0}$

Definition of $\mathcal{C}^r$-functions

We now specialize to the case that the function's domain X is a nonempty subset of $\mathbf{K}$ without isolated points and takes values in $\mathbf{K}$; our aim is a general definition of fractional differentiability under these circumstances. A good hint of the strong dependence of the common differentiability notion over the real numbers on the intermediate value theorem is given by the proof of the completeness of the continuously differentiable real-valued functions $\mathcal{C}^1(I, \mathbb{R})$ defined on an open interval I, which already uses the fundamental theorem of calculus. This shows that over general base fields we have to put stronger assumptions on our class of continuously differentiable functions to yield e.g. their completeness.

Definition. Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{K}$ a mapping thereon. For $\nu \in \mathbb{N}$ put

$$X^{[\nu]} = X^{\{0,\dots,\nu\}} \quad \text{and} \quad X^{]\nu[} := \nabla X^{[\nu]} = \{(x_0, \dots, x_\nu) : x_i = x_j \text{ only if } i = j\}.$$

The **ν-th difference quotient** $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ of a function $f : X \to \mathbf{K}$ is inductively given by $f^{]0[} := f$ and for $n \in \mathbb{N}$ and $(x_0, \dots, x_\nu) \in X^{]\nu[}$ by

$$f^{]\nu[}(x_0, \dots, x_\nu) := \frac{f^{]\nu-1[}(x_0, x_2, \dots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \dots, x_\nu)}{x_0 - x_1}.$$

Having already defined $\mathcal{C}^\rho$-functions for $\rho \in [0, 1[$, we add up our definitions to obtain our notion of fractional differentiability over (non-Archimedeanly valued) complete fields:

Definition. Fix $r = \nu + \rho \in \mathbb{R}_{\geq 0}$. Let $X \subseteq \mathbf{K}$ and $f : X \to \mathbf{K}$ a mapping thereon.

(i) We will say that f is $\mathcal{C}^r$ (or r **times continuously differentiable**) at a point $a \in X$ if $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ is $\mathcal{C}^\rho$ at $\vec{a} = (a, \dots, a) \in X^{[\nu]}$.

(ii) Then f will be a $\mathcal{C}^r$**-function** (or an r**-times continuously differentiable function**) if f is $\mathcal{C}^r$ at all points $a \in X$. The set of all $\mathcal{C}^r$-functions $f : X \to \mathbf{K}$ will be denoted by $\mathcal{C}^r(X, \mathbf{K})$.

Lemma 2.1. *Let $X \subseteq \mathbf{K}$. Then a function $f : X \to \mathbf{K}$ is $\mathcal{C}^r$ at a point $a \in X$ if and only if for every $\varepsilon > 0$, there exists a neighborhood $U \ni a$ in X such that*

$$|f^{]\nu[}(x_0, x_1, \dots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \dots, x_\nu)| \leq \varepsilon |x_0 - \tilde{x}_0|^\rho \quad \text{for distinct } x_0, \tilde{x}_0, x_1, \dots, x_\nu \in U.$$

Proof. By Example 1.15(i), the set $A := X^{]\nu[} \subseteq \mathbf{K}^{[\nu]}$ is telescopic. We find $f : A \to \mathbf{K}$ and $\vec{a} \in X^{[\nu]}$ to be both symmetric in all their coordinates. By Corollary 1.20 applied to the telescopic subset $A \subseteq X^{[\nu]} \subseteq \mathbf{K}^{[\nu]}$, the function $f^{]\nu[}$ is $\mathcal{C}^\rho$ at $\vec{a}$ if and only if, given $\varepsilon > 0$, there exists a neighborhood $U \ni \vec{a}$ in $X^{[\nu]}$ such that

$$|f^{]\nu[}(x) - f^{]\nu[}(y)| \leq \varepsilon |t|^\rho \quad \text{for all } x, y \in U \cap X^{]\nu[} \text{ with } y = x + t \cdot \boldsymbol{e}_1 \text{ and } t \in \mathbf{K}.$$

Excluding the trivial case $t = 0$ above, this translates to the proposition. ■

Remark. (i) We observe that the differentiability at some point a may vanish if the function's domain expands in $\mathbf{K}$ - as long as there is no neighborhood U of a in $\mathbf{K}$ lying in the domain.

(ii) Let a be some accumulation point in X. Then $\vec{a}$ is an accumulation point of $X^{]\nu[}$. As $\mathbf{K}$ is complete, we find by Remark 1.4 that $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ is $\mathcal{C}^0$ at $\vec{a} \in X^{[\nu]}$ if and only if their exists a limit $D_\nu f(a) \in \mathbf{K}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f^{]\nu[}(x) - D_\nu f(a)| \leq \varepsilon \quad \text{for all } x \in X^{]\nu[} \text{ with } |x_0 - a|, \ldots, |x_\nu - a| \leq \delta.$$

(iii) The previous point shows that our notion coincides with the common notion of ν-fold differentiability of f at an accumulation point a in the domain of f, as considered e.g. in [Schikhof, 1984, Section 29] in case $r = \nu \in \mathbb{N}$.

Lemma 2.2. *Let $X \subseteq \mathbf{K}$ be a nonempty subset.*

(i) The mapping $\mathbf{K}^X \to \mathbf{K}^{X^{]\nu[}}$ given by $f \mapsto f^{]\nu[}$ is $\mathbf{K}$-linear.

(ii) Let $f : X \to \mathbf{K}$ be a mapping on X. Then the function $f^{]\nu[}$ is symmetric in its $\nu + 1$ arguments.

Proof. Ad (i): This is quickly checked to hold.
Ad (ii): Conferring [Schikhof, 1984, Lemma 29.2(ii)], an induction shows that the difference quotients are symmetric in their arguments. ■

Properties of $\mathcal{C}^r$-functions

Lemma 2.3. *Let $X \subseteq \mathbf{K}$ be a subset, a some point in X and $f : X \to \mathbf{K}$ a mapping thereon. If f is $\mathcal{C}^r$ at a, then f will be $\mathcal{C}^s$ at a for every $s \leq r$.*

Proof. If f is $\mathcal{C}^r$ at a, then clearly f will be $\mathcal{C}^s$ at a for every $\nu \leq s \leq r$. By transitivity, it therefore suffices to prove that f is $\mathcal{C}^s$ at r with $s = \nu - 1 + \eta$ for $\eta \in [0, 1[$. We use the characterization by Lemma 2.1: On $X^{]\nu-1[}$ holds for distinct $x_0, \tilde{x}_0 \in X$ by definition

$$\begin{aligned}
&|f^{]\nu-1[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu-1[}(\tilde{x}_0, x_1, \ldots, x_\nu)| \\
=&|x_0 - \tilde{x}_0||f^{]\nu[}(x_0, \tilde{x}_0, x_1, \ldots, x_\nu)| \\
=&|x_0 - \tilde{x}_0|^\eta |x_0 - \tilde{x}_0|^{1-\eta} |f^{]\nu[}(x_0, \tilde{x}_0, x_1, \ldots, x_\nu)|.
\end{aligned}$$

If now f is $\mathcal{C}^r$ at $a \in X$, then $f^{]\nu[}$ will be $\mathcal{C}^0$ at $\vec{a}$ and in particular locally bounded by a constant $C > 0$ there. Fix $\varepsilon > 0$. Then there exists a neighborhood $V \ni a$ in X with $|x - \tilde{x}|^{1-\eta} \leq \varepsilon/C$ for all $x, \tilde{x} \in U$. Hence on the neighborhood $U \cap X^{]\nu-1[}$ with $U := V^{[\nu-1]} \ni \vec{a}$ in $X^{[\nu]}$ holds

$$|f^{]\nu-1[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu-1[}(\tilde{x}_0, x_1, \ldots, x_\nu)| \leq |x_0 - \tilde{x}_0|^\eta \varepsilon.$$

By Lemma 2.1, this proves f to be $\mathcal{C}^s$ at a. ■

Lemma 2.4. *Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points and $f : X \to \mathbf{K}$ a mapping thereon. Assume that for $r = \nu + \rho \in \mathbb{R}_{\geq 1}$, the map $f^{]\nu-1[} : X^{]\nu-1[} \to \mathbf{K}$ is $\mathcal{C}^\rho$ on all of $X^{[\nu-1]}$. Then $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ can be extended to a $\mathcal{C}^\rho$-function $f^{<\nu>} : X^{[\nu]} - \triangle X^{[\nu]} \to \mathbf{K}$.*

Proof. For $i, j \in \{0, \ldots, \nu\}$ with $i \neq j$ set

$$U_{ij} = \{(x_0, \ldots, x_\nu) \in X^{[\nu]} : x_i \neq x_j\}.$$

Then each U_{ij} is open in $X^{[\nu]}$ and their union is $X^{[\nu]} - \triangle X^{[\nu]}$. Because of our assumption on $f^{]\nu-1[}$, we find by Proposition 1.6 that $f^{]\nu-1[}$ extends to a $\mathcal{C}^\rho$-function $f^{[\nu-1]} : X^{[\nu-1]} \to \mathbf{K}$. We can hence define $h_{ij} : U_{ij} \to \mathbf{K}$ by

$$\begin{aligned} &h_{ij}(x_0, \ldots, x_\nu) \\ =&\frac{f^{[\nu-1]}(x_0, \ldots, \breve{x}_j, \ldots, x_\nu) - f^{[\nu-1]}(x_0, \ldots, \breve{x}_i, \ldots, x_\nu)}{x_i - x_j}; \end{aligned}$$

here the arguments beneath the breves being omitted. By the symmetry of $f^{]\nu[}$ and $f^{]\nu-1[}$, we find for $x \in X^{]\nu[}$ that

$$\begin{aligned} f^{]\nu[}(x) &= f^{]\nu[}(x_i, x_j, x_2, \ldots, \overbrace{x_0}^{i\text{-th place}}, \ldots, \overbrace{x_1}^{j\text{-th place}}, \ldots, x_\nu) \\ &= [\quad f^{]\nu-1[}(x_i, x_2, \ldots, \overbrace{x_0}^{i\text{-th place}}, \ldots, \overbrace{x_1}^{j\text{-th place}}, \ldots, x_\nu) \\ &\quad - f^{]\nu-1[}(x_j, x_2, \ldots, \underbrace{x_0}_{i\text{-th place}}, \ldots, \underbrace{x_1}_{j\text{-th place}}, \ldots, x_\nu)]/[x_i - x_j] \\ &= \frac{f^{]\nu-1[}(x_0, x_2, \ldots, \overbrace{x_1}^{j\text{-th place}}, \ldots, x_\nu) - f^{]\nu-1[}(x_1, x_2, \ldots, \overbrace{x_0}^{i\text{-th place}}, \ldots, x_\nu)}{x_i - x_j} \\ &= \frac{f^{]\nu-1[}(x_0, \ldots, \breve{x}_j, \ldots, x_\nu) - f^{]\nu-1[}(x_0, \ldots, \breve{x}_i, \ldots, x_\nu)}{x_i - x_j} = h_{ij}(x); \end{aligned} \tag{2.1}$$

hence each h_{ij} extends $f^{]\nu[}$. As $(x_i - x_j)^{-1}$ is a $\mathcal{C}^\rho$-function on U_{ij} and also $f^{[\nu-1]}$ on $X^{[\nu-1]}$, the same holds for our map h_{ij} by Proposition 1.7(ii). We glue these functions together by putting

$$f^{<\nu>}(x) = h_{ij}(x) \quad \text{if } x \in U_{ij}.$$

Then $f^{<\nu>} : X^{[\nu]} - \triangle X^{[\nu]} \to \mathbf{K}$ is a well-defined function as all the continuous functions h_{ij} coincide on the common dense subset $X^{]\nu[}$ of their domains. For $\mathcal{C}^\rho$ being a local property, $f^{<\nu>}$ is also a $\mathcal{C}^\rho$-function. ■

Proposition 2.5. *Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points and $f : X \to \mathbf{K}$ a mapping thereon. Then $f \in \mathcal{C}^r(X, \mathbf{K})$ if and only if $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ extends to a $\mathcal{C}^\rho$-function $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$.*

Proof. Firstly we note that if $f^{]\nu[}$ extends to a $\mathcal{C}^\rho$-function $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$, then it will be in particular $\mathcal{C}^\rho$ at all $\vec{a} \in X^{[\nu]}$, so that we only have to show the "only if"-part. This is proved by induction on ν. For $\nu = 0$, this holds by definition, so let us assume that $\nu \geq 1$ and that the statement is true for $n - 1$. By Lemma 2.3, we find $f \in C^{r-1}(X, \mathbf{K})$. By our induction hypothesis, we know that $f^{]\nu-1[}$ extends to a $\mathcal{C}^\rho$-function $f^{[\nu-1]} : X^{[\nu]} \to \mathbf{K}$. By Lemma 2.4, the function $f^{]\nu[}$ extends to a $\mathcal{C}^\rho$-function $f^{<\nu>}$ on all of $X^{[\nu]} - \triangle X^{[\nu]}$. We can now extend $f^{]\nu[}$ to $X^{[\nu]}$ by setting

$$f^{[\nu]}(a_0, \ldots, a_\nu) = \begin{cases} f^{<\nu>}(a_0, \ldots, a_\nu), & \text{if } (a_0, \ldots, a_\nu) \in X^{[\nu]} - \triangle X^{[\nu]}, \\ \lim_{y \to \vec{a}} f^{]\nu[}(v), & \text{if } \vec{a} = (a, \ldots, a) \in \triangle X^{[\nu]}, \end{cases}$$

with y running through $X^{]\nu[}$. Thus if we let $A := X^{]\nu[}$ and $A \subseteq B := X^{[\nu]} \subseteq \overline{A}$, then in particular $f^{]\nu[} : A \to \mathbf{K}$ will be a function which is $\mathcal{C}^\rho$ on the whole of B and hence its unique continuous extension $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$ is a $\mathcal{C}^\rho$-function by Proposition 1.6. ■

Corollary 2.6. *Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points and $f \in \mathcal{C}^r(X, \mathbf{K})$. Then the functions*

$$D_i f(a) := f^{[i]}(\vec{a}) \quad \text{for } a \in X$$

are in $\mathcal{C}^{r-i}(X, \mathbf{K})$ for $i = 0, \ldots, \nu$.

Proof. By [Schikhof, 1984, Lemma 78.1], it holds for all $x = (x_0, \ldots, x_i) \in X^{]i[}$ that

$$(D_{\nu-i} f)^{]i[}(x) = \sum_{y \in S_{i,\nu}} f^{[\nu]}(y),$$

where $S_\nu(x)$ is the set of all tuples $(x_{m_0}, \ldots, x_{m_\nu}) \in X^{[\nu]}$ for which $m_0 \leq \ldots \leq m_\nu$ and $\{m_0, \ldots, m_\nu\} = \{0, \ldots, i\}$. Because each tuple $y(x) \in S_\nu(x)$ as a function $X^{]i[} \to X^{[\nu]}$ is just repetition of coordinates, it is locally Lipschitzian, and Proposition 1.7(i),(ii) tells us that the equation's right hand side defines a $\mathcal{C}^\rho$-function on $X^{]i[}$, yielding $D_{\nu-i} f \in \mathcal{C}^{i+\rho}(X, \mathbf{K})$. In other words $D_i f \in \mathcal{C}^{r-i}(X, \mathbf{K})$ for $i = 0, \ldots, \nu$. ■

Remark 2.7. (cf. [Schikhof, 1984, Theorem 29.5]) Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points. If $f \in \mathcal{C}^\nu(X, \mathbf{K})$ then f will be ν-times continuously differentiable in the Archimedean sense and we have $\nu! D_\nu f = f^{(\nu)}$, where $f^{(\nu)}$ denotes the Archimedean ν-fold derivative of f.
There are some subtleties in characteristic $p > 0$, though: The function $f(x) = x^2$ is a $\mathcal{C}^2$-mapping on $\mathbf{K} = \mathbb{F}_2((t))$ which satisfies $D_1 f \equiv 0$, but $D_2 f \equiv 1$.

The locally convex $\mathbf{K}$-algebra of $\mathcal{C}^r$-functions

In the following, we want to endow the $\mathbf{K}$-vector space of r-times continuously differentiable functions with a complete locally convex topology.

Definition. Let $f \in \mathcal{C}^r(X, \mathbf{K})$. Then $f^{]0[}, \dots, f^{]\nu-1[}$ and $f^{]\nu[}$ extend to continuous functions $f^{[0]}, \dots, f^{[\nu-1]}$ and a $\mathcal{C}^\rho$-function $f^{[\nu]}$. For a compact subset $C \subseteq X$, we can thence define the seminorm $\|\cdot\|_{\mathcal{C}^r,C}$ on $\mathcal{C}^r(X, \mathbf{K})$ by

$$\|f\|_{\mathcal{C}^r,C} := \|f^{[0]}\|_C \vee \dots \vee \|f^{[\nu-1]}\|_{C^{[\nu-1]}} \vee \|f^{[\nu]}\|_{\mathcal{C}^\rho,C^{[\nu]}}.$$

We provide $\mathcal{C}^r(X, \mathbf{K})$ with the locally convex topology induced through this family of seminorms $\{\|\cdot\|_{\mathcal{C}^r,C}\}$ with C running through all compact subsets $C \subseteq X$.

Lemma 2.8. *We have for $s \le r$ a norm-nonincreasing inclusion of locally convex $\mathbf{K}$-vector spaces $\mathcal{C}^r(X, \mathbf{K}) \subseteq \mathcal{C}^s(X, \mathbf{K})$.*

Proof. The inclusion holds by Lemma 2.3. It remains to show that $\|\cdot\|_{\mathcal{C}^s,C} \le \|\cdot\|_{\mathcal{C}^r,C}$ on $\mathcal{C}^r(X, \mathbf{K})$ for every compact subset $C \subseteq X$.
Let $f \in \mathcal{C}^r(X, \mathbf{K}) \subseteq \mathcal{C}^s(X, \mathbf{K})$. Then clearly $\|f\|_{\mathcal{C}^s,C} \le \|f\|_{\mathcal{C}^r,C}$ for every $\nu \le s \le r$. By transitivity, it therefore suffices to prove $\|f\|_{\mathcal{C}^s,C} \le \|f\|_{\mathcal{C}^r,C}$ with $s = \nu - 1 + \eta$ for $\eta \in [0, 1[$. For this, it suffices to prove

$$\|f^{[\nu-1]}\|_{\mathcal{C}^\eta,C^{[\nu-1]}} \le \|f^{[\nu-1]}\|_{C^{[\nu-1]}} \vee \|f^{[\nu]}\|_{C^{[\nu]}}.$$

Let $\bar{C} = C^{[\nu-1]}$ and $\bar{F} := f^{[\nu-1]}$. For the last equality, we observe that we have a natural identification of $x = (x_0, x_1, x_2, \dots, x_\nu) \in C^{[\nu]}$ with $(x_0, x_1; x_2; \dots; x_\nu) \in \bar{C}^{[e_1]}$, and so, if $x_0 \neq x_1$, we have

$$f^{[\nu]}(x) = \frac{\bar{F}(x_0, x_2, \dots, x_\nu) - F(x_1, x_2, \dots, x_\nu)}{x_0 - x_1} = F^{[e_1]}(x_0, x_1; x_2; \dots; x_\nu).$$

In particular we see that if $f^{[\nu]}$ exists, so does $\bar{F}^{[e_k]}$, and we deduce

$$\begin{aligned} \|\bar{F}\|_{\mathcal{C}^\eta,\bar{C}} &\le \|\bar{F}\|_{\mathcal{C}^1,\bar{C}} \\ &= \|\bar{F}\|_{\mathcal{C}^{e_1},\bar{C}} \\ &= \|\bar{F}\|_{\bar{C}} \vee \|\bar{F}^{[e_1]}\|_{\bar{C}^{[e_1]}} \\ &= \|f^{[\nu-1]}\|_{C^{[\nu-1]}} \vee \|f^{[\nu]}\|_{C^{[\nu]}}; \end{aligned}$$

here the first inequality holding true by Lemma 1.37. For the following equality, that by Corollary 1.40, it holds $\|\bar{F}\|_{\mathcal{C}^1,\bar{C}} = \|\bar{F}\|_{\mathcal{C}^{e_1},\bar{C}}$ by symmetry of $f^{[\nu-1]} : X^{[\nu-1]} \to \mathbf{K}$. ∎

Lemma 2.9. *Let $x \in X^{]\nu[}$ with $X \subseteq \mathbf{K}$. Then there is a constant $C(x) \ge 1$ such that for every function $f : X \to \mathbf{K}$, it holds*

$$|f^{]\nu[}(x)| \le C(x) \max_{i=0,\dots,\nu} |f(x_i)|.$$

Proof. We know by [Schikhof, 1984, Exercise 29.A] a direct expression for $f^{]\nu[}(x)$ as follows:

$$f^{]\nu[}(x_0, \dots, x_\nu) = \sum_{i=0,\dots,\nu} f(x_i) \prod_{j=0,\dots,\nu \text{ s.t. } j\neq i} (x_i - x_j)^{-1} \quad \text{for } x \in X^{]\nu[}.$$

Hence we can put $C(x) := \max_{i=0,\dots,\nu} C_i(x) \vee 1$ with $C_i(x) := \prod_{j\neq i} |x_i - x_j|^{-1}$. ∎

Proposition 2.10. *Let X be a non-empty subset of $\mathbf{K}$ without isolated points. The space $\mathcal{C}^r(X,\mathbf{K})$ is a complete locally convex $\mathbf{K}$-algebra.*

Proof. It is clear that $\mathcal{C}^r(X,\mathbf{K})$ is a locally convex $\mathbf{K}$-vector space. To convince ourselves that it is also a locally convex $\mathbf{K}$-algebra, we show firstly its closure under products and secondly that $\|fg\|_{\mathcal{C}^r,C} \leq \|f\|_{\mathcal{C}^r,C}\|g\|_{\mathcal{C}^r,C}$ for all $f,g \in \mathcal{C}^r(X,\mathbf{K})$. By [Schikhof, 1984, Lemma 29.2 (v)], for $n = 0,\ldots,\nu$, we find

$$(fg)^{]n[}(x_0,\ldots,x_n) = \sum_{j=0,\ldots,n} f^{]j[}(x_0,\ldots,x_j)g^{]n-j[}(x_j,\ldots,x_n) \quad \text{for all } x \in \nabla X^{]n[}.$$

Firstly, let $f,g \in \mathcal{C}^r(X,\mathbf{K})$. By Lemma 2.3 and then Proposition 2.5, the functions $f^{]j[}$ and $g^{]j[}$ extend both to $\mathcal{C}^\rho$-functions for $j = 0,\ldots,\nu$. By Proposition 1.7(ii), this sum again extends to a $\mathcal{C}^\rho$-function. Hence $(fg)^{]\nu[}$ extends to a $\mathcal{C}^\rho$-function, i.e. $fg \in \mathcal{C}^r(X,\mathbf{K})$.
Regarding the claimed continuity, Lemma 1.10 shows that

$$\|(fg)^{]n[}\|_{\mathcal{C}^\varrho} \leq M \max_{j=0,\ldots,n} \|f^{]j[}\|_{\mathcal{C}^\varrho}\|g^{]n-j[}\|_{\mathcal{C}^\varrho} \leq M\|f\|_{\mathcal{C}^{n+\varrho}}\|g\|_{\mathcal{C}^{n+\varrho}} \leq \|f\|_{\mathcal{C}^r}\|g\|_{\mathcal{C}^r}$$

for $\varrho = 0$ if $n = 0,\ldots,\nu-1$ respectively $\varrho = \rho$ if $n = \nu$. Here for the last inequality, we used Lemma 2.8. Consequently $\|fg\|_{\mathcal{C}^r} \leq \|f\|_{\mathcal{C}^r}\|g\|_{\mathcal{C}^r}$.

We prove completeness. Firstly note that the locally convex topology on $\mathcal{C}^r(X,\mathbf{K})$ given by the family of seminorms $\{\|\cdot\|_{\mathcal{C}^r,C} : C \subseteq X \text{ compact}\}$ is equivalent to the one given by

$$\{\|\cdot^{[n]}\|_C : C \subseteq X^{[n]} \text{ compact and } n \in \{0,\ldots,\nu-1\}\} \cup \{\|\cdot^{[\nu]}\|_{\mathcal{C}^\rho,C} : C \subseteq X^{[\nu]} \text{ compact}\};$$

namely given $C \subseteq X^{[n]}$ compact, let $\tilde{C} := \mathrm{p}_0 C \cup \ldots \cup \mathrm{p}_\nu C$ compact. Then $C \subseteq \tilde{C}^{[n]}$ and hence $\|\cdot^{[n]}\|_C \leq \|\cdot\|_{\mathcal{C}^r,\tilde{C}}$ if $n \in \{0,\ldots,\nu-1\}$ and $\|\cdot^{[\nu]}\|_{\mathcal{C}^\rho,C} \leq \|\cdot\|_{\mathcal{C}^r,\tilde{C}}$ in case $n = \nu$.

Hence as a locally convex $\mathbf{K}$-vector space, the space $\mathcal{C}^r(X,\mathbf{K})$ is canonically isomorphic to the subspace

$$\begin{aligned} A := \{(g_0,\ldots,g_{\nu-1},g_\nu) \in \mathcal{C}^0(X^{[0]},\mathbf{K}) \times \cdots \times \mathcal{C}^0(X^{[\nu-1]},\mathbf{K}) \times \mathcal{C}^\rho(X^{[\nu]},\mathbf{K}) :\\ g_0 = f,\ldots,g_{\nu-1|X^{]\nu-1[}} = f^{]\nu-1[}, g_{\nu|X^{]\nu[}} = f^{]\nu[}\}\\ \subseteq \mathcal{C}^0(X^{[0]},\mathbf{K}) \times \cdots \times \mathcal{C}^0(X^{[\nu-1]},\mathbf{K}) \times \mathcal{C}^\rho(X^{[\nu]},\mathbf{K}) =: P. \end{aligned}$$

By Corollary 1.3, each factor $\mathcal{C}^0(X^{[0]},\mathbf{K}),\ldots,\mathcal{C}^{\nu-1}(X^{[\nu-1]},\mathbf{K})$ is complete, and the factor $\mathcal{C}^\rho(X^{[\nu]},\mathbf{K})$ is complete by Proposition 1.9. Hence it remains to prove that A is closed in P. For this, let $\boldsymbol{f} = (f_0,\ldots,f_{\nu-1},f_\nu)$ be in the boundary of A in P, i.e. in any neighborhood $U \ni \boldsymbol{f}$ of X lies another element $\boldsymbol{g} \in A$. We have to prove that $\boldsymbol{f} \in A$; in other words necessarily $f_{k|X^{]k[}} = f^{]k[}$ for $k = 0,\ldots,\nu$, putting $f := f_0$.

Now fix $\varepsilon > 0$, an order $n \in \{0,\ldots,\nu\}$ and $x \in X^{]n[}$. We must show $|f_n(x) - f^{]n[}(x)| \leq \varepsilon$.

Let $C \supseteq \{x_0,\ldots,x_n\}$ be compact. We can find another $\boldsymbol{g} \in A$ such that

$$\|\boldsymbol{f} - \boldsymbol{g}\| := \max_{n=0,\ldots,\nu-1} \|f_n - g_n\|_{C^{[n]}} \vee \|f_\nu - g_\nu\|_{\mathcal{C}^\rho,C^{[\nu]}} \leq \varepsilon/C(x)$$

with $C(x)$ is as in Lemma 2.9. So in particular $|(f-g)(x_i)| \leq \varepsilon/C(x)$ for $i = 0, \ldots, n$ with $g := g_0$. Since $g_{n|X^{]n[}} = f^{]n[}$, we find

$$\begin{aligned} |f_n(x) - f^{]n[}(x)| &\leq |f_n(x) - g_n(x)| \vee |g_n(x) - f^{]n[}(x)| \\ &= |f_n(x) - g_n(x)| \vee |f^{]n[}(x) - g^{]n[}(x)| \\ &= |f_n(x) - g_n(x)| \vee |(g-f)^{]n[}(x)| \leq \varepsilon; \end{aligned}$$

the last inequality by Lemma 2.9. ■

Description through iterated difference quotients

Lemma. *Let $X \subseteq \mathbf{K}$ be a nonempty subset without isolated points. Then the function $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ extends to a $\mathcal{C}^\rho$-function $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$ if and only if $f^{]\nu[}$ on $X^{]\nu[}$ extends to a $\mathcal{C}^{\rho\cdot e_1}$-function $f^{[\nu]}$ on $X^{[\nu]}$ and moreover $\|f^{[\nu]}\|_{\mathcal{C}^\rho, C} = \|f^{[\nu]}\|_{\mathcal{C}^{\rho\cdot e_1}, C}$ for compact $C \subseteq X$.*

Proof. By continuous extension of Lemma 2.2(ii) the function $f^{[\nu]}$ is symmetric. By Corollary 1.33 holds $f^{[\nu]} \in \mathcal{C}^{\rho\cdot e_1}(X^{[\nu]}, \mathbf{K})$ if and only if $f^{[\nu]} \in \mathcal{C}^\rho(X^{[\nu]}, \mathbf{K})$ and moreover $\|f^{[\nu]}\|_{\mathcal{C}^\rho, C^{[\nu]}} = \|f^{[\nu]}\|_{\mathcal{C}^{\rho\cdot e_1}, C^{[\nu]}}$ for compact $C \subseteq X$. ■

Corollary 2.11. *Let $f : X \to \mathbf{K}$ be a mapping defined on a nonempty subset $X \subseteq \mathbf{K}$ without isolated points. Define a function $|f^{]r[}| : X^{]\nu+1[} \to \mathbb{R}_{\geq 0}$ by*

$$|f^{]r[}|(x_0, \tilde{x}_0, x_1, \ldots, x_\nu) := \frac{|f^{]\nu[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \ldots, x_\nu)|}{|x_0 - \tilde{x}_0|^\rho}.$$

Then $f \in \mathcal{C}^r(X, \mathbf{K})$ if and only if $|f^{]r[}| : X^{]\nu+1[} \to \mathbb{R}_{\geq 0}$ extends to a continuous function $|f^{[r]}| : X^{[\nu+1]} \to \mathbb{R}_{\geq 0}$ which will vanish if $x_0 = \tilde{x}_0$. In this case, $\|f^{[\nu]}\|_{\mathcal{C}^\rho, C^{[\nu]}} = \|f^{[\nu]}\|_{C^{[\nu]}} \vee \||f^{[r]}|\|_{C^{[\nu+1]}}$ for compact $C \subseteq X$.

Proof. Let $F = f^{]\nu[}$ and $\bar{F} = f^{[\nu]} : \bar{X} \to \mathbf{K}$ with $\bar{X} = X^{[\nu]}$. By Proposition 2.5, if $f \in \mathcal{C}^r(X, \mathbf{K})$, then $|F^{]\rho[}|$ extends to a continuous function $|\bar{F}^{[\rho]}| : \bar{X} \times \bar{X} \to \mathbb{R}_{\geq 0}$ vanishing on $\triangle \bar{X} \times \bar{X}$. In particular $|f^{]r[}|$ extends to a continuous function $|f^{[r]}|(\tilde{x}_0, x_1, \ldots, x_\nu)$ vanishing if $x_0 = \tilde{x}_0$.
In the other direction, we assume $|f^{]r[}|$ to extend to a continuous function $|f^{[r]}|(\tilde{x}_0, x_1, \ldots, x_\nu)$ vanishing if $x_0 = \tilde{x}_0$. Then in particular for every $a \in X^{[\nu]}$ and $\varepsilon > 0$, there exists a ball $U \ni a$ in $X^{[\nu]}$ such that

$$|f^{]\nu[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \ldots, x_\nu)| \leq \varepsilon \cdot |x_0 - \tilde{x}_0|^\rho$$

for all $(x_0, x_1, \ldots, x_\nu), (\tilde{x}_0, x_1, \ldots, x_\nu) \in U \cap X^{]\nu[}$.
Because $f^{]\nu[}$ is by Lemma 2.2(ii) symmetric, we find $f^{]\nu[}$ by Corollary 1.20 to be $\mathcal{C}^\rho$ on all of $X^{[\nu]}$. In particular $f^{]\nu[} : X^{]\nu[} \to \mathbf{K}$ is $\mathcal{C}^\rho$ on all of $\triangle X^{[\nu]}$, i.e. $f \in \mathcal{C}^r(X, \mathbf{K})$.

Regarding the asserted equality of norms, let $C \subseteq X$ be a compact subset. We put as before $\bar{F} = f^{[\nu]} : \bar{X} \to \mathbf{K}$ with $\bar{X} = X^{[\nu]}$ and let $\bar{C} = C^{[\nu]}$. Then

$$\|f^{[\nu]}\|_{\mathcal{C}^\rho, C^{[\nu]}} = \|f^{[\nu]}\|_{\mathcal{C}^{\rho\cdot e_1}, C^{[\nu]}} = \|\bar{F}\|_{\bar{C}} \vee \||\bar{F}^{[\rho\cdot e_1]}|\|_{\bar{C}^{[\rho\cdot e_1]}}; \qquad (*)$$

here the first equality by the preceding Lemma and the second equality by Lemma 1.27. Let $(x_0, \tilde{x}_0, x_1, \dots, x_\nu) \in X^{[\nu+1]}$. Then $((x_0, x_1, \dots, x_\nu), (\tilde{x}_0, x_1, \dots, x_\nu)) \in \bar{X}^{[\rho \cdot e_1]}$ and

$$|f^{[r]}|(x_0, \tilde{x}_0, x_1, \dots, x_\nu) = |\bar{F}^{[\rho \cdot e_1]}|((x_0, x_1, \dots, x_\nu), (\tilde{x}_0, x_1, \dots, x_\nu)).$$

We can thereby together with Equality $(*)$ infer the asserted equality of norms $\|f^{[\nu]}\|_{\mathcal{C}^\rho, C^{[\nu]}} = \|f^{[\nu]}\|_{C^{[\nu]}} \vee \||f^{[r]}|\|_{C^{[\nu+1]}}$. ∎

Lemma 2.12. *For any permutation σ of mutually distinct $x_0, x_1, \dots, x_{\nu+1} \in X$, we find*

$$|f^{]r[}|(x^\sigma) = \left|\frac{x_0^\sigma - x_1^\sigma}{x_0 - x_1}\right|^{1-\rho} |f^{]r[}|(x) \quad \text{for } x := (x_0, \dots, x_{\nu+1}) \in X^{]\nu+1[}.$$

Proof. Let σ swap the indices $0, 1$ with $i, j \in \{0, \dots, \nu+1\}$. We notice

$$|f^{]r[}|(x) = |f^{]\nu+1[}(x)||x_0 - x_1|^{1-\rho}.$$

By symmetry of the latter function therefore holds

$$\begin{aligned} |f^{]r[}|(x^\sigma) =& |f^{]\nu+1[}(x^\sigma)||x_i - x_j|^{1-\rho} = |f^{]\nu+1[}(x)||x_0 - x_1|^{1-\rho} \frac{|x_i - x_j|^{1-\rho}}{|x_0 - x_1|^{1-\rho}} \\ =& |f^{]r[}|(x) \frac{|x_i - x_j|^{1-\rho}}{|x_0 - x_1|^{1-\rho}} = |f^{]r[}|(x) \left|\frac{x_0^\sigma - x_1^\sigma}{x_0 - x_1}\right|^{1-\rho}. \end{aligned}$$

∎

Corollary 2.13. *Let $(x_0, x_1, \dots, x_{\nu+1}) \in X^{]\nu+1[}$. Let σ be the mapping on $X^{]\nu+1[}$ swapping the entries with coordinate indices $0, 1$ with those with coordinate indices $i, j \in \{0, \dots, \nu+1\}$. Then we find $|f^{]r[}|(x) \leq |f^{]r[}|(x^\sigma)$ if $|x_0 - x_1| \leq |x_i - x_j|$. In particular if $|x_0 - x_1| = \delta\{x_0, x_1, \dots, x_{\nu+1}\}$, then $|f^{]r[}(x)| \geq |f^{]r[}|(x^\sigma)$ for any permutation σ of $(x_0, x_1, \dots, x_{\nu+1}) \in X^{]\nu+1[}$.*

2.2 Characterization through Taylor polynomials

Assumption. Throughout this subsection $X \subseteq \mathbf{K}$ will denote a nonempty subset without isolated points, if not explicitly mentioned otherwise.

The Taylor polynomial of $\mathcal{C}^r$-functions

We turn to the Taylor expansion of a $\mathcal{C}^r$-function. By a straightforward induction over $\nu \geq 0$, we find that all $\mathcal{C}^r$-functions have a unique Taylor-polynomial expansion:

Corollary 2.14 (Taylor-polynomial)**.** *Let $f \in \mathcal{C}^r(X, \mathbf{K})$. Then*

$$f(x) = \sum_{i=0,\dots,\nu-1} D_i f(y)(x-y)^i + f^{[\nu]}(x, y, \dots, y)(x-y)^\nu \quad \text{for all } x, y \in X$$

with $\mathcal{C}^{r-i}$-functions $D_i f : X \to \mathbf{K}$ for $i = 0, \dots, \nu - 1$ given in Corollary 2.6 and a $\mathcal{C}^\rho$-function $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$.

Proof. This is proven by induction on $\lfloor r \rfloor \geq 0$, the case $\lfloor r \rfloor = 0$ being trivial. So let $\lfloor r \rfloor = \nu + 1 \geq 1$ and $f \in \mathcal{C}^r(X, \mathbf{K}) \subseteq \mathcal{C}^{r-1}(X, \mathbf{K})$, the inclusion by Lemma 2.3. By the induction hypothesis, we have a unique Taylor-polynomial expansion

$$f(x) = \sum_{i=0,\dots,\nu-1} D_i f(y)(x-y)^i + f^{[\nu]}(x, y, \dots, y)(x-y)^\nu \quad \text{for all } x, y \in X$$

with $\mathcal{C}^{r-i}$-functions $D_i f : X \to \mathbf{K}$ for $i = 0, \dots, \nu - 1$ and $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$ a $\mathcal{C}^\rho$-function. Now by the definition of $f^{[\nu+1]}(x, y, \dots, y)$ for distinct $x, y \in X$ holds

$$f^{[\nu]}(x, y, \dots, y) = D_\nu f(y) + (x-y) f^{[\nu+1]}(x, y, \dots, y).$$

This furnishes the existence of our Taylor-polynomial expansion up to degree ν.

■

Let $f \in \mathcal{C}^r(X, \mathbf{K})$. We define the Taylor-polynomial's scaled rest-function by

$$\Delta_\nu f(x, y) := f^{[\nu]}(x, y, \dots, y) - D_\nu f(y) \quad \text{for } x, y \in X.$$

Then $\Delta_\nu f : X \times X \to \mathbf{K}$ is a $\mathcal{C}^\rho$-function vanishing on the diagonal. In particular we find by definition that for every $\varepsilon > 0$ and $a \in X$, there exists a neighborhood $U \ni (a, a)$ such that

$$|\Delta_\nu f(x, y) - \Delta_\nu f(y, y)| \leq \varepsilon |(x, y) - (y, y)|^\rho \quad \text{for all } (x, y), (y, y) \in U^2.$$

This yields to $|\Delta_\nu f(x, y)| \leq \varepsilon |x - y|^\rho$ for all $x, y \in U$. Thus in particular Corollary 2.14 entails:

Corollary (2.14'). *Let $f \in \mathcal{C}^r(X, \mathbf{K})$. Then there is a polynomial of degree ν whose coefficients are functions $D_0 f, \dots, D_\nu f : X \to \mathbf{K}$ such that*

$$f(x+y) = \sum_{i=0,\dots,\nu} D_i f(x) y^i + R_\nu f(x+y, x) \quad \text{for all } x+y, x \in X,$$

and for every $a \in X$ and $\varepsilon > 0$ exists a neighborhood $U \ni a$ such that

$$|R_\nu f(x+y, x)| \leq \varepsilon |y|^r \quad \text{for all } x+y, x \in U.$$

In the following we will study the relation of the functions admitting a Taylor-polynomial expansion and the continuously differentiable ones. We will see that the property of Corollary 2.14' is equivalent to being $\mathcal{C}^r$ on a large class of subsets $X \subseteq K$.

Characterizing $\mathcal{C}^r$-functions through Taylor polynomials on general domains

Definition 2.15. A function $f : X \to \mathbf{K}$ will be in $\mathcal{C}^r_{\mathrm{T}^+}(X, \mathbf{K})$ if there are *continuous* functions $\mathcal{D}_i f : X \to \mathbf{K}$ for $i = 0, \dots, \nu$ such that if one defines $R_\nu f : X \times X \to \mathbf{K}$ by

$$R_\nu f(x, y) := f(x) - \sum_{i=0,\dots,\nu} \mathcal{D}_i f(y)(x-y)^i,$$

then for every point $a \in X$ and any $\varepsilon > 0$, there will exist a neighborhood $U \ni a$ with

$$|R_\nu f(x, y)| \leq \varepsilon |x - y|^r \quad \text{for all } x, y \in U.$$

Since $R_\nu f : X \times X \to \mathbf{K}$ vanishes on the diagonal $\triangle X \times X$, we see that $f = \mathcal{D}_0 f$. Keeping $y = y_0$ fixed, the convergence condition shows that $\mathcal{D}_0 f$ is in any case continuous (and by a more elaborate argument $\mathcal{D}_1 f$, too). Moreover the continuity of $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X \to K$ implies the continuity of $R_\nu f : X \times X \to \mathbf{K}$.

Lemma 2.16. *The functions $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$ in Definition 2.15 are unique.*

Proof. This is proven by induction on $\nu \geq 0$. The case $\nu = 0$ holds as $\mathcal{D}_0 f = f$. Let $\nu \geq 1$. Let $f \in \mathcal{C}^r_{\mathrm{T+}}(X, \mathbf{K})$, so that

$$f(x) = \sum_{i=0,\ldots,\nu} \mathcal{D}_i f(y)(x-y)^i + \Delta_\nu f(x,y)(x-y)^\nu \quad \text{for all } x, y \in X$$

with continuous functions $D_0 f, \ldots, D_\nu f : X \to \mathbf{K}$ and $\Delta_\nu f : X \times X \to \mathbf{K}$, the last one vanishing on the diagonal. We moreover assume that this equality is likewise fulfilled for continuous functions $\mathfrak{D}_0 f, \ldots, \mathfrak{D}_\nu f : X \to \mathbf{K}$ and $\mathfrak{d}_\nu f : X \times X \to \mathbf{K}$ instead. Then

$$\begin{aligned} f(x) &= \sum_{i=0,\ldots,\nu} \mathfrak{D}_i f(y)(x-y)^i + \mathfrak{d}_\nu f(x,y)(x-y)^\nu \\ &= \sum_{i=0,\ldots,\nu-1} \mathfrak{D}_i f(y)(x-y)^i + \mathfrak{d}_{\nu-1} f(x,y)(x-y)^\nu \quad \text{for all } x, y \in X \end{aligned}$$

with $\mathfrak{d}_{\nu-1} f(x,y) := \mathfrak{D}_\nu f(y) + \mathfrak{d}_\nu f(x,y)(x-y)$. Since $\mathfrak{D}_\nu f$ and $\mathfrak{d}_\nu f$ are continuous maps, so is $\mathfrak{d}_{\nu-1} f$. Likewise for $\mathcal{D}_\nu f$, $\Delta_\nu f$ and the mapping $\Delta_{\nu-1} f(x,y) := \mathcal{D}_\nu f(y) + \Delta_\nu f(x,y)(x-y)$. By the assumed uniqueness up to degree $\nu - 1$, we obtain $\mathfrak{D}_0 f = \mathcal{D}_0 f, \ldots, \mathfrak{D}_{\nu-1} f = \mathcal{D}_{\nu-1} f$ and $(x-y)\mathfrak{d}_{\nu-1} f(x,y) = (x-y)\Delta_{\nu-1} f(x,y)$ for all $x, y \in X$. Hence

$$\mathfrak{d}_{\nu-1} f(x,y) = \Delta_{\nu-1} f(x,y) \quad \text{for all distinct } x, y \in X.$$

As X has no isolated points, we know that $\triangledown X \times X$ is dense in $X \times X$. Now both sides are continuous functions on $\triangledown X \times X$ and we find by Remark 1.36 that this equality holds for all $x, y \in X$. By definition of both sides in the equation above, we find $\mathfrak{D}_\nu f = D_\nu f$ as $\mathfrak{d}_\nu f$ and $\Delta_\nu f$ vanish on the diagonal. ■

Definition 2.17. (i) Let $f \in \mathcal{C}^r_{\mathrm{T+}}(X, \mathbf{K})$. We define functions $\Delta_\nu f : \triangledown X \times X \to \mathbf{K}$ and $|\Delta_r f| : \triangledown X \times X \to \mathbb{R}_{\geq 0}$ by putting

$$\Delta_\nu f(x,y) := \frac{R_\nu f(x,y)}{(x-y)^\nu} \quad \text{and} \quad |\Delta_r f|(x,y) := \frac{|R_\nu f(x,y)|}{|x-y|^r}.$$

Since $f \in \mathcal{C}^r_{\mathrm{T+}}(X, \mathbf{K})$, we can by definition extend these functions onto $X \times X$ such that they continuously vanish on the diagonal $\triangle X \times X$ and denote these prolongations likewise. By the comment following Definition 2.15, they are also continuous on $X \times X - \triangle X \times X$ and thus on all of $X \times X$.

(ii) By Lemma 2.16, the functions $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$ of Definition 2.15 are uniquely determined continuous functions. So it makes sense to endow the space $\mathcal{C}^r_{\mathrm{T+}}(X, \mathbf{K})$ with the locally convex topology induced from the family of seminorms $\{\|\cdot\|_{\mathcal{C}^r_{\mathrm{T+}}, C}\}$ running through all compact subsets $C \subseteq X$ defined by

$$\|f\|_{\mathcal{C}^r_{\mathrm{T+}}, C} := \|\mathcal{D}_0 f\|_C \vee \ldots \vee \|\mathcal{D}_\nu f\|_C \vee \||\Delta_r f|\|_{C \times C}.$$

Under our hitherto imposed assumptions, the inverse of Corollary 2.14' turns out to be true for $r \leq 2$, see [Schikhof, 1984, Proposition 28.4]. But a counterexample for $r = 3$ is given in loc.cit., which we will quote here:

Example. (cf. [Schikhof, 1984, Example 83.2]) Let $X := \{\sum_{\nu\in\mathbb{N}} a_\nu p^{n!} \in \mathbb{Z}_p : a_\nu \in \{0,1\}\} \subseteq \mathbb{Z}_p$ and define the map $f : X \to \mathbb{Z}_p$ by

$$f(\sum_{\nu\in\mathbb{N}} a_\nu p^{n!}) = \sum_{\nu\in\mathbb{N}} a_\nu p^{3n!}.$$

Then $f \in \mathcal{C}^3_{\mathrm{T}+}(X,\mathbb{Q}_p) - \mathcal{C}^3(X,\mathbb{Q}_p)$.

Proof. Note first that X is a closed subset of $\mathbb{Z}_p$ without isolated points. We prove that

$$\lim_{(x,y)\to(x_0,x_0)} \frac{f(x)-f(y)}{(x-y)^3} = 1 \quad \text{with distinct } x,y \in X$$

for every $x_0 \in X$. If we set $\mathcal{D}_0 f = f$, $\mathcal{D}_1 f = \mathcal{D}_2 f \equiv 0$, $\mathcal{D}_3 f \equiv 1$, then for any $x_0 \in X$ and $\varepsilon > 0$ will hold $|R_3 f(x,y)| = |f(x) - f(y) - (x-y)^3| \leq \varepsilon |x-y|^3$ for x, y in a suitable neighborhood of x_0. Thus these functions testify $f \in \mathcal{C}^3_{\mathrm{T}+}(X,\mathbb{Q}_p)$. But $f \notin \mathcal{C}^3(X,\mathbb{Q}_p)$, as $\mathcal{D}_3 f = 1 \neq 0 = f^{(3)}/3!$, which it should equal in case that f is a $\mathcal{C}^3$-function by Corollary 2.14 and Remark 2.7.

So let $k \in \mathbb{N}$. We shall prove that $x, y \in X$ and $v(x-y) = k!$ implies $v((f(x)-f(y))/(x-y)^3 - 1) \geq k \cdot k!$. Write $x = \sum_{\nu\in\mathbb{N}} a_\nu p^{n!}$, $y = \sum_{\nu\in\mathbb{N}} b_\nu p^{n!}$. Then $a_j = b_j$ for $j < k$ and $a_k \neq b_k$. We see that

$$f(x) - f(y) = (a_k - b_k)p^{3k!} + u_k \quad \text{and} \quad (x-y)^3 = (a_k - b_k)^3 p^{3k!} + v_k$$

with $v(u_k) \geq 3(k+1)!$ and $v(v_k) \geq 3k!(k+1)$ so that $\min(v(u_k), v(v_k)) \geq (k+3)k!$. Since $a_k, b_k \in \{0,1\}$ we have

$$(a_k - b_k)^3 = a_k - b_k$$

and we get

$$v((f(x)-f(y)) - (x-y)^3) = v(u_k - v_k) \geq (k+3)k! = 3v(x-y) + k \cdot k!.$$

Therefore $v((f(x)-f(y))/(x-y)^3 - 1) = v((f(x)-f(y)) - (x-y)^3) - 3v(x-y) \geq k \cdot k!$, which finishes the proof. ■

Definition. For a subset $C \subseteq \mathbf{K}$, we define

$$C^d_{\leq 1} := \{x = (x_1,\ldots,x_d) \in C^d : \delta\{x_1,\ldots,x_d\} \leq 1\}.$$

Lemma 2.18. *Let $f \in \mathcal{C}^r(X,\mathbf{K})$. Then for all $(x_0,\ldots,x_\nu), (y_0,\ldots,y_\nu) \in X^{[\nu]}$, we have*

$$\begin{aligned} &|f^{[\nu]}(x_0,\ldots,x_\nu) - f^{[\nu]}(y_0,\ldots,y_\nu)| \\ \leq &\max_{i=0,\ldots,\nu} |x_i - y_i|^\rho |f^{[r]}|(y_i, x_i, y_0, \ldots, y_{i-1}, x_{i+1}, \ldots, x_\nu). \end{aligned}$$

Proof. We write as a telescope sum

$$\begin{aligned} & f^{[\nu]}(x_0,\dots,x_\nu) - f^{[\nu]}(y_0,\dots,y_\nu) \\ = \quad & f^{[\nu]}(x_0,\dots,x_\nu) - f^{[\nu]}(y_0,x_1,\dots,x_\nu) \\ & + f^{[\nu]}(y_0,x_1,\dots,x_\nu) - f^{[\nu]}(y_0,y_1,x_2,\dots,x_\nu) \\ & + \cdots \\ & + f^{[\nu]}(y_0,\dots,y_\nu,x_\nu) - f^{[\nu]}(y_0,\dots,y_\nu). \end{aligned}$$

By the symmetry of $f^{[\nu]} : X^{[\nu]} \to \mathbf{K}$, we have

$$\begin{aligned} & f^{[\nu]}(y_0,\dots,y_{i-1},x_i,x_{i+1},\dots,x_\nu) - f^{[\nu]}(y_0,\dots,y_{i-1},y_i,x_{i+1},\dots,x_\nu) \\ =& f^{[\nu]}(x_i,y_0,\dots,y_{i-1},x_{i+1},\dots,x_\nu) - f^{[\nu]}(y_i,y_0,\dots,y_{i-1},x_{i+1},\dots,x_\nu). \end{aligned}$$

The result follows from the definition of $|f^{[r]}|$, as we plug the above exposed equality into the telescope sum on top. ■

Lemma 2.19. *Let $f \in \mathcal{C}^r(X,\mathbf{K})$. For any compact subset $C \subseteq X$ holds*

$$\|f\|_{\mathcal{C}^r,C} = \|D_0 f\|_C \vee \ldots \vee \|D_\nu f\|_C \vee \||f^{[r]}|\|_{C^{[\nu+1]}_{\leq 1}}.$$

Proof. We only have to prove that $\|f\|_{\mathcal{C}^r,C}$ is not greater than the right hand side. By Corollary 2.11, we have $\|f\|_{\mathcal{C}^r,C} = \|f^{[0]}\|_C \vee \ldots \vee \|f^{[\nu]}\|_{C^{[\nu]}} \vee \||f^{[r]}|\|_{C^{[\nu+1]}}$. Firstly, we prove by downward induction on $n = \nu,\dots,0$ that

$$\|f^{[n]}\|_{C^{[n]}_{\leq 1}} \leq \|f^{[\nu]}\|_{C^{[\nu]}_{\leq 1}} \vee \|D_\nu f\|_C \vee \ldots \vee \|D_n f\|_C.$$

In case $n = \nu$ there is nothing to show. Let $n < \nu$. Then for any $(x_0,\dots,x_n) \in C^{[n]}$ with $|x_i - x_j| \leq 1$ for all i,j we have

$$\begin{aligned} |f^{[n]}(x_0,\dots,x_n)| \leq & |f^{[n]}(x_0,\dots,x_n) - f^{[n]}(x_0,\dots,x_0)| \vee |f^{[n]}(x_0,\dots,x_0)| \\ =& |\sum_{j=1,\dots,n} (x_j - x_0) f^{[n+1]}(x_0,\dots,x_0,x_j,\dots,x_n)| \vee |D_n f(x_0)| \\ \leq & \|f^{[n+1]}\|_{C^{[n+1]}_{\leq 1}} \vee \|D_n f\|_C; \end{aligned}$$

the middle equality by [Schikhof, 1984, Lemma 29.2(iii)]. Thus $\|f^{[n]}\|_{C^{[n]}_{\leq 1}} \leq \|f^{[n+1]}\|_{C^{[n+1]}_{\leq 1}} \vee \|D_n f\|_C$, and the induction hypothesis for $n+1$ yields the desired inequality.

Now for any $(x_0,\dots,x_\nu) \in C^{[\nu]}$ with $|x_i - x_j| \leq 1$ for all i,j, we have

$$\begin{aligned} |f^{[\nu]}(x_0,\dots,x_\nu)| \leq & |f^{[\nu]}(x_0,\dots,x_\nu) - f^{[\nu]}(x_0,\dots,x_0)| \vee |f^{[\nu]}(x_0,\dots,x_0)| \\ \leq & \max_{j=1,\dots,\nu} |x_j - x_0|^\rho |f^{[r]}(x_0,x_j,x_0,\dots,x_0,x_{j+1},\dots,x_\nu)| \vee |D_\nu f(x_0)| \\ \leq & \||f^{[r]}|\|_{C^{[\nu+1]}_{\leq 1}} \vee \|D_\nu f\|_C; \end{aligned}$$

the middle inequality by the preceding Lemma 2.18. We see $\|f^{[\nu]}\|_{C^{[\nu]}_{\leq 1}} \leq |||f^{[r]}|||_{C^{[\nu+1]}_{\leq 1}} \vee \|D_\nu f\|_C$. Plugging both results together, we saw for $n = 0, \dots, \nu$ that

$$\begin{aligned}\|f^{[n]}\|_{C^{[n]}_{\leq 1}} &\leq \|D_0 f\|_C \vee \dots \vee \|D_{\nu-1} f\|_C \vee \|f^{[\nu]}\|_{C^{[\nu]}_{\leq 1}} \\ &\leq \|D_0 f\|_C \vee \dots \vee \|D_{\nu-1} f\|_C \vee \|D_\nu f\|_C \vee |||f^{[r]}|||_{C^{[\nu+1]}_{\leq 1}}.\end{aligned}$$

By [Schikhof, 1978, Theorem 8.3], we find $\max_{n=0,\dots,\nu}\|f^{[n]}\|_{C^{[n]}} = \max_{n=0,\dots,\nu}\|f^{[n]}\|_{C^{[n]}_{\leq 1}}$ and so

$$\max_{n=0,\dots,\nu}\|f^{[n]}\|_{C^{[n]}} \leq \|D_0 f\|_C \vee \dots \vee \|D_\nu f\|_C \vee |||f^{[r]}|||_{C^{[\nu+1]}_{\leq 1}}.$$

It solely remains to show $|||f^{[r]}|||_{C^{[\nu+1]}} \leq \|f^{[\nu]}\|_{C^{[\nu]}} \vee |||f^{[r]}|||_{C^{[\nu+1]}_{\leq 1}}$. To this end, let $x = (x_0, \tilde{x}_0, x_1, \dots, x_\nu) \in C^{[\nu+1]}$ with $\delta\{x_0, \tilde{x}_0, x_1, \dots, x_\nu\} > 1$. By continuous extension of Corollary 2.13, we find

$$|f^{[r]}|(x) = \max_{\sigma \in \{\text{permutations of } x\}} |f^{[r]}|(x^\sigma) \quad \text{if and only if} \quad |x_0 - \tilde{x}_0| = \delta\{x_0, \tilde{x}_0, x_1, \dots, x_\nu\}.$$

We may therefore assume $|x_0 - \tilde{x}_0| > 1$. By the definition and continuous extension, we find

$$|f^{[r]}|(x) = |f^{[\nu]}(x_0, x_1, \dots, x_\nu) - f^{[\nu]}(\tilde{x}_0, x_1, \dots, x_\nu)|/|x_0 - \tilde{x}_0| \leq \|f^{[\nu]}\|_{C^{[\nu]}}.$$

■

Lemma 2.20. *Let $f \in \mathcal{C}^r(X, \mathbf{K})$. Then for compact $C \subseteq X$ holds*

$$\|f\|_{\mathcal{C}^r, C} = \max_{n=0,\dots,\nu}(\|D_n f\|_C \vee |||\Delta_{r-n} D_n f|||_{C^2}).$$

Proof. Foremost for this statement to be meaningful, we note that by Corollary 2.6 and Corollary 2.14, we find $D_n f \in \mathcal{C}^{r-n}(X, \mathbf{K}) \subseteq \mathcal{C}^{r-n}_{\mathrm{T}^+}(X, \mathbf{K})$ for $n = 0, \dots, \nu$. So the above right hand side is well defined.
By the preceding Lemma 2.19, it suffices to prove $|||f^{[r]}|||_{C^{[\nu+1]}} \leq |||\Delta_{r-n} D_n f|||_{C^2}$ for $n = 0, \dots, \nu$. Conferring the reader to Lemma 2.4 for the definition of $f^{<\nu>}$, we let

$$\varphi_n f(x, y) := f^{<\nu>}(\underbrace{x, \dots, x}_{n-\text{times}}, y, \dots, y) \quad \text{for all distinct } x, y \in X.$$

By [Schikhof, 1984, Lemma 78.3], we have for distinct $x, y \in X$

$$\begin{pmatrix} (D_{\nu-1} f)^{<1>}(x, y) \\ \vdots \\ (D_0 f)^{<\nu>}(x, y, \dots, y) \end{pmatrix} = \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & & \cdots & \binom{\nu-1}{0} \\ & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{\nu-1}{1} \\ & & \ddots & & \vdots \\ & & & \binom{\nu-2}{\nu-2} & \binom{\nu-1}{\nu-2} \\ & & & & \binom{\nu-1}{\nu-1} \end{pmatrix} \begin{pmatrix} \varphi_\nu f(x, y) \\ \vdots \\ \varphi_1 f(x, y) \end{pmatrix}.$$

Denote the upper $\nu \times \nu$-square matrix by M. We note that inductively

$$\binom{i}{i} + \binom{i+1}{i} + \cdots + \binom{\nu}{i} = \binom{\nu}{i+1} + \binom{\nu}{i} = \binom{\nu+1}{i+1}.$$

Therefore

$$M \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \binom{\nu}{1} \\ \vdots \\ \binom{\nu}{\nu} \end{pmatrix} = \begin{pmatrix} \binom{\nu}{\nu-1} \\ \vdots \\ \binom{\nu}{0} \end{pmatrix}.$$

Because M has determinant 1, it is invertible in $\mathbb{Z}$, and thus

$$\begin{aligned}
\begin{pmatrix} \varphi_\nu f(x,y) - D_\nu f(y) \\ \vdots \\ \varphi_1 f(x,y) - D_\nu f(y) \end{pmatrix} &= M^{-1} \cdot \begin{pmatrix} (D_{\nu-1} f)^{<1>}(x,y) \\ \vdots \\ (D_0 f)^{<\nu>}(x,y,\ldots,y) \end{pmatrix} - \begin{pmatrix} D_\nu f(y) \\ \vdots \\ D_\nu f(y) \end{pmatrix} \\
&= M^{-1} \cdot \begin{pmatrix} (D_{\nu-1} f)^{<1>}(x,y) - \binom{\nu}{\nu-1} D_\nu f(y) \\ \vdots \\ (D_0 f)^{<\nu>}(x,y,\ldots,y) - \binom{\nu}{0} D_\nu f(y) \end{pmatrix} \\
&= M^{-1} \cdot \begin{pmatrix} (D_{\nu-1} f)^{<1>}(x,y) - D_1 D_{\nu-1} f(y) \\ \vdots \\ (D_0 f)^{<\nu>}(x,y,\ldots,y) - D_\nu D_0 f(y) \end{pmatrix} \\
&= M^{-1} \cdot \begin{pmatrix} \Delta_1 D_{\nu-1} f(x,y) \\ \vdots \\ \Delta_\nu D_0 f(x,y) \end{pmatrix};
\end{aligned}$$

the penultimate equality by [Schikhof, 1984, Theorem 78.2]. So for short, we see that we may express $\varphi_n f(x,y) - D_\nu f(y)$ for $n = 1, \ldots, \nu$ as a $\mathbb{Z}$-linear combination of the values $\Delta_1 D_{\nu-1} f(x,y), \ldots, \Delta_\nu D_0 f(x,y)$.

By [Schikhof, 1978, Lemma 8.18], we may express $f^{]\nu[} = f^{<\nu>}_{|\nabla X^{\nu+1}}$ at some $x \in \nabla X^{\nu+1}$ as a convex combination of the $\varphi_1 f, \ldots, \varphi_\nu f : \nabla X \times X \to \mathbf{K}$. More exactly

$$f^{]\nu[}(x_0, \ldots, x_\nu) = \sum_{\substack{n=1,\ldots,\nu, \\ i,j=0,\ldots,\nu \text{ s.t. } i \neq j}} \lambda_{i,j}^{(n)}(x) \varphi_n f(x_i, x_j)$$

for some elements $\lambda_{i,j}^{(n)}(x) \in \mathbf{K}$ for distinct $i, j = 0, \ldots, \nu$ and $n = 1, \ldots, \nu$ such that

$$\sum_{\substack{n=1,\ldots,\nu, \\ i,j=0,\ldots,\nu \text{ s.t. } i \neq j}} \lambda_{i,j}^{(n)}(x) = 1 \quad \text{and} \quad |\lambda_{i,j}^{(n)}(x)| \leq 1 \quad \text{for all these } i, j \text{ and } n.$$

For notational convenience, let $(x_0, x_1, \ldots, x_\nu)$ and $(\tilde{x}_0, x_1, \ldots, x_\nu)$ in $X^{]\nu[}$ be denoted by x

and x'. We obtain $\lambda^{(n)}_{i,j}(x)$ resp. $\lambda^{(n')}_{i',j'}(x')$ in $\mathbf{K}$ such that

$$\begin{aligned} &f^{]\nu[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \ldots, x_\nu) \\ =& \sum_{\substack{n=1,\ldots,\nu,\\ (i,j)\in\nabla\{0,\ldots,\nu\}^2}} \lambda^{(n)}_{i,j}(x)\varphi_n f(x_i, x_j) - \sum_{\substack{n'=1,\ldots,\nu,\\ (i',j')\in\nabla\{0,\ldots,\nu\}^2}} \lambda^{(n')}_{i',j'}(x')\varphi_{n'} f(x'_{i'}, x'_{j'}). \end{aligned} \tag{$*$}$$

As $\sum_{n,i,j} \lambda^{(n)}_{i,j}(x) = 1$ and likewise $\sum_{n',i',j'} \lambda^{(n')}_{i',j'}(x') = 1$, this equals

$$\begin{aligned} &\sum_{\substack{n,n'=1,\ldots,\nu,\\ (i,j),(i',j')\in\nabla\{0,\ldots,\nu\}^2}} \lambda^{(n)}_{i,j}(x)\lambda^{(n')}_{i',j'}(x')\varphi_n f(x_i, x_j) \\ &- \sum_{\substack{n',n=1,\ldots,\nu,\\ (i',j'),(i,j)\in\nabla\{0,\ldots,\nu\}^2}} \lambda^{(n')}_{i',j'}(x')\lambda^{(n)}_{i,j}(x)\varphi_{n'} f(x'_{i'}, x'_{j'}) \\ =& \sum_{n,n'=1,\ldots,\nu} \Big(\sum_{(i,j),(i',j')\in\nabla\{0,\ldots,\nu\}^2} \lambda^{(n)}_{i,j}(x)\lambda^{(n')}_{i',j'}(x')(\varphi_n f(x_i, x_j) - \varphi_{n'} f(x'_{i'}, x'_{j'}))\Big). \end{aligned}$$

Let $n \in \{0, \ldots, \nu\}$ and denote the $\mathbb{Z}$-coefficients of $\Delta_1 D_{\nu-1} f, \ldots, \Delta_\nu D_0 f$ summing to $\varphi_n f(x, y) - D_\nu f(y)$ by $\mu^{(n)}_1, \ldots, \mu^{(n)}_\nu$. We find

$$\begin{aligned} &\varphi_n f(x, y) - \varphi_{n'} f(x', y') \\ =&(\varphi_n f(x, y) - D_\nu f(y)) + (D_\nu f(y) - D_\nu f(y')) - (\varphi_{n'} f(x', y') - D_\nu f(y')) \\ =& \sum_{l=1,\ldots,\nu} \mu^{(n)}_l \Delta_l D_{\nu-l} f(x, y) + (D_\nu f(y) - D_\nu f(y')) - \sum_{l'=1,\ldots,\nu} \mu^{(n')}_{l'} \Delta_{l'} D_{\nu-l'} f(x', y'). \end{aligned}$$

Plugging this into Equation ($*$) and noting $|\lambda^{(n)}_{i,j}|, |\mu^{(n)}_l| \le 1$, we find

$$\begin{aligned} &|f^{]\nu[}(x_0, x_1, \ldots, x_\nu) - f^{]\nu[}(\tilde{x}_0, x_1, \ldots, x_\nu)| \\ \le& \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\ldots,x_\nu\}} (|D_\nu f(x) - D_\nu f(y)| \vee \max_{n=1,\ldots,\nu} |\Delta_n D_{\nu-n} f(x, y)|). \end{aligned}$$

Let $x = (x_0, \tilde{x}_0, x_1, \ldots, x_\nu) \in X^{]\nu+1[}$. By Corollary 2.13, we find for any coordinate permutation map $\sigma : X^{]\nu+1[} \to X^{]\nu+1[}$ with $|x'_0 - \tilde{x}'_0| = \delta\{x_0, \tilde{x}_0, x_1, \ldots, x_\nu\}$, where we put $x^\sigma = (x'_0, \tilde{x}'_0, x'_1, \ldots, x'_\nu)$, that $|f^{]r[}|(x) \le |f^{]r[}|(x^\sigma)$. Hence for such a mapping σ, we find

$$\begin{aligned} |f^{]r[}|(x) \le& |f^{]r[}|(x^\sigma) \\ \le& \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\ldots,x_\nu\}} \max_{n=1,\ldots,\nu} \frac{|\Delta_n D_{\nu-n} f(x, y)|}{|x-y|^\rho} \vee \frac{|D_\nu f(x) - D_\nu f(y)|}{|x-y|^\rho} \\ =& \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\ldots,x_\nu\}} \max_{n=0,\ldots,\nu} |\Delta_{r-n} D_n f|(x, y). \end{aligned}$$

This extends continuously to

$$|f^{[r]}|(x) \le \max_{x,y\in\{x_0,\tilde{x}_0,x_1,\ldots,x_\nu\}} \max_{n=0,\ldots,\nu} |\Delta_{r-n} D_n f|(x, y) \quad \text{for all } x \in X^{[\nu+1]}.$$

In particular for all compact $C \subseteq X$ holds $|||f^{[r]}|||_{C^{[\nu+1]}} \le \max_{n=0,\ldots,\nu} |||\Delta_{r-n} D_n f|||_{C^2}$. ■

Definition. We will define a map $f : X \to \mathbf{K}$ to lie in $\mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ if there are functions $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$ such that $\binom{n}{n}\mathcal{D}_n f, \binom{n+1}{n}\mathcal{D}_{n+1} f, \ldots, \binom{\nu}{n}\mathcal{D}_\nu f$ prove $\mathcal{D}_n f$ to be in $\mathcal{C}^{r-n}_{\mathrm{T}+}(X, \mathbf{K})$ for $n = 0, \ldots, \nu$. We endow the space $\mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\mathcal{C}^r_{\mathrm{T}++}, C}\}$ on each compact subset $C \subseteq X$ defined by

$$\|f\|_{\mathcal{C}^r_{\mathrm{T}++}, C} := \max_{n=0,\ldots,\nu} (\|\mathcal{D}_n f\|_C \vee \||\Delta_{r-n}\mathcal{D}_n f|\|_{C^2})$$

for any $f \in \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$.

Remark 2.21. In this case the functions $f = \mathcal{D}_0 f, \mathcal{D}_1 f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$ are automatically continuous.

Proof. This is proven by downward induction on $n = \nu, \ldots, 0$. Fix $a \in X$ and $\varepsilon > 0$. Let $n = \nu$. We find a neighborhood $U \ni a$ such that

$$|\mathcal{D}_\nu f(x) - \mathcal{D}_\nu f(y)| = |R_0 \mathcal{D}_\nu f(x, y)| \leq \varepsilon |x - y|^\rho \quad \text{for all } x, y \in U.$$

Hence $\mathcal{D}_\nu f : X \to \mathbf{K}$ is $\mathcal{C}^\rho$ and a fortiori continuous at the point a. Let $n < \nu$. We compute

$$\begin{aligned}
&|\mathcal{D}_n f(x) - \mathcal{D}_n f(y)| \\
=&|R_{\nu-n}\mathcal{D}_n f(x, y) + \binom{n+1}{n}\mathcal{D}_{n+1} f(y)(x - y) + \cdots + \binom{\nu}{n}\mathcal{D}_\nu f(y)(x - y)^{\nu-n}| \\
\leq&|R_{\nu-n}\mathcal{D}_n f(x, y)| \vee |\mathcal{D}_{n+1} f(y)||x - y| \vee \ldots \vee |\mathcal{D}_\nu f(y)||x - y|^{\nu-n}.
\end{aligned}$$

We find a neighborhood $U \ni a$ such that

$$|R_{\nu-n}\mathcal{D}_n f(x, y)| \leq \varepsilon |x - y|^{r-n} \quad \text{for all } x, y \in U.$$

Since $\mathcal{D}_{n+1} f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$ are by induction hypothesis continuous, they are in particular locally bounded by $M := |\mathcal{D}_{n+1} f(a)| \vee \ldots \vee |\mathcal{D}_\nu f(a)| \vee 1 > 0$. Hence we can find a neighborhood $\tilde{U} \subseteq U$ of a with diameter $\delta\, U \leq \varepsilon/M \wedge 1$ such that

$$|\mathcal{D}_n f(x) - \mathcal{D}_n f(y)| \leq \varepsilon\, \delta\, U^{r-n} \vee M \cdot \delta\, U \leq \varepsilon \quad \text{for all } x, y \in \tilde{U}.$$

■

Definition. Let $X \subseteq \mathbf{K}$ be a subset.

1. Let $i \in \mathbb{N}$. Then we will denote by $*^i : X \to \mathbf{K}$ the monomial function $x \mapsto x^i$.
2. We will call a function $p : X \to \mathbf{K}$ of the form $p = \sum_{i=0,\ldots,g} a_i *^i$ with scalars $a_0, \ldots, a_g \in \mathbf{K}$ a polynomial function.
3. We will call a function $f : X \to \mathbf{K}$ **locally polynomial of degree at most** g if X can be covered by open sets $\{U\}$ such that $f_{|U} = p$ for a polynomial function p.
   We will write $\mathcal{C}^{\mathrm{pol}}_{\leq g}(X, \mathbf{K})$ for the $\mathbf{K}$-vector space of all locally polynomial functions $f : X \to \mathbf{K}$ of degree at most g.

Lemma 2.22. *For $i \in \mathbb{Z}$, extend the above definition $*^i : X \to \mathbf{K}$ by letting $*^i \equiv 0$ if $i < 0$. Then $*^i \in \mathcal{C}^\nu(X, \mathbf{K})$ for any nonnegative integer ν with $D_\nu *^i = \binom{i}{\nu} *^{i-\nu}$.*

Proof. By induction on i. For $i = 0$, the statement surely holds, so let us assume that $i \geq 1$. With $g(x) = x^{i-1}$, so is the product $f(x) = x^{i-1} \cdot x$ in $\mathcal{C}^\nu(X, \mathbf{K})$ as the identity mapping $h(x) = x$ is a $\mathcal{C}^\nu$-function and these are closed under multiplication. Also note that $D_0 h = h, D_1 h \equiv 1$ and $D_m h \equiv 0$ for $m > 1$. Then by continuous extension of [Schikhof, 1984, Lemma 29.2 (v)], we find

$$D_\nu f = \sum_{j=0,\ldots,\nu} D_j g D_{\nu-j} h = D_{\nu-1} g D_1 h + D_\nu g D_0 h = D_{\nu-1} g + * \cdot D_\nu g.$$

By the induction hypothesis, the last term equals

$$\binom{i-1}{\nu-1} *^{(i-1)-(\nu-1)} + * \cdot \binom{i-1}{\nu} *^{(i-1)-\nu} = (\binom{i-1}{\nu-1} + \binom{i-1}{\nu}) *^{i-\nu} = \binom{i}{\nu} *^{i-\nu}.$$

■

Remark. For the following, we note that every monomial function $*^i : X \to \mathbf{K}$ is arbitrarily often continuously differentiable with $D_n *^i = \binom{i}{n} *^{i-n}$ if $n \leq i \in \mathbb{N}$ - and zero otherwise - by Lemma 2.22. As being $\mathcal{C}^r$ is a $\mathbf{K}$-linear local property, this extends to all locally polynomial functions.
Then by Corollary 2.14 and [Schikhof, 1984, Theorem 78.2], there is an inclusion of locally convex $\mathbf{K}$-vector spaces $\mathcal{C}^r(X, \mathbf{K}) \subseteq \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$. Therefore $\mathcal{C}^{\mathrm{pol}}(X, \mathbf{K}) \subseteq \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$.

Corollary 2.23. *Let $p : X \to \mathbf{K}$ be a polynomial function of degree at most i. If $j \geq i$, then $R_j p \equiv 0$.*

Proof. By linearity, it will suffice to prove $R_j *^i \equiv 0$ if $i \leq j$. By the preceding Lemma 2.22, we have $D_n *^i = \binom{i}{n} *^{i-n}$ for $n = 0, \ldots, j$. Therefore by binomial expansion of $x^i = (y + (x - y))^i$, we obtain

$$R_j *^i (x, y) = x^i - \sum_{n=0,\ldots,i} \binom{i}{n} y^{i-n} (x-y)^i = 0 \quad \text{for all } x, y \in X.$$

■

Lemma 2.24. *The locally polynomial functions of degree at most ν lie dense in the $\mathcal{C}^r_{\mathrm{T}++}$-functions.*

Proof. Fix $f \in \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ and $\varepsilon > 0$. We find a covering $\{U_\alpha\}$ of X with $\delta U_\alpha \leq 1$ such that

$$|\Delta_{r-0} \mathcal{D}_0 f(x, y)|, \ldots, |\Delta_{r-\nu} \mathcal{D}_\nu f(x, y)| \leq \varepsilon \quad \text{for all } x, y \in U_\alpha.$$

Since X is totally disconnected, we can refine this covering to one whose sets are pairwise disjoint, again denoted by $\{U_\alpha\}$. We choose $a_\alpha \in U_\alpha$ and define the locally polynomial function $g : X \to \mathbf{K}$ by putting

$$g(x) = f(a_\alpha) + (x - a_\alpha)\mathcal{D}_1 f(a_\alpha) + \cdots + (x - a_\alpha)^\nu \mathcal{D}_\nu f(a_\alpha) \quad \text{if } x \in U_\alpha.$$

By Lemma 2.22, we have

$$D_n g(x) = \mathcal{D}_n f(a_\alpha) + \binom{n+1}{n} \mathcal{D}_{n+1} f(a_\alpha)(x-a_\alpha) + \cdots + \binom{\nu}{n} \mathcal{D}_\nu f(a_\alpha)(x-a_\alpha)^{\nu-n} \text{ if } x \in U_\alpha.$$

Therefore $\mathcal{D}_n f(x) - D_n g(x) = R_{\nu-n} \mathcal{D}_n f(x, a_\alpha) = (x - a_\alpha)^{\nu-n} \Delta_{\nu-n} \mathcal{D}_n f(x, a_\alpha)$. Hence

$$\|\mathcal{D}_n f - D_n g\|_{U_\alpha} \leq \||\Delta_{r-n} \mathcal{D}_n f(x, a_\alpha)|\|_{U_\alpha^2} (\delta\, U_\alpha)^{r-n} \leq \varepsilon (\delta\, U_\alpha)^{r-n}. \qquad (*)$$

As the U_α cover X with $\delta\, U_\alpha \leq 1$, we see $\|\mathcal{D}_n f - D_n g\|_X \leq \varepsilon$ for $n = 0, \ldots, \nu$.

By Corollary 2.23, we find $R_{\nu-n}(\mathcal{D}_n g_{|U_\alpha}) \equiv 0$ and consequently

$$\||\Delta_{r-n} \mathcal{D}_n (f - g)|\|_{U_\alpha^2} = \||\Delta_{r-n} \mathcal{D}_n f|\|_{U_\alpha^2} \leq \varepsilon.$$

Since $X = \cup_{\alpha,\beta} U_\alpha \times U_\beta$, it remains to show that $\||\Delta_{r-n} \mathcal{D}_n (f-g)|\|_{U_\alpha \times U_\beta} \leq \varepsilon$ in case $\alpha \neq \beta$. So let $x \in U_\alpha, y \in U_\beta$. By disjointness, we have $|x - y| \geq \delta\, U_\alpha \vee \delta\, U_\beta =: \delta > 0$. It follows

$$\begin{aligned}
&|\Delta_{r-n} \mathcal{D}_n (f - g)(x, y)| \\
=&|R_{\nu-n} \mathcal{D}_n f(x, y) - R_{\nu-n} D_n g(x, y)| / |x - y|^{r-n} \\
=&|(\mathcal{D}_n f(x) - D_n g(x)) - \sum_{i=0,\ldots,\nu-n} \binom{i+n}{n} (\mathcal{D}_{i+n} f(y) - D_{i+n} g(y))(x - y)^i| / |x - y|^{r-n} \\
\leq&(\|\mathcal{D}_n f - D_n g\|_{U_\alpha} \vee \max_{i=0,\ldots,\nu-n} \|\mathcal{D}_{i+n} f - D_{i+n} g\|_{U_\beta} (\delta\, U_\beta)^i) / \delta^{r-n} \\
\leq&(\varepsilon (\delta\, U_\alpha)^{r-n} \vee \max_{i=0,\ldots,\nu-n} \varepsilon (\delta\, U_\beta)^{r-(i+n)}) (\delta\, U_\beta)^i / \delta^{r-n} \qquad \text{(by Inequality } (*)) \\
\leq&\varepsilon.
\end{aligned}$$

■

Corollary 2.25. *The canonical inclusion $\mathcal{C}^r(X, \mathbf{K}) \hookrightarrow \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ is an isomorphism of locally convex $\mathbf{K}$-vector spaces with $\|\cdot\|_{\mathcal{C}^r, C} = \|\cdot\|_{\mathcal{C}^r_{\mathrm{T}++}, C}$ for all compact $C \subseteq X$.*

Proof. Foremost, the inclusion map $\iota : \mathcal{C}^r(X, \mathbf{K}) \hookrightarrow \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ is an injective homomorphism of $\mathbf{K}$-vector spaces. By Lemma 2.20, it satisfies $\|\iota(\cdot)\|_{\mathcal{C}^r_{\mathrm{T}++}, C} = \|\cdot\|_{\mathcal{C}^r, C}$ on $\mathcal{C}^r(X, \mathbf{K})$ for all compact $C \subseteq X$ and is therefore an isomorphism of locally convex $\mathbf{K}$-vector spaces onto its image. It therefore suffices to prove its surjectivity. By Lemma 2.24, we have a dense inclusion $\mathcal{C}^{\mathrm{pol}}_{\leq g}(X, \mathbf{K}) \subseteq \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ and we are hence reduced to showing that the image $\mathcal{C}^r(X, \mathbf{K}) \subseteq \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ is closed with respect to the locally convex topology of $\mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$. By Proposition 2.10, it is complete with respect to the locally convex topology of $\mathcal{C}^r(X, \mathbf{K})$. Because ι is an isomorphism of topological $\mathbf{K}$-vector spaces onto its image, $\mathcal{C}^r(X, \mathbf{K})$ is also complete with respect to the subspace topology in $\mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$. Therefore $\mathcal{C}^r(X, \mathbf{K}) \subseteq \mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ is closed as $\mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K})$ is Hausdorff. ■

Sufficiency of the Taylor polynomial expansion on B_ν-sets for $\mathcal{C}^r$-functions

Definition 2.26. We will say that, for a natural number $\nu > 1$, a subset $X \subseteq \mathbf{K}$ has the **B_ν-property** if there is a positive constant $c \leq 1$ such that fixing any $x_0 \in X$ and another point x_1 around x_0, a **"c-regular" ν-gon snuggles into the circle spanned by x_1 around x_0**; i.e. there are $x_2, \dots, x_\nu \in \mathrm{B}_{\leq\delta}(x_0) \subseteq X$ with $\delta := |x_0 - x_1|$ such that

$$c_{(x_0,\dots,x_\nu)} := \min_{i,j=0,\dots,\nu \text{ distinct}} |x_i - x_j| \geq c \cdot \delta.$$

By convention, every subset $X \subseteq \mathbf{K}$ has the property B_0 and B_1. We will say that a subset $X \subseteq \mathbf{K}$ has the **local B_ν-property** if it can be covered by open B_ν-sets.

Remark. The definition of a $B_{\nu+1}$-set in [Schikhof, 1984, Section 83] implies our notion of a B_ν-set: Let x_0, x_1 be distinct points in X and $\delta = |x_0 - x_1|$. Then there is a constant $C \geq 1$ and points $x_2, \dots, x_\nu$ such that $|x_i - x_j| \leq C|x_k - x_l|$ for all $(i,j),(k,l) \in \nabla\{1,\dots,\nu\}$. This means $|x_k - x_l| \geq c \cdot \delta\{x_1,\dots,x_\nu\} \geq c \cdot \delta$ with $c := C^{-1} \leq 1$.

Lemma 2.27. *All balls in* $\mathbf{K}$ *have the* B_ν*-property and consequently all open subsets of* $\mathbf{K}$ *have the local* B_ν*-property for every* $\nu \in \mathbb{N}$.

Proof. We firstly prove that for any complete non-trivially non-Archimedeanly valued field $\mathbf{K}$ and natural number $\nu > 1$ exists a positive constant $c \leq 1$ such that some c-regular ν-gon snuggles into $\mathbf{o}$, the circle of the closed unit disc: Fix distinct x_0 and x_1 therein and put $\delta := |x_0 - x_1|$. Up to scaling, we may assume $|x_0| = 1$. Because $|\cdot|$ is nontrivial, we find positive $c \leq 1$ so small that $\#\mathbf{o}/\mathbf{o}_{\leq c} \geq \nu$. Then we find $\tilde{c} \leq c$ such that also $x_0 \not\equiv x_1 \mod o_{\leq\tilde{c}}$. We then choose $x_0, \dots, x_\nu \in \mathbf{o}$ in different residue classes of $\mathbf{o}/\mathbf{o}_{\leq\tilde{c}}$. Then $x_0, \dots, x_\nu$ constitute a c-regular ν-gon. This proves the first proposition.
Let $\nu > 1$ and assume $B \subseteq \mathbf{K}$ to be a ball. Fix a point $x_0 \in B$ and distinct $x_1 \in B$. Let $D := \mathrm{B}_{\leq\delta}(x_0) \subseteq B$ be a closed disc around x_0 with $\delta = |x_0 - x_1|$. Since B a ball, we have $D = \mathrm{B}_{\leq\delta}(x_0) \subseteq \mathbf{K}$. Now there is the homothety $(x_1 - x_0) \cdot \_$ plus the translation $x_0 + \_$ which transform the closed unit disc $\mathrm{B}_{\leq 1}(0)$ into $\mathrm{B}_{\leq\delta}(x_0)$. We apply their composed affine linear map to the ν-gon $\{y_0, \dots, y_\nu\}$ with $y_0 := 0$ and $y_1 := 1$ in the unit disc found above, yielding the ν-gon $\{x_0, \dots, x_\nu\} \subseteq B$. Then $c_{(y_0,\dots,y_\nu)} = \delta \cdot c_{(x_0,\dots,x_\nu)} \geq c \cdot \delta$. This proves the proposition. ■

Definition 2.28. Let $X \subseteq \mathbf{K}$ be a local B_ν-subset without isolated points. A function $f : X \to \mathbf{K}$ will be in $\mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$ if there are functions $\mathcal{D}_i f : X \to \mathbf{K}$ for $i = 0, \dots, \nu$ such that if one defines $R_\nu f : X \times X \to \mathbf{K}$ by

$$R_\nu f(x,y) := f(x) - \sum_{i=0,\dots,\nu} \mathcal{D}_i f(y)(x-y)^i,$$

then for every point $a \in X$ and any $\varepsilon > 0$, there will exist a neighborhood $U \ni a$ with

$$|R_\nu f(x,y)| \leq \varepsilon |x-y|^r \quad \text{for all } x, y \in U.$$

Notice that - in comparison to $\mathcal{C}^r_{\mathrm{T}+}(X, \mathbf{K})$ of Definition 2.15 - together with the plus, we have dropped the continuity assumption on $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$. In the following we want to show that this is automatically implied by the B_ν-property of X. For this, we will investigate these functions more closely.

Remember that any polynomial of degree ν is determined by $\nu + 1$ values of it. The next Lemma 2.29 makes this somewhat more explicit.

Lemma 2.29. *Let $x_0, \ldots, x_\nu \in \mathbf{K}$ be pairwise distinct points. Let $R =< x_0, \ldots, x_\nu >\subseteq \mathbf{K}$ be the subring in $\mathbf{K}$ generated by $x_0, \ldots, x_\nu$. Then we can find coefficients $c_{0,i}, \ldots, c_{\nu,i}$ in the principal fractional ideal $1/\prod_{k\neq l}(x_k - x_l) \cdot R$ for $i = 0, \ldots, \nu$ such that for any polynomial $P(X) = \sum_{i=0,\ldots,\nu} a_i X^i \in \mathbf{K}[X]$ of degree ν, it holds*

$$a_i X^i = c_{0,i} P(x_0 X) + c_{1,i} P(x_1 X) + \cdots + c_{\nu,i} P(x_\nu X).$$

Proof. Let W and D denote the $(\nu + 1) \times (\nu + 1)$-square matrices

$$W := \begin{pmatrix} 1 & x_0 X & \cdots & (x_0 X)^\nu \\ 1 & x_1 X & \cdots & (x_1 X)^\nu \\ \vdots & & & \vdots \\ 1 & x_\nu X & \cdots & (x_\nu X)^\nu \end{pmatrix} \quad \text{and} \quad D := \begin{pmatrix} 1 & & & \\ & X^{-1} & & \\ & & \ddots & \\ & & & X^{-\nu} \end{pmatrix}.$$

Denote by $V = W \cdot D$ the product of these. This is the matrix with coefficients $(x_i^j)_{i,j=0,\ldots,\nu}$ in $\mathbf{K}$, which is invertible as can be seen by its Vandermonde-determinant

$$\det V = \prod_{\substack{i,j \in \{0,\ldots,\nu\} \\ \text{with } i>j}} (x_i - x_j) \neq 0.$$

Because

$$W \begin{pmatrix} a_0 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} 1 & x_0 X & \cdots & (x_0 X)^\nu \\ 1 & x_1 X & \cdots & (x_1 X)^\nu \\ \vdots & & & \vdots \\ 1 & x_\nu X & \cdots & (x_\nu X)^\nu \end{pmatrix} \begin{pmatrix} a_0 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} P(x_0 X) \\ \vdots \\ P(x_\nu X) \end{pmatrix},$$

we find by right-multiplication with V^{-1} that

$$V^{-1} \begin{pmatrix} P(x_0 X) \\ P(x_1 X) \\ \vdots \\ P(x_\nu X) \end{pmatrix} = D^{-1} W^{-1} \begin{pmatrix} P(x_0 X) \\ P(x_1 X) \\ \vdots \\ P(x_\nu X) \end{pmatrix} = D^{-1} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_\nu \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 X^1 \\ \vdots \\ a_\nu X^\nu \end{pmatrix}.$$

By the Cramer rule and the shape of V, this spelled out is the proposition. ∎

Corollary 2.30. *Let $P(X) = \sum_{i=0,\ldots,\nu} a_i X^i \in \mathbf{K}[X]$ be a polynomial of degree ν. Then for pairwise distinct points $x_0, \ldots, x_\nu \in \mathbf{K}$ of maximal norm $\delta \in |\mathbf{K}|$ we find*

$$|a_i| \leq M(x_0, \ldots, x_\nu) \delta^{-i} (|P(x_0)| \vee \ldots \vee |P(x_\nu)|)$$

with an upper bound $M(x_0, \ldots, x_\nu) := \prod_{i \neq j} \delta / |x_i - x_j| \geq 1$.

Proof. Given these pairwise distinct $x_0, \dots, x_\nu \in \mathbf{K}$ of maximal norm δ, let w.l.o.g. $|x_0| = \delta$. Then apply the preceding Lemma 2.29 for the $\nu + 1$ points $x_0/x_0, \dots, x_\nu/x_0$ of norm at most 1 and $X = x_0$. ■

Lemma 2.31. *Let $X \subseteq \mathbf{K}$ be a B_ν-subset. Then there is a constant $C \geq 1$ such that for all $f \in \mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$ and distinct $x, y \in X$ there is a set P of $\nu + 1$ points in $U := \mathrm{B}_{\leq\delta}(x) \subseteq X$ with $\delta = |x - y|$ such that*

$$|R_{\nu-n}\mathcal{D}_n f(x,y)| \leq C \cdot \delta^{-n} \max_{z \in P}(|R_\nu f(z,x)| \vee |R_\nu f(z,y)|) \quad \text{for } n = 0, \dots, \nu;$$

here $R_{\nu-n}\mathcal{D}_n f$ is given through the functions $\binom{n}{n}\mathcal{D}_n f, \binom{n+1}{n}\mathcal{D}_{n+1} f, \dots, \binom{\nu}{n}\mathcal{D}_\nu f$.

Proof. Let $x, x+y$ and $x+y+z \in X$. We have

$$\begin{aligned} &R_\nu f(x+y+z, x) - R_\nu f(x+y+z, x+y) \\ =&f(x+y+z) - \sum_{k=0,\dots,\nu} \mathcal{D}_k f(x)(y+z)^k - (f(x+y+z) - \sum_{k=0,\dots,\nu} \mathcal{D}_k f(x+y) z^k) \\ =& \sum_{k=0,\dots,\nu} \mathcal{D}_k f(x)(y+z)^k - \sum_{k=0,\dots,\nu} \mathcal{D}_k f(x+y) z^k. \end{aligned}$$

By the binomial identity and then altering the order of summation we calculate

$$\begin{aligned} \sum_{k=0,\dots,\nu} \mathcal{D}_k f(x)(y+z)^k &= \sum_{k=0,\dots,\nu} \sum_{i+j=k} \mathcal{D}_k f(x) \binom{k}{i} y^i z^j \\ &= \sum_{j=0,\dots\nu} z^j (\sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathcal{D}_{i+j} f(x) y^i). \end{aligned}$$

Together this yields

$$\begin{aligned} &R_\nu f(x+y+z, x) - R_\nu f(x+y+z, x+y) \\ =& \sum_{j=0,\dots\nu} z^j (\mathcal{D}_j f(x+y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathcal{D}_{i+j} f(x) y^i). \end{aligned} \tag{$*$}$$

This is a polynomial function $Q(z)$ of degree ν in z. By Corollary 2.30, we obtain for its coefficients the inequality

$$|\mathcal{D}_j f(x+y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathcal{D}_{i+j} f(x) y^i| \leq M(z_0, \dots, z_\nu) \delta^{-j} \max_{z \in P} |Q(z)|$$

for any collection of $\nu + 1$ points $\tilde{P} := \{z_0, \dots, z_\nu\}$ in $\mathbf{K}$ of maximal norm δ. If we can find these points such that $x + y + P \subseteq X$, then Equality $(*)$ will yield

$$\begin{aligned} &|\mathcal{D}_j f(x+y) - \sum_{i=0,\dots,\nu-j} \binom{i+j}{i} \mathcal{D}_{i+j} f(x) y^i| \\ \leq& M(z_0, \dots, z_\nu) \delta^{-j} \max_{z \in \tilde{P}} (|R_\nu f(x+y+z, x)| \vee |R_\nu f(x+y+z, x+y)|). \end{aligned}$$

Now since X satisfies the B_ν-property, we can indeed extend the two distinct points $z_0 := x$ and $z_1 := x + y$ to a collection of $\nu + 1$ points $P := \{z_0, \ldots, z_\nu\} \subseteq U := \mathrm{B}_{\leq\delta}(x) \subseteq X$ with $c_{(z_0,\ldots,z_\nu)} \geq c \cdot \delta$ as in Definition 2.26. Then

$$M(z_0, \ldots, z_\nu) = \prod_{i \neq j} \delta/|z_i - z_j| \leq c^{-\binom{\nu+1}{2}} =: C,$$

which will be our sought positive constant. If we let

$$\tilde{P} := P - (x + y) = \{z_0 - (x + y), \ldots, z_\nu - (x + y)\},$$

we find therefore

$$\begin{aligned}&|\mathcal{D}_j f(x + y) - \sum_{i=0,\ldots,\nu-j} \binom{i+j}{i} \mathcal{D}_{i+j} f(x) y^i| \\ \leq& C \cdot \delta^{-j} \max_{z \in \tilde{P}} (|R_\nu f(x + y + z, x)| \vee |R_\nu f(x + y + z, x + y)|).\end{aligned}$$

This proves the proposition. ■

Definition. Cf. Definition 2.28, we will prove below that $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X \to \mathbf{K}$ and accordingly $R_\nu f : X \times X \to \mathbf{K}$ are continuous functions. Keeping the notations of Definition 2.17(i), we endow the space $\mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$ with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\mathcal{C}^r_{\mathrm{T}}, C}\}$ on $\mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$ running through all compact subsets $C \subseteq X$ defined by

$$\|f\|_{\mathcal{C}^r_{\mathrm{T}}, C} := \|\mathcal{D}_0 f\|_C \vee \ldots \vee \|\mathcal{D}_\nu f\|_C \vee \||\Delta_r f|\|_{C^2}.$$

In other words: In case $X \subseteq \mathbf{K}$ is a B_ν-subset, we find $\mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K}) = \mathcal{C}^r_{\mathrm{T}+}(X, \mathbf{K})$ and we give $\mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$ the same locally convex topology.

Corollary 2.32. *Let $X \subseteq \mathbf{K}$ be a nonempty local B_ν-subset without isolated points. Then the canonical inclusion $\mathcal{C}^r_{\mathrm{T}++}(X, \mathbf{K}) \hookrightarrow \mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$ is an isomorphism of locally convex $\mathbf{K}$-algebras. It will be an isomorphism of locally convex $\mathbf{K}$-algebras if $\mathbf{K}$ is locally compact.*

Proof. Let $f \in \mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$. Firstly, we have to show that $\binom{n}{n}\mathcal{D}_n f, \ldots, \binom{\nu}{n}\mathcal{D}_\nu f$ prove $\mathcal{D}_n f$ to be in $\mathcal{C}^{r-n}_{\mathrm{T}+}(X, \mathbf{K})$ for $n = 0, \ldots, \nu$. Fix $\varepsilon > 0$ and $a \in X$. Find a B_ν-neighborhood $U \ni a$ such that $|R_\nu f(x, y)| \leq \varepsilon |x - y|^r$ for all $x, y \in U$. If $x \neq y$, we find by Lemma 2.31 a constant $C = C(U) \geq 1$ solely depending on U, a finite subset $P \subseteq \mathrm{B}_{\leq\delta}(x) \subseteq U$ with $\delta := |x - y|$ such that

$$\begin{aligned}|R_{\nu-n}\mathcal{D}_n f(x, y)| \leq& C|x - y|^{-n} \max_{z_0 = x,y \text{ and } z \in P} |R_\nu f(z, z_0)| \\ \leq& C|x - y|^{-n} \max_{z_0 = x,y \text{ and } z \in P} \varepsilon |z_0 - z|^r \\ \leq& C\varepsilon |x - y|^{r-n};\end{aligned}$$

the last inequality since $|z_0 - z| \leq \delta$, the points $z_0 = x, y$ both being centers of $\mathrm{B}_{\leq\delta}(x)$. If $x = y$, the above inequality will trivially hold. By Remark 2.21, the functions $\mathcal{D}_n f$ for

$n = 0, \ldots, \nu$ are in particular automatically continuous.
Secondly, we prove that $|||\Delta_{r-n}\mathcal{D}_n f|||_{K^2} \leq C \cdot |||\Delta_r f|||_{K^2}$ for all compact B_ν-subsets $K \subseteq X$. By Lemma 2.31, we find for distinct $x, y \in K$ a finite subset $P \subseteq \mathrm{B}_{\leq\delta}(x) \subseteq K$ with $\delta := |x - y|$ such that

$$\begin{aligned}|\Delta_{r-n}\mathcal{D}_n f(x,y)| =& \frac{|R_{\nu-n}\mathcal{D}_n f(x,y)|}{|x-y|^{r-n}} \\ \leq& C \cdot \max_{z_0 = x,y \text{ and } z\in P} |R_\nu f(z,z_0)| \frac{1}{|x-y|^r} \\ =& C \cdot \max_{z_0 = x,y \text{ and } z\in P} |\Delta_r f(z,z_0)| \left|\frac{z_0 - z}{x-y}\right|^r \\ \leq& C \cdot \max_{z_0 = x,y \text{ and } z\in P} |\Delta_r f(z,z_0)|. \qquad (*)\end{aligned}$$

If $x = y$, then the left hand side will vanish as $\mathcal{D}_n f$ was seen to be in $\mathcal{C}^{r-n}_{\mathrm{T}}(X, \mathbf{K})$. As $P \subseteq K$, we find $|||\Delta_{r-n}\mathcal{D}_n f|||_{K^2} \leq C \cdot |||\Delta_r f|||_{K^2}$ for all compact subsets $K \subseteq X$.
Let now $\mathbf{K}$ be locally compact. The continuity of this inclusion holds true by definition. Regarding its openness, we observe that any compact subset $K \subseteq X$ is contained in a closed, hence compact, ball B which has by Lemma 2.27 the B_ν-property. Therefore the locally convex topology on $\mathcal{C}^r(X, \mathbf{K})$ is induced by all seminorms $\|\cdot\|_{\mathcal{C}^r,B}$ for closed balls B and we have

$$\begin{aligned}\|f\|_{\mathcal{C}^r,B} &= \max_{n=0,\ldots,\nu} (\|D_n f\|_B \vee |||\Delta_{r-n}D_n f|||_{B^2}) \\ &\leq \max_{n=0,\ldots,\nu} \|D_n f\|_B \vee C \cdot |||\Delta_r f(z,z_0)|||_{B^2} \leq C \cdot \|f\|_{\mathcal{C}^r_{\mathrm{T}},B};\end{aligned}$$

here the first inequality by B having the B_ν-property and the above Estimate $(*)$. ■

Another characterization of $\mathcal{C}^r$-functions on compact sets and an application

We show another equivalence of differentiability notions: In [Colmez, 2008], the author gave a definition of r-times differentiable functions on $\mathbb{Z}_p$ (into a closed subfield of $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$). We canonically generalize this to functions on nonempty subsets $X \subseteq \mathbf{K}$ (into $\mathbf{K}$) and show that these two notions coincide on compact B_ν-subsets $X \subseteq \mathbf{K}$ without isolated points.

Definition 2.33 (Colmez)**.** If $r \in \mathbb{R}_{\geq 0}$, we will say that $f : X \to \mathbf{K}$ is of class $\tilde{\mathcal{C}}^r_{\mathrm{T}}$, if there exist functions $\mathcal{D}_i f : X \to \mathbf{K}$ for $i = 0, \ldots, \lfloor r \rfloor$ such that if we define $R_{\lfloor r \rfloor} f : X \times X \to \mathbf{K}$ by

$$R_{\lfloor r\rfloor} f(x,y) = f(x) - \sum_{i=0,\ldots,\lfloor r\rfloor} \mathcal{D}_i f(y)(x-y)^i,$$

then

$$\tilde{C}_r f(\delta) := \sup_{x_0 \in X} \sup_{y \in \mathrm{B}_{\leq\delta}(x_0)} \frac{|R_{\lfloor r\rfloor} f(x_0,y)|}{\delta^r}$$

is a well-defined function $\tilde{C}_r f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ which converges to 0 as δ does. We denote the set of functions $f : X \to \mathbf{K}$ which are of class $\tilde{\mathcal{C}}^r_{\mathrm{T}}$ by $\tilde{\mathcal{C}}^r_{\mathrm{T}}(X, \mathbf{K})$.

Proposition. *Let r be a nonnegative real number. Then Definition 2.28 and Definition 2.33 coincide on compact B_ν-subsets $X \subseteq \mathbf{K}$ without isolated points, i.e.*

$$\mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K}) = \tilde{\mathcal{C}}^r_{\mathrm{T}}(X, \mathbf{K}).$$

Proof. Given a function $f : X \to \mathbf{K}$, it will suffice to show that the conditions on the functions $\mathcal{D}_i f$ in Definition 2.28 respectively Definition 2.33 for $i = 0, \ldots, \nu := \lfloor r \rfloor$ are equivalent.

Recall by Definition 2.33 that $f \in \tilde{\mathcal{C}}^r_{\mathrm{T}}(X, \mathbf{K})$ if $\tilde{C}_r f(\delta) \to 0$ for $\delta \to 0$, i.e. for any $\varepsilon > 0$ there exists a $\delta_0 > 0$ such that $\tilde{C}'_r f(\delta_0) := \sup_{0<\delta\leq\delta_0} \tilde{C}_r f(\delta) < \varepsilon$.

On the other hand assume that $f \in \mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$. Since $X \subseteq \mathbf{K}$ is a B_ν-subset, we find $f \in \mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K}) \subseteq \mathcal{C}^r_{\mathrm{T+}}(X, \mathbf{K})$, which holds if and only if $|\Delta_r f| : \nabla X \times X \to \mathbb{R}_{\geq 0}$ extends to a continuous function on $X \times X$ vanishing on the diagonal. As X is a compact metric space, $|\Delta_r f|$ is continuous on $X \times X$ if and only if it is uniformly so. In particular on $\triangle X \times X$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$C'_r f(\delta) := \sup_{a \in X} \sup_{\substack{x,y \in \mathrm{B}_{\leq\delta}(a),\\ x \neq y}} |\Delta_r f(x, y)| < \varepsilon.$$

It suffices to show that $C'_r f(\delta) = \tilde{C}'_r f(\delta)$. Plugging in the Definitions of $C'_r f(\delta)$ resp. $\tilde{C}'_r f(\delta)$ we thus have to show that

$$\sup_{a \in X} \sup_{\substack{x,y \in \mathrm{B}_{\leq\delta}(a),\\ x \neq y}} \frac{|R_\nu f(x, y)|}{|x - y|^r} = \sup_{0<\delta'\leq\delta} \sup_{x \in X} \sup_{y \in \mathrm{B}_{\leq\delta'}(x)} \frac{|R_\nu f(x, y)|}{\delta'^r}. \qquad (*)$$

We note that for any $x, y \in X$ and $\delta > 0$ holds $|x - y| \leq \delta$ if and only if there exists an $x_0 \in X$ such that $\max\{|x - x_0|, |y - x_0|\} \leq \delta$ by the strong triangle inequality. Thus the left hand side of $(*)$ equals

$$\sup_{x \in X} \sup_{\substack{y \in \mathrm{B}_{\leq\delta}(x),\\ x \neq y}} \frac{|R_\nu f(x, y)|}{|x - y|^r}. \qquad (**)$$

Furthermore we note that for any $x, y \in X$ we have $x \neq y$ and $|x - y| \leq \delta$ if and only if $|x - y| = \delta'$ for some $0 < \delta' \leq \delta$. Thus

$$\sup_{x \in X} \sup_{\substack{y \in \mathrm{B}_{\leq\delta}(x),\\ x \neq y}} \frac{|R_\nu f(x, y)|}{|x - y|^r} = \sup_{x \in X} \sup_{0<\delta'\leq\delta} \sup_{y \text{ s.t. } |x-y|=\delta'} \frac{|R_\nu f(x, y)|}{\delta'^r}$$

Now keeping x fixed,

$$\sup_{0<\delta'\leq\delta} \sup_{y \in X \text{ s.t. } |x-y|=\delta'} \frac{|R_\nu f(x, y)|}{\delta'^r} = \sup_{0<\delta'\leq\delta} \sup_{y \in X \text{ s.t } |x-y|\leq\delta'} \frac{|R_\nu f(x, y)|}{\delta'^r}$$

as $R_\nu f(x,y) = 0$ for any $y = x$. But then

$$\sup_{x\in X} \sup_{\substack{y\in \mathrm{B}_{\leq\delta}(x),\\ x\neq y}} \frac{|R_\nu f(x,y)|}{|x-y|^r} = \sup_{x\in X} \sup_{0<\delta'\leq\delta} \sup_{y\in \mathrm{B}_{\leq\delta'}(x)} \frac{|R_\nu f(x,y)|}{\delta'^r}.$$

This will be the claimed Equality $(*)$ if we substitute the left hand side by $(**)$. ■

Here is an application of Corollary 2.32:

Example 2.34 (Berger, Breuil)**.** Let $\mathbf{F}$ be a closed subfield of $\mathbf{K}$ with value group $\mathbb{Z}$ and fix two elements $\alpha, \beta \in \mathbf{K}^*$. We will show that the function f extending

$$\chi : \mathbf{F}^* \ni x \mapsto (\frac{\alpha}{\beta})^{v(x)} x^k \in \mathbf{K},$$

onto $\mathbf{F}$ by putting $f(0) = 0$, will be in $\mathcal{C}^r(\mathbf{F}, \mathbf{K})$ if $k > v(\beta)$, where $r := v(\alpha)$.

Proof. We note that f is the product of the unramified character $\chi_\gamma : \mathbf{F}^* \to \mathbf{K}^*$ defined by

$$\chi_\gamma : x \mapsto \gamma^{v(x)}$$

for $\gamma := \alpha/\beta \in \mathbf{K}^*$, extended onto $\mathbf{F}$ by $\chi_\gamma(0) = 0$, and the monomial function $x \mapsto x^k$. Now χ_γ is locally constant on $\mathbf{F}^*$ and thus arbitrarily often differentiable there; the same holds for the monomial function x^k by Lemma 2.22.

We are just left to show that f is r-times differentiable at 0, too. We note that the domain $X = \mathbf{F}$ has the B_ν-property by Lemma 2.27 (and has no isolated points). By Corollary 2.25 and Corollary 2.32, it therefore suffices to check that f is $\mathcal{C}^r_{\mathrm{T}}$ at 0. First let us assume that $r \geq k$. We set $D_0 f = f$ and $D_i f(x) = \chi_\gamma(x)\binom{k}{i}x^{k-i}$ for $i = 1, \ldots, k$ and $D_i f = 0$ for $i = k+1, \ldots, \lfloor r \rfloor$. We thus have

$$\begin{aligned} R_{\lfloor r \rfloor} f(x+y, x) =& f(x+y) - \sum_{i=0,\ldots,k} D_i f(x) y^i \\ =& \chi_\gamma(x+y)(x+y)^k - \sum_{i=0,\ldots,k} \chi_\gamma(x) \binom{k}{i} x^{k-i} y^i \\ =& (\chi_\gamma(x+y) - \chi_\gamma(x))(x+y)^k. \end{aligned}$$

As $\chi_\gamma(x+y) - \chi_\gamma(x) = 0$ if $v(y) > v(x)$, we may assume that $v(y) \leq v(x)$ checking the convergence condition on $R_{\lfloor r \rfloor} f$. We calculate

$$\begin{aligned} |R_{\lfloor r \rfloor} f(x+y, x)| =& |(\chi_\gamma(x+y) - \chi_\gamma(x))(x+y)^k| \\ \leq& |\chi_\gamma(x+y) - \chi_\gamma(x)||y|^k \\ \leq& |\chi_\gamma(y)||y|^k \\ =& |y|^{v(\alpha)-v(\beta)}|y|^k \\ =& |y|^{k-v(\beta)}|y|^{v(\alpha)} \\ \leq& \varepsilon|y|^r \quad \text{for } x+y, x \text{ in a neighborhood } U \ni 0 \text{ with } \delta U \leq \varepsilon^{1/(k-v(\beta))}. \end{aligned}$$

If $r < k$, then

$$R_{\lfloor r \rfloor} f(x+y,x) = (\chi_\gamma(x+y) - \chi_\gamma(x))(x+y)^k + \chi_\gamma(x) \sum_{i=\lfloor r \rfloor+1,\dots,k} \binom{k}{i} x^{k-i} y^i.$$

We already showed above that for the first summand holds $|(\chi_\gamma(x+y) - \chi_\gamma(x))(x+y)^k| \leq \varepsilon |y|^r$ for $x+y, x$ in a neighborhood of 0. The same holds for the second summand as $i > r$. Thus $f \in \mathcal{C}^r_{\mathrm{T}}(\mathbf{F}, \mathbf{K})$ in this case, too. ■

2.3 Orthogonal bases on $\mathbb{Z}_p$

Assumption. We will throughout this subsection assume that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a normed field.

Definition. We define $\mathrm{l} : \mathbb{R}_{\geq 0} \to \mathbb{N}$ by $\mathrm{l}(0) = 0$ and otherwise $\mathrm{l}(i)$ as the largest $n \in \mathbb{N}$ such that $p^n \leq i$.

So $\mathrm{l}(i) = \left\lfloor \log_p i \right\rfloor$ with $\log_p := \log / \log p$ for $i \geq 1$. Recall that in Definition 2.1, we gave a norm on $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ by

$$\|f\|_{\mathcal{C}^\rho} = \|f\|_{\sup} \vee \||f^{[\rho]}|\|_{\sup}.$$

Here $|f^{[\rho]}| : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{R}_{\geq 0}$ is the continuous function extending the mapping $|f(x) - f(y)|/|x-y|^\rho$, defined for all distinct $x, y \in \mathbb{Z}_p$, by 0 on the diagonal $\triangle \mathbb{Z}_p^2$. Thus

$$\||f^{[\rho]}|\|_{\sup} = \max_{x,y \in \mathbb{Z}_p \text{ distinct}} \frac{|f(x) - f(y)|}{|x-y|^\rho}.$$

Definition. We will denote the normed $\mathbf{K}$-linear subspace of locally constant functions $f : \mathbb{Z}_p \to \mathbf{K}$ in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ by $\mathcal{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. Then $\mathcal{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho = \cup_{n \geq 0} \mathcal{C}^{\mathrm{cst}}_n(\mathbb{Z}_p, \mathbf{K})^\rho$, where $\mathcal{C}^{\mathrm{cst}}_n(\mathbb{Z}_p, \mathbf{K})^\rho$ is the $\mathbf{K}$-Banach subspace of functions $f : \mathbb{Z}_p \to \mathbf{K}$ constant on the $p^n\mathbb{Z}_p$-cosets.

Definition. We will call a countable subset $\{e_0, e_1, \dots\}$ of a $\mathbf{K}$-Banach space $\mathbf{E}$ **orthogonal** if $\|\sum_{i \geq 0} \lambda_i e_i\| = \max_{i \geq 0} |\lambda_i| \|e_i\|$ for all scalars λ_i such that this series converges. It will be called **orthonormal** if it is orthogonal and $\|e_i\| = 1$ for all i. It will be an **orthogonal base** if every $x \in \mathbf{E}$ can be written $x = \sum_i \lambda_i e_i$ for some scalars λ_i.

Lemma 2.35. *A countable subset $\{e_1, e_2, \dots\}$ of a $\mathbf{K}$-Banach space $\mathbf{E}$ is orthogonal if and only if*

$$\|\sum_{i=m,\dots,n} \lambda_i e_i\| \geq |\lambda_m| \|e_m\| \quad \text{for all scalars } \lambda_m, \dots, \lambda_n.$$

Proof. Firstly note that $\{e_1, e_2, \dots\}$ is orthogonal if and only if

$$\|\sum_{i=m,\dots,n} a_i e_i\| \geq |a_m| \|e_m\| \vee \dots \vee |a_n| \|e_n\| \quad \text{for all } a_m, \dots, a_n \in \mathbf{K}.$$

We fix some n and proceed by downward induction on $m \leq n$. If $m = n$ there will be nothing to prove, so assume $m < n$. By assumption $\|a_m e_m + \dots + a_n e_n\| \geq \|a_m e_m\|$. Therefore the triangle inequality yields

$$\|a_m e_m + \dots + a_n e_n\| \geq |a_m| \|e_m\| \vee \|a_{m+1} e_{m+1} + \dots + a_n e_n\|.$$

The induction hypothesis renders

$$\|a_{m+1}e_{m+1} + \cdots + a_n e_n\| \geq |a_{m+1}|\|e_{m+1}\| \vee \ldots \vee |a_n|\|e_n\|.$$

Hence $\|a_m e_m + \cdots + a_n e_n\| \geq |a_m|\|e_m\| \vee \ldots \vee |a_n|\|e_n\|$, completing the induction step. ■

Corollary 2.36. *Let $\{e_0, e_1, \ldots\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ be such that $e_i(i) = 1$ and $e_i(m) = 0$ for any nonnegative integer $m < i$ and $\|e_i\|_{\mathcal{C}^\rho} = p^{\mathrm{l}(i)\rho}$. Then $\{e_0, e_1, \ldots\}$ is an orthogonal system of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.*

Proof. By the preceding Lemma 2.35, we must show that $\|\sum_{i=m,\ldots,n} a_i e_i\|_{\mathcal{C}^\rho} \geq |a_m|\|e_m\|_{\mathcal{C}^\rho}$ for any $m \leq n \in \mathbb{N}$ and $a_m, \ldots, a_n \in \mathbf{K}$. Firstly, for $m = 0$, we find

$$\|\sum_{i=0,\ldots,n} a_i e_i\|_{\mathcal{C}^\rho} \geq \|\sum_{i=0,\ldots,n} a_i e_i\|_{\sup} \geq |\sum_{i=0,\ldots,n} a_i e_i(0)| = |a_0| = |a_0|\|e_0\|_{\mathcal{C}^\rho}.$$

If instead $m > 0$, we calculate

$$\begin{aligned}\|\sum_{i=m,\ldots,n} a_i e_i\|_{\mathcal{C}^\rho} &\geq \||(\sum_{i=m,\ldots,n} a_i e_i)^{[\rho]}|\|_{\sup} \\ &\geq \frac{|\sum_{i=m,\ldots,n} a_i(e_i(m) - e_i(m - p^{\mathrm{l}(m)}))|}{|p^{\mathrm{l}(m)}|^\rho} \\ &= |a_m| p^{\mathrm{l}(m)\rho} = |a_m|\|e_m\|_{\mathcal{C}^\rho}.\end{aligned}$$

■

The van der Put base of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$

This brief interlude is motivated by the remark following [Schikhof, 1984, Theorem 63.2] about the similarity between the description of the $\mathcal{C}^{\mathrm{lip}}$-functions by its coefficients with respect to the Mahler- and van der Put-base.

Definition. We define the **van der Put characteristic function** $P_i : \mathbb{Z}_p \to \mathbf{K}$ for $i \in \mathbb{N}$ by $P_0 \equiv 1$ and for $i \geq 1$ through

$$P_i(x) = \begin{cases} 1, & \text{if } a_0 + a_1 p + \cdots + a_n p^n = i \text{ for some } n, \text{ where } x = \sum_{j \geq 0} a_j p^j, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.37. *(i) $\|P_i\|_{\mathcal{C}^\rho} = p^{\mathrm{l}(i)\rho}$ for all $i \in \mathbb{N}$.*

(ii) The family $\{P_0, P_1, \ldots\}$ is orthogonal.

Proof. Ad (i): Obviously $\|P_i\|_{\sup} = 1$ for all $i \in \mathbb{N}$. Since P_i is constant on $p^{\mathrm{l}(i)}\mathbb{Z}_p$-cosets, we therefore find $\||P_i^{[\rho]}|\|_{\sup} \leq p^{\mathrm{l}(i)\rho}$. Firstly, if $i = 0$, we thus find $1 = \|P_0\|_{\sup} \leq \|P_0\|_{\mathcal{C}^\rho} \leq p^{\mathrm{l}(0)\rho} = 1$, i.e. $\|P_0\|_{\mathcal{C}^\rho} = 1$. If instead $i > 0$, we achieve the postulated equality by

$$\frac{|P_i(i) - P_i(i - p^{\mathrm{l}(i)})|}{|i - (i - p^{\mathrm{l}(i)})|^\rho} = \frac{|1 - 0|}{|p^{\mathrm{l}(i)}|^\rho} = p^{\mathrm{l}(i)\rho}.$$

Ad (ii): Since $P_i(i) = 1$, as well as $P_i(m) = 0$ if $m < i$ and we just saw $\|P_i\|_{\mathcal{C}^\rho} = p^{\mathrm{l}(i)\rho}$, Corollary 2.36 applies. ■

Proposition. *The family* $\{P_i\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ *is an orthogonal base of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof. It is a general fact that an orthogonal system of a $\mathbf{K}$-Banach space $\mathbf{E}$ over a non-trivially non-Archimedeanly valued field $\mathbf{K}$, whose $\mathbf{K}$-linear span is dense therein is an orthonormal base (cf. [Schikhof, 1984, Exercise 50.F]). We must therefore show that the $\mathbf{K}$-linear span of $\{P_0, P_1, \dots\}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. By definition, we find $\{P_0, \dots, P_{p^n-1}\} \subseteq \mathcal{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. By the orthogonality of $\{P_0, P_1, \dots\}$ through Lemma 2.37, the coefficients a_i of each linear combination $\sum_i a_i P_i$ are unique. That is, $P_0, P_1, \dots$ are all linearly independent. Since $\dim_{\mathbf{K}} \mathcal{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho = p^n = \#\{P_0, \dots, P_{p^n-1}\}$, the family $\{P_0, \dots, P_{p^n-1}\}$ is a maximal linearly independent subset and in particular spans $\mathcal{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. Because $\mathcal{C}^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho = \cup_n \mathcal{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$, and this space is by Corollary 1.12 dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$, we are done. ∎

The Mahler base of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$

Remark. We want to show in which respect the yet to be introduced orthogonal basis of $\mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$ of Mahler polynomials relates to the domain's topological group structure: Let $\mathbf{o}[[X]]$ be the topological ring of formal power series endowed with its weak topology, described by a sequence $f_n \to f$ converging in $\mathbf{o}[[X]]$ if at any fixed index i, the i-th coefficient of f_n for $n \geq 0$ converges to the i-th coefficient of f. Let $\mathbf{o}[[\mathbb{Z}_p]] = \varprojlim \mathbf{o}[\mathbb{Z}/p^n\mathbb{Z}]$ be the completed group algebra. Then we have the Iwasawa isomorphism $\mathbf{o}[[X]] \tilde{\to} \mathbf{o}[[\mathbb{Z}_p]]$ of topological $\mathbf{o}$-algebras given by $X \mapsto 1 + \mathbf{1}$; here $\mathbf{1} \in \mathbb{Z}_p$ denoting the canonical generator of the topological abelian group $\mathbb{Z}_p$. Let $\mathcal{D}(\mathbb{Z}_p, \mathbf{o})$ be the continuous $\mathbf{o}$-linear dual of $\mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$ equipped with the topology of pointwise convergence. Then we have a natural identification $\mathbf{o}[[\mathbb{Z}_p]] \tilde{\to} \mathcal{D}(\mathbb{Z}_p, \mathbf{o})$.
We conclude that the Iwasawa isomorphism yields by Schikhof duality (see [Schikhof, 1995, Theorem 4.6]) the isomorphism of $\mathbf{K}$-Banach spaces $c_0(\mathbb{N}, \mathbf{K}) \tilde{\to} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$ with $c_0(\mathbb{N}, \mathbf{K})$ denoting all zero sequences in $\mathbf{K}$. We will then subsequently define the images of the canonical orthogonal basis in $c_0(\mathbb{N}, \mathbf{K})$ as the *Mahler polynomials* and let $f \in \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$ correspond to its *Mahler coefficients* $(a_n)_{n\in\mathbb{N}}$.

Definition 2.38. We define the i-th **Mahler polynomial** $\binom{*}{i} : \mathbb{Z}_p \to \mathbf{K}$ for $i \in \mathbb{N}$ by

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}.$$

Lemma 2.39. *(i)* $\|\binom{*}{i}\|_{\mathcal{C}^\rho} = p^{\mathrm{l}(i)\rho}$ *for all* $i \in \mathbb{N}$.

(ii) The family $\{\binom{*}{0}, \binom{*}{1}, \dots\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ *is orthogonal.*

Proof. Ad (i): By [Schikhof, 1984, Lemma 47.4], we find

$$\left|\binom{x}{i} - \binom{y}{i}\right| \leq |x-y| p^{\mathrm{l}(i)} \quad \text{for all } x, y \in \mathbb{Z}_p.$$

Since the left hand side is bounded by 1, this implies $|\binom{x}{i} - \binom{y}{i}| \leq |x-y|^\rho p^{\mathrm{l}(i)\rho}$ for $0 \leq \rho \leq 1$. Hence

$$\frac{|\binom{x}{i} - \binom{y}{i}|}{|x-y|^\rho} \leq p^{\mathrm{l}(i)\rho} \quad \text{for all distinct } x, y \in \mathbb{Z}_p.$$

We note that by continuity $\|\binom{*}{i}\|_{\sup} \leq 1$ since $\binom{j}{i} \in \mathbb{Z}_{\geq 0}$ for all $j \in \mathbb{Z}$. If $i = 0$, then $1 = \|\binom{*}{0}\|_{\sup} \leq \|\binom{*}{0}\|_{\mathcal{C}^\rho} \leq p^{\mathrm{l}(0)\rho} = 1$, i.e. $\|\binom{*}{0}\|_{\mathcal{C}^\rho} = 1$. It therefore remains to prove that $\||f^{[\rho]}(x,y)|\| = p^{\mathrm{l}(i)\rho}$ for distinct $x, y \in \mathbb{Z}_p$ for $i \geq 1$. Since $\binom{i}{i} - \binom{i-p^{\mathrm{l}(i)}}{i} = 1$, we achieve the craved equality by

$$\frac{|\binom{i}{i} - \binom{i-p^{\mathrm{l}(i)}}{i}|}{|p^{\mathrm{l}(i)}|^\rho} = p^{\mathrm{l}(i)\rho}.$$

Ad (ii): Since $\binom{i}{i} = 1$, as well as $\binom{m}{i} = 0$ if $m < i$ and since we just saw $\|\binom{*}{i}\|_{\mathcal{C}^\rho} = p^{\mathrm{l}(i)\rho}$, Corollary 2.36 applies. ■

Our aim is to prove that $\{\binom{*}{i}\}$ is an orthogonal basis of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. At this point, by the general criterion [Schikhof, 1984, Exercise 50.F] already mentioned, it remains to show that the $\mathbf{K}$-linear span of $\{\binom{*}{i}\}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. This will be initially only proven in the special case of a complete non-Archimedeanly non-trivially valued field $\mathbf{K}$ such that $v(\mathbf{K}) \ni \rho$ and $v(\mathbf{K}^*)$ is a discrete subgroup of $\mathbb{R}$. Afterwards this case will be reduced to.

Definition. Let $(E, \|\cdot\|)$ be a normed $\mathbf{K}$-vector space over a discretely non-Archimedeanly non-trivially valued field $\mathbf{K}$ such that $\|E\| \subseteq |K|$. We define the $\mathbf{o}$-module $E_{\leq 1} := \{f \in E : \|f\| \leq 1\}$ and its submodule $E_{<1} := \{f \in E : \|f\| < 1\}$. We set $\overline{E} := E_{\leq 1}/E_{<1}$.

Note that $\overline{E}$ is naturally a $\mathbf{k}$-vector space for the residue field $\mathbf{k}$ of $\mathbf{K}$. The importance of $\overline{E}$ stems from the following Lemma.

Lemma 2.40. *Let* $\mathbf{E}$ *be a* $\mathbf{K}$*-Banach space over a discretely non-Archimedeanly non-trivially valued field* $\mathbf{K}$ *such that* $\|\mathbf{E}\| \subseteq |\mathbf{K}|$. *Then a* $\mathbf{K}$*-linear subspace* $D \subseteq \mathbf{E}$ *is dense in* $\mathbf{E}$ *if and only if* $\overline{D} = \overline{\mathbf{E}}$.

Proof. If D is dense in $\mathbf{E}$, then a fortiori there will exist for any $x \in \mathbf{E}$ with $\|x\| = 1$ some $y \in D$ such that $\|x - y\| < 1$. Then $\|y\| = 1$ and therefore $\overline{D} = \overline{\mathbf{E}}$.
In the other direction, fix $\varepsilon > 0$ and some $x \in \mathbf{E}$. We have to find $y \in D$ such that $\|x-y\| < \varepsilon$. If $C := \|x\| = 0$, we will be done. Otherwise, because $\|\mathbf{E}\| \subseteq |\mathbf{K}|$, we find some nonzero scalar a_1 with $\|a_1 x\| = 1$. As $\mathbf{K}$ is discretely valued we find the largest absolute value $\theta := |\pi| \in |\mathbf{K}|$ less than 1. By assumption, there exists $y_1 \in D$ such that $\|a_1 x - y_1\| \leq \theta$ and therefore $\|x - y_1/a_1\| \leq \theta C$. Put $x_1 := x - y_1/a_1 \in \mathbf{E}$. If $C_1 := \|x_1\| = 0$, we will be done. Otherwise, there again exists some $a_2 \in \mathbf{K}^*$ such that $\|a_2 x_1\| = 1$. Once more, there exists some $y_2 \in D$ such that $\|a_2 x_1 - y_2\| \leq \theta$. Therefore

$$\|a_2 x_1 - y_2\| = \|a_2(x - y_1/a_1) - y_2\| = |a_2|\|x - y_1/a_1 - y_2/a_2\| \leq \theta.$$

Thence $\|x - y_1/a_1 - y_2/a_2\| \leq \theta C_1 \leq \theta^2 C$. Inductively, we find $y = y_1/a_1 + \cdots + y_n/a_n \in D$ such that $\|x - y\| \leq \theta^n C \leq \varepsilon$ for $n \geq 0$ big enough. ■

Lemma. *Let* $\mathbf{K}$ *be a discretely non-Archimedeanly non-trivially valued field with* $\rho \in v(\mathbf{K})$. *Then* $\|\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})\|_{\mathcal{C}^\rho} \subseteq |\mathbf{K}|$.

Proof. Let $f \in \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Since $\mathbb{Z}_p$ is compact, the supremum

$$\|f\|_{\mathcal{C}^\rho} = \|f\|_{\sup} \vee \||f^{[\rho]}|\|_{\sup} = \sup_{x \in \mathbb{Z}_p} |f(x)| \vee \sup_{x,y \in \mathbb{Z}_p \text{ distinct}} \frac{|f(x) - f(y)|}{|x - y|^\rho}$$

is attained. If $\|f\|_{\sup} \geq \||f^{[\rho]}|\|_{\sup}$, there will be nothing to show. So assume that there exists distinct $x, y \in \mathbb{Z}_p$ such that $\|f\|_{\mathcal{C}^\rho} = |f(x) - f(y)|/|x - y|^\rho$. We must show $|x - y|^\rho \in |\mathbf{K}|$ or equivalently $\rho \cdot v(x - y) \in v(\mathbf{K})$. But $\rho \in v(\mathbf{K})$ and $v(x - y) \in \mathbb{Z}$ by assumption. ∎

Corollary 2.41. *Let* $\mathbf{K}$ *be a discretely non-Archimedeanly non-trivially valued field with* $\rho \in v(\mathbf{K})$. *Let* $\{e_0, e_1, \dots\}$ *be an orthonormal family of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. *If* $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$ *for all* $n \geq 0$, *then* $\{e_0, e_1, \dots\}$ *will be an orthonormal basis of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof. It is a general fact that an orthonormal system of a $\mathbf{K}$-Banach space $\mathbf{E}$ over a complete non-trivially non-Archimedeanly valued field $\mathbf{K}$, whose $\mathbf{K}$-linear span is dense therein is an orthonormal base (cf. [Schikhof, 1984, Theorem 50.7]). We must therefore show that the K-linear span of $\{e_0, e_1, \dots\}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. By the preceding Lemma, the conditions on $\mathbf{E} = \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ of Lemma 2.40 apply. We are hence reduced to proving that

$$\overline{\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})} = \overline{\oplus_{i \geq 0} \mathbf{K} \cdot e_i} = \oplus_{i \geq 0} \mathbf{k} \cdot \overline{e_i},$$

where the last equality stems from the orthogonality of $\{e_i\}$. By Corollary 1.12 (and the obvious implication of Lemma 2.40), we find

$$\overline{\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})} = \overline{\mathcal{C}^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho} = \cup_{n \geq 0} \overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}.$$

Let $n \in \mathbb{N}$. By assumption $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$, and it suffices therefore to show that $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\}$ is a basis of this subspace. By orthonormality, the $\overline{e}_0, \dots, \overline{e}_{p^n-1}$ are linearly independent over $\mathbf{k}$. On the other hand, $\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$ is $\mathbf{K}$-linearly generated by p^n-many functions living on the $p^n\mathbb{Z}_p$-cosets, so that $\dim_{\mathbf{k}} \overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho} \leq p^n$. Therefore

$$\dim_{\mathbf{k}} \overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho} = p^n = \#\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\}.$$

Hence $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\}$ is a maximal linearly independent subset of $\overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$, i.e. a basis. ∎

Recall that we firstly prove $\{\binom{*}{i}\}$ to be an orthogonal basis of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ only in the case of a discretely valued $\mathbf{K}$ with $\rho \in v(\mathbf{K})$. Since $\rho \in v(\mathbf{K})$, the $\binom{*}{i}$ can be rescaled to yield an ortho*normal* system $\{e_i\}$ of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Also, $\mathbf{K}$ will fulfill the assumptions of Corollary 2.41, so that we are reduced to proving that $\{\overline{e_0}, \dots, \overline{e_{p^n-1}}\} \subseteq \overline{\mathcal{C}^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$. To this end, the following criterion will be helpful.

Definition. (i) Let $f \in \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Then there exists a smallest $n \geq 0$ such that $|f(x) - f(y)| < |x - y|^\rho$ for all distinct x, y in the same $p^n\mathbb{Z}_p$ coset. We denote this unique number by $o(f)$, the **oscillation index** of f.

(ii) We have a well defined $\mathbf{o}$-linear **reduction map** π_n from the $\mathbf{o}$-module of functions $f \in \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{o})$ of oscillation index $o(f) \leq n$ to the finite dimensional $\mathbf{o}/\mathbf{o}_{<p^{-n\rho}}$-module of functions $f : \mathbb{Z}_p/p^n\mathbb{Z}_p \to \mathbf{o}/\mathbf{o}_{<p^{-n\rho}}$.

Lemma. *Let $f \in \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{o})$ with $o(f) \leq n$ for $n \in \mathbb{N}$. If $\pi_n f = 0$, then $\overline{f} = 0$.*

Proof. Firstly $\|f\|_{\sup} < p^{-\rho n} \leq 1$ because $\pi_n f = 0$. Therefore it remains to show that $\||f^{[\rho]}|\|_{\sup} < 1$. We calculate

$$\||f^{[\rho]}|\|_{\sup} = \max_{x,y \in \mathbb{Z}_p \text{ distinct}} \frac{|f(x) - f(y)|}{|x-y|^\rho}$$
$$= \max_{\substack{x,y \in \mathbb{Z}_p \text{ distinct} \\ \text{s.t. } |x-y| \leq p^{-n}}} \frac{|f(x) - f(y)|}{|x-y|^\rho} \vee \max_{\substack{x,y \in \mathbb{Z}_p \text{ s.t.} \\ |x-y| > p^{-n}}} \frac{|f(x) - f(y)|}{|x-y|^\rho}$$

As $o(f) \leq n$, we find $|f(x) - f(y)| < |x-y|^\rho$ for all distinct $x, y \in \mathbb{Z}_p$ with $|x - y| \leq p^{-n}$, so that the first maximum is less than 1. Secondly,

$$\max_{\substack{x,y \in \mathbb{Z}_p \\ \text{s.t. } |x-y| > p^{-n}}} \frac{|f(x) - f(y)|}{|x-y|^\rho} \leq \max_{\substack{x,y \in \mathbb{Z}_p \\ \text{s.t. } |x-y| > p^{-n}}} \frac{|f(x)| \vee |f(y)|}{|x-y|^\rho} < \|f\|_{\sup}/p^{n\rho} < p^{n\rho}/p^{n\rho} = 1;$$

here we used $1/|x-y|^\rho < p^{n\rho}$ and $\|f\|_{\sup} \leq p^{-n\rho}$. ■

Corollary 2.42. *Let $f \in \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})_{\leq 1}$. If $o(f) \leq n$ for $n \in \mathbb{N}$, then $\overline{f} \in \overline{\mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$.*

Proof. Consider the mapping

$$\pi_n : \mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{o})^\rho \to \{f : \mathbb{Z}_p/p^n\mathbb{Z}_p \to \mathbf{o}/\mathbf{o}_{<p^{-n\rho}}\}.$$

It is well defined and surjective. Therefore $\pi_n f = \pi_n g$, i.e. $\pi_n(f - g) = 0$ for some $g \in \mathcal{C}_n^{\text{cst}}(\mathbb{Z}_p, \mathbf{o})^\rho$. By the preceding Lemma $\overline{f} = \overline{g}$. ■

We will now prove the oscillation index of the normalized $\binom{*}{i}$ to equal $\mathrm{l}(i) + 1$. This will allows us to apply the above Corollary 2.42 for $i = 0, \ldots, p^n - 1$.

Lemma 2.43. *Assume $\rho \in v(\mathbf{K})$. Then we can define $e_i = \lambda_i \binom{*}{i}$ for a scalar $\lambda_i \in \mathbf{K}$ such that $\|e_i\|_{\mathcal{C}^\rho} = 1$. Moreover $o(e_0) = 0$ and $o(e_i) = \mathrm{l}(i) + 1$ if $i \geq 1$.*

Proof. Let us find such scalar λ_i: Let $\alpha \in \mathbf{K}$ such that $v(\alpha) = \rho$. Then $\|\binom{*}{i}\|_{\mathcal{C}^\rho} = p^{\mathrm{l}(i)\rho}$ by Lemma 2.39, so that we put $\lambda_i = \alpha^{\mathrm{l}(i)}$.

Clearly $o(e_0) = 0$. Let $i \geq 1$. We now want to prove that $o(e_i) = \mathrm{l}(i) + 1$, i.e. $n = \mathrm{l}(i) + 1$ is the smallest $n \in \mathbb{N}$ such that

$$|e_i(x) - e_i(y)| < |x-y|^\rho \quad \text{for all distinct } x, y \text{ with } |x-y| \leq p^{-n}.$$

Firstly observe that $\binom{i}{i} - \binom{i-p^{\mathrm{l}(i)}}{i} = 1$ if $i \geq 1$, so $|e_i(i) - e_i(i - p^{\mathrm{l}(i)})| = |\lambda_i| = |p^{\mathrm{l}(i)}|^\rho$. Hence necessarily $o(e_i) > \mathrm{l}(i)$.

Let us prove $o(e_i) \leq \mathrm{l}(i) + 1$. By [Schikhof, 1984, Lemma 47.4], we have

$$\left|\binom{x}{i} - \binom{y}{i}\right| \leq |x - y|p^{\mathrm{l}(i)} = |x - y|^\rho p^{\mathrm{l}(i)\rho}(|x - y|p^{\mathrm{l}(i)})^{1-\rho}.$$

By definition of e_i thus $|e_i(x) - e_i(y)| \leq (|x - y|p^{\mathrm{l}(i)})^{1-\rho}|x - y|^\rho$. So if $|x - y| < p^{-\mathrm{l}(i)}$, then $|e_i(x) - e_i(y)| < |x - y|^\rho$. ■

Corollary 2.44. *Assume that* $v(\rho) \in \mathbf{K}$ *and let* $\{e_0, e_1, \dots\}$ *be as in Lemma 2.43. Then* $\{\overline{e}_0, \dots, \overline{e}_{p^n-1}\} \subseteq \overline{\mathcal{C}_n^{\mathrm{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho}$ *for all* $n \geq 0$.

Proof. We have $\mathrm{l}(i) < n$ for $i = 0, \dots, p^n - 1$. By Lemma 2.43 therefore $o(f) = \mathrm{l}(i) + 1 \leq n$. Now Corollary 2.42 applies. ■

Proposition 2.45. *Let* $\mathbf{K}$ *be a complete non-Archimedeanly non-trivially discretely valued field with* $\rho \in v(\mathbf{K})$. *Let* $e_0, e_1, \dots$ *be the normalized Mahler polynomials* $\binom{*}{i}$ *as in Lemma 2.43. Then* $\{e_0, e_1, \dots\}$ *is an orthonormal basis of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof. By Corollary 2.44, this is a direct application of Corollary 2.41. ■

Finally we show that for general $\mathbf{K}$, we still obtain that $\{\binom{*}{i}\}$ is an orthogonal basis of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ by reducing to the special case of a discretely valued $\mathbf{K}$ with $v(\mathbf{K}) \ni \rho$. For this, the following property of theirs is crucial:

Lemma 2.46. *Let* $f : \mathbb{Z}_p \to \mathbf{K}$ *be a continuous mapping and assume* $f = \sum_{i \geq 0} a_i \binom{*}{i}$ *with respect to* $\|\cdot\|_{\sup}$ *for coefficients* $a_i \in \mathbf{K}$. *Then* $\operatorname{im} f \subseteq \mathbb{Q}_p$ *if and only if* $\{a_i\} \subseteq \mathbb{Q}_p$.

Proof. Define the $\mathbf{K}$-linear endomorphism Δ of the $\mathbf{K}$-vector space $\mathbf{K}^{\mathbb{Z}_p}$ by $g \mapsto g(\cdot + 1) - g$. Since $\Delta\binom{*}{0} = 0$ and $\Delta\binom{*}{i} = \binom{*}{i-1}$, we find $\Delta f(x) = \sum_{i \geq 0} a_{i+1}\binom{x}{i}$. Transitively, we obtain

$$\Delta^{\circ n} f(x) := \underbrace{\Delta \circ \dots \circ \Delta}_{n\text{-times}} f(x) = \sum_{i \geq 0} a_{i+n}\binom{x}{i}.$$

In particular $a_n = \Delta^{\circ n} f(0)$ and hence the result. ■

Lemma 2.47. *Let* $\{b_i\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ *be such that for any complete non-Archimedeanly non-trivially valued field* $\mathbf{K} \supseteq \mathbb{Q}_p$ *we have:*

(i) $\{b_i\}$ *is an orthogonal system of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$,

(ii) for every continuous function $f : \mathbb{Z}_p \to \mathbf{K}$ *with* $f = \sum_{i \geq 0} a_i b_i$ *with respect to* $\|\cdot\|_{\sup}$ *for coefficients* $a_i \in \mathbf{K}$, *we have* $\operatorname{im} f \subseteq \mathbb{Q}_p$ *if and only if* $\{a_i\} \subseteq \mathbb{Q}_p$.

Then for any complete non-Archimedeanly non-trivially valued field $\mathbf{K} \supseteq \mathbb{Q}_p$ *we have:* $\{b_i\}$ *is an orthogonal base of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ *if and only if it is one of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$.

Proof. Let $\{b_i\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ be as above and let $\mathbf{K} \supseteq \mathbb{Q}_p$ be a complete non-Archimedeanly non-trivially valued field. By [Schikhof, 1984, Exercise 50.F], an orthogonal system of a $\mathbf{K}$-Banach space $\mathbf{E}$ will be an orthogonal basis if its $\mathbf{K}$-linear span is dense in $\mathbf{E}$. Since $\{b_i\}$ is assumed to be orthogonal, it remains to prove that the $\mathbb{Q}_p$-linear span of $\{b_i\}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ if and only if its $\mathbf{K}$-linear one is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. We will denote by $< \{b_i\} >_{\mathbb{Q}_p}$ the $\mathbb{Q}_p$-linear span of the b_i and define $< \{b_i\} >_{\mathbf{K}}$ likewise.

Firstly, suppose $< \{b_i\} >_{\mathbf{K}}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. Fix a function $f \in \mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$. As a convergent sum in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$, we have

$$f = \sum_{i \geq 0} a_i b_i \quad \text{for coefficients } a_i \in \mathbf{K}.$$

By assumption $\{a_i\} \subseteq \mathbb{Q}_p$. So $< \{b_i\} >_{\mathbb{Q}_p}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$.

Contrariwise, suppose $< \{b_i\} >_{\mathbb{Q}_p}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$. By Corollary 1.12, the locally constant functions $f : \mathbb{Z}_p \to \mathbf{K}$ are dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. It will thus suffice to prove that $< \{b_i\} >_{\mathbf{K}}$ is dense in $\mathcal{C}^{\text{cst}}(\mathbb{Z}_p, \mathbf{K})^\rho$. Fix $\varepsilon > 0$ and some locally constant $f : \mathbb{Z}_p \to \mathbf{K}$. Then f is constant on the $p^n\mathbb{Z}_p$ cosets for some n. Hence we may write

$$f = \sum_{i=0,\dots,p^n-1} a_i \mathbf{1}_{i+p^n\mathbb{Z}_p} \quad \text{for coefficients } a_0, \dots, a_{p^n-1} \in \mathbf{K}.$$

Let $C := \max_i |a_i| \vee 1$ and $\varepsilon' = \varepsilon / C$. By the density of $< \{b_i\} >_{\mathbb{Q}_p}$ in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$, we find $\{f_0, \dots, f_{p^n-1}\} \subseteq < \{b_i\} >_{\mathbb{Q}_p}$ such that $\|f_i - \mathbf{1}_{i+p^n\mathbb{Z}_p}\|_{\mathcal{C}^\rho} \leq \varepsilon'$ for $i = 0, \dots, p^n - 1$. Then

$$\|f - \sum_i a_i f_i\|_{\mathcal{C}^\rho} = \|\sum_i a_i(\mathbf{1}_{i+p^n\mathbb{Z}_p} - f_i)\|_{\mathcal{C}^\rho} \leq \max_i |a_i| \|\mathbf{1}_{i+p^n\mathbb{Z}_p} - f_i\|_{\mathcal{C}^\rho} \leq C\varepsilon' = \varepsilon.$$

Therefore $< \{b_i\} >_{\mathbf{K}}$ is dense in $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$. ∎

Lemma (2.47'). *Let $\{b_i\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbb{Q}_p)$ satisfy the assumptions of Lemma 2.47. If $\{b_i\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{F})$ is an orthogonal base of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{F})$ for* one *complete non-Archimedeanly valued field $\mathbf{F} \supseteq \mathbb{Q}_p$, then $\{b_i\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ will be one of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ for* every *complete non-Archimedeanly valued field $\mathbf{K} \supseteq \mathbb{Q}_p$.*

Proof. This is a reformulation of the conclusion of Lemma 2.47. ∎

In particular we could choose in Lemma 2.47' our field $\mathbf{F} \supseteq \mathbb{Q}_p$ to be discretely valued with $\rho \in v(\mathbf{F})$ if such $\mathbf{F}$ existed. The next Lemma constructs such $\mathbf{F}$.

Lemma 2.48. *There exists a complete non-Archimedeanly non-trivially discretely valued field $\mathbf{F} \supseteq \mathbb{Q}_p$ with $\rho \in v(\mathbf{F})$.*

Proof. Let $R = \mathbb{Q}_p[t]$ endowed with the valuation $v_{\mathbf{F}}(\sum_i a_i t^i) := \inf_i v(a_i) + \rho i$. Then its induced norm, denoted $|\cdot|_{\mathbf{F}}$, is quickly checked to be multiplicative on R. It extends to the completed fraction field $\mathbf{F}$ of R, denoted likewise. ∎

Remark. As a set, $\mathbf{F}$ consists of all formal Laurent series $\sum_{i\in\mathbb{Z}} a_i t^i$ with coefficients in $\mathbb{Q}_p$ such that, putting $c := p^{-\rho}$, we have $|a_i|c^i \to 0$ as $i \to -\infty$ and $\{|a_i|c^i : i \geq 0\}$ bounded - with norm $|\sum_{i\in\mathbb{Z}} a_i t^i|_{\mathbf{F}} = \max_{i\in\mathbb{Z}}|a_i|c^i$. This is known as the *field of bounded Laurent series over* $\mathbb{Q}_p$.

Theorem 2.49. *The family* $\{\binom{*}{0}, \binom{*}{1}, \dots\} \subseteq \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ *is an orthogonal basis of* $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ *with* $\|\binom{*}{i}\| = p^{\mathrm{l}(i)\rho}$.

Proof. The norms of the $\binom{*}{i}$ were calculated in Lemma 2.39. Then Lemma 2.39(ii) yields the first and Lemma 2.46 the second assumption of Lemma 2.47 regarding this family. By Lemma 2.47', it suffices to prove this theorem for some discretely valued field $\mathbf{F}$ with $\rho \in v(\mathbf{F})$, which exists by the preceding Lemma 2.48. By Proposition 2.45, we can rescale the $\binom{*}{i}$ such that they form an orthonormal basis of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{F})$. But then $\{\binom{*}{i}\}$ is still an orthogonal basis of $\mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{F})$. ■

The Mahler Base of $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$

Definition 2.50. Let $f : \mathbb{Z}_p \to \mathbf{K}$ be an arbitrary mapping. Then we define its **n-th Mahler coefficient** a_n for $n \in \mathbb{N}$ by

$$a_n = \Delta^{\circ n} f(0);$$

here we refer to Lemma 2.46 for the definition of the $\mathbf{K}$-linear endomorphism Δ on $\mathbf{K}^{\mathbb{Z}_p}$.

Lemma 2.51. *Let* $f : \mathbb{Z}_p \to \mathbf{K}$ *and* $a_0, a_1, \dots$ *its Mahler Coefficients. For* $x_1, \dots, x_\nu \in \mathbb{Z}_{\geq 1}$ *and* $y \in \mathbb{Z}_{\geq 0}$*, put* $z = (x_1 + \cdots + x_\nu + y, \dots, x_1 + y, y) \in \nabla\mathbb{Z}_{\geq 0}^{\nu+1}$. *Then*

$$f^{]\nu[}(z) = \sum_{j\geq 0} \sum_{m_1,\dots,m_\nu \geq 1} \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1})\cdots(m_\nu + \cdots + m_1)} \binom{x_1 - 1}{m_1 - 1} \cdots \binom{x_\nu - 1}{m_\nu - 1}\binom{y}{j}.$$

Proof. This is proven by induction on $\nu \geq 0$. If $\nu = 0$, there will be nothing to show. Let $\nu \geq 1$. By the induction hypothesis, we can compute

$$\begin{aligned}
&f^{]\nu[}(z) \\
=&1/x_\nu \cdot (\quad f^{]\nu-1[}(x_1 + \cdots + x_{\nu-1} + x_\nu + y, x_1 + \cdots + x_{\nu-2} + y, \dots, y) \\
&\qquad - f^{]\nu-1[}(x_1 + \cdots + x_{\nu-1} + y \qquad , x_1 + \cdots + x_{\nu-2} + y, \dots, y)) \\
=&\sum_{j\geq 0} \sum_{m_1,\dots,m_{\nu-1}\geq 1} \frac{a_{j+m_1+\cdots+m_{\nu-1}}}{m_{\nu-1}(m_{\nu-1} + m_{\nu-2})\cdots(m_{\nu-1} + \cdots + m_1)} \\
&\qquad \binom{x_1 - 1}{m_1 - 1} \cdots \binom{x_{\nu-2} - 1}{m_{\nu-2} - 1} \cdot (\binom{x_{\nu-1} - 1 + x_\nu}{m_{\nu-1} - 1} - \binom{x_{\nu-1} - 1}{m_{\nu-1} - 1})/x_\nu \binom{y}{j} \\
=&\sum_{j\geq 0} \sum_{m_1,\dots,m_{\nu-1}\geq 1} \frac{a_{j+m_1+\cdots+m_{\nu-1}}}{m_{\nu-1}(m_{\nu-1} + m_{\nu-2})\cdots(m_{\nu-1} + \cdots + m_1)} \\
&\binom{x_1 - 1}{m_1 - 1} \cdots \binom{x_{\nu-2} - 1}{m_{\nu-2} - 1} (\sum_{m_\nu=1}^{m_{\nu-1}-1} \binom{x_{\nu-1} - 1}{m_\nu - 1}\binom{x_\nu - 1}{m_{\nu-1} - m_\nu - 1}/(m_{\nu-1} - m_\nu))\binom{y}{j}
\end{aligned}$$

$$= \sum_{j \geq 0} \sum_{m_1,\ldots,m_{\nu-2} \geq 1} \Bigg(\sum_{m'_{\nu-1}+m'_\nu \geq 1} \frac{a_{j+m_1+\cdots+m'_{\nu-1}+m'_\nu}}{m'_\nu(m'_\nu + m'_{\nu-1}) \cdots (m'_\nu + m'_{\nu-1} + \cdots + m_1)}$$

$$\binom{x_1 - 1}{m_1 - 1} \cdots \binom{x_{\nu-2} - 1}{m_{\nu-2} - 1} \sum_{m'_{\nu-1}=1}^{m'_{\nu-1}+m'_\nu-1} \binom{x_{\nu-1} - 1}{m'_{\nu-1} - 1} \binom{x_\nu - 1}{m'_\nu - 1} \Bigg) \binom{y}{j};$$

here we put $m'_{\nu-1} = m_\nu, m'_\nu = m_{\nu-1} - m_\nu$ and noted $m_{\nu-1} = m'_{\nu-1} + m'_\nu$. As $x_1, \ldots, x_\nu, y \in \mathbb{Z}_{\geq 0}$, this is a finite sum and we can rearrange the double sum in parentheses to yield the proposed sum above. ■

For the next result we use Corollary 3.45 computing the Mahler coefficients of multivariate $\mathcal{C}^\rho$-functions and for chronological consistency the reader is advised to read the proof below and the remainder of this paragraph thereafter.

Lemma 2.52. *Let* $f : \mathbb{Z}_p \to \mathbf{K}$. *Then*

(i) $f \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ *if and only if* $|b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1-1) \vee \ldots \vee \mathrm{l}(m_\nu-1) \vee \mathrm{l}(j)] \cdot \rho} \to 0$ *as* $|\boldsymbol{m}| \to \infty$,

(ii) $\|f^{[\nu]}\|_{\mathcal{C}^\rho} = \max_{\boldsymbol{m} \geq 0} |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1-1) \vee \ldots \vee \mathrm{l}(m_\nu-1) \vee \mathrm{l}(j)] \cdot \rho}$;

here $b_{\boldsymbol{m}} \in \mathbf{K}$ *for* $\boldsymbol{m} = (m_1 - 1, \ldots, m_\nu - 1, j) \in \mathbb{N}^\nu \times \mathbb{N}$, *is given by*

$$b_{(m_1-1,\ldots,m_\nu-1,j)} = \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)}.$$

Proof. Ad (i): On $\mathbb{Z}_p^{\nu+1}$, consider the bijection φ defined by

$$(x_1, \ldots, x_\nu, y) \mapsto ((x_1 + 1) + \cdots + (x_\nu + 1) + y, \cdots, (x_1 + 1) + y, y).$$

We denote likewise its restriction onto the preimage of $\nabla \mathbb{Z}_p^{\nu+1}$. By Lemma 2.51, for $z = (x_1 + \cdots + x_\nu + y, \ldots, x_1 + y, y) \in \nabla \mathbb{Z}_{\geq 0}^{\nu+1}$, we find

$$f^{]\nu[}(z) = \sum_{j \geq 0} \sum_{m_1,\ldots,m_\nu \geq 1} \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)} \binom{x_1 - 1}{m_1 - 1} \cdots \binom{x_\nu - 1}{m_\nu - 1} \binom{y}{j}.$$

Therefore the $\boldsymbol{m} = (m_1 - 1, \ldots, m_\nu - 1, j)$-th coefficient $b_{\boldsymbol{m}}$ of $\widetilde{f^{]\nu[}} := f^{]\nu[} \circ \varphi$ is given by

$$b_{(m_1-1,\ldots,m_\nu-1,j)} = \frac{a_{j+m_1+\cdots+m_\nu}}{m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)}.$$

By Corollary 3.45, we find that

$$\widetilde{f^{]\nu[}} \text{ extends to } \widetilde{f^{[\nu]}} \in \mathcal{C}^\rho(\mathbb{Z}_p^{\nu+1}, \mathbf{K}) \text{ if and only if } |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1) \vee \ldots \vee \mathrm{l}(m_\nu) \vee \mathrm{l}(j)] \cdot \rho} \to 0 \text{ as } |\boldsymbol{m}| \to \infty.$$

For this note that the above expansion determines $\widetilde{f^{[\nu]}}$ on the dense subset $\mathbb{Z}_{\geq 1}^\nu \times \mathbb{Z}_{\geq 0} \subseteq \mathbb{Z}_p^{\nu+1}$ and hence everywhere by continuity. By Proposition 2.5, we find

$$f \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) \quad \text{if and only if} \quad f^{]\nu[} \text{ extends to } f^{[\nu]} \in \mathcal{C}^\rho(\mathbb{Z}_p^{\nu+1}, \mathbf{K}).$$

Since φ is a locally Lipschitzian automorphism, $f^{]\nu[}$ extends to a $\mathcal{C}^\rho$-function $f^{[\nu]}$ on $\mathbb{Z}_p^{\nu+1}$ if and only if $\widetilde{f^{]\nu[}}$ extends to a $\mathcal{C}^\rho$-function $\widetilde{f^{[\nu]}} = f^{[\nu]} \circ \varphi$ on $\mathbb{Z}_p^{\nu+1}$. The proposition follows.
Ad (ii): Since $\|\widetilde{f^{[\nu]}}\|_{\sup} = \|f^{[\nu]}\|_{\sup}$, we find $\|f^{[\nu]}\|_{\mathcal{C}^\rho} = \max_{\boldsymbol{m} \geq \boldsymbol{0}} |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1) \vee \ldots \vee \mathrm{l}(m_{\nu+1})] \cdot \rho}$. ■

Lemma 2.53. *Let* $f : \mathbb{Z}_p \to \mathbf{K}$ *and* $b_{\boldsymbol{m}}$ with $\boldsymbol{m} = (m_1, \ldots, m_{\nu+1}) \in \mathbb{N}^{\nu+1}$ *as in Lemma 2.52. Then*

$$\max_{|\boldsymbol{m}|=m} |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1)\vee\ldots\vee\mathrm{l}(m_{\nu+1})]\cdot\rho} = |a_{m+\nu}| p^{v_r(m+\nu)};$$

here $v_r(n)$ *for* $n \geq \nu$ *is given by*

$$v_r(n) = \max_{0\leq l_1<\ldots<l_\nu\leq n} v(l_1)+\cdots+v(l_\nu)+\rho\cdot[\mathrm{l}(l_1-1)\vee\mathrm{l}(l_2-l_1-1)\vee\ldots\vee\mathrm{l}(l_\nu-l_{\nu-1}-1)\vee\mathrm{l}(n-l_\nu)].$$

Proof. Fix $m \geq 0$ and let $\boldsymbol{m} = (m_1 - 1, \ldots, m_\nu - 1, j) \in \mathbb{N}^{\nu+1}$ with $m_1, \ldots, m_\nu \in \mathbb{Z}_{\geq 1}, j \in \mathbb{Z}_{\geq 0}$ such that $|\boldsymbol{m}| = m$. Then $b_{\boldsymbol{m}}$ is given by

$$b_{\boldsymbol{m}} = \frac{a_{m+\nu}}{m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)}.$$

We find

$$\begin{aligned}
&\max_{|\boldsymbol{m}|=m} |b_{\boldsymbol{m}}| p^{\rho\cdot[\mathrm{l}(m_1-1)\vee\ldots\vee\mathrm{l}(m_\nu-1)\vee\mathrm{l}(j)]} \\
&= \max_{\substack{m_1,\ldots,m_\nu\in\mathbb{Z}_{\geq1}, j\in\mathbb{Z}_{\geq0} \\ \text{with } m_1+\cdots+m_\nu+j=m+\nu}} \frac{|a_{m+\nu}|}{|m_\nu(m_\nu + m_{\nu-1}) \cdots (m_\nu + \cdots + m_1)|} p^{\rho\cdot[\mathrm{l}(m_1-1)\vee\ldots\vee\mathrm{l}(m_\nu-1)\vee\mathrm{l}(j)]} \\
&= \max_{0\leq l_1<\ldots<l_\nu\leq m+\nu} \frac{|a_{m+\nu}|}{|l_1 \cdots l_\nu|} p^{\rho\cdot[\mathrm{l}(l_1)\vee\mathrm{l}(l_2-l_1)\vee\ldots\vee\mathrm{l}(l_\nu-l_{\nu-1})\vee\mathrm{l}(m-l_\nu)]};
\end{aligned}$$

the last equality by the bijection $l_n \mapsto m_1 + \cdots + m_n$ for $n = 1, \ldots, \nu$ and noting that given $i \geq 0$, there exists $j \geq 0$ with $i + j = m$ if and only if $i \leq m$ and unique $j = m - i$. ■

Corollary 2.54. *Let* $f : \mathbb{Z}_p \to \mathbf{K}$. *Then*

(i) $f \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ *if and only if* $|a_m| p^{v_r(m)} \to 0$ *as* $m \to \infty$,

(ii) $\|f\|_{\mathcal{C}^r} = |a_0| \vee |a_1/1!| \vee \ldots \vee |a_{\nu-1}/(\nu-1)!| \vee \max_{m\geq\nu} |a_m| p^{v_r(m)}$.

Proof. Ad (i): By definition,

$$|b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1)\vee\ldots\vee\mathrm{l}(m_{\nu+1})]\cdot\rho} \to 0 \quad \text{as } |\boldsymbol{m}| \to \infty$$

if and only if

$$\max_{|\boldsymbol{m}|=m} |b_{\boldsymbol{m}}| p^{[\mathrm{l}(m_1)\vee\ldots\vee\mathrm{l}(m_{\nu+1})]\cdot\rho} = |a_{m+\nu}| p^{v_r(m+\nu)} \to 0 \quad \text{as } m \to \infty.$$

Ad (ii): Let $f \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) \subseteq \mathcal{C}^n(\mathbb{Z}_p, \mathbf{K})$ for $n = 0, \ldots, \nu - 1$, the inclusion by Lemma 2.3. Then Lemma 2.52 for $r = n + \rho$ with $n = 0, \ldots, \nu - 1$ and $\rho = 0$ yields $\|f^{[\nu]}\|_{\sup} = \max_{m\geq n} |a_m| p^{v_n(m)}$ with $v_n(m)$ defined by $v_n(m) = \max_{0\leq l_1<\ldots<l_\nu\leq m} v(l_1) + \cdots + v(l_\nu)$, whereas for $n = \nu$ and ρ, Lemma 2.52 yields $\|f^{[\nu]}\|_{\mathcal{C}^\rho} = \max_{m\geq\nu} |a_m| p^{v_r(m)}$. We observe that if $n \leq n' \leq m$, then by definition $v_n(m) \leq v_{n'}(m)$. We thus obtain

$$\begin{aligned}
\|f\|_{\mathcal{C}^r} &= \|f\|_{\sup} \vee \|f^{[1]}\|_{\sup} \vee \ldots \vee \|f^{[\nu-1]}\|_{\sup} \vee \|f^{[\nu]}\|_{\mathcal{C}^\rho} \\
&= \max_{m\geq0} |a_m| \vee \max_{m\geq0} |a_{m+1}| p^{v_1(m)} \vee \ldots \vee \max_{m\geq0} |a_{m+\nu-1}| p^{v_{\nu-1}(m)} \vee \max_{m\geq0} |a_{m+\nu}| p^{v_r(m)} \\
&= |a_0| \vee |a_1| p^{v_1(1)} \vee \ldots \vee |a_{\nu-1}| p^{v_{\nu-1}(\nu-1)} \vee \max_{m\geq0} |a_{m+\nu}| p^{v_r(m)}.
\end{aligned}$$

We can therefore conclude by

$$v_n(n) = \max_{0 \le l_1 < \ldots < l_n \le n} v(l_1) + \cdots + v(l_n) = v(1) + \ldots + v(n) = v(n!).$$

■

Theorem 2.55. *The family* $\{\binom{*}{0}, \binom{*}{1}, \ldots\} \subseteq \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ *is an orthogonal basis of* $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ *with* $\|\binom{*}{m}\|_{\mathcal{C}^r} = p^{w_r(m)}$*; here*

$$w_r(m) = \begin{cases} v(m!), & \text{if } m < \nu, \\ v_r(m), & \text{otherwise.} \end{cases}$$

Proof. By Corollary 2.54(ii) applied to the mapping $e_m = \binom{*}{m}$, we find $\|e_m\|_{\mathcal{C}^r} = p^{w_r(m)}$. Moreover by the same token, if $f = \sum_{m \ge 0} a_m \binom{*}{m} \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$, then

$$\|f\|_{\mathcal{C}^r} = \max_{m \ge 0} |a_m| p^{w_r(m)} = \max_{m \ge 0} |a_m| \|e_m\|_{\mathcal{C}^r}.$$

In other words, $\{\binom{*}{m}\} \subseteq \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ is an orthogonal family with $\|\binom{*}{m}\|_{\mathcal{C}^r} = p^{w_r(m)}$. Since $w_r(m) = v_r(m)$ for $m \ge \nu$, we find

$$|a_m| p^{v_r(m)} \to 0 \text{ as } m \to \infty \quad \text{if and only if} \quad |a_m| \left\| \binom{*}{m} \right\|_{\mathcal{C}^r} \to 0 \text{ as } m \to \infty.$$

By Corollary 2.54(i), we see $f \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ if and only if $|a_m| \|\binom{*}{m}\|_{\mathcal{C}^r} \to 0$ as $m \to \infty$. I.e. $\{\binom{*}{m}\}$ is an orthogonal basis of $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$. ■

Lemma 2.56. *For* $m \ge \nu$ *with* $\nu \ge 1$ *holds*

$$v_\nu(m) := \max_{0 \le l_1 < \ldots < l_\nu \le m} v(l_1) + \cdots + v(l_\nu) = \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu).$$

Proof. Let $L \subset \{1, \ldots, m\}$ with $\#L = \nu$ and $b := \mathrm{l}(m)$, the maximal exponent e such that $p^e \le m$. Then

$$v(l_1) + \cdots + v(l_\nu) = \#\{l \in L : v(l) \ge 1\} + \cdots + \#\{l \in L : v(l) \ge b\}.$$

Let $a = \mathrm{l}(m/\nu)$ be the maximal exponent e such that $p^a \nu \le m$. Then the subsets $L \subset \{1, \ldots, m\}$ with $\#L = \nu$ for which the right hand side above is maximal are precisely those with

$$\{x \le m : v(x) > a\} \subseteq L \subseteq \{x \le m : v(x) \ge a\}.$$

For such $L = \{l_1 < \ldots < l_\nu\}$, we find

$$v(l_1) + \cdots + v(l_\nu) = a \cdot \nu + \#\{l = 1, \ldots, m : v(l) \ge a+1\} + \cdots + \#\{l = 1, \ldots, m : v(l) \ge b\}.$$

We likewise add together

$$
\begin{aligned}
&\mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu) \\
=&\#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq 1\} + \cdots + \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq b\} \\
=& \ \nu \cdot a + \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq a+1\} \\
&+ \cdots + \#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq b\}.
\end{aligned}
$$

Observe that $\mathrm{l}(m/k) > a = \mathrm{l}(m/\nu)$ implies in particular $k \in \{1, \ldots, \nu\}$. Hence for $c \in \mathbb{Z}_{>a}$, we find $\#\{x = m, m/2, \ldots, m/\nu : \mathrm{l}(x) \geq c\} = \#\{k = 1, \ldots, \nu : \mathrm{l}(m/k) \geq c\}$. To obtain the proposed equality, we are thus reduced to: For any $h \in \mathbb{Z}_{>a}$, it holds

$$
\#\{l = 1, \ldots, m : v(l) \geq h\} = \#\{k = 1, \ldots, m : \mathrm{l}(m/k) \geq h\}.
$$

The left hand side is the number of elements $l \leq m$ divisible by p^h. Since $\mathrm{l}(x) \geq h$ if and only if $x \geq p^h$, the right hand side equals the number of elements $k \leq m$ with $m \geq p^h k$. This is also the number of elements below m divisible by p^h. ∎

Lemma 2.57. *For $m \geq \nu$, we find*

$$
v_r(m) = \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu) + \rho \cdot \begin{cases} \mathrm{l}(m/\nu), & \text{if } q(\nu+1) \leq m, \\ \mathrm{l}(m/\nu) - 1, & \text{otherwise;} \end{cases}
$$

here $q = \max\{x : x = p^h \text{ for some } h \in \mathbb{N} \text{ and } x\nu \leq m\}$.

Proof. For $0 \leq l_1 < \ldots \leq l_\nu \leq m$ with $m \geq \nu$, let $\breve{w} := v(l_1) + \cdots + v(l_\nu)$ and

$w = \breve{w} + \rho \cdot [\mathrm{l}(k_1 - 1) + \cdots + \mathrm{l}(k_\nu - 1) + \mathrm{l}(m - l_\nu)]$ with $k_1 := l_1, k_2 := l_2 - l_1, \ldots, k_\nu := l_\nu - l_{\nu-1}$.

Let $L = \{0 \leq l_1 < \ldots < l_\nu \leq m\}$ be such that $\breve{w} = v_\nu(m)$ is maximal. Let $q = p^a$ be the maximal p-power such that $q\nu \leq m$. Then $\{x \leq m : v(x) > a\} \subseteq L \subseteq \{x \leq m : v(x) \geq a\}$. If and only if $q\nu \leq m - q$, we can find an index $n \in \{1, \ldots, \nu\}$ with $k_n > q$. In this case, we can assume $l_\nu = q\nu$ and $m - l_\nu \geq q$. Therefore

$$
\tilde{a} := \mathrm{l}(k_1 - 1) \vee \ldots \vee \mathrm{l}(k_\nu - 1) \vee \mathrm{l}(m - l_\nu) = \begin{cases} a, & \text{if } q(\nu+1) \leq m, \\ a - 1, & \text{otherwise.} \end{cases}
$$

We prove that if $l_1, \ldots, l_\nu$ are such that $w(l_1, \ldots, l_\nu)$ is maximal for all possible $\{0 \leq l_1 < \ldots < l_\nu \leq m\}$, so will be $\breve{w}(l_1, \ldots, l_\nu)$. As $a = \mathrm{l}(m/\nu)$ this will by the above consideration prove the proposition.

Let $0 \leq l_1 < \ldots < l_\nu \leq m$. As $\rho < 1$, it suffices to prove that $\mathrm{l}(k_n - 1) = \tilde{a} + c$ for some $n \in \{1, \ldots, \nu\}$ or $\mathrm{l}(m - l_\nu) = \tilde{a} + c$ for some $c > 0$ implies $\breve{w}(l_1, \ldots, l_\nu) + c \leq v_\nu(m)$. Let us define $k_{\nu+1} := m - l_\nu$, so that $k_1 + \cdots + k_{\nu+1} = m$. Since $\nu \cdot q \leq m < \nu \cdot qp$, there must by the pigeonhole principle exist $s := p^c - p + 1$ indices $n_1, \ldots, n_s$ with $k_{n_1}, \ldots, k_{n_s} < q$. Hence $v(k_{n_1}), \ldots, v(k_{n_s}) < a$. For $n = 1, \ldots, \nu$, if $v(k_n) < a$, either $v(l_n) < a$ or $v(l_{n+1}) = v(l_n + k_n) < a$. Hence there must exist $\lceil s/2 \rceil := \min\{i \in \mathbb{N} : i \geq s/2\} \geq c$ elements $l_n \in \{l_1, \ldots, l_\nu\}$ with $v(l_n) < a$. But if $\breve{w}(L) = v_\nu(m)$ is maximal, then $\{x \leq m : v(x) > a\} \subseteq L \subseteq \{x \leq m : v(x) \geq a\}$. Therefore $\breve{w}(l_1, \ldots, l_\nu) \leq v_\nu(m) - c$. ∎

Lemma 2.58. *There exist positive constants $c \leq 1 \leq C$ with $c \cdot m^r \leq p^{w_r(m)} \leq C \cdot m^r$.*

Proof. By Lemma 2.57, up to a possible deduction of the constant $\rho > 0$ holds

$$v_r(m) = \mathrm{l}(m) + \mathrm{l}(m/2) + \cdots + \mathrm{l}(m/\nu) + \rho\,\mathrm{l}(m/\nu).$$

As $\mathrm{l}(xy) \leq \mathrm{l}(x) + \mathrm{l}(y) + 1$ implies $\mathrm{l}(x) - \mathrm{l}(y) - 1 \leq \mathrm{l}(x/y)$, we find accordingly

$$r \cdot \mathrm{l}(m) - \tilde{c} \leq v_r(m) \leq r \cdot \mathrm{l}(m) \quad \text{with } \tilde{c} := \mathrm{l}(2) + \cdots + \mathrm{l}(\nu) + \rho\,\mathrm{l}(\nu) + r.$$

Since $p^{\mathrm{l}(m)} \leq m \leq p^{\mathrm{l}(m)+1}$, we find $c \cdot m^r \leq p^{v_r(m)} \leq m^r$ with $c := 1/p^{(\tilde{c}+\rho)\cdot r} > 0$. Recall that v_r differs from w_r only in the finitely many nonzero values $w_r(m)$ for $m = 0, \ldots, \nu - 1$. Hence we can decrease $c > 0$ and increase $C := 1$ such that this inequality holds for $w_r(m)$ instead of $v_r(m)$. ■

Corollary 2.59. *Let $f : \mathbb{Z}_p \to \mathbf{K}$ and $a_0, a_1, \ldots$ its Mahler coefficients.*

(i) We have $f \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ if and only if $|a_m| m^r \to 0$ as $m \to \infty$,

(ii) The norm $\|\cdot\|_{\mathcal{C}^r}$ is equivalent to the one given by $|a_0| \vee \max_{m \geq 1} |a_m| m^r \in \mathbb{R}_{\geq 0}$.

Proof. By Corollary 2.54, it suffices to see that there exist positive constants $c \leq 1 \leq C$ with $c \cdot m^r \leq p^{w_r(m)} \leq C \cdot m^r$ and Lemma 2.58 yields the existence of these. ■

3 Fractional differentiability in many variables

Assumption. Throughout this section, we will fix a real number $r = \nu + \rho \in \mathbb{R}_{\geq 0}$ with $\nu = \lfloor r \rfloor \in \mathbb{N}$ and $\rho = \{r\} \in [0, 1[$.

Notation. For a multi-index $\boldsymbol{n} \in \mathbb{N}^d$, we put $|\boldsymbol{n}| = n_1 + \ldots + n_d$. We define for $\nu \in \mathbb{N}$ finite sets of multi-indices

$$\mathbb{N}^d_{=\nu} = \{\boldsymbol{n} \in \mathbb{N}^d : |\boldsymbol{n}| = \nu\}$$

and accordingly $\mathbb{N}^d_{<\nu}$ or $\mathbb{N}^d_{\leq\nu}$ by replacing $=$ with $\leq$ or $<$. For multi-indices $\boldsymbol{i}, \boldsymbol{j} \in \mathbb{N}^d$, we define their natural partial ordering by

$$\boldsymbol{i} \leq \boldsymbol{j} \quad \text{if} \quad i_1 \leq j_1, \ldots, i_d \leq j_d.$$

We denote by $[\boldsymbol{i}, \boldsymbol{j}] \subseteq \mathbb{N}^d$ the finite block

$$[\boldsymbol{i}, \boldsymbol{j}] = \{\boldsymbol{k} \in \mathbb{N}^d \text{ s.t. } \boldsymbol{i} \leq \boldsymbol{k} \leq \boldsymbol{j}\}.$$

During a chain of (in)equalities, placeholders such as dots or hyphenations for a function's arguments will throughout replace the same omitted variables.

3.1 $\mathcal{C}^r$-functions for $r \in \mathbb{R}_{\geq 0}$

Definition of $\mathcal{C}^r$-functions

Let $\boldsymbol{X} \subseteq \mathbf{K}^d$ be a subset. We recall that $\boldsymbol{X}$ is called *cartesian* if $\boldsymbol{X} = X_1 \times \cdots \times X_d$ with $X_1, \ldots, X_d \subset \mathbf{K}$. We also recall that for a subset $X \subseteq \mathbf{K}$ and $n \in \mathbb{N}$, we defined

$$X^{[n]} = X^{\{0,\ldots,n\}} \quad \text{and} \quad X^{]n[} = \triangledown X^{[n]} = \{(x_0, \ldots, x_n) : x_i = x_j \text{ only if } i = j\}.$$

Assumption. We will from now on let $\mathbf{E}$ denote a $\mathbf{K}$-Banach space.

Definition. Let $X \subseteq \mathbf{K}^d$ be a cartesian subset and $f : X \to \mathbf{E}$ a mapping thereon. Let $\boldsymbol{n} \in \mathbb{N}^d$. Put

$$X^{]\boldsymbol{n}[} := X_1^{]n_1[} \times \cdots \times X_d^{]n_d[} \quad \text{and} \quad X^{[\boldsymbol{n}]} := X_1^{[n_1]} \times \cdots \times X_d^{[n_d]}.$$

Write elements $x \in X^{[\boldsymbol{n}]}$ as $x = ({}^1x; -; {}^dx)$ with ${}^1x \in X_1^{[n_1]}, \ldots, {}^dx \in X_d^{[n_d]}$. Through recursion on $n = |\boldsymbol{n}|$ we define functions $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ by

$$f^{]\mathbf{0}[} = f,$$

and if $\boldsymbol{n}^+ = \boldsymbol{n} + \boldsymbol{e}_k$ for $k \in \{1, \ldots, d\}$, then

$$\begin{aligned}&f^{]\boldsymbol{n}^+[}(-; {}^kx_0, {}^kx_1, {}^kx_2, \ldots, {}^kx_{n_k+1}; -)\\ =&\frac{f^{]\boldsymbol{n}[}(-; {}^kx_0, {}^kx_2, \ldots, {}^kx_{n_k+1}; -) - f^{]\boldsymbol{n}[}(-; {}^kx_1, {}^kx_2, \ldots, {}^kx_{n_k+1}; -)}{{}^kx_0 - {}^kx_1};\end{aligned}$$

here the hyphenations to the left and right of the semicolons representing the same omitted arguments ${}^1x; -; {}^{k-1}x$ and ${}^{k+1}x; -; {}^dx$.

We remark that this definition does not depend on the order of summation of $\boldsymbol{n} = n_1\boldsymbol{e}_1 + \cdots + n_d\boldsymbol{e}_d \in \mathbb{N}^d$ by $\mathbf{K} \subseteq \mathbf{E}$ being central.

Example. For notational convenience, we consider the case of two variables and a function $f : X \times Y \to \mathbf{E}$ for $X, Y \subseteq \mathbf{K}$.

(i) We have $X^{]1,0[} = \{(x, x'; y) : x, x' \in X, y \in Y \text{ with } x \neq x'\}$ and

$$f^{]1,0[}(x, x'; y) = \frac{f(x, y) - f(x', y)}{x - x'}.$$

In other words the $f^{]1,0[}$, $f^{]0,1[}$ are the first partial difference quotients of f.

(ii) We have $X^{]1,1[} = \{(x + u, x; y + v, y) : x + u, x \in X, y + v, y \in Y \text{ with } u, v \neq 0\}$ and

$$f^{]1,1[}(x + u, x; y + v, y) = \frac{[f(x + u, y + v) - f(x, y + v)] - [f(x + u, y) - f(x, y)]}{u \cdot v}.$$

In other words $f^{]1,1[}$ is the first mixed partial difference quotient of f.

Definition. Let $V = V_1 \times \cdots \times V_d$ be a topological space. Then we will call a subset $X \subseteq V$ **locally cartesian** if every point $x \in X$ has a cartesian neighborhood with respect to the relative topology in X.

Definition 3.1. (i) Let $X \subseteq \mathbf{K}^d$ be a cartesian subset; we will say that a mapping $f : X \to \mathbf{E}$ is $\mathcal{C}^r$ or r**-times continuously differentiable** at some point $a \in X$ if $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ is $\mathcal{C}^\rho$ at $\vec{a} := (\vec{a}_1; -; \vec{a}_d) \in X^{[\boldsymbol{n}]}$ for all $\boldsymbol{n} \in \mathbb{N}^d$ with $n_1 + \cdots + n_d = \nu$.

(ii) Let $X \subseteq \mathbf{K}^d$ be locally cartesian and $f : X \to \mathbf{E}$ a map thereon. We will say that f is $\mathcal{C}^r$ at a if $f_{|U}$ is $\mathcal{C}^r$ at a for some cartesian neighborhood $U \subseteq X$.

(iii) Let $X \subseteq \mathbf{K}^d$ be a locally cartesian subset. Then f will be a $\mathcal{C}^r$**-function** or an r**-times continuously differentiable function** if f is $\mathcal{C}^r$ at all points $a \in X$. The set of all $\mathcal{C}^r$-functions $f : X \to \mathbf{E}$ is denoted by $\mathcal{C}^r(X, \mathbf{E})$.

We will make the following terminological convention: Let $X_1, \ldots, X_d \subseteq \mathbf{K}$ be subsets and put $X = X_1 \times \cdots \times X_d$. Then by definition

$$X^{[\boldsymbol{n}]} = X_1^{\{0,\ldots,n_1\}} \times \cdots \times X_d^{\{0,\ldots,n_d\}}.$$

Let $x = (x_{k,i_k}) \in X^{[\boldsymbol{n}]}$ with $k = 1, \ldots, d$ and $i_k = 0, \ldots, n_k$. Then we will call $x_{k,i}$ a *coordinate of* x *valued in* X_k or for short an X_k*-coordinate*.
We will say that a function $f : X^{[\boldsymbol{n}]} \to \mathbf{E}$ is *symmetric in its* X_k*-coordinates* if f is symmetric in its coordinates indexed by $\{(k, 0), \ldots, (k, n_k)\}$.

Lemma 3.2. *Let* $X \subseteq \mathbf{K}^d$ *be cartesian. Then a mapping* $f : X \to \mathbf{E}$ *is* $\mathcal{C}^r$ *at a point* $a \in X$ *if and only if for every* $\varepsilon > 0$, $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ *and* $k = 1, \ldots, d$, *there exists a cartesian neighborhood* $U \ni a$ *in* X *such that*

$$\|f^{]\boldsymbol{n}[}(-; {}^k x_0, {}^k x_1, \ldots, {}^k x_{n_k}; -) - f^{]\boldsymbol{n}[}(-; {}^k \tilde{x}_0, {}^k x_1, \ldots, {}^k x_{n_k}; -)\| \leq \varepsilon \cdot |{}^k x_0 - {}^k \tilde{x}_0|^\rho \quad \text{on } U^{]\boldsymbol{n}[}; \quad (*)$$

here the hyphenations to the left and right of the semicolons representing the same omitted arguments ${}^1x; -; {}^{k-1}x$ *and* ${}^{k+1}x; -; {}^d x$.

Proof. By Example 1.15(ii), the set $A := X^{]\boldsymbol{n}[} \subseteq X^{[\boldsymbol{n}]}$ is telescopic. For $k = 1, \ldots, d$, denote by

$$I_k = \{(k,0), \ldots, (k,n_k)\} = \{\ X_k\text{-coordinate indices of } X^{[\boldsymbol{n}]}\ \}.$$

Let us define ${}^k\boldsymbol{e}_0 := (\boldsymbol{0}; \ldots; \boldsymbol{e}_0; \ldots; \boldsymbol{0}) \in \mathbf{K}^{[n_1]} \times \cdots \times \mathbf{K}^{[n_d]} = \mathbf{K}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0 = (1, 0, \ldots)$ at the k-th place. Then the only nonzero entry of ${}^k\boldsymbol{e}_0$ is at the i_k-th coordinate for the representative $i_k = (k, 0)$ of I_k. We find $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ and $\vec{a} \in X^{[\boldsymbol{n}]}$ to be both symmetric in their X_k-coordinates for $k = 1, \ldots, d$, i.e. those indexed by $I_1, \ldots, I_d$. By Corollary 1.20 applied to the telescopic subset $A \subseteq X^{[\boldsymbol{n}]}$, the function $f^{]\boldsymbol{n}[}$ is $\mathcal{C}^\rho$ at $\vec{a}$ if and only if, given $\varepsilon > 0$, there exists a neighborhood $U \ni \vec{a}$ in $X^{[\boldsymbol{n}]}$ such that

$$\|f^{]\boldsymbol{n}[}(x) - f^{]\boldsymbol{n}[}(y)\| \leq \varepsilon|t|^\rho \text{ for all } x, y \in U \cap X^{]\boldsymbol{n}[} \qquad (**)$$

with $y = x + t \cdot {}^k\boldsymbol{e}_0$ for $t \in \mathbf{K}$ and $k = 1, \ldots, d$.

The family

$$\{U^{[\boldsymbol{n}]} \subseteq X^{[\boldsymbol{n}]} : U \text{ some cartesian neighborhood of } a \text{ in } X\}$$

forms a basis of neighborhoods around $\vec{a} \in X^{[\boldsymbol{n}]}$. We conclude that f is $\mathcal{C}^r$ at a, if and only if Inequality $(**)$ above holds for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $k = 1, \ldots, d$, if and only if Condition $(*)$ holds. ■

Remark 3.3. (i) By definition, being $\mathcal{C}^r$ is a local property. In the following we will therefore formulate local results on $\mathcal{C}^r$-functions solely for cartesian subsets $X \subseteq \mathbf{K}^d$.

(ii) We observe that the differentiability at some point a may vanish if the function's domain expands in $\mathbf{K}^d$ - as long as there is no neighborhood $U \ni a$ in $\mathbf{K}^d$ lying in the domain.

(iii) If $\mathbf{E} = \mathbf{E}_1 \times \cdots \times \mathbf{E}_d$ with $\mathbf{K}$-Banach spaces $\mathbf{E}_k$ for $k = 1, \ldots, d$, then $f : X \to \mathbf{E}$ will be $\mathcal{C}^\rho$ at $a \in X$ if and only if $f_k := \mathrm{p}_k \circ f : X \to \mathbf{E}_k$ will be $\mathcal{C}^\rho$ at a for $k = 1, \ldots, d$. Hence $\mathcal{C}^r(X, \mathbf{E}) = \mathcal{C}^r(X, \mathbf{E}_1) \times \cdots \times \mathcal{C}^r(X, \mathbf{E}_d)$.

(iv) Let $f : X \to \mathbf{K}$ some mapping, $\boldsymbol{n} \in \mathbb{N}^d$ and $a_1, \ldots, a_d$ accumulation points in $X_1, \ldots, X_d$. Then $\vec{a}$ is an accumulation point of $X^{]\boldsymbol{n}[}$. As $\mathbf{E}$ is complete, we find by Remark 1.4 that $f^{]\boldsymbol{n}[}$ is $\mathcal{C}^0$ at $\vec{a}$ if and only if there exists a limit $D_{\boldsymbol{n}} f(a) \in \mathbf{E}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\|f^{]\boldsymbol{n}[}(x) - D_{\boldsymbol{n}} f(a)\| \leq \varepsilon \quad \text{for all } x \in X^{]\boldsymbol{n}[} \text{ with } \|x - \vec{a}\| \leq \delta.$$

Lemma 3.4. *Let $X \subseteq \mathbf{K}^d$ be a cartesian subset.*

(i) The mapping $\mathbf{E}^X \to \mathbf{E}^{X^{]\boldsymbol{n}[}}$ given by $f \mapsto f^{]\boldsymbol{n}[}$ is $\mathbf{K}$-linear.

(ii) Let $f : X \to \mathbf{E}$ be a mapping on X. Then the mapping $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ is symmetric in its X_k-coordinates for each $k = 1, \ldots, d$.

(iii) Let $f, g : X \to \mathbf{E}$ be two mappings on X. Then for all $({}^1x; -; {}^dx) \in X^{]\boldsymbol{n}[}$, we find

$$(f \cdot g)^{]\boldsymbol{n}[}({}^1x; -; {}^dx) = \sum_{\boldsymbol{j} \in [\boldsymbol{0}, \boldsymbol{n}]} f^{]\boldsymbol{j}[}({}^1x_0, \ldots, {}^1x_{j_1}; -; {}^dx_0, \ldots, {}^dx_{j_d}) \cdot g^{]\boldsymbol{n}-\boldsymbol{j}[}({}^1x_{j_1}, \ldots, {}^1x_{n_1}; -; {}^dx_{j_d}, \ldots, {}^dx_{n_d}).$$

Proof. Ad (i): This is quickly checked to hold.
Ad (ii): By [Schikhof, 1984, Lemma 29.2(ii)] applied to $f^{]\boldsymbol{n}[}({}^1x; -; {}^{k-1}x; \_; {}^{k+1}x; -; {}^dx) : X_k^{]n_k[} \to \mathbf{K}$ for fixed arguments ${}^lx \in X_l^{]n_l[}$ with $l = 1, \ldots, d$ distinct from k.
Ad (iii): In case $d = 1$, this is proven in [Schikhof, 1984, Lemma 29.2(v)]. In the general case $d > 1$, we restrict for notational convenience to $d = 2$ and a function $f : X \times Y \to \mathbf{E}$ with $X, Y \subseteq \mathbf{K}$. Then for $d > 2$ the result follows by induction through the argument below. Fixing $y \in Y^{]m[}$, we find by the case $d = 1$ that

$$(f \cdot g)^{]n,m[}(x; y) = \Big(\sum_{i=0,\ldots,n} f^{]i[}(x_0, \ldots, x_i; y) g^{]n-i[}(x_i, \ldots, x_n; y)\Big)^{]m[} \quad \text{for } x \in X^{]n[}.$$

For $x \in X^{]n[}, y \in Y^{]m[}$, let us denote

$$f^{]i[}(x_0, \ldots, x_i; y) = f_{x_0,\ldots,x_i}(y) \quad \text{and} \quad g_{x_i,\ldots,x_n}(y) = g^{]n-i[}(x_i, \ldots, x_n; y).$$

Then

$$\begin{aligned}
(f \cdot g)^{]n,m[}(x; y) &= \Big(\sum_{i=0,\ldots,n} f_{x_0,\ldots,x_i}(y) g_{x_i,\ldots,x_n}(y)\Big)^{]m[} \\
&= \sum_{i=0,\ldots,n} \big(f_{x_0,\ldots,x_i}(y) g_{x_i,\ldots,x_n}(y)\big)^{]m[} \\
&= \sum_{i=0,\ldots,n} \sum_{j=0,\ldots,m} f_{x_0,\ldots,x_i}(y_0, \ldots, y_j)^{]j[} g_{x_i,\ldots,x_n}(y_j, \ldots, y_m)^{]m-j[} \\
&= \sum_{(i,j) \in [\mathbf{0}, (n,m)]} f^{]i,j[}(x_0, \ldots, x_i; y_0, \ldots, y_j) g^{]n-i,m-j[}(x_i, \ldots, x_n; y_j, \ldots, y_n).
\end{aligned}$$

■

Properties of $\mathcal{C}^r$-functions

We will make the following terminological conventions:

- Let $X = X_1 \times \cdots \times X_d \subseteq \mathbf{K}^d$ be a cartesian subset. Then we say that *its factors contain no isolated points* if each subset $X_1, \ldots, X_d \subseteq \mathbf{K}$ contains no isolated points.
- Let $X = X_1 \times \cdots \times X_d \subseteq \mathbf{K}^d$ be a locally cartesian subset. Then we say that *its local factors contain no isolated points* if for every cartesian neighborhood $U = U_1 \times \cdots \times U_d$ in X, each subset $U_1, \ldots, U_d \subseteq \mathbf{K}$ contains no isolated points.

Lemma 3.5. *Let $X \subseteq \mathbf{K}^d$ be a cartesian subset and $f : X \to \mathbf{E}$ a mapping thereon. Let $a \in X$. If f is $\mathcal{C}^r$ at a, then f will be $\mathcal{C}^s$ at a for every $s \leq r$.*

Proof. If f is $\mathcal{C}^r$ at a, then clearly f will be $\mathcal{C}^s$ at a for every $\nu \leq s \leq r$. By transitivity, it therefore suffices to prove that f is $\mathcal{C}^s$ at a with $s = \nu - 1 + \eta$ for $\eta \in [0,1[$. We use the characterization of Lemma 3.2: Let $\boldsymbol{m} \in \mathbb{N}^d$ and $m = \nu - 1$. Then for $x \in X^{]\boldsymbol{m}[}$ and ${}^k\tilde{x}_0 \in X_k$ distinct from ${}^kx_0, \ldots, {}^kx_{m_k}$, we find

$$\begin{aligned}
&\|f^{]\boldsymbol{m}[}(-; {}^kx_0, {}^kx_1, \ldots, {}^kx_{m_k}; -) - f^{]\boldsymbol{m}[}(-; {}^k\tilde{x}_0, {}^kx_1, \ldots, {}^kx_{m_k}; -)\| \\
=&|{}^kx_0 - {}^k\tilde{x}_0| \|f^{]\boldsymbol{m}+\boldsymbol{e}_k[}(-; {}^kx_0, {}^k\tilde{x}_0, {}^kx_1, \ldots, {}^kx_{m_k}; -)\| \\
=&|{}^kx_0 - {}^k\tilde{x}_0|^\eta (|{}^kx_0 - {}^k\tilde{x}_0|^{1-\eta} \|f^{]\boldsymbol{m}+\boldsymbol{e}_k[}(-; {}^kx_0, {}^k\tilde{x}_0, {}^kx_1, \ldots, {}^kx_{m_k}; -)\|).
\end{aligned}$$

Let $\boldsymbol{m} + \boldsymbol{e}_k =: \boldsymbol{n}$ and observe $|\boldsymbol{n}| = \nu$. By assumption $f^{]\boldsymbol{n}[}$ is $\mathcal{C}^\rho$, hence $\mathcal{C}^0$ at $\vec{a}$ and in particular locally bounded by a constant $C > 0$ there. Given $\varepsilon > 0$, we find a cartesian neighborhood $U \ni a$ in X such that $\|f^{]\boldsymbol{n}[}\| \leq C$ on $U^{]\boldsymbol{n}[}$ and $\|x - \tilde{x}\|^{1-\eta} \leq \varepsilon / C$ for all $x, \tilde{x} \in U$. Hence on $U^{]\boldsymbol{m}[}$ holds by the above exposed equality

$$\|f^{]\boldsymbol{m}[}(-; {}^kx_0, {}^kx_1, \ldots, {}^kx_{m_k}; -) - f^{]\boldsymbol{m}[}(-; {}^k\tilde{x}_0, {}^kx_1, \ldots, {}^kx_{m_k}; -)\| \leq |{}^kx_0 - {}^k\tilde{x}_0|^\eta \varepsilon.$$

By Lemma 3.2, this proves f to be $\mathcal{C}^s$ at a. ∎

Lemma 3.6. *Let $X \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point and $f : X \to \mathbf{E}$ a mapping thereon. Assume that $f^{]\boldsymbol{m}[} : X^{]\boldsymbol{m}[} \to \mathbf{E}$ is $\mathcal{C}^\rho$ on all of $X^{[\boldsymbol{m}]}$ for $\boldsymbol{m} \in \mathbb{N}^d$. Then if $\boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_k$ for $k \in \{1, \ldots, d\}$, the mapping $f^{]\boldsymbol{n}[}$ can be extended to a $\mathcal{C}^\rho$-function $f^{<\boldsymbol{n}>_k} : X^{<\boldsymbol{n}>_k} \to \mathbf{E}$; here $X^{<\boldsymbol{n}>_k}$ is defined by*

$$X^{<\boldsymbol{n}>_k} := X_1^{[n_1]} \times \cdots \times X_{k-1}^{[n_{k-1}]} \times (X_k^{[n_k]} - \triangle X_k^{[n_k]}) \times X_{k+1}^{[n_{k+1}]} \times \cdots \times X_d^{[n_d]}.$$

Proof. For $i, j \in \{0, \ldots, n_k\}$ with $i \neq j$ put

$$U_{ij} = \{x \in X^{[\boldsymbol{n}]} : {}^kx_i \neq {}^kx_j\} \subseteq X^{[\boldsymbol{n}]}.$$

Then each U_{ij} is open in $X^{[\boldsymbol{n}]}$ and their union is $X^{<\boldsymbol{n}>_k}$. By our assumption on $f^{]\boldsymbol{m}[}$, we find by Proposition 1.6 that $f^{]\boldsymbol{m}[}$ extends to a $\mathcal{C}^\rho$-function $f^{[\boldsymbol{m}]}$ on all of $X^{[\boldsymbol{m}]}$. We can hence define $h_{ij} : U_{ij} \to \mathbf{E}$ by

$$\begin{aligned}
&h_{ij}(-; {}^kx_0, \ldots, {}^kx_{n_k}; -) \\
=&\frac{f^{[\boldsymbol{m}]}(-; {}^kx_0, \ldots, \breve{{}^kx_j}, \ldots, {}^kx_{n_k}; -) - f^{[\boldsymbol{m}]}(-; {}^kx_0, \ldots, \breve{{}^kx_i}, \ldots, {}^kx_{n_k}; -)}{{}^kx_i - {}^kx_j};
\end{aligned}$$

here the arguments beneath the breves being omitted. By Lemma 3.4(ii), the functions $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ and $f^{]\boldsymbol{m}[} : X^{]\boldsymbol{m}[} \to \mathbf{E}$ are symmetric in its X_k-coordinates, and we find by Equality (2.1) in Lemma 2.4 that $f^{]\boldsymbol{n}[}(x) = h_{ij}(x)$; hence each h_{ij} extending $f^{]\boldsymbol{n}[}$ onto U_{ij}.

Since $x \mapsto 1/({}^kx_i - {}^kx_j)$ is $\mathcal{C}^\rho$ on U_{ij} and $f^{[\boldsymbol{m}]}$ is $\mathcal{C}^\rho$ on $X^{[\boldsymbol{m}]}$, the function h_{ij} is by Proposition 1.7(ii) $\mathcal{C}^\rho$ on U_{ij}. We glue these functions h_{ij} for distinct $i, j \in \{1, \ldots, n_k\}$ together by defining $f^{<\boldsymbol{n}>_k} : X^{<\boldsymbol{n}>_k} \to \mathbf{E}$ through

$$f^{<\boldsymbol{n}>_k}(x) = h_{ij}(x) \quad \text{if } x \in U_{ij}.$$

As all the continuous h_{ij} coincide on the common dense subset $X^{]\boldsymbol{n}[}$ of their domains, this assignment is well-defined. Because each h_{ij} was seen to be $\mathcal{C}^\rho$ on its open domain $U_{ij} \subseteq X^{<\boldsymbol{n}>_k}$, and since this is a local property, we conclude $f^{<\boldsymbol{n}>_k}$ to be $\mathcal{C}^\rho$. ■

Lemma 3.7. *Let $X \subseteq \mathbf{K}^d$ be a cartesian, nonempty subset whose factors contain no isolated point and $f : X \to \mathbf{E}$ a mapping thereon. Assume that for all $\boldsymbol{m} \in \mathbb{N}^d_{=\nu-1}$ with $\nu \geq 1$, the mapping $f^{]\boldsymbol{m}[} : X^{]\boldsymbol{m}[} \to \mathbf{E}$ is $\mathcal{C}^\rho$ on all of $X^{[\boldsymbol{m}]}$. Then for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$, the mapping $f^{]\boldsymbol{n}[}$ can be extended to a $\mathcal{C}^\rho$-function $f^{<\boldsymbol{n}>} : X^{<\boldsymbol{n}>} \to \mathbf{E}$; here $X^{<\boldsymbol{n}>}$ is defined by*

$$X^{<\boldsymbol{n}>} := X^{[\boldsymbol{n}]} - \triangle X_1^{[n_1]} \times \cdots \times \triangle X_d^{[n_d]}$$

Proof. Fix $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. For every coordinate $k = 1, \ldots, d$ with $n_k \geq 1$, by Lemma 3.6 applied to $\boldsymbol{m}_k = \boldsymbol{n} - \boldsymbol{e}_k$, the function $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a $\mathcal{C}^\rho$-function $f^{<\boldsymbol{n}>_k} : X^{<\boldsymbol{n}>_k} \to \mathbf{E}$. We notice

$$\begin{aligned}
&\bigcup_{k=1,\ldots,d \text{ with } n_k \geq 1} X^{<\boldsymbol{n}>_k} \\
=&\bigcup_{k=1,\ldots,d \text{ with } n_k \geq 1} X_1^{[n_1]} \times \cdots \times X_{k-1}^{[n_{k-1}]} \times (X_k^{[n_k]} - \triangle X_k^{[n_k]}) \times X_{k+1}^{[n_{k+1}]} \times \cdots \times X_d^{[n_d]} \\
=&\{x \in X^{[\boldsymbol{n}]} : \quad \exists k = 1, \ldots, d \text{ with } {}^k x_i \neq {}^k x_j \text{ for some } i, j \in \{1, \ldots, n_k\}\} \\
=&\{x \in X^{[\boldsymbol{n}]} : \neg \forall k = 1, \ldots, d \text{ holds } {}^k x_1 = \ldots = {}^k x_{n_k}\} \\
=&X^{[\boldsymbol{n}]} - \triangle X_1^{[n_1]} \times \cdots \times \triangle X_d^{[n_d]} = X^{<\boldsymbol{n}>}.
\end{aligned}$$

We can therefore extend $f^{]\boldsymbol{n}[}$ to a function $f^{<\boldsymbol{n}>} : X^{<\boldsymbol{n}>} \to \mathbf{E}$ by

$$f^{<\boldsymbol{n}>}(x) = f^{<\boldsymbol{n}>_k}(x) \quad \text{if } x \in X^{<\boldsymbol{n}>_k}.$$

As all the continuous $f^{<\boldsymbol{n}>_k}$ coincide on the common dense subset $X^{]\boldsymbol{n}[}$ of their domains, this assignment is well-defined. Since all $f^{<\boldsymbol{n}>_k}$ were seen to be $\mathcal{C}^\rho$ and $X^{<\boldsymbol{n}>_k} \subseteq X^{<\boldsymbol{n}>}$ is open, we conclude $f^{<\boldsymbol{n}>}$ to be $\mathcal{C}^\rho$, as this is a local property. ■

Proposition 3.8. *Let $X \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point and $f : X \to \mathbf{E}$ a mapping thereon. Then $f \in \mathcal{C}^r(X, \mathbf{E})$ if and only if for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$, the function $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a unique $\mathcal{C}^\rho$-function $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{E}$.*

Proof. Firstly, let $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $a \in X$. If $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a $\mathcal{C}^\rho$-function $f : X^{[\boldsymbol{n}]} \to \mathbf{E}$, then its restriction $f^{]\boldsymbol{n}[}$ onto $X^{]\boldsymbol{n}[}$ will be in particular $\mathcal{C}^\rho$ at all points $\vec{a} \in X^{[\boldsymbol{n}]}$. Therefore $f^{]\boldsymbol{n}[}$ is $\mathcal{C}^\rho$ at all points $\vec{a}$ for $a \in X$, i.e. $f \in \mathcal{C}^r(X, \mathbf{E})$.
Contrariwise, let $f \in \mathcal{C}^r(X, \mathbf{E})$. If $\nu = 0$, then by definition f will be a $\mathcal{C}^\rho$-function on all of X. Let $\nu \geq 1$ and assume by induction on $\nu \geq 0$ that, as soon as f is $\mathcal{C}^{r-1}$, the function $f^{]\boldsymbol{m}[}$ extends to a $\mathcal{C}^\rho$-function on all of $X^{[\boldsymbol{m}]}$ for all $\boldsymbol{m} \in \mathbb{N}^d_{=\nu-1}$.
By Lemma 3.5, the mapping f is $\mathcal{C}^{r-1}$. Therefore $f^{]\boldsymbol{m}[}$ extends to a $\mathcal{C}^\rho$-function on all of $X^{[\boldsymbol{m}]}$ for all $\boldsymbol{m} \in \mathbb{N}^d_{=\nu-1}$. Let $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. By Lemma 3.7, the mapping $f^{]\boldsymbol{n}[}$ extends to a $\mathcal{C}^\rho$-function $f^{<\boldsymbol{n}>}$ on all of $X^{<\boldsymbol{n}>}$. Since f is $\mathcal{C}^r$, the mapping $f^{]\boldsymbol{n}[}$ is $\mathcal{C}^\rho$ at all $\vec{a}$ for $a \in X$. We extend $f^{]\boldsymbol{n}[}$ to $X^{[\boldsymbol{n}]}$ by

$$f^{[\boldsymbol{n}]}(x) = \begin{cases} f^{<\boldsymbol{n}>}(x), & \text{if } x \in X^{<\boldsymbol{n}>}, \\ \lim_{y \to \vec{a}} f^{]\boldsymbol{n}[}(y), & \text{if } x = \vec{a} \quad \text{for } a \in X; \end{cases}$$

here y running through $X^{]\boldsymbol{n}[}$. Thus if we let $A := X^{]\boldsymbol{n}[}$ and $A \subseteq B := X^{[\boldsymbol{n}]} \subseteq \overline{A}$, then in particular $f^{]\boldsymbol{n}[} : A \to \mathbf{K}$ will be a function which is $\mathcal{C}^\rho$ on the whole of B and hence its unique continuous extension $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{E}$ is a $\mathcal{C}^\rho$-function by Proposition 1.6. ∎

The locally convex $\mathbf{K}$-algebra of $\mathcal{C}^r$-functions

Assumption. In this subsection's paragraph on the locally convex $\mathbf{K}$-algebra of $\mathcal{C}^r$-functions, we will by $X \subseteq \mathbf{K}^d$ denote a nonempty locally cartesian subset whose local factors contain no isolated point, if not explicitly mentioned otherwise.

Lemma 3.9. *Each compact cartesian subset $C \subseteq X$ is contained in some nonempty open cartesian subset $P \subseteq X$ whose factors contain no isolated point.*

Proof. We can cover each compact cartesian subset $C \subseteq X$ by finitely many nonempty open cartesian subsets $U_1, \ldots, U_n \subseteq X$ whose factors contain no isolated point. Let $W = U_1 \cup \ldots \cup U_n$. Let $k \in \{1, \ldots, d\}$ and define $P_k \subseteq \mathbf{K}$ by

$$P_k = \bigcap_{\breve{x} \in \in \mathbf{K}^{\{1,\ldots,\breve{k},\ldots,d\}}} W_{\breve{x}}, \quad \text{where } W_{\breve{x}} := \{x_k \in \mathbf{K} : (x_1, \ldots, x_k, \ldots, x_d) \in W\};$$

here and in the following the breve always denoting the omittance of the element below it. Then $P_k \supseteq C_k$ as $C_k \subseteq W_{\breve{x}}$ for all $\breve{x} \in \mathbf{K}^{\{1,\ldots,\breve{k},\ldots,d\}}$ since $C_1 \times \cdots \times C_d \subseteq W$. Moreover $P_1 \times \cdots \times P_d \subseteq W \subseteq X$ as

$$W_{\breve{x}} \supseteq P_k = \bigcap_{\breve{x} \in \mathbf{K}^{d-1}} W_{\breve{x}} \quad \text{for all } k = 1, \ldots, d \text{ and each } \breve{x} \in \mathbf{K}^{d-1}.$$

We propose that $P_k \subseteq \mathrm{p}_k W$ is open and without isolated points: As $W = U_1 \cup \ldots \cup U_n$, we find for each $\breve{x} \in \mathbf{K}^{\{1,\ldots,\breve{k},\ldots,d\}}$ that

$$W_{\breve{x}} = \bigcup_{U \in \mathfrak{U}} \mathrm{p}_{1,\ldots,\breve{k},\ldots,d} U \quad \text{with } \mathfrak{U} = \{U \in \{U_1, \ldots, U_n\} : \mathrm{p}_{1,\ldots,\breve{k},\ldots,d} U \ni \breve{x}\} \subseteq \{U_1, \ldots, U_n\};$$

here we let $\mathrm{p}_{1,\ldots,\breve{k},\ldots,d} : X_1 \times \cdots \times X_d \to X_1 \times \cdots \times \breve{X}_k \times \cdots \times X_d$. Each $W_{\breve{x}}$ is therefore open in $\mathrm{p}_k W$. Since there are only finitely many families $\mathfrak{U} \subseteq \{U_1, \ldots, U_n\}$, the intersection P_k is finite and hence open in $\mathrm{p}_k W$. Since $\mathrm{p}_k U$ for $U = U_1, \ldots, U_n$ has by assumption no isolated points, neither does the open subset $P_k \subseteq \mathrm{p}_k W$ with

$$\mathrm{p}_k W = \bigcup_{U = U_1,\ldots,U_n} \mathrm{p}_k U.$$

We conclude that $P_k \subseteq \mathrm{p}_k W$ is a nonempty open subset without isolated points. Thus each compact cartesian $C \subseteq X$ is contained in some nonempty open cartesian subset $P := P_1 \times \cdots \times P_d \subseteq X$ whose factors contain no isolated point. ∎

Definition. Let $f \in \mathcal{C}^r(X, \mathbf{E})$ and $C \subseteq U$ compact cartesian for some nonempty cartesian neighborhood U in X with factors free of isolated points, supplied by Lemma 3.9. By Lemma 3.5 and Proposition 3.8, for all $\boldsymbol{n} \in \mathbb{N}^d$ with $|\boldsymbol{n}| < \nu$ respectively $|\boldsymbol{n}| = \nu$, the mapping $f_{|U}^{]\boldsymbol{n}[}$

extends to a continuous respectively $\mathcal{C}^\rho$-function $f^{[\boldsymbol{n}]}_{|U}$. Hence we can define for each compact cartesian $C \subseteq X$ the seminorm $\|\cdot\|_{\mathcal{C}^r,C}$ by

$$\|f\|_{\mathcal{C}^r,C} := \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}|<\nu} \|f^{[\boldsymbol{n}]}_{|U}\|_{C^{[\boldsymbol{n}]}} \vee \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}|=\nu} \|f^{[\boldsymbol{n}]}_{|U}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{n}]}}.$$

We provide $\mathcal{C}^r(X,\mathbf{E})$ with the locally convex topology induced through this family of seminorms $\{\|\cdot\|_{\mathcal{C}^r,C}\}$ with $C \subseteq X$ compact cartesian - the **topology of compact (cartesian) convergence**.

Remark 3.10. If $\mathbf{K}$ is locally compact, then $\mathcal{C}^r(X,\mathbf{E})$ will be the initial locally convex $\mathbf{K}$-vector space with respect to all restriction mappings

$$\mathcal{C}^r(X,\mathbf{E}) \to \mathcal{C}^r(C,\mathbf{E}),$$
$$f \mapsto f_{|C};$$

here C running through the family of all balls $C \subseteq X$.

Proof. By definition, a function $f : X \to \mathbf{E}$ is $\mathcal{C}^r$ if and only if its restrictions onto a basis of open subsets $U \subseteq X$ are $\mathcal{C}^r$. This basis is given by all balls in X. Therefore we have an equality of $\mathbf{K}$-vector spaces.
Since X is locally compact, we can cover each compact subset by a finite number of balls. These are compact, nonempty and as open also free of isolated points. Therefore the locally convex topology given by these seminorms on balls coincides with the one given by the family of seminorms on all compact subsets. ■

Lemma 3.11. *We have for $s \leq r$ a norm-nonincreasing inclusion of locally convex $\mathbf{K}$-vector spaces $\mathcal{C}^r(X,\mathbf{E}) \subseteq \mathcal{C}^s(X,\mathbf{E})$.*

Proof. The inclusion holds by Lemma 3.5. It remains to show that $\|\cdot\|_{\mathcal{C}^s,C} \leq \|\cdot\|_{\mathcal{C}^r,C}$ on $\mathcal{C}^r(X,\mathbf{E})$ for every compact cartesian $C \subseteq X$.
Let $f \in \mathcal{C}^r(X,\mathbf{E}) \subseteq \mathcal{C}^s(X,\mathbf{E})$. Then clearly $\|f\|_{\mathcal{C}^s,C} \leq \|f\|_{\mathcal{C}^r,C}$ for every $\nu \leq s \leq r$. By transitivity, it therefore suffices to prove $\|f\|_{\mathcal{C}^s,C} \leq \|f\|_{\mathcal{C}^r,C}$ with $s = \nu - 1 + \eta$ for $\eta \in [0,1[$.
For this, it suffices to prove

$$\|f^{[\boldsymbol{m}]}\|_{\mathcal{C}^\eta,C^{[\boldsymbol{m}]}} \leq \|f^{[\boldsymbol{m}]}\|_{C^{[\boldsymbol{m}]}} \vee \max_{\boldsymbol{n}=\boldsymbol{m}+\boldsymbol{e}_1,\ldots,\boldsymbol{m}+\boldsymbol{e}_d} \|f^{[\boldsymbol{n}]}\|_{C^{[\boldsymbol{n}]}} \quad \text{for any } \boldsymbol{m} \in \mathbb{N}^d_{=\nu-1}.$$

Let $\bar{C} = C^{[\boldsymbol{m}]}$ and $\bar{F} := f^{[\boldsymbol{m}]}$. Recall $\mathbb{N}^{[\boldsymbol{n}]} = \mathbb{N}^{[n_1]} \times \cdots \times \mathbb{N}^{[n_d]}$ with $\mathbb{N}^{[n]} = \mathbb{N}^{0,\ldots,n}$ for $\boldsymbol{n} \in \mathbb{N}^d$ and ${}^k\boldsymbol{e}_0 := (\mathbf{0};\ldots;\boldsymbol{e}_0;\ldots;\mathbf{0}) \in \mathbb{N}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0 \in \mathbb{N}^{[n_k]}$ at the k-th place. Then we can naturally identify $x = (-;{}^kx_0,{}^kx_1,{}^kx_2,\ldots;-) \in C^{[\boldsymbol{m}+\boldsymbol{e}_k]}$ with $(- : {}^kx_0,{}^kx_1;{}^kx_2;\ldots : -) \in \bar{C}^{[{}^k\boldsymbol{e}_0]}$ and so, if ${}^kx_0 - {}^kx_1 \neq 0$, we have

$$\begin{aligned}
&f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(-;{}^kx_0,{}^kx_1,{}^kx_2,\ldots;-)\\
=&\frac{\bar{F}(-;{}^kx_0,{}^kx_2,\ldots;-) - \bar{F}(-;{}^kx_1,{}^kx_2,\ldots;-)}{{}^kx_0 - {}^kx_1}\\
=&\bar{F}^{[{}^k\boldsymbol{e}_0]}(- : {}^kx_0,{}^kx_1;{}^kx_2;\ldots : -).
\end{aligned}$$

In particular we see that if $f^{[\boldsymbol{m}+e_k]}$ exists, so does $\bar{F}^{[^k e_0]}$, and we deduce

$$\begin{aligned}\|\bar{F}\|_{\mathcal{C}^\eta,\bar{C}} &\leq \|\bar{F}\|_{\mathcal{C}^1,\bar{C}}\\ &= \max_{k=1,\dots,d} \|\bar{F}\|_{\mathcal{C}^{^k e_0},\bar{C}}\\ &= \max_{k=1,\dots,d} (\|\bar{F}\|_{\bar{C}} \vee \|\bar{F}^{[^k e_0]}\|_{\bar{C}^{[^k e_0]}})\\ &= \|f^{[\boldsymbol{m}]}\|_{C^{[\boldsymbol{m}]}} \vee \max_{k=1,\dots,d} \|f^{[\boldsymbol{m}+e_k]}\|_{C^{[\boldsymbol{m}+e_k]}};\end{aligned}$$

here the first inequality holding true by Lemma 1.37. For the following equality, we find by Corollary 1.40 by symmetry of $f^{[\boldsymbol{m}]} : X_1^{[m_1]} \times \dots \times X_d^{[m_d]} \to \mathbf{E}$ on its X_k-coordinates for $k = 1, \dots, d$ moreover $\|\bar{F}\|_{\mathcal{C}^1,\bar{C}} = \|\bar{F}\|_{\mathcal{C}^{^1 e_0},\bar{C}} \vee \dots \vee \|\bar{F}\|_{\mathcal{C}^{^d e_0},\bar{C}}$. ■

Lemma 3.12. *Let $x \in X^{]\boldsymbol{n}[}$ with $X \subseteq \mathbf{K}^d$ a cartesian subset whose factors contain no isolated point. Then there is a constant $C(x) \geq 1$ such that for every $f : X \to \mathbf{E}$ it holds*

$$\|f^{]\boldsymbol{n}[}(x)\| \leq C(x) \max_{\boldsymbol{i}\in[\boldsymbol{0},\boldsymbol{n}]} \|f({}^1x_{i_1}, \dots, {}^dx_{i_d})\|.$$

Proof. By induction and applying [Schikhof, 1984, Exercise 29.A] to the n_d-th difference quotient of the one-variable function $f^{]\check{\boldsymbol{n}}[}({}^1x; \dots; {}^{d-1}x; \_) : X_d \to \mathbf{E}$ with $\check{\boldsymbol{n}} = n_1 \cdot \boldsymbol{e}_1 + \dots + n_{d-1} \cdot \boldsymbol{e}_{d-1}$, we obtain for $x \in X^{]\boldsymbol{n}[}$ a closed formula

$$f^{]\boldsymbol{n}[}({}^1x; -; {}^dx) = \sum_{\boldsymbol{i}\in[\boldsymbol{0},\boldsymbol{n}]} f({}^1x_{i_1}, \dots, {}^dx_{i_d}) \prod_{\substack{\boldsymbol{j}\in[\boldsymbol{0},\boldsymbol{n}] \text{ s.t.}\\ j_1\neq i_1,\dots,j_d\neq i_d}} (x_{i_1} - x_{j_1})^{-1} \cdots (x_{i_d} - x_{j_d})^{-1}.$$

Thus $C(x) := \max_{\boldsymbol{i}\in[\boldsymbol{0},\boldsymbol{n}]} C_i(x) \vee 1$ with $C_i(x) := |\prod_{\substack{\boldsymbol{j}\in[\boldsymbol{0},\boldsymbol{n}] \text{ s.t.}\\ j_1\neq i_1,\dots,j_d\neq i_d}} (x_{i_1} - x_{j_1})^{-1} \cdots (x_{i_d} - x_{j_d})^{-1}|$ fulfills the claim. ■

Recall that a locally convex $\mathbf{K}$-vector space will be called **Fréchet** if its topology can be induced by a countable family of seminorms. A topological space X will be called **σ-compact** if it is the (ascending) countable union of compact subsets.

Proposition 3.13. *The space $\mathcal{C}^r(X, \mathbf{E})$ is a complete locally convex $\mathbf{K}$-algebra. It is also a Fréchet space if and only if X is σ-compact.*

Proof. It is clear that $\mathcal{C}^r(X, \mathbf{E})$ is a locally convex $\mathbf{K}$-vector space. Let $\mathbf{E}$ be a $\mathbf{K}$-Banach algebra whose multiplication has operator norm $M \geq 1$. To convince ourselves that it is also a locally convex $\mathbf{K}$-algebra, we show firstly its closure under products and secondly that $\|fg\|_{\mathcal{C}^r,C} \leq M \cdot \|f\|_{\mathcal{C}^r,C}\|g\|_{\mathcal{C}^r,C}$ for all $f, g \in \mathcal{C}^r(X, \mathbf{E})$ and $C \subseteq X$ compact cartesian. Because these are local properties, we may assume X to be cartesian with factors free of isolated points. Foremost by Lemma 3.4(iii), we find for all $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$ that

$$\begin{aligned}&(f \cdot g)^{]\boldsymbol{n}[}({}^1x; -; {}^dx)\\ = &\sum_{\boldsymbol{j}\in[\boldsymbol{0},\boldsymbol{n}]} f^{]\boldsymbol{j}[}({}^1x_0, \dots, {}^1x_{j_1}; -; {}^dx_0, \dots, {}^dx_{j_d}) \cdot g^{]\boldsymbol{n}-\boldsymbol{j}[}({}^1x_{j_1}, \dots, {}^1x_{n_1}; -; {}^dx_{j_d}, \dots, {}^dx_{n_d}). \qquad (*)\end{aligned}$$

Firstly, let $f, g \in \mathcal{C}^r(X, \mathbf{E})$. By Proposition 3.8 and Lemma 3.5, the functions $f^{]\boldsymbol{n}[}, g^{]\boldsymbol{n}[}$ extend to $\mathcal{C}^\rho$-functions for $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$. By Proposition 1.7(ii), this sum $(f \cdot g)^{]\boldsymbol{n}[}$ again extends to a $\mathcal{C}^\rho$-function $(f \cdot g)^{[\boldsymbol{n}]}$. Since this holds in particular for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$, we find $f \cdot g \in \mathcal{C}^r(X, \mathbf{E})$. To show $\|fg\|_{\mathcal{C}^r,C} \leq M \cdot \|f\|_{\mathcal{C}^r,C}\|g\|_{\mathcal{C}^r,C}$, we must prove $\|(fg)^{[\boldsymbol{n}]}\|_{\sup} \leq M \cdot \|f\|_{\mathcal{C}^r}\|g\|_{\mathcal{C}^r}$ for all $\boldsymbol{n} \in \mathbb{N}^d_{<\nu}$ and $\|fg^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{n}]}} \leq M \cdot \|f\|_{\mathcal{C}^r}\|g\|_{\mathcal{C}^r}$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. The first inequality follows directly by Equation $(*)$. Regarding the latter inequality, by Lemma 1.10 applied to Equation $(*)$ we find

$$\|(fg)^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{n}]}} \leq M \cdot \max_{\boldsymbol{j}\in[\boldsymbol{0},\boldsymbol{n}]} \|f^{[\boldsymbol{j}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{j}]}} \|g^{[\boldsymbol{n}-\boldsymbol{j}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{n}-\boldsymbol{j}]}}.$$

For the latter inequality, let $h \in \mathcal{C}^r(X, \mathbf{E})$ be arbitrary and $\boldsymbol{j} \in \mathbb{N}^d_{\leq\nu}$. If $|\boldsymbol{j}| = \nu$, then $\|h^{[\boldsymbol{j}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{j}]}} \leq \|h\|_{\mathcal{C}^r,C}$ by definition of $\|h\|_{\mathcal{C}^r,C}$. If $|\boldsymbol{j}| < \nu$, then $\|h^{[\boldsymbol{j}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{j}]}} \leq \|h\|_{\mathcal{C}^{j+\rho},C} \leq \|h\|_{\mathcal{C}^r,C}$, the last inequality by Lemma 3.11. Applying this to $h = f, g$, we can conclude $\|(fg)^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho,C^{[\boldsymbol{n}]}} \leq M \cdot \|f\|_{\mathcal{C}^r,C}\|g\|_{\mathcal{C}^r,C}$.

We prove completeness. Let $\{U_i : i \in I\}$ be the cover of X by all nonempty open cartesian subsets whose factors contain no isolated point. As being $\mathcal{C}^r$ is a local property, the $\mathcal{C}^r$-functions form a sheaf on X: In other words the locally convex $\mathbf{K}$-vector space $\mathcal{C}^r(X, \mathbf{E})$ is canonically isomorphic to the subspace

$$A := \{(f_i) \in \prod_{i\in I} \mathcal{C}^r(U_i, \mathbf{E}) : f_{i|U_i\cap U_j} = f_{j|U_i\cap U_j} \text{ for all } i, j \in I\} \subseteq \prod_{i\in I} \mathcal{C}^r(U_i, \mathbf{E}) := P.$$

Then A is closed in P, as convergence in $\mathcal{C}^r(U_i, \mathbf{E})$ implies in particular pointwise convergence. As P is complete if and only if each factor is complete, we are reduced to the case that $X \subseteq \mathbf{K}^d$ is cartesian.

We note that the locally convex topology on $\mathcal{C}^r(X, \mathbf{K})$ given by $\{\|\cdot\|_{\mathcal{C}^r,C} : C \subseteq X$ compact cartesian$\}$ is equivalent to the one given by

$$\{\|\cdot^{[\boldsymbol{n}]}\|_C : C \subseteq X^{[\boldsymbol{n}]} \text{ compact and } \boldsymbol{n} \in \mathbb{N}^d_{<\nu}\}$$
$$\cup \{\|\cdot^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho,C} : C \subseteq X^{[\boldsymbol{n}]} \text{ compact and } \boldsymbol{n} \in \mathbb{N}^d_{=\nu}\}.$$

Namely, given $C \subseteq X^{[\boldsymbol{n}]}$ compact, denote the projections of $X^{[\boldsymbol{n}]} = X_1^{[n_1]} \times \cdots \times X_d^{[n_d]}$ onto its components by $\mathrm{p}_k : X^{[\boldsymbol{n}]} \to X_k^{[n_k]}$ for $k = 1, \ldots, d$ and put $C_k = \mathrm{p}_k\, C$ compact. Then for $i_k = 0, \ldots, n_k$, denote by $\mathrm{p}_{k,i_k} : X_k^{[n_k]} \to X_k$ the projection onto the i_k-th copy of X_k. We let $\tilde{C}_k := \mathrm{p}_{k,0}\, C_k \cup \ldots \cup \mathrm{p}_{k,n_k}\, C_k \subseteq X_k$ - which is compact - and put $\tilde{C} = \tilde{C}_1 \times \cdots \times \tilde{C}_d \subseteq X$ compact cartesian. Then $C \subseteq \tilde{C}^{[\boldsymbol{n}]}$ and hence $\|\cdot^{[\boldsymbol{n}]}\|_C \leq \|\cdot\|_{\mathcal{C}^r,\tilde{C}}$ if $|\boldsymbol{n}| < \nu$ respectively $\|\cdot^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho,C} \leq \|\cdot\|_{\mathcal{C}^r,\tilde{C}}$ if $|\boldsymbol{n}| = \nu$.

Hence as a locally convex $\mathbf{K}$-vector space, the space $\mathcal{C}^r(X, \mathbf{E})$ is canonically isomorphic to the subspace

$$\begin{aligned} A :=& \{(g_{\boldsymbol{n}}) \in \prod_{n<\nu} \mathcal{C}^0(X^{[\boldsymbol{n}]}, \mathbf{E}) \times \prod_{n=\nu} \mathcal{C}^\rho(X^{[\boldsymbol{n}]}, \mathbf{E}) : g_{\boldsymbol{n}|X^{]\boldsymbol{n}[}} = f^{]\boldsymbol{n}[} \text{ for } \boldsymbol{n} \in \mathbb{N}^d \text{ with } n \leq \nu\} \\ &\subseteq \prod_{n<\nu} \mathcal{C}^0(X^{[\boldsymbol{n}]}, \mathbf{E}) \times \prod_{n=\nu} \mathcal{C}^\rho(X^{[\boldsymbol{n}]}, \mathbf{E}) =: P. \end{aligned}$$

Each factor $\mathcal{C}^0(X^{[n]}, \mathbf{E})$ for $n < \nu$ is complete by Corollary 1.3. The factors $\mathcal{C}^\rho(X^{[n]}, \mathbf{E})$ for $n = \nu$ are complete by Proposition 1.9. Hence it remains to prove that A is closed in P.

For this, let $\boldsymbol{f} = (f_{\boldsymbol{n}})_{\boldsymbol{n} \in \mathbb{N}^d_{\leq \nu}}$ be in the boundary of A in P, i.e. in any neighborhood $U \ni \boldsymbol{f}$ of P lies another element $\boldsymbol{g} \in A$. We have to prove that $\boldsymbol{f} \in A$; in other words putting $f := f_{\mathbf{0}}$, necessarily $f_{\boldsymbol{n}|X^{]\boldsymbol{n}[}} = f^{]\boldsymbol{n}[}$ for $\boldsymbol{n} \in \mathbb{N}^d_{\leq \nu}$.

Fix $\varepsilon > 0$, an order $\boldsymbol{n} \in \mathbb{N}^d$ with $n \leq \nu$ and $\boldsymbol{x} \in X^{]\boldsymbol{n}[}$. We must show $\|f_{\boldsymbol{n}}(\boldsymbol{x}) - f^{[\boldsymbol{n}]}(\boldsymbol{x})\| \leq \varepsilon$.

Let $C \supseteq \{{}^1x_0, \dots, {}^1x_{n_1}\} \times \dots \times \{{}^dx_0, \dots, {}^1x_{n_d}\}$ be compact cartesian. With $C(\boldsymbol{x}) \geq 1$ as in Lemma 3.12, we find another $\boldsymbol{g} \in A$ such that $\|\boldsymbol{f} - \boldsymbol{g}\|_{\boldsymbol{n},\{\boldsymbol{x}\}} \leq \varepsilon$ and $\|\boldsymbol{f} - \boldsymbol{g}\|_{\mathbf{0},C} \leq \varepsilon / C(\boldsymbol{x})$; here $\|\cdot\|_{\boldsymbol{n},\{\boldsymbol{x}\}}$ and $\|\cdot\|_{\mathbf{0},C}$ denoting the seminorms on P given by $\|\boldsymbol{h}\|_{\boldsymbol{n},\{\boldsymbol{x}\}} := \|\boldsymbol{h}_{\boldsymbol{n}}(\boldsymbol{x})\|$ and $\|\boldsymbol{h}\|_{\mathbf{0},C} := \|\boldsymbol{h}_{\mathbf{0}}\|_C$. Hence with $g := g_0$, it holds $\|(f - g)(x)\| \leq \varepsilon / C(\boldsymbol{x})$ for all $x \in C$. By Lemma 3.12, we find $\|(g - f)^{]\boldsymbol{n}[}(\boldsymbol{x})\| \leq \varepsilon$. Since $g_{\boldsymbol{n}|X^{]\boldsymbol{n}[}} = g^{]\boldsymbol{n}[}$, we find

$$\begin{aligned}
\|f_{\boldsymbol{n}}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| &\leq \|f_{\boldsymbol{n}}(\boldsymbol{x}) - g_{\boldsymbol{n}}(\boldsymbol{x})\| \vee \|g_{\boldsymbol{n}}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| \\
&= \|f_{\boldsymbol{n}}(\boldsymbol{x}) - g_{\boldsymbol{n}}(\boldsymbol{x})\| \vee \|g^{]\boldsymbol{n}[}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| \\
&= \|f_{\boldsymbol{n}}(\boldsymbol{x}) - g_{\boldsymbol{n}}(\boldsymbol{x})\| \vee \|(g - f)^{]\boldsymbol{n}[}(\boldsymbol{x})\| \leq \varepsilon.
\end{aligned}$$

Ad the Fréchet property: We have to show that $\mathcal{C}^r(X, \mathbf{E})$'s locally convex topology can be induced by a countable family of seminorms. The union of finitely many compact subsets is again compact and every compact subset is contained in the cartesian compact subset given through the product of its projections. Because $\|\cdot\|_{\mathcal{C}^r,C} \leq \|\cdot\|_{\mathcal{C}^r,D}$ pointwise on $\mathcal{C}^r(X, \mathbf{E})$ for compact cartesian subsets $C \subseteq D$, the defining family of seminorms is therefore directed. Hence we see that the claim holds if and only if X can be exhausted by a countable family of compact subsets. ■

Remark. We recall that complete metric spaces are in particular Baire spaces (see for example [Schikhof, 1984, Appendix A.1]). Remember that these can be defined by the property that whenever the union of countably many closed subsets of X has an interior point, then one of these closed subsets must have an interior point. So a Hausdorff Baire space which is σ-compact, is at least at one point locally compact. Thus for a closed subset of a complete normed group, σ-compactness implies local compactness. As a compact metric space is separable, so are σ-compact ones.
On the other hand, a locally compact separable metric space X is σ-compact: If S denotes a countable dense subset of X, we let $A := \bigcup_{s \in S} U_s$ where U_s denotes a compact neighborhood of $s \in S$. Let $C \subseteq X$ be compact. By compactness, finitely many U_s cover C and therefore $A \cap C = (U_{s_1} \cup \ldots \cup U_{s_n}) \cap C$ for a finite collection $s_1, \dots, s_n \in S$. So $A \cap C$ is again compact and therefore closed. For X is as a metric space compactly generated, A is closed and equals therefore all of X. Hence $\{U_s\}$ is the desired countable cover by compact subsets. We conclude that a closed subset of a complete normed group is σ-compact if and only if it is separable and locally compact. E.g., $\mathcal{C}^r(X, \mathbf{E})$ is not Fréchet for a ball $X \subseteq \mathbb{C}_p^d$.

Locally analytic functions in $\mathcal{C}^r(X, \mathbf{K})$ on an open domain

Definition. Let $X \subseteq \mathbf{K}^d$ be an open subset. A function $f : X \to \mathbf{K}$ will be called **locally analytic** if for each point $\boldsymbol{a} \in X$, there exists a closed ball $\mathrm{B} \ni a$ in X such that

$$f(\boldsymbol{x} - \boldsymbol{a}) = \sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} (\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{i}} \quad \text{for all } \boldsymbol{x} \in \mathrm{B};$$

here $a_{\boldsymbol{i}} \in \mathbf{K}$ and $(\boldsymbol{x} - \boldsymbol{a})^{\boldsymbol{i}} := (x_1 - a_1)^{i_1} \cdots (x_d - a_d)^{i_d}$ for $\boldsymbol{i} \in \mathbb{N}^d$.

Definition. For two sets A, B, a ring R and mappings $\phi : A \to R, \psi : B \to R$, define **their tensor product** $\phi \odot \psi : A \times B \to R$ by

$$\phi \odot \psi(a, b) := \phi(a)\psi(b).$$

Lemma 3.14. *(i) Let $X \subseteq \mathbf{K}^d, Y \subseteq \mathbf{K}^e$ be cartesian and $f : X \to \mathbf{E}, g : Y \to \mathbf{E}$ two mappings. If $\boldsymbol{n} = (\boldsymbol{n}', \boldsymbol{n}'') \in \mathbb{N}^d \times \mathbb{N}^e$, then*

$$(f \odot g)^{]\boldsymbol{n}[} = f^{]\boldsymbol{n}'[} \odot g^{]\boldsymbol{n}''[}.$$

(ii) Let $X_1, \ldots, X_d \subseteq \mathbf{K}$ and $f_1 : X_1 \to \mathbf{E}, \ldots, f_d : X_d \to \mathbf{E}$ be d mappings. If $\boldsymbol{n} \in \mathbb{N}^d$, then

$$(f_1 \odot \cdots \odot f_d)^{]\boldsymbol{n}[} = f_1^{]n_1[} \odot \cdots \odot f_d^{]n_d[}.$$

Proof. Ad (i): This is proven by induction on $n = |\boldsymbol{n}|$. If $n = 0$ there is nothing to show. Let $n > 0$. We may assume w.l.o.g. $n' > 0$. Then $\boldsymbol{n}' = \tilde{\boldsymbol{n}}' + \boldsymbol{e}_k$ for some coordinate $k \in \{1, \ldots, d\}$. Put $h := f \odot g$ and let $(x : y) \in X^{]\boldsymbol{n}'[} \times X^{]\boldsymbol{n}''[}$. By induction

$$\begin{aligned}
&h^{]\boldsymbol{n}[}(x : y) \\
=&h^{]\tilde{\boldsymbol{n}}' + \boldsymbol{e}_k, \boldsymbol{n}''[}(x : y) \\
=&\frac{h^{]\tilde{\boldsymbol{n}}', \boldsymbol{n}''[}(-; {}^k x_0, {}^k x_2, \ldots, {}^k x_{\tilde{n}'_k + 1}; - : y) - h^{]\tilde{\boldsymbol{n}}', \boldsymbol{n}''[}(-; {}^k x_1, {}^k x_2, \ldots, {}^k x_{\tilde{n}'_k + 1}; - : y)}{{}^k x_0 - {}^k x_1} \\
=&\frac{f^{]\tilde{\boldsymbol{n}}'[}(-; {}^k x_0, {}^k x_2, \ldots, {}^k x_{\tilde{n}'_k + 1}; -) \odot g^{]\boldsymbol{n}''[}(y) - f^{]\tilde{\boldsymbol{n}}'[}(-; {}^k x_1, {}^k x_2, \ldots, {}^k x_{\tilde{n}'_k + 1}; -) \odot g^{]\boldsymbol{n}''[}(y)}{{}^k x_0 - {}^k x_1} \\
=&f^{]\tilde{\boldsymbol{n}}' + \boldsymbol{e}_k[}(x) \odot g^{]\boldsymbol{n}''[}(y) = f^{]\boldsymbol{n}'[}(x) \odot g^{]\boldsymbol{n}''[}(y).
\end{aligned}$$

Ad (ii): Let $\boldsymbol{n} = (\boldsymbol{n}', n_d) \in \mathbb{N}^{d-1} \times \mathbb{N}$. By (i) and induction on d, we find

$$(f_1 \odot \cdots \odot f_d)^{]\boldsymbol{n}[} = (f_1 \odot \cdots \odot f_{d-1})^{]\boldsymbol{n}'[} \odot f_d^{]n_d[} = f_1^{]n_1[} \odot \cdots \odot f_d^{]n_d[}.$$

∎

Lemma 3.15. *Let $X \subseteq \mathbf{K}^d$ be open cartesian, $\boldsymbol{n} \in \mathbb{N}^d$ and $\delta > 0$. We define*

$$X^{[\boldsymbol{n}]}_{\leq \delta} := \{x = ({}^1 x; -; {}^d x) \in X^{[\boldsymbol{n}]} : \delta\, {}^1 x, \ldots, \delta\, {}^d x \leq \delta\},$$

where $\delta\,{}^k x := \delta\{{}^k x_0, \dots, {}^k x_{n_k}\}$ *for* $k = 1, \dots, d$. *Let* $p \in X$ *and put* $P := \mathrm{B}_{\leq\delta}(p) \subseteq X$. *Then* $P^{[\boldsymbol{n}]} = \mathrm{B}_{\leq\delta}(\vec{p}) \subseteq X^{[\boldsymbol{n}]}$ *with* $\vec{p} = (\vec{p}_1; \dots; \vec{p}_d) \in X^{[\boldsymbol{n}]}$ *and*

$$X^{[\boldsymbol{n}]}_{\leq\delta} = \bigcup_{p\in X} P^{[\boldsymbol{n}]}.$$

Likewise for $X^{]\boldsymbol{n}[}_{\leq\delta} := X^{[\boldsymbol{n}]}_{\leq\delta} \cap X^{]\boldsymbol{n}[}$.

Proof. The first assertion holds by definition. We have

$$X^{[\boldsymbol{n}]}_{\leq\delta} = \bigcup_{\boldsymbol{p}\in X^{[\boldsymbol{n}]}} \mathrm{B}_{\leq\delta}(\boldsymbol{p}).$$

Let $a, b \in X^{[\boldsymbol{n}]}_{\leq\delta}$ and $\|\boldsymbol{b} - \boldsymbol{a}\| \leq \delta$. Then by the ultrametric triangle inequality holds $\mathrm{B}_{\leq\delta}(\boldsymbol{a}) = \mathrm{B}_{\leq\delta}(\boldsymbol{b})$ in $X^{[\boldsymbol{n}]}_{\leq\delta}$. Let $\boldsymbol{p} \in X^{[\boldsymbol{n}]}_{\leq\delta}$. Then we can find $p \in X$ such that $\|\boldsymbol{p} - \vec{p}\| \leq \delta$ where we put $\vec{p} = (\vec{p}_1; \dots; \vec{p}_d) \in X^{[\boldsymbol{n}]}$. (E.g. $p = ({}^1p_0, \dots, {}^dp_0) \in X$ for $\boldsymbol{p} = ({}^1\boldsymbol{p}; -; {}^d\boldsymbol{p}) \in X^{[\boldsymbol{n}]}_{\leq\delta}$.) We can therefore conclude $\mathrm{B}_{\leq\delta}(\boldsymbol{p}) = P^{[\boldsymbol{n}]}$ with $P = \mathrm{B}_{\leq\delta}(x)$. ■

Definition. Let $\boldsymbol{i} \in \mathbb{N}^d$ and $X \subseteq \mathbf{K}^d$. Then we will denote by $*^{\boldsymbol{i}} : X \to \mathbf{K}$ the function $\boldsymbol{x} \mapsto \boldsymbol{x}^{\boldsymbol{i}} := x_1^{i_1} \cdots x_d^{i_d}$.

Lemma 3.16. *Let* $f(x) = *^i : X \to \mathbf{K}$ *defined on a subset* $X \subseteq \mathbf{K}$. *Then*

$$f^{]n[} \equiv \begin{cases} 1, & \text{if } i = n, \\ 0, & \text{if } i < n. \end{cases}$$

Proof. It will suffice to prove by induction on n that $f^{]n[} \equiv 1$ if $f = *^n$. If $n = 0$, there will be nothing to show. For $n = 1$, let $h(x) = x$. then $h^{]1[} \equiv 1$. Let $n > 1$. Then $f(x) = g(x)h(x)$ with $g(x) = x^{n-1}$. We note that if $g^{]m[}$ is constant, then

$$g^{]m+1[}(x_0, \dots, x_{m+1}) = (x_0 - x_1)^{-1}(g^{]m[}(x_0, x_2, \dots, x_{m+1}) - g^{]m[}(x_1, x_2, \dots, x_{m+1})) = 0$$

for all $x_0, \dots, x_{m+1}$. By [Schikhof, 1984, Lemma 29.2 (v)], we find

$$\begin{aligned} f^{]n[}(x_0, \dots, x_n) &= \sum_{j=0,\dots,n} g^{]j[}(x_0, \dots, x_j) h^{]n-j[}(x_j, \dots, x_n) \\ &= g^{]n-1[}(x_0, \dots, x_{n-1}) h^{]1[}(x_{n-1}, x_n) + g^{]n[}(x_0, \dots, x_n) h^{]0[}(x_n) \\ &= g^{]n-1[}(x_0, \dots, x_{n-1}) + g^{]n[}(x_0, \dots, x_n) x_n. \end{aligned}$$

By induction, $g^{]n-1[} \equiv 1$ and thus $g^{]n[} = 0$. Hence $f^{]n[} \equiv 1$. ■

Corollary 3.17. *Let* $X \subseteq \mathbf{K}^d$ *be a cartesian subset and* $\boldsymbol{i}, \boldsymbol{n} \in \mathbb{N}^d$.

(i) Let $f(x) = *^{\boldsymbol{i}} : X \to \mathbf{K}$. *Then* $f^{]\boldsymbol{n}[} \equiv \begin{cases} 1, & \text{if } \boldsymbol{i} = \boldsymbol{n}, \\ 0, & \text{if } \boldsymbol{i} \not\geq \boldsymbol{n}. \end{cases}$

(ii) *Let $f, g : X \to \mathbf{E}$ be two mappings with g being δ-constant. Then $(g \cdot f)^{]\boldsymbol{n}[}({}^1x; -; {}^dx) = g(p) \cdot f^{]\boldsymbol{n}[}({}^1x; -; {}^dx)$ on $P^{]\boldsymbol{n}[}$ with $P = \mathrm{B}_{\leq\delta}(p)$ independent of the representative $p \in P$. In particular on $P^{]\boldsymbol{n}[}$ holds* $(g \cdot *^{\boldsymbol{i}})^{]\boldsymbol{n}[} \equiv \begin{cases} g(p), & \text{if } \boldsymbol{i} = \boldsymbol{n}, \\ 0, & \text{if } \boldsymbol{i} \not\geq \boldsymbol{n}. \end{cases}$

Proof. Ad (i): By Lemma 3.14(ii) and Lemma 3.16, we have

$$f^{]\boldsymbol{n}[} = *^{i_1]n_1[} \odot \cdots \odot *^{i_d]n_d[} \equiv \begin{cases} 1, & \text{if } i_1 = n_1, \ldots, i_d = n_d, \\ 0, & \text{if } i_k < n_k \text{ for some } k \in \{1, \ldots, d\}. \end{cases}$$

Ad (ii): We have $g_{|P} \equiv g(p)$ for some representative $p \in P$. By linearity of $f \mapsto f^{]\boldsymbol{n}[}$, we get $(g \cdot f)^{]\boldsymbol{n}[}({}^1x; -; {}^dx) = g(p) \cdot f^{]\boldsymbol{n}[}(x)$. Now apply (i). ■

Proposition 3.18. *Let $X \subseteq \mathbf{K}^d$ be an open subset. A locally analytic function $f : X \to \mathbf{K}$ is a $\mathcal{C}^r$-function for any $r \in \mathbb{R}_{\geq 0}$.*

Proof. By Lemma 3.5, we find $\mathcal{C}^r(X, \mathbf{K}) \supseteq \mathcal{C}^{\nu+1}(X, \mathbf{K})$. Hence we may assume $r = \nu \in \mathbb{N}$. Since being $\mathcal{C}^r$ is a local property, it suffices to prove this for an analytic function $f : X \to \mathbf{K}$ on a closed ball $X \subseteq \mathbf{K}^d$ whose radius we may assume to lie in $|\mathbf{K}^*|_{\leq 1}$. Let $X = \mathrm{B}_{\leq\varepsilon}(\boldsymbol{a})$ with $\boldsymbol{a} \in X$ and $\varepsilon > 0$. For notational convenience, let us assume $\boldsymbol{a} = 0$. Altogether our function f is defined as

$$f(\boldsymbol{x}) = \sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} \boldsymbol{x}^{\boldsymbol{i}} \quad \text{for all } \boldsymbol{x} \in \mathbf{K}^d \text{ with } \|\boldsymbol{x}\| \leq \varepsilon.$$

Since this power series converges for all $\boldsymbol{x}$ with $\|x\| = \varepsilon \in |\mathbf{K}^*|$, we find $|a_{\boldsymbol{i}}|\varepsilon^{|\boldsymbol{i}|} \to 0$ as $|\boldsymbol{i}| \to \infty$. It suffices to prove $|*^{\boldsymbol{i}[\boldsymbol{n}]}(\boldsymbol{x})| \leq \varepsilon^{|\boldsymbol{i}|-\nu}$ for all $\boldsymbol{x} \in X^{[\boldsymbol{n}]}$ with $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$: Then uniformly in all compact cartesian $C \subseteq X$ holds $\|a_{\boldsymbol{i}} *^{\boldsymbol{i}}\|_{\mathcal{C}^\nu, C} \leq |a_{\boldsymbol{i}}|\varepsilon^{|\boldsymbol{i}|}/\varepsilon^\nu \to 0$ as $|\boldsymbol{i}| \to \infty$, hence $f = \sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$ as a convergent sum in $\mathcal{C}^\nu(X, \mathbf{K})$ by completeness.
Let $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$. By Corollary 3.17(i), we find $*^{\boldsymbol{i}[\boldsymbol{n}]} = 0$ if $\boldsymbol{i} \not\geq \boldsymbol{n}$. Otherwise, by Lemma 3.14(ii) holds

$$|*^{\boldsymbol{i}[\boldsymbol{n}]}(\boldsymbol{x})| = |*^{i_1[n_1]}({}^1\boldsymbol{x})| \cdots |*^{i_d[n_d]}({}^d\boldsymbol{x})| \quad \text{for all } \boldsymbol{x} = ({}^1\boldsymbol{x}; -; {}^d\boldsymbol{x}) \in X^{[\boldsymbol{n}]}.$$

We are hence reduced to proving $|*^{i[n]}(x)| \leq \varepsilon^{i-n}$ for $i \geq n \in \mathbb{N}$ and $x \in X^{[n]}$ with $X := \mathrm{B}_{\leq\varepsilon}(0)$, as then

$$|*^{\boldsymbol{i}[\boldsymbol{n}]}(\boldsymbol{x})| \leq \varepsilon^{i_1-n_1} \cdots \varepsilon^{i_d-n_d} = \varepsilon^{|\boldsymbol{i}|-|\boldsymbol{n}|} \leq \varepsilon^{|\boldsymbol{i}|-\nu} \quad \text{if } \boldsymbol{i} \geq \boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}.$$

Let $*^i = g \cdot h$ with $g = *^{i-1}$ and $h = *$. By the proof of Lemma 3.16, we find

$$*^{i]n[}(x_0, \ldots, x_n) = g^{]n-1[}(x_0, \ldots, x_{n-1}) + g^{]n[}(x_0, \ldots, x_n)x_0.$$

As $|x| \leq \varepsilon$, it follows by induction on $i \geq 0$ that

$$|*^{i[n]}(x)| \leq \varepsilon^{(i-1)-(n-1)} \vee \varepsilon^{i-1-n} \cdot \varepsilon = \varepsilon^{i-n} \quad \text{for all } x \in X^{[n]}.$$

■

Composition properties of $\mathcal{C}^r$-functions

Lemma 3.19. *Let $X \subseteq \mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point and $f : X \to \mathbf{E}$ a mapping thereon. Let $n \leq r$ be a nonnegative integer. Then $f \in \mathcal{C}^r(X, \mathbf{E})$ if and only if $f \in \mathcal{C}^n(X, \mathbf{E})$ and $f^{[\boldsymbol{n}]} \in \mathcal{C}^{r-n}(X^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$. Moreover $\|f\|_{\mathcal{C}^r,C} = \|f\|_{\mathcal{C}^n,C} \vee \max_{\boldsymbol{n}\in\mathbb{N}^d_{=n}} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{r-n},C^{[\boldsymbol{n}]}}$.*

Proof. We make a preliminary observation: Put $\mathbb{N}^{[n]} = \mathbb{N}^{\{0,\dots,n\}}$ and $\mathbb{N}^{[\boldsymbol{n}]} = \mathbb{N}^{[n_1]} \times \cdots \times \mathbb{N}^{[n_d]}$ for $\boldsymbol{n} \in \mathbb{N}^d$. Write $\boldsymbol{m} = ({}^1\boldsymbol{m}; -; {}^d\boldsymbol{m}) \in \mathbb{N}^{[\boldsymbol{n}]}$ with ${}^k\boldsymbol{m} \in \mathbb{N}^{[n_k]}$ for $k = 1, \dots, d$. Define $\bar{\boldsymbol{m}} \in \mathbb{N}^d$ by $\bar{m}_k = {}^k m_0 + \cdots + {}^k m_{n_k}$ for $k = 1, \dots, d$. Then we have an identification $(X^{[\boldsymbol{n}]})^{[\boldsymbol{m}]} = X^{[\boldsymbol{n}+\bar{\boldsymbol{m}}]}$ as follows: We will consider the left and right hand side to be a product of subsets of $\mathbf{K}$ indexed by

$$I = \{((k,i),j) : k \in \{1,\dots,d\}, i = 0,\dots,n_k, j = 0,\dots,{}^k m_i\}$$

respectively

$$J = \{(k,i) : k \in \{1,\dots,d\}, i = 0,\dots,(n+\bar{m})_k\}.$$

Then we have a bijection $\phi : I \to J$ via

$$((k,i),j) \mapsto (k, {}^k m_0 + \cdots + {}^k m_{i-1} + j).$$

This yields the equally labeled bijection $\phi : (X^{[\boldsymbol{n}]})^{[\boldsymbol{m}]} \xrightarrow{\sim} X^{[\boldsymbol{n}+\bar{\boldsymbol{m}}]}$ by

$$(X^{[\boldsymbol{n}]})^{[\boldsymbol{m}]} = \prod_{i\in I} X_i \ni x = (x_i)_{i\in I} \mapsto \phi(x) = (x_{\phi(i)}) \in \prod_{i\in J} X_j = X^{[\boldsymbol{n}+\bar{\boldsymbol{m}}]}.$$

We show that this identification yields an equality of mappings

$$f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[} \circ \phi = (f^{]\boldsymbol{n}[})^{]\boldsymbol{m}[} \quad \text{restricted onto } \phi^{-1}(X^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}). \tag{$*$}$$

We proceed by induction on $|\boldsymbol{m}^+|$, the starting case $\boldsymbol{m}^+ = \boldsymbol{0}$ holding true by definition. Assume $|\boldsymbol{m}^+| \geq 1$, say $\boldsymbol{m}^+ = \boldsymbol{m} + {}^k\boldsymbol{e}_i$. Let $\boldsymbol{x} = (-; {}^{k,i}\boldsymbol{x}; -) \in \phi^{-1}(X^{]\boldsymbol{n}+\bar{\boldsymbol{m}}^+[}) \subseteq (X^{[\boldsymbol{n}]})^{[\boldsymbol{m}]}$. Then we have

$$\begin{aligned}
&(f^{]\boldsymbol{n}[})^{]\boldsymbol{m}^+[}(\boldsymbol{x}) \\
=&(f^{]\boldsymbol{n}[})^{]\boldsymbol{m}+{}^k\boldsymbol{e}_i[}(-; {}^{k,i}x_0, {}^{k,i}x_1, {}^{k,i}x_2, \dots; -) \\
=&\frac{(f^{]\boldsymbol{n}[})^{]\boldsymbol{m}[}(-; {}^{k,i}x_0, {}^{k,i}x_2, \dots; -) - (f^{]\boldsymbol{n}[})^{]\boldsymbol{m}[}(-; {}^{k,i}x_1, {}^{k,i}x_2, \dots; -)}{{}^{k,i}x_0 - {}^{k,i}x_1} \\
=&\frac{f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[} \circ \phi(-; {}^{k,i}x_0, {}^{k,i}x_2, \dots; -) - f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[} \circ \phi(-; {}^{k,i}x_1, {}^{k,i}x_2, \dots; -)}{{}^{k,i}x_0 - {}^{k,i}x_1} \\
=&\frac{f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}(-; \cdots, {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+0}, \dots; -) - f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}(-; \cdots, {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+1}, \dots; -)}{{}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+0} - {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+1}} \\
=&\frac{f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}(-; {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+0}, {}^k x_2, \dots; -) - f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}(-; {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+1}, {}^k x_2, \dots; -)}{{}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+0} - {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+1}} \\
=&f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}+\boldsymbol{e}_k[}(-; {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+0}, {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+1}, {}^k x_2, \dots; -) \\
=&f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}+\boldsymbol{e}_k[}(-; \dots, {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+0}, {}^k x_{{}^k m_0+\cdots+{}^k m_{i-1}+1}, \dots; -) \\
=&f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}^+[} \circ \phi(\boldsymbol{x}).
\end{aligned}$$

Here we used the (recursive) definition of the iterated difference quotient for the second and sixth equality, the symmetry of the iterated difference quotients in all X_k-coordinates for the fifth and seventh equality and the induction hypothesis for the fourth equality.

Because $\phi : (X^{[\boldsymbol{n}]})^{[\boldsymbol{m}]} \xrightarrow{\sim} X^{[\boldsymbol{n}+\bar{\boldsymbol{m}}]}$ is a topological isomorphism, we find by Equality $(*)$ thus $f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}$ to extend to a continuous mapping $f^{[\boldsymbol{n}+\bar{\boldsymbol{m}}]}$ if and only if $(f^{[\boldsymbol{n}]})^{]\boldsymbol{m}[}$ extends to a continuous mapping $(f^{[\boldsymbol{n}]})^{]\boldsymbol{m}[}$. This shows $f \in \mathcal{C}^r(X, \mathbf{E})$ if and only if $f \in \mathcal{C}^n(X, \mathbf{E})$ and $f^{[\boldsymbol{n}]} \in \mathcal{C}^{r-n}(X^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$, as follows: Firstly, assume that $f \in \mathcal{C}^r(X, \mathbf{E})$.
Let $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ and $\boldsymbol{m} \in \mathbb{N}^{[\boldsymbol{n}]}_{=m}$ with $n + m = \nu$; here $\mathbb{N}^{[\boldsymbol{n}]} = \mathbb{N}^{[n_1]} \times \cdots \times \mathbb{N}^{[n_d]}$ with $\mathbb{N}^{[n_k]} = \mathbb{N}^{\{0,\ldots,n_k\}}$ for $k = 1, \ldots, d$. Because $f \in \mathcal{C}^r(X, \mathbf{E})$ and $|\boldsymbol{n} + \bar{\boldsymbol{m}}| = \nu$, these $f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[}$ extend by Proposition 3.8 to $\mathcal{C}^\rho$-functions $f^{[\boldsymbol{n}+\bar{\boldsymbol{m}}]}$ for all $\bar{\boldsymbol{m}} \in \mathbb{N}^{[\boldsymbol{n}]}$. By Equality $(*)$ therefore $f^{[\boldsymbol{n}]} \in \mathcal{C}^{r-n}(X^{[\boldsymbol{n}]}, \mathbf{E})$.
Contrariwise, assume that $f \in \mathcal{C}^n(X, \mathbf{E})$ and $f^{[\boldsymbol{n}]} \in \mathcal{C}^{r-n}(X^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$. For $\boldsymbol{\nu} \in \mathbb{N}^d_{=\nu}$ write $\boldsymbol{\nu} = \boldsymbol{n} + \tilde{\boldsymbol{m}}$ with $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ and $\tilde{\boldsymbol{m}} \in \mathbb{N}^d_{=m}$. Choose $\boldsymbol{m} \in \mathbb{N}^{[\boldsymbol{n}]}_{=m}$ with $\bar{\boldsymbol{m}} = \tilde{\boldsymbol{m}}$. Because $f^{[\boldsymbol{n}]} \in \mathcal{C}^{r-n}(X^{[\boldsymbol{n}]}, \mathbf{E})$, regarding Equality $(*)$, we find $f^{]\boldsymbol{n}+\bar{\boldsymbol{m}}[} = f^{]\boldsymbol{\nu}[}$ by Proposition 3.8 to extend to a $\mathcal{C}^\rho$-function $f : X^{[\boldsymbol{\nu}]} \to \mathbf{E}$. As this holds for arbitrary $\boldsymbol{\nu} \in \mathbb{N}^d_{=\nu}$, we find $f \in \mathcal{C}^r(X, \mathbf{E})$.

The equality of norms then follows directly by the bijectivity of ϕ, as follows: By Equality $(*)$ holds $\|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{r-n}, C^{[\boldsymbol{n}]}} \leq \|f\|_{\mathcal{C}^r, C}$ for every compact cartesian subset $C \subseteq X$ and $\boldsymbol{n} \in \mathbb{N}^d_{=n}$, thus $\|f\|_{\mathcal{C}^n, C} \vee \max_{\boldsymbol{n} \in \mathbb{N}^d_{=n}} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{r-n}, C^{[\boldsymbol{n}]}} \leq \|f\|_{\mathcal{C}^r, C}$ by Lemma 3.11. Regarding the inverse inequality, for $\boldsymbol{l} \in \mathbb{N}^d_{=l}$ with $n \leq l \leq \nu$ write $\boldsymbol{l} = \boldsymbol{n} + \tilde{\boldsymbol{m}}$ with $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ and $\tilde{\boldsymbol{m}} \in \mathbb{N}^d_{=m}$. Choose $\boldsymbol{m} \in \mathbb{N}^{[\boldsymbol{n}]}_{=m}$ with $\bar{\boldsymbol{m}} = \tilde{\boldsymbol{m}}$. Then by Equality $(*)$ holds $\|f^{[\boldsymbol{l}]}\|_C = \|(f^{[\boldsymbol{n}]})^{[\boldsymbol{m}]}\|_{C^{[\boldsymbol{n}]}} \leq \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{r-n}, C}$ respectively $\|f^{[\boldsymbol{l}]}\|_{\mathcal{C}^\rho, C} \leq \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{r-n}, C}$ in case $|\boldsymbol{l}| = \nu$. Therefore

$$\begin{aligned}
\|f\|_{\mathcal{C}^r, C} &= \max_{\boldsymbol{n} \in \mathbb{N}^d_{<\nu}} \|f^{[\boldsymbol{n}]}\|_{C^{[\boldsymbol{n}]}} \vee \max_{\boldsymbol{n} \in \mathbb{N}^d_{=\nu}} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho, C^{[\boldsymbol{n}]}} \\
&\leq \max_{\boldsymbol{n} \in \mathbb{N}^d_{\leq n}} \|f^{[\boldsymbol{n}]}\|_{C^{[\boldsymbol{n}]}} \vee \max_{\boldsymbol{l} \in \mathbb{N}^d \text{ with } n \leq |\boldsymbol{l}| < \nu} \|f^{[\boldsymbol{l}]}\|_{\mathcal{C}^\rho, C^{[\boldsymbol{l}]}} \vee \max_{\boldsymbol{l} \in \mathbb{N}^d_{=\nu}} \|f^{[\boldsymbol{l}]}\|_{\mathcal{C}^\rho, C^{[\boldsymbol{l}]}} \\
&= \|f\|_{\mathcal{C}^n, C} \vee \max_{\boldsymbol{n} \in \mathbb{N}^d_{=n}} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{r-n}, C^{[\boldsymbol{n}]}}.
\end{aligned}$$

■

Remark. The Equality $(*)$ in the preceding proof is also shown in [Glöckner, 2007, Remark 2.5]. It is informed by the viewpoint of $f^{[\boldsymbol{n}]}$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ as the $\boldsymbol{n}$-th column-vector of the the ν-th iterated difference quotient $f^{[\nu]}$ of $f : X \to \mathbf{K}$, up to reduction by symmetry in the coordinates of $f^{[\nu]}$. Here e.g. $f^{[0]} = f$, then $f^{[1]}$ is to be understood in the sense of Proposition 1.34, and if $f^{[1]} : X^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E}) = \mathbf{E}^d$ exists, we let

$$f^{]2[} = (f^{[1]})^{]1[} : (X^{[1]})^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(\mathbf{K}^d, \mathbf{E}), \mathbf{E}) = \mathbf{E}^{d^2}.$$

If existent, the unique continuous extension of $f^{]2[} : (X^{[1]})^{]1[} \to \mathbf{E}$ to all of $X^{[2]} = X^{[1]} \times X^{[1]}$ is then denoted $f^{[2]}$. This was also discussed in the introduction.

Notation. We switch notation by writing $(x; t)$ for $(x_1; \ldots; x_{k-1}; x_k + t, x_k; x_{k+1}; \ldots; x_d) \in X^{[\boldsymbol{e}_k]}$.

In the following, we will tacitly use that projection functions are $\mathcal{C}^r$-functions for any $r \geq 0$, a convenient criterion for this given by Proposition 3.18. Recall that by Remark 3.3(iii) a cartesian product of $\mathcal{C}^r$-functions is again a $\mathcal{C}^r$-function.

Lemma 3.20. *Let $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ be nonempty cartesian subsets whose factors contain no isolated point. Let $f \in \mathcal{C}^1(X, \mathbf{K}^e)$ and $g \in \mathcal{C}^1(Y, \mathbf{E})$ be two functions with* $\operatorname{im} f \subseteq Y$. *Then $g \circ f \in \mathcal{C}^1(X, \mathbf{E})$, seen by the matrix product*

$$g \circ f^{[e_k]}(x;t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[e_k]}(x;t) \quad \text{for } k = 1, \dots, d \text{ and } (x;t) \in X^{[e_k]}.$$

Proof. Let $k \in \{1, \dots, d\}$. We have to prove that the above equation's right hand side composed linear map $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[e_k]}(x;t) : \mathbf{K} \to \mathbf{E}$ sends any $t \in \mathbf{K}^*$ such that $x + t \cdot \boldsymbol{e}_k, x \in X$ to $g \circ f(x + t\boldsymbol{e}_k) - g \circ f(x)$. Then by continuity this equality extends to all of $X^{[e_k]}$.

By definition $f^{[e_k]}(x;t) \cdot t = f(x + t \cdot \boldsymbol{e}_k) - f(x)$ and

$$g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot (f(x + t \cdot \boldsymbol{e}_k) - f(x)) = g(f(x + t \cdot \boldsymbol{e}_k)) - g(f(x)),$$

where we recall Proposition 1.34 for the definition of $g^{]1[} : Y^{[1]} \to \mathbf{E}$ (and use continuous extension). Together, we find

$$\begin{aligned} g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) f^{[e_k]}(x;t) \cdot t &= g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot (f(x + t \cdot \boldsymbol{e}_k) - f(x)) \\ &= g(f(x + t \cdot \boldsymbol{e}_k)) - g(f(x)). \end{aligned}$$

■

Lemma 3.21. *Let $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ be nonempty cartesian subsets whose factors contain no isolated point. For $r \geq 1$, let $f \in \mathcal{C}^r(X, \mathbf{K}^e)$ and $g \in \mathcal{C}^r(Y, \mathbf{E})$ be two functions with* $\operatorname{im} f \subseteq Y$. *Then $g \circ f^{]e_k[}(x, y)$ extends to $g \circ f^{[e_k]} \in \mathcal{C}^{r-1}(X^{[e_k]}, \mathbf{E})$ for $k = 1, \dots, d$.*

Proof. We proceed by induction on $\nu \geq 1$. If $\nu = 1$, i.e. $r = 1 + \rho$, then by Lemma 3.20 will hold

$$g \circ f^{[e_k]}(x;t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[e_k]}(x;t) \quad \text{for } k = 1, \dots, d;$$

here the right hand side meaning the matrix product. By Lemma 1.37, we firstly find $f \in \mathcal{C}^1(X, \mathbf{K}^e) \subseteq \mathcal{C}^{\text{lip}}(X, \mathbf{K}^e)$ and by Proposition 1.34, it also holds $g^{[1]} \in \mathcal{C}^\rho(Y^{[1]}, \mathbf{E})$. The function $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x))$ is therefore again a $\mathcal{C}^\rho$-function by Proposition 1.7(i). Also $f^{[e_k]}$ is a $\mathcal{C}^\rho$-function as $f \in \mathcal{C}^{1+\rho}(X, \mathbf{K}^e) \subseteq \mathcal{C}^\rho(X, \mathbf{K}^e)$ by Lemma 3.5.
If $B = (b_j) \in \mathrm{M}_{1 \times e}(\mathbf{E})$ and $A = (a_i) \in \mathrm{M}_{e \times 1}(\mathbf{K})$ are matrices whose coordinate entries are $\mathcal{C}^\rho$-functions on $X^{[e_k]}$ into $\mathbf{E} \supseteq \mathbf{K}$, then their matrix product $C = B \cdot A : X^{[e_k]} \to \mathbf{E}$ will be again a $\mathcal{C}^\rho$-function: For this, note that $C = a_1 b_1 + \cdots + a_e b_e$. By Proposition 1.7(ii), this sum of products is again a $\mathcal{C}^\rho$-function. Therefore with $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x))$ and $f^{[e_k]}(x;t)$, so is their matrix-product $g \circ f^{[e_k]}(x;t)$ a $\mathcal{C}^\rho$-function.
If $\nu > 1$, then we just saw $g \circ f \in \mathcal{C}^1(X, \mathbf{E})$ and by Lemma 3.19, we must prove $g \circ f^{[e_k]}$ to be a $\mathcal{C}^{r-1}$-function for $k = 1, \dots, d$. By Lemma 3.20 holds

$$g \circ f^{[e_k]}(x;t) = g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) \cdot f^{[e_k]}(x;t). \tag{$*$}$$

We have $g^{[1]} = (g^{[e_1]} \circ p_1, \ldots, g^{[e_e]} \circ p_e)$ with projection functions $p_l : Y^{[1]} \to Y^{[e_l]}$ for $l = 1, \ldots, e$ and $g^{[e_l]} \in \mathcal{C}^{r-1}(X^{[e_l]}, \mathbf{E})$ by Lemma 3.19. By the induction hypothesis, $g^{[e_l]} \circ p_l \in \mathcal{C}^{r-1}(Y^{[1]}, \mathbf{E})$ and hence $g^{[1]} \in \mathcal{C}^{r-1}(Y^{[1]}, \mathbf{E}^e)$. Moreover $f \in \mathcal{C}^r(X, \mathbf{K}^e) \subseteq \mathcal{C}^{r-1}(X, \mathbf{K}^e)$. Again by the induction hypothesis, we find $g^{[1]}(f(x + t \cdot \boldsymbol{e}_k), f(x)) : X^{[e_k]} \to \mathbf{E}^e$ to be a $\mathcal{C}^{r-1}$-function. By Lemma 3.19 holds $f^{[e_k]}(x;t) \in \mathcal{C}^{r-1}(X^{[e_k]}, \mathbf{K}^e)$. By Proposition 3.13, we find their matrix product $(*)$ to be a $\mathcal{C}^{r-1}$-function. Hence $g \circ f^{[e_k]}$ is a $\mathcal{C}^{r-1}$-function. ■

Corollary 3.22. *Let $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ be nonempty cartesian subsets whose factors contain no isolated point. For $r \geq 1$, let $f \in \mathcal{C}^r(X, \mathbf{K}^e)$ and $g \in \mathcal{C}^r(Y, \mathbf{E})$ be two functions with $\operatorname{im} f \subseteq Y$. Then $g \circ f \in \mathcal{C}^r(X, \mathbf{E})$.*

Proof. By Lemma 3.19, we find that $f^{]e_k[}$ and $g^{]e_k[}$ extend to $f^{[e_k]} \in \mathcal{C}^{r-1}(X^{[e_k]}, \mathbf{K})$ and $g^{[e_k]} \in \mathcal{C}^{r-1}(Y^{[e_k]}, \mathbf{E})$ for $k = 1, \ldots, d$. By the same token, $f \circ g \in \mathcal{C}^{r-1}(X, \mathbf{E})$ if and only if $g \circ f^{]e_k[} : X^{]e_k[} \to \mathbf{E}$ extends to a $\mathcal{C}^{r-1}$-function $g \circ f^{[e_k]} : X^{[e_k]} \to \mathbf{E}$ for $k = 1, \ldots, d$. We can conclude by Lemma 3.21. ■

Proposition 3.23. *Let $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ be nonempty locally cartesian subsets whose local factors contain no isolated point. Let $f : X \to \mathbf{K}^e$ and $g : Y \to \mathbf{E}$ be two functions with $\operatorname{im} f \subseteq Y$. Let r be a nonnegative real number. If $r \geq 1$ and f and g are both $\mathcal{C}^r$-functions, so will be their composition $g \circ f : X \to \mathbf{E}$. If $r < 1$, then the same will hold true provided either f or g is locally Lipschitzian.*

Proof. Foremost if $r < 1$, this will hold by Proposition 1.7(i). In case $r \geq 1$, we can by assumption cover Y by nonempty open cartesian subsets $V \subseteq Y$ whose factors are free of isolated points. Since g is in particular continuous, their preimages $\tilde{U} \subseteq X$ are again open. By assumption on X, we can find a nonempty open cartesian $U \subseteq \tilde{U}$ whose factors contain no isolated point. Such U covering X and since being $\mathcal{C}^r$ is a local property, we can restrict to the case X and Y being nonempty cartesian with factors free of isolated points. In this case, Corollary 3.22 yields the result. ■

Proposition 3.24. *Let $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ be nonempty locally cartesian subsets whose local factors contain no isolated point. Let r be a nonnegative real number. Let $f : X \to Y$ be either of class $\mathcal{C}^r$ if $r \geq 1$ or locally Lipschitzian if $r < 1$. Then the precomposition operator $\mathcal{C}^r(Y, \mathbf{E}) \ni g \mapsto g \circ f \in \mathcal{C}^r(X, \mathbf{E})$ is continuous.*

Proof. The mapping is well defined by Proposition 3.23. Since the norms are defined on compact cartesian subsets inside nonempty open cartesian subsets whose factors contain no isolated point, we can reduce to the case that $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ are nonempty cartesian subsets whose factors contain no isolated point. Let $C \subseteq X$ be compact cartesian. Then $f(C) \subseteq Y$ is again compact and we let $D \supseteq f(C)$ be compact cartesian in Y, which exists as Y is cartesian. Then in any case $\|g \circ f\|_C \leq \|g\|_D$.
Foremost if $r = \rho < 1$, then $f \in \mathcal{C}^{\text{lip}}(X, Y)$ and by Lemma 1.38 will hold

$$\|g \circ f\|_{\mathcal{C}^\rho, C} \leq (\|f\|^\rho_{\mathcal{C}^{\text{lip}}, C} \vee 1) \cdot \|g\|_{\mathcal{C}^\rho, D},$$

proving continuity in case $r < 1$. If $r \geq 1$, we will prove by induction on $\nu \geq 1$ that $\|g \circ f\|_{\mathcal{C}^r, C} \leq M \cdot \|g\|_{\mathcal{C}^r, D}$ for a constant $M = M(f, C, r) \geq 1$ depending solely on $r \geq 0$,

$C \subseteq X$ and $f \in \mathcal{C}^r(X,Y)$. First off, we find by Lemma 3.20 that

$$g \circ f^{[e_k]}(x;t) = g^{[1]}(f(x+t\cdot \boldsymbol{e}_k), f(x)) \cdot f^{[e_k]}(x;t) \quad \text{for } k=1,\dots,d.$$

We assume for convenience the operator norm of the multiplication mapping in $\mathbf{E}$ to be equal to 1. Defining $F^{[e_k]} \in \mathcal{C}^r(X^{[e_k]}, Y^{[1]})$ by $(x;t) \mapsto (f(x+t\cdot\boldsymbol{e}_k), f(x))$, it therefore holds by the proof of the continuity of multiplication in Proposition 3.13 that

$$\max_{k=1,\dots,d} \|[g\circ f]^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \leq \max_{k=1,\dots,d} \|g^{[1]} \circ F^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \|f^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \tag{$*$}$$

We also have by Lemma 3.19, for a general cartesian subset $X \subseteq \mathbf{K}^d$ with factors free of isolated points and $C \subseteq X$ compact cartesian

$$\|h\|_{\mathcal{C}^r,C} = \|h\|_C \vee \max_{k=1,\dots,d} \|h^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \quad \text{for any } h \in \mathcal{C}^r(X,\mathbf{E}). \tag{$**$}$$

We can now turn to the starting case $\nu = 1$, i.e. $r = 1+\rho$. We compute

$$\begin{aligned}
\|g\circ f\|_{\mathcal{C}^r,C} =& \|g\circ f\|_C \vee \max_{k=1,\dots,d} \|[g\circ f]^{[e_k]}\|_{\mathcal{C}^\rho,C^{[e_k]}} \\
\leq& \|g\|_D \vee \max_{k=1,\dots,d} \|g^{[1]} \circ F^{[e_k]}\|_{\mathcal{C}^\rho,C^{[e_k]}} \|f^{[e_k]}\|_{\mathcal{C}^\rho,C^{[e_k]}} \\
\leq& \|g\|_D \vee [\max_{k=1,\dots,d} (\|F^{[e_k]}\|^\rho_{\mathcal{C}^{\text{lip}},C^{[e_k]}} \vee 1)] \cdot \|g^{[1]}\|_{\mathcal{C}^\rho,D^{[1]}} \|f\|_{\mathcal{C}^r,C} \\
=& \|g\|_D \vee [\max_{k=1,\dots,d} M(F^{[e_k]}, C^{[e_k]}, \rho)] \|f\|_{\mathcal{C}^r,C} \cdot \|g^{[1]}\|_{\mathcal{C}^\rho,D^{[1]}} \\
\leq& M \cdot \|g\|_{\mathcal{C}^r,D};
\end{aligned}$$

where we put $M(F^{[e_k]}, C^{[e_k]}, \rho) := \|F^{[e_k]}\|^\rho_{\mathcal{C}^{\text{lip}},C^{[e_k]}} \vee 1 \geq 1$ for $k=1,\dots,d$ and accordingly $M := 1 \vee [\max_{k=1,\dots,d} M(F^{[e_k]}, C^{[e_k]}, \rho)] \cdot \|f\|_{\mathcal{C}^r,C} \geq 1$.
Here the first equality by definition, the following inequality by Inequality $(*)$ and the next one by the case $r = \rho < 1$ just observed (as well as $\|f^{[e_k]}\|_{\mathcal{C}^\rho,C^{[e_k]}} \leq \|f\|_{\mathcal{C}^r,C}$ for $k=1,\dots,d$ by definition).
Finally the last inequality follows through Proposition 1.34 by $\|g^{[1]}\|_{\mathcal{C}^\rho,D^{[1]}} = \|g^{[e_1]}\|_{\mathcal{C}^\rho,D^{[e_1]}} \vee \dots \vee \|g^{[e_e]}\|_{\mathcal{C}^\rho,D^{[e_e]}} \leq \|g\|_{\mathcal{C}^r,D}$. This settles the case $\nu = 1$.

Let $\nu > 1$. Then we compute similarly

$$\begin{aligned}
\|g\circ f\|_{\mathcal{C}^r,C} =& \|g\circ f\|_C \vee \max_{k=1,\dots,d} \|[g\circ f]^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \\
\leq& \|g\|_D \vee \max_{k=1,\dots,d} \|g^{[1]} \circ F^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \|f^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}} \\
\leq& \|g\|_D \vee (\max_{k=1,\dots,d} \|g^{[1]} \circ F^{[e_k]}\|_{\mathcal{C}^{r-1},C^{[e_k]}}) \|f\|_{\mathcal{C}^r,C} \\
\leq& \|g\|_D \vee ([\max_{k=1,\dots,d} M(F^{[e_k]}, C^{[e_k]}, r-1)] \cdot \|f\|_{\mathcal{C}^r,C}) \|g^{[1]}\|_{\mathcal{C}^{r-1},D^{[1]}} \\
\leq& M \cdot \|g\|_{\mathcal{C}^r,D},
\end{aligned}$$

with $M := 1 \vee ([\max_{k=1,\ldots,d} M(F^{[e_k]}, C^{[e_k]}, r-1)] \cdot \|f\|_{\mathcal{C}^r,C}) \cdot \tilde{M} \geq 1$ and the constant $\tilde{M} \geq 1$ defined below. Here the first equality by Equality $(**)$, the following inequality by Inequality $(*)$ and the one thereafter by Equality $(**)$.
The penultimate inequality is obtained by the induction hypothesis for $\nu - 1$.
The last inequality follows by

$$\begin{aligned}\|g^{[1]}\|_{\mathcal{C}^{r-1},D^{[1]}} &= \max_{l=1,\ldots,e} \|g^{[e_l]} \circ p_l\|_{\mathcal{C}^{r-1},D^{[1]}} \\ &\leq \max_{l=1,\ldots,e} \tilde{M}(p_l, Y^{[1]}, r-1) \cdot \|g^{[e_l]}\|_{\mathcal{C}^{r-1},D^{[e_l]}} \leq \tilde{M} \cdot \|g\|_{\mathcal{C}^r,D}\end{aligned}$$

for projection functions $p_l : Y^{[1]} \to Y^{[e_l]}$ for $l = 1, \ldots, e$ and $\tilde{M}(p_l, Y^{[1]}, r-1) \geq 1$ given the induction hypothesis, and putting $\tilde{M} := \max_{l=1,\ldots,d} \tilde{M}(p_l, Y^{[1]}, r-1)$. The last inequality by Equality $(**)$. ■

Density of (locally) polynomial functions in $\mathcal{C}^r(X, \mathbf{K})$

Definition. Let $X \subseteq \mathbf{K}^d$ be a subset.

1. We will call a function $p : X \to \mathbf{K}$ of the form $p = \sum_{\boldsymbol{i} \in \mathbb{N}^d} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$, whose scalars $a_{\boldsymbol{i}} \in \mathbf{K}$ are all zero but for a finite number, a polynomial function.
2. We will call a function $f : X \to \mathbf{K}$ a **locally polynomial function of total degree at most** g if for every point $a \in X$, there exists a neighborhood $U \ni a$ such that $f_{|U} = p_{|U}$ for a polynomial function $p = \sum_{\boldsymbol{i} \in \mathbb{N}^d_{\leq g}} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$.

Remark 3.25. We remark that on open domains locally polynomial functions are in particular locally analytic. By Proposition 3.18, these are $\mathcal{C}^r$-functions for any $r \in \mathbb{R}_{\geq 0}$. Hence a fortiori locally polynomial functions defined on a general subset $X \subseteq \mathbf{K}$ lie in $\mathcal{C}^r(X, \mathbf{K})$ for any $r \in \mathbb{R}_{\geq 0}$.

Assumption. Throughout this subsection's paragraph on the density of polynomial functions, we will denote by $X \subseteq \mathbf{K}^d$ a nonempty compact cartesian subset whose factors contain no isolated point.

Lemma 3.26. *Let $f \in \mathcal{C}^n(X, \mathbf{E})$. Fix $\delta, \varepsilon > 0$. If for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ holds*

$$\|f^{[\boldsymbol{n}]}(x)\| \leq \varepsilon \quad \text{for all } x \in X^{[\boldsymbol{n}]}_{\leq \delta},$$

then for all $\boldsymbol{m} \in \mathbb{N}^d_{=n-1}$ will hold

$$\|f^{[\boldsymbol{m}]}(x) - f^{[\boldsymbol{m}]}(\vec{a})\| \leq \varepsilon \cdot \delta \quad \text{for all } x, \vec{a} \in X^{[\boldsymbol{m}]} \text{ with } \|x - \vec{a}\| \leq \delta.$$

Proof. Fix $\boldsymbol{m} \in \mathbb{N}^d_{=n-1}$ and let $k \in \{1, \ldots, d\}$. Then for all $x \in X^{[\boldsymbol{m}]}_{\leq \delta}$ and $t \in \mathbf{K}$ with ${}^k x_0 \in X_k$ and $|t| \leq \delta$ holds by assumption

$$\begin{aligned}&\|f^{[\boldsymbol{m}]}(-; {}^k x + t \cdot \boldsymbol{e}_1; -) - f^{[\boldsymbol{m}]}(-; {}^k x; -)\| \\ =&\|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(-; {}^k x_0 + t, {}^k x_0, {}^k x_1, \ldots, {}^k x_{m_k}; -) \cdot t\| \leq \varepsilon \cdot |t| \leq \varepsilon\delta. \qquad (*)\end{aligned}$$

By Lemma 3.15, we have

$$X^{[\boldsymbol{m}]}_{\leq\delta} = \bigcup_{a\in X} A^{[\boldsymbol{m}]} \tag{$**$}$$

with $A = \mathrm{B}_{\leq\delta}(a) \subseteq X$ for $a \in X$. The set $A^{[\boldsymbol{m}]}$ is symmetric in its A_k-coordinates for $k = 1,\dots,d$ and, as cartesian, also telescopic. By Lemma 1.19 for $\rho = 1$, Inequality $(*)$ for $k = 1,\dots,d$ implies $\|f^{[\boldsymbol{m}]}(x) - f^{[\boldsymbol{m}]}(y)\| \leq \varepsilon\|x-y\|$ for all $x, y \in A^{[\boldsymbol{m}]}$.
We notice that $\vec{a} \in X^{[\boldsymbol{m}]}_{\leq\delta}$ for any $a \in X$. Moreover, if $\|x - \vec{a}\| \leq \delta$ for $x \in X^{[\boldsymbol{m}]}$, then by the non-Archimedean triangle inequality $x \in A^{[\boldsymbol{m}]}$ with $A = \mathrm{B}_{\leq\delta}(a)$. By Equality $(**)$, we thus find for all $\vec{a}, x \in X^{[\boldsymbol{m}]}$ with $\|x - \vec{a}\| \leq \delta$ that

$$\|f^{[\boldsymbol{m}]}(x) - f^{[\boldsymbol{m}]}(\vec{a})\| \leq \varepsilon\|x - \vec{a}\| \leq \varepsilon\cdot\delta.$$

■

Lemma 3.27. *Let $f \in \mathcal{C}^n(X, \mathbf{K})$ and $\boldsymbol{n} \in \mathbb{N}^d_{=n}$. Fix $\delta, \varepsilon > 0$. If*

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \quad \text{for all } x, \vec{a} \in X^{[\boldsymbol{n}]} \text{ with } \|x - \vec{a}\| \leq \delta,$$

then there will exist δ-constant $g : X \to \mathbf{K}$ such that $\tilde{f} := f - g*^{\boldsymbol{n}}$ satisfies

$$\|\tilde{f}^{[\boldsymbol{n}]}(x)\|_{X^{[\boldsymbol{n}]}_{\leq\delta}} \leq \varepsilon.$$

Proof. For all $x, \vec{a} \in X^{[\boldsymbol{n}]}$, we have by assumption

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \quad \text{if } \|x - \vec{a}\| \leq \delta.$$

In particular for all $a, b \in X$, we have $|\mathrm{D}_{\boldsymbol{n}} f(a) - \mathrm{D}_{\boldsymbol{n}} f(b)| \leq \varepsilon$ if $\|a - b\| \leq \delta$. By Lemma 1.11, there exists δ-constant $g : X \to \mathbf{K}$ such that $\|D_{\boldsymbol{n}} f - g\|_{\sup} \leq \varepsilon$. By Corollary 3.17(ii) we find $D_{\boldsymbol{n}}(g*^{\boldsymbol{n}}) = g$. Hence $\tilde{f} := f - g*^{\boldsymbol{n}}$ satisfies

$$\|\mathrm{D}_{\boldsymbol{n}} \tilde{f}\|_{\sup} = \|\mathrm{D}_{\boldsymbol{n}} f - \mathrm{D}_{\boldsymbol{n}}(g*^{\boldsymbol{n}})\|_{\sup} = \|\mathrm{D}_{\boldsymbol{n}} f - g\|_{\sup} \leq \varepsilon.$$

Let $x, \vec{a} \in X^{[\boldsymbol{n}]}$. Then $\|x - \vec{a}\| \leq \delta$ implies

$$\begin{aligned}
|\tilde{f}^{[\boldsymbol{n}]}(x) - \tilde{f}^{[\boldsymbol{n}]}(\vec{a})| =& |(f - g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(x) - (f - g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(\vec{a})| \\
=& |(f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})) - ((g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(x) - (g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(\vec{a}))| \\
\leq& |f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \vee |(g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(x) - (g*^{\boldsymbol{n}})^{[\boldsymbol{n}]}(\vec{a})| \\
\leq& \varepsilon \vee |g(a) - g(a)| = \varepsilon;
\end{aligned}$$

the last equality by Corollary 3.17(ii) as $x \in \mathrm{B}_{\leq\delta}(a)^{[\boldsymbol{n}]}$. Since $\|\mathrm{D}_{\boldsymbol{n}} \tilde{f}\|_{\sup} \leq \varepsilon$, it follows

$$|\tilde{f}^{[\boldsymbol{n}]}(x)| \leq |\tilde{f}^{[\boldsymbol{n}]}(x) - \tilde{f}^{[\boldsymbol{n}]}(\vec{a})| \vee |\tilde{f}^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon.$$

By Corollary 3.17(ii), we find $X^{[\boldsymbol{n}]}_{\leq\delta} = \cup_{a\in X} A^{[\boldsymbol{n}]}$ with $A := \mathrm{B}_{\leq\delta}(a) \subseteq X$. Hence $|\tilde{f}^{[\boldsymbol{n}]}(x)| \leq \varepsilon$ for all $x \in X^{[\boldsymbol{n}]}_{\leq\delta}$. ■

Lemma 3.28. *Let $f \in \mathcal{C}^\nu(X, \mathbf{E})$ and $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. For $k = 1, \dots, d$, we define the function $|f^{[\boldsymbol{n}+\rho\cdot \boldsymbol{e}_k]}| : X^{[\boldsymbol{n}+\boldsymbol{e}_k]} \to \mathbb{R}_{\geq 0}$ by*

$$\begin{aligned} &|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-; {}^k\tilde{x}_0, {}^kx_0, {}^kx_1, \dots, {}^kx_{n_k}; -) \\ :=&\|f^{[\boldsymbol{n}]}(-; {}^k\tilde{x}_0, {}^kx_1, \dots, {}^kx_{n_k}; -) - f^{[\boldsymbol{n}]}(-; {}^kx_0, {}^kx_1, \dots, {}^kx_{n_k}; -)\|/|{}^k\tilde{x}_0 - {}^kx_0|^\rho \end{aligned}$$

if ${}^k\tilde{x}_0 \neq {}^kx_0$ and zero otherwise; here the hyphenations to the left and right of the semicolons representing the omitted arguments ${}^1x; \dots ; {}^{k-1}x$ and ${}^{k+1}x; \dots ; {}^dx$. Then $f^{[\boldsymbol{n}]} \in \mathcal{C}^\rho(X^{[\boldsymbol{n}]}, \mathbf{E})$ implies $|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|$ to be a continuous function for $k = 1, \dots, d$ and it holds

$$\|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} = \|f^{[\boldsymbol{n}]}\|_{\sup} \vee \||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_1]}|\|_{\sup} \vee \dots \vee \||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|\|_{\sup}.$$

Proof. Recall $X^{[\boldsymbol{n}]} = X_1^{[n_1]} \times \cdots \times X_d^{[n_d]}$ and ${}^k\boldsymbol{e}_0 := (\boldsymbol{0}; \dots ; \boldsymbol{e}_0; \dots ; \boldsymbol{0}) \in \mathbb{N}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0 = (1, 0, \dots) \in \mathbb{N}^{[n_k]}$ with $N^{[n_k]} = \mathbb{N}^{\{0,\dots,n_k\}}$ at the k-th place. We view $\tilde{X} = X^{[\boldsymbol{n}]} \subseteq \mathbf{K}^{[\boldsymbol{n}]}$. Denote by

$$I_k = \{(k, 0), \dots, (k, n_k)\} = \{\ X_k\text{-coordinate indices of } X^{[\boldsymbol{n}]}\ \}.$$

Then the only nonzero entry of ${}^k\boldsymbol{e}_0$ is at the i_k-th coordinate for a representative $i_k \in I_k$. By Lemma 3.4(ii), the function $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{K}$ is symmetric in its coordinates indexed by $I_1, \dots, I_d$. By Corollary 1.33, we find

$$\|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} = \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{\rho\cdot{}^1\mathrm{e}_0}} \vee \dots \vee \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{\rho\cdot{}^d\mathrm{e}_0}}.$$

By Definition 1.21, we have $\|\tilde{f}\|_{\mathcal{C}^{\rho\cdot{}^k\mathrm{e}_0},\tilde{X}} = \|\tilde{f}\|_{\tilde{X}} \vee \||\tilde{f}^{[\rho\cdot{}^k\mathrm{e}_0]}|\|_{\tilde{X}^{[\rho\cdot{}^k\mathrm{e}_0]}}$ for $\tilde{X} := X^{[\boldsymbol{n}]}$ and $\tilde{f} := f^{[\boldsymbol{n}]} : \tilde{X} \to \mathbf{E}$. Now identifying $x = (-; {}^k\tilde{x}_0, {}^kx_0; {}^kx_1; -) \in \tilde{X}^{[{}^k\mathrm{e}_0]}$ with the element $x = (-; {}^k\tilde{x}_0, {}^kx_0, {}^kx_1, \cdots ; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$, it holds

$$|\tilde{f}^{[\rho\cdot{}^k\mathrm{e}_0]}|(x) = |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x).$$

Therefore $\|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} = \|f^{[\boldsymbol{n}]}\|_{\sup} \vee \||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_0]}|\|_{\sup} \vee \dots \vee \||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|\|_{\sup}$. ∎

Lemma 3.29. *Given $f \in \mathcal{C}^r(X, \mathbf{E})$, let $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $k \in \{1, \dots, d\}$. If $x \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $|{}^lx_i - {}^lx_j| > \delta$ for some coordinate $l \in \{1, \dots, d\}$ and $i, j \in \{0, \dots, n_l + \delta_{kl}\}$, then for all $\tilde{\boldsymbol{n}} \in \mathbb{N}^d_{=\nu}$, we have*

$$|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x) < \|f^{[\tilde{\boldsymbol{n}}]}\|_{\sup}/\delta^\rho.$$

Proof. We distinguish three cases in ascending generality.
Case 1: $|{}^kx_0 - {}^kx_1| > \delta$. Then by definition

$$\begin{aligned} &|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x) \\ =&|f^{[\boldsymbol{n}]}(-; {}^kx_0, {}^kx_2, \dots, {}^kx_{n_k+1}; -) - f^{[\boldsymbol{n}]}(-; {}^kx_1, {}^kx_2, \dots, {}^kx_{n_k+1}; -)|/|{}^kx_0 - {}^kx_1|^\rho \\ <&\|f^{[\boldsymbol{n}]}\|_{\sup}/\delta^\rho. \end{aligned}$$

Case 2: $|{}^lx_0 - {}^lx_1| > \delta$ for some $l \in \{1, \dots, d\}$. If this holds for $l = k$, we will be in Case 1. If this only holds for $l \neq k$, then we can write $\boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_l$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu-1}$ and we will assume w.l.o.g. $l < k$. Let $(-; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{n_k}; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $s \in \mathbf{K}$, and put $x = (-; {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{n_k}; -) \in X^{[\boldsymbol{n}]}$. Then

$$\begin{aligned}
&|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{n_k}; -) \\
=&|f^{[\boldsymbol{n}]}(-; {}^k\!x_0 + s, {}^k\!x_1, \dots, {}^k\!x_{n_k}; -) - f^{[\boldsymbol{n}]}(-; {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{n_k}; -)|/|s|^\rho \\
=&|f^{[\boldsymbol{n}]}(x + s\cdot {}^k\boldsymbol{e}_0) - f^{[\boldsymbol{n}]}(x)|/|s|^\rho
\end{aligned}$$

with ${}^k\boldsymbol{e}_0 := (\mathbf{0}; \dots; \boldsymbol{e}_0; \dots; \mathbf{0}) \in \mathbf{K}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0 = (1, 0, \dots) \in \mathbf{K}^{[n_k]} = \mathbf{K}^{\{0,\dots,n_k\}}$ at the k-th place. Let $(-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -) \in X^{[\boldsymbol{n}]}$ with $t \in \mathbf{K}$. Put $x = (-; {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -) \in X^{[\boldsymbol{m}]}$ and $\tilde{f} := f^{[\boldsymbol{m}]}$. Then by definition, we have

$$\begin{aligned}
&f^{[\boldsymbol{n}]}(-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -) \\
=&f^{[\boldsymbol{m}+\boldsymbol{e}_l]}(-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -) \\
=&[f^{[\boldsymbol{m}]}(-; {}^l\!x_0 + t, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -) - f^{[\boldsymbol{m}]}(-; {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -)]/t \\
=&[\tilde{f}(x + t\cdot {}^l\boldsymbol{e}_0) - \tilde{f}(x)]/t.
\end{aligned}$$

Let $(-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{m_k}; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $t, s \in \mathbf{K}$, and put as before $x = (-; {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -; {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{m_k}; -) \in X^{[\boldsymbol{m}]}$. Combining both obtained equalities, we infer

$$\begin{aligned}
&|f^{[\boldsymbol{m}+\boldsymbol{e}_l+\rho\cdot\boldsymbol{e}_k]}|(-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{m_k}; -) \\
=&|[\tilde{f}(x + t\cdot {}^l\boldsymbol{e}_0 + s\cdot {}^k\boldsymbol{e}_0) - \tilde{f}(x + s\cdot {}^k\boldsymbol{e}_0)] - [\tilde{f}(x + t\cdot {}^l\boldsymbol{e}_0) - \tilde{f}(x)]|/|t||s|^\rho \\
\leq&[|\tilde{f}(x + t\cdot {}^l\boldsymbol{e}_0 + s\cdot {}^k\boldsymbol{e}_0) - \tilde{f}(x + t\cdot {}^l\boldsymbol{e}_0)|/|s|^\rho \vee |\tilde{f}(x + s\cdot {}^k\boldsymbol{e}_0) - \tilde{f}(x)|/|s|^\rho]/|t|. \qquad (*)
\end{aligned}$$

We notice that for $x + s\cdot {}^k\boldsymbol{e}_0, x \in X^{[\boldsymbol{m}]}$ holds

$$\begin{aligned}
|\tilde{f}(x + s\cdot {}^k\boldsymbol{e}_0) - \tilde{f}(x)|/|s|^\rho =&|[\tilde{f}(x + s\cdot {}^k\boldsymbol{e}_0) - \tilde{f}(x)]/s||s|^{1-\rho} \\
=&|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(-; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{m_k}; -)||s|^{1-\rho}. \qquad (**)
\end{aligned}$$

Let $x = (-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{m_k}; -) \in X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ with $|t| > \delta$ for $l \in \{1, \dots, d\}$. By Case 1, we may assume $|s| \leq \delta$. Then under these assumptions $|s| \leq \delta < |t|$, so Inequalities $(*)$ and $(**)$ yield

$$\begin{aligned}
&|f^{[\boldsymbol{m}+\boldsymbol{e}_l+\rho\cdot\boldsymbol{e}_k]}|(-; {}^l\!x_0 + t, {}^l\!x_0, {}^l\!x_1, \dots, {}^l\!x_{m_l}; -; {}^k\!x_0 + s, {}^k\!x_0, {}^k\!x_1, \dots, {}^k\!x_{m_k}; -) \\
<&\|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}\|_{\sup} \cdot \delta^{1-\rho}/\delta = \|f^{[\tilde{\boldsymbol{n}}]}\|_{\sup}/\delta^\rho.
\end{aligned}$$

Case 3: $|{}^l\!x_i - {}^l\!x_j| > \delta$ for some $l \in \{1, \dots, d\}$ and $i, j \in \{0, \dots, n_l\}$. We want to reduce to the second case.
Case 3.1: If $l \neq k$, then by the symmetry of $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{K}$ in its X_l-coordinates, we may assume $i, j = 0, 1$ and the result follows by Case 2.
Case 3.2: If $l = k$, we may assume that $|{}^k\!x_0 - {}^k\!x_1| \leq \delta < |{}^k\!x_i - {}^k\!x_j|$ as otherwise the result will follow by Case 2. Let σ be the permutation on $X_k^{[n_k+1]} = X_k^{\{0,\dots,n_k+1\}}$ swapping the i-th and j-th coordinate with the first and second one. We notice that by definition,

$$|f_{\boldsymbol{x}}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(x) = |f^{[\boldsymbol{n}+\boldsymbol{e}_k]}(x)||{}^k\!x_0 - {}^k\!x_1|^{1-\rho}.$$

Then (as in the proof of Lemma 2.12), by symmetry of $f^{[\boldsymbol{n}+e_k]}$ in its X_k-coordinates, we find

$$\begin{aligned}|f_{\boldsymbol{x}}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^k x^\sigma;-) &= |f^{[\boldsymbol{n}+\boldsymbol{e}_k]}(-;{}^k x^\sigma;-)||{}^k x_i - {}^k x_j|^{1-\rho} \\ &= |f^{[\boldsymbol{n}+\boldsymbol{e}_k]}(-;{}^k x;-)||{}^k x_0 - {}^k x_1|^{1-\rho}\frac{|{}^k x_i - {}^k x_j|^{1-\rho}}{|{}^k x_0 - {}^k x_1|^{1-\rho}} \\ &= |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}(-;{}^k x;-)|\frac{|{}^k x_i - {}^k x_j|^{1-\rho}}{|{}^k x_0 - {}^k x_1|^{1-\rho}} \\ &= |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}(-;{}^k x;-)|\left|\frac{{}^k x_0^\sigma - {}^k x_1^\sigma}{{}^k x_0 - {}^k x_1}\right|^{1-\rho}.\end{aligned}$$

Because $|{}^k x_0 - {}^k x_1| < |{}^k x_0^\sigma - {}^k x_1^\sigma|$, we obtain $|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^k x;-) < |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|(-;{}^k x^\sigma;-)$. We can therefore conclude

$$\begin{aligned}|f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|({}^1 x;-;{}^{k-1}x;{}^k x;{}^{k+1}x;-;{}^d x) &< |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|({}^1 x;-;{}^{k-1}x;{}^k x^\sigma;{}^{k+1}x;-;{}^d x) \\ &\le \|f^{[\boldsymbol{n}]}\|_{\sup}/\delta^\rho;\end{aligned}$$

the last inequality by Case 1 as $|{}^k x_0^\sigma - {}^k x_1^\sigma| = |{}^k x_i - {}^k x_j| > \delta$. ■

Proposition 3.30 (the case d>1)**.** *The locally polynomial functions of total degree at most ν are dense in $\mathcal{C}^r(X,\mathbf{K})$.*

Proof. Fix $\varepsilon > 0$ and $f \in \mathcal{C}^r(X,\mathbf{K})$. Then $f^{[\boldsymbol{n}]} \in \mathcal{C}^\rho(X^{[\boldsymbol{n}]},\mathbf{K})$. By compactness of X, there exists by Proposition 3.8 some $0 < \delta \le 1$ such that for all $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $x, y \in X^{[\boldsymbol{n}]}$ with $\|x - y\| \le \delta$ holds

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(y)| \le \varepsilon \cdot \|x-y\|^\rho. \qquad (*)$$

We will fix this $\delta > 0$ for the rest of the proof and recall $X^{[\boldsymbol{n}]}_{\le\delta} := \{x = ({}^1 x;-;{}^d x) \in X^{[\boldsymbol{n}]} \text{ with } \delta\,{}^1 x,\dots,\delta\,{}^d x \le \delta\}$ for $\boldsymbol{n} \in \mathbb{N}^d_{\le\nu}$.

Step 1.: By downward induction on $n = \nu,\dots,0$, we will construct δ-constant functions $g_{\boldsymbol{i}} : X \to \mathbf{K}$ for $\boldsymbol{i} \in \mathbb{N}^d$ with $n \le |\boldsymbol{i}| \le \nu$ such that $f_n = f - \sum_{n\le|\boldsymbol{i}|\le\nu} g_{\boldsymbol{i}} *^{\boldsymbol{i}}$ for all $\boldsymbol{n} \in \mathbb{N}^d_{=n}$ satisfies

$$|f_n^{[\boldsymbol{n}]}(x)| \le \varepsilon\delta^{r-n} \quad \text{for all } x \in X^{[\boldsymbol{n}]}_{\le\delta}. \qquad (**)$$

Let $n = \nu$ and $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. By Inequality $(*)$, in particular for all $x, \vec{a} \in X^{[\boldsymbol{n}]}$ with $\|x - \vec{a}\| \le \delta$, it holds

$$|f^{[\boldsymbol{n}]}(x) - f^{[\boldsymbol{n}]}(\vec{a})| \le \varepsilon \cdot \delta^\rho.$$

By Lemma 3.27, we find δ-constant $g_{\boldsymbol{n}}$ such that $f_{\boldsymbol{n}} = f - g_{\boldsymbol{n}} *^{\boldsymbol{n}}$ satisfies

$$|f_{\boldsymbol{n}}^{[\boldsymbol{n}]}(x)| \le \varepsilon\delta^\rho \quad \text{for all } x \in X^{[\boldsymbol{n}]}_{\le\delta}.$$

Then we put $f_\nu := f - \sum_{\boldsymbol{n}\in\mathbb{N}^d_{=\nu}} g_{\boldsymbol{n}} *^{\boldsymbol{n}}$. We will prove Inequality $(**)$ for fixed $\boldsymbol{n}_0 \in \mathbb{N}^d_{=\nu}$. Let $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ be different from $\boldsymbol{n}_0$, so that in particular $\boldsymbol{n} \not\ge \boldsymbol{n}_0$. As $g_{\boldsymbol{n}} : X \to \mathbf{K}$ is δ-constant, we find by Corollary 3.17(ii) that $(g_{\boldsymbol{n}} *^{\boldsymbol{n}})^{[\boldsymbol{n}_0]} \equiv 0$ and thus $f_\nu^{[\boldsymbol{n}_0]} = f_{\boldsymbol{n}_0}^{[\boldsymbol{n}_0]}$ on $X_{\leq\delta}^{[\boldsymbol{n}_0]}$. Therefore by construction of $g_{\boldsymbol{n}} : X \to \mathbf{K}$, we obtain

$$|f_\nu^{[\boldsymbol{n}_0]}(x)| \leq \varepsilon\delta^\rho \quad \text{for all } \ x \in X_{\leq\delta}^{[\boldsymbol{n}_0]}.$$

Let $n < \nu$ and put $m = n + 1$. By induction hypothesis we have constructed δ-constant functions $g_{\boldsymbol{i}} : X \to \mathbf{K}$ for all $\boldsymbol{i} \in \mathbb{N}$ with $m \leq |\boldsymbol{i}| \leq \nu$ such that $f_m = f - \sum_{m \leq |\boldsymbol{i}| \leq \nu} g_{\boldsymbol{i}} *^{\boldsymbol{i}}$ for all $\boldsymbol{m} \in \mathbb{N}_{=m}^d$ satisfies

$$|f_m^{[\boldsymbol{m}]}(x)| \leq \varepsilon\delta^{r-m} \quad \text{for all } \ x \in X_{\leq\delta}^{[\boldsymbol{m}]}.$$

Let $\boldsymbol{n} \in \mathbb{N}_{=n}^d$. By Lemma 3.26, for all $x, \vec{a} \in X^{[\boldsymbol{n}]}$ with $\|x - \vec{a}\| \leq \delta$, we have

$$|f_m^{[\boldsymbol{n}]}(x) - f_m^{[\boldsymbol{n}]}(\vec{a})| \leq \varepsilon \cdot \delta^{r-m} \cdot \delta = \varepsilon \cdot \delta^{r-n}.$$

By Lemma 3.27, there exists δ-constant $g_{\boldsymbol{n}} : X \to \mathbf{K}$ such that $f_{\boldsymbol{n}} = f_m - g_{\boldsymbol{n}} *^{\boldsymbol{n}}$ satisfies

$$|f_{\boldsymbol{n}}^{[\boldsymbol{n}]}(x)| \leq \varepsilon \cdot \delta^{r-n} \quad \text{for all } \ x \in X_{\leq\delta}^{[\boldsymbol{n}]}.$$

Then we put $f_n := f_m - \sum_{\boldsymbol{n} \in \mathbb{N}_{=n}^d} g_{\boldsymbol{n}} *^{\boldsymbol{n}}$. We will prove Inequality $(**)$ for fixed $\boldsymbol{n}_0 \in \mathbb{N}_{=n}^d$. Let $\boldsymbol{n} \in \mathbb{N}_{=n}^d$ be different from $\boldsymbol{n}_0$, so that in particular $\boldsymbol{n} \not\geq \boldsymbol{n}_0$. As $g_{\boldsymbol{n}} : X \to \mathbf{K}$ is δ-constant, by Corollary 3.17(ii), we find $(g_{\boldsymbol{n}} *^{\boldsymbol{n}})^{[\boldsymbol{n}_0]} \equiv 0$ and thus $f_n^{[\boldsymbol{n}_0]} = f_{\boldsymbol{n}_0}^{[\boldsymbol{n}_0]}$ on $X_{\leq\delta}^{[\boldsymbol{n}_0]}$. Therefore by construction of $g_{\boldsymbol{n}} : X \to \mathbf{K}$, we obtain

$$|f_n^{[\boldsymbol{n}_0]}(x)| \leq \varepsilon\delta^\rho \quad \text{for all } \ x \in X_{\leq\delta}^{[\boldsymbol{n}_0]}.$$

This finishes the construction of the $g_{\boldsymbol{i}}$ for $\boldsymbol{i} \in \mathbb{N}_{\leq\nu}^d$.

Step 2.1.: We will prove by induction on $|\boldsymbol{n}| =: n = 0, \ldots, \nu$ that $\|f_0^{[\boldsymbol{n}]}\|_{\text{sup}} \leq \varepsilon\delta^{r-n}$ for $\boldsymbol{n} \in \mathbb{N}_{\leq\nu}^d$.

Let $n = 0$. Then $\delta\{{}^1x_0\} \vee \ldots \vee \delta\{{}^dx_0\} = 0 \leq \delta$ for all $({}^1x_0, \ldots, {}^dx_0) \in X$. Hence $|f_0^{[\mathbf{0}]}({}^1x_0, \ldots, {}^dx_0)| \leq \varepsilon\delta^r$ for all $({}^1x_0, \ldots, {}^dx_0) \in X$, i.e. $\|f_0^{[\mathbf{0}]}\|_{\text{sup}} \leq \varepsilon\delta^r$. Let $n \geq 1$. Then we split up

$$\|f_0^{[\boldsymbol{n}]}\|_{\text{sup}} = \|f_0^{[\boldsymbol{n}]}\|_{X_{\leq\delta}^{[\boldsymbol{n}]}} \vee \|f_0^{[\boldsymbol{n}]}\|_{\{x \in X^{[\boldsymbol{n}]} \text{ s.t. } |{}^kx_i - {}^kx_j| > \delta \text{ for some coordinate } k \text{ and } i,j\}}.$$

Ad $\|f_0^{[\boldsymbol{n}]}\|_{X_{\leq\delta}^{[\boldsymbol{n}]}} \leq \varepsilon\delta^{r-n}$:
Let $\boldsymbol{i} \in \mathbb{N}_{<n}^d$, so that in particular $\boldsymbol{i} \not\geq \boldsymbol{n}$. As $g_{\boldsymbol{i}} : X \to \mathbf{K}$ is δ-constant, by Corollary 3.17(ii), we find $(g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} \equiv 0$ and thus $f_0^{[\boldsymbol{n}]} = f_n^{[\boldsymbol{n}]}$ on $X_{\leq\delta}^{[\boldsymbol{n}]}$. Therefore restricted onto $X_{\leq\delta}^{[\boldsymbol{n}]}$, it holds

$$f_0^{[\boldsymbol{n}]} = (f - \sum_{\boldsymbol{i} \in \mathbb{N}_{\leq\nu}^d} g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} = (f - \sum_{n \leq |\boldsymbol{i}| \leq \nu} g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} = f_n^{[\boldsymbol{n}]}.$$

By construction of the $g_{\boldsymbol{i}} : X \to \mathbf{K}$ for $\boldsymbol{i} \in \mathbb{N}^d$ with $n \leq |\boldsymbol{i}| \leq \nu$, we have $\|f_n^{[\boldsymbol{n}]}\|_{X_{\leq\delta}^{[\boldsymbol{n}]}} \leq \varepsilon\delta^{r-n}$.

Ad $\|f_0^{[\boldsymbol{n}]}\|_{\{x\in X^{[\boldsymbol{n}]} \text{ s.t. } |^k x_i - ^k x_j|>\delta \text{ for some coordinate } k \text{ and } i,j\}} \leq \varepsilon\delta^{r-n}$:
Let $x \in X^{[\boldsymbol{n}]}$ with $|^k x_i - ^k x_j| > \delta$ for some coordinate $k \in \{1, \dots, d\}$ and $i, j \in \{0, \dots, n_k\}$. Assume we have shown that

$$|f_0^{[\boldsymbol{n}]}(x)| \leq \varepsilon\delta^{r-n} \quad \text{for all } \ x \in X^{[\boldsymbol{n}]} \text{ with } |^k x_0 - ^k x_1| > \delta.$$

Let σ be the permutation on $X_k^{[n_k]} = X_k^{\{0,\dots,n_k\}}$ swapping the i-th and j-th coordinate with the first and second one. Then by symmetry of $f_0^{[\boldsymbol{n}]}$ in its X_k-coordinates, we have

$$\begin{aligned}\varepsilon\delta^{r-n} &\geq |f_0^{[\boldsymbol{n}]}(-; {^k x^\sigma}; -)| \\ &= |f_0^{[\boldsymbol{n}]}(-; {^k x_i}, {^k x_j}, \dots, {^k x_{n_k}}; -)| \\ &= |f_0^{[\boldsymbol{n}]}(-; {^k x_0}, {^k x_1}, \dots, {^k x_{n_k}}; -)|;\end{aligned}$$

here the hyphenations to the left and right of the semicolons representing the omitted arguments ${^1 x}; \dots ; {^{k-1} x}$ and ${^{k+1} x}; \dots ; {^d x}$. Hence we are reduced to the case $|^k x_0 - ^k x_1| > \delta$. Since $n_k \geq 1$, we can write $\boldsymbol{n} = \boldsymbol{m} + \boldsymbol{e}_k$ for $\boldsymbol{m} \in \mathbb{N}^d_{=n-1}$. We compute

$$\begin{aligned}&|f^{[\boldsymbol{m}+\boldsymbol{e}_k]}(x)| \\ =&|({^k x_0} - {^k x_1})^{-1}[f_0^{[\boldsymbol{m}]}(-; {^k x_0}, {^k x_2}, \dots, {^k x_{n_k}}; -) - f_0^{[\boldsymbol{m}]}(-; {^k x_1}, {^k x_2}, \dots, {^k x_{n_k}}; -)]| \\ <&\delta^{-1} \cdot \|f_0^{[\boldsymbol{m}]}\|_{\sup} \leq \delta^{-1} \cdot \varepsilon\delta^{r-m} = \varepsilon\delta^{r-n};\end{aligned}$$

the last inequality by the induction hypothesis for $|\boldsymbol{m}| = n - 1$. This finishes the proof of $\|f_0^{[\boldsymbol{n}]}\|_{\sup} \leq \varepsilon\delta^{r-n}$ for $\boldsymbol{n} \in \mathbb{N}^d_{\leq\nu}$.

Step 2.2.: It remains to prove $\|f_0^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} \leq \varepsilon$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$. We have already proven $\|f_0^{[\boldsymbol{n}]}\|_{\sup} \leq \varepsilon\delta^\rho \leq \varepsilon$, so by Lemma 3.28, it remains to show $|||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\sup} \leq \varepsilon$ for $k = 1, \dots, d$. We split its domain $X^{[\boldsymbol{n}+\boldsymbol{e}_k]}$ up via

$$|||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\sup} = |||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{X^{[\boldsymbol{n}+\boldsymbol{e}_k]}_{\leq\delta}} \vee |||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{\{x\in X^{[\boldsymbol{n}+\boldsymbol{e}_k]} \text{ s.t. } |^k x_i - ^k x_j|>\delta \text{ for some } k \text{ and } i,j\}}.$$

Ad $|||f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}|||_{X^{[\boldsymbol{n}+\boldsymbol{e}_k]}_{\leq\delta}} \leq \varepsilon$:
Let $\boldsymbol{i} \in \mathbb{N}^d_{\leq\nu}$. As $g_{\boldsymbol{i}} : X \to \mathbf{K}$ is δ-constant, we find by Corollary 3.17(ii) that

$$(g_{\boldsymbol{i}} *^{\boldsymbol{i}})^{[\boldsymbol{n}]} \equiv \begin{cases} g(p), & \text{if } \boldsymbol{i} = \boldsymbol{n}, \\ 0, & \text{if } \boldsymbol{i} \neq \boldsymbol{n} \end{cases}$$

on $P^{[\boldsymbol{n}]}$ for a representative $p \in P := \mathrm{B}_{\leq\delta}(p) \subseteq X$. Therefore $|\tilde{g}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| \equiv 0$ on $P^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}$ with $\tilde{g} := g_{\boldsymbol{i}} *^{\boldsymbol{i}}$ for $\boldsymbol{i} \in \mathbb{N}^d_{\leq\nu}$. By Lemma 3.15, we find $X^{[\boldsymbol{n}]}_{\leq\delta} = \cup_{p\in X} P^{[\boldsymbol{n}]}$ with $P := \mathrm{B}_{\leq\delta}(p) \subseteq X$ for $p \in X$ and hence $|\tilde{g}^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| \equiv 0$ on $X^{[\boldsymbol{n}+\boldsymbol{e}_k]}_{\leq\delta}$. Therefore $|f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| = |f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}| \leq \varepsilon$ on $X^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}_{\leq\delta}$ by Inequality $(*)$.

Ad $\|f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}\|_{\{x \text{ s.t. } |^k x_i - ^k x_j|>\delta \text{ for some coordinate } k \text{ and } i,j\}} \leq \varepsilon$:
Then it holds

$$\|f_0^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_k]}\|_{\{x \text{ s.t. } |^k x_i - ^k x_j|>\delta \text{ for some coordinate } k \text{ and } i,j\}} \leq \|f_0^{[\tilde{\boldsymbol{n}}]}\|_{\sup}/\delta^\rho \leq \varepsilon\delta^\rho/\delta^\rho;$$

the first inequality for $\tilde{\boldsymbol{n}} \in \mathbb{N}^d_{=\nu}$ by Lemma 3.29 and the second one by Step 2.1. This completes the proof of $\|f_0^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} \leq \varepsilon$ for $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$.

Step 3.: Finally put $g := \sum_{\boldsymbol{i}\in\mathbb{N}^d_{\leq\nu}} g_{\boldsymbol{i}} *^{\boldsymbol{i}}$. Then g is a locally polynomial function of total degree at most ν and $f_0 = f - g$. Then $\|f-g\|_{\mathcal{C}^r} = \max_{\boldsymbol{n}\in\mathbb{N}^d_{<\nu}} \|f_0^{[\boldsymbol{n}]}\|_{\sup} \vee \max_{\boldsymbol{n}\in\mathbb{N}^d_{=\nu}} \|f_0^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} \leq \varepsilon$, q.e.d. ■

During the following proof we will use terminology introduced in the next subsection's paragraph about topological tensor products.

Lemma 3.31. *The closure of the set of all polynomial functions inside the* $\mathbf{K}$*-Banach space* $\mathcal{C}^r(X, \mathbf{K})$ *contains all locally constant functions.*

Proof. The proof is divided into two steps.

(i) Fix a characteristic function $\mathbf{1}_{\mathrm{B}} : X \to \mathbf{K}$ of a closed ball $\mathrm{B} \subseteq X$ of positive radius and $\varepsilon > 0$. Then there exists a polynomial function $p : X \to \mathbf{K}$ such that $\|\mathbf{1}_{\mathrm{B}} - p\|_{\mathcal{C}^r} \leq \varepsilon$.

(ii) The polynomial functions are dense in the locally constant functions inside $\mathcal{C}^r(X, \mathbf{K})$.

Ad (i): By Lemma 3.11 and Remark 3.39, it suffices to prove that there exists a polynomial function $p : X \to \mathbf{K}$ such that $\|\mathbf{1}_{\mathrm{B}} - p\|_{\mathcal{C}^{\vec{\nu}}} \leq \varepsilon$ for $\vec{\nu} := (\nu, \ldots, \nu) \in \mathbb{N}^d$ with $\nu \geq r$. This is done by induction on $d \geq 1$. If $d = 1$, then this will be taken care of by [Araujo and Schikhof, 1993, Corollary 1.3]. Let $d > 1$. Let $\mathrm{B} = \mathrm{B}' \times \mathrm{B}'' \subseteq X$ with $\mathrm{B}' = \mathrm{B}_1 \times \cdots \times \mathrm{B}_{d-1} \subseteq X_1 \times \cdots \times X_{d-1} =: X'$ and $\mathrm{B}'' := \mathrm{B}_d \subseteq X_d =: X''$. By induction, there exists a polynomial function $p' : X' \to \mathbf{K}$ with $\|\mathbf{1}_{\mathrm{B}'} - p'\|_{\mathcal{C}^{\vec{\nu}}} \cdot M'' \leq \varepsilon$ with $M'' = \|\mathbf{1}_{\mathrm{B}''}\|_{\mathcal{C}^\nu} \geq 0$. Then by the case $d = 1$, there exists a polynomial function $p'' : X'' \to \mathbf{K}$ with $\|\mathbf{1}_{\mathrm{B}''} - p''\|_{\mathcal{C}^\nu} \cdot M' \leq \varepsilon$ with $M' = \|p'\|_{\mathcal{C}^{\vec{\nu}}} \geq 0$. We put $p := p' \odot p'' : X \to \mathbf{K}$ and compute, using first the bilinearity and then the norm preservation, both stated Property 1 in Lemma 3.40,

$$\begin{aligned}
\|\mathbf{1}_{\mathrm{B}} - p\|_{\mathcal{C}^{\vec{\nu}}} &= \|\mathbf{1}_{\mathrm{B}'} \odot \mathbf{1}_{\mathrm{B}''} - p' \odot p''\|_{\mathcal{C}^{\vec{\nu}}} \\
&\leq \|\mathbf{1}_{\mathrm{B}'} \odot \mathbf{1}_{\mathrm{B}''} - p' \odot \mathbf{1}_{\mathrm{B}''}\|_{\mathcal{C}^{\vec{\nu}}} \vee \|p' \odot \mathbf{1}_{\mathrm{B}''} - p' \odot p''\|_{\mathcal{C}^{\vec{\nu}}} \\
&= \|(\mathbf{1}_{\mathrm{B}'} - p') \odot \mathbf{1}_{\mathrm{B}''}\|_{\mathcal{C}^{\vec{\nu}}} \vee \|p' \odot (\mathbf{1}_{\mathrm{B}''} - p'')\|_{\mathcal{C}^{\vec{\nu}}} \\
&= \|\mathbf{1}_{\mathrm{B}'} - p'\|_{\mathcal{C}^{\vec{\nu}}} \cdot \|\mathbf{1}_{\mathrm{B}''}\|_{\mathcal{C}^\nu} \vee \|p'\|_{\mathcal{C}^{\vec{\nu}}} \cdot \|\mathbf{1}_{\mathrm{B}''} - p''\|_{\mathcal{C}^\nu} \leq \varepsilon.
\end{aligned}$$

Ad (ii): The closed balls $\mathrm{B} \subseteq X$ constitute a base of the topological space $X \subseteq \mathbf{K}^d$. Hence by compactness of X, every locally constant function g is the finite sum $f = \sum_i \lambda_i \mathbf{1}_{\mathrm{B}_i}$ with $\lambda_i \in \mathbf{K}$ and characteristic functions $\mathbf{1}_{\mathrm{B}_i}$ of closed balls $\mathrm{B}_i \subseteq X$ for $i \in I$. By (i), for every $\varepsilon > 0$, there exist polynomial functions $p_i : X \to \mathbf{K}$ such that $\|p_i - \mathbf{1}_{\mathrm{B}_i}\|_{\mathcal{C}^r} M_i \leq \varepsilon$ with $M_i := |\lambda_i| \geq 0$. Then $p := \sum_i \lambda_i p_i : X \to \mathbf{K}$ satisfies $\|p - f\|_{\mathcal{C}^r} \leq \max_i |\lambda_i| \|p_i - \mathbf{1}_{\mathrm{B}_i}\|_{\mathcal{C}^r} \leq \varepsilon$. ■

Corollary 3.32. *The polynomial functions are dense in* $\mathcal{C}^r(X, \mathbf{K})$.

Proof. Fix $f \in \mathcal{C}^r(X, \mathbf{K})$ and $\varepsilon > 0$. By Proposition 3.30, there exists a locally polynomial function $g = \sum_{\boldsymbol{i} \in \mathbb{N}^d_{\leq \nu}} g_{\boldsymbol{i}} *^{\boldsymbol{i}} : X \to \mathbf{K}$ with locally constant $g_{\boldsymbol{i}}$ such that $\|f - g\|_{\mathcal{C}^r} \leq \varepsilon$. By Lemma 3.31, there exist polynomial functions $p_{\boldsymbol{i}} : X \to \mathbf{K}$ with $\|p_{\boldsymbol{i}} - g_{\boldsymbol{i}}\| \cdot M_{\boldsymbol{i}} \leq \varepsilon$ with $M_{\boldsymbol{i}} = \|*^{\boldsymbol{i}}\|_{\mathcal{C}^r} > 0$ for all $\boldsymbol{i} \in \mathbb{N}^d_{\leq \nu}$. Then the polynomial function $p := \sum_{\boldsymbol{i} \in \mathbb{N}^d_{\leq \nu}} p_{\boldsymbol{i}} *^{\boldsymbol{i}} : X \to \mathbf{K}$ satisfies

$$\|p - g\|_{\mathcal{C}^r} \leq \max_{\boldsymbol{i} \in \mathbb{N}^d_{\leq \nu}} \|p_{\boldsymbol{i}} - g_{\boldsymbol{i}}\|_{\mathcal{C}^r} \cdot \|*^{\boldsymbol{i}}\|_{\mathcal{C}^r} \leq \varepsilon$$

and therefore $\|p - f\|_{\mathcal{C}^r} \leq \|p - g\|_{\mathcal{C}^r} \vee \|g - f\|_{\mathcal{C}^r} \leq \varepsilon$. ■

3.2 Orthogonal bases of $\mathcal{C}^r$-functions on a compact domain

Interlude: Orthogonal bases of $\mathbf{K}$-Banach spaces

Given a topological Hausdorff abelian group X, recall that a series $\sum_{i \in I} x_i$ over an arbitrary index set I is defined as the unique element $x \in X$ such that for every neighborhood $U \ni x$ in X, there exists a finite subset $F \subseteq I$ such that $\sum_{i \in F} x_i \in U$.

Definition. (i) For a sequence $(w_i)_{i \in I}$ of weights in $\mathbb{R}_{>0}$, define the $\mathbf{K}$-Banach space of **weighted zero sequences with respect to** (w_i) by

$$\begin{aligned} c_0((w_i)_{i \in I}) := \{&\text{all sequences } (\lambda_i) \text{ in } \mathbf{K} \text{ such that, for any} \\ &\varepsilon > 0, \text{ only finitely often } |\lambda_i| w_i \geq \varepsilon \text{ for } i \in I\} \end{aligned}$$

with the maximum-norm

$$\|(\lambda_i)\| := \max_{i \in I} |\lambda_i| w_i.$$

(ii) Given a $\mathbf{K}$-Banach space $\mathbf{E}$, we will call the subset $\{e_i\} \subseteq \mathbf{E}$ an **orthogonal basis** if the following map is an (isometric) isomorphism of $\mathbf{K}$-Banach spaces:

$$\begin{aligned} c_0((w_i)_{i \in I}) &\to \mathbf{E} \\ (\lambda_i) &\mapsto \sum_i \lambda_i e_i, \end{aligned}$$

were $w_i := \|e_i\|$ is the canonical weight associated to the basis vector e_i.

For the notion of the **completed tensor product** $V \hat{\otimes} W$ of two $\mathbf{K}$-Banach spaces V and W, we refer the reader to [van Rooij, 1978, Chapter IV, Section "The Tensor Product"].

Lemma 3.33. *Let I and J be two index sets. Given weights $(w_i)_{i \in I}$ and $(w_j)_{j \in J}$, consider the mapping given by $\mathbf{K}$-linear continuous extension of*

$$\begin{aligned} c_0((w_i)_{i \in I}) \times c_0((w_j)_{j \in J}) &\to c_0((w_{i,j})_{(i,j) \in I \times J}), \\ (e_i, e_j) &\mapsto e_{i,j} \end{aligned}$$

with $w_{i,j} := w_i w_j$. It induces an (isometric) isomorphism of $\mathbf{K}$-Banach spaces

$$c_0((w_i)_{i \in I}) \hat{\otimes} c_0((w_j)_{j \in J}) \to c_0((w_{i,j})_{(i,j) \in I \times J}).$$

Proof. Denote the above mapping by Ψ. By the criterion of [van Rooij, 1978, Comment following Cor. 4.31], we have to check the following:

1. The mapping Ψ is bilinear and norm-nonincreasing.
2. The $\mathbf{K}$-linear span of $\operatorname{im}\Psi$ is dense in $c_0((w_{i,j})_{(i,j)\in I\times J})$.
3. Let $0 < t \leq 1$. If $f_1, \cdots, f_n \in c_0((w_i)_{i\in I})$ are t-orthogonal, then for any $g_1, \ldots, g_n \in c_0((w_j)_{j\in J})$ we find $\Psi(f_1, g_1), \ldots, \Psi(f_n, g_n) \in c_0((w_{i,j})_{(i,j)\in I\times J})$ to be t-orthogonal.

Ad 1.: The mapping is quickly checked to be bilinear. We have

$$\|\Psi(e_i, e_j)\|_{c_0((w_{i,j})_{(i,j)\in I\times J})} = \|e_{i,j}\|_{c_0((w_{i,j})_{(i,j)\in I\times J})} = \|e_i\|_{c_0((w_i)_{i\in I})} \cdot \|e_j\|_{c_0((w_j)_{j\in J})}.$$

If $f = \sum_{i\in I} a_i e_i$ and $g = \sum_{j\in I} b_j e_j$, then

$$\begin{aligned}\|\Psi(f,g)\|_{c_0((w_{i,j})_{(i,j)\in I\times J})} &= \|\sum_{i,j} a_i b_j e_{i,j}\| \\ &= \max_{i,j} |a_i||b_j|\|e_i\|\|e_j\| \\ &= \|\sum_i a_i e_i\|\|\sum_j b_j e_j\| = \|f\|_{c_0((w_i)_{i\in I})}\|g\|_{c_0((w_j)_{j\in J})}.\end{aligned}$$

Ad 2.: By definition of $c_0((w_{i,j})_{(i,j)\in I\times J})$, we have $< \{e_{i,j}\} >_K \subseteq c_0((w_{i,j})_{(i,j)\in I\times J})$ densely.
Ad 3.: Let $g_1 = (g_{1,j}), \ldots, g_n = (g_{n,j})$. We compute

$$\begin{aligned}\|\Psi(f_1, g_1) + \cdots + \Psi(f_n, g_n)\|_{c_0((w_{i,j})_{(i,j)\in I\times J})} &\geq w_j \cdot \|f_1 \cdot g_{1,j} + \cdots + f_n \cdot g_{n,j}\|_{c_0((w_i)_{i\in I})} \\ &\geq w_j \cdot t \cdot (|g_{1,j}|\|f_1\| \vee \ldots \vee |g_{n,j}|\|f_n\|)\end{aligned}$$

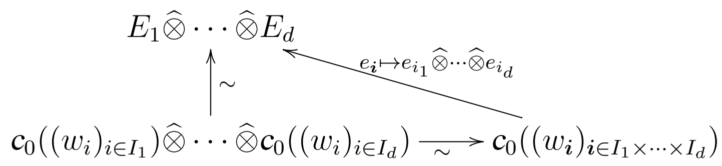
for all $j \in J$. Consequently,

$$\begin{aligned}\|\Psi(f_1, g_1) + \cdots + \Psi(f_n, g_n)\|_{c_0((w_{i,j})_{(i,j)\in I\times J})} &\geq t \cdot \max_{j\in J}(w_j|g_{1,j}|\|f_1\| \vee \ldots \vee w_j|g_{n,j}|\|f_n\|) \\ &= t \cdot (\|g_1\|\|f_1\| \vee \ldots \vee \|g_n\|\|f_n\|) \\ &= t \cdot (\|\Psi(f_1, g_1)\| \vee \ldots \vee \|\Psi(f_n, g_n)\|).\end{aligned}$$

■

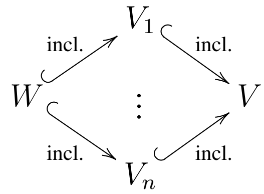
Corollary 3.34. *If* $\{e_{i_1}\} \subseteq E_1, \ldots, \{e_{i_d}\} \subseteq E_d$ *are orthogonal bases, then* $\{e_{i_1} \otimes \cdots \otimes e_{i_d}\} \subseteq E_1 \hat{\otimes} \cdots \hat{\otimes} E_d$ *will be an orthogonal basis.*

Proof. Consider the canonical commuting diagram



with $w_i := w_{i_1} \cdots w_{i_d}$. The bottom map is an (isometric) isomorphism of $\mathbf{K}$-Banach spaces by an induction over $d \geq 1$ through Lemma 3.33. The left-hand map is an (isometric) isomorphism of $\mathbf{K}$-Banach spaces by functoriality of the completed tensor product. Consequently the right-hand map is also an (isometric) isomorphism. ■

Lemma 3.35. *Let W be the initial $\mathbf{K}$-Banach space with respect to finitely many inclusion mappings*



for $\mathbf{K}$-Banach spaces $V_1, \ldots, V_n$ and V. I.e. $W = V_1 \cap \ldots \cap V_n$ as an abstract $\mathbf{K}$-vector space and its norm $\|\cdot\|_W$ on W is given by the pointwise maximum $\|\cdot\|_W = \|\cdot\|_{V_1} \vee \ldots \vee \|\cdot\|_{V_n}$.

(i) If $\{e_i\} \subseteq W$ is an orthogonal family of $V_1, \ldots, V_n$, then $\{e_i\}$ will be an orthogonal family of W.

(ii) If $\{e_i\} \subseteq W$ is an orthogonal basis of $V_1, \ldots, V_n$ and V, then $\{e_i\}$ will be an orthogonal basis of W.

Proof. Ad (i): Let $\{e_i\}$ be orthogonal in $V_1, \ldots, V_n$. We prove $\{e_i\} \subseteq W$ to be orthogonal by the following computation:

$$\begin{aligned}\|\sum_i \lambda_i e_i\|_W &= \|\sum_i \lambda_i e_i\|_{V_1} \vee \ldots \vee \|\sum_i \lambda_i e_i\|_{V_n} \\ &= \max_i |\lambda_i| \|e_i\|_{V_1} \vee \ldots \vee \max_i |\lambda_i| \|e_i\|_{V_n} \\ &= \max_i |\lambda_i| \|e_i\|_W .\end{aligned}$$

Ad (ii): Let $x \in W$. Then we can write $x = \sum_{i \geq 0} \lambda_i e_i$ in V_j for $j \in \{1, \ldots, n\}$. This implies $x = \sum_{i \geq 0} \lambda_i e_i$ in V. By orthogonality of $\{e_i\} \subseteq V$, the coefficients λ_i are uniquely determined, so the same equality holds in $V_1, \ldots, V_n$ and therefore as well in W. The orthogonality of $\{e_i\} \subseteq W$ has been proven in (i). ■

The initial $\mathbf{K}$-Banach algebra $\mathcal{C}^r(X, \mathbf{K})$ of thought topological tensor products $\mathcal{C}^{\boldsymbol{r}}(X, \mathbf{K})$ for $\boldsymbol{r} \in \mathbb{N}^d_{=r}$

Assumption. Throughout this subsection's paragraph about the initial $\mathbf{K}$-Banach algebra of thought topological tensor products, we will by $X \subseteq \mathbf{K}^d$ denote a nonempty compact cartesian subset whose factors contain no isolated point.

Notation. For a d-tuple $\boldsymbol{s} \in \mathbb{R}^d_{\geq 0}$, we put $|\boldsymbol{s}| = s_1 + \cdots + s_d$. For $r \in \mathbb{R}_{\geq 0}$, we define finite sets of d-tuples

$$\mathbb{N}^d_{=r} = \{\boldsymbol{s} \in \mathbb{R}^d_{\geq 0} : |\boldsymbol{s}| = r \text{ and } s_k \in \mathbb{N} \text{ for all but possibly one coordinate } k \in \{1, \ldots, d\}\}.$$

Definition 3.36. Let $\boldsymbol{r} = \boldsymbol{\nu} + \rho \cdot \boldsymbol{e}_k \in \mathbb{N}^d_{=r}$ with $\boldsymbol{\nu} \in \mathbb{N}^d$ and $k \in \{1, \ldots, d\}$. Then we define a mapping $f : X \to \mathbf{E}$ to be a **$\mathcal{C}^{\boldsymbol{r}}$-function** if the following holds:

(i) For all $\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]$ with $n_k < \nu_k$, the mapping $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a continuous function $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{E}$.

(ii) For all $\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]$ with $n_k = \nu_k$, the mapping $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a $\mathcal{C}^{\rho \cdot {}^k\boldsymbol{e}}$-function $f^{[\boldsymbol{n}]} : X^{[\boldsymbol{n}]} \to \mathbf{E}$; here we consider $X^{[\boldsymbol{n}]} = {}^1X \times \cdots \times {}^dX$ for $\boldsymbol{n} \in \mathbb{N}^d_{\geq 0}$ as the cartesian product of the metric spaces ${}^kX := X_k^{[n_k]}$, and so put ${}^k\boldsymbol{e} = \boldsymbol{e}_k$ for $k = 1, \ldots, d$.

The $\mathbf{K}$-vector space of all $\mathcal{C}^r$-functions $f : X \to \mathbf{E}$ will be denoted by $\mathcal{C}^r(X, \mathbf{E})$. We equip it with the norm $\|\cdot\|_{\mathcal{C}^r}$ defined by

$$\|f\|_{\mathcal{C}^r} = \max_{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}] \text{ with } n_k < \nu_k} \|f^{[\boldsymbol{\nu}]}\|_{\sup} \vee \max_{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}] \text{ with } n_k = \nu_k} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{\rho \cdot {}^k\boldsymbol{e}}}.$$

Proposition 3.37. *The space $\mathcal{C}^r(X, \mathbf{E})$ is a $\mathbf{K}$-Banach space.*

Proof. It is clear that $\mathcal{C}^r(X, \mathbf{E})$ is a normed $\mathbf{K}$-vector space. We prove completeness. As a normed $\mathbf{K}$-vector space, the space $\mathcal{C}^r(X, \mathbf{E})$ is canonically isomorphic to the subspace

$$\begin{aligned} A :=& \{(g_{\boldsymbol{n}}) \in \prod_{\substack{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}] \\ \text{with } n_k < \nu_k}} \mathcal{C}^0(X^{[\boldsymbol{n}]}, \mathbf{E}) \times \prod_{\substack{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}] \\ \text{with } n_k = \nu_k}} \mathcal{C}^{\rho \cdot {}^k\boldsymbol{e}}(X^{[\boldsymbol{n}]}, \mathbf{E}) : g_{\boldsymbol{n}|X^{]\boldsymbol{n}[}} = f^{]\boldsymbol{n}[} \text{ for } \boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]\} \\ \subseteq& \prod_{\substack{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}] \\ \text{with } n_k < \nu_k}} \mathcal{C}^0(X^{[\boldsymbol{n}]}, \mathbf{E}) \times \prod_{\substack{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}] \\ \text{with } n_k = \nu_k}} \mathcal{C}^{\rho \cdot {}^k\boldsymbol{e}}(X^{[\boldsymbol{n}]}, \mathbf{E}) =: P. \end{aligned}$$

Each factor $\mathcal{C}^0(X^{[\boldsymbol{n}]}, \mathbf{E})$ for $n_k < \nu_k$ is complete by Corollary 1.3. The factors $\mathcal{C}^{\rho \cdot {}^k\boldsymbol{e}}(X^{[\boldsymbol{n}]}, \mathbf{E})$ for $n_k = \nu_k$ are complete by Proposition 1.25. Hence it remains to prove that A is closed in P.

For this, let $\boldsymbol{f} = (\boldsymbol{f}_{\boldsymbol{n}})_{\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]}$ be in the boundary of A in P, i.e. in any neighborhood $U \ni \boldsymbol{f}$ of P lies another element $\boldsymbol{g} \in A$. We have to prove that $\boldsymbol{f} \in A$; in other words putting $f := \boldsymbol{f}_{\boldsymbol{0}}$, necessarily $\boldsymbol{f}_{\boldsymbol{n}|X^{]\boldsymbol{n}[}} = f^{]\boldsymbol{n}[}$ for $\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]$.

Fix $\varepsilon > 0$, an order $\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]$ and $\boldsymbol{x} \in X^{]\boldsymbol{n}[}$. We must show $\|\boldsymbol{f}_{\boldsymbol{n}}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| \leq \varepsilon$.

With $C(\boldsymbol{x}) \geq 1$ as in Lemma 3.12, we find another $\boldsymbol{g} \in A$ such that $\|\boldsymbol{f} - \boldsymbol{g}\| \leq \varepsilon / C(\boldsymbol{x})$. Hence with $g := \boldsymbol{g}_0$, it holds in particular

$$\|(f - g)(x)\| \leq \varepsilon / C(\boldsymbol{x}) \quad \text{for } x \in \{x_{i_1} : i_1 = 0, \ldots, n_1\} \times \cdots \times \{x_{i_d} : i_d = 0, \ldots, n_d\}.$$

By Lemma 3.12, we find $\|(g - f)^{]\boldsymbol{n}[}(\boldsymbol{x})\| \leq \varepsilon$. Since $\boldsymbol{g}_{\boldsymbol{n}|X^{]\boldsymbol{n}[}} = g^{]\boldsymbol{n}[}$, we find

$$\begin{aligned} \|\boldsymbol{f}_{\boldsymbol{n}}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| &\leq \|\boldsymbol{f}_{\boldsymbol{n}}(\boldsymbol{x}) - \boldsymbol{g}_{\boldsymbol{n}}(\boldsymbol{x})\| \vee \|\boldsymbol{g}_{\boldsymbol{n}}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| \\ &= \|\boldsymbol{f}_{\boldsymbol{n}}(\boldsymbol{x}) - \boldsymbol{g}_{\boldsymbol{n}}(\boldsymbol{x})\| \vee \|g^{]\boldsymbol{n}[}(\boldsymbol{x}) - f^{]\boldsymbol{n}[}(\boldsymbol{x})\| \\ &= \|\boldsymbol{f}_{\boldsymbol{n}}(\boldsymbol{x}) - \boldsymbol{g}_{\boldsymbol{n}}(\boldsymbol{x})\| \vee \|(g - f)^{]\boldsymbol{n}[}(\boldsymbol{x})\| \leq \varepsilon. \end{aligned}$$

■

Proposition 3.38. *The $\mathbf{K}$-Banach space $\mathcal{C}^r(X, \mathbf{E})$ is the initial $\mathbf{K}$-Banach space with respect to the inclusion mappings $\mathcal{C}^r(X, \mathbf{E}) \overset{\text{incl.}}{\hookrightarrow} \mathcal{C}^{\boldsymbol{r}}(X, \mathbf{E})$ for $\boldsymbol{r} \in \mathbb{N}^d_{=r}$.*

Proof. Firstly by Lemma 3.5, we find $f \in \mathcal{C}^r(X, \mathbf{K})$ if and only if $f \in \mathcal{C}^{n+\rho}(X, \mathbf{K})$ for $n = 0, \ldots, \nu$. Hence by Proposition 3.8, the mapping $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a mapping $f^{[\boldsymbol{n}]} \in \mathcal{C}^\rho(X^{[\boldsymbol{n}]}, \mathbf{E})$ for every $\boldsymbol{n} \in \mathbb{N}^d_{\leq \nu}$. By Corollary 1.31, we find $f^{[\boldsymbol{n}]} \in \mathcal{C}^\rho(X^{[\boldsymbol{n}]}, \mathbf{E})$ if and only if $f^{[\boldsymbol{n}]} \in \mathcal{C}^{\rho \cdot {}^k e}(X, \mathbf{E})$ for $k = 1, \ldots, d$. Thus $f \in \mathcal{C}^r(X, \mathbf{E})$ if and only if for every $\boldsymbol{n} \in \mathbb{N}^d_{\leq \nu}$ and $k = 1, \ldots, d$, the mapping $f^{]\boldsymbol{n}[} : X^{]\boldsymbol{n}[} \to \mathbf{E}$ extends to a mapping $f^{[\boldsymbol{n}]} \in \mathcal{C}^{\rho \cdot {}^k e}(X, \mathbf{E})$. Hence by Definition 3.36, we find $f \in \mathcal{C}^r(X, \mathbf{E})$ if and only if $f \in \mathcal{C}^{\boldsymbol{r}}(X, \mathbf{E})$ for every $\boldsymbol{r} = \boldsymbol{n} + \rho \cdot \boldsymbol{e}_k \in \mathbb{N}^d_{=r}$ with $\boldsymbol{n} \in \mathbb{N}^d_{=\nu}$ and $k = 1, \ldots, d$.
Moreover for $f \in \mathcal{C}^r(X, \mathbf{E})$ holds

$$
\begin{aligned}
\|f\|_{\mathcal{C}^r} &= \|f\|_{\mathcal{C}^\rho} \vee \ldots \vee \|f\|_{\mathcal{C}^{\nu+\rho}} \\
&= \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| \leq \nu} \|f^{[\boldsymbol{n}]}\|_{\sup} \vee \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| \leq \nu} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho} \\
&= \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| \leq \nu} \|f^{[\boldsymbol{n}]}\|_{\sup} \vee \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| \leq \nu} \max_{k=1,\ldots,d} \|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^{\rho \cdot {}^k e}} \\
&= \max_{\boldsymbol{n} \text{ with } |\boldsymbol{n}| = \nu} \max_{k=1,\ldots,d} \|f\|_{\mathcal{C}^{\boldsymbol{n}+\rho \cdot \boldsymbol{e}_k}} = \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} \|f\|_{\mathcal{C}^{\boldsymbol{r}}};
\end{aligned}
$$

the first equality by Lemma 3.11, the second one by definition, the third one by Corollary 1.31 and the penultimate one by Definition 3.36. ■

Remark 3.39. In the other direction, we have by definition a norm-nonincreasing inclusion of $\mathbf{K}$-Banach spaces $\mathcal{C}^{\vec{\nu}}(X, \mathbf{K}) \subseteq \mathcal{C}^\nu(X, \mathbf{K})$ with $\vec{\nu} := (\nu, \ldots, \nu)$ for $\nu \in \mathbb{N}$.

Lemma 3.40. *Let $X' \subseteq \mathbf{K}^{d'}, X'' \subseteq \mathbf{K}^{d''}$ be nonempty compact cartesian subsets whose factors contain no isolated point. For $\boldsymbol{r} = (\boldsymbol{r}', \boldsymbol{r}'') \in (\mathbb{N}^{d'} \times \mathbb{N}^{d''})_{=r}$, consider the mapping*

$$
\begin{gathered}
\mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K}) \times \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K}) \to \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K}), \\
(f, g) \mapsto f \odot g.
\end{gathered}
$$

1. *It is bilinear and norm-preserving.*

2. *Let $0 < t \leq 1$. If $f_1, \ldots, f_n \in \mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K})$ are t-orthogonal, then for any $g_1, \ldots, g_n \in \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K})$ their products $f_1 \odot g_1, \ldots, f_n \odot g_n \in \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$ will be t-orthogonal.*

Proof. Let $\boldsymbol{r} = (\boldsymbol{r}', \boldsymbol{r}'') = \boldsymbol{\nu} + \rho \cdot \boldsymbol{e}_k \in (\mathbb{N}^{d'} \times \mathbb{N}^{d''})_{=r}$ with $\boldsymbol{\nu} = (\boldsymbol{\nu}', \boldsymbol{\nu}'') \in \mathbb{N}^{d'} \times \mathbb{N}^{d''}$ and $k \in \{1, \ldots, d+e\}$. We may assume w.l.o.g. $k \in \{1, \ldots, d\}$. Let us denote the above mapping by Ψ.

Firstly, we prove $\operatorname{im} \Psi \subseteq \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$. Assume $f \in \mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K}), g \in \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K})$. Let $h = f \odot g$ be the image of (f, g). By Lemma 3.14, if $\boldsymbol{n} = (\boldsymbol{n}', \boldsymbol{n}'') \in [\boldsymbol{0}, \boldsymbol{\nu}] \subseteq \mathbb{N}^{d'} \times \mathbb{N}^{d''}$, then $h^{]\boldsymbol{n}[} = f^{]\boldsymbol{n}'[} \odot g^{]\boldsymbol{n}''[}$. Hence for $\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]$ with $n_k < \nu_k$, by Proposition 1.7(iii) the functions $f^{]\boldsymbol{n}'[}$ and $g^{]\boldsymbol{n}''[}$will extend to continuous functions $f^{[\boldsymbol{n}']}$ and $g^{[\boldsymbol{n}'']}$ only if $h^{]\boldsymbol{n}[}$ extends to a continuous function $h^{[\boldsymbol{n}]}$, and for $\boldsymbol{n} \in [\boldsymbol{0}, \boldsymbol{\nu}]$ with $n_k = \nu_k$, by Proposition 1.24(ii) the function $f^{]\boldsymbol{n}'[}$ will extend to a $\mathcal{C}^{\rho \cdot {}^k e}$-function $f^{[\boldsymbol{n}']}$ and $g^{]\boldsymbol{n}''[}$ will extend to a $\mathcal{C}^0$-function $g^{[\boldsymbol{n}'']}$ only if $h^{]\boldsymbol{n}[}$ extends to a $\mathcal{C}^{\rho \cdot {}^k e}$-function $h^{[\boldsymbol{n}]}$.

Ad 1.: The map Ψ is quickly checked to be bilinear. We find

$$\begin{aligned}
&\|f \odot g\|_{\mathcal{C}^r} \\
=& \max_{\boldsymbol{n}\in[\boldsymbol{0},\boldsymbol{\nu}] \text{ with } n_k<\nu_k} \|(f \odot g)^{[\boldsymbol{n}]}\|_{\sup} \vee \max_{\boldsymbol{n}\in[\boldsymbol{0},\boldsymbol{\nu}] \text{ with } n_k=\nu_k} \|(f \odot g)^{[\boldsymbol{n}]}\|_{\mathcal{C}^{\rho\cdot k_e}} \\
=& \max_{\substack{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k<\nu'_k,\\ \boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']}} \|f^{[\boldsymbol{n}']} \odot g^{[\boldsymbol{n}'']}\|_{\sup} \vee \max_{\substack{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k=\nu'_k,\\ \boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']}} \|f^{[\boldsymbol{n}']} \odot g^{[\boldsymbol{n}'']}\|_{\mathcal{C}^{\rho\cdot k_e}} \\
=& \max_{\substack{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k<\nu'_k,\\ \boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']}} \|f^{[\boldsymbol{n}']}\|_{\sup} \cdot \|g^{[\boldsymbol{n}'']}\|_{\sup} \vee \max_{\substack{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k=\nu'_k,\\ \boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']}} \|f^{[\boldsymbol{n}']}\|_{\mathcal{C}^{\rho\cdot k_e}} \cdot \|g^{[\boldsymbol{n}'']}\|_{\sup} \\
=& \|f\|_{\mathcal{C}^{r'}} \|g\|_{\mathcal{C}^{r''}};
\end{aligned}$$

here the second equality by Lemma 3.14(ii) and the following one by Proposition 1.24(ii).
Ad 2.: We compute

$$\begin{aligned}
&\|f_1 \odot g_1 + \cdots + f_n \odot g_n\|_{\mathcal{C}^r} \\
=& \max_{\boldsymbol{n}\in[\boldsymbol{0},\boldsymbol{\nu}] \text{ with } n_k<\nu_k} \|(f_1 \odot g_1 + \cdots + f_n \odot g_n)^{[\boldsymbol{n}]}\|_{\sup} \\
&\vee \max_{\boldsymbol{n}\in[\boldsymbol{0},\boldsymbol{\nu}] \text{ with } n_k=\nu_k} \|(f_1 \odot g_1 + \cdots + f_n \odot g_n)^{[\boldsymbol{n}]}\|_{\mathcal{C}^{\rho\cdot k_e}} \\
=& \max_{\substack{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k<\nu'_k,\\ \boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']}} \|f_1^{[\boldsymbol{n}']} \odot g_1^{[\boldsymbol{n}'']} + \cdots + f_n^{[\boldsymbol{n}']} \odot g_n^{[\boldsymbol{n}'']}\|_{\sup} \\
&\vee \max_{\substack{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k=\nu'_k,\\ \boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']}} \|f_1^{[\boldsymbol{n}']} \odot g_1^{[\boldsymbol{n}'']} + \cdots + f_n^{[\boldsymbol{n}']} \odot g_n^{[\boldsymbol{n}'']}\|_{\mathcal{C}^{\rho\cdot k_e}} \\
=& \max_{\boldsymbol{n}''\in[\boldsymbol{0},\boldsymbol{\nu}'']} \Big(\max_{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k<\nu'_k} \|f_1^{[\boldsymbol{n}']} \odot g_1^{[\boldsymbol{n}'']} + \cdots + f_n^{[\boldsymbol{n}']} \odot g_n^{[\boldsymbol{n}'']}\|_{\sup} \\
&\qquad \vee \max_{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k=\nu'_k} \|f_1^{[\boldsymbol{n}']} \odot g_1^{[\boldsymbol{n}'']} + \cdots + f_n^{[\boldsymbol{n}']} \odot g_n^{[\boldsymbol{n}'']}\|_{\mathcal{C}^{\rho\cdot k_e}} \Big) \\
\geq& \max_{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k<\nu_k} \|f_1^{[\boldsymbol{n}']} \cdot g_1^{[\boldsymbol{n}'']}(x'') + \cdots + f_n^{[\boldsymbol{n}']} \cdot g_n^{[\boldsymbol{n}'']}(x'')\|_{\sup} \\
&\vee \max_{\boldsymbol{n}'\in[\boldsymbol{0},\boldsymbol{\nu}'] \text{ with } n'_k=\nu'_k} \|f_1^{[\boldsymbol{n}']} \cdot g_1^{[\boldsymbol{n}'']}(x'') + \cdots + f_n^{[\boldsymbol{n}']} \cdot g_n^{[\boldsymbol{n}'']}(x'')\|_{\mathcal{C}^{\rho\cdot k_e}},
\end{aligned}$$

for any fixed $\boldsymbol{n}'' \in [\boldsymbol{0}, \boldsymbol{\nu}'']$ and $x'' \in X''^{[\boldsymbol{n}'']}$. Here the second equality by Lemma 3.14(ii). Then the last term, fixing $\boldsymbol{n}'' \in [\boldsymbol{0}, \boldsymbol{\nu}'']$ and $x'' \in X''^{[\boldsymbol{n}'']}$, equals $\|f_1 \cdot g_1^{[\boldsymbol{n}'']}(x'') + \cdots + f_n \cdot g_n^{[\boldsymbol{n}'']}(x'')\|_{\mathcal{C}^{r'}}$. Since $f_1, \ldots, f_n$ are t-orthogonal with respect to $\|\cdot\|_{\mathcal{C}^{r'}}$, we find for all $\boldsymbol{n}'' \in [\boldsymbol{0}, \boldsymbol{\nu}'']$ and $x'' \in X''^{[\boldsymbol{n}'']}$ that

$$\|f_1 \cdot g_1^{[\boldsymbol{n}'']}(x'') + \cdots + f_n \cdot g_n^{[\boldsymbol{n}'']}(x'')\|_{\mathcal{C}^{r'}}/t \geq |g_1^{[\boldsymbol{n}'']}(x'')| \cdot \|f_1\|_{\mathcal{C}^{r'}} \vee \ldots \vee |g_n^{[\boldsymbol{n}'']}(x'')| \cdot \|f_n\|_{\mathcal{C}^{r'}}.$$

In particular for fixed $\boldsymbol{n}'' \in [\boldsymbol{0}, \boldsymbol{\nu}'']$, it holds for $i = 1, \ldots, n$ that

$$\|f_1 \cdot g_1^{[\boldsymbol{n}'']}(x'') + \cdots + f_n \cdot g_n^{[\boldsymbol{n}'']}(x'')\|_{\mathcal{C}^{r'}}/t \geq \sup_{x''\in X''^{[\boldsymbol{n}'']}} |g_i^{[\boldsymbol{n}'']}(x'')| \|f_i\|_{\mathcal{C}^{r'}} = \|g_i^{[\boldsymbol{n}'']}\|_{\sup} \cdot \|f_i\|_{\mathcal{C}^{r'}}.$$

Consequently,

$$
\begin{aligned}
\|f_1 \odot g_1 + \cdots + f_n \odot g_n\|_{\mathcal{C}^{\boldsymbol{r}}} &\geq t \cdot \max_{i=1,\ldots,n} \max_{\boldsymbol{n}'' \in [\boldsymbol{0},\boldsymbol{\nu}'']} \|g_i^{[\boldsymbol{n}'']}\|_{\sup} \|f_i\|_{\mathcal{C}^{\boldsymbol{r}'}} \\
&= t \cdot \max_{i=1,\ldots,n} \|g_i\|_{\mathcal{C}^{\boldsymbol{\nu}''}} \|f_i\|_{\mathcal{C}^{\boldsymbol{r}'}} \\
&= t \cdot \max_{i=1,\ldots,n} \|f_i \odot g_i\|_{\mathcal{C}^{\boldsymbol{r}}}.
\end{aligned}
$$

■

Corollary 3.41. *Let $X_1, \ldots, X_d \subseteq \mathbf{K}$ be nonempty compact subsets without isolated points. For $\boldsymbol{r} \in \mathbb{N}^d_{=r}$, consider the mapping*

$$
\mathcal{C}^{r_1}(X_1, \mathbf{K}) \times \cdots \times \mathcal{C}^{r_d}(X_d, \mathbf{K}) \xrightarrow{\Psi} \mathcal{C}^{\boldsymbol{r}}(X_1 \times \cdots \times X_d, \mathbf{K}), \\
(f_1, \ldots, f_d) \mapsto f := [(x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)].
$$

If $\{e_{i_1}\} \subseteq \mathcal{C}^{r_1}(X_1, \mathbf{K}), \ldots, \{e_{i_d}\} \subseteq \mathcal{C}^{r_d}(X_d, \mathbf{K})$ are orthogonal families, so $\{e_{i_1} \odot \cdots \odot e_{i_d}\} \subseteq \mathcal{C}^{\boldsymbol{r}}(X_1 \times \cdots \times X_d, \mathbf{K})$ will be an orthogonal family with $\|e_{i_1} \odot \cdots \odot e_{i_d}\|_{\mathcal{C}^{\boldsymbol{r}}} = \|e_{i_1}\|_{\mathcal{C}^{r_1}} \cdots \|e_{i_d}\|_{\mathcal{C}^{r_d}}$.

Proof. First off we revisit the situation of Lemma 3.40: Let $X' \subseteq \mathbf{K}^{d'}, X'' \subseteq \mathbf{K}^{d''}$ and let $\{e'_i\} \subseteq \mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K})$ as well as $\{e''_j\} \subseteq \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K})$ be orthogonal families. We want to prove $\{e_{i,j} = e'_i \odot e''_j\} \subseteq \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$ to be an orthogonal family. That is, if $f = \sum_{i,j} a_{i,j} e_{i,j} \in \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$ is a finite sum of such, then $\|f\|_{\mathcal{C}^{\boldsymbol{r}}} = \max_{i,j} |a_{i,j}| \|e_{i,j}\|_{\mathcal{C}^{\boldsymbol{r}}}$.

Firstly $\|e'_i \odot e''_j\|_{\mathcal{C}^{\boldsymbol{r}}} = \|e'_i\|_{\mathcal{C}^{\boldsymbol{r}'}} \|e''_j\|_{\mathcal{C}^{\boldsymbol{r}''}}$ by the norm-preservation in Property 1 of Lemma 3.40. By the bilinearity in Property 1, we can write

$$
f = \sum_i \sum_j a_{i,j} e_i \odot e_j = \sum_i e_i \odot \left(\sum a_{i,j} e_j\right) = \sum_i e_i \odot f_i
$$

with $f_i := \sum a_{i,j} e_j \in \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K})$. By Property 2, we find by orthogonality of the finitely many $e_i \in \mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K})$ in this sum that $\|f\|_{\mathcal{C}^{\boldsymbol{r}}} = \max_i \|e'_i\|_{\mathcal{C}^{\boldsymbol{r}'}} \|f_i\|_{\mathcal{C}^{\boldsymbol{r}''}}$. Fixing the index i, it holds by orthogonality of the finitely many $e''_j \in \mathcal{C}^{\boldsymbol{r}''}(X'')$ in the f_i-sum that $\|f_i\|_{\mathcal{C}^{\boldsymbol{r}''}} = \max_j |a_{i,j}| \|e''_j\|_{\mathcal{C}^{\boldsymbol{r}''}}$. Together, we find

$$
\|f\|_{\mathcal{C}^{\boldsymbol{r}}} = \max_i \|e'_i\|_{\mathcal{C}^{\boldsymbol{r}'}} \|f_i\|_{\mathcal{C}^{\boldsymbol{r}''}} = \max_{i,j} |a_{i,j}| \|e'_i\|_{\mathcal{C}^{\boldsymbol{r}'}} \|e''_j\|_{\mathcal{C}^{\boldsymbol{r}''}} = \max_{i,j} |a_{i,j}| \|e_{i,j}\|_{\mathcal{C}^{\boldsymbol{r}}}.
$$

I.e. $\{e_{i,j}\} \subseteq \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$ is an orthogonal family.

Now let $d = d' + d''$ for $d' \geq 1$ and $X = X' \times X''$ with $X' = X_1 \times \cdots \times X_{d'} \subseteq \mathbf{K}^{d'}$ and $\boldsymbol{r} = (\boldsymbol{r}', \boldsymbol{r}'') \in \mathbb{N}^d_{=r}$ with $\boldsymbol{r}' = (r_1, \ldots, r_{d'})$. Then our mapping Ψ coincides with the mapping $\mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K}) \times \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K}) \to \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$ given in Lemma 3.40. By induction on $d \geq 1$, we find $\{e'_{\boldsymbol{i}'} = e_{i_1} \odot \cdots \odot e_{i_{d'}}\} \subseteq \mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K})$ to be an orthogonal family and likewise for $\{e''_{\boldsymbol{j}''}\} \subseteq \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K})$. Then in this situation, we have shown above that

$$
\{e_{i_1} \odot \cdots \odot e_{i_d}\} = \{e_{\boldsymbol{i}',\boldsymbol{j}''} = e'_{\boldsymbol{i}'} \odot e''_{\boldsymbol{j}''}\} \subseteq \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K}) = \mathcal{C}^{\boldsymbol{r}}(X_1 \times \cdots \times X_d, \mathbf{K})
$$

is an orthogonal family. Moreover we find by induction on $d \geq 1$ the norm to comport with the tensor product:

$$\|e_{i_1} \odot \cdots \odot e_{i_d}\|_{\mathcal{C}^{\boldsymbol{r}}} = \|e'_{\boldsymbol{i}'} \odot e''_{\boldsymbol{j}''}\|_{\mathcal{C}^{\boldsymbol{r}}} = \|e_{\boldsymbol{i}'}\|_{\mathcal{C}^{\boldsymbol{r}'}} \|e_{\boldsymbol{j}''}\|_{\mathcal{C}^{\boldsymbol{r}''}} = \|e_{i_1}\|_{\mathcal{C}^{r_1}} \cdots \|e_{i_d}\|_{\mathcal{C}^{r_d}}.$$

■

Lemma 3.42. *Let $X_1, \ldots, X_d \subseteq \mathbf{K}$ be nonempty compact subsets without isolated points. For $\boldsymbol{r} \in \mathbb{N}^d_{=r}$, consider the mapping*

$$\mathcal{C}^{r_1}(X_1, \mathbf{K}) \times \cdots \times \mathcal{C}^{r_d}(X_d, \mathbf{K}) \xrightarrow{\Psi} \mathcal{C}^{\boldsymbol{r}}(X_1 \times \cdots \times X_d, \mathbf{K}),$$
$$(f_1, \ldots, f_d) \mapsto f := [(x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)].$$

Assume the $\mathbf{K}$-linear span of $\operatorname{im}\Psi$ *to be dense in $\mathcal{C}^{\boldsymbol{r}}(X_1 \times \cdots \times X_d, \mathbf{K})$. Then Ψ induces an (isometric) isomorphism of $\mathbf{K}$-Banach spaces*

$$\mathcal{C}^{r_1}(X_1, \mathbf{K}) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{C}^{r_d}(X_d, \mathbf{K}) \to \mathcal{C}^{\boldsymbol{r}}(X_1 \times \cdots \times X_d, \mathbf{K}).$$

Proof. Let $X' = X_1 \times \cdots \times X_{d-1} \subseteq \mathbf{K}^{d-1}, X'' = X_d \subseteq \mathbf{K}$ and $\boldsymbol{r}' = (r_1, \ldots, r_{d-1}), \boldsymbol{r}'' = r_d$ so that $\boldsymbol{r} = (\boldsymbol{r}', \boldsymbol{r}'') \in \mathbb{N}^d_{=r}$. Then Ψ coincides with the mapping

$$\mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K}) \times \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K}) \to \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$$

given in Lemma 3.40. By our density assumption, the following three premisses are fulfilled:

1. The mapping Ψ is bilinear and norm-nonincreasing.
2. Let $0 < t \leq 1$. If $f_1, \cdots, f_n \in \mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K})$ are t-orthogonal, then for any $g_1, \ldots, g_n \in \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K})$, their products $f_1 \odot g_1, \ldots, f_n \odot g_n \in \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$ will be t-orthogonal.
3. The $\mathbf{K}$-linear span of $\operatorname{im}\Psi$ is dense in $\mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$.

By the criterion of [van Rooij, 1978, Comment following Cor. 4.31], the map Ψ induces an isomorphism of $\mathbf{K}$-Banach spaces $\mathcal{C}^{\boldsymbol{r}'}(X', \mathbf{K}) \widehat{\otimes} \mathcal{C}^{\boldsymbol{r}''}(X'', \mathbf{K}) \to \mathcal{C}^{\boldsymbol{r}}(X' \times X'', \mathbf{K})$. Then an induction over $d \geq 1$ yields the result. ■

The following Corollary 3.43 and Lemma 3.44 hold for an arbitrary coordinate index $k \in \{1, \ldots, d\}$ but will for notational convenience only be stated and proven for $k = 1$.

Corollary 3.43. *Let $X_1, \ldots, X_d \subseteq \mathbf{K}$ be nonempty compact subsets without isolated points. Consider the mapping*

$$\mathcal{C}^{\rho}(X_1, \mathbf{K}) \times \mathcal{C}^0(X_2, \mathbf{K}) \times \cdots \times \mathcal{C}^0(X_d, \mathbf{K}) \xrightarrow{\Psi} \mathcal{C}^{(\rho,0,\ldots,0)}(X_1 \times \cdots \times X_d, \mathbf{K}),$$
$$(f_1, \ldots, f_d) \mapsto f := [(x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)].$$

Then Ψ induces an (isometric) isomorphism of $\mathbf{K}$-Banach spaces

$$\mathcal{C}^{\rho}(X_1, \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X_2, \mathbf{K}) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{C}^0(X_d, \mathbf{K}) \to \mathcal{C}^{(\rho,0,\ldots,0)}(X_1 \times \cdots \times X_d, \mathbf{K}).$$

Proof. By Corollary 1.29, the locally constant functions are dense in $\mathcal{C}^{(\rho,0,\ldots,0)}(X_1 \times \cdots \times X_d, \mathbf{K})$. For any characteristic function $\mathbf{1}_{\mathrm{B}} : X_1 \times \cdots \times X_d \to \mathbf{K}$ of a ball $\mathrm{B} = \mathrm{B}_1 \times \cdots \times \mathrm{B}_d$ holds $\mathbf{1}_{\mathrm{B}} = \mathbf{1}_{\mathrm{B}_1} \odot \cdots \odot \mathbf{1}_{\mathrm{B}_d}$. By compactness, these characteristic functions $\mathbf{K}$-linearly span all locally constant functions. Therefore $\operatorname{im}\Psi \subseteq \mathcal{C}^{(\rho,0,\ldots,0)}(X_1 \times \cdots \times X_d, \mathbf{K})$ densely and the result follows by Lemma 3.42. ■

The Mahler base of $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$

Assumption. We will throughout this subsection's paragraph on the Mahler base of $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ assume that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a normed field.

The argument given in Subsection 2.3 on the distinguished orthogonal basis of $\mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$ by Mahler polynomials extends verbatim to the multivariate case, yielding an isomorphism of $\mathbf{K}$-Banach spaces $c_0(\mathbb{N}^d, \mathbf{K}) \xrightarrow{\sim} \mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$. We will in the following adopt the terminology used there in the multivariate case:

Definition. We define the $\boldsymbol{i}$-th **Mahler polynomial** $\binom{*}{\boldsymbol{i}} : \mathbb{Z}_p^d \to \mathbf{K}$ for $\boldsymbol{i} \in \mathbb{N}^d$ by $\binom{*}{\boldsymbol{i}} := \binom{*}{i_1} \odot \cdots \odot \binom{*}{i_d}$, cf. Definition 2.38.

Lemma 3.44. *The family* $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^{\rho \cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal basis with* $\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^{\rho \cdot \boldsymbol{e}_1}} = p^{\rho \, \mathrm{l}(i_1)}$.

Proof. By Theorem 2.49, the family $\{\binom{*}{i}\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K})$ is an orthogonal basis for arbitrary $\rho \in [0, 1[$. Hence we find $\{\binom{*}{i_1} \otimes \cdots \otimes \binom{*}{i_d}\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K}) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$ to be an orthogonal basis by Corollary 3.34. Then by Corollary 3.43 holds

$$\mathcal{C}^{\rho \cdot \boldsymbol{e}_1}(\mathbb{Z}_p \times \cdots \times \mathbb{Z}_p, \mathbf{K}) = \mathcal{C}^\rho(\mathbb{Z}_p, \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K}) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K}).$$

This (isometric) isomorphism maps $\binom{*}{\boldsymbol{i}}$ to $\binom{*}{i_1} \otimes \cdots \otimes \binom{*}{i_d}$. Therefore $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^{\rho \cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K})$ is an orthogonal basis. By [van Rooij, 1978, Theorem 4.27(i)], we find

$$\left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathcal{C}^{\rho \cdot \boldsymbol{e}_1}} = \left\| \binom{*}{i_1} \right\|_{\mathcal{C}^\rho} \cdot \left\| \binom{*}{i_d} \right\|_{\mathcal{C}^0} \cdots \left\| \binom{*}{i_d} \right\|_{\mathcal{C}^0} = p^{\rho \cdot \mathrm{l}(i_1)};$$

the last equality by Theorem 2.49. ■

Corollary 3.45. *The family* $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal basis with* $\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^\rho} = p^{\rho \cdot [\mathrm{l}(i_1) \vee \ldots \vee \mathrm{l}(i_d)]}$.

Proof. By Lemma 3.44 for $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$, we find $\{\binom{*}{\boldsymbol{i}}\}$ to be an orthogonal basis of the $\mathbf{K}$-Banach spaces $\mathcal{C}^{\rho \cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}), \ldots, \mathcal{C}^{\rho \cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K})$ and for $\rho = 0$ one of $\mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$. Consequently by Corollary 1.31 and Lemma 3.35(ii), we find $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^\rho(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal basis with

$$\left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathcal{C}^r} = \left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathcal{C}^{\rho \cdot \boldsymbol{e}_1}} \vee \ldots \vee \left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathcal{C}^{\rho \cdot \boldsymbol{e}_d}} = p^{\rho \cdot [\mathrm{l}(i_1) \vee \ldots \vee \mathrm{l}(i_d)]};$$

the last equality by Theorem 2.49. ■

Lemma 3.46. *For* $\boldsymbol{r} \in \mathbb{N}^d_{=r}$ *the subset* $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal family with* $\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^r} = p^{w_{r_1}(i_1) + \cdots + w_{r_d}(i_d)}$*; here* $w_{r_1}(i_1), \ldots, w_{r_d}(i_d)$ *as in Theorem 2.55.*

Proof. By Theorem 2.55, the family $\{\binom{*}{i}\} \subseteq \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})$ is in particular an orthogonal family. By Corollary 3.41, we find $\{\binom{*}{\boldsymbol{i}} = \binom{*}{i_1} \odot \cdots \odot \binom{*}{i_d}\} \subseteq \mathcal{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with

$$\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathcal{C}^{\boldsymbol{r}}} = \left\|\binom{*}{i_1}\right\|_{\mathcal{C}^{r_1}} \cdots \left\|\binom{*}{i_d}\right\|_{\mathcal{C}^{r_d}} = p^{w_{r_1}(i_1)} \cdots p^{w_{r_d}(i_d)};$$

the last equality by Theorem 2.55. ■

Theorem 3.47. *The family* $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal basis and*

$$\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathcal{C}^r} = p^{w_r(\boldsymbol{i})} \quad \text{with } w_r(\boldsymbol{i}) = \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} w_{r_1}(i_1) + \cdots + w_{r_d}(i_d);$$

here $w_{r_1}(i_1), \ldots, w_{r_d}(i_d)$ *as in Theorem 2.55.*

Proof. By Lemma 3.46, we find $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^{\boldsymbol{r}}(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with valuations $\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^{\boldsymbol{r}}} = p^{w_{r_1}(i_1)+\cdots+w_{r_d}(i_d)}$ for all $\boldsymbol{r} \in \mathbb{N}^d_{=r}$. Consequently by Proposition 3.38 and Lemma 3.35(i), we find $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ to be an orthogonal family with $\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^r} = \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} \|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^{\boldsymbol{r}}}$. By [Schikhof, 1984, Exercise 50.F], an orthogonal family whose $\mathbf{K}$-linear span is dense is an orthogonal base. It thus remains to show that the $\mathbf{K}$-linear span of $\{\binom{*}{\boldsymbol{i}}\}$ is dense in $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$.
In the case of one variable, the family $\{\binom{*}{i}\}$ is by orthogonality of $\{\binom{*}{i}\} \subseteq \mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$ in particular linearly independent. As

$$< \{\binom{*}{0}, \ldots, \binom{*}{g}\} >_{\mathbf{K}\text{-vctsp.}} \subseteq \{ \text{ polynomial functions } p : \mathbb{Z}_p \to \mathbf{K} \text{ of degree at most } g\},$$

and the right hand side has dimension $g+1$, the $\mathbf{K}$-linear span of $\{\binom{*}{i}\}$ consists of all polynomial functions $p : \mathbb{Z}_p \to \mathbf{K}$. By multilinearity, the $\mathbf{K}$-linear span of $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ consists of all polynomial functions $p : \mathbb{Z}_p^d \to \mathbf{K}$. By Corollary 3.32, these are indeed dense inside $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$. ■

Definition. For sequences (u_i) and (w_i) with values in $\mathbb{R}_{>0}$, running over the same index set I, we introduce the equivalence relation

$$(u_i) \sim (w_i) \text{ if there exist constants } 0 < c \le 1 \le C \text{ with } c \cdot u_i \le w_i \le C \cdot u_i \text{ for all } i \in I.$$

Lemma 3.48. *We have* $(\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^r})_{\boldsymbol{i} \in \mathbb{N}^d} \sim (i_1^r \vee \ldots \vee i_d^r)_{\boldsymbol{i} \in \mathbb{N}^d}$.

Proof. By Lemma 2.58, we find for every $r \in \mathbb{R}_{\ge 0}$ positive constants $c(r) \le 1 \le C(r)$ with $c(r) \cdot m^r \le p^{w_r(m)} \le C(r) \cdot m^r$ for every $m \in \mathbb{N}$.
For $\boldsymbol{r} \in \mathbb{N}^d_{=r}$, define the positive constants

$$c(\boldsymbol{r}) := c(r_1) \cdots c(r_d) \le 1 \le C(\boldsymbol{r}) := C(r_1) \cdots C(r_d).$$

Then by Lemma 3.46, for all $\boldsymbol{i} \in \mathbb{N}^d$ holds

$$c(\boldsymbol{r}) \cdot i_1^{r_1} \cdots i_d^{r_d} \leq \left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathcal{C}^r} \leq C(\boldsymbol{r}) \cdot i_1^{r_1} \cdots i_d^{r_d}.$$

Assume that $i_k = i_1 \vee \ldots \vee i_d$. Then $i_1^{r_1} \cdots i_d^{r_d} \leq i_k^{r_1} \cdots i_k^{r_d} = i_k^r$. Hence $i_1^{r_1} \cdots i_d^{r_d}$ is maximal among $\{i_1^{r_1} \cdots i_d^{r_d} : \boldsymbol{r} \in \mathbb{N}^d_{=r}\}$ if and only if $r_k = r$. Therefore $\max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} i_1^{r_1} \cdots i_d^{r_d} = i_1^r \vee \ldots \vee i_d^r$. Defining the positive constants $c(r) := \min_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} c(\boldsymbol{r}) \leq 1 \leq C(r) := \max_{\boldsymbol{r} \in \mathbb{N}^d_{=r}} C(\boldsymbol{r})$, it follows by the preceding Theorem 3.47 in particular that for all $\boldsymbol{i} \in \mathbb{N}^d$ holds

$$c(r) \cdot (i_1 \vee \ldots \vee i_d)^r \leq \left\| \binom{*}{\boldsymbol{i}} \right\|_{\mathcal{C}^r} \leq C(r) \cdot (i_1 \vee \ldots \vee i_d)^r.$$

■

Corollary 3.49. *We have* $(\| \binom{*}{\boldsymbol{i}} \|_{\mathcal{C}^r})_{\boldsymbol{i} \in \mathbb{N}^d} \sim (|\boldsymbol{i}|^r)_{\boldsymbol{i} \in \mathbb{N}^d}$.

Proof. By Lemma 3.48, there exist positive constants $\tilde{c} \leq 1 \leq C$ with $\tilde{c} \cdot (i_1 \vee \ldots \vee i_d)^r \leq \| \binom{*}{\boldsymbol{i}} \|_{\mathcal{C}^r} \leq C \cdot (i_1 \vee \ldots \vee i_d)^r$. Since $i_1 \vee \ldots \vee i_d \leq i_1 + \cdots + i_d \leq d \cdot (i_1 \vee \ldots \vee i_d)$, the asserted constriction holds for the positive constants $c := \tilde{c}/d^r \leq 1 \leq C$. ■

3.3 Description of $\mathcal{C}^r(X, \mathbf{K})$ for open $X \subseteq \mathbb{Q}_p^d$ through Taylor polynomials

Assumption. We will throughout this subsection assume $\mathbf{K}$ to be a complete non-trivially non-Archimedeanly valued locally compact field.

Definition 3.50. Let $X \subseteq \mathbf{K}^d$ be an open subset and $k \in \{1, \ldots, d\}$. We will speak of a $\mathcal{C}_{\mathrm{T}}^{r \cdot e_k}$**-function** $f : X \to \mathbf{K}$ if there are *continuous* functions $\mathcal{D}_{\mathbf{0}} f, \mathcal{D}_{1 \cdot \boldsymbol{e}_k} f, \ldots, \mathcal{D}_{\nu \cdot \boldsymbol{e}_k} f : X \to \mathbf{K}$ such that if one defines $R_{\nu \cdot \boldsymbol{e}_k} f : X^{[e_k]} \to \mathbf{K}$ on $X^{[e_k]} := \{(x; t) \in X \times \mathbf{K}$ with $x + t \cdot \boldsymbol{e}_k \in X\}$ by

$$R_{\nu \cdot \boldsymbol{e}_k} f(x; t) := f(x + t \cdot \boldsymbol{e}_k) - \sum_{i=0,\ldots,\nu} \mathcal{D}_{i \cdot \boldsymbol{e}_k} f(x) t^i,$$

then for every point $a \in X$ and any $\varepsilon > 0$, there will exist a neighborhood $U \ni a$ such that

$$|R_{\nu \cdot \boldsymbol{e}_k} f(x; t)| \leq \varepsilon |t|^r \quad \text{for all } x + t \cdot \boldsymbol{e}_k, x \in U.$$

We will denote the set of all $\mathcal{C}_{\mathrm{T}}^{r \cdot e_k}$-function $f : X \to \mathbf{K}$ by $\mathcal{C}_{\mathrm{T}}^{r \cdot e_k}(X, \mathbf{K})$.

Since $R_{\nu \cdot \boldsymbol{e}_k} f : X^{[e_k]} \to \mathbf{K}$ vanishes on $X \times \{0\}$, we see that $f = \mathcal{D}_0 f$. Moreover the continuity of $\mathcal{D}_0 f, \mathcal{D}_{1 \cdot \boldsymbol{e}_k} f, \ldots, \mathcal{D}_{\nu \cdot \boldsymbol{e}_k} f : X \to K$ implies the continuity of $R_{\nu \cdot \boldsymbol{e}_k} f : X^{[e_k]} \to \mathbf{K}$. By the above convergence condition, we even have a continuous mapping $\Delta_{\nu \cdot \boldsymbol{e}_k} f : X^{[e_k]} \to \mathbf{K}$, defined as the extension of the function $\Delta_{\nu \cdot \boldsymbol{e}_k} f(x; t) := R_{\nu \cdot \boldsymbol{e}_k} f(x; y)/t^\nu$ with domain $X^{]e_k[} := \{(x; t) \in X \times \mathbf{K}^*$ with $x + t \cdot \boldsymbol{e}_k \in X\}$ and which will vanish if t does.

Lemma 3.51. *The functions* $\mathcal{D}_{\mathbf{0}} f, \mathcal{D}_{1 \cdot \boldsymbol{e}_k} f, \ldots, \mathcal{D}_{\nu \cdot \boldsymbol{e}_k} f : X \to \mathbf{K}$ *in Definition 3.50 are unique.*

Proof. Let $f \in \mathcal{C}^{r\cdot e_k}(X, \mathbf{K})$. This is proven by induction on $\nu \geq 0$. If $\nu = 0$, we noticed above that necessarily $\mathcal{D}_{\mathbf{0}} f = f$ will be uniquely determined. If $\nu \geq 1$, let us assume that f has another Taylor-polynomial expansion

$$f(x + t \cdot \boldsymbol{e}_k) = \sum_{i=0,\ldots,\nu} \mathfrak{D}_{i\cdot \boldsymbol{e}_k} f(x) t^i + \mathfrak{d}_{\nu\cdot \boldsymbol{e}_k} f(x;t) t^\nu \quad \text{for } (x;t) \in X^{[e_k]}$$

with continuous functions $\mathfrak{D}_{0\cdot \boldsymbol{e}_k} f, \ldots, \mathfrak{D}_{\nu\cdot \boldsymbol{e}_k} f : X \to \mathbf{K}$ and $\mathfrak{d}_{\nu\cdot \boldsymbol{e}_k} f : X^{[e_k]} \to \mathbf{K}$. With $\mathfrak{d}_{\nu-1\cdot \boldsymbol{e}_k} f(x;t) := \mathfrak{D}_{\nu\cdot \boldsymbol{e}_k} f(x) + \mathfrak{d}_{\nu\cdot \boldsymbol{e}_k} f(x;t)t$, we find

$$\begin{aligned} f(x + t \cdot \boldsymbol{e}_k) &= \sum_{i=0,\ldots,\nu} \mathfrak{D}_{i\cdot \boldsymbol{e}_k} f(x) t^i + \mathfrak{d}_{\nu\cdot \boldsymbol{e}_k} f(x;t) t^\nu \\ &= \sum_{i=0,\ldots,\nu-1} \mathfrak{D}_{i\cdot \boldsymbol{e}_k} f(x) t^i + \mathfrak{d}_{\nu-1\cdot \boldsymbol{e}_k} f(x;t) t^{\nu-1} \quad \text{for } (x;t) \in X^{[e_k]}. \end{aligned}$$

Since $\mathfrak{D}_{\nu\cdot \boldsymbol{e}_k} f$ and $\mathfrak{d}_{\nu\cdot \boldsymbol{e}_k} f$ are continuous maps, so is $\mathfrak{d}_{\nu-1\cdot \boldsymbol{e}_k} f$. Likewise for $\mathcal{D}_{\nu\cdot \boldsymbol{e}_k} f$, $\Delta_{\nu\cdot \boldsymbol{e}_k} f$ and the mapping $\Delta_{\nu-1\cdot \boldsymbol{e}_k} f(x;t) := \mathcal{D}_{\nu\cdot \boldsymbol{e}_k} f(x) + \Delta_{\nu\cdot \boldsymbol{e}_k} f(x;t)t$. By the assumed uniqueness up to degree $\nu - 1$, we obtain $\mathfrak{D}_{\mathbf{0}} f = \mathcal{D}_{\mathbf{0}} f, \mathfrak{D}_{1\cdot \boldsymbol{e}_k} f = \mathcal{D}_{1\cdot \boldsymbol{e}_k} f, \ldots, \mathfrak{D}_{\nu-1\cdot \boldsymbol{e}_k} f = \mathcal{D}_{\nu-1\cdot \boldsymbol{e}_k} f$ and thus

$$\mathfrak{d}_{\nu-1\cdot \boldsymbol{e}_k} f(x;t) = \Delta_{\nu-1\cdot \boldsymbol{e}_k} f(x;t) \quad \text{for all } (x;t) \in X^{[e_k]}.$$

By definition, the above equality for $t = 0$ yields $\mathfrak{D}_{\nu\cdot \boldsymbol{e}_k} f = \mathcal{D}_{\nu\cdot \boldsymbol{e}_k} f$, as $\mathfrak{d}_{\nu\cdot \boldsymbol{e}_k} f$ and $\Delta_{\nu\cdot \boldsymbol{e}_k} f$ vanish for $t = 0$. ■

Definition. Fix a coordinate index $k \in \{1, \ldots, d\}$.

(i) Let $f \in \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$. We define functions $\Delta_{\nu\cdot \boldsymbol{e}_k} f : X^{]e_k[} \to \mathbf{K}$ and $|\Delta_{r\cdot \boldsymbol{e}_k} f| : X^{]\nu\cdot e_k[} \to \mathbb{R}_{\geq 0}$ by putting

$$\Delta_{\nu\cdot \boldsymbol{e}_k} f(x;t) := \frac{R_{\nu\cdot \boldsymbol{e}_k} f(x;t)}{t^\nu} \quad \text{and} \quad |\Delta_{r\cdot \boldsymbol{e}_k} f|(x;t) := \frac{|R_{\nu\cdot \boldsymbol{e}_k} f(x;t)|}{|t|^r}.$$

Since $f \in \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$, these functions will extend continuously onto $X^{[e_k]}$ if we let them vanish for $t = 0$. We denote these extensions likewise.

(ii) By Lemma 3.51, the functions $\mathcal{D}_{0\cdot \boldsymbol{e}_k}, \ldots, \mathcal{D}_{\nu\cdot \boldsymbol{e}_k} f : X \to \mathbf{K}$ of Definition 3.51 are uniquely determined continuous functions. So it makes sense to endow $\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_k}(X, \mathbf{K})$ with the locally convex topology induced by the family of seminorms $\{\|\cdot\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_k}, C}\}$ running through all compact subsets $C \subseteq X$ defined by

$$\|f\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_k}, C} := \|\mathcal{D}_{\mathbf{0}} f\|_C \vee \|\mathcal{D}_{1\cdot \boldsymbol{e}_k} f\|_C \vee \ldots \vee \|\mathcal{D}_{\nu\cdot \boldsymbol{e}_k} f\|_C \vee \||\Delta_{r\cdot \boldsymbol{e}_k} f|\|_{C\times C}.$$

The next five lemmata hold for a general coordinate index $k \in \{1, \ldots, d\}$, but will for notational convenience only be stated and proven for $k = 1$.

Lemma 3.52. *Let $f \in \mathcal{C}^{r\cdot e_1}(X, \mathbf{K})$ for a ball $X \subseteq \mathbf{K}^d$. Then for $(x;t) \in X^{[e_1]}$ holds*

$$f(x + t \cdot \boldsymbol{e}_1) = \sum_{i=0,\ldots,\nu-1} D_{i\cdot e_1} f(x) t^i + f^{[\nu\cdot e_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d) t^\nu$$

with continuous *functions $D_{i\cdot e_1} f : X \to \mathbf{K}$ and the $\mathcal{C}^{\rho\cdot e_1}$-function $f^{[\nu\cdot e_1]} : X^{[\nu\cdot e_1]} \to \mathbf{K}$.*

Proof. This is proven bx induction on $\lfloor r \rfloor$, the case $\lfloor r \rfloor = 0$ being trivial. So let $\lfloor r \rfloor = \nu + 1 \geq 1$ and $f \in \mathcal{C}^{r\cdot e_1}(X, \mathbf{K}) \subseteq \mathcal{C}^{r-1\cdot e_1}(X, \mathbf{K})$ (the inclusion holding by Definition 3.36). By the induction hypothesis, we have a unique Taylor-polynomial expansion

$$f(x + t \cdot \boldsymbol{e}_1) = \sum_{i=0,\ldots,\nu-1} D_{i\cdot e_1} f(x) t^i + f^{[\nu\cdot e_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d) t^\nu$$

for all $(x;t) \in X^{[e_1]}$ with *continuous* functions $D_{i\cdot e_1} f : X \to \mathbf{K}$ and a $\mathcal{C}^{\rho\cdot e_1}$-function $f^{[\nu\cdot e_1]} : X^{[\nu\cdot e_1]} \to \mathbf{K}$. The definition of $f^{[\nu+1\cdot e_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d)$ for nonzero t yields

$$f^{[\nu\cdot e_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d) = D_{\nu\cdot e_1} f(x) + t f^{[\nu+1\cdot e_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d).$$

This furnishes the existence of our Taylor-polynomial expansion up to degree ν. As $f \in \mathcal{C}^{r\cdot e_1}(X, \mathbf{K})$, we see that $D_{\nu\cdot e_1} f$ is continuous since $f^{[\nu\cdot e_1]}$ is so and that $f^{[\nu+1\cdot e_1]}$ is a $\mathcal{C}^{\rho\cdot e_1}$-function. ■

Remark. We note that the preceding Lemma 3.52 yields an inclusion of locally convex $\mathbf{K}$-vector spaces $\mathcal{C}^{r\cdot e_k}(X, \mathbf{K}) \subseteq \mathcal{C}_{\mathrm{T}}^{r\cdot e_k}(X, \mathbf{K})$ for any coordinate index $k \in \{1, \ldots, d\}$. For this, note that by uniqueness necessarily $\mathcal{D}_{i\cdot e_1} f = D_{i\cdot e_1} f$ for $i = 0, \ldots, \nu$ and $\Delta_{\nu\cdot e_k} f(x;t) = f^{[\nu\cdot e_1]}(x_1 + t, x_1, \ldots, x_1; x_2; \ldots; x_d)$.

Lemma 3.53. *Let $X \subseteq \mathbf{K}^d$ be open and $f \in \mathcal{C}_{\mathrm{T}}^{r\cdot e_1}(X, \mathbf{K})$. Then $\binom{j}{j}\mathcal{D}_{j\cdot e_1} f$, $\binom{j+1}{j}\mathcal{D}_{j+1\cdot e_1} f$, $\ldots$, $\binom{\nu}{j}\mathcal{D}_{\nu\cdot e_1} f$ prove $\mathcal{D}_{j\cdot e_1} f$ to be in $\mathcal{C}_{\mathrm{T}}^{r-j\cdot e_1}(X, \mathbf{K})$ for $j = 0, \ldots, \nu$.*

Proof. Let $f \in \mathcal{C}_{\mathrm{T}}^{r\cdot e_1}(X, \mathbf{K})$. We show that the continuous functions

$$\binom{j}{j}\mathcal{D}_{j\cdot e_1} f, \binom{j+1}{j}\mathcal{D}_{j+1\cdot e_1} f, \ldots, \binom{\nu}{j}\mathcal{D}_{\nu\cdot e_1} f$$

prove $\mathcal{D}_{j\cdot e_1} f$ to be in $\mathcal{C}_{\mathrm{T}}^{r-j\cdot e_1}(X, \mathbf{K})$ for fixed $j \in \{0, \ldots, \nu\}$. Fix $\varepsilon > 0$ and $a \in X$. We find a ball $U = U_1 \times \cdots \times U_d \ni a$ such that

$$|R_\nu f(x; x_1 - y_1)| \leq \varepsilon |x_1 - y_1|^r \quad \text{for all } x = (x_1, x_2, \ldots, x_d), (y_1, x_2, \ldots, x_d) \in U.$$

As U_1 is likewise a ball, it has the B_ν-property by Lemma 2.27. If $t := x_1 - y_1 \neq 0$, then we will fix $x_2, \ldots, x_d$ and find by Lemma 2.31 applied to $f_{x_2,\ldots,x_d} := f(\_, x_2, \ldots, x_d) \in \mathcal{C}_{\mathrm{T}}^r(U_1, \mathbf{K})$ a uniform constant $C > 0$ (only depending on U_1), a finite subset $P \subseteq \mathrm{B}_{\leq\delta}(x_1) \subseteq U_1$ with $\delta := |t| > 0$ such that

$$\begin{aligned} |R_{\nu-j\cdot e_1}\mathcal{D}_{j\cdot e_1} f(x;t)| \leq & C|t|^{-j} \max_{z_0 = x_1, y_1 \text{ and } z \in P} |R_{\nu\cdot e_1} f(z, x_2, \ldots, x_d; z_0 - z)| \\ \leq & C|t|^{-j} \max_{z_0 = x_1, y_1 \text{ and } z \in P} \varepsilon |z_0 - z|^r \leq C\varepsilon |t|^{r-j}; \end{aligned}$$

the middle inequality as $(z_0, x_2, \ldots, x_d), (z, x_2, \ldots, x_d) \in U$ and the last one since $|z_0 - z| \leq \delta$, both points $z_0 = x_1, y_1$ being the centers of $\mathrm{B}_{\leq\delta}(x_1)$. If $t = 0$, this inequality will hold trivially. ■

Definition. Let $\boldsymbol{g} \in \mathbb{N}^d$. Then we define a locally polynomial function $f : X \to \mathbf{K}$ to have **degree at most $\boldsymbol{g}$** if for every point $a \in X$, there will exist a neighborhood $U \ni a$ such that $f_{|U} = p_{|U}$ for a polynomial function $p = \sum_{\boldsymbol{i} \in [\boldsymbol{0}, \boldsymbol{g}]} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$.
We will denote the set of all locally polynomials functions $f : X \to \mathbf{K}$ of degree at most $\boldsymbol{g}$ by $\mathcal{C}^{\mathrm{pol}}_{\leq \boldsymbol{g}}(X, \mathbf{K})$.

Lemma 3.54. *For a ball $X \subseteq \mathbf{K}^d$, we have a dense inclusion $\mathcal{C}^{\mathrm{pol}}_{\leq \nu \cdot \boldsymbol{e}_1}(X, \mathbf{K}) \subseteq \mathcal{C}^{r \cdot \boldsymbol{e}_1}_{\mathrm{T}}(X, \mathbf{K})$ of the locally polynomial functions of degree $\boldsymbol{g} \leq \nu \cdot \boldsymbol{e}_1$ into the $\mathcal{C}^{r \cdot \boldsymbol{e}_1}_{\mathrm{T}}$-functions.*

Proof. For this statement to be meaningful, recall that by Remark 3.25 and Lemma 3.52 above, we have a chain of inclusions

$$\mathcal{C}^{\mathrm{pol}}_{\leq \nu \cdot \boldsymbol{e}_1}(X, \mathbf{K}) \subseteq \mathcal{C}^{r}(X, \mathbf{K}) \subseteq \mathcal{C}^{r \cdot \boldsymbol{e}_1}(X, \mathbf{K}) \subseteq \mathcal{C}^{r \cdot \boldsymbol{e}_1}_{\mathrm{T}}(X, \mathbf{K}).$$

Fix $f \in \mathcal{C}^{r \cdot \boldsymbol{e}_1}_{\mathrm{T}}(X, \mathbf{K})$ and $\varepsilon > 0$. By Lemma 3.53, we find $\mathcal{D}_{n \cdot \boldsymbol{e}_1} f \in \mathcal{C}^{r - n \cdot \boldsymbol{e}_1}(X, \mathbf{K})$ for $n = 0, \ldots, \nu$. By compactness, we find $0 < \delta_1 \leq 1$ such that for $n = 0, \ldots, \nu$ holds

$$|R_{r - n \cdot \boldsymbol{e}_1} \mathcal{D}_{n \cdot \boldsymbol{e}_1} f(x; t)| \leq \varepsilon |t|^{r-n} \quad \text{for all } \ x + t \cdot \boldsymbol{e}_1, x \in X \text{ with } |t| \leq \delta_1. \qquad (*)$$

We will fix $\delta := \delta_1 > 0$ and $\boldsymbol{\delta} := (\delta_1, 0, \ldots, 0) \in [0, 1]^d$ until the end of this proof. By downward induction on $n = \nu, \ldots, 0$, we will inductively construct locally $\boldsymbol{\delta}$-constant (see Definition 1.26) functions $g_{\nu \cdot \boldsymbol{e}_1}, \ldots, g_{n \cdot \boldsymbol{e}_1} : X \to \mathbf{K}$ such that

$$\|\mathcal{D}_{n \cdot \boldsymbol{e}_1} f - \binom{\nu}{n} g_{\nu \cdot \boldsymbol{e}_1} *^{\nu - n \cdot \boldsymbol{e}_1} - \cdots - \binom{n+1}{n} g_{n+1 \cdot \boldsymbol{e}_1} *^{\boldsymbol{e}_1} - g_{n \cdot \boldsymbol{e}_1}\|_{\sup} \leq \varepsilon \delta^{r-n}.$$

Let $n = \nu$. By $(*)$ for $n = \nu$, it holds

$$|\mathcal{D}_{\nu \cdot \boldsymbol{e}_1} f(x'_1, x_2, \ldots, x_d) - \mathcal{D}_{\nu \cdot \boldsymbol{e}_1} f(x_1, x_2, \ldots, x_d)| \leq \varepsilon |x'_1 - x_1|^{\rho}$$

for all $(x'_1, x_2, \ldots, x_d), (x_1, x_2, \ldots, x_d) \in X$ with $|x'_1 - x_1| \leq \delta_1$. By Lemma 1.28 applied to $\boldsymbol{\delta} = (\delta, 0, \ldots, 0)$, we find locally $\boldsymbol{\delta}$-constant $g_{\nu \cdot \boldsymbol{e}_1} : X \to \mathbf{K}$ such that

$$\|\mathcal{D}_{\nu \cdot \boldsymbol{e}_1} f - g_{\nu \cdot \boldsymbol{e}_1}\|_{\sup} \leq \varepsilon \delta^{\rho}.$$

Let $n < \nu$ and assume we have constructed locally $\boldsymbol{\delta}$-constant functions $g_{\nu \cdot \boldsymbol{e}_1}, \ldots, g_{n+1 \cdot \boldsymbol{e}_1} : X \to \mathbf{K}$ such that

$$\|\mathcal{D}_{m \cdot \boldsymbol{e}_1} f - \binom{\nu}{m} g_{\nu \cdot \boldsymbol{e}_1} *^{\nu - m \cdot \boldsymbol{e}_1} - \cdots - g_{m \cdot \boldsymbol{e}_1}\|_{\sup} \leq \varepsilon \delta^{r-m} \quad \text{for } m = \nu, \ldots, n+1.$$

We put $\breve{f}_{n \cdot \boldsymbol{e}_1} := \mathcal{D}_{n \cdot \boldsymbol{e}_1} f - \binom{\nu}{n} g_{\nu \cdot \boldsymbol{e}_1} *^{\nu - n \cdot \boldsymbol{e}_1} - \cdots - \binom{n+1}{n} g_{n+1 \cdot \boldsymbol{e}_1} *^{\boldsymbol{e}_1}$. Let $(x'_1, x_2, \ldots, x_d)$ and $(x_1, x_2, \ldots, x_d)$ be two points in X. We will prove

$$|\breve{f}_{n \cdot \boldsymbol{e}_1}(x'_1, x_2, \ldots, x_d) - \breve{f}_{n \cdot \boldsymbol{e}_1}(x_1, x_2, \ldots, x_d)| \leq \varepsilon \delta^{r-n} \quad \text{if } \ |x'_1 - x_1| \leq \delta.$$

Then by Lemma 1.28, there exists locally $\boldsymbol{\delta}$-constant $g_{n\cdot e_1} : X \to \mathbf{K}$ such that $f_{n\cdot e_1} := \breve{f}_{n\cdot e_1} - g_{n\cdot e_1}$ has norm $\|f_{n\cdot e_1}\|_{\sup} \leq \varepsilon\delta^{r-n}$. This will complete the n-th construction step since

$$f_{n\cdot e_1} = \breve{f}_{n\cdot e_1} - g_{n\cdot e_1} = \mathcal{D}_{n\cdot e_1} f - \binom{\nu}{n} g_{\nu\cdot e_1} *^{\nu - n\cdot e_1} - \cdots - \binom{n+1}{n} g_{n+1\cdot e_1} *^{e_1} - g_{n\cdot e_1}.$$

Put $X = X' \times X''$ with $X' = X_1$ and $X'' = X_2 \times \cdots \times X_d$ and let $(x'+h', x''), (x', x'') \in X' \times X''$ with $h' \in \mathbf{K}$. We compute

$$\begin{aligned}
&|\breve{f}_{n\cdot e_1}(x'+h', x'') - \breve{f}_{n\cdot e_1}(x', x'')| \\
=&|\mathcal{D}_{n\cdot e_1} f(x'+h', x'') - \mathcal{D}_{n\cdot e_1} f(x', x'') - \sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} g_{n+i\cdot e_1}(x', x'')((x'+h')^i - x'^i)| \\
\leq \quad & |\mathcal{D}_{n\cdot e_1} f(x'+h', x'') - \mathcal{D}_{n\cdot e_1} f(x', x'') - \sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} \mathcal{D}_{i\cdot e_1} f(x', x'') h'^i| \\
&\vee |\sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} \mathcal{D}_{n+i\cdot e_1} f(x', x'') h'^i - \sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} g_{n+i\cdot e_1}(x', x'')((x'+h')^i - x'^i)|;
\end{aligned}$$

the first equality by $g_{n+1\cdot e_1}, \ldots, g_{\nu\cdot e_1} : X \to \mathbf{K}$ being locally $\boldsymbol{\delta}$-constant. To prove the claimed inequality above, we will assume from now on $|h'| \leq \delta$. We can then estimate by Inequality $(*)$ the above maximum's first absolute value through

$$\begin{aligned}
&|\mathcal{D}_{n\cdot e_1} f(x'+h', x'') - \mathcal{D}_{n\cdot e_1} f(x', x'') - \sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} \mathcal{D}_{n+i\cdot e_1} f(x', x'') h'^i| \\
=&|R_{r-n\cdot e_1} \mathcal{D}_{n\cdot e_1} f((x', x''); h)| \leq \varepsilon |h'|^{r-n} \leq \varepsilon \delta^{r-n}.
\end{aligned}$$

Regarding the second term, let us fix $n \in \mathbb{N}$. We use the binomial identity and rearrange the summation order to obtain

$$\begin{aligned}
&\sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} g_{n+i\cdot e_1}(x', x'')((x'+h')^i - x'^i) \\
=&\sum_{i=1,\ldots,\nu-n} \binom{n+i}{n} g_{n+i\cdot e_1}(x', x'') \sum_{j=1,\ldots,i} \binom{i}{j} x'^{i-j} h'^j \\
=&\sum_{j=1,\ldots,\nu-n} h'^j \sum_{i=0,\ldots,\nu-n-j} \binom{i+j}{j} \binom{n+i+j}{n} g_{n+i+j\cdot e_1}(x', x'') x'^i \\
=&\sum_{j=1,\ldots,\nu-n} h'^j \sum_{i=0,\ldots,\nu-(n+j)} \binom{n+j}{n} \binom{n+j+i}{n+j} g_{n+j+i\cdot e_1}(x', x'') x'^i.
\end{aligned}$$

We obtain

$$
\begin{aligned}
&|\sum_{i=1,\dots,\nu-n}\binom{n+i}{n}\mathcal{D}_{n+i\cdot e_1}f(x',x'')h'^i - \sum_{i=1,\dots,\nu-n}\binom{n+i}{n}g_{n+i\cdot e_1}(x',x'')((x'+h')^i - x'^i)|\\
=&|\sum_{j=1,\dots,\nu-n} h'^j\binom{n+j}{n}[\mathcal{D}_{n+j\cdot e_1}f(x',x'') - \sum_{i=0,\dots,\nu-(n+j)}\binom{n+j+i}{n+j}g_{n+j+i\cdot e_1}(x',x'')x'^i]|\\
\leq& \max_{m=n+1,\dots,\nu}\|\mathcal{D}_{m\cdot e_1}f - \binom{\nu}{m}g_{\nu\cdot e_1} *^{\nu-m\cdot e_1} - \dots - g_{m\cdot e_1}\|_{\sup}|h'|^{m-n}\\
\leq& \max_{m=n+1,\dots\nu}\varepsilon\delta^{r-m}\delta^{m-n} = \varepsilon\delta^{r-n};
\end{aligned}
$$

the last inequality by the induction hypothesis for $m = n+1,\dots,\nu$ (and since $|h'| \leq \delta$).

Having found $g_{0\cdot e_1},\dots,g_{\nu\cdot e_1} : X \to \mathbf{K}$, we claim that the locally ($\boldsymbol{\delta}$-)polynomial function $g := g_{\nu\cdot e_1} *^{\nu\cdot e_1} + \dots + g_{1\cdot e_1} *^{e_1} + g_{0\cdot e_1}$ accomplishes $\|\tilde{f}\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot e_1}} \leq \varepsilon$ with $\tilde{f} := f - g$. For this, we prove firstly $\|\mathcal{D}_{n\cdot e_1}\tilde{f}\|_{\sup} \leq \varepsilon\delta^{r-n}$ for $n = 0,\dots,\nu$.

By a multi-variable version of Lemma 2.22 through Lemma 3.14, we find

$$
D_{n\cdot e_1}g = \binom{\nu}{n}g_{\nu\cdot e_1} *^{\nu-n\cdot e_1} - \dots - \binom{n+1}{n}g_{n+1\cdot e_1} *^{e_1} - g_{n\cdot e_1}.
$$

By construction of $g_{\nu\cdot e_1},\dots,g_{0\cdot e_1} : X \to \mathbf{K}$, it holds

$$
\|\mathcal{D}_{n\cdot e_1}\tilde{f}\|_{\sup} = \|\mathcal{D}_{n\cdot e_1}f - D_{n\cdot e_1}g\|_{\sup} \leq \varepsilon\delta^{r-n} \quad \text{for } n = 0,\dots,\nu. \tag{$**$}
$$

It now remains to prove

$$
|\Delta_{r\cdot e_1}\tilde{f}|(x;t) \leq \varepsilon \quad \text{for all } x + t\cdot e_1, x \in X.
$$

First off, assume $|t| \leq \delta$. Then by construction, the $g_{\nu\cdot e_1},\dots,g_{0\cdot e_1} : X \to \mathbf{K}$ are locally $\boldsymbol{\delta}$-constant. Hence $R_{\nu\cdot e_1}g(x;t) = 0$ for all $x + t\cdot e_1, x \in X$ with $|t| \leq \delta$. Therefore $|\Delta_{r\cdot e_1}\tilde{f}|(x;t) = |\Delta_{r\cdot e_1}f|(x;t) \leq \varepsilon$ by Inequality ($*$) for $n = 0$. Otherwise $|t| > \delta$. Then we estimate

$$
\begin{aligned}
|\Delta_{r\cdot e_1}\tilde{f}|(x;t) =&|R_{\nu\cdot e_1}\tilde{f}(x;t)|/|t|^r\\
=&|f(x+t\cdot e_1) - g(x+t\cdot e_1) - \sum_{i=0,\dots,\nu}(\mathcal{D}_{i\cdot e_1}f(x) - D_{i\cdot e_1}g(x))t^i|/|t|^r\\
\leq&(\|\tilde{f}\|_{\sup} \vee \max_{i=0,\dots,\nu}\|\mathcal{D}_{i\cdot e_1}f - D_{i\cdot e_1}g\|_{\sup}\delta^i)/\delta^r\\
\leq&(\varepsilon\delta^r \vee \varepsilon\max_{i=0,\dots,\nu}\delta^{r-i}\delta^i)/\delta^r = \varepsilon. && \text{(by ($**$))}
\end{aligned}
$$

■

Lemma 3.55. *Let $X_1, \ldots, X_d \subseteq \mathbf{K}$ be compact open subsets. Consider the mapping*

$$\mathcal{C}^r_{\mathrm{T}}(X_1, \mathbf{K}) \times \mathcal{C}^0(X_2, \mathbf{K}) \times \cdots \times \mathcal{C}^0(X_d, \mathbf{K}) \to \mathcal{C}^{r\cdot e_1}_{\mathrm{T}}(X_1 \times \cdots \times X_d, \mathbf{K}),$$
$$(f_1, \ldots, f_d) \mapsto f := [(x_1, \ldots, x_d) \mapsto f_1(x_1) \cdots f_d(x_d)].$$

It induces an (isometric) isomorphism of $\mathbf{K}$-Banach spaces

$$\mathcal{C}^r_{\mathrm{T}}(X_1, \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X_2, \mathbf{K}) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{C}^0(X_d, \mathbf{K}) \to \mathcal{C}^{r\cdot e_1}_{\mathrm{T}}(X_1 \times \cdots \times X_d, \mathbf{K}).$$

Proof. Firstly, notice that putting $X' := X_1 \subseteq \mathbf{K}$ and $X'' := X_2 \times \cdots X_d \subseteq \mathbf{K}^{d-1}$, the above mapping is given by

$$\mathcal{C}^r_{\mathrm{T}}(X', \mathbf{K}) \times \mathcal{C}^0(X'', \mathbf{K}) \xrightarrow{\Psi} \mathcal{C}^{r\cdot e_1}_{\mathrm{T}}(X' \times X'', \mathbf{K}),$$
$$(f, g) \mapsto f \odot g.$$

We prove $\operatorname{im} \Psi \subseteq \mathcal{C}^{r\cdot e_1}_{\mathrm{T}}(X, \mathbf{K})$ with $X := X' \times X''$. Let us assume $f \in \mathcal{C}^r_{\mathrm{T}}(X', \mathbf{K}), g \in \mathcal{C}^0(X'', \mathbf{K})$. Let $h = f \odot g$ be the image of (f, g). We suppose that the continuous functions $\mathcal{D}_0 f, \ldots, \mathcal{D}_\nu f : X' \to \mathbf{K}$ prove f to be a $\mathcal{C}^r_{\mathrm{T}}$-function. We claim that the maps $\mathcal{D}_{n\cdot e_1} h := \mathcal{D}_n f \odot g : X \to \mathbf{K}$ for $n = 0, \ldots, \nu$ prove $h : X \to \mathbf{K}$ to be a $\mathcal{C}^{r\cdot e_1}$-function: The maps $\mathcal{D}_{0\cdot e_1} h, \ldots, \mathcal{D}_{\nu\cdot e_1} h$ are continuous. It suffices to prove that for every $\varepsilon > 0$, there exists some $\delta > 0$ such that

$$|R_{\nu\cdot e_1} f(x; t)| \leq \varepsilon |t|^r \quad \text{for all } (x; t) \in X^{[e_1]} \text{ with } |t| < \delta.$$

Since $f \in \mathcal{C}^r_{\mathrm{T}}(X, \mathbf{K})$, there exists such $\delta > 0$ such that $|R_\nu f(x'; t)| \|g\|_{\sup} \leq \varepsilon |t|^r$ for $x' + t, x' \in X'$ with $|t| < \delta$. Then for $x + t\cdot e_1, x \in X$ with $|t| < \delta$ and $x = (x', x'') \in X = X' \times X''$, we compute

$$\begin{aligned}|R_{\nu\cdot e_1} f(x; t)| &= |f(x' + t) \odot g(x'') - \sum_{i=0,\ldots,\nu} \mathcal{D}_i f(x') \odot g(x'')| \\ &= |R_\nu f(x'; t)||g(x'')| \leq |R_\nu f(x'; t)| \|g\|_{\sup} \leq \varepsilon |t|^r.\end{aligned}$$

Secondly, by the criterion of [van Rooij, 1978, Comment following Cor. 4.31], we have to check the following:

1. The mapping Ψ is bilinear and norm-nonincreasing.
2. The $\mathbf{K}$-linear span of $\operatorname{im} \Psi$ is dense in $\mathcal{C}^{r\cdot e_1}_{\mathrm{T}}(X' \times X'', \mathbf{K})$.
3. Let $0 < t \leq 1$. If $f_1, \cdots, f_n \in \mathcal{C}^r_{\mathrm{T}}(X', \mathbf{K})$ are t-orthogonal and $g_1, \ldots, g_n \in \mathcal{C}^0(X'', \mathbf{K})$, then their products $f_1 \odot g_1, \ldots, f_n \odot g_n \in \mathcal{C}^{r\cdot e_1}_{\mathrm{T}}(X' \times X'', \mathbf{K})$ will be t-orthogonal.

Ad 1.: The map Ψ is quickly checked to be bilinear. We find

$$\begin{aligned}\|f \odot g\|_{\mathcal{C}^{r\cdot e_1}_{\mathrm{T}}} &= \|\mathcal{D}_{0\cdot e_1} f \odot g\|_{\sup} \vee \ldots \vee \|\mathcal{D}_{\nu\cdot e_1} f \odot g\|_{\sup} \vee \||\Delta_{r\cdot e_1} f \odot g|\|_{\sup} \\ &= \|\mathcal{D}_0 f \odot g\|_{\sup} \vee \ldots \vee \|\mathcal{D}_\nu f \odot g\|_{\sup} \vee \||\Delta_r f| \odot |g|\|_{\sup} \\ &= (\|\mathcal{D}_0 f\|_{\sup} \vee \ldots \vee \|\mathcal{D}_\nu f\|_{\sup} \vee \||\Delta_r f|\|_{\sup}) \cdot \|g\|_{\sup} \\ &= \|f\|_{\mathcal{C}^r_{\mathrm{T}}} \cdot \|g\|_{\mathcal{C}^0}.\end{aligned}$$

Ad 2.: All locally monomial functions with a ball as support lie in $\operatorname{im}\Psi$. Hence all locally polynomial functions with a ball as support are in the $\mathbf{K}$-linear span of $\operatorname{im}\Psi$. Since the balls form a basis of the topological space X and this space is compact, we find all locally polynomial functions to be in the $\mathbf{K}$-linear span of $\operatorname{im}\Psi$. By Lemma 3.54 those of degree $\vec{g} \leq \boldsymbol{\nu} \cdot \boldsymbol{e}_1$ are already dense in $\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}(X' \times X'', \mathbf{K})$.

Ad 3.: We compute

$$\begin{aligned} &\|f_1 \odot g_1 + \cdots + f_n \odot g_n\|_{\mathcal{C}_{\mathrm{T}}^{re_1}} \\ &= \max_{i=0,\dots,\nu} \|\mathcal{D}_{i \cdot \boldsymbol{e}_1}(f_1 \odot g_1 + \cdots + f_n \odot g_n)\|_{\sup} \vee \||\Delta_{r \cdot \boldsymbol{e}_1}(f_1 \odot g_1 + \cdots + f_n \odot g_n)|\|_{\sup} \\ &\geq \quad \max_{i=0,\dots,\nu} \|\mathcal{D}_{i \cdot \boldsymbol{e}_1} f_1 \cdot g_1(x'') + \cdots + \mathcal{D}_{i \cdot \boldsymbol{e}_1} f_n \cdot g_n(x'')\|_{\sup} \\ &\quad \vee \||\Delta_{r \cdot \boldsymbol{e}_1}(f_1 \cdot g_1(x'') + \cdots + f_n \cdot g_n(x''))|\|_{\sup} \qquad \text{for any fixed } x'' \in X''. \end{aligned}$$

We see that the last term for fixed $x'' \in X''$ equals $\|f_1 \cdot g_1(x'') + \cdots + f_n \cdot g_n(x'')\|_{\mathcal{C}_{\mathrm{T}}^r}$. Since $f_1, \dots, f_n$ are t-orthogonal with respect to $\|\cdot\|_{\mathcal{C}_{\mathrm{T}}^r}$, we find for all $x'' \in X''$ that

$$\|f_1 \cdot g_1(x'') + \cdots + f_n \cdot g_n(x'')\|_{\mathcal{C}_{\mathrm{T}}^r}/t \geq |g_1(x'')| \cdot \|f_1\|_{\mathcal{C}_{\mathrm{T}}^r} \vee \ldots \vee |g_n(x'')| \cdot \|f_n\|_{\mathcal{C}_{\mathrm{T}}^r}.$$

In particular it holds for $j = 1, \dots, n$ that

$$\|f_1 \cdot g_1(x'') + \cdots + f_n \cdot g_n(x'')\|_{\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}}/t \geq \sup_{x'' \in X''} |g_j(x'')| \|f_j\|_{\mathcal{C}_{\mathrm{T}}^r} = \|g_j\|_{\sup} \cdot \|f_j\|_{\mathcal{C}_{\mathrm{T}}^r}.$$

We conclude

$$\|f_1 g_1 + \cdots + f_n g_n\|_{\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}} \geq t \cdot \max_{j=1,\dots,n} \|g_j\|_{\sup} \|f_j\|_{\mathcal{C}_{\mathrm{T}}^r} = t \cdot \max_{j=1,\dots,n} \|f_j \odot g_j\|_{\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}}.$$

∎

Assumption. We will assume until the end of this subsection that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a normed field.

Lemma 3.56. *The family* $\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}_{\mathrm{T}}^{r \cdot e_1}(\mathbb{Z}_p^d, \mathbf{K})$ *is an orthogonal basis with*

$$(\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}})_{\boldsymbol{i} \in \mathbb{N}^d} \sim (i_1^r)_{\boldsymbol{i} \in \mathbb{N}^d}.$$

Proof. Denote $X' = \mathbb{Z}_p$ and $X'' = \mathbb{Z}_p^{d-1}$. We consider the composition of morphisms

$$\mathcal{C}^r(X', \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X'', \mathbf{K}) \to \mathcal{C}_{\mathrm{T}}^r(X', \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X'', \mathbf{K}) \to \mathcal{C}_{\mathrm{T}}^{r \cdot e_1}(X' \times X'', \mathbf{K}).$$

The first arrow is the induced map $\iota \widehat{\otimes} \mathrm{id} : \mathcal{C}^r(X', \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X'', \mathbf{K}) \to \mathcal{C}_{\mathrm{T}}^r(X', \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X'', \mathbf{K})$ of the topological $\mathbf{K}$-vector space morphisms given by inclusion $\iota : \mathcal{C}^r(X', \mathbf{K}) \hookrightarrow \mathcal{C}_{\mathrm{T}}^r(X', \mathbf{K})$ and the identity on $\mathcal{C}^0(X'', \mathbf{K})$. Since $X' \subseteq \mathbb{Q}_p$ is a ball, it has the B_ν-property by Lemma 2.27. By Corollary 2.32, noting $\mathbb{Q}_p$ being locally compact, and Corollary 2.25, the canonical inclusion $\mathcal{C}^r(X', \mathbf{K}) \hookrightarrow \mathcal{C}_{\mathrm{T}}^r(X', \mathbf{K})$ is a topological isomorphism of $\mathbf{K}$-vector spaces. By

functoriality, the first map is therefore an isomorphism of topological $\mathbf{K}$-vector spaces. The right hand isomorphism of $\mathbf{K}$-Banach spaces is given by the preceding Lemma 3.55. Therefore its composition is an isomorphism of topological $\mathbf{K}$-vector spaces.
We conclude

$$\{\binom{*}{\boldsymbol{i}}\} \subseteq \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \simeq \mathcal{C}^r(X', \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(X'', \mathbf{K}) \simeq \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) \widehat{\otimes} \mathcal{C}^0(\mathbb{Z}_p) \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{C}^0(\mathbb{Z}_p, \mathbf{K})$$

by Theorem 2.55 and Corollary 3.34 to be an orthogonal basis with

$$(\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}})_{\boldsymbol{i}\in\mathbb{N}^d} \sim (\|\binom{*}{i_1}\|_{\mathcal{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d} = (p^{w_r(i_1)})_{\boldsymbol{i}\in\mathbb{N}^d}.$$

Then by Lemma 2.58, it holds $(p^{w_r(i_1)})_{i_1\in\mathbb{N}} \sim (i_1^r)_{i_1\in\mathbb{N}}$. ■

Lemma 3.57. *The $\mathbf{K}$-Banach space $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ is the initial topological $\mathbf{K}$-vector space with respect to the inclusion mappings*

$$\begin{array}{ccc} & & \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \\ & \overset{\text{incl.}}{\nearrow} & \\ \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K}) & & \vdots \\ & \underset{\text{incl.}}{\searrow} & \\ & & \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K}) \end{array}$$

That is $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K}) = \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \cap \ldots \cap \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K})$ as an abstract $\mathbf{K}$-vector space and its norm $\|\cdot\|_{\mathcal{C}^r}$ on $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ is equivalent to $\|\cdot\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}} \vee \ldots \vee \|\cdot\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}}$.

Proof. We consider the canonical commutative diagram

$$\begin{array}{ccc} c_0((\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d}) & \xrightarrow{\sim} & c_0((\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}} \vee \ldots \vee \|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}})_{\boldsymbol{i}\in\mathbb{N}^d}) \\ \downarrow\sim & & \downarrow\sim \\ \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K}) & \xhookrightarrow{\text{incl.}} & \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \cap \ldots \cap \mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K}) \end{array}$$

Here the $\mathbf{K}$-Banach space at the bottom right is defined as the initial $\mathbf{K}$-Banach space of $\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_k}(\mathbb{Z}_p^d, \mathbf{K})$ for $k = 1, \ldots, d$ (inside $\mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$) and the lower inclusion mapping is given by Lemma 3.52. By Theorem 3.47, the left-hand map is an isomorphism of $\mathbf{K}$-Banach spaces and by the preceding Lemma 3.56 for $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_d$ together with Lemma 3.35(ii) the right-hand map is an isomorphism of $\mathbf{K}$-Banach spaces. By Lemma 3.48 and Lemma 3.56 for $k = 1, \ldots, d$, we find

$$(\left\|\binom{*}{\boldsymbol{i}}\right\|_{\mathcal{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d} \sim (i_1^r \vee \ldots \vee i_d^r)_{\boldsymbol{i}\in\mathbb{N}^d} \sim (\left\|\binom{*}{i}\right\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}} \vee \ldots \vee \left\|\binom{*}{i}\right\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}})_{\boldsymbol{i}\in\mathbb{N}^d}.$$

So $(\|\binom{*}{i}\|_{\mathcal{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d} \sim (\|\binom{*}{i}\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_1}} \vee \ldots \vee \|\binom{*}{i}\|_{\mathcal{C}_{\mathrm{T}}^{r\cdot \boldsymbol{e}_d}})_{\boldsymbol{i}\in\mathbb{N}^d}$ and the upper map is an isomorphism of topological $\mathbf{K}$-vector spaces. The commutativity of the diagram can by $\mathbf{K}$-linearity and continuity be checked on all $\boldsymbol{e}_{\boldsymbol{i}} \in c_0((\|\binom{*}{\boldsymbol{i}}\|_{\mathcal{C}^r})_{\boldsymbol{i}\in\mathbb{N}^d})$ whose only nonzero entry is 1 at the $\boldsymbol{i}$-th position. There it holds by definition of the above maps. All together, the bottom map is also an isomorphism of topological $\mathbf{K}$-vector spaces. ■

Lemma 3.58. *Let C and D be balls in $\mathbb{Q}_p^d$ and $k \in \{1, \ldots, d\}$.*

(i) Every $\mathbb{Q}_p$-scalar mapping $s : C \overset{\sim}{\to} D$ induces by precomposition a morphism of $\mathbf{K}$-Banach spaces $\mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(D, \mathbf{K}) \to \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(C, \mathbf{K})$.

(ii) Every $\mathbb{Q}_p$-translate mapping $t : C \overset{\sim}{\to} D$ induces by precomposition a morphism of $\mathbf{K}$-Banach spaces $\mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(D, \mathbf{K}) \to \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(C, \mathbf{K})$.

(iii) Every $\mathbb{Q}_p$-affine scalar mapping $m : C \overset{\sim}{\to} D$ induces by precomposition an isomorphism of topological $\mathbf{K}$-vector spaces $\mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(D, \mathbf{K}) \to \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(C, \mathbf{K})$.

Proof. Ad (i): Let $f \in \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(D, \mathbf{K})$ and assume $\mathcal{D}_{n \cdot \boldsymbol{e}_k} f : D \to \mathbf{K}$ for $n = 0, \ldots, \nu$ to prove this. We claim that the maps $\lambda^n \mathcal{D}_{n \cdot \boldsymbol{e}_k} f \circ s : D \to \mathbf{K}$ prove $f \circ s \in \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(C, \mathbf{K})$. We compute

$$\begin{aligned} R_{\nu \cdot \boldsymbol{e}_k} f \circ s(x; t) &= f \circ s(x + t \cdot \boldsymbol{e}_k) - \sum_{n=0,\ldots,\nu} \lambda^n \mathcal{D}_{n \cdot \boldsymbol{e}_k} f \circ s(x) t^n \\ &= f(\lambda \cdot (x + t \cdot \boldsymbol{e}_k)) - \sum_{n=0,\ldots,\nu} \mathcal{D}_{n \cdot \boldsymbol{e}_k} f(\lambda \cdot x)(\lambda t)^n \\ &= f(\lambda \cdot x + \lambda t \cdot \boldsymbol{e}_k) - \sum_{n=0,\ldots,\nu} \mathcal{D}_{n \cdot \boldsymbol{e}_k} f(\lambda \cdot x)(\lambda t)^n \\ &= R_{\nu \cdot \boldsymbol{e}_k} f(\lambda \cdot x; \lambda t). \end{aligned}$$

This shows $\|f \circ s\|_{\mathcal{C}^{r \cdot \boldsymbol{e}_k}} \leq \|\mathcal{D}_{0 \cdot \boldsymbol{e}_k} f \circ s\|_{\sup} \vee \ldots \vee \|\lambda^\nu \mathcal{D}_{\nu \cdot \boldsymbol{e}_k} f \circ s\|_{\sup} \vee \||\Delta_{r \cdot \boldsymbol{e}_k} f|\|_{\sup} \cdot |\lambda|^r \leq M \cdot \|f\|_{\mathcal{C}^{r \cdot \boldsymbol{e}_k}}$ with $M := 1 \vee |\lambda|^r > 0$ and therefore continuity of precomposition.
Ad (ii): Let $f \in \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(D, \mathbf{K})$ and assume $\mathcal{D}_{n \cdot \boldsymbol{e}_k} f : D \to \mathbf{K}$ for $n = 0, \ldots, \nu$ to prove this. Then by a computation as above $\mathcal{D}_{n \cdot \boldsymbol{e}_k} f \circ t : D \to \mathbf{K}$ prove $f \circ t \in \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_k}(C, \mathbf{K})$. Moreover this mapping is quickly checked to be norm-preserving.
Ad (iii): Since the inverses of scalar and translation mappings are again such, they are by (i) and (ii) isomorphisms of topological $\mathbf{K}$-vector spaces. Then claim (iii) follows. ■

Lemma 3.59. *For any ball $C \subseteq \mathbb{Q}_p^d$, the $\mathbf{K}$-Banach space $\mathcal{C}^r(C, \mathbf{K})$ is the initial topological $\mathbf{K}$-vector space with respect to the inclusion mappings*

$$\begin{array}{ccc} & & \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_1}(C, \mathbf{K}) \\ & \overset{\text{incl.}}{\nearrow} & \\ \mathcal{C}^r(C, \mathbf{K}) & & \vdots \\ & \underset{\text{incl.}}{\searrow} & \\ & & \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_d}(C, \mathbf{K}) \end{array}$$

Proof. We consider the canonical commutative diagram

$$\begin{array}{ccc} \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K}) & \xrightarrow{\sim} & \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_1}(\mathbb{Z}_p^d, \mathbf{K}) \cap \ldots \cap \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_d}(\mathbb{Z}_p^d, \mathbf{K}) \\ \downarrow \wr & & \downarrow \wr \\ \mathcal{C}^r(C, \mathbf{K}) & \hookrightarrow & \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_1}(C, \mathbf{K}) \cap \ldots \cap \mathcal{C}_{\mathrm{T}}^{r \cdot \boldsymbol{e}_d}(C, \mathbf{K}). \end{array}$$

The lower and upper right $\mathbf{K}$-Banach spaces are the initial $\mathbf{K}$-Banach spaces of $\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}(C, \mathbf{K})$ and $\mathcal{C}_{\mathrm{T}}^{r \cdot e_1}(\mathbb{Z}_p^d, \mathbf{K})$ for $k = 1, \ldots, d$ (inside $\mathcal{C}^0(\mathbb{Z}_p^d, \mathbf{K})$) and the lower and upper arrows are the inclusion maps provided by Lemma 3.52. The left and right hand map are given by precomposition with the $\mathbb{Q}_p$-affine scalar mapping $t : C \xrightarrow{\sim} \mathbb{Z}_p^d$. By definition, the diagram commutes. Regarding the left hand map, the precomposed function $t : C \to \mathbb{Z}_p^d$ is an invertible $\mathcal{C}^r$-function for every $r \geq 0$. Therefore the left hand map is an isomorphism of topological $\mathbf{K}$-vector spaces $\mathcal{C}^r(C, \mathbf{K}) \to \mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K})$ by Proposition 3.24. The top map is an isomorphism of topological $\mathbf{K}$-vector spaces by Lemma 3.57. Concerning the right hand map, by the preceding Lemma 3.58 applied to $D = \mathbb{Z}_p^d$ for $k = 1, \ldots, d$, the above initial topological $\mathbf{K}$-vector space with respect to $\mathcal{C}_{\mathrm{T}}^{r \cdot e_k}(C, \mathbf{K})$ is isomorphic to the one with respect to $\mathcal{C}^{r \cdot e_k}(\mathbb{Z}_p^d, \mathbf{K})$ for $k = 1, \ldots, d$. All together the bottom map is an isomorphism of topological $\mathbf{K}$-vector spaces. ■

Definition. Let $X \subseteq \mathbb{Q}_p^d$ be an open subset. We define $\mathcal{C}_{\mathrm{T}}^r(X, \mathbf{K})$ as the initial locally convex $\mathbf{K}$-vector space with respect to the inclusion mappings given by

$$\mathcal{C}_{\mathrm{T}}^r(X, \mathbf{K}) \xhookrightarrow{\text{incl.}} \mathcal{C}_{\mathrm{T}}^{r \cdot e_1}(X, \mathbf{K}), \quad \ldots, \quad \mathcal{C}_{\mathrm{T}}^r(X, \mathbf{K}) \xhookrightarrow{\text{incl.}} \mathcal{C}_{\mathrm{T}}^{r \cdot e_d}(X, \mathbf{K}).$$

Corollary 3.60. *Let $X \subseteq \mathbb{Q}_p^d$ be an open subset. Then the canonical inclusion $\mathcal{C}^r(X, \mathbf{K}) \hookrightarrow \mathcal{C}_{\mathrm{T}}^r(X, \mathbf{K})$ is an isomorphism of locally convex $\mathbf{K}$-vector spaces.*

Proof. First off it is an injective homomorphism of locally convex $\mathbf{K}$-vector spaces by the Remark following Lemma 3.52. Let $f \in \mathcal{C}_{\mathrm{T}}^r(X, \mathbf{K})$. Then $f_{|C} \in \mathcal{C}_{\mathrm{T}}^r(C, \mathbf{K}) = \mathcal{C}^r(C, \mathbf{K})$ for all balls $C \subseteq X$ by Lemma 3.59. As being $\mathcal{C}^r$ is a local property, we find $f \in \mathcal{C}^r(X, \mathbf{K})$, proving surjectivity. The defining families of seminorms are given by all $\|\cdot\|_{\mathcal{C}_{\mathrm{T}}^r, C}$ on $\mathcal{C}_{\mathrm{T}}^r(X, \mathbf{K})$ respectively $\|\cdot\|_{\mathcal{C}^r, C}$ on $\mathcal{C}^r(X, \mathbf{K})$ for balls $C \subseteq X$. By Lemma 3.59, these seminorms for a fixed ball $C \subseteq X$ are equivalent. Thence we have an equality of locally convex $\mathbf{K}$-vector spaces. ■

3.4 The space $\mathcal{D}^r(X, \mathbf{K})$ of distributions on $\mathcal{C}^r(X, \mathbf{E})$ for a compact group X

Definition. For a compact locally cartesian subset $X \subseteq \mathbf{K}^d$ with local factors free of isolated points, we define the $\mathbf{K}$-vector space $\mathcal{D}^r(X, \mathbf{K})$ of **distributions** by

$$\mathcal{D}^r(X, \mathbf{K}) = \{\text{all continuous } \mathbf{K}\text{-linear mappings } \mu : \mathcal{C}^r(X, \mathbf{E}) \to \mathbf{K}\}.$$

We endow $\mathcal{D}^r(X, \mathbf{K})$ with the structure of a complete topological $\mathbf{K}$-vector space by the **operator norm** $\|\cdot\|_{\mathcal{D}^r}$ defined on $\mathcal{D}^r(X, \mathbf{K})$ by

$$\|\mu\|_{\mathcal{D}^r} = \inf\{C \in \mathbb{R}_{\geq 0} : |\mu(f)| \leq C \cdot \|f\|_{\mathcal{C}^r} \quad \text{for all } f \in \mathcal{C}^r(X, \mathbf{E})\}.$$

We remark firstly that by [van Rooij, 1978, Chapter III, Section "Linear Operators"], the operator norm $\|\cdot\|_{\mathcal{D}^r}$ is well defined, as a $\mathbf{K}$-linear operator is continuous if and only if it is *bounded* - meaning the existence of such a largest lower bound C. Secondly, by [van Rooij, 1978, Exercise 3.M(i)], this normed $\mathbf{K}$-vector space is complete with respect to $\|\cdot\|_{\mathcal{D}^r}$ as $\mathbf{K}$ is.

We want to define a convolution product for $\mathcal{C}^r$-distributions on compact groups. For this a couple of technical preparations are in order.

Definition. Let $X = X' \times X'' \subseteq \mathbf{K}^{d'} \times \mathbf{K}^{d''}$ be a nonempty compact cartesian subset whose factors have no isolated points and $f : X' \times X'' \to \mathbf{E}$ a mapping thereon. We consider $\mathbf{K}^{d'} \times \mathbf{K}^{d''}$ as direct sum $V' \oplus V''$ with $V' = \mathbf{K}^{d'}$ and $V'' = \mathbf{K}^{d''}$. Then we define f to lie in $\mathcal{C}^{\rho'\otimes\rho''}(X' \times X'', \mathbf{E})$ for $\rho', \rho'' \in [0, 1[$, if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in X$ and $h' \in V', h'' \in V''$ of norm at most δ holds, where defined,

$$\|[f(x + h' + h'') - f(x + h'')] - [f(x + h') - f(x)]\| \leq \varepsilon \|h'\|^{\rho'} \|h''\|^{\rho''}.$$

Lemma 3.61. *Let $X = X' \times X'' \subseteq \mathbf{K}^{d'} \times \mathbf{K}^{d''} = V' \oplus V''$ be a compact cartesian subset whose factors have no isolated points and $f \in \mathcal{C}^\rho(X, \mathbf{E})$ for $\rho \in [0, 2[$. Then $f \in \mathcal{C}^{\rho'\otimes\rho''}(X' \times X'', \mathbf{E})$ for all $\rho', \rho'' \in [0, 1[$ with $\rho' + \rho'' \leq \rho$.*

Proof. We distinguish two cases.
Case 1: $\rho < 1$. By compactness, there exists a $\delta > 0$ such that $\|f(x + h) - f(x)\| \leq \varepsilon\|h\|^\rho$ for all $x + h, x \in X$ with $\|h\| \leq \delta$. Applying this to $h = h'$ in V' respectively $h = h'' \in V''$, the non-Archimedean triangle inequality yields

$$\|[f(x + h' + h'') - f(x + h'')] - [f(x + h') - f(x)]\| \leq \varepsilon\|h'\|^\rho \wedge \varepsilon\|h''\|^\rho \leq \varepsilon\|h'\|^{\rho'} \|h''\|^{\rho''}$$

for all $x \in X$ and $h' \in V', h'' \in V''$ of norm at most δ such that the above term is defined.
Case 2: $\rho \geq 1$. By Proposition 1.34, we found $f \in \mathcal{C}^{1+\rho}(X, \mathbf{E})$ if and only if $f^{]1[} : X^{]1[} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V' \oplus V'', \mathbf{E})$ extends to a $\mathcal{C}^\rho$-function $f^{[1]} : X^{[1]} \to \mathrm{Hom}_{\mathbf{K}\text{-vctsp.}}(V' \oplus V'', \mathbf{E})$. By definition, we find by continuous extension $f^{[1]}(x + h, x) \cdot h = f(x + h) - f(x)$ for all $x + h, x \in X$. In particular for $h' \in V', h'' \in V''$ holds, where defined,

$$\begin{aligned}
&[f(x + h' + h'') - f(x + h'')] - [f(x + h') - f(x)] \\
=&f^{[1]}(x + h' + h'', x + h'') \cdot h' - f^{[1]}(x + h', x) \cdot h' \\
=&[f^{[1]}(x + h' + h'', x + h'') - f^{[1]}(x + h', x)] \cdot h'.
\end{aligned}$$

By compactness, there exists $\delta > 0$ such that $\|f^{[1]}(\tilde{x}, x) - f^{[1]}(\tilde{y}, y)\| \leq \varepsilon\|(\tilde{x}, x) - (\tilde{y}, y)\|^\rho$ for all $(\tilde{x}, x), (\tilde{y}, y) \in X^{[1]} = X \times X$ with $\|(\tilde{x}, x) - (\tilde{y}, y)\| \leq \delta$. In particular

$$\begin{aligned}
&\|[f(x + h' + h'') - f(x + h'')] - [f(x + h') - f(x)]\| \\
\leq&\|f^{[1]}(x + h' + h'', x + h'') - f^{[1]}(x + h', x)\|\|h'\| \\
\leq&\varepsilon \cdot \|h''\|^\rho \|h'\| && \text{if } \|h''\| \leq \delta.
\end{aligned}$$

By symmetry of h' and h'', we even have

$$\begin{aligned}&\|[f(x+h'+h'')-f(x+h'')]-[f(x+h')-f(x)]\|\\ \le&\varepsilon\cdot\|h'\|^{\rho}\|h''\|\wedge\varepsilon\cdot\|h'\|\|h''\|^{\rho}\le\varepsilon\cdot\|h'\|^{\rho'}\|h''\|^{\rho''}\qquad \text{if } \|h'\|,\|h''\|\le\delta.\end{aligned}$$

■

Lemma 3.62. *Let $X\subseteq\mathbf{K}^d$ be a nonempty cartesian subset whose factors contain no isolated point and $f\in\mathcal{C}^{\nu-1}(X,\mathbf{E})$. Then $f\in\mathcal{C}^r(X,\mathbf{E})$ if and only if $f^{[\boldsymbol{n}]}\in\mathcal{C}^{1+\rho}(X^{[\boldsymbol{n}]},\mathbf{E})$ for all $\boldsymbol{n}\in\mathbb{N}^d_{=\nu-1}$.*

Proof. Let $\boldsymbol{n}\in\mathbb{N}^d_{=\nu-1}$. By continuous extension of Lemma 3.5(i), the mapping $f^{[\boldsymbol{n}]}:X^{[\boldsymbol{n}]}\to\mathbf{E}$ is symmetric on $X_1^{[n_1]},\dots,X_d^{[n_d]}$. Recall $\mathbb{N}^{[\boldsymbol{n}]}=\mathbb{N}^{[n_1]}\times\dots\times\mathbb{N}^{[n_d]}$ for $\boldsymbol{n}\in\mathbb{N}^d$ with $\mathbb{N}^{[n]}=\mathbb{N}^{\{0,\dots,n\}}$ and ${}^k\boldsymbol{e}_0:=(\mathbf{0};\dots;\boldsymbol{e}_0;\dots;\mathbf{0})\in\mathbb{N}^{[\boldsymbol{n}]}$, whose only nonzero vector entry is $\boldsymbol{e}_0=(1,0,\dots)\in\mathbb{N}^{[n_k]}$ at the k-th place. Denote by

$$I_k=\{(k,0),\dots,(k,n_k)\}=\{\ X_k\text{-coordinate indices of } X^{[\boldsymbol{n}]}\ \}.$$

Then the only nonzero entry of ${}^k\boldsymbol{e}_0$ is at the i_k-th coordinate for a representative $i_k\in I_k$. By Corollary 1.40, we find $f^{[\boldsymbol{n}]}\in\mathcal{C}^{1+\rho}(X^{[\boldsymbol{n}]},\mathbf{E})$ if and only if $f^{[\boldsymbol{n}]}\in\mathcal{C}^{(1+\rho)\cdot{}^k\boldsymbol{e}_0}(X^{[\boldsymbol{n}]},\mathbf{E})$ for $k=1,\dots,d$. By Lemma 3.19, this holds for $k=1,\dots,d$ if and only if $f^{]\boldsymbol{n}+\boldsymbol{e}_k[}:X^{]\boldsymbol{n}+\boldsymbol{e}_k[}\to\mathbf{E}$ extends to $f^{[\boldsymbol{n}+\boldsymbol{e}_k]}\in\mathcal{C}^{\rho}(X^{[\boldsymbol{n}+\boldsymbol{e}_k]},\mathbf{E})$. Since $\mathbb{N}^d_{=\nu}=\{\boldsymbol{n}+\boldsymbol{e}_k:\boldsymbol{n}\in\mathbb{N}^d_{=\nu-1}\text{ and }k=1,\dots,d\}$, this holds by Proposition 3.8 if and only if $f\in\mathcal{C}^r(X,\mathbf{E})$. ■

Remark 3.63. Let $X\subseteq\mathbf{K}^d$ be a compact locally cartesian subset with local factors free of isolated points. For a finite covering $\mathfrak{U}=\{U_1,\dots,U_n\}$ by balls of X (which are closed and hence compact), define the norm $\|\cdot\|_{\mathcal{C}^r,\mathfrak{U}}:=\|\cdot\|_{\mathcal{C}^r,U_1}\vee\dots\vee\|\cdot\|_{\mathcal{C}^r,U_n}$ on $\mathcal{C}^r(X,\mathbf{E})$. Then the locally convex topology on $\mathcal{C}^r(X,\mathbf{E})$ is induced by any such norm.

Proof. Given two such coverings $\mathfrak{U}$ and $\tilde{\mathfrak{U}}$ by balls of X, we have to prove that $\|\cdot\|_{\mathcal{C}^r,\mathfrak{U}}$ and $\|\cdot\|_{\mathcal{C}^r,\tilde{\mathfrak{U}}}$ are equivalent. Then we can jointly refine these coverings. Since equivalence of norms is transitive, it thus suffices to prove that if $\tilde{\mathfrak{U}}=\{\tilde U_1,\dots,\tilde U_m\}$ refines $\mathfrak{U}$, then the norms $\|\cdot\|:=\|\cdot\|_{\mathcal{C}^r,\mathfrak{U}}$ and $\widetilde{\|\cdot\|}_{\mathcal{C}^r}:=\|\cdot\|_{\mathcal{C}^r,\tilde{\mathfrak{U}}}$ will be equivalent. We have $\widetilde{\|\cdot\|}\le\|\cdot\|$ as $\|\cdot\|_{\mathcal{C}^r,\tilde U}\le\|\cdot\|_{\mathcal{C}^r,U}$ for $U\supseteq\tilde U$. For the inverse estimate, let $\delta:=\delta\,\tilde U_1\wedge\dots\wedge\delta\,\tilde U_m\in]0,1]$, w.l.o.g.

Let for the time being $X\subseteq\mathbf{K}^d$ be compact cartesian with factors free of isolated points. Let $X^{[\boldsymbol{m}]}_{\le\delta}$ be as in Lemma 3.15. By induction on $|\boldsymbol{n}|=:n=0,\dots,\nu$, it holds for any $f\in\mathcal{C}^r(X,\mathbf{E})$ by symmetry of $f^{[\boldsymbol{n}]}:X^{[\boldsymbol{n}]}\to\mathbf{E}$ in its X_k-coordinates for $k=1,\dots,d$ that

$$\|f^{[\boldsymbol{n}]}\|_{X^{[\boldsymbol{n}]}}\le\|f^{[\boldsymbol{n}]}\|_{X^{[\boldsymbol{n}]}_{\le\delta}}\vee\max_{\boldsymbol{m}\in\mathbb{N}^d_{\le n}}\|f^{[\boldsymbol{m}]}\|_{X^{[\boldsymbol{m}]}}/\delta^{n-|\boldsymbol{m}|}.$$

Likewise for $\boldsymbol{n}\in\mathbb{N}^d_{=\nu}$, we find

$$\begin{aligned}\|f^{[\boldsymbol{n}]}\|_{\mathcal{C}^\rho}=&\|f^{[\boldsymbol{n}]}\|_{\sup}\vee|||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_1]}|||_{\sup}\vee\dots\vee|||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|||_{\sup}\\ \le&\max_{\boldsymbol{n}\in\mathbb{N}^d_{=\nu}}\|f^{[\boldsymbol{n}]}\|_{X^{[\boldsymbol{n}]}}/\delta^{\rho}\vee|||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_1]}|||_{X^{[\boldsymbol{n}+\boldsymbol{e}_1]}_{\le\delta}}\vee\dots\vee|||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|||_{X^{[\boldsymbol{n}+\boldsymbol{e}_d]}_{\le\delta}}\\ \le&\max_{\boldsymbol{m}\in\mathbb{N}^d_{\le\nu}}\|f^{[\boldsymbol{m}]}\|_{X^{[\boldsymbol{m}]}_{\le\delta}}/\delta^{r-|\boldsymbol{m}|}\vee|||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_1]}|||_{X^{[\boldsymbol{n}+\boldsymbol{e}_1]}_{\le\delta}}\vee\dots\vee|||f^{[\boldsymbol{n}+\rho\cdot\boldsymbol{e}_d]}|||_{X^{[\boldsymbol{n}+\boldsymbol{e}_d]}_{\le\delta}};\end{aligned}$$

the first equality by Lemma 3.28, the following inequality by Lemma 3.29 and the last one we just have shown. By Lemma 3.15, recall $X^{[\boldsymbol{n}]}_{\leq\delta} = \cup_{p\in X} P^{[\boldsymbol{n}]}$ with $P := \mathrm{B}_{\leq\delta}(p) \subseteq X$ for $\boldsymbol{n} \in \mathbb{N}^d$. If $P_1, \dots, P_t \subseteq X$ are covering balls of diameter δ, thus

$$\begin{aligned}\|f\|_{\mathcal{C}^r} \leq & \max_{\boldsymbol{m}\in\mathbb{N}^d_{\leq\nu}} |||f^{[\boldsymbol{m}]}|||_{X^{[\boldsymbol{m}]}}/\delta^{r-|\boldsymbol{m}|} \vee \max_{k=1,\dots,d} \max_{\boldsymbol{n}\in\mathbb{N}^d_{=\nu}} |||f^{[\boldsymbol{n}+\rho\cdot e_k]}|||_{X^{[\boldsymbol{n}+e_k]}_{\leq\delta}} \\ \leq & 1/\delta^r \cdot (\|f\|_{\mathcal{C}^r,P_1} \vee \dots \vee \|f\|_{\mathcal{C}^r,P_t}).\end{aligned}$$

Let $U \in \mathfrak{U}$ be compact cartesian with factors free of isolated points. Then $\{P_1, \dots, P_t\}$ refines $\tilde{\mathfrak{U}}$, and we therefore obtain $\|\cdot\|_{\mathcal{C}^r,U} \leq 1/\delta^r \cdot (\|\cdot\|_{\mathcal{C}^r,\tilde{U}_1} \vee \dots \vee \|\cdot\|_{\mathcal{C}^r,\tilde{U}_m}) = 1/\delta^r \cdot \tilde{\|\cdot\|}$. As $U \in \mathfrak{U}$ was arbitrary, thus $\|\cdot\| \leq M\tilde{\|\cdot\|}$ with $M := 1/\delta^r \geq 1$. ■

Lemma 3.64. *Let r', r'' and $r = r' + r''$ be nonnegative real numbers and $X \subseteq \mathbf{K}^d$ a compact locally cartesian group with $\mathcal{C}^r$-multiplication (respectively $\mathcal{C}^{\mathrm{lip}}$-multiplication if $r < 1$) whose local factors contain no isolated point. Then for $\mu \in \mathcal{D}^{r'}(X, \mathbf{K})$ and $f \in \mathcal{C}^r(X, \mathbf{E})$, their* ***convolution*** *$\mu \star f : X \to \mathbf{K}$, defined by $y \mapsto \mu \cdot f(\_ \cdot y)$, is a $\mathcal{C}^{r''}$-function.*

Proof. Assume $r' = \nu' + \rho'$ and $r'' = \nu'' + \rho''$ with $\nu', \nu'' \in \mathbb{N}$ and $\rho', \rho'' \in [0, 1[$ so that $r = \nu + \rho$ with $\nu = \nu' + \nu'' \in \mathbb{N}$ and $\rho = \rho' + \rho'' \in [0, 2[$. Let $X'' \subseteq X$ be an open (and w.l.o.g. closed, hence compact) cartesian subset whose factors contain no isolated point. For every $\varepsilon > 0$ and $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$, we want to find a $\delta'' > 0$ such that

$$\|\mu \star f^{]\boldsymbol{n}''[}(y + h'') - \mu \star f^{]\boldsymbol{n}''[}(y)\| \leq \varepsilon \|h''\|^{\rho''} \quad \text{for all } \; y + h'', y \in X^{]\boldsymbol{n}''[} \text{ with } \|h''\| \leq \delta''.$$

Then for every $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$, the function $\mu \star f^{]\boldsymbol{n}''[} : X''^{]\boldsymbol{n}''[} \to \mathbf{K}$ extends by Proposition 1.6 to a $\mathcal{C}^{\rho''}$-function $\mu \star f^{[\boldsymbol{n}'']} : X''^{[\boldsymbol{n}'']} \to \mathbf{K}$. That is, $\mu \star f_{|X''} \in \mathcal{C}^{r''}(X'', \mathbf{K})$ and so $\mu \star f \in \mathcal{C}^{r''}(X, \mathbf{K})$.

Consider the composed mapping $F : X \times X \to \mathbf{E}$ given by

$$\begin{aligned} X \times X \to \; & X \quad \to \mathbf{E} \\ (x, y) \mapsto \; & x \cdot y \mapsto f(x \cdot y). \end{aligned}$$

By Proposition 3.23, this is again a $\mathcal{C}^r$-function on $X \times X$. Let $X' \subseteq X$ be an open (and closed, hence compact) cartesian subset whose factors contain no isolated point. By Lemma 3.62, we find $F^{[\boldsymbol{n}]} \in \mathcal{C}^{\rho}((X' \times X'')^{[\boldsymbol{n}]}, \mathbf{E})$ for all $\boldsymbol{n} = (\boldsymbol{n}', \boldsymbol{n}'') \in (\mathbb{N}^d \times \mathbb{N}^d)_{\leq\nu}$. Then $(X' \times X'')^{[\boldsymbol{n}]} = X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}$ and by Lemma 3.61 holds $F^{[\boldsymbol{n}]} \in \mathcal{C}^{\rho'\otimes\rho''}(X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}, \mathbf{E})$. We consider $X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']} \subseteq V' \oplus V''$ with $V' := V^{[\boldsymbol{n}']}$ and $V'' := V^{[\boldsymbol{n}'']}$ with $V = \mathbf{K}^d$. Let $\varepsilon > 0$. By compactness, there exists $\delta > 0$ such that for all $\boldsymbol{n}', \boldsymbol{n}'' \in \mathbb{N}^d$ with $|\boldsymbol{n}'| + |\boldsymbol{n}''| = \nu$, all $x \in X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}$ and $h' \in V', h'' \in V''$ of norm at most δ holds, where defined,

$$\|[F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x + h' + h'') - F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x + h'')] - [F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x + h') - F^{[\boldsymbol{n}',\boldsymbol{n}'']}(x)]\| \leq \varepsilon \|h'\|^{\rho'} \|h''\|^{\rho''}. \tag{$*$}$$

In particular this holds for all $\boldsymbol{n}' \in \mathbb{N}^d_{=\nu'}$ and $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$.

Fix $X'' \subseteq X$ compact cartesian with factors free of isolated points, $\varepsilon > 0$ and $\boldsymbol{n}'' \in \mathbb{N}^d_{=\nu''}$.

We have $\mu \star f(y) = \mu \cdot F(\_, y)$ for all $y \in X$. Hence for all $y + h'', y \in X''^{]\boldsymbol{n}''[}$ holds by $\mathbf{K}$-linearity of $\mu : \mathcal{C}^{r'}(X, \mathbf{E}) \to \mathbf{K}$ that

$$\begin{aligned}|(\mu \star f)^{]\boldsymbol{n}''[}(y + h'') - (\mu \star f)^{]\boldsymbol{n}''[}(y)| =&|\mu \cdot F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y + h'') - \mu \cdot F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y)| \\ =&|\mu \cdot (F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y + h'') - F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y))| \\ \leq&\|H\|_{\mathcal{C}^{r'}}, \qquad (**)\end{aligned}$$

where $H := F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y+h'') - F^{]\mathbf{0},\boldsymbol{n}''[}(\_, y) \in \mathcal{C}^r(X, \mathbf{E}) \subseteq \mathcal{C}^{r'}(X, \mathbf{E})$; and up to multiplying the distribution $\mu : \mathcal{C}^(X, \mathbf{K})^{r'} \to \mathbf{K}$ by a scalar $\lambda \in \mathbf{K}^*$, we assumed $\|\mu\|_{\mathcal{D}^{r'}} \leq 1$.
We just saw in Remark 3.63 that the topology of $\mathcal{C}^{r'}(X, \mathbf{E})$ is up to equivalence given by some norm $\|\cdot\|_{\mathcal{C}^{r'}} := \max_{X'} \|\cdot\|_{\mathcal{C}^{r'}, X'}$ for a finite covering of compact cartesian open subsets $X' \subseteq X$ whose factors have no isolated points. We may assume their diameters to be at most δ.
We recall the above definition of the function $H = H(y + h'', y) \in \mathcal{C}^{r'}(X, \mathbf{E})$. Then by the above Inequality $(**)$, to conclude the proof, it remains in the following to find $\delta'' > 0$ such that

$$\|H(y + h'', y)\|_{\mathcal{C}^{r'}} \leq \varepsilon \|h''\|^{\rho''} \quad \text{for all } y + h'', y \in X''^{]\boldsymbol{n}''[} \text{ with } \|h''\| \leq \delta''.$$

Since a fortiori $F^{[\boldsymbol{n}', \boldsymbol{n}'']} \in \mathcal{C}^{\rho''}(X'^{[\boldsymbol{n}']} \times X''^{[\boldsymbol{n}'']}, \mathbf{E})$ for all $X' \subseteq X$ compact cartesian without isolated points and $\boldsymbol{n}' \in \mathbb{N}^d_{\leq \nu'}$, we find for $\tilde{\varepsilon} = \varepsilon \delta^{\rho'} \leq \varepsilon$ by compactness $\tilde{\delta} > 0$ such that

$$\|h^{[\boldsymbol{n}']}\|_{X'^{[\boldsymbol{n}']}} \leq \|F^{[\boldsymbol{n}', \boldsymbol{n}'']}(\_, y + h'') - F^{[\boldsymbol{n}', \boldsymbol{n}'']}(\_, y)\|_{X'^{[\boldsymbol{n}']}} \leq \tilde{\varepsilon} \cdot \|h''\|^{\rho''} \qquad (\dagger)$$

for all $y + h'', y \in X''^{]\boldsymbol{n}''[}$ with $\|h''\| \leq \tilde{\delta}$.

Fix $y + h'', y \in X''^{]\boldsymbol{n}''[}$ with $\|h''\| \leq \delta'' := \delta \wedge \tilde{\delta}$. Let moreover $X' \subseteq X$ be a compact cartesian subset with factors free of isolated points, $\boldsymbol{n}' \in \mathbb{N}^d_{=\nu'}$ and $x + h', x \in X'^{[\boldsymbol{n}']}$. Then

$$\begin{aligned}&H^{[\boldsymbol{n}']}(x + h') - H^{[\boldsymbol{n}']}(x) \\ =&[F^{[\boldsymbol{n}', \boldsymbol{n}'']}(x + h', y + h'') - F^{[\boldsymbol{n}', \boldsymbol{n}'']}(x + h', y)] - [F^{[\boldsymbol{n}', \boldsymbol{n}'']}(x, y + h'') - F^{[\boldsymbol{n}', \boldsymbol{n}'']}(x, y)].\end{aligned}$$

As $\|h''\| \leq \delta$, we find by $(*)$ that

$$\|H^{[\boldsymbol{n}']}(x + h') - H^{[\boldsymbol{n}']}(x)\| \leq M \cdot \|h'\|^{\rho'} \quad \text{for all } x + h', x \in X'^{[\boldsymbol{n}']} \text{ with } \|h'\| \leq \delta; \quad (\dagger\dagger)$$

where we set $M := \varepsilon \cdot \|h''\|^{\rho''}$. Putting $\tilde{H} := H^{[\boldsymbol{n}']}$, we see moreover

$$\begin{aligned}&\|H^{[\boldsymbol{n}']}\|_{\mathcal{C}^{\rho'}, X'^{[\boldsymbol{n}']}} \\ =&\|\tilde{H}\|_{X'^{[\boldsymbol{n}']}} \vee \||\tilde{H}^{[\rho']}|\|_{X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']}} \\ =&\|\tilde{H}\|_{X'^{[\boldsymbol{n}']}} \vee \||\tilde{H}^{[\rho']}|\|_{\{(x+h', x) \in X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']} \,:\, \|h'\| \leq \delta\}} \vee \||\tilde{H}^{[\rho']}|\|_{\{(x+h', x) \in X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']} \,:\, \|h'\| > \delta\}} \\ \leq&\|\tilde{H}\|_{X'^{[\boldsymbol{n}']}} \vee \||\tilde{H}^{[\rho']}|\|_{\{(x+h', x) \in X'^{[\boldsymbol{n}']} \times X'^{[\boldsymbol{n}']} \,:\, \|h'\| \leq \delta\}} \vee \|\tilde{H}\|_{X'^{[\boldsymbol{n}']}} / \delta^{\rho'} \\ \leq&\tilde{\varepsilon} \cdot \|h''\|^{\rho''} \vee M \vee \tilde{\varepsilon} \cdot \|h''\|^{\rho''} / \delta^{\rho'} = \varepsilon \|h''\|^{\rho''};\end{aligned}$$

the last inequality as because of $\|h''\| \leq \tilde{\delta}$, we can invoke Inequality (†) for the outer terms, and as $\|h'\| \leq 1$, Inequality (††) for the one in-between.

We can therefore conclude the proof by

$$\begin{aligned}\|H\|_{\mathcal{C}^{r'}} &= \max_{X' \subseteq X \text{ cpt. cart.}} \|H\|_{\mathcal{C}^{r'}, X'} \\ &= \max_{X' \subseteq X \text{ cpt. cart.}} \big(\max_{\boldsymbol{n}' \in \mathbb{N}^d_{<\nu'}} \|H^{[\boldsymbol{n}']}\|_{X'[\boldsymbol{n}']} \vee \max_{\boldsymbol{n}' \in \mathbb{N}^d_{=\nu'}} \|H^{[\boldsymbol{n}']}\|_{\mathcal{C}^{\rho'}, X'[\boldsymbol{n}']} \big) \leq \varepsilon \cdot \|h''\|^{\rho''}.\end{aligned}$$

■

Definition. Let $r, s \in \mathbb{R}_{\geq 0}$ and $X \subseteq \mathbf{K}^d$ a compact locally cartesian group with $\mathcal{C}^{r+s}$-multiplication whose local factors contain no isolated point. We define the **convolution product** $\mu \star \lambda \in \mathcal{D}^{r+s}(X, \mathbf{K})$ of two distributions $\mu \in \mathcal{D}^r(X, \mathbf{K})$ and $\lambda \in \mathcal{D}^s(X, \mathbf{K})$ as the continuous $\mathbf{K}$-linear form on $\mathcal{C}^{r+s}(X, \mathbf{E})$ given by

$$(\mu \star \lambda) \cdot f = \lambda \cdot (\mu \star f).$$

By Proposition 3.30, we have for any $r \in \mathbb{R}_{\geq 0}$ a dense inclusion

$$\{ \text{ locally polynomial functions } f : X \to \mathbf{K} \} \subseteq \mathcal{C}^r(X, \mathbf{K}).$$

Hence the restriction map $\mathcal{D}^s(X, \mathbf{K}) \to \mathcal{D}^r(X, \mathbf{K})$ for $s \leq r$ is injective. We consider therefore $\mathcal{D}^s(X, \mathbf{K}) \subseteq \mathcal{D}^r(X, \mathbf{K})$ to be inclusions for $s \leq r$ by fixing such a system of injections.

Proposition. *Let* $\mathbf{K} \supseteq \mathbb{Q}_p$ *as a normed field and* $r, s \in \mathbb{R}_{\geq 0}$.

(i) The mapping

$$\begin{aligned}\mathcal{D}^r(\mathbb{Z}_p^d, \mathbf{K}) &\to \mathbf{K}[[\boldsymbol{X}]]_{r\text{-bdd.}} := \{\sum_{\boldsymbol{i} \geq \boldsymbol{0}} a_{\boldsymbol{i}} \boldsymbol{X}^{\boldsymbol{i}} \in \mathbf{K}[[\boldsymbol{X}]] \text{ with } \{|a_{\boldsymbol{i}}|/|\boldsymbol{i}|^r\} \text{ bounded}\} \\ \mu &\mapsto \sum_{\boldsymbol{i} \in \mathbb{N}^d} \mu \binom{*}{\boldsymbol{i}} \cdot \boldsymbol{X}^{\boldsymbol{i}}\end{aligned}$$

is an isomorphism of topological $\mathbf{K}$*-vector spaces.*

(ii) By the inclusions of $\mathcal{D}^r(\mathbb{Z}_p^d, \mathbf{K})$ *and* $\mathcal{D}^s(\mathbb{Z}_p^d, \mathbf{K})$ *into* $\mathcal{D}^{r+s}(\mathbb{Z}_p^d, \mathbf{K})$*, the convolution product* $\mu * \lambda$ *of* $\mu \in \mathcal{D}^r(\mathbb{Z}_p^d, \mathbf{K})$ *and* $\lambda \in \mathcal{D}^s(\mathbb{Z}_p^d, \mathbf{K})$ *corresponds to the product of their corresponding power series in* $\mathbf{K}[[\boldsymbol{X}]]_{r\,+\,s\text{-bdd.}}$.

Proof. Ad (i): By Theorem 3.47 and Corollary 3.49, we have $\mathcal{C}^r(\mathbb{Z}_p^d, \mathbf{K}) = c_0((|\boldsymbol{i}|^r)_{\boldsymbol{i} \in \mathbb{N}^d})$ as a topological $\mathbf{K}$-vector space. As the dual $\mathbf{K}$-Banach space to $c_0((w_i)_{i \in I})$ for weights $w_i \in \mathbb{R}_{\geq 0}$ running through an index set I consists of all $\mathbf{K}$-sequences $(a_i)_{i \in I}$ with $\{|a_i| w_i\}$ bounded, we have the above identification.
Ad (ii): Since $\binom{x+y}{n} = \sum_{j+k=n} \binom{x}{j}\binom{y}{k}$ for $x, y \in \mathbb{Z}_p$, we find $\mu \star \lambda \cdot \binom{*}{n} = \sum_{j+k=n} \mu\binom{*}{j} \cdot \lambda\binom{*}{k}$.
Therefore the mapping in (i) respects products. ■

3.5 Applications

Example of an induced $\mathcal{C}^r$-representation

In this paragraph, we want to describe the representation $\Pi(V)$ constructed in [Berger and Breuil, 2010, Section 4] as a quotient of principal series representation given by $\mathcal{C}^r$-functions.

Assumption. We will throughout this subsection's paragraph on the example of an induced $\mathcal{C}^r$-representation assume that $\mathbf{K} \supseteq \mathbb{Q}_p$ as a finite extension of valued fields.

We let $G := \mathrm{GL}_2(\mathbb{Q}_p)$ and $B \subseteq G$ its Borel subgroup of upper triangular matrices, $w = \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) \in G$. We put $N := \{\left(\begin{smallmatrix} & 1 \\ -1 & z \end{smallmatrix}\right) | z \in \mathbb{Q}_p\}$ and $N' = Nw$. We have canonical identifications of $\mathbb{Q}_p$ with N respectively N' which we denote by ι respectively ι'.

We let $\pi : B \to \mathbf{K}^*$ be a one-dimensional $\mathbf{K}$-linear representation of B. Let us assume that it will be of class $\mathcal{C}^r$ if we view B as an open subset of $\mathbb{Q}_p^3$. We define

$$I := \mathrm{Ind}_B^G \pi = \{f : G \to \mathbf{K} | f(bg) = \pi(b)f(g) \text{ for all } b \in B, g \in G\}.$$

If J is any set of functions on a domain $X \subseteq \mathbf{K}^d$ into some $\mathbf{K}$-Banach space $\mathbf{E}$, we will denote by $J^{\mathcal{C}^r}$ its subset of elements of class $\mathcal{C}^r$.

Proposition 3.65. *We have an isomorphism of* $\mathbf{K}$*-vector spaces*

$$\begin{aligned} I^{\mathcal{C}^r} \quad \to \quad & \{f : \mathbb{Q}_p \to \mathbf{K} : f_{|\mathbb{Z}_p} \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) \text{ and} \\ & \pi(\left(\begin{smallmatrix} 1/z & -1 \\ & -z \end{smallmatrix}\right))f(1/z)_{|\mathbb{Z}_p - \{0\}} \text{ extends to a function in } \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})\}, \end{aligned}$$

given by the restriction of a function in $I^{\mathcal{C}^r}$ *onto* N.

Proof. The proof is carried out in four steps.

1. Let $J := \{f : N \coprod N' \to \mathbf{K} : f(\iota(z)) = \pi(\left(\begin{smallmatrix} 1/z & -1 \\ & -z \end{smallmatrix}\right))f(\iota'(1/z)) \text{ for } z \neq 0 \in \mathbb{Q}_p\}$. Then the restriction of a function $f : G \to \mathbf{K}$ onto the two subgroups N and N' yields a $\mathbf{K}$-vector space isomorphism $\varphi : I \to J$.

   Proof: First of all we note that

   $$\begin{pmatrix} & 1 \\ -1 & z \end{pmatrix} w = \begin{pmatrix} 1/z & -1 \\ & -z \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & 1/z \end{pmatrix} \quad \text{for } z \neq 0,$$

   so that the image of φ actually lies in J. We find

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{-c} & -a \\ & -c \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \frac{d}{-c} \end{pmatrix} \quad \text{for } ad - bc \neq 0, c \neq 0.$$

   As G is the union of all matrices such that $ad - bc \neq 0$ and either $c \neq 0$ or $d \neq 0$, this shows the injectivity of this map, whereas its surjectivity is clear.

2. We define a function $f : N \coprod N' \to \mathbf{K}$ to be of class $\mathcal{C}^r$ if $f \circ \iota$ and $f \circ \iota'$ are in $\mathcal{C}^r(\mathbb{Q}_p, \mathbf{K})$. With this definition, the restriction of φ onto $I^{\mathcal{C}^r}$ yields an isomorphism of $\mathbf{K}$-vector spaces $\phi : I^{\mathcal{C}^r} \to J^{\mathcal{C}^r}$.

   Proof: First of all we note that the image lies in $J^{\mathcal{C}^r}$ as the restriction of a $\mathcal{C}^r$-function onto N respectively Nw is again in $\mathcal{C}^r$ by Proposition 3.23 since ι respectively ι' and their inclusions into G are surely $\mathcal{C}^r$. It is left to show that its inverse map $\varphi^{-1}_{|J^{\mathcal{C}^r}}$ has image in $I^{\mathcal{C}^r}$. We have $\varphi^{-1}(f) = F$ with

$$F(g) = \begin{cases} \pi(b)f(n) & \text{if } g = bn \text{ with } b \in B, n \in N, \\ \pi(b)f(n) & \text{if } g = bnw \text{ with } b \in B, n \in N'. \end{cases}$$

   We have to show that F is $\mathcal{C}^r$ on either open subset of the cover formed by $BN = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G | c \neq 0\}$ respectively $BNw = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G | d \neq 0\}$ of G. We will do so for BN, the proof for BNw carries through similarly. Now

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{ad-bc}{-c} & -a \\ & -c \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \frac{d}{-c} \end{pmatrix} \quad \text{for } ad - bc \neq 0, c \neq 0,$$

   so that all matrix-entries on the right hand side are well defined fractional polynomial functions in the entries of the matrix on the left hand side, in particular the well defined map $\mathbb{Q}_p^4 \supseteq BN \ni g \mapsto (b, n) \in \mathbb{Q}_p^3 \times \mathbb{Q}_p$ is in $\mathcal{C}^r$. As π and $f_{|N}$ are by assumption in $\mathcal{C}^r$, our map F is in $\mathcal{C}^r$ on BN by the multiplicative closure proven in Proposition 3.13.

3. We let

$$\tilde{J} := \{f : \mathbb{Q}_p \to \mathbf{K} : \; f \in \mathcal{C}^r(\mathbb{Q}_p, \mathbf{K}) \text{ and } \pi(\left(\begin{smallmatrix} 1/z & -1 \\ & -z \end{smallmatrix}\right)) f(1/z)_{|\mathbb{Q}_p - \{0\}} \text{ extends to a function in } \mathcal{C}^r(\mathbb{Q}_p, \mathbf{K})\}.$$

   Then we have an isomorphism of $\mathbf{K}$-vector spaces $J^{\mathcal{C}^r} \to \tilde{J}$ given by the restriction of a function $f \in J^{\mathcal{C}^r}$ onto $N = \mathbb{Q}_p$.

   Proof: Let f be in $J^{\mathcal{C}^r}$. We show that this map is injective. By the property of f being in J, we find that f is already determined on $\iota'(\mathbb{Q}_p - \{0\}) \subseteq N'$. As f is $\mathcal{C}^r$ on N', it is in particular continuous there and thus already determined by its values on this dense subset of N'. The surjectivity is clear.

4. We have

$$\tilde{J} = \{f : \mathbb{Q}_p \to \mathbf{K} : \; f_{|\mathbb{Z}_p} \in \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) \text{ and } \pi(\left(\begin{smallmatrix} 1/z & -1 \\ & -z \end{smallmatrix}\right)) f(1/z)_{|\mathbb{Z}_p - \{0\}} \text{ extends to a function in } \mathcal{C}^r(\mathbb{Z}_p, \mathbf{K})\}.$$

   Proof: We have to show that every function in the right hand side is already in $\tilde{J}$. Let

$f : \mathbb{Q}_p \to \mathbf{K}$ be such a mapping. As π is of class $\mathcal{C}^r$, the map $\mathbb{Q}_p - \{0\} \ni z \mapsto \pi(\begin{pmatrix} 1/z & -1 \\ & -z \end{pmatrix}) \in \mathbf{K}^*$ also is by Proposition 3.23. Therefore $f(1/z)_{|\mathbb{Z}_p - \{0\}}$ is in $\mathcal{C}^r$ and thus through Proposition 3.23 the function $f_{|(\mathbb{Z}_p - \{0\})^{-1}}$, too. Since $\mathbb{Q}_p = \mathbb{Z}_p \cup (\mathbb{Z}_p - \{0\})^{-1}$, the function f is thence in $\mathcal{C}^r(\mathbb{Q}_p, \mathbf{K})$. By the same argument $\pi(\begin{pmatrix} 1/z & -1 \\ & -z \end{pmatrix}) f(1/z)_{|\mathbb{Q}_p - \{0\}}$ extends to a function in $\mathcal{C}^r(\mathbb{Q}_p, \mathbf{K})$.

Then running through the equalities held in each step, we obtain the proposition. ■

Let $P := \iota(p\mathbb{Z}_p) \dot\cup \iota'(\mathbb{Z}_p) \subseteq G$. As P is compact and open, we can endow $I^{\mathcal{C}^r}$ with the seminorm $\|\cdot\|_{\mathcal{C}^r, P}$. Let us denote the range of the isomorphism in Proposition 3.65 by $\tilde{J}$ and endow it with the induced seminorm through this bijection. Explicitly $\|f\|_{\tilde{J}} := \max(\|f_1\|_{\mathcal{C}^r}, \|f_2\|_{\mathcal{C}^r})$, where $f_1 = f_{|p\mathbb{Z}_p}$ and $f_2(z) = \pi(\begin{pmatrix} 1/z & -1 \\ & -z \end{pmatrix}) f(1/z)$ for $z \in \mathbb{Z}_p$. On the other hand, we endow $I^{\mathcal{C}^r}$ with its natural locally convex topology given by the norms $\|\cdot\|_{\mathcal{C}^r, C}$ for compact and open $C \subseteq G$.

Proposition 3.66. *For any compact and open $C \subseteq G$ containing P exists $M > 0$ such that $\|\cdot\|_{\mathcal{C}^r, C} \leq M \|\cdot\|_{\mathcal{C}^r, P}$. Thence the locally convex topology on $I^{\mathcal{C}^r}$ is given by the norm $\|\cdot\|_{\mathcal{C}^r, P}$, so that the isomorphism of Proposition 3.65 is a homeomorphism.*

Proof. Let $C \subseteq G$ be open, compact and containing P. As G is totally disconnected, the projection map $P \to B\backslash G$ has a continuous section and so the natural map $B \times P \to G$ is a continuous bijection. Let C_B be the image of the continuous map

$$C \overset{\text{inclusion}}{\hookrightarrow} G \overset{\sim}{\to} B \times P \overset{\text{projection}}{\twoheadrightarrow} B.$$

Thus $C \subseteq C_B \times P$. As C_B is compact, $\|\pi\|_{\mathcal{C}^r, C_B} =: M < \infty$. Therefore, for any function $f \in I^{\mathcal{C}^r}$, we have

$$\|f\|_{\mathcal{C}^r, C} \leq \|f\|_{\mathcal{C}^r, P \times C_B} = M \cdot \|f\|_{\mathcal{C}^r, P}.$$

■

Remark. One usually endows $I^{\mathcal{C}^r}$ with the norm $\|\cdot\|_{\mathcal{C}^r, K}$, where $K = \mathrm{GL}_2(\mathbb{Z}_p) \supseteq P$.

Example 3.67. (Berger, Breuil) As in Example 2.34 we denote by $\chi_\gamma : \mathbb{Q}_p^* \to \mathbf{K}^*$ for $\gamma \in \mathbf{K}^*$ the unramified character defined by

$$\chi_\gamma : x \mapsto \gamma^{v(x)}.$$

Define $\pi : \mathbb{Q}_p^* \times \mathbb{Q}_p^* \to K^*$ by $\pi(x, y) = \chi_{\alpha^{-1}}(x) \otimes y^{k-2} \chi_{p\beta^{-1}}(y)$, where $\alpha, \beta \in \mathbf{K}^*$ are algebraic elements over $\mathbb{Q}_p$. Let $T = \{\begin{pmatrix} a & \\ & d \end{pmatrix} | a, d \in \mathbb{Q}_p^*\}$ be the diagonal torus of G. By the canonical projection and group homomorphism $B \twoheadrightarrow T$ we can view $\pi : B \to \mathbf{K}^*$ as a character on B. As any character χ_γ is locally constant on $\mathbb{Q}_p^*$ and the monomial $*^{k-2}$ is by Lemma 2.22 arbitrarily often differentiable, we find that π is a $\mathcal{C}^r$-function for any $r \in \mathbb{R}_{\geq 0}$. Remember that we have by Corollary 2.54 an equality $\mathcal{C}^r(\mathbb{Z}_p, \mathbf{K}) = \mathcal{C}^r_{\mathrm{B}}(\mathbb{Z}_p, \mathbf{K})$ of topological $\mathbf{K}$-vector spaces, where

$$\mathcal{C}^r_{\mathrm{B}}(\mathbb{Z}_p, \mathbf{K}) := \{f : \mathbb{Z}_p \to \mathbf{K} : |a_n| n^r \to 0 \text{ as } n \to \infty\}$$

with the Mahler coefficients $(a_n)_{n\in\mathbb{N}}$ from Definition 2.50 and $\|f\|_{\mathcal{C}^r_{\mathrm{B}}} := |a_0| \vee \max_{n\geq 1}|a_n|n^r$. Then Proposition 3.65 together with Proposition 3.66 state that we have a topological $\mathbf{K}$-vector-space-isomorphism

$$(\mathrm{Ind}_B^G \chi_{\alpha^{-1}} \otimes d^{k-2}\chi_{p\beta^{-1}})^{\mathcal{C}^r} \overset{\eta}{\to} \{f : \mathbb{Q}_p \to \mathbf{K} | f \in \mathcal{C}^r_{\mathrm{B}}(\mathbb{Z}_p, \mathbf{K}) \text{ and } (\frac{\alpha p}{\beta})^{v(z)} z^{k-2} \cdot f(1/z)_{|\mathbb{Z}_p - \{0\}} \text{ extends to a function in } \mathcal{C}^r_{\mathrm{B}}(\mathbb{Z}_p, \mathbf{K})\}$$

given by the restriction of a function $f : G \to \mathbf{K}$ in the left hand side onto N.

We now let $r = v(\alpha)$. If $v(\alpha) \geq v(\beta)$ and $v(\alpha) + v(\beta) = k - 1$ for a natural number $k \geq 2$, we can apply Example 2.34 with $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta p^{-1}$ and $\tilde{k} = k - 2 - j (= v(\tilde{\beta}) + v(\tilde{\alpha}) - j)$ for $0 \leq j < v(\tilde{\alpha})$ to find that $(\frac{\alpha p}{\beta})^{v(z)} z^{k-2} f(1/z)_{|\mathbb{Q}_p - \{0\}}$ with $f(z) := z^j$ for $0 \leq j < r$ extends to a function in $\mathcal{C}^r(\mathbb{Q}_p, \mathbf{K})$ by sending 0 to the value 0. Therefore f is an element of the range of η, which we will refer to as $B(\alpha)$.

We denote for any $g \in G$ and $F \in (\mathrm{Ind}_B^G \chi_{\alpha^{-1}} \otimes d^{k-2}\chi_{p\beta^{-1}})^{\mathcal{C}^r}$ by $\rho_g\ F$ the right-translation $\rho_g\ F := F(\cdot g)$ of F by g. As matrix multiplication by a fixed element $g \in G$ is for any $r \in \mathbb{R}_{\geq 0}$ a $\mathcal{C}^r$-function from G into itself, we find that $\rho_g\ F$ is again in $(\mathrm{Ind}_B^G \chi_{\alpha^{-1}} \otimes d^{k-2}\chi_{p\beta^{-1}})^{\mathcal{C}^r}$. Now let $a \in \mathbb{Q}_p$. As $(\frac{\alpha p}{\beta})^{v(z-a)}(z-a)^{k-2-j} = \eta(\rho_g \eta^{-1}(f))$ for $f = z^j$ and $g = \begin{pmatrix} -a & -1 \\ -1 & \end{pmatrix}$, we find that the $\mathbf{K}$-vector-space $L(\alpha)$ generated by all functions z^j for $0 \leq j < r$ respectively $(\frac{\alpha p}{\beta})^{v(z-a)}(z-a)^{k-2-j}$ for $a \in \mathbb{Q}_p$ and $0 \leq j < r$ lies in $B(\alpha)$.

$\mathcal{C}^r$-manifolds

Let $\mathbf{K}$ as usual denote a complete non-Archimedeanly non-trivially valued field. Let M be a Hausdorff topological space. In this paragraph we want to introduce the notion of a $\mathcal{C}^r$-manifold. The presentation follows [Schneider, 2007/08, Section 7,8].

Definition. We say that M is a **topological manifold** of dimension d or a topological d-manifold if for every point $x \in M$, we can find a **chart** (U, ϕ) consisting of:

1. an open set $U \subseteq M$ containing x, and
2. a map $\phi : U \to \mathbf{K}^d$ such that $\phi(U)$ is open in $\mathbf{K}^d$ and $\phi : U \to \phi(U)$ is a homeomorphism.

Assume that we have a notion of $\mathcal{C}^*$-function on open subsets in $\mathbf{K}^d$, i.e. for all open subsets $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$ we have the subset $\mathcal{C}^*(X, Y) \subseteq Y^X$ of all $\mathcal{C}^*$-functions $f : X \to Y$.

Definition. We will say that two charts (U, ϕ) and (V, ψ) are **($\mathcal{C}^*$-)compatible** if both maps

$$\mathbf{K}^d \supseteq \phi(U \cap V) \underset{\phi\circ\psi^{-1}}{\overset{\psi\circ\phi^{-1}}{\rightleftarrows}} \psi(V \cap U) \subseteq \mathbf{K}^d$$

are $\mathcal{C}^*$-functions.

Definition. An **atlas** for M is a set $\mathcal{A} = \{(U_i, \phi_i)\}$ of charts on M such that any two of these are compatible and which covers M.

Definition. Let M be a topological d-manifold and $\mathcal{A} = \{(U_i, \phi_i) : i \in I\}$ an atlas of M. Then we will say that a function $f : M \to \mathbf{K}^e$ is a $\mathcal{C}^*$-function with respect to $\mathcal{A}$, if

$$f \circ \phi_i^{-1} \in \mathcal{C}^*(\phi_i(U_i), \mathbf{K}^e) \quad \text{for all } \ i \in I.$$

Definition. We will call an atlas **maximal** if it is not contained in any strictly larger atlas.

Remark. Equivalently an atlas $\mathcal{A}_0$ is maximal, if each chart on M compatible with every chart in $\mathcal{A}_0$ will be already in $\mathcal{A}_0$.

Proposition 3.68. *Let M be a topological d-manifold endowed with an atlas. We assume that:*

- *The class of $\mathcal{C}^*$-functions is closed under composition: I.e. if $U \subseteq \mathbf{K}^d, V \subseteq \mathbf{K}^e$ and $W \subseteq \mathbf{K}^f$ are open subsets, and $f \in \mathcal{C}^*(U, V), g \in \mathcal{C}^*(V, W)$ then $g \circ f \in \mathcal{C}^*(U, W)$.*
- *The $\mathcal{C}^*$-property is local: I.e. if $\{U_i : i \in I\}$ is a cover by open sets of $U \subseteq \mathbf{K}^d$ open and $f : U \to \mathbf{K}^e$ is such that $f_{|U_i}$ is a $\mathcal{C}^*$-function for all $i \in I$, then f is a $\mathcal{C}^*$-function.*

Then we find that:

1. *The manifold M has a maximal atlas $\mathcal{A}_0$.*
2. *The domains $\{U\}$ of all charts (U, ϕ) in $\mathcal{A}_0$ form a topological basis of M.*
3. *A function $f : M \to \mathbf{K}^e$ is a $\mathcal{C}^*$-function with respect to* any *atlas $\mathcal{A} \subseteq \mathcal{A}_0$ if and only if it is a $\mathcal{C}^*$-function with respect to* the maximal *atlas $\mathcal{A}_0$.*

Proof. Ad 1.: We firstly show the existence of the maximal atlas $\mathcal{A}_0$. Let $\mathcal{A}$ be an atlas on M whose existence we assume. We put

$$\mathcal{A}_0 = \{ \text{ all charts } (U, \phi) \text{ compatible with every chart in } \mathcal{A} \ \}.$$

We will show that $\mathcal{A}_0$ is an atlas on M. Then by definition, it will be maximal. For this, it remains to prove that every two charts (U, ϕ) and (V, ψ) on M which are compatible with every chart in $\mathcal{A}$ are itself compatible: Let $x \in U \cap V$. By localness, we have to show that there exists a neighborhood $W \subseteq U \cap V$ of x such that

$$\mathbf{K}^d \supseteq \phi(W) \underset{\phi\circ\psi^{-1}}{\overset{\psi\circ\phi^{-1}}{\rightleftarrows}} \psi(W) \subseteq \mathbf{K}^d$$

are $\mathcal{C}^*$-functions. Let $(\tilde{W}, \tilde{\theta})$ be a chart in $\mathcal{A}$ with $\tilde{W} \ni x$. Put $W = \tilde{W} \cap (U \cap V)$. Then the maps $\phi \circ \theta^{-1}$ and $\theta \circ \psi^{-1}$ are by the assumed compatibility $\mathcal{C}^*$-functions. Therefore $\phi \circ \psi^{-1} = (\phi \circ \tilde{\theta}^{-1}) \circ (\theta^{-1} \circ \psi)$ is by closure under composition a $\mathcal{C}^*$-function on $\psi(W)$. By symmetry, we also have that $\psi \circ \phi^{-1}$ is a $\mathcal{C}^*$-function on $\phi(W)$.

Ad 2.: We now show that the domains of the charts in $\mathcal{A}_0$ form a topological basis of M. Let $U \subseteq M$ be an open subset. We have to show that for any point $x \in U$ we find a chart (U_x, ϕ_x) in $\mathcal{A}_0$ such that $x \in U_x \subseteq U$. Let $(\tilde{U}_x, \tilde{\phi}_x) \in \mathcal{A}_0$ be a chart with $\tilde{U}_x \ni x$. Then we put $U_x := \tilde{U}_x \cap U$ and $\phi_x := \tilde{\phi}_{x|U_x}$. Then clearly (U_x, ϕ_x) is a chart such that $x \in U_x \subseteq U$. Because (U_x, ϕ_x) is the restriction of a chart $(\tilde{U}_x, \tilde{\phi}_x)$ in the atlas $\mathcal{A}_0$ and hence being compatible with every chart in $\mathcal{A}_0$, we find it to be compatible with any chart in $\mathcal{A}_0$. Since $\mathcal{A}_0$ is maximal we just observed above that $(U_x, \phi_x) \in \mathcal{A}_0$.

Ad 3.: We assume that $f : M \to \mathbf{K}^e$ is a $\mathcal{C}^*$-function with respect to an atlas $\mathcal{A} \subseteq \mathcal{A}_0$. We have to show that $f \circ \phi$ is a $\mathcal{C}^*$-function with respect to any chart $(U, \phi) \in \mathcal{A}_0$ on M. By assumption, we find a cover by charts $\{(U_i, \psi_i) : i \in I\} \subseteq \mathcal{A}$ of U. We may assume $U_i \subseteq U$. By localness, it suffices to check that $f \circ \phi_{|U_i}$ is a $\mathcal{C}^*$-function for every $i \in I$. We have $\phi_{|U_i} = \psi_i \circ (\psi^{-1} \circ \phi_{|U_i})$. Because (ψ_i, U_i) and (ϕ, U) in $\mathcal{A}_0$ are compatible, we find the right hand map $(\psi^{-1} \circ \phi_{|U_i})$ to be a $\mathcal{C}^*$-function. By assumption $f \circ \psi_i$ is a $\mathcal{C}^*$-function. By closure under composition, therefore $f \circ \phi_{|U_i}$ is a $\mathcal{C}^*$-function. ∎

Definition. We will call the pair $(M, \mathcal{A}_0)$ of a topological manifold M and a maximal atlas $\mathcal{A}_0$ on M a **$\mathcal{C}^*$-manifold**.

Remark. The preceding proposition says that the property of a function $f : M \to \mathbf{K}^e$ to be $\mathcal{C}^*$ does not depend on the particular choice of atlas on the $\mathcal{C}^*$-manifold M inside $\mathcal{A}_0$. Calling two atlases $\mathcal{A}$ and $\mathcal{B}$ *equivalent* if $\mathcal{A} \cup \mathcal{B}$ is again an atlas, we find the maximal atlas $\mathcal{A}_0$ therefore to be an maximal element in its (equivalence) class of all coverings of M allowing for the notion of a $\mathcal{C}^*$-function.

Example. Let $r \in \mathbb{R}_{\geq 0}$ and let $\mathcal{C}^r$ be the notion of r-fold differentiability, i.e. for all open subsets $X \subseteq \mathbf{K}^d$ and $Y \subseteq \mathbf{K}^e$, we let $\mathcal{C}^r(X, Y) \subseteq Y^X$ be the subset of $\mathcal{C}^r$-functions as given in Definition 3.1. Then in the sense of Proposition 3.68, we find this notion for $r \geq 1$ to be local and closed under composition.

Proof. We check that:

1. By Definition 3.1, the $\mathcal{C}^r$-property is defined pointwise, in particular it is local.
2. If $r \geq 1$, the $\mathcal{C}^r$-functions are by Corollary 3.22 closed under composition.

∎

Remark. If $r < 1$, then we still have a good notion of $\mathcal{C}^r$-functions on $\mathcal{C}^{\text{lip}}$-manifolds: That is, let $(M, \mathcal{A}_0)$ be a $\mathcal{C}^{\text{lip}}$-manifold. Then as in Proposition 3.68, by the same arguments one can characterize a function $f : M \to \mathbf{K}^e$ to be a $\mathcal{C}^r$-function if it is a $\mathcal{C}^r$-function with respect to any atlas $\mathcal{A} \subseteq \mathcal{A}_0$.

References

Araujo, J., Schikhof, W. H., 1993. The Weierstrass-Stone approximation theorem for p-adic $\mathcal{C}^n$-functions. Report, Mathematisch Institut, Katholieke Universiteit, Nijmegen 9309.

Berger, L., Breuil, C., 2010. Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbb{Q}_p)$. Astérisque 330, 155–211.

Bertram, W., Glöckner, H., Neeb, K., 2004. Differential calculus over general base fields and rings. Expositiones Mathematicae 22 (3), 213–282.

Colmez, P., 2008. Fonctions d'une variable p-adique. Astérisque 319, 187–212.

Glöckner, H., 2007. Comparison of some notions of $\mathcal{C}^k$-maps in multivariable non-Archimedean analysis. Bull. Belg. Math. Soc. Simon Stevin 14, 877–904.

Kelley, J., 1975. General Topology. Springer-Verlag, New York, 2nd edition.

Schikhof, W., 1978. Non-Archimedean calculus. Report, Mathematisch Institut, Katholieke Universiteit, Nijmegen 7812.

Schikhof, W., 1984. Ultrametric Calculus: An Introduction to p-Adic Analysis. Cambridge University Press.

Schikhof, W., 1995. A perfect duality between p-adic Banach spaces and compactoids. Indagationes Mathematicae 6 (3), 325–339.

Schneider, P., 2007/08. p-Adic Analysis and Lie Groups. Lecture Notes.

van Rooij, A., 1978. Non-Archimedean functional analysis. Marcel Dekker.

The intertwined open cells in the universal unitary lattice of an unramified algebraic principal series

Introduction

We let G be (the rational points of) a connected reductive group over a local field $\mathbf{F}$. Let $\mathbf{K}$ be a complete non-Archimedeanly non-trivially valued field of characteristic 0 with valuation ring $\mathbf{o}$. Let $\bar{P} \subseteq G$ be a minimal parabolic subgroup and let $\theta : \bar{P} \to \mathbf{K}^*$ be an unramified character. We will consider the $\mathbf{K}$-linear G-representation $V = I(\theta) \otimes_{\mathbf{K}} U$ for an unramified principal series $I(\theta) = \operatorname{Ind}_{\bar{P}}^{G} \theta^{\mathrm{lc}}$ and an algebraic representation U, an example of a *locally algebraic representation* as defined in Subsection 0.2. (For this to be meaningful, we assume here and in the following that $\mathbf{K} \supseteq \mathbf{F}$ in the case that U is a general nontrivial algebraic representation.)
Let V be a locally algebraic G-representation and endow V with its finest locally convex topology. Then V is a locally convex $\mathbf{K}$-vector space with a continuous G-action. We will call a continuous $\mathbf{K}$-Banach space representation of G *unitary* it its topology can be defined by a G-invariant norm. The *universal unitary completion* of V is then defined as the unitary $\mathbf{K}$-Banach space representation $\widehat{V}$ of G which is universal with respect to the morphism of locally convex $\mathbf{K}$-vector spaces $V \to \widehat{V}$.

We want to describe the *universal unitary* $\mathbf{o}$*-lattice* $\mathfrak{L} \subseteq V$ given as the preimage of the unit ball in $\widehat{V}$.

Assumption. We remark that because G is reductive and $\operatorname{char} \mathbf{F} = 0$, we may assume U to be irreducible. If U is trivial or equivalently if V is a smooth representation, we will assume G to be a general connected reductive group, and if U is allowed to be a general irreducible algebraic representation, we will assume G to split (to invoke the theory of algebraic representations of split reductive groups). Then U is parameterized by a *dominant* (cf. Subsection 0.2) algebraic character $\psi : \bar{P} \to \mathbf{K}^*$ and we will write $I(\chi)$ with $\chi = \theta\psi$ for V. It is given by the *locally algebraic vectors* (cf. Subsection 0.2) in the abstract $\mathbf{K}$-linear principal series representation $\operatorname{Ind}_{\bar{P}}^{G} \chi$ with G acting by right translation.

Let P be the minimal parabolic subgroup opposite to $\bar{P}$. In Section 1 we will regard $I(\chi)$ as a P-representation and give a distinguished set of generators of the $\mathbf{K}[P]$-module $I(\chi)$ indexed by the Weyl group W of G.

Let $I(\chi)(N)$ be the P-representation of functions in $I(\chi)$ with support in $N \subseteq \bar{P}\backslash G$. Let

M be the centralizer of a maximal $\mathbf{F}$-split torus $A \subseteq P$ and M^+ its dominant submonoid (cf. Interlude in Section 2). In Section 2 we will then show that the the seminorm attached to the universal unitary lattice $\mathfrak{L}(N) \subseteq I(\chi)(N)$ is nonzero if and only if $|\chi(M^+)| \leq 1$. We will then describe a norm $\|\cdot\|$ on $I(\chi)(N)$, which can be viewed as a norm of $\boldsymbol{r}$-fold differentiable functions for $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$. Its unit ball contains (a nonzero scalar multiple of) $\mathfrak{L}(N)$ as a subset and we infer $\mathfrak{L}(N)$ to be a free $\mathbf{o}$-module.

In the final Section 3, we will then by a general argument observe that $\mathfrak{L} \subseteq I(\chi)$ is a universal unitary lattice of $I(\chi)$ as a G-representation if and only if it is a universal unitary lattice of $I(\chi)$ as a P-representation. We will then work under the assumption that θ is regular, so that we can make use of the theory of intertwining operators between smooth principal series: Then by the results in Section 1, we can so describe the universal unitary lattice $\mathfrak{L} \subseteq I(\chi)$ as $\mathfrak{L} = \sum_{w \in W} \mathfrak{L}_w$ with cyclic $\mathbf{o}[P]$-modules $\mathfrak{L}_w$, and moreover by the results in Section 2 - at least if $I(\chi)$ is absolutely irreducible, i.e. θ fulfills the conditions of Remark 3.13 - we can show each $\mathfrak{L}_w$ to be free as an $\mathbf{o}$-module.

In the context of the existing literature, in the article [Berger and Breuil, 2010] the authors showed the universal unitary completion of $I(\chi)$ for the connected reductive group $G = \mathrm{GL}_2(\mathbb{Q}_p)$ among other results to be nonzero under the necessary assumptions on the character χ given here (which is, as we do here, also assumed to be regular), implying the whole universal unitary lattice in $I(\chi)$ to be free as an $\mathbf{o}$-module. To obtain these results, they make noteworthy use of the very shape of this connected reductive group.

0 Prerequisites

Let $\mathbf{E}$ be a complete non-Archimedeanly non-trivially valued field with ring of integers $\mathfrak{o}_{\mathbf{E}}$, maximal ideal $\mathfrak{m}_{\mathbf{E}}$ and residue field $\mathbf{k}_{\mathbf{E}} = \mathfrak{o}_{\mathbf{E}}/\mathfrak{m}_{\mathbf{E}}$, additive valuation $v_{\mathbf{E}}$ and multiplicative absolute value $|\cdot|_{\mathbf{E}}$ defined by $|x|_{\mathbf{E}} := c_{\mathbf{E}}^{v_{\mathbf{E}}(x)}$ for a constant $c_{\mathbf{E}} < 1$ which we chose to be $c_{\mathbf{E}} = p_{\mathbf{E}}^{-1}$ in case of nonzero residue field characteristic $p_{\mathbf{E}}$.
If $\mathbf{E}$ is a local field (i.e. a complete non-Archimedeanly non-trivially valued field with discrete valuation and finite residue field), we assume the additive valuation $v_{\mathbf{E}}$ to take values in $\mathbb{Z}$. We denote by $\pi_{\mathbf{E}}$ a fixed element such that $v_{\mathbf{E}}(\pi_{\mathbf{E}}) = 1$ and let $q_{\mathbf{E}}$ be the cardinality of its residue field.
We will drop the subscripts whenever confusion is unlikely.

We will fix a local field $\mathbf{F}$ of residue field characteristic p and a complete non-Archimedeanly non-trivially valued field $\mathbf{K}$ of characteristic 0.

0.1 The groups

We will call an affine group scheme of finite type over a field an **affine (or linear) algebraic group**. Then the field $\mathbf{F}$ will serve as the coefficients of the rational points of our affine algebraic groups.

We will assume all affine algebraic groups to be defined over $\mathbf{F}$ and denote these by boldface letters. We will denote the rational points of a linear algebraic group by the corresponding letter in ordinary type. Then the topology of $\mathbf{F}$ turns this into a topological group and we will denote by an additional subscript naught either, if existent, its maximal compact open subgroup or, otherwise, a suitably chosen compact open subgroup to be specified: E.g. if $\mathbf{A}$ is a split torus, then $A = \mathbf{A}(\mathbf{F})$ and $A_0 \subseteq A$ will be its maximal compact open subgroup. We will denote elements of these groups by their corresponding small roman letters in ordinary type, e.g. $a \in A$. The Lie algebra over $\mathbf{F}$ of a linear algebraic group will then be denoted by the corresponding small gothic letter: E.g. if $\mathbf{N}$ is an affine algebraic group, $\mathfrak{n}$ will be the Lie algebra over $\mathbf{F}$ of $\mathbf{N}$.

If we let $\mathbf{G}$ be a connected reductive group defined over $\mathbf{F}$, then we will denote by $\bar{\mathbf{P}}$ a minimal parabolic subgroup of $\mathbf{G}$, by $\mathbf{A}$ a maximal split torus in $\bar{\mathbf{P}}$ and by K a special, good, maximal compact open subgroup in G, chosen such that its Iwahori subgroup $\bar{B} \subseteq K$ is of the same type as $\bar{\mathbf{P}}$.
We let $\mathbf{P}$ be the parabolic subgroup opposite to $\bar{\mathbf{P}}$ and $\mathbf{N}$ respectively $\bar{\mathbf{N}}$ the unipotent radical of $\mathbf{P}$ respectively $\bar{\mathbf{P}}$. Similarly we define B as the Iwahori subgroup opposite to $\bar{B}$. We denote by $\mathbf{M} = \mathbf{C}_{\mathbf{G}}(\mathbf{A})$ the centralizer of $\mathbf{A}$ in $\mathbf{G}$. Then $\mathbf{M}$ normalizes $\mathbf{N}$ respectively $\bar{\mathbf{N}}$ by conjugation and we have $\mathbf{P} = \mathbf{N}\mathbf{M}$ and $\bar{\mathbf{P}} = \bar{\mathbf{N}}\mathbf{M}$. Denote by $\mathbf{N}_{\mathbf{G}}(\mathbf{A})$ the normalizer of $\mathbf{A}$ inside $\mathbf{G}$ and let $W = \mathbf{N}_{\mathbf{G}}(\mathbf{A})/\,\mathbf{C}_{\mathbf{G}}(\mathbf{A})$ be the Weyl group of $\mathbf{G}$. We let $\mathbf{Z} = \mathbf{Z}_{\mathbf{G}}$ the center in $\mathbf{G}$. For an affine algebraic group $\mathbf{G}$, we will denote its maximal connected subgroup by $\mathbf{G}^{\circ}$; e.g. $\mathbf{Z}^{\circ}$ will be the connected identity component of the center $\mathbf{Z}$.

The choice of the maximal $\mathbf{F}$-split torus $\mathbf{A}$ determines a (*relative*) root system Φ and the choice of the minimal parabolic subgroup $\bar{\mathbf{P}}$ a basis Δ of simple roots inside Φ. Then $\Phi = (\sum_{\alpha\in\Delta} \mathbb{Z}_{\geq 0} \cdot \alpha \cap \Phi)\dot{\cup}(\sum_{\alpha\in\Delta} \mathbb{Z}_{\leq 0} \cdot \alpha \cap \Phi)$ and we will say that $\alpha \in \Phi$ is **positive/negative** or write $\alpha \gtrless 0$ if it lies in the left/right hand segment. We will denote the set of nondivisible roots by Φ^{red} and the set of positive/negative nondivisible roots by $\Phi^{\pm} \subseteq \Phi^{\mathrm{red}}$.
There exists by [Borel, 1991, Proposition 21.9] for each $\alpha \in \Phi$ a unique root subgroup $\mathbf{N}_\alpha$ normalized by $\mathbf{M}$ and such that its Lie algebra is the sum of all $\mathbf{F}$-vector spaces inside $\mathfrak{n}$ whereupon $\mathbf{A}$ acts through the adjoint action by the characters α or 2α.
The reflections $W_\Delta := \{w_\alpha : \alpha \in \Delta\}$ generate W and there is a well defined length function ℓ on W assigning to w the shortest length of any of its expressions through products of elements in W_Δ. We denote by w_0 the unique element of maximal length in W.

0.2 The representations

The field $\mathbf{K}$ will serve as the coefficients of the vectors spaces our groups act on. We will call a $\mathbf{K}$-vector space V together with a $\mathbf{K}$-linear action of a group G a (**$\mathbf{K}$-linear**) **G-representation**. We say that representation V of a topological group G is **smooth** if the natural map $G\times V \to V$ is continuous for the discrete topology on V. This holds if and only if every vector is smooth, i.e. its stabilizer is open.
We say that a representation of (the rational points of) an affine algebraic group G on a finite dimensional $\mathbf{F}$-vector space V is **algebraic (or rational)**, if the natural map $G \times V \to V$ is given by (the rational points of) a morphism of affine $\mathbf{F}$-schemes $\mathbf{G} \times \mathbf{V} \to \mathbf{V}$; here $\mathbf{V}$ is the affine $\mathbf{F}$-scheme defined by $\mathbf{V}(R) = V \otimes_{\mathbf{F}} R$ for any $\mathbf{F}$-algebra R. If $\mathbf{K} \supseteq \mathbf{F}$ and V is a $\mathbf{K}$-vector space, then we call a representation of G upon V algebraic if there exists a G-stable $\mathbf{F}$-vector subspace $V_{\mathbf{F}} \subseteq V$ with $V_{\mathbf{F}} \otimes_{\mathbf{F}} \mathbf{K} = V$ such that the induced G-representation on $V_{\mathbf{F}}$ is algebraic.
Here every vector in V is algebraic, i.e. the orbit map $o_v : g \mapsto g \cdot v$ is algebraic in the above sense. This can be paraphrased by saying that a representation of a linear algebraic group G on V is algebraic if the action of G on V is given by a rational function in the coordinate entries of G and V.
We say that a representation V of (the rational points of) an affine algebraic group G over a topological field $\mathbf{F}$ is **locally algebraic** if every vector is locally algebraic: Fixing any $v_0 \in V$, there exists a finite dimensional $\mathbf{K}$-vector subspace $V_0 \ni v_0$ and a compact open subgroup $G_0 \subseteq G$ such that the natural map $G_0 \times V_0 \to V_0$ is the restriction of an algebraic representation of G. (See [Emerton, To appear, Comment succeeding Corollary 4.2.9].)
We remark that the tensor product $V \otimes U$ of a smooth representation V with an algebraic representation U is always locally algebraic.

Let $\mathbf{G}$ be a connected reductive group over $\mathbf{F}$ and let $\chi : M \to \mathbf{K}^*$ be a character. By precomposition with the projection $\bar{P} \twoheadrightarrow M$, it induces a character $\chi : \bar{P} \to \mathbf{K}^*$. We can then construct the $\mathbf{K}$-linear G-representation

$$\mathrm{Ind}_{\bar{P}}^G \chi := \{f : G \to \mathbf{K} : f(\bar{p}g) = \chi(\bar{p}) \cdot f(g) \text{ for all } \bar{p} \in \bar{P}, g \in G\},$$

where G acts upon by right translation denoted by $f^g := f(\cdot g)$ and inconsistently - by matters of convention - also by $g \cdot f = f^g$.

We call a character $\theta : M \to \mathbf{K}^*$ **unramified** if it is trivial on the maximal compact open subgroup $M_0 \subseteq M$. Then we can define the **unramified principal series** as the smooth G-representation given by all the smooth vectors inside $\mathrm{Ind}_P^G \theta$, which we denote by $\mathrm{Ind}_P^G \theta^{\mathrm{lc}}$. It is nonzero and, as the action of G is by translation, consists of all the locally constant functions therein.
We call a character $\psi : M \to \mathbf{K}^*$ **algebraic** if it is an algebraic representation on the $\mathbf{K}$-vector space $V = \mathbf{K}$. We say that ψ is **dominant** if $< \psi, \check{\alpha} > \geq 0$ for all $\alpha \in \Phi^+$ (See Section 2 for the notation used here).

For the following, we assume G to split: In every irreducible algebraic G-representation U, there exists a unique one dimensional subspace fixed by P and we call the corresponding algebraic character $\psi : P \to \mathbf{F}^*$ the **highest weight** of U. Then for every dominant algebraic character ψ, there exists a unique - up to isomorphism - irreducible algebraic G-representation U_ψ with highest weight ψ. Because $\mathbf{F} \subseteq \mathbf{K}$ is of characteristic 0, it is constructed by all the algebraic vectors inside the G-representation $\mathrm{Ind}_P^G \psi$, which we denote by $\mathrm{Ind}_P^G \psi^{\mathrm{alg}}$. It is nonzero and, as the action of G is by translation, consists of all the algebraic functions therein.
We call $\chi : M \to \mathbf{K}^*$ an **unramified dominant** character, if it is the product $\chi = \theta\psi$ of an unramified character θ and a dominant algebraic character ψ. Then we can define the **unramified dominant principal series** $I(\chi)$ for the unramified dominant character χ as the locally algebraic G-representation $I(\chi) = \mathrm{Ind}_P^G \theta^{\mathrm{lc}} \otimes_{\mathbf{K}} U_\psi$. It is also given by all the locally algebraic vectors $\mathrm{Ind}_P^G \chi^{\mathrm{lp}}$ inside $\mathrm{Ind}_P^G \chi$, and we have the following isomorphism of G-representations:

$$\begin{aligned} I(\theta\psi) = \mathrm{Ind}_P^G \theta^{\mathrm{lc}} \otimes U_\psi = \mathrm{Ind}_P^G \theta^{\mathrm{lc}} \otimes_{\mathbf{K}} \mathrm{Ind}_P^G \psi^{\mathrm{alg}} &\xrightarrow{\sim} \mathrm{Ind}_P^G \theta\psi^{\mathrm{lp}}, \\ f \otimes u &\mapsto f \cdot u := [g \mapsto f(g)u(g)]. \end{aligned}$$

Assumption. We will throughout tacitly make the following assumption: If $\chi = \theta$ is unramified, then we assume G to be a general connected reductive group and $I(\theta) = \mathrm{Ind}_P^G \theta^{\mathrm{lc}}$ is the unramified principal series as defined above.
If χ is assumed to be a general unramified dominant character, we will always assume G to split and therefore $I(\chi) = I(\theta) \otimes_{\mathbf{K}} U_\psi$ with U_ψ the unique irreducible algebraic representation of highest weight ψ.

0.3 The universal unitary completion of a locally algebraic representation

We let G be a topological group and V a $\mathbf{K}$-vector space equipped with a G-action.
Notation. Let R be a ring and $X \subseteq M$ a subset of an R-module M. Then we denote by $< X >_{R\text{-mod.}}$ the minimal R-module containing X inside M.

Definition. A lattice $\mathfrak{L}$ in V is an $\mathbf{o}$-submodule such that for any $v \in V$, there exists $\lambda \in \mathbf{K}^*$ such that $\lambda v \in \mathfrak{L}$. It has a corresponding seminorm $\|\cdot\|_{\mathfrak{L}}$ on V given by

$$\|v\|_{\mathfrak{L}} := \inf\{|\lambda| : \lambda \in \mathbf{K}^* \text{ with } \lambda v \in \mathfrak{L}\}.$$

We firstly recall the definition of the universal unitary completion of a continuous G-representation on a locally convex topological $\mathbf{K}$-vector space equipped with its finest locally convex topology. We will restrict to this case as by [Emerton, To appear, Corollary 6.3.7] any admissible (cf. Subsection 3.2) locally algebraic representation of V such as $I(\chi)$ is necessarily equipped with its finest locally convex topology.

Definition. 1. We call a G-representation on a $\mathbf{K}$-Banach space U **unitary** if the topology of U may be defined by a G-invariant norm.

2. Let V be a G-representation. Then the unitary $\mathbf{K}$-Banach space representation $\widehat{V}$ is the **universal unitary completion** of V if any $\mathbf{K}[G]$-linear map $V \to W$ into a unitary $\mathbf{K}$-Banach space representation W factors uniquely over $\widehat{V}$.

Lemma 0.1. *Let $\mathfrak{L}$ be a minimal element in the set of commensurability classes of G-invariant lattices in V ordered by inclusion. Then the completion with respect to $\mathfrak{L}$ is the universal unitary unitary completion of V.*

Proof. By definition, see [Emerton, 2005, Lemma 1.3]. ■

Remark. (i) Here two lattices $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ are defined to be **commensurable** if there exists scalars λ and Λ in $\mathbf{K}^*$ such that $\lambda\mathfrak{L} \subseteq \tilde{\mathfrak{L}} \subseteq \Lambda\mathfrak{L}$. Their induced seminorms are therefore equivalent and yield thus the same completion so that the above notion of universal unitary unitary completion is indeed well-defined. Because G-invariant lattices are closed under finite intersections, this minimal commensurability class is unique and we will by abuse of language denote any lattice therein as the **universal unitary lattice**.

(ii) We will call a lattice $\mathfrak{L}$ **Hausdorff** if its induced topology is Hausdorff. Equivalent characterizations of this property are as follows:

a) The lattice does not contain any $\mathbf{K}$-line.

b) The induced seminorm is a norm.

c) If V is countably infinite dimensional: The lattice is commensurable to a free $\mathbf{o}$-module.

Proof. The implications 1. $\Rightarrow$ 2. and 3. $\Rightarrow$ 1. are clear. For 2. $\Rightarrow$ 3., we refer the reader to the structure theory of non-Archimedean Banach spaces: Let $\hat{V}$ be the completion of V with respect to the norm $\|\cdot\|$ attached to $\mathfrak{L}$. Let $c_0(\mathbb{N})$ be the $\mathbf{K}$-Banach space of sequences in $\mathbf{K}$ converging to 0. Then the proof of [Schneider, 2002, Proposition 10.4] shows that we can find a topological isomorphism $\hat{V} \xrightarrow{\sim} c_0(\mathbb{N})$ such that the preimage $\{v_n\} \subseteq \hat{V}$ of all the zero sequences $\{(0, \dots, 0, 1, 0, \dots) : n \in \mathbb{N}\} \subseteq c_0(\mathbb{N})$ whose sole nonzero entry is 1 at the n-th place actually lies in $V \hookrightarrow \hat{V}$. Then $\{v_n\}$ is basis of V whose $\mathbf{o}$-linear span is commensurable to $\mathfrak{L}$. ■

We add that, if $\|V\| \subseteq |K|$, (this happens e.g. if $\mathbf{K}$ is discretely valued), then we can scale our basis vectors of $\mathfrak{L}$ to have norm 1. Therefore $\mathfrak{L}$ is already a free $\mathbf{o}$-module (not only commensurable).

(iii) We will call a nonzero lattice $\mathfrak{L} \subseteq V$ **proper** if $\mathfrak{L} \subset V$ or equivalently its induced seminorm is nonzero.

Proposition 0.2. *If V is a finitely generated G-representation, then any finite type lattice of V gives rise to the universal unitary unitary completion of V. Here the finiteness conditions refer to finite generation as a $\mathbf{K}[G]$- respectively $\mathbf{o}[G]$-module.*

Proof. Firstly we note that being finitely generated as an $\mathbf{o}[G]$-module, any two such lattices are commensurable and hence induce the same topology. The proposition follows by the above characterization in Lemma 0.1. ■

We remark that we do not require (universal unitary) lattices by our definition to be proper or even Hausdorff. Thus the universal unitary completion - being always equipped with a proper norm through factoring over the quotient space by the kernel of the seminorm - can vanish. We observe the following, though.

Remark. If V is irreducible as a G-representation and the lattice $\mathfrak{L} \subseteq V$ is G-stable, then $\mathfrak{L}$ is proper if and only if it is Hausdorff. In particular this applies to any universal unitary lattice.

Proof. We have to show that if $\mathfrak{L}$ is proper, then it is Hausdorff. By contraposition, assume that there is nonzero $v \in \mathfrak{L}$ with $\mathbf{K} \cdot v \subseteq \mathfrak{L}$. Then also $\mathbf{K} \cdot gv \subseteq \mathfrak{L}$ for all $g \in G$, as $\mathfrak{L}$ is G-stable, i.e. $\mathbf{K}[G] \cdot v \subseteq \mathfrak{L}$. Because V is an irreducible $\mathbf{K}[G]$-module, we conclude $V = \mathbf{K}[G] \cdot v \subseteq \mathfrak{L}$. ■

1 The unramified dominant principal series as a representation of P

Notation. Let G be a group.

1. We have a left respectively right action of G on itself through group automorphisms by left respectively right conjugation which we will denote by ${}^{g}\cdot$ respectively $\cdot^{g}$ for $g \in G$.

2. Let G act on a $\mathbf{K}$-vector space V. Then we put $V^G = \{v \in V : g \cdot x = x \text{ for all } g \in G\}$.

Notation. For a topological space X and a set Y, we denote by $\mathcal{C}^{\mathrm{lc}}(X, Y)$ all locally constant functions $f : X \to Y$.
Let $Y \ni 0, 1$. Then for $f : X \to Y$ we define $\operatorname{supp} f = \{x \in X : f(x) \neq 0\}$ and write $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(X, Y) \subseteq \mathcal{C}^{\mathrm{lc}}(X, Y)$ for all locally constant functions of compact support. For a subset $W \subset X$ we let $1_W : X \to Y$ be the *indicator function of* W defined by $\operatorname{supp} f = W$ and $1_{W|W} = 1$.

We let $\mathbf{G}$ be a connected reductive group over $\mathbf{F}$ and $\theta : M \to \mathbf{K}^*$ an unramified character. We remark that we have a well-defined notion of support inside $\mathfrak{F} := \bar{P}\backslash G$ for functions in $I(\theta)$ as $\bar{P}$ acts on the left through multiplication with invertible scalars on $I(\theta)$. Then we can view N as an open subset in $\mathfrak{F}$ via the image of the open immersion $N \subseteq G \overset{\text{can.}}{\to} \bar{P}\backslash G$ and accordingly every Nw for $w \in W$ as an open subset of $\mathfrak{F}$ via the image of the open immersion $N \hookrightarrow \bar{P}\backslash G \overset{\cdot w}{\to} \bar{P}\backslash G$. Thus we can define $I(\theta)(Nw)$ to be the functions inside $I(\theta)$ whose support lies in the open subset $Nw \subseteq \mathfrak{F}$. Then the support is automatically compact: Because $\operatorname{supp} f = \mathfrak{F} - f^{-1}\{0\}$ is an open subset of $\mathfrak{F}$ and $\mathfrak{F}$ is compact, hence bounded, we find by total disconnectedness of $\mathfrak{F}$ the support of f to be closed and thus compact.

Lemma 1.1. *By restriction onto $Nw \subseteq G$, we have an isomorphism of $\mathbf{K}$-vector spaces*

$$\operatorname{Ind}_{\bar{P}}^{G} \theta^{\mathrm{lc}}(Nw) \overset{\sim}{\to} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(Nw, \mathbf{K}).$$

Proof. It is clearly injective and it rests to be seen to be surjective: Since $1 \in N$ has a neighborhood basis of compact open subgroups, the $\mathbf{K}$-vector space $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(Nw, \mathbf{K})$ is generated by all indicator functions $1_{N_c nw}$ for $n \in N$ and $\{N_c\}$ some neighborhood basis of compact open subgroups in N. We want to construct their preimages.
Fix $n \in N$ and let $I \subseteq G$ be a compact open subgroup with Iwahori factorization $I = I_{\bar{P}} I_N$ with $I_{\bar{P}} := I \cap \bar{P}$ and $I_N := I \cap N$, chosen sufficiently small to ensure θ to be trivial on $I_{\bar{P}}$. These form by [Casselman, 1995, Proposition 1.4.4] a neighborhood basis of the identity.
As θ is trivial on $I_{\bar{P}}$ and I is a group, the function f defined by bearing support $\bar{P}Inw$ and $f(\bar{p}inw) = \theta(\bar{p})$ for $\bar{p} \in \bar{P}, i \in I$, is quickly checked to be well-defined.
By construction f is constant on all right I-cosets and so in particular smooth. Thus $f \in I(\theta)$ and $\operatorname{supp} f = \bar{P} I_{\bar{P}} I_N nw = \bar{P} I_N nw \subseteq \bar{P} Nw$, i.e. $f \in I(\theta)(Nw)$. Finally $f_{|Nw} = 1_{I_N nw}$, where the compact open subgroup $I_N \subseteq N$ can be made arbitrarily small by choosing sufficiently small compact open $I \subseteq G$ in the neighborhood basis of 1 consisting of all compact open subgroups with Iwahori factorization. ■

Lemma 1.2. *(i) We find M to leave $I(\theta)(Nw)$ stable and operate on the $\mathbf{K}$-vector space $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K}) \cong \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(Nw,\mathbf{K}) \cong I(\theta)(Nw)$ by*

$$f^m = \theta({}^w m) f(\cdot^{{}^w m}).$$

In particular $1_U^m = \theta({}^w m) 1_{{}^{{}^w m}U}$ for $U \subseteq N$ compact open.

(ii) We find P to leave $I(\theta)(N)$ stable and operate on $I(\theta)(N) \cong \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K})$ by

$$f^p = \theta(m) f(\cdot^m n) \quad \text{for } p = mn \in P \text{ with } m \in M, n \in N.$$

In particular $1_U^{n^{-1}m} = \theta(m) 1_{{}^m U n}$ for $U \subseteq N$ compact open.

(iii) The $\mathbf{K}[P]$-module $I(\theta)(N) \cong \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K})$ is generated by any $f = 1_U$ with $U \subseteq N$ compact open.

(iv) Let $\eta_w : I(\theta)(N_w) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K})$ be the shifted restriction morphism given by $f \mapsto f_{|Nw}(\cdot w)$. For any $w \in W$, let $U_w \ni 1$ be a compact open neighborhood in N and $f_w = \eta_w^{-1}(1_{U_w})$. Then $\{f_w : w \in W\} \subseteq I(\theta)$ generates $I(\theta)$ as a $\mathbf{K}[P]$-module.

Proof. Ad (i): For $f \in I(\theta)$ and $m \in M$, we find for $n \in N$ that

$$f^m(nw) = f(nwm) = f(n^w mw) = \theta({}^w m) f(({}^w m)^{-1} n^w mw) = \theta({}^w m) f(n^{{}^w m} w),$$

where we recall M to normalize N. In particular, again $f^m \in I(\theta)(Nw)$. The formula for $f = 1_U$ with $U \subseteq N$ compact open follows directly.

Ad (ii): The action by right translation of $N \subseteq P = MN$ on $I(\theta)(N)$ translates trivially. Then the formula for $f = 1_U$ with $U \subseteq N$ compact open follows directly.

Ad (iii): By right translation through suitable $n \in N$, we obtain $Un = N_c$ for a compact open neighborhood $N_c \ni 1$. Let $f := 1_{N_c}$. By [Casselman, 1995, Proposition 1.4.3] there exists an element $a \in A$ with $|\alpha(a)|_{\mathbf{F}}$ sufficiently small for all $\alpha \in \Delta$, such that $\{{}^{a^i}N_c : i \in \mathbb{N}\}$ constitutes a system of neighborhoods of $1 \in N$. We just saw $f^m = \theta(m) 1_{{}^m N_c}$ for $m \in M$. The group N acts by right translation on these (scaled) indicator functions, therefore $\mathbf{K} \cdot \{f^p : p \in P\} \supseteq \mathbf{K} \cdot \{1_U\}$ for a topological basis of compact open subsets $\{U\}$ in N. Every $f \in \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K})$ is by definition a linear combination of such indicator functions 1_U of compact open subsets, so $\mathbf{K}[P] \cdot f \supseteq \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K})$.

Ad (iv): Regarding the operation of M on $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N,\mathbf{K}) \cong I(\theta)(Nw)$ given in (i), we just saw in (iii) above that there exists $a \in A$ such that $\mathbf{K}^{w^{-1}}\{a^i\} \cdot 1_{U_w} = \mathbf{K}\{1_{V_w} : V_w \in \mathcal{V}_w\}$, with $\mathcal{V}_w$ a basis of neighborhoods of $1 \in N$. Let $\phi_{V_w} = \eta_w^{-1}(1_{V_w})$ be their preimages in $I(\theta)(Nw)$, determined by carrying support $V_w w \subseteq \bar{P}\backslash G$ and being equal to 1 on $V_w w$. (Here and in the following we will identify a subset in G with its canonical image in $\bar{P}\backslash G$.) By letting N act on these through right translation, we obtain all functions $\phi_{V_w n} \in I(\theta)$ with support in $V_w wn \subseteq \bar{P}\backslash G$ and equal to 1 on $V_w wn$ for $V_w \in \mathcal{V}_w, w \in W$ and $n \in N$.

Now there is a decomposition $\bar{P}\backslash G = \dot{\bigcup}_{w\in W} wN$ as follows: By [Borel, 1991, IV.14.12], we have the Bruhat decomposition $G = \dot{\cup}_{w\in W}\bar{P}w\bar{P}$. By definition W normalizes M, thus $\bar{P}\backslash G = \dot{\cup}_{w\in W} w\bar{N}$. By conjugation with the longest element $w_0 \in W$ giving $\bar{N}^{w_0} = N$, we obtain $\bar{P}\backslash G = \dot{\cup}_{w\in W} wN$.
Thereof and since $\mathcal{V}_w$ forms for all $w \in W$ a basis of neighborhoods of 1 in the open subset $N \subseteq \bar{P}\backslash G$, we find $\{V_w wn : V_w \in \mathcal{V}_w \text{ for } w \in W, n \in N\}$ to be a topological basis of $\bar{P}\backslash G$. Thus

$$I(\theta) = \sum_{V_w\in\mathcal{V}_w \text{ for } w\in W} \mathbf{K}[N]\phi_{V_w} = \sum_{w\in W} \mathbf{K}[P] f_w.$$

This proves the proposition. ■

Definition 1.3. Conferring [Casselman, 1980, Section 2], let $\phi_w \in I(\theta)$ for $w \in W$ be defined by having support $\bar{P}w\bar{B}$ and being equal to 1 on $w\bar{B}$.

Remark 1.4. By the Bruhat-Tits decomposition $K = \dot{\bigcup}_{w\in W}\bar{B}w\bar{B}$ for K with respect to $\bar{B}$, the ϕ_w constitute a basis of the Iwahori invariants $I(\theta)^{\bar{B}}$ of $I(\theta)$, the $\mathbf{K}$-vector subspace of all elements in $I(\theta)$ fixed by $\bar{B}$.

Corollary 1.5. *We have* $I(\theta) = \sum_{w\in W} \mathbf{K}[P]\phi_w$.

Proof. We show ϕ_w to fulfill the conditions of Lemma 1.2(iv). We put $\bar{P}_{w\bar{B}} := \bar{P} \cap {}^w\bar{B}$ and $N_{w\bar{B}} := N \cap {}^w\bar{B}$. Then by the following Lemma 1.6, we find ${}^w\bar{B} = \bar{P}_{w\bar{B}} N_{w\bar{B}}$, so

$$\bar{P}w\bar{B} = \bar{P}{}^w\bar{B}w = \bar{P}N_{w\bar{B}}w.$$

Therefore $\phi_w \in I(\theta)(Nw)$, identifying to $1_{N_{w\bar{B}}} \in \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \cong I(\theta)(Nw)$, where we recall $N_{w\bar{B}} = N \cap {}^w\bar{B} \ni 1$ to be a compact open neighborhood in N, as $\bar{B}$ is compact open in G. ■

Lemma 1.6. *Let* $\bar{P}_{w\bar{B}} := \bar{P} \cap {}^w\bar{B}$ *and* $N_{w\bar{B}} := N \cap {}^w\bar{B}$. *Then* ${}^w\bar{B} = \bar{P}_{w\bar{B}} N_{w\bar{B}}$

Proof. By [Tits, 1979, Section 3.1.1 (with $\Omega = \{x_0\}$)], we have the *Iwahori factorization*, a bijection for every ordering of the following product:

$$\bar{B} = \bar{N}_0 M_0 N_1 \quad \text{with} \quad \bar{N}_0 = \prod_{a\in\Phi_0^-} N(a) \text{ and } N_1 = \prod_{a\in\Phi_0^+} N(a^+);$$

here $\Phi_0^\pm \xrightarrow{\sim} \Phi^\pm$ are bijections denoted by $a \mapsto \alpha$, and $N(a)$ respectively $N(a^+)$ being compact open subgroups of N_α. Thus

$${}^w\bar{B} = \prod_{\substack{a\in\Phi_0^-,\\ w^{-1}\alpha\prec 0}} N(a) \prod_{\substack{a\in\Phi_0^-,\\ w^{-1}\alpha\succ 0}} N(a^+) M_0 \prod_{\substack{a\in\Phi_0^+,\\ w^{-1}\alpha\succ 0}} N(a^+) \prod_{\substack{a\in\Phi_0^+,\\ w^{-1}\alpha\prec 0}} N(a).$$

We find ${}^w\bar{B} = \bar{P}_{w\bar{B}} N_{w\bar{B}}$ with

$$\bar{P}_{w\bar{B}} = \prod_{\substack{a\in\Phi_0^-,\\ w^{-1}\alpha\prec 0}} N(a) \prod_{\substack{a\in\Phi_0^-,\\ w^{-1}\alpha\succ 0}} N(a^+) M_0 = \bar{P} \cap {}^w\bar{B}$$

and

$$N_{^w\bar{B}} = \prod_{\substack{a\in\Phi_0^+,\\ w^{-1}\alpha\succ 0}} N(a^+) \prod_{\substack{a\in\Phi_0^+,\\ w^{-1}\alpha\prec 0}} N(a) = N \cap {}^w\bar{B}.$$

■

Corollary 1.7. *We let $\chi : M \to \mathbf{K}^*$ be an unramified dominant character. Let $U = U_\psi$ be an irreducible algebraic G-representation and denote by $\bar{u}$ its unique - up to a scalar - vector fixed by $\bar{N}$. Then*

$$I(\chi)(N) = \mathbf{K}[P] \cdot \phi_1 \otimes \bar{u} \quad \text{and} \quad I(\chi) = \sum_{w\in W} \mathbf{K}[P] \cdot \phi_w \otimes \bar{u}.$$

Proof. We firstly point out the following fact: Let G split and assume U to be an irreducible algebraic G-representation. Denote by $\bar{u}$ its unique - up to a scalar - vector fixed by $\bar{N}$. Let $N_0 \subseteq N$ be any open subgroup. By [Borel, 1991, Theorem 21.20(i)] we find $N \subseteq \mathbf{N}$ to be Zariski dense, and this equally holds by the same token together with the Taylor expansion for the inclusion $N_0 \subseteq N$. Therefore the proof of [Humphreys, 1975, Proposition 31.2] shows that the $K[P_0]$-module U is generated by $\bar{u}$ for any open subgroup $P_0 \subseteq P$.
Now let $P_{\bar{B}} = P \cap \bar{B}$. Since $P_{\bar{B}} \subseteq P$ is open, we find

$$\mathbf{K}[P_{\bar{B}}] \cdot \phi_1 \otimes \bar{u} = \mathbf{K} \cdot \phi_1 \otimes \mathbf{K}[P_{\bar{B}}] \cdot \bar{u} \overset{\text{fact}}{=} \mathbf{K} \cdot \phi_1 \otimes U.$$

Therefore

$$\begin{aligned} \mathbf{K}[P] \cdot \phi_1 \otimes \bar{u} &= \mathbf{K}[P] \cdot (\mathbf{K}[P_{\bar{B}}] \cdot \phi_1 \otimes \bar{u}) \\ &= \mathbf{K}[P] \cdot (\mathbf{K} \cdot \phi_1 \otimes U) \\ &= I(\theta)(N) \otimes U = I(\chi)(N); \end{aligned}$$

the last equality by Lemma 1.2.(iii), and likewise

$$\begin{aligned} \mathbf{K}[P] \cdot \{\phi_w \otimes \bar{u}\} &= \mathbf{K}[P] \cdot (\mathbf{K}[P_{\bar{B}}] \cdot \{\phi_w \otimes \bar{u}\}) \\ &= \mathbf{K}[P] \cdot (I(\theta)^{\bar{B}} \otimes U) \\ &= I(\theta) \otimes U = I(\chi); \end{aligned}$$

the last equality by Corollary 1.5. ■

2 The universal unitary lattice of the P-representation on an open cell and a norm of differentiable functions

Interlude: The dominant submonoid acting on the affine root factors

Let **G** be a connected reductive group over **F**. We recall both $A \subseteq M$ to contain maximal compact open subgroups $A_0 \subseteq M_0$. Let $N_0 \subseteq N$ be a compact open subgroup given as the product of open root subgroups, and define $M^+ := \{m \in M : {}^m N_0 \subseteq N_0\}$. We will see that $|\chi(\cdot)|_{\mathbf{F}}$ naturally extends from A/A_0 to M/M_0 for any algebraic character χ on **A**. Then we deduce $M^+ = \{m \in M : |\alpha(m)|_{\mathbf{F}} \leq 1 \quad \text{for all } \alpha \in \Delta\}$, by looking at the action of **M** on the root factors $\mathbf{N}_\alpha \subseteq \mathbf{N}$ for $\alpha \in \Phi^+$.

The order morphism into the cocharacter group of M

Definition. Let **G** be any affine algebraic group over **F** .

(i) We denote by $X^*(\mathbf{G}) := \mathrm{Hom}_{\mathbf{F}\text{-grp.schm.}}(\mathbf{G}, \mathbb{G}_m)$ the abelian group of characters of **G**.

(ii) We define the order morphism $v : G \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{G}), \mathbb{Z})$ for any $g \in G$ by

$$v(g) := [\lambda \mapsto -v_{\mathbf{F}}(\lambda(g))] \quad \text{for all } \lambda \in X^*(\mathbf{G}).$$

Here $\lambda(g)$ denotes evaluation at g of the morphism obtained by application of the functor of rational points to the morphism λ (and $v_{\mathbf{F}}$ the normalized valuation of **F**).

Remark 2.1. 1. If $\mathbf{G} = \mathbf{S}$ is an **F**-split torus, then $v : S \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{S}), \mathbb{Z})$ is readily seen to be surjective.

2. The kernel of v is the maximal compact open subgroup $M_0 \subset M$. In particular M/M_0 is a free $\mathbb{Z}$-module.

Definition. We let **G** be a connected reductive group over **F**.

(i) Define $\Lambda := v(M/Z)$ as the image of the group morphism

$$v : M/Z \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M}\,/\,\mathbf{Z}), \mathbb{Z}).$$

(ii) Let $\Lambda^+ \subseteq \Lambda$ be the **dominant submonoid** defined by

$$\Lambda^+ := \{\lambda \in \Lambda :< \alpha, \lambda > \geq 0 \quad \text{for all } \alpha \in \Delta\}.$$

We define $M^+ \subseteq M$ as the preimage of Λ^+ under $M \overset{\text{can.}}{\twoheadrightarrow} M/Z \overset{v}{\to} \Lambda$.

Remark 2.2. The submonoid Λ^+ generates the $\mathbb{Z}$-module Λ.

Proof. Let $\lambda_0 \in \Lambda^+$ be such that $< \alpha, \lambda_0 >\geq C$ for all $\alpha \in \Delta$ for a constant $C > 0$. Then since $< \alpha, \lambda - i \cdot \lambda_0 >=< \alpha, \lambda > -i\cdot < \alpha, \lambda_0 >$, we find $-i \cdot \lambda_0 + \Lambda^+ = \{\lambda \in \Lambda :< \alpha, \lambda >\geq -iC\}$, exhausting Λ for $i \in \mathbb{N}$ tending towards infinity. ■

We let **G** be a connected reductive group over **F**.

Lemma 2.3. *Let $\mathbf{A_Z} \subseteq \mathbf{Z}$ be the maximal* **F**-*split central torus of* **G**. *Then the embedding $\mathbf{A_Z} \subseteq \mathbf{G}$ induces an inclusion of finite free $\mathbb{Z}$-modules $X^*(\mathbf{G}) \hookrightarrow X^*(\mathbf{A_Z})$ of finite index.*

Proof. Cf. [Borel, 1966, Section 2.2], the $\mathbb{Z}$-module $X^*(\mathbf{G})$ is finitely generated (and if **G** is connected also torsion-free, hence free).
By [Borel, 1966, Theorem 5.2(2)], we find **G** to be reductive if and only if $\mathbf{G} = \mathbf{Z}^\circ[\mathbf{G}, \mathbf{G}]$ and by [Borel, 1991, Proposition 14.2(3)] this is an almost direct product, i.e. $\mathbf{Z}^\circ \cap [\mathbf{G}, \mathbf{G}]$ is finite. Therefore each character factors over a finite quotient of $\mathbf{Z}^\circ$. Hence we have an inclusion of finite index $X^*(\mathbf{G}) \hookrightarrow X^*(\mathbf{Z}^\circ)$. Cf. [Borel, 1966, Theorem 5.2], the center $\mathbf{Z}^\circ =: \mathbf{T}$ is a torus over **F**. By [Borel, 1966, Theorem 3.3 et pre.], the torus **T** is the almost direct product of its maximal **F**-split torus $\mathbf{T}_s = \mathbf{A_Z}$ and maximal anisotropic torus $\mathbf{T}_a$ over **F**, i.e. $\mathbf{T} = \mathbf{T}_s \mathbf{T}_a$ with $\mathbf{T}_s \cap \mathbf{T}_a$ finite; also recall that a torus **T** is called **anisotropic** if $X^*(\mathbf{T}) = \{0\}$. Thus we obtain an inclusion of finite index $X^*(\mathbf{T}) = X^*(\mathbf{T}_s \,/\, \mathbf{T}_s \cap \mathbf{T}_a) \hookrightarrow X^*(\mathbf{T}_s)$. We conclude that we obtain by concatenation an inclusion $X^*(\mathbf{G}) \hookrightarrow X^*(\mathbf{A_Z})$ of finite index. ■

Corollary 2.4. *The abelian group Λ is a finite free $\mathbb{Z}$-module of rank $\#\Delta$.*

Proof. Because $X^*(\mathbf{M}) \subseteq X^*(\mathbf{A})$ is an inclusion of finite index by Lemma 2.3, we have an inclusion of finite free $\mathbb{Z}$-modules $\mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{A} \,/\, \mathbf{Z}), \mathbb{Z}) \subseteq \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M} \,/\, \mathbf{Z}), \mathbb{Z})$ of finite index. Thence by the sandwiching

$$\mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{A} \,/\, \mathbf{Z}), \mathbb{Z}) = v(A/Z) \subseteq v(M/Z) = \Lambda \subseteq \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M} \,/\, \mathbf{Z}), \mathbb{Z}),$$

and the elementary divisor theorem Λ is a finite free $\mathbb{Z}$-module of rank $\#\Delta$. ■

The normalizer of the maximal split torus acting on the apartment

Definition. For any **F**-split torus **S**, we denote by

$$X_*(\mathbf{S}) := \mathrm{Hom}_{\mathbf{F}\text{-grp.schm.}}(\mathbb{G}_m, \mathbf{S})$$

the finitely generated free abelian group of cocharacters of **S**. It is dual to the abelian group of characters $X^*(\mathbf{S})$ by the natural pairing

$$X_*(\mathbf{S}) \times X^*(\mathbf{S}) \to \mathrm{End}_{\mathbf{F}\text{-grp.schm.}}(\mathbb{G}_m) = \mathbb{Z}$$

through composition (cf. [Borel, 1991, Proposition 8.6]) and we may thus identify $X_*(\mathbf{S}) = \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{S}), \mathbb{Z})$.

If **G** is as always connected reductive, by the above Lemma 2.3, we find $X^*(\mathbf{G}) \subseteq X^*(\mathbf{Z_A})$ to be an inclusion of finite index. Therefore dually an inclusion of finite index $X_*(\mathbf{Z_A}) \cong \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{Z_A}), \mathbb{Z}) \subseteq \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{G}), \mathbb{Z})$. Thus we can view $\mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{G}), \mathbb{Z})$ as a lattice inside $X_*(\mathbf{Z_A}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Then the above morphism $v : G \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{G}), \mathbb{Z})$ identifies to the morphism $v : G \to X^*(\mathbf{Z_A}) \otimes_{\mathbb{Z}} \mathbb{Q}$ characterized by

$$< v(g), \lambda >= v_{\mathbf{F}}(\lambda(g)) \quad \text{for all } g \in \mathbf{G} \text{ and } \lambda \in X^*(\mathbf{G}) \subseteq X_*(\mathbf{Z_A}).$$

We want to apply this in the following setting: We observe that $\mathbf{M} = \mathbf{C_G}(\mathbf{A})$ is a connected reductive group over **F**: By [Borel, 1966, Theorem 5.3(1)] we find **M** to be connected and by [Borel, 1966, Section 5.4(1)] it is defined over **F**. It is also nilpotent, in particular reductive. Hence we may put $\mathbf{G} = \mathbf{M}$ (and then $\mathbf{Z_A} = \mathbf{A}$) and obtain $v : M \to X_*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Following [Schneider and Stuhler, 1997, Section I.1], define $\mathcal{A} := X_*(\mathbf{A})/X_*(\mathbf{Z}^\circ) \otimes_{\mathbb{Z}} \mathbb{R}$, the **apartment** corresponding to **A** in the Bruhat-Tits building of G. Then we have:

1. A translation action of $M = C_G(A)$ on $\mathcal{A}$ as follows: Recall that we have a morphism of groups

$$v : M \xrightarrow{v} \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M}), \mathbb{Z}) \hookrightarrow X_*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow X_*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R} \twoheadrightarrow \mathcal{A}.$$

   Then $m \in M$ acts on $\mathcal{A}$ by the translation $x \mapsto x + v(m)$ (cf. [Schneider and Stuhler, 1997, Section I.1], this also explains the minus sign in the definition of v).

2. A linear action of $W = N_G(A)/C_G(A)$ on $\mathcal{A}$ through conjugation.

By a general argument given in [Tits, 1979, Section 1.2], these actions combine to an action by all of $N_G(A)$ of $\mathcal{A}$ through affine linear maps.

The valuation functions on the root factors

The Weyl group W is generated by reflections at the hyperplanes $H_\alpha := \ker \alpha$ for $\alpha \in \Phi$. These are hence of the form

$$x \mapsto x - \alpha(x)\check{\alpha}$$

for some $\check{\alpha} \in \mathcal{A}$ with $\alpha(\check{\alpha}) =< \alpha, \check{\alpha} >= 2$. Because W permutes the finite set of generators Φ of the $\mathbb{R}$-vector space $\mathcal{A}$, a general argument shows this reflection w_α and so $\check{\alpha}$ to be uniquely determined.
Fix $\alpha \in \Phi^{\text{red}}$ and let $n \in N_\alpha$, the root subgroup belonging to α. Then NnN lies in a unique double coset of the Bruhat-Tits decomposition $G = \dot{\cup}_{m \in N_G(A)} \bar{N} \times \{m\} \times \bar{N}$, parameterized by $\tilde{m}(n) \in N_G(A)$ say. We put $m(n) = \tilde{m}(n)$ if $\alpha \succ 0$ and $m(n) = {}^{w_0}\tilde{m}(n)$ if $\alpha \prec 0$. If $n \neq 1$, then the coset of $m(n)$ in W can be seen to be w_α and more exactly, the affine linear map through which $m(n)$ acts on $\mathcal{A}$ is given by

$$x \mapsto w_\alpha(x) - \varphi_\alpha(n)\check{\alpha} = x - (\alpha(x) + \varphi_\alpha(n)) \cdot \check{\alpha},$$

for some real number $\varphi_\alpha(n)$.

We conclude that we have for each $\alpha \in \Phi^{\text{red}}$ a **valuation function** $\varphi_a : N_\alpha^* \to \mathbb{R}$ with $N_\alpha^* = N_\alpha - \{1\}$ and we let $\Gamma_\alpha := \operatorname{im} \varphi_\alpha$ be its image. It is a discrete and unbounded subset in $\mathbb{R}$. Then the set of affine linear forms $\Phi_{\text{aff}} := \coprod_{\alpha \in \Phi^{\text{red}}} \alpha + \Gamma_\alpha$ on $\mathcal{A}$ is called the **affine root system** and, given $a = \alpha + i \in \Phi_{\text{aff}}$, we denote by α its **vectorial part**.
It will be convenient to extend φ_α to all of N_α by sending 1 to ∞. Then we define for each $a = \alpha + i \in \Phi_{\text{aff}}$ the affine root subgroup

$$N(a) := \{n \in N_\alpha : \varphi_\alpha(n) \geq i\}.$$

By [Bruhat and Tits, 1972, I.6.2.12b] this is a separated filtration of N_α. We note that even though $a = \alpha + i$ for $\alpha \in \Phi^{\text{red}}$ is not necessarily an affine root for general $i \in \mathbb{R}$, this definition is still meaningful.

The Cartan subgroup acting on the affine root factors

Notation. For the remainder of this subsection, we will denote by $v : M \to \mathcal{A}$ the above constructed morphism of groups

$$v : M \xrightarrow{v} \operatorname{Hom}_{\mathbb{Z}}(X^*(\mathbf{M}), \mathbb{Z}) \hookrightarrow X_*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{Q} \hookrightarrow X_*(\mathbf{A}) \otimes_{\mathbb{Z}} \mathbb{R} \twoheadrightarrow \mathcal{A}.$$

We note that $\Phi \subseteq X^*(\mathbf{A} / \mathbf{Z})$ and therefore can evaluate every $\alpha \in \Phi$ on the appartement $\mathcal{A}$.

Lemma 2.5. *Let $m \in M$. We have ${}^mN(a) = N(a + \alpha(v(m)))$ for all $a \in \Phi_{\text{aff}}$ with vectorial part $\alpha \in \Phi^{\text{red}}$.*

Proof. For $m \in M$, we have ${}^mN(a) = N(m \cdot a)$ (with $m \cdot a := a(m \cdot))$, see [Tits, 1979, Section 1.4]. Each $m \in M$ acts on $\mathcal{A}$ by the translation $x \mapsto x + v(m)$. By linearity of α thus $m \cdot \alpha = \alpha + \alpha(v(m))$ for any $\alpha \in \Phi$ and the proposition follows. ∎

Corollary 2.6. *Let $N(a)$ for $a = \alpha + i \in \Phi_{\text{aff}}$ be an affine root group. Then for $m \in M$, it holds ${}^mN(a) \subseteq N(a)$ if and only if $\alpha(v(m)) \geq 0$.*

Proof. If $\alpha(v(m)) \geq 0$, then by the above Lemma 2.5, we find ${}^mN(a) = N(a + \alpha(v(m))) \subseteq N(a)$.
Conversely, if $c := \alpha(v(m)) < 0$, then $i + \alpha(v(m^k)) = i + k(\alpha(v(m)) \leq j < i$ with $j \in \Gamma_\alpha$ for $k >> 0$, since Γ_α is unbounded in both directions by [Schneider and Stuhler, 1997, Section I.1]. This implies ${}^{m^k}N(a) = N(a + \alpha(v(m^k))) \supset N(a)$, as $N(\alpha + i) = \varphi_\alpha^{-1}[i, \infty]$ by definition. But ${}^{m^k}N(a) \supset N(a)$ only if ${}^mN(a) \supset N(a)$. ∎

Recall that we chose our maximal compact open subgroup **K** to be **special**: This means that its corresponding fixed point $x_0 \in \mathcal{A}$ satisfies $\alpha(x) \in -\Gamma_\alpha$ for all $\alpha \in \Phi^{\text{red}}$ (see [Schneider and Stuhler, 1997, Section I.3]).

Definition. We put $\Phi_0 := \{\alpha - \alpha(x_0) : \alpha \in \Phi^{\mathrm{red}}\} \subseteq \Phi_{\mathrm{aff}}$; it can be described as the set of all affine roots vanishing at x_0. Correspondingly we denote the translates of Φ^+, Φ^- and Δ in Φ_{aff} vanishing at x_0 by Φ_0^+, Φ_0^- and Δ_0.

We will from now on fix compact open subgroups $N_0 \subseteq N$ and $\bar{N}_1 \subseteq \bar{N}$ already occurring in the proof of Lemma 1.6.

Definition. We let $\bar{N}_1 = \prod_{a\in\Phi_0^-} N(a^+)$ and $N_0 = \prod_{a\in\Phi_0^+} N(a)$.

Remark. By [Tits, 1979, Section 3.1.1 with $\Omega = \{x_0\}$], we have the Iwahori factorization

$$B = \bar{N}_1 M_0 N_0 \quad \text{with} \quad \bar{N}_1 = \prod_{a\in\Phi_0^-} N(a^+) \ \text{ and } \ N_0 = \prod_{a\in\Phi_0^+} N(a).$$

Lemma 2.7. *Let $m \in M$. We have ${}^mN_0 \subseteq N_0$ if and only if $\alpha(v(m)) \geq 0$ for all $\alpha \in \Delta$.*

Proof. This follows by Corollary 2.6 as m stabilizes N_0 if and only if it stabilizes each factor $N(\alpha + i)$, and noting that $\alpha(v(m)) \geq 0$ for all $\alpha \in \Delta$ if and only if $\alpha(v(m)) \geq 0$ for all $\alpha \in \Phi^+$. ■

Corollary 2.8. *We find $M^+ = \{m \in M : {}^mN_0 \subseteq N_0\}$.*

Proof. Fix $m \in M$. By the above Lemma 2.7, we find ${}^mN_0 \subseteq N_0$ if and only if $\alpha(v(m)) =< \alpha, v(m) >\geq 0$ for all $\alpha \in \Delta$, which holds by definition if and only if $m \in M^+$. ■

2.1 The necessity criterion

We let $\mathbf{G}$ be a connected reductive group over $\mathbf{F}$ and let $\chi = \theta\psi : M \to \mathbf{K}^*$ be an unramified dominant character.

Notation. If $f : X \to Y$ is mapping, from a set X into a normed space $(Y, |\cdot|)$. Then we put, if defined, $\|f\|_{\sup} = \sup_{x\in X}|f(x)|$ and $\|f\|_{\sup} = \infty$ otherwise. For a subset $W \subseteq X$, we will abbreviate $\|f\|_W = \|f_{|W}\|_{\sup}$.

Assumption. We recall that by definition $I(\chi) = \mathrm{Ind}_P^G \theta^{\mathrm{lc}} \otimes_{\mathbf{K}} U_\psi$.

Throughout this subsection, we will by Lemma 1.1 identify

$$I(\chi) \supseteq I(\chi)(N) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_\psi$$

as $\mathbf{K}[P]$-modules without further mention. We recall that the action of P on the right hand side is given by $f \otimes u^m = \theta(m) \cdot f(\cdot^m) \otimes u^m$ for all $m \in M$ and by $f \otimes u^n = f(\cdot n) \otimes u^n$ for all $n \in N$.

We firstly show that the criteria $|\chi(M^+)| \leq 1$ and $|\chi(Z)| = 1$ are necessary for the universal unitary completion not to vanish, equivalently the inclusion of the universal unitary $\mathbf{o_K}$-lattice $\mathfrak{L} \subseteq I(\chi)(N)$ to be proper.

Lemma 2.9. *The universal unitary $o_{\mathbf{K}}$-lattice of $I(\chi)(N)$ is given by $\mathfrak{L} = \mathbf{o_K}[P] \cdot \phi_1 \otimes \bar{u}$.*

Proof. By Corollary 1.7, we find $I(\chi)(N)$ to be generated by $\phi_1 \otimes \bar{u}$ as a $\mathbf{K}[P]$-module. Therefore

$$\mathfrak{L} = \mathbf{o_K}[P] \cdot \phi_1 \otimes \bar{u} \subseteq I(\chi)(N)$$

is an $\mathbf{o_K}$-lattice of $I(\chi)(N)$ and of course finitely generated as an $\mathbf{o_K}[P]$-module. We conclude that it is by Proposition 0.2 the universal unitary unitary lattice of $I(\chi)(N)$ as a $\mathbf{K}[P]$-module. ∎

Lemma 2.10. *Let $\mathfrak{L} \subseteq I(\chi)(N)$ be the universal unitary lattice and $\|\cdot\|_{\mathfrak{L}}$ its associated norm.*

(i) If $\|\cdot\|_{\mathfrak{L}} \neq 0$, then $|\chi(Z)| = \{1\}$.

(ii) Let $|\chi(Z)| = \{1\}$. If $|\chi(m)| > 1$ for some $m \in M^+$, then $\mathfrak{L} = \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_\psi$.

Proof. Ad (i): The action of G on $I(\chi) \xrightarrow{\sim} \mathrm{Ind}_P^G \chi^{\mathrm{lp}}$ (as $\mathbf{K}[G]$-modules) is given on the right hand side by $f^g = f(\cdot g)$ and therefore $f^z := f(\cdot z) = f(z \cdot) = \chi(z) \cdot f$ for all $z \in Z$. Therefore

$$\|f\| = |\chi(z)| \cdot \|f\| \quad \text{for all } \ z \in Z, f \in I(\chi)(N). \tag{$*$}$$

We can by assumption find f such that $\|f\| \neq 0$. Then Equality $(*)$ holds if and only if $|\chi(z)| = 1$ for all $z \in Z$.

Ad (ii): We firstly assume that $\chi = \theta$ is unramified. Because $|\theta| : M^+ \to |\mathbf{K}^*|$ is unramified and trivial on the center $Z \subseteq G$ by assumption, we find $|\theta|_{|M^+}$ to factor over $M^+ \twoheadrightarrow M^+/M_0Z$. Let $m \in M^+/M_0Z$ such that $|\theta(m)| > 1$. We show that $\mathfrak{L} \supseteq \mathbf{K} \cdot 1_{N_0}$; since $\mathfrak{L} = \mathbf{o_K}[P] \cdot 1_{N_0}$ and $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) = \mathbf{K}[P] \cdot \phi_1$, this proves $\mathfrak{L} = \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$.
Because $\mathbf{G}/\mathbf{Z}$ is semi-simple adjoint, the root basis Δ spans $X^*(\mathbf{A}/\mathbf{Z})$. By Lemma 2.3, there is a canonical inclusion $X^*(\mathbf{M}/\mathbf{Z}) \subseteq X^*(\mathbf{A}/\mathbf{Z})$. Hence we find $v(m) \in v(M/Z) \subseteq \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M}/\mathbf{Z}), \mathbb{Z})$ to be zero if and only if $v(m)(\alpha) = 0$ for all $\alpha \in \Delta$. There is therefore $\alpha \in \Delta$ with $\alpha(v(m)) > 0$. Thus ${}^mN(a) = N(a + \alpha(v(m))) \subset N(a)$, if $a \in \Delta_0$ is the affine root with vectorial part α vanishing at x_0. As $N_0^m \subseteq N_0$ and $N_0 = \prod_{a \in \Phi_0^+} N(a)$, thus ${}^mN_0 \subset N_0$. We obtain

$$\theta(m) \cdot 1_{N_0} = \theta(m) \cdot \sum_{n \in N_0/{}^mN_0} 1_{{}^mN_0n} = \sum_{n \in N_0/{}^mN_0} 1_{N_0}^{n^{-1}m} \in \mathfrak{L};$$

conferring to Lemma 1.2(ii) for the second equality given by definition of the P-action. As $|\theta(m)| > 1$, we see that $\mathbf{K} \cdot 1_{N_0} \subseteq \mathbf{o_K}[P] \cdot 1_{N_0} = \mathfrak{L}$.

We let $\chi = \theta\psi$ with θ unramified and ψ an arbitrary dominant algebraic character. In particular, we assume G to split. By [Jantzen, 2003, Proposition II.2.4(b)], we find as an M-representation

$$U_\psi = \bigoplus_{\xi \in \mathrm{wt}(\psi)} U(\xi) \quad \text{with } \{w_0 \cdot \psi, \psi\} \subseteq \mathrm{wt}\, \psi \subseteq \{\xi : w_0 \cdot \psi \leq \xi \leq \psi\}.$$

Here M acts on the one dimensional $\mathbf{K}$-vector space $U(\xi) = \mathbf{K} \cdot u_\xi$ through the algebraic character $\xi : M \to \mathbf{K}^*$, and for two algebraic characters ζ and η on M, we have by definition

$\zeta \leq \eta$ if $\eta - \zeta = \sum_{\alpha \in \Delta} i_\alpha \alpha$ for integers $i_\alpha \geq 0$. Because the universal unitary lattice is by Proposition 0.2 up to commensurability given by any lattice finitely generated as an $\mathbf{o_K}[P]$-module, we may assume

$$\mathfrak{L} = \sum_{\xi \in \mathrm{wt}(\psi)} \mathbf{o_K}[P] \cdot 1_{N_0} \otimes u_\xi \subseteq \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_\psi.$$

We prove by downward induction on $\#\{\xi \in \mathrm{wt}(\psi) : \xi \geq \eta\}$ that for any algebraic character $\eta : M \to \mathbf{K}^*$ in $\mathrm{wt}(\psi)$, we have

$$\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_{[\eta,\psi]} \subseteq \mathfrak{L}$$

with

$$U_{[\eta,\psi]} = \bigoplus_{\xi \in [\eta,\psi]} U(\xi) \quad \text{and} \quad [\eta, \psi] := \{\xi \in \mathrm{wt}(\psi) : \xi \geq \eta\}.$$

Because $U_{[w_0 \cdot \psi, \psi]} = U_\psi$, we can then conclude $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_\psi \subseteq \mathfrak{L}$.

To begin the induction, let η be maximal, i.e. $\eta = \psi$. Then $U(\psi) = \mathbf{K} \cdot u_\psi$ and u_ψ is a highest weight vector of U_ψ, i.e. N acts trivially on it. So we can reason as above in the unramified case and obtain

$$\begin{aligned} \chi(m) \cdot 1_{N_0} \otimes u_\psi &= \sum_{n \in {}^m N_0} \theta(m) \cdot 1_{{}^m N_0 n} \otimes \psi(m) \cdot u_\psi \\ &= \sum_{n \in N / {}^m N_0} 1_{N_0}^{n^{-1}m} \otimes u_\psi^{n^{-1}m} \\ &= \sum_{n \in N / {}^m N_0} n^{-1}m \cdot [1_{N_0} \otimes u_\psi] \in \mathfrak{L}. \end{aligned}$$

As $|\chi(m)| > 1$, we see that $\mathbf{K} \cdot 1_{N_0} \otimes u_\psi \subseteq \mathbf{o_K}[P] \cdot 1_{N_0} \otimes u_\psi \subseteq \mathfrak{L}$.

We now let $\eta \lneqq \psi$ and may assume by the induction hypothesis

$$\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_{]\eta,\psi]} \subseteq \mathfrak{L}$$

with

$$U_{]\eta,\psi]} = \bigoplus_{\xi \in]\eta,\psi]} U(\xi) \quad \text{and} \quad]\eta, \psi] := \{\xi \in \mathrm{wt}(\psi) : \xi \gneqq \eta\}.$$

Then we write

$$\begin{aligned} &\theta\eta(m) \cdot 1_{N_0} \otimes u_\eta \\ &= \sum_{n \in N_0 / {}^m N_0} \theta(m) \cdot 1_{{}^m N_0 n} \otimes \eta(m) \cdot u_\eta \\ &= \sum_{n \in N_0 / {}^m N_0} 1_{N_0}^{n^{-1}m} \otimes [u_\eta^{n^{-1}m} + (u_\eta^m - u_\eta^{n^{-1}m})] \\ &= \sum_{n \in N_0 / {}^m N_0} n^{-1}m \cdot [1_{N_0} \otimes u_\eta] + \sum_{n \in N_0 / {}^m N_0} 1_{N_0}^{n^{-1}m} \otimes (u_\eta^m - u_\eta^{n^{-1}m}). \end{aligned}$$

Then for the left hand summand holds

$$\sum_{n \in N_0/{}^m N_0} n^{-1}m \cdot [1_{N_0} \otimes u_\eta] \in \mathbf{o_K}[P] \cdot 1_{N_0} \otimes u_\eta \subseteq \mathfrak{L}.$$

Regarding the right hand summand, by [Humphreys, 1975, Remark at the beginning of the proof of Proposition 31.2], we have $u_\eta^{n^{-1}m} - u_\eta^m \in U_{]\eta,\psi]}$. Therefore

$$\sum_{n \in N_0/{}^m N_0} 1_{N_0}^{n^{-1}m} \otimes (u_\eta^m - u_\eta^{n^{-1}m}) \subseteq \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_{]\eta,\psi]} \subseteq \mathfrak{L},$$

the last inclusion by the induction hypothesis.

Hence $\theta\eta(m) \cdot 1_{N_0} \otimes u_\eta \in \mathfrak{L}$. We have $\eta = \psi - \sum_{\alpha \in \Delta} i_\alpha \cdot \alpha$ with integers $i_\alpha \geq 0$. By Lemma 2.7, we have $|\alpha(m)| \leq 1$ for any $m \in M^+$. Therefore

$$|\theta\eta(m)| = |\chi(m)| \cdot |-\sum_{\alpha \in \Delta} i_\alpha \cdot \alpha(m)| > 1.$$

As $|\theta\eta(m)| > 1$, we see that $\mathbf{K} \cdot 1_{N_0} \otimes u_\eta \subseteq \mathfrak{L}$. Therefore

$$\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes [U_{]\eta,\psi]} \oplus U(\eta)] = \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes U_{[\eta,\psi]} \subseteq \mathfrak{L},$$

completing the induction step. ■

Corollary 2.11. *For the universal unitary completion of the* $\mathbf{K}[P]$*-module* $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$ *to be nonzero, necessarily* $|\chi(M^+)| \leq 1$ *and in particular* $|\chi(Z)| = 1$.

Proof. This is a reformulation of the preceding Lemma 2.10. ■

2.2 The smooth case

Assumption. Throughout this subsection, we identify, by Lemma 1.2(ii), through the restriction morphism $I(\theta)(N) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$ as $\mathbf{K}[P]$-modules. We recall that, on the right hand side, N acts by right-translation and M by $f^m := \theta(m) f(\cdot^m)$. Then under this identification, ϕ_1 restricts to 1_{N_0}.

In this subsection, we want show that the, a priori, *seminorm* $\|\cdot\|$ induced by the universal unitary lattice in $I(\theta)(N)$ - given by Lemma 2.9 through $\mathfrak{L} := \mathbf{o_K}[P] \cdot 1_{N_0} \subseteq \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$ - is Hausdorff.

Since $\mathfrak{L}$ is by definition the smallest $\mathbf{o_K}$-lattice containing all $1_{N_0}^p$ for $p \in P$, its associated norm $\|\cdot\|$ can be characterized as the pointwise greatest seminorm fulfilling

$$\|1_{{}^p N_0}\| \leq 1/|\theta(p)| \quad \text{for all } \; p \in P.$$

In order to prove $\|\cdot\|$ to be an actual norm and not to vanish, the preceding subsection has shown that we have to impose on our unramified character further conditions:

Assumption. Throughout this subsection, we assume $\theta : M \to \mathbf{K}^*$ to fulfill the condition of Corollary 2.11.

1. It holds $|\theta(M^+)| \leq 1$.

2. In particular $|\theta(Z)| = 1$.

Definition of the valuation on N_Δ

We firstly remind ourselves that by Corollary 2.4 the abelian group $\Lambda = v(M/Z)$ is a finite free $\mathbb{Z}$-module of rank $\#\Delta$.

We recall that the image of the conjugation action of M on the compact open subgroup $N_0 \subseteq N$ by $m \cdot N_0 := {}^m N_0$ only depends on its image $\Lambda = v(M/Z)$ under $M \overset{\text{can.}}{\to} M/Z \overset{v}{\to} \Lambda$. This is seen by Lemma 2.5 together with the isomorphism of affine $\mathbf{F}$-varieties $\prod_{\alpha \in \Phi^+} \mathbf{N}_\alpha \overset{\sim}{\to} \mathbf{N}$, given by the multiplication map (cf. [Borel, 1991, Proposition 21.9(ii)]).

Definition. Let $\Psi = \Phi^+ - \Delta$ and put $N_\Psi = \prod_{\alpha \in \Psi} N_\alpha$. Then we define

$$N_\Delta := N/N_\Psi.$$

Remark 2.12. We have an isomorphism of groups $N_\Delta = \prod_{\alpha \in \Delta} N_\alpha$.

Proof. Because $\prod_{\alpha \in \Phi^+} N_\alpha \overset{\sim}{\to} N$ and $[N_\alpha, N_\beta] \subseteq N_\Psi$ for all distinct $\alpha, \beta \in \Phi^+$ by [Borel, 1991, Remark 21.10(1)], we have an isomorphism of groups $N_\Delta = \prod_{\alpha \in \Delta} N_\alpha$, the right hand side being endowed with componentwise multiplication. ■

We want to define a filtration of N_Δ indexed by $\mathbb{Z}^\Delta$. To prepare this, recall firstly that Λ is a finite free $\mathbb{Z}$-module of rank $\#\Delta$. Moreover its submonoid Λ^+ generates Λ by Remark 2.2.

Definition. We choose a basis $\{\lambda_\alpha\} \subseteq \Lambda^+$ of Λ and fix an identification

$$\begin{aligned} \Lambda &\to \mathbb{Z}^\Delta, \\ \lambda &\mapsto (i_\alpha); \end{aligned}$$

so that λ_α is mapped to $\boldsymbol{e}_\alpha = (0, \ldots, 1, \ldots, 0) \in \mathbb{Z}^\Delta$, the canonical basis vector whose sole nonzero coordinate with entry 1 has index α. We let $m_\alpha \in M$ be any element mapping to λ_α.

This amounts to fixing a choice of lexicographic ordering on Λ. We will use this identification in the following without further mention.

Definition. Let $N_{\Delta,0} := N_0/(N_\Psi \cap N_0)$. We then let M act on the subgroup $N_{\Delta,0} \subseteq N_\Delta$ naturally by ${}^m N_{\Delta,0} = {}^m N_0 / N_\Psi \cap {}^m N_0$.

1. We define the descending filtration $(N_\Delta)_{\boldsymbol{i} \in \mathbb{Z}^\Delta}$ on N_Δ by

   $$N_{\Delta,\boldsymbol{i}} := {}^m N_{\Delta,0}.$$

   for any $m \in M$ with $v(m) = \boldsymbol{i}$.

2. Let $N_\Delta^* = \prod_{\alpha \in \Delta} N_\alpha^* \subseteq N_\Delta$. We define $\boldsymbol{i} : N_\Delta^* \to \mathbb{Z}^\Delta$ by letting $\boldsymbol{i}(n)$ be the unique element which is maximal for the lexicographic ordering under all $\boldsymbol{i} \in \mathbb{Z}^\Delta$ with $n \in N_{\Delta,\boldsymbol{i}}$.

Remark. We want to show that $\boldsymbol{i} : N_\Delta^* \to \mathbb{Z}^\Delta$ is indeed well defined, i.e. given any $n \in {N^*}_\Delta$, we want to affirm the existence of a unique maximal $\boldsymbol{i} \in \mathbb{Z}^\Delta$ such that $n \in N_{\Delta,\boldsymbol{i}}$.

1. We have $N_{\Delta,\boldsymbol{i}} \cap N_{\Delta,\boldsymbol{j}} = N_{\Delta,\boldsymbol{k}}$ where $\boldsymbol{k} = \max\{\boldsymbol{i}, \boldsymbol{j}\}$ is defined as the componentwise maximum, i.e. $\boldsymbol{k} = (k_\alpha)_{\alpha\in\Delta}$ with $k_\alpha = \max\{i_\alpha, j_\alpha\}$. This follows by ${}^m N(a) = N(a + \alpha(v(m)))$ for any $a \in \Delta_0$ and $N_{\Delta,0} = \prod_{\alpha\in\Delta_0} N(a)$.

2. Let $n \in N^*{}_\Delta$. Then we define $\boldsymbol{i} = (i_\alpha)_{\alpha\in\Delta}$ by

$$i_\alpha := \max\{j_\alpha : n \in N_{\Delta,\boldsymbol{j}} \text{ for some } \boldsymbol{j} \in \mathbb{Z}^\Delta\}.$$

By separateness, we have

$$\bigcap_{i\geq 0} {}^{i\cdot \boldsymbol{e}_\alpha} N(b) = \bigcap_{i\geq 0} N(b + i < \beta, \boldsymbol{e}_\alpha >) = \{1\}$$

if and only if $< \beta, \boldsymbol{e}_\alpha >> 0$. We deduce that $i_\alpha = \infty$ if and only if $n_\beta = 1$ for some $\beta \in \Delta$ with $< \beta, \boldsymbol{e}_\alpha >> 0$. Because $n \in N^*_\Delta$ therefore $i_\alpha < \infty$.

3. We want to show that $\boldsymbol{i}$ is the sought for maximal element (and then unique by construction): We surely have $\boldsymbol{i} \geq \boldsymbol{j}$ for all $\boldsymbol{j}$ such that $N_{\boldsymbol{j}} \ni n$. Contrariwise, let $n \in N_{\boldsymbol{j}}$ for some $\boldsymbol{j} \in \mathbb{Z}^\Delta$ with $\boldsymbol{j} \geq \boldsymbol{i}$. Then $\boldsymbol{i} \geq \boldsymbol{j}$ by construction of $\boldsymbol{i}$. Hence $\boldsymbol{i}$ is maximal.

We observe that $< \beta, \sum \lambda_\alpha >> 0$ for all $\beta \in \Delta$: Indeed, as $\{\lambda_\alpha\} \subseteq \Lambda^+$, it would otherwise hold $\Lambda^+ \subseteq \ker \beta$ for some $\beta \in \Delta$, and thus $\dim_{\mathbb{Z}} \Lambda < \#\Delta$. But we have equality.

Definition. Put $r_\alpha := v_{\mathbf{K}}(\theta(\boldsymbol{e}_\alpha))$. We will define a valuation $v : N_\Delta \to \mathbb{R} \cup \{\infty\}$, dependent on θ as follows: Fix $n \in N_\Delta$. Then we put

$$v(n) := \sum_{\alpha\in\Delta} r_\alpha \cdot (i(n)_\alpha + b_\alpha);$$

here $\boldsymbol{b} = (b_\alpha)$ is an element in $\{\boldsymbol{b} \in \{0,1\}^\Delta :< \beta, \boldsymbol{b} >> 0 \quad \text{for all } \beta \in \Delta\}$ (which is nonempty since $< \beta, \mathbf{1} >> 0$ with $\mathbf{1} = (1, \ldots, 1)$ by the above observation) with $\sum r_\alpha \cdot b_\alpha$ minimal. (The shift by $\boldsymbol{b}$ is only for normalization purposes.)

Definition of the norm

We will now define a norm on $\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$. Recall the isomorphism of $\mathbf{K}$-vector spaces $\otimes_{\alpha\in\Phi^+} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N_\alpha, \mathbf{K}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$, induced by the multiplication map $\prod_{\alpha\in\Phi^+} \mathbf{N}_\alpha \xrightarrow{\sim} \mathbf{N}$.

Definition 2.13. Let $n_\alpha \in N_\alpha$. We define the $\mathbf{K}$-linear difference operator $\varDelta\_(\cdot; (n_\alpha)) : \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \circlearrowleft$ for any $f = \otimes f_\alpha \in \otimes_{\alpha\in\Phi^+} \mathcal{C}^{\mathrm{lc}}(N_\alpha, \mathbf{K}) = \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$ by

$$\varDelta f(\cdot; (n_\alpha)) = \bigotimes_{\alpha\in\Delta} \varDelta_\alpha f_\alpha(\cdot; n_\alpha) \otimes \bigotimes_{\alpha\in\Phi^+ - \Delta} f_\alpha;$$

here $\varDelta_\alpha \_(\cdot; n_\alpha) : \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N_\alpha, \mathbf{K}) \circlearrowleft$ being defined by $\varDelta_\alpha f(\cdot; n_\alpha) := f - f(\cdot n_\alpha)$.

Definition. We put

$$\|f\| := \sup_{n\in N, (n^*_\alpha)\in N^*_\Delta} |\varDelta f(n; (n^*_\alpha))| / |(n^*_\alpha)|;$$

here $|\cdot|$ is the multiplicative valuation on N_Δ attached to the additive valuation v defined above (i.e. $|n| := c_{\mathbf{K}}^{v(n)}$ for any $n \in N_\Delta$).

Lemma. *The above defined map* $\|\cdot\| : \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \to \mathbb{R}_{\geq 0}$ *is indeed a norm.*

Proof. Firstly, because $\operatorname{supp} f$ is compact, we find $\|\Delta f\|_{\sup} < C$ to be bounded. Because $\operatorname{supp} f$ is compact and f is locally constant, we find f to be constant on the cosets of a compact open neighborhood of $U \ni 1$ in N, whose image in N_Δ has diameter, say $\delta := \sup\{|n| : n \in U\}$. Therefore $\||\Delta f(n;(n^*_\alpha))|/|(n^*_\alpha)|\|_{\sup} \leq C/\delta$.
It follows readily that $\|\cdot\|$ is a seminorm. Since the support of each $f \in \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$ is by definition compact, $\|f\| = 0$ only if $f = 0$ and so $\|\cdot\|$ is an actual norm. ■

Let $\tilde{\mathfrak{L}} := \{f \in \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) : \|f\| \leq 1\}$ be the lattice attached to our norm $\|\cdot\|$. In order for $\mathfrak{L} \subseteq \tilde{\mathfrak{L}}$ to hold, it suffices to prove the following.

Proposition 2.14. *The norm* $\|\cdot\|$ *fulfills the following two conditions:*

1. *It is invariant under translation by* N.
2. *We have* $\|1_{^m N_0}\| \leq 1/|\theta(m)|$ *for any* $m \in M$.

Proof. Ad 1.: This holds by definition.

Ad 2.: Let $v(m) \in \Lambda$ correspond to $\boldsymbol{l} = (l_\alpha) \in \mathbb{Z}^\Delta$. Because $f := 1_{^m N_0}$ is constant on $^m N_0$-cosets in N, we have $\Delta f(n;(n^*_\alpha)) \neq 0$ if and only if $n^*_\alpha \notin N(a_0 + <\alpha, \boldsymbol{l}>)$ for all $\alpha \in \Delta$. This holds if and only if $<\alpha, \boldsymbol{i}(n^*_\alpha)> << \alpha, \boldsymbol{l}>$ for all $\alpha \in \Delta$. We also have $|\Delta f(n;(n^*_\alpha))| \neq 0$ if and only if $|\Delta f(n;(n^*_\alpha))| = 1$. Together we find:

$$|\Delta f(n;(n^*_\alpha))|/|(n^*_\alpha)| \quad \text{is maximal}$$

if and only if $n = 1$ and

$$<\alpha, \boldsymbol{i}(n^*_\alpha)> << \alpha, \boldsymbol{l}> \quad \text{for all } \alpha \in \Delta \quad \text{and} \quad v((n^*_\alpha)) \text{ is maximal.} \tag{$*$}$$

This maximum of $v((n^*_\alpha))$ for (n^*_α) as in $(*)$ is attained if $\boldsymbol{i}(n) = \boldsymbol{l}^- := \boldsymbol{l} - \boldsymbol{b}$ (see the definition of $\boldsymbol{b}$). Then

$$\begin{aligned} v((n^*_\alpha)) &= \sum_{\alpha \in \Delta} r_\alpha \cdot (l^-_\alpha + b_\alpha) = \sum_{\alpha \in \Delta} r_\alpha \cdot l_\alpha \\ &= \sum_{\alpha \in \Delta} v_{\mathbf{K}}(\theta(m_\alpha)) \cdot l_\alpha = v_{\mathbf{K}}(\theta(\prod_{\alpha \in \Delta} m_\alpha^{l_\alpha})) = v_{\mathbf{K}}(\theta(m)). \end{aligned}$$

We conclude $|\Delta f(n;(n^*_\alpha))|/|(n^*_\alpha)| \leq 1/|\theta(m)|$. ■

Corollary. *We have* $\|1^p_{N_0}\| = 1$ *for all* $p \in P$.

Proof. Let $f = 1_{N_0}$. Because M is a group, we have $\|f^m\| \leq \|f\|$ for all $m \in M$ if and only if $\|f^m\| = \|f\|$ for all $m \in M$: Indeed, given any $m \in M$, we have $\|f\| = \|f^{mm^{-1}}\| \leq \|f^m\|$, hence $\|f\| = \|f^m\|$. Because $P = MN$, we infer $\|f^p\| = \|f\|$ for all $p \in P$.
It rests to show that $\|f\| = 1$. We have already seen above that $\|f\| \leq 1$. On the other hand, choosing $(n^*_\alpha) \in N_\Delta$ with $\boldsymbol{i}((n_\alpha)) = -\boldsymbol{b}$, we have $\Delta f(1;(n^*_\alpha)) = 1$ and $v((n_\alpha)) = 1$, and so

$$\|f\| \geq |\Delta f(1;(n^*_\alpha))|/|(n^*_\alpha)| = 1.$$

■

Corollary 2.15. *The lattice* $\mathfrak{L}$ *is Hausdorff.*

Example: The smooth case of small order image

Generators of the dominant submonoid

Assumption 2.16. In this paragraph, we will assume that

$$\Lambda = \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{A} \,/\, \mathbf{Z}), \mathbb{Z}).$$

This means that the mapping $\mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{A} \,/\, \mathbf{Z}), \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M} \,/\, \mathbf{Z}), \mathbb{Z})$ induced by the inclusion $\mathbf{A} \hookrightarrow \mathbf{M}$ has image Λ. This holds e.g. if $\mathbf{G}$ splits over a field extension unramified over $\mathbf{F}$ (see [Tits, 1979, Remark 1.3] for the statement and [Borel, 1979, Section 9.5] for an argument in the quasi-split case).

Proposition 2.17. *There exists a unique basis* $\{\lambda_\alpha : \alpha \in \Delta\} \subseteq \Lambda$ *which is orthonormal with respect to* $\Delta \subseteq X^*(\mathbf{A} \,/\, \mathbf{Z})$*, i.e.*

$$< \lambda_\alpha, \beta >= \begin{cases} 1, & \text{if } \alpha = \beta, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } \alpha, \beta \in \Delta$$

with $< \, , \, >$ *denoting the natural pairing by evaluation between the* $\mathbb{Z}$*-modules* $X^*(\mathbf{A} \,/\, \mathbf{Z})$ *and* $\mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{A} \,/\, \mathbf{Z}), \mathbb{Z})$.

Proof. We may assume $\mathbf{G} = \mathbf{G} \,/\, \mathbf{Z}$. Then in particular $\mathbf{Z}^\circ = 1$ and, since $\mathbf{G}$ is reductive, it is therefore semi-simple. Moreover, by [Borel, 1991, Section 3.15], we find $\ker \mathrm{Ad} = \mathbf{Z}$ and thus $\mathbf{G} = \mathbf{G} \,/\, \mathbf{Z}$ to be adjoint.
Then by [Borel, 1966, Section 6.5(2)], using the semi-simplicity of $\mathbf{G}$, the root system Φ or equivalently its base Δ spans a finite free $\mathbb{Z}$-lattice $Q \subseteq X^*(\mathbf{A})$ in V^*. Because $\mathbf{G}$ is adjoint, we have $Q = X^*(\mathbf{A})$. I.e. Δ is a basis of the free $\mathbb{Z}$-module $X^*(\mathbf{A})$. Now for each $\alpha \in \Delta \subseteq X^*(\mathbf{A})$, we (must) define $\lambda_\alpha \in \Lambda = \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{A}), \mathbb{Z})$ by the Kronecker-Delta $\lambda_\alpha(\beta) = \delta_\beta$ on the basis $\{\beta : \beta \in \Delta\} \subseteq X_*(\mathbf{A})$. ■

Corollary. *We have* $\Lambda^+ = \oplus \mathbb{N} \lambda_\alpha$.

Proof. By orthogonality of $\{\lambda_\alpha\}$ with respect to Δ. ■

By its definition, $v : M \to \mathrm{Hom}_{\mathbb{Z}}(X^*(\mathbf{M}), \mathbb{Z})$ has to be trivial on any bounded subgroup of M and therefore the kernel of v is the maximal compact open subgroup M_0 of M.

Definition 2.18. We define $m_\alpha \in M$ as a representative of the coset corresponding to λ_α under the isomorphism of groups $M/M_0Z \to \Lambda$.

Corollary 2.19. *We have* $M^+/M_0Z = \oplus \mathbb{N} m_\alpha$.

Proof. By the above Corollary and the isomorphism of groups $M^+/M_0Z \cong \Lambda^+$. ■

Action on the affine root groups

Corollary 2.20. *We have $M^+/M_0Z = \oplus \mathbb{N}m_\alpha$ and these generators act on the root factors by*

$$^{m_\alpha}N(b) = N(b+\nu_\alpha), \quad \text{if } b = \beta + i \in \Phi_{\text{aff}} \text{ with } \beta = \sum_{\gamma\in\Delta} \nu_\gamma \cdot \gamma \in \Phi^{\text{red}}.$$

Proof. The first statement was already given in Corollary 2.19.
For the second, recall $<\beta, \lambda_\alpha> = \delta_{\alpha,\beta}$ for all $\beta \in \Delta$. Thence $\beta(v(m_\alpha)) = \beta(\lambda_\alpha) = <\beta, \lambda_\alpha> = \delta_{\alpha,\beta}$. We conclude by Lemma 2.5. ■

Definition of the norm

Recall the isomorphism of $\mathbf{K}$-vector spaces $\otimes_{\alpha\in\Phi^+}\mathcal{C}^{\text{lc}}_{\text{cpt}}(N_\alpha, \mathbf{K}) \tilde{\to} \mathcal{C}^{\text{lc}}_{\text{cpt}}(N, \mathbf{K})$ induced by the multiplication map $\prod_{\alpha\in\Phi^+} \mathbf{N}_\alpha \tilde{\to} \mathbf{N}$.

The existence of such an orthogonal basis of the submonoid Λ^+ allows us to define the filtration and valuation on N_Δ more conveniently:

Definition 2.21. Let $\alpha \in \Delta$. We define a valuation on $i_\alpha : N^*_\alpha \to \mathbb{Z}$ by

$$i_\alpha(n_\alpha) := \max\{i \in \mathbb{Z} : n_\alpha \in N(a+i)\}.$$

To define our norm $\|\cdot\|$, we employ as before the isomorphism of $\mathbf{K}$-vector spaces

$$\otimes_{a\in\Phi^+}\mathcal{C}^{\text{lc}}(N_\alpha, \mathbf{K}) \tilde{\to} \mathcal{C}^{\text{lc}}_{\text{cpt}}(N, \mathbf{K}).$$

Then the ($\mathbf{K}$-linear) operator $\varDelta\_(\cdot; (n_\alpha)) : \mathcal{C}^{\text{lc}}_{\text{cpt}}(N, \mathbf{K}) \circlearrowleft$ is defined as before in Definition 2.13.

Definition 2.22. Let $\alpha \in \Delta$ and put $r_\alpha := v_{\mathbf{K}}(\theta(m_\alpha))$. We define

$$\|f\| := C \cdot \max_{n\in N, (n^*_\alpha)\in\prod N^*_\alpha} |\varDelta f(n; (n^*_\alpha))| / \prod_{\alpha\in\Delta} |n^*_\alpha|^{r_\alpha}_\alpha;$$

here $|\cdot|_\alpha$ being the multiplicative valuation on N^*_α attached to the additive valuation i_α defined above (i.e. $|n^*_\alpha| := c_{\mathbf{K}}^{i_\alpha(n^*_\alpha)}$ for any $n^*_\alpha \in N^*_\alpha$), and $C = p^{\#\Delta}$ is a normalization constant.

Example: The smooth split case

Assumption. In this paragraph, we will assume G to split.

Definition of the norm

The assumption that G splits allows for further simplifications in the definition of the norm attached to the universal unitary lattice $\mathfrak{L} \subseteq \mathcal{C}^{\text{lc}}_{\text{cpt}}(N, \mathbf{K}) \otimes U$.

We firstly observe that, since $\mathbf{M} = \mathbf{A}$ is a torus, we find $v : A \twoheadrightarrow X_*(\mathbf{A})$ and we are in the case of small order image, as defined in Assumption 2.16. Secondly, because G splits, we have by [Tits, 1979, Section 1.1] an isomorphism of groups $\mathbf{F} \cong N_\alpha$ which identifies $\varphi_\alpha : N_\alpha^* \to \mathbb{R}$ with $v_{\mathbf{F}} : \mathbf{F}^* \to \mathbb{Z}$. We may assume $N(a_0) \xrightarrow{\sim} \mathbf{o_F}$ under this isomorphism. We summarize:

Lemma 2.23. *We have an isomorphism of abelian groups $N_\Delta = \mathbf{F}^\Delta$ and under this identification $i_\alpha(n_\alpha^*) = v_{\mathbf{F}}(n_\alpha^*)$ for all $n_\alpha^* \in N_\alpha^*$.*

Proof. By Remark 2.12, the multiplication map $\prod_{\alpha\in\Delta} N_\alpha \to N$ yields a group isomorphism $\prod_{\alpha\in\Delta} N_\alpha \xrightarrow{\sim} N_\Delta$, the left hand side endowed with the componentwise multiplication. By our assumption on the choice of group isomorphism $\mathbf{F} \cong N_\alpha$ for all $\alpha \in \Delta$, we have $n_\alpha^* \in N(a+i) - N(a+i+1)$ if and only if $v_{\mathbf{F}}(n_\alpha^*) = i$, and thus $i_\alpha(n_\alpha^*) = v_F(n_\alpha^*)$, looking at Definition 2.21. ■

Recall that the multiplication map $\prod_{\alpha\in\Phi^+} \mathbf{N}_\alpha \xrightarrow{\sim} \mathbf{N}$ induces an isomorphism of $\mathbf{K}$-vector spaces $\otimes_{\alpha\in\Phi^+}\mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N_\alpha, \mathbf{K}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K})$. Then Definition 2.22 reduces as follows:

Definition. Identify $N_\Delta = \mathbf{F}^\Delta$ and recall the difference operator $\mathit{\Delta}\_(\cdot; (n_\alpha)) : \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \circlearrowleft$ given in Definition 2.13. Let $\alpha \in \Delta$ and put $r_\alpha := v_{\mathbf{K}}(\theta(m_\alpha))$. Then

$$\|f\| := C \cdot \max_{n\in N, (n_\alpha^*)\in \mathbf{F}^{*\Delta}} \frac{|\mathit{\Delta} f(n; (n_\alpha^*))|_{\mathbf{K}}}{\prod_{\alpha\in\Delta} |n_\alpha^*|_{\mathbf{F}}^{r_\alpha}}$$

with the normalization constant $C := p^{\#\Delta}$.

Remark 2.24. Define the tuple $\boldsymbol{r} = (r_\alpha)_{\alpha\in\Phi^+}$ by

$$r_\alpha = \begin{cases} v_{\mathbf{K}}(\theta(m_\alpha)), & \text{if } \alpha \in \Delta, \\ 0, & \text{otherwise.} \end{cases}$$

Identifying $\mathbf{F}^{\Phi^+} \cong \prod_{\alpha\in\Phi^+} N_\alpha \xrightarrow{\sim} N$, this norm can hence be seen as the one for $\boldsymbol{r}$-times differentiable functions $f : \mathbf{F}^{\Phi^+} \to \mathbf{K}$ with vanishing derivative.

Example. Let $\#\Delta = 1$ and $N \cong \mathbf{F}$. Put $r := v(\theta(m_\alpha))$ if $M^+/ZM_0 = \mathbb{N} \cdot m_\alpha$. Then we find

$$\|f\| = \|f\|_{\mathcal{C}_0^r} := \||\mathit{\Delta} f|\|_{\sup} \quad \text{with } |\mathit{\Delta} f(x,y)| := \frac{|f(x) - f(y)|}{|x-y|^r}.$$

Interlude: Locally polynomial differentiable functions

Notation. For $a \in \mathbf{F}^d$ and $\boldsymbol{\delta} = (\delta_1, \dots, \delta_d) \in \mathbb{R}^d_{>0}$, we define the polydisc around a of radius $\boldsymbol{\delta}$ by $\mathrm{B}_{\leq\boldsymbol{\delta}}(a) = \{x \in \mathbf{F}^d : |x_1 - x_1| \leq \delta_1, \dots, |x_d - a_d| \leq \delta_d\}$ and the pointed polydisc by $\mathrm{B}^\bullet_{\leq\boldsymbol{\delta}}(a) = \mathrm{B}_{\leq\boldsymbol{\delta}}(a) - \{a\}$.

Definition. We denote by $\mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K})$ all locally polynomial functions $f : \mathbf{F}^d \to \mathbf{K}$ of compact support. Here a function $f : \mathbf{F}^d \to \mathbf{K}$ is called **locally polynomial** if for any $x \in \mathbf{F}^d$, there exists an open neighborhood $U \ni x$ in $\mathbf{F}^d$ such that $f_{|U}$ is given by the restriction of a polynomial function.
A function $f : \mathbf{F}^d \to \mathbf{K}$ is called **δ-polynomial** if each neighborhood $U \ni x$ such that $f_{|U}$ is polynomial can be chosen as $U = \mathrm{B}_{\leq \delta}(x)$.
We say that $f \in \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K})$ has **degree at most $\boldsymbol{n} \in \mathbb{N}^d$** if all $f_{|U}$ are given by the restriction of a polynomial function $p_U = \sum_{\boldsymbol{i} \leq \boldsymbol{n}} a_{\boldsymbol{i}} *^{\boldsymbol{i}}$ (where we write $\boldsymbol{i} \leq \boldsymbol{n}$ if $i_1 \leq n_1, \ldots, i_d \leq n_d$).

Lemma 2.25. *Fix $n \in \mathbb{N}$. There exists a positive constant $c \leq 1$ such that for any compact open $U = \pi^k \cdot \mathbf{o}_F \subseteq \mathbf{F}$ and any polynomial function $f = \sum_{i=0,\ldots,n} a_i *^i$ of degree at most n holds*

$$c \cdot \max_{i=0,\ldots,n} |a_i| \|*^i\|_U \leq \| \sum_{i=0,\ldots,n} a_i *^i \|_U \leq \max_{i=0,\ldots,n} |a_i| \|*^i\|_U.$$

Proof. Because the $\mathbf{K}$-vector space of polynomial functions $f : \mathbf{o}_{\mathbf{K}} \to \mathbf{K}$ of degree at most n is finite dimensional and $\mathbf{K}$ is complete, we find $\|\cdot\|_{\mathbf{o}_{\mathbf{F}}}$ to be equivalent to the norm $\|\cdot\|$ given by the orthonormal basis $*^i$, i.e. defined by $\|f\| = \max_{i=0,\ldots,n} |a_i|$ for $f = \sum_{i=0,\ldots,n} a_i *^i$. (See [Schneider, 2002, Proposition 4.13].) In particular $c \cdot \max_{i=0,\ldots,n} |a_i| \leq \|f\|_{\mathbf{o}_{\mathbf{F}}}$ for some positive constant $c \leq 1$.

If now more generally $U = \pi^k \cdot \mathbf{o}_{\mathbf{F}}$, then we observe

$$\begin{aligned} \|f\|_U &= \| \sum_{i=0,\ldots,n} a_i \pi^{ki} *^i \|_{\mathbf{o}_{\mathbf{F}}} \\ &\geq c \cdot \max_{i=0,\ldots,n} |a_i| |\pi|^{ki} \\ &= c \cdot \max_{i=0,\ldots,n} |a_i| \|*^i\|_{\pi^k \cdot \mathbf{o}_{\mathbf{F}}} = c \cdot \max_{i=0,\ldots,n} |a_i| \|*^i\|_U. \end{aligned}$$

■

Remark. By [Chabert and Cahen, 2006, Proposition 1.3], we find more exactly $c = |\pi|^{w(n)}$ with $w(n) := \sum_{i \geq 1} \lfloor n / q^i_{\mathbf{F}} \rfloor$.

Lemma 2.26. *There exists a constant $C \geq 1$ such that for every polynomial function $p : \mathbf{F} \to \mathbf{K}$ of degree at most n holds*

$$\left\| \frac{|p(x+h) - p(x)|}{|h|^\rho} \right\|_{\mathrm{B}_{\leq \delta}(0) \times \mathrm{B}^{\bullet}_{\leq \delta}(0)} \leq C \cdot 1/\delta^\rho \cdot \|p\|_{\mathrm{B}_{\leq \delta}(0)}.$$

Proof. Write $p = \sum_{i=0,\ldots,n} a_i *^i$. We have

$$(x+h)^i - x^i = \sum_{j=0,\ldots,i-1} \binom{i}{j} x^j h^{i-j}$$

and therefore

$$|p(x+h) - p(x)| \leq \max_{i=1,\ldots,n} |a_i| \cdot (\max_{j=0,\ldots,i-1} |x^j| |h^{i-j}|).$$

We obtain

$$\Big\|\frac{|p(x+h)-p(x)|}{|h|^\rho}\Big\|_{\mathrm{B}_{\le\delta}(0)\times\mathrm{B}^\bullet_{\le\delta}(0)} \le \max_{i=1,\dots,n}|a_i|\delta^i/\delta^\rho$$
$$= 1/\delta^\rho\cdot\max_{i=1,\dots,n}|a_i|\|*^i\|_{\mathrm{B}_{\le\delta}(0)}.$$

By the preceding Lemma 2.25, there exist a constant $C\ge 1$ such that

$$\max_{i=1,\dots,n}|a_i|\|*^i\|_{\mathrm{B}_{\le\delta}(0)} \le \max_{i=0,\dots,n}|a_i|\|*^i\|_{\mathrm{B}_{\le\delta}(0)}$$
$$\le C\cdot\|\sum_{i=0,\dots,n}a_i*^i\|_{\mathrm{B}_{\le\delta}(0)}.$$

We conclude

$$\Big\|\frac{|p(x+h)-p(x)|}{|h|^\rho}\Big\|_{\mathrm{B}_{\le\delta}(0)\times\mathrm{B}^\bullet_{\le\delta}(0)} \le C\cdot 1/\delta^\rho\cdot\|\sum_{i=0,\dots,n}a_i*^i\|_{\mathrm{B}_{\le\delta}(0)}$$
$$=C\cdot 1/\delta^\rho\cdot\|p\|_{\mathrm{B}_{\le\delta}(0)}.$$

■

Corollary 2.27. *There exists a constant $C\ge 1$ such that for any δ-polynomial function $f:\mathbf{F}\to\mathbf{K}$ of degree at most n of compact support holds*

$$\Big\|\frac{|f(x+h)-f(x)|}{|h|^\rho}\Big\|_{\mathbf{F}\times\mathbf{F}^*}\le C\cdot 1/\delta^\rho\cdot\|f\|_{\sup}.$$

Proof. We distinguish two cases: Firstly fix $|h|>\delta$. Then we find

$$\Big\|\frac{|f(\cdot+h)-f|}{|h|^\rho}\Big\|_{\mathbf{F}}<1/\delta^\rho\cdot\|f\|_{\sup}.$$

Now let $h\in\mathbf{F}^*$ with $|h|\le\delta$. We can write $f=\sum_i 1_{\mathrm{B}_{\le\delta}(x_i)}p_i$ with polynomial functions p_i. Since $|h|\le\delta$, we thus find

$$\Big\|\frac{|f(x+h)-f(x)|}{|h|^\rho}\Big\|_{\sup}=\Big\|\frac{|\sum_i 1_{\mathrm{B}_{\le\delta}(x_i)}[p_i(x+h)-p_i(x)]|}{|h|^\rho}\Big\|_{\sup}$$
$$\le\max_i\Big\|\frac{|1_{\mathrm{B}_{\le\delta}(x_i)}[p_i(x+h)-p_i(x)]|}{|h^\rho|}\Big\|_{\sup}.$$

We also have

$$\|f\|_{\sup}=\|\sum_i 1_{\mathrm{B}_{\le\delta}(x_i)}p_i\|_{\sup}=\max_i\|1_{\mathrm{B}_{\le\delta}(x_i)}p_i\|_{\sup}.$$

It hence suffices to prove

$$\Big\|\frac{|p_i(x+h)-p_i(x)|}{|h^\rho|}\Big\|_{\mathrm{B}_{\le\delta}(x_i)\times\mathrm{B}^\bullet_{\le\delta}(0)}\le C\cdot 1/\delta^\rho\cdot\|p_i\|_{\mathrm{B}_{\le\delta}(x_i)}.$$

Let $q = p_i(\cdot + x_i)$. By the preceding Lemma 2.26, we obtain

$$\begin{aligned}\left\|\frac{|p_i(x+h) - p_i(x)|}{|h^\rho|}\right\|_{\mathrm{B}_{\leq\delta}(x_i)\times\mathrm{B}^\bullet_{\leq\delta}(0)} &= \left\|\frac{|q(x+h) - q(x)|}{|h|^\rho}\right\|_{\mathrm{B}_{\leq\delta}(0)\times\mathrm{B}^\bullet_{\leq\delta}(0)} \\ &\leq C \cdot 1/\delta^\rho \cdot \|q\|_{\mathrm{B}_{\leq\delta}(0)} \\ &= C \cdot 1/\delta^\rho \cdot \|p_i\|_{\mathrm{B}_{\leq\delta}(x_i)}.\end{aligned}$$

■

Definition. Let $i \geq 0$ and $(h_1, \dots, h_i) \in \mathbf{F}^i$. Then we define the $\mathbf{K}$-linear **iterated difference quotient** $\Delta^i\_(\cdot; h_1, \dots, h_i) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \circlearrowleft$ iteratively by $\Delta^0 f = f$ and

$$\Delta^{i+1} f(\cdot; h_1, \dots, h_i, h_{i+1}) := \Delta^i f(\cdot + h_{i+1}; h_1, \dots, h_i) - \Delta^i f(\cdot; h_1, \dots, h_i).$$

Notation. Given a real number $r \geq 0$, we split it into $r = \nu + \rho$ with an integral part $\nu := \lfloor r \rfloor \in \mathbb{N}$ and a fractional part $\rho := r - \nu \in [0, 1[$.

Definition. Let $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$.

1. Let $\boldsymbol{h} = ({}^1\boldsymbol{h}; \dots; {}^d\boldsymbol{h}) \in \mathbf{F}^{\nu_1} \times \dots \times \mathbf{F}^{\nu_d}$. We define a $\mathbf{K}$-linear **iterated partial difference operator** $\Delta^{\boldsymbol{\nu}}\_(\cdot; \boldsymbol{h}) : \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K}) \circlearrowleft$ as follows: We have an isomorphism of $\mathbf{K}$-vector spaces

$$\mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \otimes \dots \otimes \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K}).$$

   Then we define

$$\Delta^{\boldsymbol{\nu}}\_(\cdot; \boldsymbol{h}) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \otimes \dots \otimes \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \circlearrowleft$$

   by

$$\Delta^{\boldsymbol{\nu}}\_(\cdot; \boldsymbol{h}) = \Delta^{\nu_1}\_(\cdot; {}^1\boldsymbol{h}) \otimes \dots \otimes \Delta^{\nu_d}\_(\cdot; {}^d\boldsymbol{h}).$$

2. We put

$$\|f\|_{\mathcal{C}^{\boldsymbol{r}}} := \sup_{\substack{x \in \mathbf{F}^d, \\ \boldsymbol{h} \in \mathbf{F}^{*\nu_1+1} \times \dots \times \mathbf{F}^{*\nu_d+1}}} \frac{|\Delta^{\boldsymbol{\nu}+\mathbf{1}} f(x; \boldsymbol{h})|}{\prod_{k=1,\dots,d}(|{}^k h_1| \cdots |{}^k h_{\nu_k}| \cdot |{}^k h_{\nu_k+1}|^{\rho_k})};$$

   here and in the following $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}^d$.

Notation. For a tuple $\boldsymbol{n} \in \mathbb{N}^d$, we define $|\boldsymbol{n}| = n_1 + \dots + n_d \in \mathbb{N}$.

Remark 2.28. We give a direct definition of $\Delta^{\boldsymbol{\nu}}\_(\cdot; \boldsymbol{h}) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K})$ by recursion over $|\boldsymbol{\nu}|$, suited for the proofs by induction on $|\boldsymbol{\nu}|$ to come.

1. We set $\Delta^{\mathbf{0}} = \mathrm{id}_{\mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K})}$. Let $\boldsymbol{\nu}^+ \in \mathbb{N}^d$ with $|\boldsymbol{\nu}^+| \geq 1$, say $\boldsymbol{\nu}^+ = \boldsymbol{\nu} + \boldsymbol{e}_k$ and let $\boldsymbol{h}^+ \in \prod_{k=1,\dots,d} \mathbf{F}^{*\nu^+_k}$. Put

$$\boldsymbol{h} = ({}^l\boldsymbol{h})_{l=1,\dots,d} \quad \text{with} \quad {}^l\boldsymbol{h} = \begin{cases} ({}^l h^+_1, \dots, {}^l h^+_{\nu_l}), & \text{if } l = k, \\ {}^l\boldsymbol{h}^+, & \text{otherwise.} \end{cases}$$

Then

$$\Delta^{\boldsymbol{\nu}^+} f(\cdot, \boldsymbol{h}^+) := \Delta^{\boldsymbol{\nu}} f(\cdot + {}^k h_{\nu_k+1} \cdot \boldsymbol{e}_k, \boldsymbol{h}) - \Delta^{\boldsymbol{\nu}} f(\cdot, \boldsymbol{h}).$$

We notice that $\Delta^{\boldsymbol{\nu}} f(\cdot, \boldsymbol{h}) / \prod_{k=1,\dots,d} {}^k h_1 \cdots {}^k h_{\nu_k} \in \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K})$ can likewise be given by the iterated difference quotient operator $\_^{]\boldsymbol{\nu}[}(\cdot, \boldsymbol{h}) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K}) \circlearrowleft$ defined recursively over $|\boldsymbol{\nu}|$ by $f^{]\mathbf{0}[} = f$, and for $\boldsymbol{\nu}^+ \in \mathbb{N}^d$ with $|\boldsymbol{\nu}^+| = |\boldsymbol{\nu}| + 1$, say $\boldsymbol{\nu}^+ = \boldsymbol{\nu} + \boldsymbol{e}_k$, we put

$$f^{]\boldsymbol{\nu}^+[}(\cdot, \boldsymbol{h}^+) := \frac{f^{]\boldsymbol{\nu}[}(\cdot + {}^k h_{\nu_k+1} \cdot \boldsymbol{e}_k, \boldsymbol{h}) - f^{]\boldsymbol{\nu}[}(\cdot, \boldsymbol{h})}{{}^k h_{\nu_k+1}}.$$

2. Let $\boldsymbol{h} \in \mathbf{F}^{*\nu_1+1} \times \cdots \times \mathbf{F}^{*\nu_d+1}$. Then

$$\frac{|\Delta^{\boldsymbol{\nu}+\mathbf{1}} f(x; \boldsymbol{h})|}{\prod_{k=1,\dots,d}(|{}^k h_1| \cdots |{}^k h_{\nu_k}| \cdot |{}^k h_{\nu_k+1}|^{\rho_k})} = \frac{|\Delta^{\mathbf{1}} F(x, ({}^1 h_{\nu_1+1}, \dots, {}^d h_{\nu_d+1}))|}{|{}^1 h_{\nu_1+1}|^{\rho_1} \cdots |{}^d h_{\nu_d+1}|^{\rho_d}}$$

with $F = f^{]\boldsymbol{\nu}[}(\cdot, \check{\boldsymbol{h}})$ and $\check{\boldsymbol{h}} = ({}^1 h_1, \dots, {}^1 h_{\nu_1}; \dots; {}^d h_d, \dots, {}^d h_{\nu_d})$.

Remark. We firstly note that this remark only possibly affects the description of the completions of the normed spaces constructed here, not to be dealt with at this point.
We opted here for the $\mathcal{C}^{\boldsymbol{r}}$-function norms as defined in Section 2 for convenience to generalize those considered in [Barsky, 1973] with domain $\mathbb{Z}_p$ and $\boldsymbol{r} = \nu \in \mathbb{N}$ instead of the one given in [Schikhof, 1984]. These norms induce the same topologies on the locally polynomial function spaces considered here, and their completions of ν-times differentiable functions coincide on the p-adic integers, as seen by their Mahler expansions through [Barsky, 1973, Section II] and [Schikhof, 1984, Section 54]. In general however, when we look for additive differentiable (even in the most naive Archimedean sense) functions on the valuation ring $\mathbf{o}$ of a local field $\mathbf{F}$ with vanishing derivative $f' = 0$, we observe the following: In characteristic 0, due to the density of $\mathbb{Z} \subseteq \mathbb{Z}_p$, any such function has by continuity to be $\mathbb{Z}_p$-linear and thence by $f' = 0$ to be zero, whereas in characteristic $p > 0$, an additive map is only $\mathbf{F}_p$-linear and e.g. automatically differentiable with vanishing derivative if fulfilling $|f(x)| \leq |x|^{1+\varepsilon}$ with $\varepsilon > 0$. Since by additivity $\Delta^2 f = 0$, such functions will be twice differentiable in the sense of [Barsky, 1973]. But there are noticeable examples in positive characteristic not allowing for a Taylor polynomial of second order (see [Glöckner, 2007, Theorem 3.7]) and hence not being twice differentiable in the sense of Schikhof by [Schikhof, 1984, Proposition 28.4]. Thence in general the notion of $\mathcal{C}^{\nu}$-function by Schikhof is stricter than the one given by Barsky.

Notation. For the remainder of this interlude, we fix $n \in \mathbb{N}$ and denote by $C \geq 1$ the corresponding constant appearing in the formulation of Corollary 2.27.

Lemma 2.29. *For any $\boldsymbol{\delta}$-polynomial function $f : \mathbf{F}^d \to \mathbf{K}$ of compact support of degree at most $\boldsymbol{n} = (n, \dots, n)$ holds*

$$\|f^{]\boldsymbol{\nu}[}\|_{\sup} \leq C^{|\boldsymbol{\nu}|} / \delta_1^{\nu_1} \cdots \delta_d^{\nu_d} \cdot \|f\|_{\sup}.$$

Proof. This is proven by induction on $|\boldsymbol{\nu}|$. In case $|\boldsymbol{\nu}| = 0$, there is nothing to prove. Let $|\boldsymbol{\nu}^+| \geq 1$, so that we can write $\boldsymbol{\nu}^+ = \boldsymbol{\nu} + \boldsymbol{e}_k$ for some coordinate $k \in \{1, \dots, d\}$. For notational convenience, assume $k = 1$. Let $x \in \mathbf{F}^d$ and $\boldsymbol{h}^+ \in \prod_{k=1,\dots,d} \mathbf{F}^{*\nu_k}$. Put

$$\boldsymbol{h} = ({}^l\boldsymbol{h})_{l=1,\dots,d} \quad \text{with} \quad {}^l\boldsymbol{h} = \begin{cases} ({}^lh_1^+, \dots, {}^lh_{\nu_l}^+), & \text{if } l = 1, \\ {}^l\boldsymbol{h}^+, & \text{otherwise.} \end{cases}$$

Then

$$f^{]\boldsymbol{\nu}^+[}(x, \boldsymbol{h}^+) = \frac{F(x_1) - F(x_1 + {}^kh_{\nu_1+1})}{{}^kh_{\nu_1+1}} \quad \text{with} \quad F(x_1) := f^{]\boldsymbol{\nu}[}(x, \boldsymbol{h}).$$

We fix any $(\cdot, x_2, \dots, x_d) \in \mathbf{F}^{d-1}$ and $\boldsymbol{h} \in \prod_{k=1,\dots,d} \mathbf{F}^{*\nu_k}$. Then the above defined function $F : \mathbf{F} \to \mathbf{K}$ is given by $F := f^{]\boldsymbol{\nu}[}((\cdot, x_2, \dots, x_d), \boldsymbol{h})$. It is a δ_1-polynomial function. By Corollary 2.27, it thus holds

$$\left\| \frac{F(x) - F(x+h)}{h} \right\|_{\mathbf{F}\times\mathbf{F}^*} = \left\| \frac{|F(x) - F(x+h)|}{|h|} \right\|_{\mathbf{F}\times\mathbf{F}^*} \leq C/\delta_k \cdot \|F\|_{\sup}. \qquad (*)$$

Because the above inequality $(*)$ held for any choice of $(\cdot, x_2, \dots, x_d)$ and $\boldsymbol{h} \in \prod_{k=1,\dots,d} \mathbf{F}^{*\nu_k}$, we find

$$\|f^{]\boldsymbol{\nu}^+[}\| \leq C/\delta_1 \cdot \|f^{]\boldsymbol{\nu}[}\| \leq C^{|\nu^+|}/\delta_1^{\nu_1^+} \cdots \delta_d^{\nu_d^+},$$

the last inequality by the induction hypothesis. ■

Lemma 2.30. *Let $f : \mathbf{F}^d \to \mathbf{K}$ be $\boldsymbol{\delta}$-polynomial of compact support of degree at most $\boldsymbol{n} = (n, \dots, n)$. Then*

$$\|f\|_{\mathcal{C}^{\boldsymbol{\rho}}} := \sup_{\substack{x \in \mathbf{F}^d, \\ \boldsymbol{h} \in \mathbf{F}^* \times \cdots \times \mathbf{F}^*}} \frac{|\Delta^{\mathbf{1}} f(x; \boldsymbol{h})|}{|h_1|^{\rho_1} \cdots |h_d|^{\rho_d}} \leq C^d / \delta_1^{\rho_1} \cdots \delta_d^{\rho_d} \cdot \|f\|_{\sup}.$$

Proof. For $\rho \in v(\mathbf{K}^*)$, let $*^\rho : \mathbf{F} \to \mathbf{K}$ be given by $x^\rho = a^{v_{\mathbf{F}}(x)}$ for any $a \in \mathbf{K}$ with $v(a) = \rho$. Then we define $\_^{]\boldsymbol{\rho}[}(\cdot, \boldsymbol{h}) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K}) \circlearrowleft$ for $\boldsymbol{h} \in \mathbf{F}^d$ by

$$f^{]\boldsymbol{\rho}[}(\cdot, \boldsymbol{h}) = \frac{\Delta^{\mathbf{1}} f(\cdot, \boldsymbol{h})}{h_1^{\rho_1} \cdots h_d^{\rho_d}},$$

so that

$$\frac{|\Delta^{\mathbf{1}} f(x; \boldsymbol{h})|}{|h_1|^{\rho_1} \cdots |h_d|^{\rho_d}} = |f^{]\boldsymbol{\rho}[}(x, \boldsymbol{h})|.$$

For $I \subseteq \{1, \dots, d\}$ and $\boldsymbol{h} \in \mathbf{F}^{*I}$, we define $\__I^{]\boldsymbol{\rho}[}(\cdot; \boldsymbol{h}) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K}) \circlearrowleft$ on $\mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^d, \mathbf{K}) = \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \otimes \cdots \otimes \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K})$ by

$$\__I^{]\boldsymbol{\rho}[}(\cdot, \boldsymbol{h}) = \bigotimes_{k \in I} \_^{]\rho_k[}(\cdot, h_k) \otimes \bigotimes_{k \in \{1,\dots,d\} - I} \mathrm{id}_{\mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K})}$$

with $\_^{]\rho[}(\cdot, h) : \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}, \mathbf{K}) \circlearrowleft$ defined by

$$f^{]\rho[}(\cdot, h) := \frac{f(\cdot + h) - f}{h^\rho}.$$

Then in particular

$$f^{]\boldsymbol{\rho}[}(x, \boldsymbol{h}) = F^{]\rho_1[}(x_1, h_1) \quad \text{with } F := f^{]\boldsymbol{\rho}[}_{\{2,\ldots,d\}}((\cdot, x_2, \ldots, x_d), (h_2, \ldots, h_d)).$$

By induction on $\#I$ - the starting case $\#I = 0$ holding true by definition - we may assume $\|F\|_{\mathbf{F}} \leq C^{d-1}/\delta_2^{\rho_2} \cdots \delta_d^{\rho_d} \|f\|_{\sup}$. Then F is a δ_1-polynomial function in x_1 and so together with Corollary 2.27, it holds

$$\|F^{]\rho_1[}\|_{\mathbf{F}\times\mathbf{F}^*} \leq C/\delta_1^{\rho_1} \cdot \|F\|_{\mathbf{F}} \leq C^d/\delta_1^{\rho_1} \cdots \delta_d^{\rho_d} \cdot \|f\|_{\sup}.$$

Because this holds for any choice of $(\cdot, x_2, \ldots, x_d)$ and $(h_2, \ldots, h_d)$ in the definition of F, we conclude

$$\|f^{]\boldsymbol{\rho}[}(x, \boldsymbol{h})\|_{\mathbf{F}^d\times\mathbf{F}^{*d}} \leq C^d/\delta_1^{\rho_1} \cdots \delta_d^{\rho_d} \cdot \|f\|_{\sup}.$$

■

Proposition 2.31. *Let $f : \mathbf{F}^d \to \mathbf{K}$ be a $\boldsymbol{\delta}$-polynomial function of compact support of degree at most $\boldsymbol{n} = (n, \ldots, n)$. Then*

$$\|f\|_{\mathcal{C}^r} \leq C^{|\boldsymbol{\nu}+\mathbf{1}|}/\delta_1^{r_1} \cdots \delta_d^{r_d} \cdot \|f\|_{\sup}.$$

Proof. By definition, we have

$$\|f\|_{\mathcal{C}^r} = \sup_{\substack{x\in\mathbf{F}^d,\\ \boldsymbol{h}\in\mathbf{F}^{*\nu_1+1}\times\cdots\times\mathbf{F}^{*\nu_d+1}}} \frac{|\Delta^{\boldsymbol{\nu}+\mathbf{1}} f(x; \boldsymbol{h})|}{\prod_{k=1,\ldots,d}(|{}^k h_1| \cdots |{}^k h_{\nu_k}| \cdot |{}^k h_{\nu_k+1}|^{\rho_k})}.$$

Let $\boldsymbol{h} \in \mathbf{F}^{*\nu_1+1} \times \cdots \times \mathbf{F}^{*\nu_d+1}$. By Remark 2.28, we have

$$\frac{|\Delta^{\boldsymbol{\nu}+\mathbf{1}} f(x; \boldsymbol{h})|}{\prod_{k=1,\ldots,d}(|{}^k h_1| \cdots |{}^k h_{\nu_k}| \cdot |{}^k h_{\nu_k+1}|^{\rho_k})} = \frac{|\Delta^{\mathbf{1}} F_{\breve{\boldsymbol{h}}}(x, ({}^1 h_{\nu_1+1}, \ldots, {}^d h_{\nu_d+1}))|}{|{}^1 h_{\nu_1+1}|^{\rho_1} \cdots |{}^d h_{\nu_d+1}|^{\rho_d}}$$

with $F_{\breve{\boldsymbol{h}}} = f^{]\boldsymbol{\nu}[}(\cdot, \breve{\boldsymbol{h}})$ and $\breve{\boldsymbol{h}} = ({}^1 h_1, \ldots, {}^1 h_{\nu_1}; \ldots; {}^d h_d, \ldots, {}^d h_{\nu_d}) \in \mathbf{F}^{*\nu_1} \times \cdots \times \mathbf{F}^{*\nu_d}$. By the preceding Lemma 2.30, we obtain

$$\frac{|\Delta^{\mathbf{1}} F_{\breve{\boldsymbol{h}}}(x, ({}^1 h_{\nu_1+1}, \ldots, {}^d h_{\nu_d+1}))|}{|{}^1 h_{\nu_1+1}|^{\rho_1} \cdots |{}^d h_{\nu_d+1}|^{\rho_d}} \leq C^d/\delta_1^{\rho_1} \cdots \delta_d^{\rho_d} \cdot \|F_{\breve{\boldsymbol{h}}}\|_{\sup}.$$

By Lemma 2.29, we have

$$\|F_{\breve{\boldsymbol{h}}}\|_{\sup} \leq \|f^{]\boldsymbol{\nu}[}(x, \breve{\boldsymbol{h}})\|_{\mathbf{F}^d\times(\mathbf{F}^{*\nu_1}\times\cdots\times\mathbf{F}^{*\nu_d})} \leq C^{|\boldsymbol{\nu}|}/\delta_1^{\nu_1} \cdots \delta_d^{\nu_d} \|f\|_{\sup}.$$

Because $\breve{\boldsymbol{h}} \in \mathbf{F}^{*\nu_1} \times \cdots \times \mathbf{F}^{*\nu_d}$ was arbitrary, we can conclude

$$\|f\|_{\mathcal{C}^r} \leq C^{|\boldsymbol{\nu}+\mathbf{1}|}/\delta_1^{r_1} \cdots \delta_d^{r_d} \cdot \|f\|_{\sup}.$$

■

2.3 The locally algebraic case

Assumption 2.32. In this subsection, we will make the following assumptions:

1. We assume that G splits.
2. We assume the unramified dominant character $\chi : M \to \mathbf{K}^*$ to fulfill the conditions of Corollary 2.11.
   a) It holds $|\chi(M^+)| \leq 1$.
   b) In particular $|\chi(Z)| = 1$.

Remark. The assumption for G to split furnishes us with the classification theory of rational representations of split reductive groups.
Moreover, we have an identification $\mathbf{F} \cong N_\alpha$ and can therefore conveniently speak of polynomial functions on N_α.

Because $\bar{P}N \subseteq G$ is by [Borel, 1991, Corollary 14.14 and Theorem 21.20] Zariski-dense, we have by restriction onto N an injective homomorphism of $\mathbf{K}$-vector spaces $\mathrm{Ind}_{\bar{P}}^{G}(\psi)^{\mathrm{alg}} \hookrightarrow \mathcal{C}^{\mathrm{alg}}(N, \mathbf{K})$. It becomes P-equivariant by letting N act through translation and M by $f^m = \psi(m)f(\cdot^m)$ on the right hand side.

We thus obtain a P-equivariant injection

$$I(\theta)(N) \otimes U_\psi \xrightarrow{\sim} \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes_{\mathbf{K}} U_\psi \hookrightarrow \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) \otimes_{\mathbf{K}} \mathcal{C}^{\mathrm{alg}}(N, \mathbf{K}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K}).$$

Here the first isomorphism stems from Lemma 1.2(ii). The injectivity of the last map comes from the fact that by [Borel, 1991, Theorem 21.20(i)] and the Taylor expansion, any polynomial function on N is in characteristic 0 uniquely determined on an open subset in N. The surjectivity holds by compactness of support.

We denote the image of this injection by $\mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K})$. It can be described by

$$\begin{aligned}\mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K}) = \{f : N \to \mathbf{K} \text{ of cpt. supp.}: &\text{ For all } n \in N \text{ exists open } U \ni n \\ &\text{in } N \text{ such that } f_{|U} = p_{|U} \text{ for some } p \in \mathrm{Ind}_{\bar{P}}^{G}(\psi)^{\mathrm{alg}}\}.\end{aligned}$$

We conclude that there is an isomorphism of $\mathbf{K}[P]$-modules

$$I(\chi)(N) \xrightarrow{\sim} \mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K}),$$

where the right hand side is endowed with the P-action by $f^n = (\cdot n)$ for $n \in N$ and $f^m = \chi(m) \cdot f(\cdot^m)$ for $m \in M$. Under this isomorphism $\phi_1 \otimes u$ is sent to $1_{N_0} \cdot u_{|N}$ for $u \in U$, and which we will denote through abuse of notation by $1_{N_0} \otimes u$.

We will from now on use the above identification in this subsection without further mention.

Definition of the norm

By Corollary 1.7, we can describe the universal unitary lattice of the P-representation $I(\chi)(N)$ by $\mathfrak{L} := \mathbf{o_K}[P] \cdot 1_{N_0} \otimes \bar{u} \subseteq \mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K})$. In this subsection, we want show that it is Hausdorff.

Because $\mathfrak{L}$ is in particular P-stable, we find

$$\|f^p\| \leq \|(f^p)^{p^{-1}}\| = \|f\| \leq \|f^p\| \quad \text{for any } f \in \mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K}) \text{ and } p \in P.$$

In particular $\|\cdot^n\| = \|\cdot\|$ for all $n \in N$. We also compute

$$m \cdot (1_{N_0} \otimes \bar{u}) = \theta(m) 1_{{}^m N_0} \otimes \bar{u}(\cdot m) = \theta\bar{\psi}(m) \cdot 1_{{}^m N_0} \otimes \bar{u}.$$

Since $\mathfrak{L}$ is by definition the smallest $\mathbf{o_K}$-lattice containing all $p \cdot (1_{N_0} \otimes \bar{u})$ for $p \in P$, its associated norm $\|\cdot\|$ can thence be characterized as the pointwise greatest norm fulfilling the following two conditions:

1. It is invariant under translation by N.
2. We have $\|1_{{}^m N_0} \otimes \bar{u}\| = 1/|\theta\bar{\psi}(m)|$ for all $m \in M$.

Denote by $\|\cdot\|$ the norm we want to construct and let $\tilde{\mathfrak{L}} := \{f \in \mathcal{C}^{\mathrm{lc}}_{\mathrm{cpt}}(N, \mathbf{K}) : \|f\| \leq 1\}$ be the lattice attached to it. In order that $\mathfrak{L} \subseteq \Lambda \cdot \tilde{\mathfrak{L}}$ for a scalar $\Lambda \in \mathbf{K}^*$ - and thus in particular showing $\mathfrak{L}$ to be Hausdorff, it will suffice (see Corollary 2.34) to show that this norm satisfies the following two conditions:

1. It is invariant under translation by N.
2. There is a constant $C \geq 1$ such that $\|1_{{}^m N_0} \otimes \bar{u}\| \leq C \cdot 1/|\theta\bar{\psi}(m)|$ for all $m \in M$.

Definition.

1. Let $m_\alpha \in M^+/M_0 Z$ be as given in Definition 2.18. We define $\boldsymbol{r} \in \mathbb{R}^{\Phi^+}_{\geq 0}$ by

$$r_\alpha := \begin{cases} v_{\mathbf{K}}(\chi(m_\alpha)) & \text{if } \alpha \in \Delta, \\ 0 & \text{otherwise.} \end{cases}$$

   This is well defined by Assumption 2.32.2.

2. Recall that we have an isomorphism of affine algebraic varieties $\mathbf{N} \xrightarrow{\sim} \prod_{\alpha \in \Phi^+} \mathbf{N}_\alpha \xrightarrow{\sim} \mathbf{F}^{\Phi^+}$ giving rise to an injection of $\mathbf{K}$-vector spaces

$$\iota : \mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K}) \subseteq \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K}) \xrightarrow{\sim} \mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(\mathbf{F}^{\Phi^+}, \mathbf{K});$$

   here we recall $\mathcal{C}^{\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K})$ to denote the locally algebraic functions $f : N \to \mathbf{K}$ of compact support. Then we endow $\mathcal{C}^{\psi-\mathrm{lp}}_{\mathrm{cpt}}(N, \mathbf{K})$ with a norm via

$$\|\cdot\| := \|\iota(\cdot)\|_{\mathcal{C}^r}.$$

Remark. We want to recall the explicit definition of $\|\cdot\|_{\mathcal{C}^r}$. As before, given a real number $r \geq 0$, we split it into $r = \nu + \rho$ with an integral part $\nu := \lfloor r \rfloor \in \mathbb{N}$ and a fractional part $\rho := r - \nu \in [0, 1[$. For $(\boldsymbol{n}^*_\alpha) \in \prod_{\alpha\in\Phi^+} \mathbf{F}^{*\nu_\alpha}$, i.e. $\boldsymbol{n}^*_\alpha \in \mathbf{F}^{*\nu_\alpha}$ for all $\alpha \in \Phi^+$, let us define

$$\mathit{\Delta}^{\boldsymbol{\nu}}{}_{-}(\cdot; (\boldsymbol{n}^*_\alpha)) : \bigotimes_{\alpha\in\Phi^+} \mathcal{C}^{\mathrm{lp}}(\mathbf{F}, \mathbf{K}) \circlearrowleft$$

by

$$\mathit{\Delta}^{\boldsymbol{\nu}}{}_{-}(\cdot; (\boldsymbol{n}^*_\alpha)_{\alpha\in\Delta}) = \bigotimes_{\alpha\in\Phi^+} \mathit{\Delta}^{\nu_\alpha}{}_{-}(\cdot; \boldsymbol{n}^*_\alpha).$$

Then

$$\|f\|_{\mathcal{C}^r} = \sup_{n\in N, (\boldsymbol{n}^*_\alpha)\in\prod_{\alpha\in\Phi^+} \mathbf{F}^{*\nu_\alpha+1}} \frac{|\mathit{\Delta}^{\boldsymbol{\nu}+\mathbf{1}} f(n; (\boldsymbol{n}^*_\alpha))|}{\prod_{\alpha\in\Delta}(|n^*_{\alpha,1}| \cdots |n^*_{\alpha,\nu_\alpha}| \cdot |n^*_{\alpha,\nu_\alpha+1}|^{\rho_\alpha})};$$

here $\mathbf{1} = (1, \ldots, 1) \in \mathbb{N}^{\Phi^+}$.

Lemma 2.33. *The norm $\|\cdot\|$ satisfies the following two conditions:*

1. *It is invariant under translation by N.*
2. *There is a constant $C \geq 1$ such that $\|1_{^mN_0} \otimes \bar{u}\| \leq C \cdot 1/|\theta\bar{\psi}(m)|$ for all $m \in M$.*

Proof. Ad 1.: This holds by definition.

Ad 2.: For any $\alpha \in \Phi^+$, we may assume the algebraic isomorphism of groups $N_\alpha \xrightarrow{\sim} \mathbf{F}$ to be chosen such that $N(a) \xrightarrow{\sim} \mathbf{o_F}$, where we let $a \in \Phi_0^+$ correspond to $\alpha \in \Phi^+$. We have ${}^mN_0 = \prod_{\alpha\in\Phi^+} N(a+ < \alpha, v(m) >)$ and $N(a+ < \alpha, v(m) >) \xrightarrow{\sim} \pi^{<\alpha,v(m)>} \cdot o_{\mathbf{F}}$. Therefore $\iota(1_{N_0} \otimes \bar{u}) : \mathbf{F}^{\Phi^+} \xrightarrow{\sim} N \to \mathbf{K}$ is a $\boldsymbol{\delta}$-polynomial function with $\boldsymbol{\delta} = (\delta_\alpha) \in \mathbb{R}^{\Phi^+}_{>0}$ given by $\delta_\alpha := |\pi|_{\mathbf{F}}^{<\alpha,v(m)>}$. Because U_ψ is a finite dimensional $\mathbf{K}$-vector space, there exists by Proposition 2.31 a constant $\tilde{C}$ such that

$$\|1_{^mN_0} \otimes \bar{u}\| = \|\iota(1_{^mN_0} \otimes \bar{u})\|_{\mathcal{C}^r} \leq \tilde{C} / \prod_{\alpha\in\Delta} \delta_\alpha^{r_\alpha} \cdot \|\iota(1_{^mN_0} \otimes \bar{u})\|_{\sup}.$$

Firstly, within M/M_0Z we can write $m = \sum_{\alpha\in\Delta} i_\alpha \cdot m_\alpha$ with $i_\alpha \in \mathbb{Z}_{\geq 0}$. Then for any $\alpha \in \Delta$, we have $< \alpha, v(m) >= i_\alpha$. Therefore

$$\delta_\alpha^{r_\alpha} = |\pi|_{\mathbf{F}}^{i_\alpha \cdot r_\alpha} = |\pi|_{\mathbf{F}}^{i_\alpha \cdot v_{\mathbf{K}}(\chi(m_\alpha))};$$

the last equality by definition of r_α. We therefore obtain

$$\begin{aligned}\prod_{\alpha\in\Delta} \delta_\alpha^{r_\alpha} &= |\pi|_{\mathbf{F}}^{\sum_{\alpha\in\Delta} i_\alpha \cdot v_{\mathbf{K}}(\chi(m_\alpha))} \\ &= |\pi|_{\mathbf{F}}^{v_{\mathbf{K}}(\chi(\sum_{\alpha\in\Delta} i_\alpha \cdot m_\alpha))} \\ &= |\pi|_{\mathbf{F}}^{v_{\mathbf{K}}(\chi(m))} \\ &= c_{\mathbf{K}}^{v_{\mathbf{K}}(\chi(m))} = |\chi(m)|_{\mathbf{K}}.\end{aligned}$$

Secondly, we have

$$\|1_{{}^m N_0} \otimes \bar{u}\|_{\sup} = \|\bar{u}\|_{{}^m N_0} = \|\bar{u}({}^m\cdot)\|_{N_0} = \|[\psi - \bar{\psi}](m) \cdot \bar{u}\|_{N_0} = |[\psi - \bar{\psi}](m)| \cdot \|\bar{u}\|_{N_0}.$$

Combined, we compute

$$\begin{aligned} \|1_{{}^m N_0} \otimes \bar{u}\| &\leq \tilde{C} \cdot |\chi(m)|_{\mathbf{K}}^{-1} |[\psi - \bar{\psi}](m)| \cdot \|\bar{u}\|_{N_0} \\ &= \tilde{C} \cdot |\chi(m)|_{\mathbf{K}}^{-1} |[\psi - \bar{\psi}](m)| \cdot \|1_{N_0} \otimes \bar{u}\|_{\sup}. \end{aligned}$$

We have $\chi = \theta\psi$ and therefore $\chi^{-1}[\psi - \bar{\psi}] = 1/\theta\bar{\psi}$. This gives

$$\|1_{{}^m N_0} \otimes \bar{u}\| \leq \tilde{C} \cdot 1/|\theta\bar{\psi}(m)| \cdot \|1_{N_0} \otimes \bar{u}\|_{\sup}.$$

We conclude $\|1_{{}^m N_0} \otimes \bar{u})\| \leq C \cdot 1/|\theta\bar{\psi}(m)|$ with $C = \tilde{C} \cdot \|1_{N_0} \otimes \bar{u}\|_{\sup}$. ∎

Corollary 2.34. *There exist constants* $0 < c \leq 1 \leq C$ *such that* $c \leq \|p \cdot 1_{N_0} \otimes \bar{u}\| \leq C$ *for all* $p \in P$.

Proof. Let $f = 1_{N_0} \otimes \bar{u}$. Because M is a group, we have $\|f^m\| \leq C \cdot \|f\|$ for all $m \in M$ if and only if $\|f^m\| = C \cdot \|f\|$ for all $m \in M$. Indeed, given any $m \in M$, we have $\|f\| = \|f^{mm^{-1}}\| \leq \|f^m\|$, hence $\|f\| = \|f^m\|$. Put $c = 1/C$. Because $P = MN$, we infer from the preceding Lemma 2.33 that $c \cdot \|f\| \leq \|f^p\| \leq C \cdot \|f\|$ for all $p \in P$. ∎

Corollary 2.35. *The lattice* $\mathfrak{L}$ *is Hausdorff.*

3 The universal unitary lattice of an unramified dominant principal series

3.1 The universal unitary lattice of the underlying P-representation

We let G be a topological group.

Lemma 3.1. *Let V be a $\mathbf{K}$-linear G-representation and assume the following conditions to hold:*

1. *The group G is* locally profinite*, i.e. every neighborhood of* 1 *contains a compact open subgroup.*
2. *The group G has a* formal Iwasawa decomposition $G = PK$ *for a compact subgroup K of G and a subgroup $P \subseteq G$.*
3. *The representation V is* locally finite dimensional*: For every vector $v \in V$, there is a compact open subgroup $G_0 \subseteq G$ such that $\mathbf{K}[G_0] \cdot v$ is a finite dimensional $\mathbf{K}$-vector space.*

Then the universal unitary lattice of V with its finest locally convex topology is given by any lattice finitely generated as an $\mathbf{o}[P]$-module.

Proof. By Proposition 0.2, the universal unitary lattice of V is given by any lattice finitely generated as an $\mathbf{o}[G]$-module. We hence have to show that the commensurability class of lattices finitely generated as an $\mathbf{o}[G]$-module equals the one of lattices finitely generated as an $\mathbf{o}[P]$-module.
Let $\mathfrak{L} := \sum_{i \in I} \mathbf{o}[G]v_i$ with I finite be such a lattice. Then $\sum_{i \in I} \mathbf{K}[K] \cdot v_i$ is a finite dimensional $\mathbf{K}$-vector space: By assumption, there exists a compact open subgroup $K_0 \subseteq G$ such that $V_0 := \sum_i \mathbf{K}[K_0] \cdot v_i$ is a finite dimensional $\mathbf{K}$-vector space. By intersecting with K and possibly shrinking K_0, we can assume K_0 to be an open normal subgroup of K, so that the quotient K/K_0 is a finite group. Therefore $\sum_i \mathbf{K}[K]v_i = \sum_{k \in K/K_0} (\sum_i \mathbf{K}[kK_0] \cdot v_i)$ is finite dimensional.
We thus find the $\mathbf{o}$-module $\sum_{i \in I} \mathbf{o}[K] \cdot v_i$ to be finitely generated as a $\mathbf{K}$-vector space and, since K is compact, also to be bounded. Therefore it is finitely generated as an $\mathbf{o}$-module and hence finite free. Denote its basis by $\{v_j : j \in J\}$ for a finite index set J. Therefore

$$\begin{aligned}
\mathfrak{L} &= \sum_{i \in I} \mathbf{o}[G]v_i \\
&= < g \cdot v_i : g \in PK, i \in I >_{\mathbf{o}\text{-mod.}} \\
&= < k \cdot v_i : k \in K, i \in I >_{\mathbf{o}[P]\text{-mod.}} \\
&= < v_j : j \in J >_{\mathbf{o}[P]\text{-mod.}} \\
&= \sum_{j \in J} \mathbf{o}[P]v_j.
\end{aligned}$$

Conversely, assume we are given a lattice $\mathfrak{L} = \sum_{i \in I} \mathbf{o}[P]v_i \subseteq V$ with I finite. Then likewise $\sum_{i \in I} \mathbf{o}[G]v_i = \sum_{j \in J} \mathbf{o}[P]v_j$ with J finite. So by finiteness (and because $\mathfrak{L}$ is a lattice), we find $\{v_j\} \subseteq \Lambda \cdot \mathfrak{L}$ for some $\Lambda \in \mathbf{K}$ and hence by P-stability of $\mathfrak{L}$, we find $G \cdot \{v_i\} \subseteq \Lambda \cdot \mathfrak{L}$. Therefore, putting $\tilde{\mathfrak{L}} = \sum_{i \in I} \mathbf{o}[G]v_i$, we have

$$\mathfrak{L} \subseteq \tilde{\mathfrak{L}} \subseteq \Lambda \cdot \mathfrak{L}.$$

Hence $\mathfrak{L}$ and $\tilde{\mathfrak{L}}$ are commensurable. ■

Remark 3.2. We have the following examples of G and V fulfilling the hypotheses of Lemma 3.1 in mind:

1. If G is an affine algebraic group over a local field, then Condition 1 is always fulfilled, see e.g. [Cartier, 1979, Section 1.1].

2. If G is a connected reductive group over a local field with a minimal parabolic subgroup P as defined in Section 0, then Condition 2 is by definition fulfilled if $K \subseteq G$ is a *good* maximal compact open subgroup.

3. We find Condition 3 to be satisfied if V is smooth or more generally locally algebraic. As remarked in Section 0 this in particular applies to any unramified dominant principal series representation.

Corollary 3.3. *Let* **G** *be a connected reductive group and* $\chi : M \to \mathbf{K}^*$ *an unramified dominant character. Then the universal unitary lattice of the locally algebraic* G*-representation* $I(\chi)$ *is given by any lattice finitely generated as an* $\mathbf{o_K}[P]$*-module.*

Proof. By the above Remark 3.2, we can apply Lemma 3.1 to the representation $V = I(\chi)$ of our topological group G. ■

3.2 The Jacquet module

We let **G** be a connected reductive group over **F**.

Definition.

1. Recall that a smooth G-representation V is called **admissible**, if V^K is a finite dimensional **K**-vector space for every compact open subgroup in G. The **Jacquet module** $J_P(V)$ of a locally algebraic representation $V = I \otimes U$ for an admissible smooth G-representation I and an irreducible algebraic G-representation U is the $\mathbf{K}[P]$-module defined by

$$J_P(V) = I_N \otimes_{\mathbf{K}} U^N;$$

here V_N is defined as the largest quotient on which N acts trivially. So $V_N := V/V(N)$, where $V(N)$ is the N-subrepresentation generated by all $nv - v$ for $n \in N$ and $v \in V$. (See [Emerton, 2006, Proposition 4.3.6].)

2. We fix as before the compact open subgroup $N_0 := \bar{B} \cap N$ of N and let $M^+ \subseteq M$ respectively $Z_M^+ \subseteq Z_M$ be the *dominant submonoid* in M respectively Z_M consisting of all elements stabilizing N_0 by left conjugation.

3. We define $\delta_P : P \to \mathbb{Q}$ to be the **modulus character** on P given by precomposition of the projection $P \twoheadrightarrow M$ with the unramified character $\delta_P : M \to p^{\mathbb{Z}}$ with $\delta_P(m) := |\det \mathrm{Ad}_{\mathfrak{n}}(m)|$.

Remark 3.4. 1. By definition, N acts trivially on $J_P(V)$.

2. If $U = U_\psi$ is the irreducible algebraic representation parameterized by its highest weight ψ, then P acts on U^N by definition through the dominant algebraic character ψ, and we conclude

$$J_P(I \otimes_{\mathbf{K}} U_\psi) = I_N \otimes_{\mathbf{K}} \psi.$$

Remark. Let $m \in M$ and let $\delta : M \to \mathbf{K}^*$ be a character which is either unramified or algebraic. Then $\delta(\cdot^m) = \delta$.

Proof. 1. If δ is unramified, this holds as M/M_0 is by Remark 2.1 abelian.

2. If δ is algebraic, we find by Lemma 2.3 this character to be determined by its restriction onto the subgroup $A \subseteq M$, which by definition M centralizes.

■

We can therefore define the Weyl group W to act on the product $\delta : M \to \mathbf{K}^*$ of an unramified and an algebraic character through $\delta^w = \delta(\cdot^w)$.

Remark (Interlude on the square root of the modulus character). Let $\Delta_P : M \to \mathbf{F}^*$ be defined by $\Delta_P(m) = \det \mathrm{Ad}_{\mathfrak{n}}(m)$. Because $\Delta_P : M \to \mathbf{F}^*$ is an algebraic character on a connected reductive group, it is by Lemma 2.3 determined by its restriction onto $A \subseteq M$.
Recall that $\mathfrak{n} = \oplus_{\alpha \in \Phi^+} \mathfrak{n}_\alpha$ with A acting through the adjoint action on $\mathfrak{n}_\alpha$ by multiplication with the character α.
We observe the following properties:

1. We have $v(\Delta_P(a)) = \sum_{\alpha \in \Phi^+} v(\alpha(a))$ for $a \in A$.

2. We have $\Delta_{\bar{P}} = \Delta_P^{-1}$.

Recall that $\delta_P : P \to p^{\mathbb{Z}} \subseteq K^*$ was defined by $\delta_P(mn) := |\Delta_P(m)|_{\mathbf{F}}$ for $mn \in P = MN$. We deduce:

1. By continuity, the image $\delta_P(M_b)$ of any bounded subgroup $M_b \subseteq M$ has to bounded. This can by the group properties only hold if $\delta_P(M_b) = 1$. Therefore δ_P has to be trivial on the maximal compact (open) subgroup $M_0 \subseteq M$, i.e. it is unramified.

2. Let $w \in W$. We recall W to permute Φ. Therefore

$$v(\delta_P/\delta_P^w(a)) = 2 \cdot \sum_{\alpha \in \Phi^+ \cap w\Phi^+} v(\alpha(a))$$

and we see that $\delta_P/\delta_P^w(A) \subseteq p^{2\mathbb{Z}}$. Therefore the character $(\delta_P/\delta_P^w)^{1/2} : M \to p^{\mathbb{Z}}$ is well defined.

We conclude that we have a well defined unramified character $(\delta_P/\delta_P^w)^{1/2} : M \to \mathbf{K}^*$.

The following interlude shows that $\mathbb{C}$ as an abstract field is determined by being an algebraically closed field of characteristic 0 and cardinality $2^{\aleph_0}$.

Lemma. *Let C and D be two algebraically closed fields of equal characteristic and transcendence degree. Then there is an isomorphism of rings $C \cong D$.*

Proof. Let c and d be their isomorphic prime fields. Then we choose transcendence bases T of C and S of D. Because $\#T = \#S$, there is an isomorphism of rings $c(S) \cong d(T)$. Because these are maximal algebraically independent subsets, any extension of these fields is algebraic and we can extend this isomorphism to their algebraic closures $C = \overline{c(S)} \cong \overline{d(T)} = D$. ■

Notation. Let $\mathbf{E}$ be the subfield $\mathbb{Q}(\theta(M)) \subseteq \mathbf{K}$.

Lemma 3.5. *There exists a ring embedding $\mathbf{E} = \mathbb{Q}(\theta(M)) \hookrightarrow \mathbb{C}$.*

Proof. Let $\mathbb{C}(\theta(M))$ be the composite field of $\mathbf{E}$ and $\mathbb{C}$ and $\tilde{\mathbb{C}}$ its algebraic closure. By Corollary 2.4, we have an isomorphism of groups $\Lambda := M/M_0 = \mathbb{Z}^{\Delta}$ and thence in particular $\chi(M)$ is a finitely generated abelian group. We find $\#\tilde{\mathbb{C}} = \#\mathbb{C}(\theta(M)) = \#\mathbb{C}$ and thence an equality of transcendence degrees $\operatorname{td}\tilde{\mathbb{C}} = \operatorname{td}\mathbb{C}(\theta(M)) = \operatorname{td}\mathbb{C}$. By the preceding lemma we obtain a ring isomorphism $\tilde{\mathbb{C}} \cong \mathbb{C}$. Therefore

$$\mathbf{E} = \mathbb{Q}(\theta(M)) \hookrightarrow \mathbb{C}(\theta(M)) \hookrightarrow \tilde{\mathbb{C}} \cong \mathbb{C}.$$

■

Remark. Let $I(\theta)_{\mathbf{E}} \subseteq I(\theta)$ be the $\mathbf{E}[G]$-module given by all functions with image in $\mathbf{E}$. Then we have an equality of $\mathbf{K}[G]$-modules $I(\theta) = I(\theta)_{\mathbf{E}} \otimes_{\mathbf{E}[G]} \mathbf{K}[G]$.

Definition.

1. Put $\theta_w := \theta^w(\delta_{\bar{P}}/\delta_{\bar{P}}^w)^{1/2}$. We call a character $\theta : \bar{P} \to \mathbf{K}^*$ regular if $\theta_w = \theta$ only if $w = 1$.
2. Given an unramified dominant character $\chi = \theta\psi$, we put $\chi_w := \theta_w\psi$. We call an unramified dominant character χ **regular** if $\chi_w = \chi$ only if $w = 1$. This holds if and only if θ is regular.

Lemma 3.6. *We have an isomorphism of $\mathbf{K}[P]$-modules*

$$I(\theta)_N = \bigoplus_{w \in W} \tilde{\theta}_w$$

with $\tilde{\theta}_w := (\theta/\delta_{\bar{P}}^{1/2})^w \cdot \delta_P^{1/2}$.

Proof.

1. We firstly assume $\mathbf{K} \cong \mathbb{C}$ as an abstract field. By [Cartier, 1979, Theorem 3.5] and observing $\delta_{\bar{P}} = \delta_P^{-1}$ - where we note that we have taken the Jacquet module with respect to the parabolic subgroup *opposite* to the one we induced from - the semi-simplified Jacquet module $I(\theta)_N^{\mathrm{ss}}$ of $I(\theta) = \operatorname{Ind}_{\bar{P}}^G \theta^{\mathrm{lc}}$ is given by

$$I(\theta)_N^{\mathrm{ss}} = \bigoplus_{w \in W} \tilde{\theta}_w$$

with $\tilde{\theta}_w := (\theta/\delta_{\bar{P}}^{1/2})^w \cdot \delta_P^{1/2}$. By the regularity of θ, the θ_w or equivalently the $\tilde{\theta}_w$ are all pairwise distinct. Then by the Chinese remainder theorem the cyclic $\mathbf{K}[P]$-module $J_P(I(\theta))$ splits and therefore

$$I(\theta)_N = \bigoplus_{w\in W} \tilde{\theta}_w.$$

2. We now let $\mathbf{K}$ be arbitrary and set $\mathbf{E} = \mathbb{Q}(\theta(M)) \subseteq \mathbf{K}$. By the preceding Lemma 3.5, we have an embedding of rings $\mathbf{E} \hookrightarrow \mathbb{C}$. Let $R = \mathbf{E}[P]$. By the first step, we have an equality of $R \otimes_{\mathbf{E}} \mathbb{C}$-modules

$$(I(\theta)_{\mathbf{E}})_N \otimes_{\mathbf{E}} \mathbb{C} = \bigoplus_{w\in W} \tilde{\theta}_w.$$

Because $\mathbb{C}$ is faithfully flat over $\mathbf{E}$, we find $R \otimes_{\mathbf{E}} \mathbb{C}$ to be faithfully flat over R. Thence the above equality already held over R. By flatness of $\mathbf{K}[P]$ over $\mathbf{E}[P]$, we conclude $I(\theta)_N = \bigoplus_{w\in W} \tilde{\theta}_w$ as $\mathbf{K}[P]$-modules.

■

3.3 Gluing the universal unitary lattice from the intertwined open cells

We let $\mathbf{G}$ be a connected reductive group over $\mathbf{F}$ and let $\chi = \theta\psi : M \to \mathbf{K}^*$ be an unramified dominant character.

Assumption. We will from now on assume $\chi : M \to \mathbf{K}^*$ to be regular.

Lemma 3.7. *There exists for each $w \in W$ a nonzero morphism of G-representations $T_w : I(\theta) \to I(\tilde{\theta}_w)$ with $\tilde{\theta}_w = \theta^w(\delta_P/\delta_P^w)^{1/2}$.*

Proof. Let $\eta : \bar{P} \to \mathbf{K}^*$ be an unramified character. Because $\bar{N}$ lies in the commutator subgroup of $\bar{P}$ whereas $\mathbf{K}^*$ is abelian, η is necessarily trivial on $\bar{N}$. Therefore and by Frobenius reciprocity thus holds $\mathrm{Hom}_G(I(\theta), I(\eta)) = \mathrm{Hom}_{\bar{P}}(I(\theta)_{\bar{N}}, \eta)$. Note that $I(\theta)_{\bar{N}} = J_{\bar{P}}(I(\theta))$. So Lemma 3.6 with $\bar{P}$ exchanged with P shows that $I(\theta)_{\bar{N}} = \bigoplus_{w\in W} \tilde{\theta}_w$. Therefore $\mathrm{Hom}_G(I(\theta), I(\eta)) = \prod_{w\in W} \mathrm{Hom}_{\bar{P}}(\tilde{\theta}_w, \eta)$, which is nonzero if and only if $\eta = \tilde{\theta}_w$ for some $w \in W$. In particular $\mathrm{Hom}_G(I(\theta), I(\tilde{\theta}_w)) \neq 0$ for all $w \in W$. ■

Corollary 3.8. *For any unramified dominant character $\chi : M \to \mathbf{K}^*$, there exist nonzero morphisms of G-representations $T_{\chi_w} : I(\chi_w) \to I(\chi)$ for all $w \in W$.*

Proof. Fix $w \in W$. We firstly assume $\chi = \theta$ to be unramified. Using $\delta_{\bar{P}} = \delta_P^{-1}$, we see that θ_w was chosen such that $\tilde{\theta}_w = \theta_w^w(\delta_{\bar{P}}/\delta_{\bar{P}}^w)^{1/2} = \theta$. Therefore we have by the preceding Lemma 3.7 a $K[G]$-linear morphism $T_w : I(\theta_w) \to I(\theta)$.
Let us now assume $\chi = \theta\psi$ to be a general unramified dominant character. Then we just define $T_{\chi_w} : I(\chi_w) \to I(\chi)$ by

$$I(\chi_w) = I(\theta_w) \otimes U_\psi \overset{T_w \otimes \mathrm{id}_{U_\psi}}{\to} I(\theta) \otimes U_\psi = I(\chi).$$

■

Assumption. We will in the following assume that $\mathbf{K} \cong \mathbb{C}$ as an abstract field.

Lemma 3.9. *Let $\alpha \in \Delta$. Then the action of T_{w_α} on $\{\phi_w : w \in W\}$ - cf. Definition 1.3 - is described as follows:*

$$T_{w_\alpha}(\phi_w) = \begin{cases} (c_\alpha(\theta_{w_\alpha}) - 1)\phi_w + q_\alpha^{-1} q_{\alpha/2}^{-1} \phi_{w_\alpha w}, & \text{if } \ell(w_\alpha w) > \ell(w) \\ (c_\alpha(\theta_{w_\alpha}) - q_\alpha^{-1} q_{\alpha/2}^{-1})\phi_w + \phi_{w_\alpha w}, & \text{if } \ell(w_\alpha w) < \ell(w). \end{cases}$$

Here $q_\alpha, q_{\alpha/2} \in q_{\mathbf{F}}^{\mathbb{Z}}$ are constants solely dependent on α constructed in [Casselman, 1980, Equation (12) et seq.] and $c_\alpha(\theta_{w_\alpha}) \in \mathbf{E}$ is a constant solely dependent on α and θ_{w_α} (cf. Remark 3.13 below).

Proof. By [Casselman, 1980, First equality of Theorem 3.4], we have the above equality if $\ell(w_\alpha w) > \ell(w)$. Otherwise $\ell(w_\alpha w) < \ell(w)$, so that $w = w_\alpha \breve{w}$ with $\breve{w} = w_\alpha w$ and $\breve{w}$ fulfilling $\ell(w_\alpha \breve{w}) > \ell(\breve{w})$ and then

$$T_{w_\alpha}(\phi_w) = \phi_{w_\alpha w} + (c_\alpha(\theta) - q_\alpha^{-1} q_{\alpha/2}^{-1})\phi_w$$

by the second equality of [Casselman, 1980, Theorem 3.4]. ∎

Remark. Because T_w is $\mathbf{E}[G]$-linear, it preserves in particular the subspaces of Iwahori invariants, i.e. $T_w(I(\theta_w)_{\mathbf{E}}^{\bar{B}}) \subset I(\theta)_{\mathbf{E}}^{\bar{B}}$. We reasoned in Remark 1.4 these to have $\{\phi_v\}$ as a basis and so deduce that $T_{w_\alpha}(\phi_v)$ is again a $\mathbf{E}$-linear combination of $\{\phi_v\}$.
Now the only content in the above theorem relevant to us is the observation that if $w = w_\alpha$ for some $\alpha \in \Delta$, then the coefficient of ϕ_{wv} in this linear combination will be *nonzero*. By induction on the length of $w \in W$, this holds for all T_w if $v = 1$, as seen next.

Lemma 3.10. *Fix $w \in W$. Then $T_w(\phi_1) = \lambda_w \phi_w + \sum_{\ell(v)<\ell(w)} \lambda_v \phi_v$ with $\lambda_w \neq 0, \lambda_v \in \mathbf{E}$.*

Proof. By induction on $\ell(w)$. If $\ell(w) = 0$, then $w = 1$ and there is nothing to prove.
Let $l := \ell(w) \geq 1$. Then we can write $w = w_\alpha \breve{w}$ with $\ell(\breve{w}) = l - 1$ for suitable $\alpha \in \Delta$. By [Casselman, 1995, Theorem 6.4.4], we find $T_w = C_{w_\alpha,\breve{w}} \cdot T_{w_\alpha} \circ T_{\breve{w}}$ with $C_{w_\alpha,\breve{w}} \in p^{\mathbb{Z}}$. The induction hypothesis for $\breve{w}$ gives

$$T_{\breve{w}}\phi_1 = \lambda_{\breve{w}}\phi_{\breve{w}} + \sum_{\ell(v)<\ell(\breve{w})} \lambda_v \phi_v$$

with suitable scalars $\lambda_{\breve{w}} \neq 0$ and $\lambda_v \in \mathbf{E}$. By Lemma 3.9, we obtain in either case $\ell(w_\alpha v) \gtrless \ell(v)$ that

$$T_{w_\alpha}\phi_v = \sum_u \lambda_u \phi_u \quad \text{for } u \in W \text{ with } \ell(u) \leq \ell(w_\alpha v) < 1 + \ell(\breve{w}) = \ell(w)$$

and suitable scalars $\lambda_u \in \mathbf{E}$. By the same token

$$T_{w_\alpha}\phi_{\breve{w}} = \lambda_w \phi_w + \lambda_{\breve{w}}\phi_{\breve{w}} \quad \text{with } \lambda_w \neq 0, \lambda_{\breve{w}} \in \mathbf{E}.$$

Therefore

$$\begin{aligned} T_w\phi_1 &= C_{w_\alpha,\breve{w}} \cdot T_{w_\alpha} T_{\breve{w}}(\phi_1) \\ &= C_{w_\alpha,\breve{w}} \cdot T_{w_\alpha}(\lambda_{\breve{w}}\phi_{\breve{w}}) + \sum_{\ell(v)<\ell(\breve{w})} C_{w_\alpha,\breve{w}} \cdot \lambda_v T_{w_\alpha}(\phi_v) \\ &= [\mu_{w_\alpha\breve{w}}\phi_{w_\alpha\breve{w}} + \mu_{\breve{w}}\phi_{\breve{w}}] + \sum_{\ell(v)<\ell(\breve{w})} (\mu_v\phi_v + \mu_{w_\alpha v}\phi_{w_\alpha v}) \\ &= \mu_w\phi_w + \sum_{\ell(v)<\ell(w)} \mu_v\phi_v \end{aligned}$$

for suitable scalars $\mu_w = \mu_{w_\alpha\breve{w}} \neq 0$ and $\mu_v \in \mathbf{E}$. ∎

Lemma 3.11. *The* $\mathbf{E}$*-linear span of* $\{T_w\phi_1 : w \in W\}$ *contains* $\{\phi_w : w \in W\}$.

Proof. We show by induction on l that the $\mathbf{E}$-linear span of $\{T_w\phi_1 : \ell(w) \leq l\}$ contains $\{\phi_w : \ell(w) \leq l\}$. If $l = 0$, then $w = 1$ and $T_w = 1$, so that everything holds.
Let $l \geq 1$. By the induction hypothesis, we find the $\mathbf{E}$-linear span of $\{T_v\phi_1 : \ell(v) \leq l-1\} \subseteq \{T_v\phi_1 : \ell(v) \leq l\}$ to contain $\{\phi_v : \ell(v) \leq l-1\}$. But every $T_w\phi_1$ with $\ell(w) = l$ is by Lemma 3.10 the sum of a linear combination of $\{\phi_v : \ell(v) \leq l-1\}$ and a nonzero scalar multiple of ϕ_w. Therefore $\{T_w\phi_1 : \ell(w) \leq l\}$ contains also all ϕ_w with $\ell(w) = l$ and the proposition holds. ∎

Assumption. For the remainder of this section, we will again assume $\mathbf{K}$ to be an arbitrary complete non-Archimedeanly non-trivially valued field of characteristic 0.

Lemma 3.12. *The universal unitary lattice of* $I(\chi)$ *is given by*

$$\mathfrak{L} = \sum_{w\in W} \mathbf{o_K}[P] \cdot T_{\chi_w}(\phi_1 \otimes \bar{u}).$$

Proof. By Corollary 3.3, the universal unitary lattice of $I(\chi)$ is given by any $\mathbf{o_K}$-lattice finitely generated as an $\mathbf{o_K}[P]$-module. It therefore rests to show that $\mathfrak{L}$ is indeed an $\mathbf{o_K}$-lattice in $I(\chi)$.

By Lemma 3.11, we find

$$< T_w(\phi_1) \otimes \bar{u} : w \in W >_{\mathbf{E}\text{-vsp.}} \supseteq \{\phi_w \otimes \bar{u} : w \in W\}.$$

Therefore

$$\mathfrak{L} \otimes_{\mathbf{o_K}} \mathbf{K} = \sum_{w\in w} \mathbf{K}[P] \cdot T_{\chi_w}(\phi_1 \otimes \bar{u}) \supseteq \sum_{w\in W} \mathbf{K}[P] \cdot \phi_w \otimes \bar{u} = I(\chi);$$

here the last equality by Corollary 1.7. ∎

Remark 3.13. Let $\alpha \in \Phi$. If G is split, we define

$$c_\alpha(\theta) = \frac{1 - q_{\mathbf{F}}^{-1}\theta(a_\alpha)}{1 - \theta(a_\alpha)}$$

with $a_\alpha \in A$ such that $v(a_\alpha) = \check{\alpha}$. For general non-split G, we refer to [Casselman, 1980, Beginning of Section 3].

Then the following statements are equivalent:

1. It holds $c_\alpha(\theta), c_\alpha(\theta_{w_0}) \neq 0$ for all $\alpha \in \Phi$. $(*)$

2. The G-representation $I(\chi)$ is absolutely irreducible.

3. All $T_{\chi_w} : I(\chi_w) \to I(\chi)$ for $w \in W$ are bijective.

Proof. We begin by assuming $\mathbf{K} \cong \mathbb{C}$. Firstly by [Casselman, 1980, Proposition 3.5(b)], we find $(*)$ to hold if and only if $I(\theta)$ is irreducible. By [Casselman, 1995, Proposition 2.2.6], we find $I(\theta)$ to be irreducible if and only if it is absolutely irreducible.
Secondly, by [Casselman, 1980, Proposition 3.5(b)], we find $(*)$ to hold if and only if all intertwining operators $T_w : I(\theta_w) \to I(\theta)$ are isomorphisms.

If $\mathbf{K}$ is a general complete non-Archimedeanly non-trivially valued field of characteristic 0, we can by Lemma 3.5 embed $\mathbf{E} \hookrightarrow \mathbb{C}$. We recall that $I(\theta)_\mathbb{C} = I(\theta)_\mathbf{E} \otimes_\mathbf{E} \mathbb{C}$ and $\mathbb{C}[G]$ to be faithfully flat over $\mathbf{E}[G]$. Thence we find $T_w \otimes_\mathbf{E} \mathbb{C} : I(\theta_w)_\mathbb{C} \to I(\theta)_\mathbb{C}$ to be an isomorphism of $\mathbb{C}[G]$-modules if and only if $T_w : I(\theta_w)_\mathbf{E} \to I(\theta)_\mathbf{E}$ is an isomorphism of $\mathbf{E}[G]$-modules. Likewsie because $I(\theta) = I(\theta)_\mathbf{E} \otimes_\mathbf{E} \mathbf{K}$ and because $\mathbf{K}[G]$ is faithfully flat over $\mathbf{E}[G]$, this holds if and only if $T_w : I(\theta_w) \to I(\theta)$ is an isomorphism.

Then these equivalences expand to a general unramified dominant character χ as follows: Firstly by definition $U_\psi = U_\mathbf{F} \otimes_\mathbf{F} \mathbf{K}$ with an irreducible $\mathbf{F}[G]$-module $U_\mathbf{F}$. Invoking the classification theory of irreducible rational representations of split reductive groups (see [Jantzen, 2003, Section II.2]) and using the density of rational points $G \subseteq \mathbf{G}$ by [Borel, 1991, Corollary 18.3], we see that $U_\mathbf{F}$ or equivalently U_ψ is absolutely irreducible. By [Schneider and Teitelbaum, 2001, Appendix] a tensor product of an irreducible smooth and irreducible algebraic representation is again irreducible. Hence we find $I(\chi) = I(\theta) \otimes_\mathbf{K} U_\psi$ to be absolutely irreducible if and only if $I(\theta)$ is absolutely irreducible. Secondly by definition $T_{\chi_w} = T_w \otimes_\mathbf{K} \mathrm{id} : I(\theta_w) \otimes_\mathbf{K} U_\psi \to I(\theta) \otimes U_\psi$. Thence T_{χ_w} is bijective if and only if T_w is bijective. ■

Corollary. *Let $I(\chi)$ be absolutely irreducible and assume that $|\chi_w(M^+)| \leq 1$ for all $w \in W$. Then the universal unitary lattice of $I(\chi)$ is of the form*

$$I(\chi) = \sum_{w \in W} \mathfrak{L}_w \quad \text{with } \mathfrak{L}_w = \mathbf{o_K}[P] \cdot T_w(\phi_1 \otimes \bar{u}) \text{ a free } \mathbf{o_K}\text{-module.}$$

Proof. Because T_w is in particular $\mathbf{K}[P]$-linear, we have $\mathbf{o_K}[P] \cdot T_w(\phi_1 \otimes \bar{u}) = T_w(\mathbf{o_K}[P] \cdot \phi_1 \otimes \bar{u})$. By Corollary 2.15 in the smooth respectively Corollary 2.35 in the locally algebraic case, we find $\tilde{\mathfrak{L}}_w = \mathbf{o_K}[P] \cdot \phi_1 \otimes \bar{u} \subseteq I(\chi_w)(N)$ to be Hausdorff (or free). Therefore its $\mathbf{o_K}$-linear image

$$T_w(\tilde{\mathfrak{L}}_w) = o_\mathbf{K}[P] \cdot T_w(\phi_1 \otimes \bar{u}) = \mathfrak{L}_w$$

under the injection $T_w : I(\chi_w) \to I(\chi)$ is again free. ■

References

Barsky, D., 1973. Fonctions k-lipschitziennes sur un anneau local et polynômesa valeurs entieres. Bull. Soc. Math. France 101, 397–411.

Berger, L., Breuil, C., 2010. Sur quelques représentations potentiellement cristallines de $\mathrm{GL}_2(\mathbb{Q}_p)$. Astérisque 330, 155–211.

Borel, A., 1966. Linear algebraic groups. Proc. Symp. Pure Math. 9, 3-19 (1966).

Borel, A., 1979. Automorphic L-functions. In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., pp. 27–61.

Borel, A., 1991. Linear algebraic groups, 2nd Edition. Vol. 126 of Graduate Texts in Mathematics. Springer-Verlag, New York.

Bruhat, F., Tits, J., 1972. Groupes reductifs sur un corps local. Publ. math. IHES.

Cartier, P., 1979. Representations of p-adic groups: a survey. In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., pp. 111–155.

Casselman, W., 1980. The unramified principal series of p-adic groups. I. The spherical function. Compositio Math. 40 (3), 387–406.

Casselman, W., 1995. Theory of admissible representations. Preprint.

Chabert, J., Cahen, P., 2006. Old problems and new questions around integer-valued polynomials and factorial sequences. Multiplicative Ideal Theory in Commutative Algebra, 89–108.

Emerton, M., 2005. p-adic L-functions and unitary completions of representations of p-adic reductive groups. Duke Math. J. 130 (2), 353–392.

Emerton, M., 2006. Jacquet modules of locally analytic representations of p-adic reductive groups. I. Construction and first properties. Ann. Sci. École Norm. Sup. (4) 39 (5), 775–839.

Emerton, M., To appear. Locally analytic vectors in representations of non-archimedean locally p-adic analytic groups. Memoirs of the AMS.

Glöckner, H., 2007. Comparison of some notions of C^k-maps in multivariable non-Archimedean analysis. Bull. Belg. Math. Soc. Simon Stevin 14, 877–904.

Humphreys, J., 1975. Linear algebraic groups. Springer-Verlag, New York, graduate Texts in Mathematics, No. 21.

Jantzen, J., 2003. Representations of algebraic groups. American Mathematical Society.

Schikhof, W., 1984. Ultrametric Calculus: An Introduction to p-Adic Analysis. Cambridge University Press.

Schneider, P., 2002. Nonarchimedean functional analysis. Springer Verlag.

Schneider, P., Stuhler, U., 1997. Representation theory and sheaves on the Bruhat-Tits building. Inventiones.

Schneider, P., Teitelbaum, J., 2001. $U(\mathfrak{g})$-finite locally analytic representations. Represent. Theory 5, 111–128.

Tits, J., 1979. Reductive groups over local fields. In: Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1. Proc. Sympos. Pure Math., XXXIII. Amer. Math. Soc., Providence, R.I., pp. 29–69.