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Iterated asymptotic cones

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Für Katja

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Introduction

In his paper [G_1] from 1981, Gromov considered the geometry of finitely generated groups and proved a remarkable theorem. Any finitely generated group G has a natural metric, the so-called word metric, depending on the choice of a generating set and any two word metrics are bi-Lipschitz equivalent and so it makes sense to talk about the growth rate of such a group. Every ball of radius n around the unit element of G contains only finitely many elements of the group and if this number is bounded by a polynomial in n, we say that G has polynomial growth. If on the other hand this number of elements is asymptotically bigger than a^n for some $a > 1$ we say that G grows exponentially.

It was known at that time that subgroups of connected Lie groups grow exponentially, unless they are virtually nilpotent (i.e. unless they contain a nilpotent subgroup of finite index). Gromov succeeded to prove the converse:

Theorem ([G_1], Main Theorem)**.** *If a finitely generated group G is of polynomial growth, then G is virtually nilpotent.*

To prove this theorem, Gromov considered G as a metric space and associated another metric space Y to G, which should capture the "large-scale geometry" of G. To do this, he considered a sequence X_n of metric spaces, all having the same underlying set G and the metric in X_n being the word metric rescaled by the factor $1/n$. He then proved that this sequence of spaces converges to a space Y in the sense of the Gromov-Hausdorff distance, a concept of distances for metric spaces.

This associated metric space was later termed "asymptotic cone" of G. It is a useful invariant when considering finitely generated groups (or more general metric spaces) up to quasi-isometry, because the asymptotic cones of quasi-isometric spaces are bi-Lipschitz equivalent and therefore homeomorphic. However, the construction still had a serious drawback: The sequence of metric spaces does not always have a limit.

Van den Dries and Wilkie remedied this in their paper [vDW] from 1983 and rewrote Gromov's proof, defining the asymptotic cone as we do it today using concepts from model theory, namely ultrafilters and ultrapowers. The clear advantage of this new construction is that now the asymptotic cone of any metric space exists. But now one has to specify an ultrafilter and it is a priori not at all clear if (and how much) the resulting cone depends on this choice of ultrafilter.

Note at this point that the existence of non-principal ultrafilters (which are needed to give non-trivial cones) is a consequence of the axiom of choice and that it is impossible to give such an ultrafilter explicitly. However, it is still possible to show that in many cases the asymptotic cone of a group is unique, i.e independent of the choice of an ultrafilter. Easy examples are given by finite groups (here the cone is always a single point, since the group is bounded as a metric space) and infinite abelian groups (here the cone is isometric to Euclidean space with the l_1 metric). Cones of hyperbolic groups are always $2^{\aleph_0}$-universal trees, see for example [G_2], and any two such trees are isometric, so these cones are also unique.

The question whether in general the asymptotic cone of a group is independent of the choice of the ultrafilter remained open until 2000. In 1993 Gromov asked in [G_2] to give an example of a finitely generated (or better: finitely presented) group having two non-homeomorphic cones by choosing suitable ultrafilters. He also asked how many different cones such a group can have.

Thomas and Velickovic gave the desired example for a finitely generated group in 2000 in their paper [TV]. Their group is not finitely presentable, however, and it took another five years until an example of a finitely presented group with non-homeomorphic cones was given by Ol'Shanskii and Sapir in 2005 in the paper [OS].

The second question, how many different cones a finitely generated group can at most have, turned out to depend on the Continuum Hypothesis (CH), which is the question whether $\aleph_1 = 2^{\aleph_0}$. It can be shown that (CH) is independent of the usual (ZFC) axioms of set theory. In [KSTT] the authors considered the asymptotic cone of a uniform lattice in a Lie group of rank at least 2 and gave an algebraic description of this object. It turns out that the cone of such a lattice is unique when (CH) holds. On the other hand if one assumes the negation of (CH), then this lattice has $2^{2^{\aleph_0}}$ different cones. The authors also showed that the maximal cardinality of non-isometric asymptotic cones a finitely generated group can have is $2^{\aleph_0}$, provided (CH) holds.

In the same year, Druţu and Sapir constructed in [DS] a finitely generated group having this (under (CH) maximal possible) number of non-homeomorphic cones. Using small cancellation theory they constructed a group G, in the Cayley graph of which they could encode various sequences of graphs, all giving rise to asymptotic cones with different fundamental groups. Depending on which set of scaling factors an ultrafilter "selects", it is possible to distinguish the cones.

Overview

This thesis considers a slightly different question, although it is related to one of the questions of Gromov above. We consider iterated cones. Starting with any metric space (for example a finitely generated group with the word metric), taking the asymptotic cone gives another metric space, which is in some sense "more saturated". It is for example relatively easy to prove that the asymptotic cone of any space is always complete. The question now is what happens if one takes the asymptotic cone of the cone. Will this always be isometric to the original cone or are there cases in which something new can happen?

Again, looking at simple examples the answer seems to be that indeed nothing new can occur. If the original space is bounded, the cone will be a point, as will be all iterated cones. If the space is $\mathbb{R}^n$ (or any net in there, for example $\mathbb{Z}^n$ or $\mathbb{Q}^n$), the asymptotic cone will (again) be $\mathbb{R}^n$ as well as in any further iteration. Also any cone of a hpyerbolic group is a universal tree and here again iterating the process doesn't change anything.

In fact it turns out that the question of iterating cones is related to the question whether a given space can have different cones. One can show that the cone of a cone will also be a cone of the original space and from this it follows that if a space X has a unique cone independent of the choice of the ultrafilter, then all iterations of the cone will again be isometric to this unique cone of X.

On the other hand, it is possible to give examples of metric spaces with different iterated cones and this is the goal of this thesis.

In Section 1 we start with some preliminaries and define all notions we need. We also discuss the dependence of the cone on the data and gather some basic facts about asymptotic cones of metric spaces.

In section 2 we briefly consider cones of group extensions and give sufficient conditions under which the metric on a split extension is the product metric, even though the group itself might not be a direct product of groups. More precisely, we prove the following:

Theorem 1. *Let N and H be finitely generated groups and $G = N \rtimes H$. Fix generating sets S_N and S_H for N and H and consider the word metric on G given by the generating set $S_N \cup S_H$. Suppose that H acts on N by isometries. Then the metric on G is the product metric, in particular N is undistorted in G.*

In Section 3 we show that cones of scaling invariant spaces (like $\mathbb{R}^n$ or universal $\mathbb{R}$-trees) do not depend on the choice of the scaling factor and after that we give examples of metric spaces with different iterated cones. We show:

Theorem 2. *Let (Y, d) be a proper metric space. Then there is a proper metric space $(X, \overline{d})$ with basepoint $e \in X$ and a sequence of scaling factors (α_n), such that for any non-principal ultrafilter μ on $\mathbb{N}$, there is an isometry*

$$\mathrm{Cone}_\mu(X, e, \alpha) \cong Y.$$

This then allows us to construct spaces for which the sequence of iterated cones gets stationary only after a finite number of steps, specified before. We then use a different construction to give an example of a space with infinitely many non-homeomorphic iterated cones:

Theorem 3. *There exists a metric space X with basepoint e and a sequence of scaling factors α, such that for any ultrafilter μ and any natural numbers i, j with $i \neq j$ the iterated cones $\mathrm{Cone}^i_\mu(X, e, \alpha)$ and $\mathrm{Cone}^j_\mu(X, e, \alpha)$ are not homeomorphic.*

In Section 4 we use the space from Section 3 and the methods from Druţu and Sapir ([DS]) to find an example of a finitely generated group having an infinite chain of different iterated cones:

Theorem 4. *There exists a finitely generated group G and a sequence of scaling factors α, such that for every ultrafilter μ and every two numbers $i, j \in \mathbb{N}$ with $i \neq j$ we have*

$$\mathrm{Cone}^i_\mu(G, e, \alpha) \ncong \mathrm{Cone}^j_\mu(G, e, \alpha)$$

In order to do this we explain the notion of tree-graded spaces which was also introduced in [DS].

Finally, in Section 5 we answer a question of Druţu and Sapir from the same paper [DS]. All previously known examples of groups (or more general metric spaces) having different cones depended on using very fast growing sequences of natural numbers as scaling factors. It is, however, possible to avoid such sequences altogether by only allowing so-called "slow ultrafilters". Their question was if this restriction would be useful to get rid of these examples. Unfortunately the answer is no. We show that any asymptotic cone that can be realised with a very fast growing sequence is isometric to some cone using a slow ultrafilter:

Theorem 5. *Let A be a thin set and μ an ultrafilter containing A. Then there is a slow ultrafilter μ', such that for every pointed metric space (X, e), there is an isometry*

$$\varphi\colon \mathrm{Cone}_\mu(X, e, \omega) \to \mathrm{Cone}_{\mu'}(X, e, \omega).$$

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1 Preliminaries

1.1 Ultrafilters and ultraproducts

Definition 1.1. Let I be a set. A **filter** μ on I is a nonempty collection of subsets of I, such that for all subsets $A, B \subseteq I$ we have

i) $\emptyset \notin \mu$.

ii) $A \in \mu, A \subseteq B \Rightarrow B \in \mu$.

iii) $A, B \in \mu \Rightarrow A \cap B \in \mu$.

The set of all filters on I can be partially ordered by inclusion. It is easy to see that totally ordered subsets have upper bounds and therefore maximal filters exist by Zorn's lemma. Those are called **ultrafilters**. They can be characterized as follows: A filter μ is an ultrafilter if and only if

iv) For all $A \subseteq I$ either $A \in \mu$ or $I \setminus A \in \mu$.

An ultrafilter on I can also be regarded as a finitely additive probability measure on I, which only takes the values 0 and 1. We say that some property of elements of I holds **μ-almost everywhere** (μ-a.e.) if the set where it holds lies in μ.

Example 1.2. Let I be a set and $i \in I$ a point. Then the collection

$$\mu_i := \{A \subseteq I : i \in A\}$$

defines an ultrafilter on I. Such an ultrafilter is called **principal**.

Note that for finite sets I each ultrafilter is of this form. Non-principal ultrafilters on I exist if and only if I is infinite: Take the collection of all cofinite sets in an infinite I. This is a filter and therefore contained in an ultrafilter, which is non-principal since it contains no finite sets.

Definition 1.3. Let X and I be sets and μ a non-principal ultrafilter on I. The **ultrapower** of X with respect to μ and I is defined as

$$^*X := \prod_\mu X := \left(\prod_I X\right)/\sim$$

where $\sim$ is the equivalence relation on the product given by

$$\begin{aligned}(x_i)_{i\in I} \sim (y_i)_{i\in I} :\iff & \quad x_i = y_i \ \mu\text{-a.e.} \\ \iff & \quad \{i \in I : x_i = y_i\} \in \mu \quad \text{for } (x_i), (y_i) \in \prod_I X.\end{aligned}$$

An equivalence class modulo $\sim$ will be denoted by $[x_n]$.

The important thing to note here is that if X carries additional structure, this can be carried over to the ultraproduct ${}^*X$. Łoš's theorem (cf. [BS]) states that any first-order sentence true in X is also true in ${}^*X$. As an example consider, the field of **hyperreal numbers**.

Example 1.4. Fix a non-prinicipal ultrafilter μ on $\mathbb{N}$ and consider

$${}^*\mathbb{R} := \prod_\mu \mathbb{R}.$$

These are just sequences (x_n) of real numbers, where two sequences are identified if they agree μ-a.e. Then ${}^*\mathbb{R}$ carries the structure of an ordered field. It is clear that the set of sequences forms a ring (but not an integral domain) if addition and multiplication is defined componentwise. After the identification $\sim$ one really obtains a field: For any sequence (x_n) of real numbers either the set $\{n \in \mathbb{N} : x_n = 0\}$ is in μ or its complement. In the first case, $[x_n] = 0$ in ${}^*\mathbb{R}$ and in the second one can define

$$y_n := \begin{cases} x_n^{-1} & \text{if } x_n \neq 0 \\ 0 & \text{if } x_n = 0 \end{cases}$$

Then $(x_n \cdot y_n)$ is μ-a.e. equal to 1, so $[x_n]^{-1} = [y_n]$.

The order $<$ can also be transferred to make ${}^*\mathbb{R}$ into an ordered field, which is real closed but not archimedean.

The field $\mathbb{R}$ can be embedded into ${}^*\mathbb{R}$ by taking constant sequences. An element $x \in {}^*\mathbb{R}$ is called **finite** if there is some $C \in \mathbb{R}$ such that $|x| < C$, otherwise it is called **infinite**. Further x is called **infinitesimal** if $x \neq 0$ but $|x| < \varepsilon$ for all $\varepsilon > 0, \varepsilon \in \mathbb{R}$.

The set of all finite elements of ${}^*\mathbb{R}$ forms a local ring with the set of all infinitesimal elements as maximal ideal. The quotient is isomorphic to $\mathbb{R}$ and the projection map st is called the **standard part**.

Another way of looking at this is the following. Any finite element $x =$

$[x_n] \in {}^*\mathbb{R}$ corresponds to a bounded sequence (x_n) which has a unique limit with respect to μ, i.e. a number $a \in \mathbb{R}$ such that every neighbourhood of a contains μ-almost every element of the sequence (x_n). Write

$$\lim_{n,\mu} x_n = a \qquad \text{or simply} \quad \lim_{\mu} x_n = a$$

for this limit. Note that it depends a lot on the choice of μ. The bounded sequence $(-1)^n$ has μ-limit 1 or -1 depending on whether the set of even or the set of odd natural numbers lies in μ.

Taking the limit of a bounded sequence is the same as taking the standard part of a finite hyperreal number:

$$\operatorname{st}\big([x_n]\big) = \lim_{\mu} x_n \qquad \text{if } [x_n] \in {}^*\mathbb{R} \text{ is finite.}$$

For later use we also need the product of ultrafilters.

Definition 1.5. Let I be a set and μ and ν ultrafilters on I. Define the **product** $\mu \times \nu$ on the set $I \times I$ by saying that for $A \subseteq I \times I$ we have

$$A \in (\mu \times \nu) \iff \{i \in I : \{j \in I : (i,j) \in A\} \in \nu\} \in \mu.$$

If I is infinite, there is a bijection $\sigma\colon I \times I \to I$ and we may regard $\mu \times \nu$ again as an ultrafilter on I by taking the preimage of a subset of I under this bijection. Of course the resulting ultrafilter will then depend on the choice of σ.

Note that the product is not commutative in general, that is, if I is infinite and μ and ν are non-principal ultrafilters on I, we might have $\mu \times \nu \neq \nu \times \mu$.

Suppose that for each pair $i, j \in \mathbb{N}$, we have a number $x_{ij} \in \mathbb{R}$ and two ultrafilters μ and ν on $\mathbb{N}$. Then it is easy to see that

$$\lim_{i,\mu} \big(\lim_{j,\nu} x_{ij} \big) = \lim_{(i,j),\mu \times \nu} x_{ij}.$$

A proof can for example be found in [DS, Lemma 3.22].

1.2 Quasi isometries and coarse isometries

Definition 1.6. Let (X, d) be a metric space, $Y \subseteq X$ and $C \geq 0$ a real number. We say that Y is C**-dense** in X if for every $x \in X$ and $\varepsilon > 0$ there is a point $y \in Y$ such that $d(x, y) \leq C + \varepsilon$, or in other words if

$$\sup_{x \in X} d(x, Y) \leq C.$$

Definition 1.7. Let (X, d) and (Y, d') be metric spaces and let $f\colon X \to Y$ be a map. This map f is called a **quasi-isometry** between X and Y if there are real numbers $L \geq 1$ and $C \geq 0$ such that $f(X)$ is C-dense in Y and for all $x, x' \in X$ we have

$$\frac{1}{L}d(x,x') - C \leq d'\big(f(x), f(x')\big) \leq Ld(x,x') + C.$$

It is easy to see that if $f\colon X \to Y$ is a quasi-isometry, then there also exists a quasi-isometry $g\colon Y \to X$, such that $g \circ f$ and $f \circ g$ are of bounded distance to the identity. The composition of two quasi-isometries is again a quasi-isometry (with different constants) and therefore "being quasi-isometric" is an equivalence relation for metric spaces.

A quasi-isometry $f\colon X \to Y$ as above is called a **coarse isometry** if L can be chosen to be 1. (Recall that f is called a bi-Lipschitz map if C can be chosen equal to 0 and an isometry if both $L = 1$ and $C = 0$).

Note that a subset $Y \subseteq X$ of a metric space X is C-dense if and only if (by definition) the embedding is a coarse isometry.

1.3 Asymptotic Cones

We start by defining the asymptotic cone of an arbitrary (pseudo)-metric space and discuss to what extend it depends on the defining data. The ideas here are not new and can be found in [R] and [DS]. Recall that in a pseudo-metric space all the axioms for metric spaces are valid except for the possibility that two different points can have distance 0.
In what follows if X and Y are metric spaces, the notation $X \cong Y$ will mean that X and Y are isometric.

Definition 1.8. Let (X, d) be a pseudo-metric space, μ a non-principal ultrafilter on $\mathbb{N}$, (e_n) a sequence of points in X (the "sequence of base-points") and (α_n) a sequence of positive real numbers tending μ-almost surely to infinity (the "sequence of scaling factors"). Consider the ultrapower ${}^*X$, which is an ${}^*\mathbb{R}$-pseudo-metric space, which means that the metric takes values in the ordered abelian group ${}^*\mathbb{R}$. This ${}^*\mathbb{R}$ pseudo-metric will be denoted by ${}^*d$.

Set $e := [e_n] \in {}^*X$ and $\alpha := [\alpha_n] \in {}^*\mathbb{R}$. The metric ${}^*d/\alpha$ is again an ${}^*\mathbb{R}$ pseudo-metric on ${}^*X$. Consider now the following set:

$${}^*X_e^\alpha := \left\{ [x_n] \in {}^*X : \frac{{}^*d\big([x_n], [e_n]\big)}{\alpha} \text{ is finite in } {}^*\mathbb{R} \right\}.$$

As stated above, for any finite non-standard real number, one can take the standard part, which is real. This makes ${}^*X_e^\alpha$ into a pseudo-metric space. Identifying points with distance 0 gives the **asymptotic cone** of X: $\mathrm{Cone}_\mu(X, e, \alpha) := {}^*X_e^\alpha / \approx$, where

$$[x_n] \approx [y_n] \iff \frac{{}^*d\big([x_n], [y_n]\big)}{\alpha} \text{ is infinitesimal.}$$

We don't want to complicate the notation even more, so we will denote an equivalence class with respect to $\approx$ again by $[x_n]$. The metric d_∞ on $\mathrm{Cone}_\mu(X, e, \alpha)$ is defined by

$$d_\infty\big([x_n], [y_n]\big) := \mathrm{st}\left(\frac{{}^*d\big([x_n], [y_n]\big)}{\alpha}\right) = \lim_\mu \frac{d(x_n, y_n)}{\alpha_n}.$$

Note that d_∞ indeed defines a metric.

Remark 1.9. Saturation properties of ultrapowers guarantee that the asymptotic cone is always a complete metric space[1]. A more direct proof can, for example, be found in [vDW], Proposition 4.2.

Remark 1.10. Sometimes it is convenient to consider more general **μ-limits** of metric spaces. Let (X_n, d_n) be a sequence of metric spaces and consider a point $x = [x_n] \in \prod_\mu X_n$. Then the ultralimit of the X_n with basepoint x is defined as

$$\lim_\mu (X_n, x) := \left\{ [y_n] \in \prod_\mu X_n : \lim_\mu d_n(x_n, y_n) < \infty \right\}.$$

This again is turned into a metric space by identifying points of distance 0. In this light, the construction of the asymptotic cone refers to the special case of setting $(X_n, d_n) := (X, \frac{d}{\alpha_n})$.

Definition 1.11. For later use we also define **iterated asymptotic cones**. Fix a non-principal ultrafilter μ on $\mathbb{N}$ and an infinite hyperreal number α. Note that if Y is any metric space and $e \in {}^*Y$ a fixed basepoint, the asymptotic cone of Y will have a canonical basepoint given by the equivalence class of e, which we will denote by $\hat{e} \in \mathrm{Cone}_\mu(Y, e, \alpha)$. Then we set $\mathrm{Cone}_\mu^0(X, e, \alpha) := X$ and for $i \in \mathbb{N}$ set

$$\mathrm{Cone}_\mu^{i+1}(X, e, \alpha) := \mathrm{Cone}_\mu\left(\mathrm{Cone}_\mu^i(X, e, \alpha), \hat{e}, \alpha\right).$$

[1] An ultraproduct over a countable set is always $\aleph_1$-saturated, cf. [M], Exercise 4.5.37. A limit of a Cauchy sequence can be written as the realization of a type over a countable set and from this, the assertion follows directly. Note that the asymptotic cone itself is a quotient of a subset of the ultrapower and will not be saturated in general.

As indicated in the notation, the definition of the asymptotic cone depends on the choices of the ultrafilter μ, the sequence of base points e, and the sequence of scaling factors α. We want to discuss how severe these dependencies are. The first, almost obvious, observation is the following.

Lemma 1.12. *Let μ be a non-principal ultrafilter, $\alpha \in {}^*\mathbb{R}$ an infinite hyperreal number and let $\beta \in {}^*\mathbb{R}$ be a finite number. Let X be a metric space with basepoint $e \in X$. Then*

$$\mathrm{Cone}_\mu(X, e, \alpha) \cong \mathrm{Cone}_\mu(X, e, \alpha + \beta).$$

Proof. This is the following simple fact about real sequences. If x_n is any sequence of real numbers, α_n another sequence tending to infinity and β_n a μ-a.s. bounded sequence, such that $\frac{x_n}{\alpha_n}$ converges with respect to μ, then

$$\lim_\mu \frac{x_n}{\alpha_n + \beta_n} = \lim_\mu \frac{x_n}{\alpha_n}.$$

□

Definition 1.13. A metric space (X, d) is called **quasi-homogeneous** if the action of $\mathrm{Isom}(X)$ has a bounded fundamental domain in X. Put another way, (X, d) is quasi-homogeneous if $\mathrm{diam}\big(X/\,\mathrm{Isom}(X)\big) < \infty$. Recall that a metric space is called homogeneous if the isometry group acts transitively on the points of X.

Lemma 1.14. *Let (X, d) be a metric space, μ a non-principal ultrafilter on $\mathbb{N}$ and $\alpha \in {}^*\mathbb{R}$ a sequence of scaling factors as in Definition 1.8. Let $e = [e_n]$ and $e' = [e'_n]$ be two basepoints in ${}^*X$. If (X, d) is quasi-homogeneous, there exists an isometry*

$$\varphi\colon\ \mathrm{Cone}_\mu(X, e, \alpha) \to \mathrm{Cone}_\mu(X, e', \alpha)$$

mapping e to e'.

Proof. By assumption, there is a constant $C > 0$ and isometries $\varphi_n \in \mathrm{Isom}(X)$ such that

$$d\big(\varphi_n(e_n), e'_n\big) < C.$$

This induces a well-defined map $\varphi\colon\ \mathrm{Cone}_\mu(X, e, \alpha) \to \mathrm{Cone}_\mu(X, e', \alpha)$, which can be seen as follows. Let $x = [x_n] \in {}^*X_e^\alpha$, then

$$\begin{aligned}\frac{{}^*d\big(\varphi(x), e'\big)}{\alpha} = \frac{{}^*d\big([\varphi_n(x_n)], [e'_n]\big)}{\alpha} &\leq \frac{{}^*d\big([\varphi_n(x_n)], [\varphi_n(e_n)]\big)}{\alpha} + \frac{{}^*d\big([\varphi_n(e_n)], [e'_n]\big)}{\alpha} \\ &\leq \underbrace{\frac{{}^*d(x, e)}{\alpha}}_{\text{finite}} + \frac{C}{\alpha}.\end{aligned}$$

This shows that $\varphi(x) \in {}^*X^\alpha_{e'}$ and therefore the map φ is indeed a map from ${}^*X^\alpha_e$ to ${}^*X^\alpha_{e'}$. It is clear, that it is an isometry and is therefore a well-defined map on the cones. The calculation above also shows $\varphi(e) = e'$, because C/α is infinitesimal and therefore $d_\infty\big(\varphi(e), e'\big) = 0$. □

Remark 1.15. This proof also shows that for two basepoints $e, e' \in {}^*X$ with finite distance (in ${}^*\mathbb{R}$) the identity map induces an isometry between the cones. In particular, this is the case if e and e' are constant, i.e. points of X.

Convention: Assume from now on unless stated otherwise the basepoint to be one point $e \in X$ embedded into ${}^*X$ via a constant sequence.

The dependence on the ultrafilter μ and the scaling factor α is more crucial. There is an example of a metric space (X, d) having non-homeomorphic cones $\mathrm{Cone}_\mu(X, e, \alpha)$ and $\mathrm{Cone}_{\mu'}(X, e, \alpha)$, where μ and μ' are distinct ultrafilters on $\mathbb{N}$, see [TV]. The construction in this paper can be adapted to give an example of non-homeomorphic cones $\mathrm{Cone}_\mu(X, e, \alpha)$ and $\mathrm{Cone}_\mu(X, e, \beta)$ for different scaling factors α and β. Indeed, the choices of the ultrafilter and the sequence of scaling factors are interrelated. We will discuss the example of [TV] in greater detail in Example 1.19.

Definition 1.16. Let α_n be a sequence of positive real numbers tending to infinity. We say that this sequence has **bounded accumulation** if there is a number $N \in \mathbb{N}$, such that for all $r \in \mathbb{N}$ the set

$$S_r = \{n \in \mathbb{N} : \alpha_n \in [r, r+1[\} = \{n \in \mathbb{N} : \lfloor \alpha_n \rfloor = r\}$$

has less than N elements.

If μ is any ultrafilter on $\mathbb{N}$, we say that α has **μ-almost surely bounded accumulation** if there is a set $T \in \mu$, such that $|T \cap S_r|$ is uniformly bounded.

Remark 1.17. Since $\mathbb{N} \in \mu$ for all ultrafilters μ, we have that if α has bounded accumulation it also has μ-almost surely bounded accumulation for all μ. Moreover, if α has μ-almost surely bounded accumulation for some ultrafilter μ, then there exists a set $A' \in \mu$ such that for each $r \in \mathbb{N}$ the set

$$A' \cap S_r = \{n \in A' : \alpha_n \in [r, r+1[\} = \{n \in A' : \lfloor \alpha_n \rfloor = r\}$$

has at most one element. Indeed, by assumption exists an $N \in \mathbb{N}$ and a set $T \in \mu$ such that for all $r \in \mathbb{N}$ we have $|T \cap S_r| \leq N$. Therefore we can write T as a finite disjoint union

$$T = A_1 \dot\cup A_2 \dot\cup \ldots \dot\cup A_N$$

with $|A_i \cap S_r| \leq 1$ for each $i \leq N$ and each $r \in \mathbb{N}$. Because the union is disjoint, exactly one of the A_i lies in μ and this can be taken as A'.

In the following, denote the hyperreal number $[n]$ by ω. This is often used as the "standard" scaling sequence.

Proposition 1.18 ([R], Appendix B)**.** *Let (X,d) be a metric space, μ a non-principal ultrafilter on $\mathbb{N}$, e a basepoint in X and α a sequence of scaling factors. Then there exists a non-principal ultrafilter μ' on $\mathbb{N}$ and an isometric embedding $\varphi\colon \mathrm{Cone}_{\mu'}(X,e,\omega) \to \mathrm{Cone}_{\mu}(X,e,\alpha)$.*
If moreover α has μ-almost surely bounded accumulation, then φ is an isometry.

Proof. Define a map $\psi\colon \mathbb{N} \to \mathbb{N}$ by setting $\psi(n) := \lfloor \alpha_n \rfloor$. Indeed it is no loss of generality to assume $\alpha_n \in \mathbb{N}$ for all n, since $\mathrm{Cone}_{\mu}(X,e,\alpha)$ and $\mathrm{Cone}_{\mu}(X,e,\lfloor\alpha\rfloor)$ are isometric by Lemma 1.12.

Define the ultrafilter μ' as follows. For any subset $A \subseteq \mathbb{N}$ set

$$A \in \mu' :\iff \psi^{-1}(A) \in \mu.$$

It is clear that this defines a non-principal ultrafilter on $\mathbb{N}$.
For every $[x_n] \in \mathrm{Cone}_{\mu'}(X,e,\omega)$ set $\varphi\big([x_n]\big) := [x_{\psi(n)}]$. This is a well-defined map to $\mathrm{Cone}_{\mu}(X,e,\alpha)$. Let $[x_n] \in {}^*X_e^{\omega}$ be a representative of any point in $\mathrm{Cone}_{\mu'}(X,e,\omega)$ and consider $\varphi(x)$:

$$\frac{{}^*d\big(\varphi([x_n]),e\big)}{[\alpha_n]} = \frac{{}^*d\big(x_{\psi(n)},e\big)}{[\psi(n)]}.$$

This is a finite hyperreal with respect to μ, because

$$\frac{{}^*d\big([x_n],e\big)}{\omega} = \frac{{}^*d\big([x_n],e\big)}{[n]}$$

is by assumption a finite hyperreal with respect to μ'. This shows $\varphi(x) \in {}^*X_e^{\alpha}$.
We also have, for any two representatives $x = [x_n]$ and $y = [y_n]$ of points in $\mathrm{Cone}_{\mu'}(X,\omega,e)$:

$$\frac{{}^*d\big(\varphi([x_n]),\varphi([y_n])\big)}{[\alpha_n]} = \frac{{}^*d\big([x_{\psi(n)}],[y_{\psi(n)}]\big)}{[\psi(n)]}$$

The μ-limit of this number is the same as the μ'-limit of ${}^*d(x,y)/\omega$ and this shows that the map φ respects the distance and is therefore an isometric embedding. Since the asymptotic cone is a metric (not a pesudo-metric) space, it follows in particular that φ is injective.

Assume now that α has μ-almost surely bounded accumulation and consider again for each $r \in \mathbb{N}$ the set

$$S_r = \{n \in \mathbb{N} : \lfloor \alpha_n \rfloor = r\} = \psi^{-1}\big(\{r\}\big).$$

By assumption, there exists a set $A \subseteq \mathbb{N}$ with $A \in \mu$ and $|A \cap S_r| \leq 1$ for each $r \in \mathbb{N}$.

Consider then $A' := \psi(A)$. By construction we have $\psi^{-1}(A') = A$ and therefore $A' \in \mu'$. The inverse of φ can then be defined on the set of indices in A' and it follows that φ is surjective and therefore an isometry. □

We will now see what this proof shows in the particular example of Thomas and Velickovic.

Example 1.19. In the paper [TV], the authors give an example of a metric space (X, d) (in this case a finitely generated group G with the word metric) and two different asymptotic cones with respect to two distinct ultrafilters. In particular they prove the following:

There is a metric space (X, d) and two disjoint subsets $A, B \subseteq \mathbb{N}$ such that for any ultrafilter μ containing A, the cone $\mathrm{Cone}_\mu(X, e, \omega)$ is simply connected whereas for any ultrafilter μ' containing B, the cone $\mathrm{Cone}_{\mu'}(X, e, \omega)$ has non-trivial fundamental group. The cones are therefore non-homeomorphic.

Note that together with the proof of Proposition 1.18 we obtain the following. Let α be the sequence of scaling factors obtained by ordering the elements of A in the natural order and β the sequence obtained from the set B. Then the asymptotic cone $\mathrm{Cone}_\mu(X, e, \alpha)$ is simply connected for **any** ultrafilter μ, because by Proposition 1.18 it is isometric to $\mathrm{Cone}_{\mu'}(X, e, \omega)$ and μ' is an ultrafilter containing A by construction.

The same argument shows that $\mathrm{Cone}_\mu(X, e, \beta)$ has non-trivial fundamental group, again independently of the choice of the ultrafilter μ.

This example shows that it is not enough to simply fix the scaling factor and vary the ultrafilter to get all possible asymptotic cones.

1.4 Basic properties

For the rest of this section, fix a non-principal ultrafilter μ on $\mathbb{N}$ and a sequence of scaling factors $\alpha = [\alpha_n]$.

Lemma 1.20. *Let (X, d) and (Y, d') be metric spaces and $f\colon X \to Y$ a quasi-isometry with constants L and C. Fix a basepoint $e \in X$. Then the* *induced map*

$$\overline{f}\colon \operatorname{Cone}_\mu(X,e,\alpha) \to \operatorname{Cone}_\mu(Y,f(e),\alpha)$$

is a bi-Lipschitz map with Lipschitz constant L. In particular the cones are homeomorphic. If f is a coarse isometry then $\overline{f}$ is an isometry.

Proof. First of all, the map $\overline{f}$ is well-defined, because for every $[x_n] \in \operatorname{Cone}_\mu(X,e,\alpha)$ we know that ${}^*d\big([x_n],e\big)/\alpha$ is finite. It follows that

$$\frac{{}^*d'\big([f(x_n)],f(e)\big)}{\alpha} \leq \frac{L\cdot {}^*d\big([x_n],e\big)}{\alpha} + \frac{C}{\alpha}$$

is also finite and therefore defining $\overline{f}\big([x_n]\big) := \big[f(x_n)\big]$ gives a point in ${}^*Y^\alpha_{f(e)}$. It is then easy to check that $\overline{f}$ is a well-defined map to $\operatorname{Cone}_\mu(Y,f(e),\alpha)$.

Consider now points $[x_n],[x'_n] \in \operatorname{Cone}_\mu(X,e,\alpha)$. For each $n \in \mathbb{N}$ we have

$$\frac{1}{L}\cdot d(x_n,x'_n) - C \leq d'\big(f(x_n),f(x'_n)\big) \leq L\cdot d(x_n,x'_n) + C$$

and therefore

$$\frac{1}{L}\cdot {}^*d\big([x_n],[x'_n]\big) - C \leq {}^*d'\left(\overline{f}([x_n]),\overline{f}([x'_n])\right) \leq L\cdot {}^*d\big([x_n],[x'_n]\big) + C.$$

Dividing by α and taking the standard part yields

$$\frac{1}{L}\cdot d_\infty\big([x_n],[x'_n]\big) \leq d'_\infty\left(\overline{f}([x_n]),\overline{f}([x'_n])\right) \leq L\cdot d_\infty\big([x_n],[x'_n]\big)$$

since C/α is infinitesimal and has standard part 0. If f is a coarse isometry, then L may be taken equal to 1 and the result follows immediately. $\square$

Note, however, that not every map f between metric spaces X and Y lifts to the cone. If a slow growing sequence in X (one that represents a point in the cone of X) is mapped to a very fast growing sequence in Y, then there is no well defined lifting. However, there are certain properties a map f can have that guarantee that it can be lifted to the cone. One, being a quasi-isometry, we have seen above. Another one is given in the following.

Definition 1.21. Let (X,d_X) and (Y,d_Y) be metric spaces and $f\colon X \to Y$. We call f **distance non increasing** (or **dni-map**) if for all $x,x' \in X$ we have

$$d_Y\big(f(x),f(x')\big) \leq d_X(x,x').$$

Note that a dni-map may not be a quasi-isometric embedding, for example if X is unbounded and f is constant.

Lemma 1.22. *Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \to Y$ a dni-map. Fix a basepoint $e \in X$. Then f induces a map*

$$\overline{f}\colon \operatorname{Cone}_\mu(X, e, \alpha) \to \operatorname{Cone}_\mu(Y, f(e), \alpha)$$

which is continuous.

Proof. Define $\overline{f}([x_n]) := [f(x_n)]$ as usual. As in the proof of Lemma 1.20, it is easy to see that $\overline{f}$ is well-defined. Let us check that it is continuous. Fix $\varepsilon > 0$ and pick any point $[x_n] \in \operatorname{Cone}_\mu(X, e, \alpha)$. Then for any $[x'_n] \in \operatorname{Cone}_\mu(X, e, \alpha)$ with ${}^*d_X\big([x_n], [x'_n]\big) < \varepsilon$ we know

$${}^*d_Y\big([f(x_n)], [f(x'_n)]\big) \le {}^*d_X\big([x_n], [x'_n]\big) < \varepsilon$$

and therefore $\overline{f}$ is continuous. □

Corollary 1.23. *Let (X, d) be a metric space, $Y \subseteq X$ a C-dense subspace for some $C \geq 0$ and $e \in Y$ a basepoint. Consider Y as a metric space in its own right by restricting the metric d on X to Y. Then the natural embedding $\iota\colon Y \to X$ induces an isometry*

$$\overline{\iota}\colon \operatorname{Cone}_\mu(Y, e, \alpha) \to \operatorname{Cone}_\mu(X, e, \alpha).$$

Proof. The embedding $\iota\colon Y \to X$ is a coarse isometry whenever Y is C-dense in X and we can apply Lemma 1.20. □

Example 1.24. Consider $X = \mathbb{R}$ as a metric space with the usual metric $d(x, y) = |x-y|$ for $x, y \in \mathbb{R}$. Fix 0 as the basepoint. Then for any ultrafilter μ and any sequence of scaling factors α, we have

$$\operatorname{Cone}_\mu(\mathbb{R}, 0, \alpha) \cong \mathbb{R}.$$

This is easy to see: Consider the map sending $x \in \mathbb{R}$ to the point $x \cdot \alpha \in {}^*\mathbb{R}$. This gives an isometry.

Corollary 1.23 then implies:

$$\operatorname{Cone}_\mu(\mathbb{Z}, 0, \alpha) \cong \operatorname{Cone}_\mu(\mathbb{Q}, 0, \alpha) \cong \mathbb{R}.$$

Remark 1.25. Note that the reverse is not true in general: If $Y \subseteq X$ is not C-dense, then the asymptotic cones need not be different. The following example should illustrate this point.

Example 1.26. Take again $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$. Define the subset Y as follows:

$$Y' := \bigcup_{k\in\mathbb{N}} \big(k\mathbb{Z} \cap [e^{k-1}, e^k]\big) \cup \{0\} \qquad \text{and} \qquad Y := Y' \cup -Y'.$$

So two points in Y in the interval between e^{k-1} and e^k have distance k. The space Y is obviously not C-dense in X for all $C \geq 0$, because the distance of two consecutive points of Y can be arbitrarily large. Take 0 as the basepoint and $\alpha = (\alpha_n)$ as the sequence of scaling factors. Consider again the embedding $\iota : Y \to X$ and the induced isometric embedding $\bar{\iota}\colon \mathrm{Cone}_\mu(Y, 0, \alpha) \to \mathrm{Cone}_\mu(X, 0, \alpha)$.

This is in fact an isometry. Take $[x_n] \in \mathrm{Cone}_\mu(\mathbb{R}, 0, \alpha)$ (without loss of generality $x_n \geq 0$ for all $n \in \mathbb{N}$) and set $y_n := \sup\{y \in Y : y \leq x_n\}$. Then obviously $[y_n] \in \mathrm{Cone}_\mu(Y, 0, \alpha)$ and it is not hard to see that these points coincide: By assumption the sequence (x_n/α_n) is μ-a.e. bounded because it defines a finite hyperreal number, so it is no loss of generality to assume $x_n < \alpha_n \cdot M$ for some $M > 0$.

The construction of Y then implies $x_n - y_n \leq \ln(\alpha_n \cdot M)$ and therefore

$$d_\infty\big([x_n], [y_n]\big) = \lim_\mu \frac{x_n - y_n}{\alpha_n} \leq \lim_\mu \frac{\ln(\alpha_n) + \ln(M)}{\alpha_n} = 0$$

and this shows that $\bar{\iota}\big([y_n]\big) = [x_n]$, so $\bar{\iota}$ is a surjection.

Note that this example works independently of the choice of μ and α. The point here is that even though the distances between points of Y can grow arbitrarily large, this happens "too slowly". If you wanted to construct a sequence (x_n) which is not in the image of $\bar{\iota}$, it would grow too fast and not be a part of the asymptotic cone of X anymore.

Also note that this gives an example of non-quasi isometric spaces X and Y having isometric asymptotic cones, although only in our restricted setting of a constant basepoint.

1.5 Word metrics and the isometry group

We start with some general and well-known facts about word metrics and length functions on finitely generated groups. We include these basic definitions here for completeness and to fix notation.

Definition 1.27. Let G be a group.

- We call G **finitely generated** if there exists some finite subset $S \subseteq G$ such that every $g \in G$ can be written as a word composed of elements of S. We will always assume S to be symmetric, that is $S = S^{-1}$ and $e \notin S$.

- A metric d on G is called **left-invariant** if for all elements $g, h_1, h_2 \in G$ we have $d(gh_1, gh_2) = d(h_1, h_2)$.

- A **length function** on G is a function $l_G \colon G \to \mathbb{R}_+$ with the following properties for all $g, h \in G$.

  i) $l_G(g) = 0 \iff g = e$.
  ii) $l_G(g) = l_G(g^{-1})$.
  iii) $l_G(gh) \leq l_G(g) + l_G(h)$.

- Let G be finitely generated. The **word metric** on G with respect to a finite generating set S is defined as the length function l_G given by

$$l_G(g) := \min\{r \in \mathbb{N} : g = s_1 s_2 \cdots s_r \text{ with } s_i \in S\}.$$

Remark 1.28. It is easy to check that the word metric is a length function in the sense of the definition above. The reason why it is called a metric comes from the following correspondence.

Lemma 1.29. *Let G be a group. The set of all length functions on G is in 1:1 correspondence with the set of all left-invariant metrics on G.*

Proof. If l_G is a length function on G we can define a metric d by setting

$$d(g, h) := l_G(h^{-1}g).$$

It is easy to see that this gives rise to a left-invariant metric. On the other hand, if a left-invariant metric d is given, then $l_G(g) := d(g, e)$ clearly defines a length function. □

Remark 1.30. To identify length functions with left-invariant metrics is of course arbitrary, one could as easily use right-invariant metrics instead. Therefore it seems more natural to just work with length functions. Also in some cases it will be convenient to drop the assumption $l_G(g) \neq 0$ for all $g \neq e$. These more general length functions give rise to left-invariant pseudo-metrics.

Lemma 1.31. *Let G be a finitely generated group. Consider two finite generating sets S and T with corresponding word metrics l_S and l_T. Then these metrics are bi-Lipschitz equivalent, which means there is a constant $L \geq 1$, such that for all $g \in G$ we have*

$$\frac{1}{L} l_S(g) \leq l_T(g) \leq L \cdot l_S(g).$$

Proof. Every $s \in S$ can be written as some word in the elements of T. Since S is finite, the number

$$L_1 := \max\{l_T(s) : s \in S\}$$

is well defined. By definition of the word metric, for every $g \in G$ we have the inequality $l_T(g) \leq L_1 \cdot l_S(g)$.
Switching the roles of S and T, we find a constant $L_2 \geq 1$ with $l_S(g) \leq L_2 \cdot l_T(g)$. Taking $L := \max\{L_1, L_2\}$ gives the desired result. □

This shows that the word metric of a finitely generated group is unique up to bi-Lipschitz equivalence.

Lemma 1.32. *Let (X, d) be a metric space and consider the isometry group $G := \mathrm{Isom}(X)$ of X. Fix a non-principal ultrafilter μ on $\mathbb{N}$, a basepoint $e \in X$, and a sequence of scaling factors $\alpha \in {}^*\mathbb{R}$ as usual. Let $H := \mathrm{Isom}\big(\mathrm{Cone}_\mu(X, e, \alpha)\big)$. Then the action of G on X induces an action on the asymptotic cone and we have a homomorphism*

$$G \to \mathrm{Stab}_H(e).$$

Proof. This is clear since for any $\varphi \in G$ we have $d_\infty\big(\varphi(e), e\big) = 0$. □

Consider now the ultraproduct ${}^*G$ of $G = \mathrm{Isom}(X)$ and the length function l defined by

$$l(\varphi) := d\big(\varphi(e), e\big)$$

and its version ${}^*l$ on ${}^*G$. The induced pseudo-metric on G is a metric if and only if $\mathrm{Stab}_G(e) = \{\mathrm{id}_X\}$.

Lemma 1.33. *Let (X, d) be a metric space as above, $G = \mathrm{Isom}(X)$ its isometry group and μ, e, and α as usual. The set*

$$\Gamma := \left\{ \varphi \in {}^*G : \frac{{}^*l(\varphi)}{\alpha} \text{ is finite in } {}^*\mathbb{R} \right\}$$

is a subgroup of ${}^*G$. Moreover, an element $\varphi = [\varphi_n] \in {}^*G$ induces an isometry of $\mathrm{Cone}_\mu(X, e, \alpha)$ if and only if $\varphi \in \Gamma$.

Proof. By the properties of the length function, it follows easily that $\mathrm{id}_X \in \Gamma$, $\Gamma = \Gamma^{-1}$ and $\Gamma \cdot \Gamma \subseteq \Gamma$ and therefore Γ is a subgroup of ${}^*G$.

Consider now an element $\varphi = [\varphi_n]$ of Γ and $x = [x_n] \in \mathrm{Cone}_\mu(X, e, \alpha)$. Then

$$\begin{aligned}\frac{{}^*d\big(\varphi(x), e\big)}{\alpha} = \frac{{}^*d\big([\varphi_n(x_n)], e\big)}{\alpha} &\leq \frac{{}^*d\big([\varphi_n(x_n)], \varphi(e)\big)}{\alpha} + \frac{{}^*d\big(\varphi(e), e\big)}{\alpha} \\ &= \frac{{}^*d(x, e)}{\alpha} + \frac{{}^*l(\varphi)}{\alpha}.\end{aligned}$$

The right hand side is finite and therefore $\varphi(x) \in \mathrm{Cone}_\mu(X, e, \alpha)$. It is clear that φ is an isometry in this case.

On the other hand, if for some $\varphi \in {}^*G$ we have $\varphi(x) \in \mathrm{Cone}_\mu(X, e, \alpha)$ for all $x \in \mathrm{Cone}_\mu(X, e, \alpha)$, we know this in particular for $x = e$:

$$\frac{{}^*l(\varphi)}{\alpha} = \frac{{}^*d\big(\varphi(e), e\big)}{\alpha}$$

and the right hand side is finite and therefore $\varphi \in \Gamma$. □

Remark 1.34. Note that the map

$$\Gamma \to \mathrm{Isom}\,\big(\,\mathrm{Cone}_\mu(X, e, \alpha)\big)$$

is in general not an embedding. However, if $\varphi \in \Gamma$ acts trivially on the cone we have

$$\frac{{}^*l(\varphi)}{\alpha} \text{ is infinitesimal in } {}^*\mathbb{R}.$$

As an example, consider the isometry $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\varphi(a) = a+1$. This is a nontrivial element of Γ that fixes every point of the asymptotic cone. On the other hand, the isometry $\psi\colon \mathbb{Z} \to \mathbb{Z}$ given by $\psi(a) = -a$ has length 0 (for $e = 0$) and acts nontrivially on the cone.

It is easy to construct an example where the map given above is not onto. Consider the space X obtained by glueing an interval of length 1 with one endpoint to a point of $\mathbb{R}$. This space will have $\mathbb{Z}/2\mathbb{Z}$ as isometry group, but the isometry group of the cone (which is just $\mathbb{R}$, because X is coarsely isometric to $\mathbb{R}$) will be much larger.

An important class of metric spaces we consider is the class of finitely generated groups with the word metric. Those are always homogeneous as metric spaces and it is not hard to see that the cones will be as well. In fact we have the following lemma.

Lemma 1.35. *Let* (X, d) *be a metric space,* μ *any non-principal ultrafilter on* $\mathbb{N}$, $\alpha \in {}^*\mathbb{R}$ *infinite and* $e \in X$ *a basepoint. If* X *is quasi-homogeneous, then the space* $Y := \mathrm{Cone}_\mu(X, e, \alpha)$ *is homogeneous.*

Proof. Let $x, x' \in Y$ be any points represented by sequences $x = [x_n]$ and $x' = [x'_n]$. Because X is quasi-homogeneous, there is a constant $C > 0$ and isometries $\varphi_n \in \mathrm{Isom}(X)$ with the property $d\big(\varphi_n(x_n), x'_n\big) \leq C$. We have to check that $\varphi := [\varphi_n]$ induces an isometry of Y, because then $\varphi(x) = x'$ is clear. We have

$$\frac{{}^*l(\varphi)}{\alpha} = \lim_\mu \frac{d\big(\varphi_n(e), e\big)}{\alpha_n} \leq \lim_\mu \frac{d\big(\varphi_n(e), \varphi_n(x_n)\big)}{\alpha_n} + \frac{d\big(\varphi_n(x_n), x'_n\big)}{\alpha_n} + \frac{d(x'_n, e)}{\alpha_n}.$$

Since x and x' are points of the asymptotic cone, the right hand side is finite. By Lemma 1.33 the assertion follows. □

2 Cones of group extensions

In this section we want to state some small results about cones of group extensions. Consider the following setting: Suppose there is an exact sequence of groups

$$1 \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow 1.$$

Assume that N and H are finitely generated and fix some finite generating sets for these groups. Call the resulting word metrics d_N and d_H respectively. Then of course G is also finitely generated and one possible choice of a generating set is taking the image of the generating set of N under ι and fixing some preimages for the generators of H under π.

This will give a word metric on G given by a length function denoted l_G. Note that even though the group N can be considered as a subgroup of G, the induced metric $l_G|_N$ might be radically different from l_N. As usual we will denote the corresponding metrics by d_G and d_N.

Definition 2.1. In the situation above, the embedding ι of N into G is called **undistorted** if ι induces a quasi-isometry between the metrics d_N and $d_G|_N$.

Example 2.2. Let A be any 2×2 matrix with determinant 1, so $A \in \mathrm{SL}_2(\mathbb{Z}) = \mathrm{Aut}(\mathbb{Z}^2)$. Consider the group $G_A = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ is given by the matrix A.

If, for example, $A = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then the subgroup $N = \mathbb{Z}^2$ is undistorted in G_A. If on the other hand A has some real eigenvalue strictly bigger than 1, the embedding will not be a quasi-isometry anymore.

Proof. Consider a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ with an eigenvalue $\alpha \in \mathbb{R}$ with $\alpha > 1$. Of course A will then have α^{-1} as the second eigenvalue. Denote the word metric on $\mathbb{Z}^2$ with respect to the "standard" generating system by d. Further denote the l_1 norm on $\mathbb{R}^2$ by $|\cdot|_1$. Then of course we have for every vector $w \in \mathbb{Z}^2$:

$$d(w, 0) = |w|_1.$$

The restricted metric on $\mathbb{Z}^2$ will be denoted by $\tilde{d}$. We want to show that these two metrics are not quasi-isometric or, in other words, that the identity map is not a quasi-isometry with respect to these two metrics.

Fix a vector $v \in \mathbb{Z}^2$, such that v does not lie in the eigenspace for the eigenvector α^{-1}. Such a v clearly exists. For $n \in \mathbb{N}$ set $v_n := A^n v$, so $v = v_0$. Observe that for every $w \in \mathbb{Z}^2$ we have

$$\tilde{d}(Aw, 0) \leq \tilde{d}(w, 0) + 2$$

by the choice of the generating set of $G = \mathbb{Z}^2 \rtimes \mathbb{Z}$. Inductively it follows

$$\tilde{d}(v_n, 0) = \tilde{d}(A^n v, 0) \leq \tilde{d}(v, 0) + 2n.$$

The right hand side grows linearly in n. On the other hand if we fix two non-zero eigenvectors $x, y \in \mathbb{R}^2$ with eigenvalue α resp. α^{-1}, then this will form a basis of $\mathbb{R}^2$. Write $v = \lambda x + \lambda' y$. Since v is not an eigenvector with eigenvalue α^{-1} we know that $\lambda \neq 0$. We have

$$\begin{aligned} d(v_n, 0) = |A^n v|_1 = |\lambda A^n x + \lambda' A^n y|_1 &= |\alpha^n \cdot \lambda x + \alpha^{-n} \cdot \lambda' y|_1 \\ &\geq \alpha^n |\lambda x|_1 - \alpha^{-n} |\lambda' y|_1. \end{aligned}$$

Since $\lambda \neq 0$ we know that $|\lambda x|_1 \neq 0$ and therefore the right hand side grows exponentially in n. But this can't be if d and $\tilde{d}$ are quasi-isometric. □

Now we want to gather some facts about asymptotic cones of groups which are given as group extensions as above. A particularly nice case occurs when the cone of G can be seen as the product of the cones of N and H. In the following, given metric spaces (X, d_X) and (Y, d_Y) we call the l_1-metric on the product $X \times Y$ given by

$$d\big((x, y), (x', y')\big) := d_X(x, x') + d_Y(y, y')$$

the **product metric** with respect to the spaces X and Y.

Some results about cones of extensions have been obtained by Ol'Shanskii, Osin and Sapir in [OOS] who considered the case of central extensions. Their result is stated below. We start with some basic facts, which can also be found in [OOS]. We give our own proof here.

Proposition 2.3 (cf. [OOS], Theorem 5.2)**.** *In the situation above, for every ultrafilter μ and scaling sequence α the maps $\iota\colon N \to G$ and $\pi\colon G \to H$ induce two continuous maps $\bar{\iota}\colon \mathrm{Cone}_\mu(N, e, \alpha) \to \mathrm{Cone}_\mu(G, e, \alpha)$ and $\bar{\pi}\colon \mathrm{Cone}_\mu(G, e, \alpha) \to \mathrm{Cone}_\mu(H, e, \alpha)$.*
If N is undistorted in G, then $\bar{\iota}$ is injective. The map $\bar{\pi}$ is always a surjection and for each $h \in \mathrm{Cone}_\mu(H, e, \alpha)$ the fiber $\bar{\pi}^{-1}(h)$ is isometric to $\mathrm{Cone}_\mu(\iota(N), e, \alpha)$.

Proof. It is clear by the definition of the metric on G that both ι and π are dni-maps and therefore they induce well-defined continuous maps $\bar{\iota}$ and $\bar{\pi}$ on the respective cones by Lemma 1.22.
Assume now that N is undistorted in G. We want to show that $\bar{\iota}$ is injective. Being undistorted means that ι is a quasi-isometry between N and $\iota(N)$,

the latter being seen as a subset of G with the induced metric. Therefore by Lemma 1.20, we know that $\bar{\iota}$ is a bi-Lipschitz map onto its image and therefore it is injective.
To show that $\overline{\pi}$ is a surjection, take $h = [h_n] \in \mathrm{Cone}_\mu(H, e, \alpha)$ arbitrarily. For each h_n choose a preimage g_n under π of minimal word-length in G. By the choice of the generating set of G we then know that $d_G(g_n, e) = d_H(h_n, e)$ and this implies that the point $g := [g_n]$ lies in $\mathrm{Cone}_\mu(G, e, \alpha)$. This shows that $\overline{\pi}$ is a surjection.
Consider now the space $X_h \subseteq \mathrm{Cone}_\mu(G, e, \alpha)$ given by

$$X_h := \{[g_n x_n] \in \mathrm{Cone}_\mu(G, e, \alpha) : x_n \in \iota(N)\}.$$

This is precisely the preimage $\overline{\pi}^{-1}(h)$. We claim that the map

$$\varphi \colon X_h \to \mathrm{Cone}_\mu(\iota(N), e, \alpha)$$

given by $\varphi([g_n x_n]) := [x_n]$ is a well-defined map of cones and is indeed our required isometry. To check that it is well-defined, it is enough to see that the sequence $[x_n]$ is a valid element of the cone. By left-invariance of the metric d_G, we see that

$$\frac{d_G(x_n, e)}{\alpha_n} = \frac{d_G(g_n x_n, g_n)}{\alpha_n} \leq \frac{d_G(g_n x_n, e)}{\alpha_n} + \frac{d_G(g_n, e)}{\alpha_n}.$$

The right hand side has a finite limit with respect to μ and thus our sequence gives a well-defined point in the cone of $\iota(N)$. The same argument reversed shows that φ is a surjection. Finally, for any points $[g_n x_n], [g_n y_n] \in X_h$ we have

$$\frac{d_G(g_n x_n, g_n y_n)}{\alpha_n} = \frac{d_G(x_n, y_n)}{\alpha_n}$$

and this shows that φ is an isometry. □

The possible distortion of metrics presents problems when one tries to do induction proofs, which rely on the structure of solvable groups, to show something about the asymptotic cones of these groups. Suppose that, in the setting above, the cones of N and H are known, then there is very little chance to deduce anything about the structure of $\mathrm{Cone}(G)$ if N is distorted. It is however possible to get some results under certain assumptions.

Theorem 2.4 ([OOS], Theorem 5.6)**.** *Suppose N is a central subgroup of G and consider the induced metric $d_G|_N$. Set $H := G/N$ as above. Suppose that for some ultrafilter μ and scaling sequence α the space* $\mathrm{Cone}_\mu(H, e, \alpha)$ *is an* $\mathbb{R}$*-tree. Then* $\mathrm{Cone}_\mu(G, e, \alpha)$ *is bi-Lipschitz equivalent to* $\mathrm{Cone}_\mu(N, e, \alpha) \times \mathrm{Cone}_\mu(H, e, \alpha)$ *endowed with the product metric.*

We want to prove something similar here, in slightly different circumstances. We do not require the subgroup N to be central and we don't need the assumption that the cone of H is an $\mathbb{R}$-tree. However, we consider only split extensions and we have a strong assumption on the action of H on N.

Theorem 2.5. *Let G, N and H be groups from an exact sequence as before and assume that the extension is split, that is $G = N \rtimes H$. Assume further that the action of H on N by conjugation is isometric with respect to the induced metric $d_G|_N$, in other words assume that for all $n \in N$ and $h \in H$ we have $l_G(n) = l_G(hnh^{-1})$.*
Then N is undistorted in G (in fact $d_N = d_G|_N$ in this case) and the metric on G is the product metric.

Remark 2.6. Note that if N happens to be a central subgroup, the action of H on N will be trivial, in particular it will be isometric. On the other hand, central extensions which are split are just direct products of groups. A non-trivial example fulfilling the requirements of the theorem is $G = \mathbb{Z}^n \rtimes \mathbb{Z}$ where the action of $\mathbb{Z}$ on $\mathbb{Z}^n$ is given by a non-trivial permutation of the basis vectors.

Proof. Since the extension is split, we can regard both N and H as subgroups of G. Both these subgroups are finitely generated and the metrics on N and H are given by some fixed generating sets of N and H, respectively. The metric on G which we consider is the word metric with respect to the (disjoint) union of these generating sets.
We claim that each $g \in G$ has a unique reduced expression of the form $g = hn$ with $h \in H$ and $n \in N$. Again since our extension splits it is clear that there exists only one such expression. We have to show that it is reduced. Consider any reduced expression of g, say $g = n_1 h_1 n_2 h_2 \cdots$ with $n_i \in N$ and $h_i \in H$. For any elements $n \in N$ and $h \in H$, we know that since N is a normal subgroup there is an element $n' \in N$ with $nh = hn'$. Our assumption states that n and n' have the same word length in N. Therefore if we consider our reduced expression for g, whenever we do this operation of "pushing an h in front" (and thereby possibly changing the element of N) we will not increase the length of the expression. A priori there might occur some cancellation afterwards, but this can't happen, because we started with a reduced expression for g and any cancellation would result in a shorter word. Therefore we will not change the length of the expression if we put all the h_i in front. This means that we have a reduced expression $g = h_1 h_2 \cdots n_1' n_2' \cdots$ of the desired form. From this we directly see that the metric on G is simply the product metric with respect to the metrics on N and H (recall that in a semi-direct product, the underlying set is just the product of the two groups),

that is for any $g \in G$ uniquely written as $g = hn$ with $h \in H$ and $n \in N$, we have

$$l_G(g) = l_N(n) + l_H(h).$$

This proves that the metric on G is just the product metric and that N is undistorted, in fact $d_N = d_G|_N$. □

Remark 2.7. Of course being a product metric on the group lifts directly to the asymptotic cone. However this fact alone can be seen more easily since any group G satisfying the assumptions of the theorem is virtually a direct product: The subgroup of H acting trivially on N has finite index (since H has to map the finite set S of generators of N to itself, so the action factors through a finite group).

3 Iterated cones of metric spaces

3.1 Iterated cones are cones

The next question we want to address is the following. Suppose (X, d) is a metric space with basepoint $e \in X$. Given a non-principal ultrafilter μ on the natural numbers and a sequence of scaling factors α_n tending to infity, we can consider the iterated asymptotic cone $Y := \mathrm{Cone}^2_\mu(X, e, \alpha)$ of X. What can we say in relation to $\mathrm{Cone}_\mu(X, e, \alpha)$? Is the doublecone of X always homeomorphic or even isometric to the asymptotic cone? In general the answer is negative as we will show, but there are cases in which the quesion can be answered affirmatively.

First we need a general lemma, which can also be found in [DS], Section 3.2. We give our own proof here.

Lemma 3.1. *Let (X, d) be a metric space, $e \in X$ a basepoint and fix two non-principal ultrafilters μ and ν on $\mathbb{N}$ and sequences of scalars α and β. Then*

$$\mathrm{Cone}_\mu\left(\mathrm{Cone}_\nu(X, e, \alpha), \hat{e}, \beta\right) \cong \mathrm{Cone}_{\mu\times\nu}(X, e, \gamma),$$

where γ is an equivalence class of a sequence of real numbers indexed by $\mathbb{N} \times \mathbb{N}$ defined as

$$\gamma_{k,n} := \alpha_n \beta_k.$$

Proof. Elements on the left hand side are given by classes of sequences of points in the asymptotic cone of X (taken with respect to the ultrafilter ν and scaling sequence α). Denote such a sequence (indexed by $k \in \mathbb{N}$) by $[x_n]^{(k)}$.
Fixing k we get

$$\lim_{n,\nu} \frac{d(x_n^{(k)}, e)}{\alpha_n} < \infty.$$

This limit is precisely the distance from the point $[x_n]^{(k)}$ to $\hat{e}$ in $\mathrm{Cone}_\nu(X, e, \alpha)$. Therefore we have

$$\lim_{k,\mu} \frac{d_\infty\left([x_n]^{(k)}, \hat{e}\right)}{\beta_k} = \lim_{k,\mu}\left(\lim_{n,\nu} \frac{d(x_n^{(k)}, e)}{\alpha_n\beta_k}\right) = \lim_{(k,n),\mu\times\nu} \frac{d(x_n^{(k)}, e)}{\gamma_{k,n}} < \infty.$$

This shows that the obvious map from the left hand side to the right hand side is well-defined. Since distances are preserved, it is also injective. Surjectivity is also clear since every element of the right hand side is a sequence $y_{n,k}$ indexed over $\mathbb{N} \times \mathbb{N}$ and the same calculation shows that it gives an element of the left hand side. $\square$

Corollary 3.2. *Suppose X is a metric space with a fixed basepoint $e \in X$ and a unique asymptotic cone, that is the asymptotic cone $Y := \mathrm{Cone}_\mu(X, e, \alpha)$ is independent of the choice of μ and α. Then $\mathrm{Cone}_\mu(Y, \hat{e}, \alpha) \cong Y$ and therefore*

$$\mathrm{Cone}^n_\mu(X, e, \alpha) \cong \mathrm{Cone}_\mu(X, e, \alpha) \qquad \textit{for all } n \in \mathbb{N}.$$

Proof. This is a direct consequence of Lemma 3.1: Choose any bijection $\sigma\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. The Lemma implies

$$\mathrm{Cone}^2_\mu(X, e, \alpha) \cong \mathrm{Cone}_{\mu\times\mu}(X, e, \overline{\alpha}),$$

where $\overline{\alpha}_{k,n} := \alpha_n \alpha_k$. Applying the bijection σ we can regard $\mu \times \mu$ as an ultrafilter over $\mathbb{N}$ and $\overline{\alpha}$ as an element of ${}^*\mathbb{R}$. Since, by assumption, the cone of X does not depend on the choice of the ultrafilter or the scaling factor, the assertion follows directly. ☐

3.2 Scaling invariant spaces

In this section, we want to give a definition of scaling invariance for a space. We will prove that the asymptotic cones of scaling invariant spaces do not depend on the choice of the scaling factor and that cones of scaling invariant spaces are again scaling invariant. Finally we will show that homogeneous and complete trees have this property.

Definition 3.3. A metric space (X, d) with basepoint $e \in X$ is called **scaling invariant** with respect to e if there is a constant $C > 0$ and for every $n \in \mathbb{N}$ a map $\varphi_n\colon X \to X$ such that φ_n is an isometric embedding from (X, d) to $(X, d/n)$ with $d\big(e, \varphi_n(e)\big) \le C$ and such that the image of φ_n is C-dense in (X, d).

Remark 3.4. The assumption on the basepoint in the definition of scaling invariant spaces can be ignored if X is quasi-homogeneous, because then the isometry group of X can move any point in a C-neighbourhood of e for a suitable choice of C.

Example 3.5. For every $n \in \mathbb{N}$ the space $\mathbb{R}^n$ with the Euclidean metric is scaling invariant (with respect to any basepoint). Note that $\mathbb{Z}^n$ is not scaling invariant. It is true that homogeneous $\mathbb{R}$-trees arising as asymptotic cones are scaling invariant, cf. Proposition 3.10 below.

Lemma 3.6. *Fix an ultrafilter μ on $\mathbb{N}$, an infinite $\alpha = [\alpha_n] \in {}^*\mathbb{R}$ as a sequence of scaling factors, and a metric space (X, d). Assume X is scaling invariant with respect to some basepoint e. Then there is an isometric* *embedding*

$$\varphi\colon X \to \mathrm{Cone}_\mu(X, e, \alpha).$$

Suppose further that X is proper. Then φ is an isometry.

Proof. Without loss of generality we can assume $\alpha_n \in \mathbb{N}$ for all $\mathbb{N}$ (again by Lemma 1.12). By assumption, for every $n \in \mathbb{N}$ there is a map $\varphi_{\alpha_n}\colon X \to X$ with $d\big(\varphi_{\alpha_n}(e), e\big) \leq C$ and

$$d(x, x') = \frac{d\big(\varphi_{\alpha_n}(x), \varphi_{\alpha_n}(x')\big)}{\alpha_n} \qquad \text{for all } x, x' \in X.$$

Define now $\varphi\colon X \to \mathrm{Cone}_\mu(X, e, \alpha)$ by

$$\varphi(x) := \big[\varphi_{\alpha_n}(x)\big].$$

First observe that $\varphi(e) = \hat{e}$, where the right hand side means the image of the point e embedded using the constant sequence. This is clear, because

$$d_\infty\big(\varphi(e), \hat{e}\big) = \lim_\mu \frac{d\big(\varphi_{\alpha_n}(e), e\big)}{\alpha_n} \leq \lim_\mu \frac{C}{\alpha_n} = 0.$$

For all $x, x' \in X$, we have

$$d_\infty\big(\varphi(x), \varphi(x')\big) = \lim_\mu \frac{d\big(\varphi_{\alpha_n}(x), \varphi_{\alpha_n}(x')\big)}{\alpha_n} = \lim_\mu d(x, x') = d(x, x').$$

From this it follows at once that φ is well-defined and an isometric embedding with the property that for all $x \in X$ we have $d_\infty\big(\varphi(x), \hat{e}\big) = d(x, e)$.

Suppose now that X is proper and therefore complete. We want to show that φ is onto in that case. Let $x \in \mathrm{Cone}_\mu(X, e, \alpha)$ be an arbitrary point. Then x is represented by some sequence (x_n) in X and by scaling invariance, for every $n \in \mathbb{N}$ there is a point $y_n \in X$ such that

$$d\big(\varphi_{\alpha_n}(y_n), x_n\big) \leq C.$$

It follows that the sequence $\big(\varphi_{\alpha_n}(y_n)\big)$ also represents the point x in the cone, so it is no loss of generality to assume $x_n = \varphi_{\alpha_n}(y_n)$. We also know

$$d(y_n, e) = \frac{d\big(\varphi_{\alpha_n}(y_n), \varphi_{\alpha_n}(e)\big)}{\alpha_n} \leq \frac{d(x_n, e)}{\alpha_n} + \frac{d\big(e, \varphi_{\alpha_n}(e)\big)}{\alpha_n}.$$

The right hand side has a limit with respect to μ, since the first summand just converges to $d_\infty(x, \hat{e})$ and the second one tends to 0. Therefore the sequence

(y_n) is μ-almost surely bounded and can be assumed to lie in a compact set since X is proper. By completeness we therefore have a point $y \in X$ with $\lim_\mu y_n = y$.

It remains to show that $\varphi(y) = x$. But this is obvious since

$$d_\infty\big(\varphi(y), x\big) = \lim_\mu \frac{d\big(\varphi_{\alpha_n}(y), x_n\big)}{\alpha_n} = \lim_\mu \frac{d\big(\varphi_{\alpha_n}(y), \varphi_{\alpha_n}(y_n)\big)}{\alpha_n} = \lim_\mu d(y, y_n) = 0.$$

Therefore φ is an isometry. □

Proposition 3.7. *Let again μ be a fixed ultrafilter on $\mathbb{N}$ and (X, d) a scaling invariant metric space with respect to some basepoint e. Then the asymptotic cone of X is independent of the choice of the scaling factor, meaning that for all infinite $\alpha, \beta \in {}^*\mathbb{R}$ we have an isometry*

$$\psi\colon\ \mathrm{Cone}_\mu(X, e, \alpha) \to \mathrm{Cone}_\mu(X, e, \beta).$$

Proof. Assume again without loss of generality $\alpha, \beta \in {}^*\mathbb{N}$. Let $C \geq 0$ be the constant in the definition of scaling invariance for X. For $n \in \mathbb{N}$ we know that φ_{α_n} is an isometric embedding from (X, d) to $(X, d/\alpha_n)$ with C-dense image. For any $x \in X$ we can therefore choose a point $\overline{x}$ in the image of φ_{α_n} such that $d(x, \overline{x}) \leq C$. Take the unique point y in the pre-image of $\overline{x}$ and denote the function $x \mapsto y$ by φ'_{α_n}. Note that φ'_{α_n} is close to an inverse of φ_{α_n} in the following sense: For any $x \in X$ we have

$$d\big((\varphi_{\alpha_n} \circ \varphi'_{\alpha_n})(x), x\big) \leq C.$$

Then we can define

$$\psi_n\colon X \to X \qquad \psi_n(x) := (\varphi_{\beta_n} \circ \varphi'_{\alpha_n})(x).$$

The map $\psi\colon\ \mathrm{Cone}_\mu(X, e, \alpha) \to \mathrm{Cone}_\mu(X, e, \beta)$ is then given by

$$\psi\big([x_n]\big) := \big[\psi_n(x_n)\big] \qquad \text{for } [x_n] \in \mathrm{Cone}_\mu(X, e, \alpha).$$

Again we first show that the basepoint $\hat{e}$ (this makes sense in both cones) is mapped to the basepoint. Set $e'_n := \varphi'_{\alpha_n}(e)$. We know that $d\big(\varphi_{\alpha_n}(e'_n), e\big) \leq C$. Then we see

$$\begin{aligned}
\frac{d\big(\psi_n(e), e\big)}{\beta_n} &= \frac{d\big(\varphi_{\beta_n}(e'_n), e\big)}{\beta_n} \leq \frac{d\big(\varphi_{\beta_n}(e'_n), \varphi_{\beta_n}(e)\big)}{\beta_n} + \frac{d\big(\varphi_{\beta_n}(e), e\big)}{\beta_n} \\
&\leq d(e'_n, e) + \frac{C}{\beta_n} = \frac{d\big(\varphi_{\alpha_n}(e'_n), \varphi_{\alpha_n}(e)\big)}{\alpha_n} + \frac{C}{\beta_n} \\
&\leq \frac{d\big(\varphi_{\alpha_n}(e'_n), e\big)}{\alpha_n} + \frac{d\big(e, \varphi_{\alpha_n}(e)\big)}{\alpha_n} + \frac{C}{\beta_n} \leq \frac{2C}{\alpha_n} + \frac{C}{\beta_n}.
\end{aligned}$$

The μ-limit of the right hand side is equal to 0, so the points $\psi(\hat{e})$ and $\hat{e}$ coincide in $\mathrm{Cone}_\mu(X,e,\beta)$. Next we consider two points $x = [x_n]$ and $x' = [x'_n]$ in $\mathrm{Cone}_\mu(X,e,\alpha)$ and their images under ψ. Denote $y_n := \varphi'_{\alpha_n}(x_n)$ and $y'_n := \varphi'_{\alpha_n}(x'_n)$.

$$\frac{d\big(\psi_n(x_n),\psi_n(x'_n)\big)}{\beta_n} = \frac{d\big(\varphi_{\beta_n}(y_n),\varphi_{\beta_n}(y'_n)\big)}{\beta_n} = d(y_n,y'_n).$$

Therefore the distance between $\psi(x)$ and $\psi(x')$ in $\mathrm{Cone}_\mu(X,e,\beta)$ is given by the μ-limit of $d(y_n,y'_n)$. The triangle inequality gives us

$$\begin{aligned} d(y_n,y'_n) &= \frac{d\big(\varphi_{\alpha_n}(y_n),\varphi_{\alpha_n}(y'_n)\big)}{\alpha_n} \\ &\le \frac{d\big(\varphi_{\alpha_n}(y_n),x_n\big)}{\alpha_n} + \frac{d(x_n,x'_n)}{\alpha_n} + \frac{d\big(x_n,\varphi_{\alpha_n}(y'_n)\big)}{\alpha_n} \\ &\le \frac{2C}{\alpha_n} + \frac{d(x_n,x'_n)}{\alpha_n}. \end{aligned}$$

By assumption the right hand side is equal to the distance between the points x and x' in $\mathrm{Cone}_\mu(X,e,\alpha)$. In particular the μ-limit of $d(y_n,y'_n)$ exists. A similar argument shows that it must be equal to the right hand side and therefore ψ is an isometric embedding and well-defined, since it respects the basepoints.

It remains to show that ψ is onto. Let $x = [x_n] \in \mathrm{Cone}_\mu(X,e,\beta)$ be an arbitrary point. For each x_n there is a point $x'_n \in X$ such that $d\big(\varphi_{\beta_n}(x'_n),x_n\big) \le C$. Set $y_n := \varphi_{\alpha_n}(x'_n)$ and $y := [y_n]$. It is easy to check that y is a point in $\mathrm{Cone}_\mu(X,e,\alpha)$. It remains to show that $\psi(y) = x$:

$$\begin{aligned} \frac{d\big(\psi_n(y_n),x_n\big)}{\beta_n} &= \frac{d\big(\varphi_{\beta_n}(\varphi'_{\alpha_n}(y_n)),x_n\big)}{\beta_n} \\ &\le \frac{d\big(\varphi_{\beta_n}(\varphi'_{\alpha_n}(y_n)),\varphi_{\beta_n}(x'_n)\big)}{\beta_n} + \frac{d\big(\varphi_{\beta_n}(x'_n),x_n\big)}{\beta_n} \\ &\le d\big(\varphi'_{\alpha_n}(y_n),x'_n\big) + \frac{C}{\beta_n} = \frac{d\big(\varphi_{\alpha_n}(\varphi'_{\alpha_n}(y_n)),\varphi_{\alpha_n}(x'_n)\big)}{\alpha_n} + \frac{C}{\beta_n} \\ &= \frac{d\big(\varphi_{\alpha_n}(\varphi'_{\alpha_n}(y_n)),y_n\big)}{\alpha_n} + \frac{C}{\beta_n} \le \frac{C}{\alpha_n} + \frac{C}{\beta_n}. \end{aligned}$$

The μ-limit of the right hand side is 0 and therefore $\psi(y) = x$ in $\mathrm{Cone}_\mu(X,e,\beta)$ and ψ is an isometry. □

Corollary 3.8. *Let μ and α be as usual and let (X,d) be scaling invariant with respect to some basepoint $e \in X$. Then $\mathrm{Cone}_\mu(X,e,\alpha)$ is also scaling invariant.*

Proof. For every $n \in \mathbb{N}$, the space $\mathrm{Cone}_\mu(X, e, \alpha)$ scaled by the factor $1/n$ is nothing but $\mathrm{Cone}_\mu(X, e, n \cdot \alpha)$. The proposition shows that there is an isometry between these spaces which fixes the basepoint and therefore we see that $\mathrm{Cone}_\mu(X, e, \alpha)$ is scaling invariant. The constant C can in this case be chosen equal to 0. □

Next we want to show that important examples of asymptotic cones are scaling invariant. We will discuss the case of homogeneous $\mathbb{R}$-trees here.

Definition 3.9. Recall some definitions for a metric space (X, d).

- The space is called **geodesic** if for every two points $x, y \in X$, there is a closed interval $[a, b] \subseteq \mathbb{R}$ and an isometric embedding $\varphi\colon [a, b] \to X$ with $\varphi(a) = x$ and $\varphi(b) = y$.
- A geodesic space is called **uniquely geodesic** if the image of φ above is unique. Denote it by $[x, y]$ if this is the case.
- A uniquely geodesic space is called an $\mathbb{R}$**-tree** if for any two geodesics $[x, y]$ and $[y, z]$ with $[x, y] \cap [y, z] = \{y\}$ we have $[x, y] \cup [y, z] = [x, z]$.
- If X is an $\mathbb{R}$-tree and $x \in X$ any point, the **valency** at x is defined as the cardinality of $\pi_0\big(X \backslash \{x\}\big)$.
- If X is an $\mathbb{R}$-tree and κ a cardinal with $\kappa \geq 2$, then X is called κ**-universal** if the valency at every point of X is equal to κ and if every $\mathbb{R}$-tree, in which the valency of the points is bounded by κ, embeds isometrically into X.

It is known that for every cardinal $\kappa \geq 2$ there exists a κ-universal tree and it is unique up to isometry, cf. [DP].

Proposition 3.10. *Let X be a homogeneous and complete $\mathbb{R}$-tree. Then X is κ-universal for some κ and scaling invariant.*

Proof. The characterisation in [MNO], Theorem 3.5, shows that a homogeneous, complete $\mathbb{R}$-tree is universal. This is because the valency at every point has to be the same (since the isometry group acts transitively) and completeness ensures that each embedding of $[0, a)$ for any $a \in \mathbb{R}_+$ can be extended to an embedding of $[0, \infty)$ into X.

For every $n \in \mathbb{N}$, the metric space $(X, d/n)$ is again an $\mathbb{R}$-tree, it is of course complete and homogeneous. By the argument above it is universal. Since universal trees are unique up to isometry, we can find an isometry between (X, d) and $(X, d/n)$, which proves scaling invariance. □

The importance of this observation lies in the fact that the asymptotic cone of any hyperbolic group is a $2^{\aleph_0}$-universal tree (cf. [D_1], Theorem 3.A.7). In this case we see that the cone is unique up to isometry, in particular it does not depend on the choice of μ and α. Further we see that iterating the cone won't change anything in this case.

3.3 Proper spaces as asymptotic cones

We will now proceed to show that many different spaces can arise as asymptotic cones. In particular the following theorem holds.

Theorem 3.11. *Let (Y, d) be a proper metric space. Then there is a proper metric space $(X, \overline{d})$ with basepoint $e \in X$ and a sequence of scaling factors (α_n), such that for any non-principal ultrafilter μ on $\mathbb{N}$, there is an isometry*

$$\mathrm{Cone}_\mu(X, e, \alpha) \cong Y.$$

Remark 3.12. Sisto obtained a similar result independently in [S] with a different construction. In his proof, the space X will be geodesic if Y is (which is not true for our construction). However, he does not use a fixed basepoint e but a changing sequence of basepoints. We found it more useful to provide an example even in the more restricted setting of a fixed basepoints.

Proof. Set $\alpha_n := n!$ and fix any non-principal ultrafilter μ on $\mathbb{N}$. Choose any point $e \in Y$ as basepoint. For $n \geq 2$, consider the following subset of Y:

$$Y_n := \left\{ y \in Y : d(y, e) \in \left[\frac{1}{\log n}, \log n \right] \right\} \cup \{e\}.$$

This is a closed subset of Y and therefore itself a complete metric space. Rescale the metric on Y_n by $n!$ and call the resulting space X_n, i.e. for $x, x' \in X_n$ we have $\overline{d}(x, x') = n! \cdot d(x, x')$.

Now define the space X as the union of the spaces X_n amalgamated along the common basepoint e. For $x \in X$ with $x \neq e$ write $x < X_n$ if $x \in X_k$ for some $k < n$ and similarly write $x > X_n$ if $x \in X_k$ for some $k > n$.

Consider now the asymptotic cone of X with respect to α, μ and the basepoint e. Suppose $[x_n]$ is any point in this cone represented by a sequence in X. Then there are three cases:

Case 1: We have μ-almost surely $x_n < X_n$. Then

$$\lim_\mu \frac{\overline{d}(x_n, e)}{\alpha_n} \leq \lim_\mu \frac{(n-1)! \log(n-1)}{n!} = \lim_\mu \frac{\log(n-1)}{n} = 0.$$

In this case (x_n) is equivalent to the constant sequence given by the basepoint.

Case 2: We have μ-almost surely $x_n \in X_n$, see below.

Case 3: We have μ-almost surely $x_n > X_n$. But then

$$\lim_\mu \frac{\overline{d}(x_n, e)}{\alpha_n} \geq \lim_\mu \frac{(n+1)!}{\log(n+1)n!} = \lim_\mu \frac{n+1}{\log n} = \infty.$$

In this case the sequence does not give a point in the asymptotic cone, contradicting the assumption.

This shows that any point in the asymptotic cone which is different from the basepoint must fulfill the condition of case 2 above.

Now let $y \in Y$ be an arbitrary point. For $n \in \mathbb{N}$ define

$$\varphi_n(y) := \begin{cases} y & \text{if } \frac{1}{\log(n)} \leq d(y, e) \leq \log(n) \\ e & \text{otherwise.} \end{cases}$$

Then $\varphi_n(y) \in X_n$ and for all $y \in Y$, there is a natural number N, such that for all $n \geq N$ we have $\varphi_n(y) = y$. Now define a map $\varphi \colon Y \to \mathrm{Cone}_\mu(X, e, \alpha)$ by setting $\varphi(y) := [\varphi_n(y)]$. The basepoint e of Y is then mapped to the class of the constant sequence $[e]$. By construction, this map is an isometric embedding, for if $y, y' \in Y$ are arbitrary points we have

$$\overline{d}_\infty\big(\varphi(y), \varphi(y')\big) = \lim_\mu \frac{\overline{d}\big(\varphi_n(y), \varphi_n(y')\big)}{\alpha_n} = \lim_\mu \frac{n! \cdot d(y, y')}{n!} = d(y, y').$$

We now have to prove that φ is a surjection to get the required isometry. For this let $[x_n]$ be an arbitrary point of the cone represented by a sequence (x_n) in X. We may assume that this sequence is not equivalent to the basepoint. By the above condition we know that μ-almost surely we have $x_n \in X_n$. Regarding the points x_n as points in Y, we get the inequality

$$\lim_\mu d(x_n, e) = \lim_\mu \frac{n! \cdot d(x_n, e)}{n!} = \lim_\mu \frac{\overline{d}(x_n, e)}{\alpha_n} < \infty$$

since the point is by assumption in the cone. It follows that the sequence (x_n) is μ-a.s. bounded in Y. Since Y is proper and therefore complete, there is a limit y of this sequence with respect to μ. And since

$$\overline{d}_\infty\big([x_n], \varphi(y)\big) = \lim_\mu \frac{\overline{d}(x_n, y)}{\alpha_n} = \lim_\mu \frac{n! \cdot d(x_n, y)}{n!} = \lim_\mu d(x_n, y) = 0$$

the point $\varphi(y)$ is equivalent to (x_n) and therefore φ is a surjection.

It remains to show that X is again a proper metric space. First observe that each of the spaces X_n is compact, since it is a rescaled version of a closed subset of the closed ball with radius $\log n$ around e in the proper space Y. Moreover, the basepoint e in each of the X_n is isolated and has distance at least $\frac{n!}{\log n}$ from any other point in X_n. Since this grows with n, it is clear that any closed ball with fixed radius around any point in X only meets finitely many of the X_n and can therefore be seen as a finite union of compact sets, which is compact itself. □

Corollary 3.13. *Let (Y, d) be a proper metric space. Then for each number $k \in \mathbb{N}$ there is a proper metric space $(X^{(k)}, \overline{d})$ with basepoint $e \in X$ and a sequence of scaling factors (α_n), such that for any non-principal ultrafilter μ on $\mathbb{N}$ there is an isometry*

$$\mathrm{Cone}^k_\mu(X^{(k)}, e, \alpha) \cong Y.$$

Proof. Since the metric space $(X, \overline{d})$ from Theorem 3.11 is again proper, the process can be iterated. □

Remark 3.14. Instead of proving Theorem 3.11 and Corollary 3.13 for fixed scaling factor and all ultrafilters, we could have stated that there is a non-principal ultrafilter μ, such that the theorem is valid for the scaling factor ω by Proposition 1.18.

Applying Corollary 3.13 to any proper space Y which is not isometric to its cone (for example $Y = \mathbb{Z}$), we obtain examples of spaces with different iterated cones up to a certain $k \in \mathbb{N}$ by using Corollary 3.13.

3.4 Infinite iteration

Next, we want to give an example of a metric space X having infinitely many pairwise non-homeomorphic iterated cones, which means that for every $i \neq j$ the space $\mathrm{Cone}^i_\mu(X, e, \alpha)$ is not homeomorphic to $\mathrm{Cone}^j_\mu(X, e, \alpha)$.
In order to obtain an example like this, we use a trick of Bowditch from [B]. The idea is to encode infinite 0-1-sequences into the space. I am grateful to A. Sisto for suggesting this method.

Definition 3.15. A set $A \subseteq \mathbb{N}$ given by $A = \{\alpha_1 < \alpha_2 < \alpha_3 < \cdots\}$ is called **thin** if

$$\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \infty.$$

Note that the set $\{n! : n \in \mathbb{N}\}$ which was used as the sequence of scaling factors in Theorem 3.11 is an easy example of a thin set.

Definition 3.16. Fix a sequence $(a_k)_{k\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$. We will encode this sequence in a metric space X, called **bullseye space** associated to the sequence (a_k). Define X to be the union of circles in $\mathbb{R}^2$ with radii 2^k, $k \in \mathbb{Z}$ all centered at the origin and add the origin to the space as a basepoint, called e. For each $k \in \mathbb{Z}$, connect the circle of radius 2^k to the circle of radius 2^{k+1} with an interval if and only if $a_k = 1$. These intervals are called **bridges**.
For later use it will be convenient if the space X is connected and has no global cut-point, therefore we add a line through the origin, disjoint from the bridges. We will consider X with the induced path metric, so X will be a geodesic metric space.
We want to be able to distinguish better between bullseye spaces up to homeomorphism and for this we will add **markings** and call the resulting space a **marked bullseye space**: The line we added can be seen as two rays from the origin to infinity. On one of the rays we put discs on every interval between two circles, rescaled in such a way that the space stays scaling invariant for powers of 2. On the other ray we do the same for 3-dimensional balls. See Figure 1 for a visualization of the space X. Small triangles stand for the discs and small squares stand for 3-dimensional balls. The dot is the origin.

Lemma 3.17. *Fix a sequence (a_k) as above and a thin set*

$$A = \{\alpha_1 < \alpha_2 < \cdots\} \subseteq \{2^n : n \in \mathbb{N}\}.$$

Set $\alpha := [\alpha_n]$ and fix any ultrafilter μ. Denote the bullseye space associated to (a_k) by X. Then $\mathrm{Cone}_\mu(X, e, \alpha)$ will be isometric to a bullseye space associated to the sequence (b_k) given by

$$b_k = \lim_\mu a_{\alpha_n + k}.$$

Proof. Disregarding the bridges it is clear by construction that the space X is scaling invariant, if rescaled with center e by any power of 2. Therefore the asymptotic cone of X will again be a bullseye space. It will have a bridge between the circle of 2^k and 2^{k+1} if and only if the set of rescaled spaces having a bridge between 2^{α_n+k} and 2^{α_n+k+1} has measure 1 with respect to μ. This proves the assertion. □

We want to be able to distinguish the spaces corresponding to sequences we use and for this we need a good invariant.

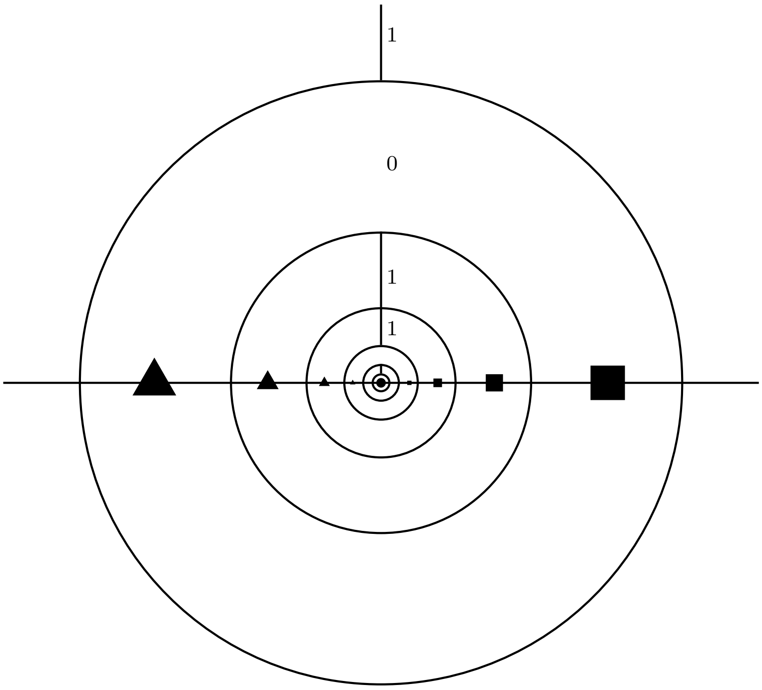

Figure 1: A bullseye space.

Definition 3.18. Let (a_k) be a sequence as above. The **asymptotic density** of (a_k) is defined as

$$\operatorname{adn}(a_k) := \limsup_{n\to\infty} \frac{1}{2n+1} \cdot \sum_{k=-n}^{n} a_k.$$

Lemma 3.19. *Let (a_k) be a sequence as above with a well-defined asymptotic density. Fix $N \in \mathbb{N}$ and consider the shifted sequence $b_k := a_{k+N}$. Then*

$$\operatorname{adn}(b_k) = \operatorname{adn}(a_k).$$

Proof. For $n > N$ we have

$$\left|\frac{1}{2n+1}\left(\sum_{k=-n}^{n} a_k - \sum_{k=-n+N}^{n+N} a_k\right)\right| = \left|\frac{1}{2n+1}\left(\sum_{k=-n}^{-n+N} a_k - \sum_{k=n}^{n+N} a_k\right)\right| \leq \frac{N}{2n+1}$$

and this tends to 0 for fixed N and $n \to \infty$, therefore $\operatorname{adn}(a_k) = \operatorname{adn}(a_{k+N})$. □

We can now state and prove the main theorem for this section.

Theorem 3.20. *There exists a metric space X with basepoint e and a sequence of scaling factors α, such that for any ultrafilter μ and any natural numbers i, j with $i \neq j$ the iterated cones* $\mathrm{Cone}^i_\mu(X, e, \alpha)$ *and* $\mathrm{Cone}^j_\mu(X, e, \alpha)$ *are not homeomorphic.*

Proof. Fix any ultrafilter μ on $\mathbb{N}$. Take again a thin set $A = \{\alpha_1 < \alpha_2 < \cdots\} \subseteq \{2^n : n \in \mathbb{N}\}$ and $\alpha := [\alpha_n]$. For any sequence $(a_k) \in \{0,1\}^{\mathbb{Z}}$, call the numbers a_k with $k \in [\alpha_n - n, \alpha_n + n]$ for some $n \in \mathbb{N}$ the **variable part** of the sequence. Its complement will be called the **fixed part**. Note that it is no loss of generality to assume $\alpha_1 \gg 0$ and to assume further that the intervals given above are disjoint.
For every $i \in \mathbb{N}$, $i \neq 0$, define now a sequence $(a_k^{(i)})$ with $\mathrm{adn}(a_k^{(i)}) = 1/i$. Note that since the set A is thin, the density will still be defined and will have the same value if you modify the sequence $(a_k^{(i)})$ on the variable part, since the relative amount of the variable part in any given interval of the form $[-n, n]$ in the sequence tends to 0 as n goes to infinity.
Next, modify the sequence $(a_k^{(1)})$ in such a way that the cone of the bullseye space X associated to $(a_k^{(1)})$ will be the bullseye space associated to $(a_k^{(2)})$. By Lemma 3.17, it is enough to modify $(a_k^{(1)})$ on the variable part, not changing its density.
Then iterate this process, modifying the variable part of $(a_k^{(i)})$ in such a way that the cone of the bullseyespace associated to this sequence is the bullseye space associated to $(a_k^{(i+1)})$. This change has to be reflected in all the $(a_k^{(j)})$ with $j < i$ as well. Since by assumption $\alpha_1 \gg 0$, this process yields a well-defined limit sequence $(a_k^{(i)})$, because every fixed entry in any given sequence is modified only finitely many times.
Now, define the space X as the marked bullseye space associated to the sequence $(a_k^{(1)})$. It is easy to see that if two marked bullseye spaces are homeomorphic, they correspond to the same underlying sequence, up to a shift, because the rays can only be sent to the same rays using a homeomorphism. From this, it follows that for any numbers $i, j \in \mathbb{N}$ with $i \neq j$, the spaces $\mathrm{Cone}^i_\mu(X, e, \alpha)$ and $\mathrm{Cone}^j_\mu(X, e, \alpha)$ can't be homeomorphic by Lemma 3.19, since the underlying sequences have different asymptotic densities. □

Remark 3.21. With the same method it is possible to construct spaces X with non-trivial periodic iterated cones. Fix any $m \geq 2$ and choose pairwise different numbers $d_1, d_2, \ldots, d_m \in [0, 1]$. For any $j \leq m$ fix a sequence $(a_k^{(j)})$ with asymptotic density d_j. Using the same method as in the proof above, you can find a marked bullseye space X, such that the underlying sequence of the bullseye space $\mathrm{Cone}^i_\mu(X, e, \alpha)$ corresponds to the sequence $(a_k^{(j)})$ whenever $i \equiv j \pmod{m}$.

3.5 A space with uncountably many cones

Using ideas from the previous section, we want to give an example of a metric space having $2^{\aleph_0}$ many pairwise non-homeomorphic asymptotic cones. This is not a new result, the paper [DS] gave an example of a finitely generated group with this property, which is much harder to do. Nevertheless we think that this simpler example in the context of metric spaces is interesting in its own right.

Definition 3.22. Let $(a_k)_{k\in\mathbb{Z}}$ be a sequence in $\{0,1\}^{\mathbb{Z}}$ as above. This sequence is called **rich** if it contains every finite sequence of the numbers 0 and 1 in its positive part $(a_k)_{k\in\mathbb{N}}$.

Clearly, rich sequences exist since there are only countably many finite sequences. Also note that a rich sequence will contain every given finite sequence infinitely many times, since any given finite sequence can be extended in infinitely many ways to different longer finite sequences.

Proposition 3.23. *There exists a metric space X with basepoint $e \in X$ and a set $\mathcal{U}$ of ultrafilters over $\mathbb{N}$ with $|\mathcal{U}| = 2^{\aleph_0}$ such that for every $\mu, \mu' \in \mathcal{U}$ with $\mu \neq \mu'$ the spaces $\mathrm{Cone}_\mu(X, e, \omega)$ and $\mathrm{Cone}_{\mu'}(X, e, \omega)$ are not homeomorphic.*

Proof. Let $(a_k)_{k\in\mathbb{Z}}$ be a rich sequence and (X, e) the marked bullseye space associated to it. For every $t \in [0,1]$ choose a sequence $(a_k^{(t)})_{k\in\mathbb{Z}}$ with $\mathrm{adn}(a_k^{(t)}) = t$.
Fix $t \in [0,1]$. We will construct a set of indices $i_1^{(t)} < i_2^{(t)} < i_3^{(t)} < \dots$ in $\mathbb{N}$, such that for all $n \in \mathbb{N}$ and $l \in \mathbb{Z}$ with $-n \leq l \leq n$ we have

$$a_{i_n^{(t)}+l} = a_l^{(t)}.$$

This is clearly possible since (a_k) is rich and these finite sequences all occur infinitely often. Let μ_t be an ultrafilter containing the set $\{i_1^{(t)}, i_2^{(t)}, \dots\}$. By construction, it follows that the space $\mathrm{Cone}_{\mu_t}(X, e, \omega)$ is again a marked bullseye space with associated sequence $(a_k^{(t)})$. Set $\mathcal{U} := \{\mu_t : t \in [0,1]\}$. Since all the resulting sequences have different asymptotic densities, the claim follows. □

Remark 3.24. In [KSTT, Theorem 1.10], the authors showed that $2^{\aleph_0}$ is the maximal number of asymptotic cones a finitely generated group can have, provided the continuum hypothesis (CH) is true. However, their proof does not use the group structure at all and works exactly the same way for arbitrary metric spaces. So, even in the more general context of arbitrary metric spaces there can only be $2^{\aleph_0}$ different cones, provided (CH) holds.

4 Iterated cones of groups

4.1 Tree-graded spaces

To give the desired example of a finitely generated group with countably many pairwise non homeomorphic iterated asymptotic cones, we use a result by Druţu and Sapir from [DS]. To state and explain this result, we first need the notion of tree-graded spaces, which is, to my knowledge, due to them.

Definition 4.1 ([DS], Definition 2.1)**.** Let X be a complete geodesic metric space and let $\mathcal{P}$ be a collection of closed geodesic subsets, called **pieces**, which cover the space X. We say that X is **tree-graded** with respect to $\mathcal{P}$ if

(T1) The intersection of any two different pieces is either empty or a single point.

(T2) Every simple geodesic triangle in X is contained in one piece.

For later use, we state some basic properties of tree-graded spaces.

Lemma 4.2 ([DS], Lemma 2.6, Corollary 2.11)**.** *Let X be tree-graded with respect to a collection of pieces $\mathcal{P}$. For every $x \in X$ and every piece $P \in \mathcal{P}$, there exists a unique point $p_x \in \mathcal{P}$, called the* **projection** *from x onto P, such that $d(x, P) = d(x, p_x)$. Moreover, if A is a connected subset of X (for example another piece), which intersects the piece P in at most one point, then A projects onto P in a unique point.*

Remark 4.3. For a collection of many other interesting properties of tree-graded spaces, see again the paper [DS], Section 2. Note that the set of pieces is in general not unique, meaning that a metric space X can be tree-graded with respect to many different sets of pieces. As an easy example, every geodesic metric space X is tree graded with respect to $\mathcal{P} = \{X\}$. On the other hand if X is an $\mathbb{R}$-tree without endpoints, then it is tree-graded with respect to the set of pieces containing all singletons in X. Also note that it is always possible to add any number of pieces, each containing just a single point, and the properties will remain valid. To avoid this, one usually assumes that a piece can't be completely contained in another piece. This assumption will only rule out superfluous singletons.

We will need another result of Druţu and Sapir, stating that while the set of pieces is in general not unique, in some cases one can characterize a minimal set of pieces.

Definition 4.4. Let X be a metric space which is tree-graded with respect to two sets of pieces called $\mathcal{P}$ and $\mathcal{P}'$. Write $\mathcal{P} \prec \mathcal{P}'$ if for every $A \in \mathcal{P}$ there is a piece $A' \in \mathcal{P}'$ such that $A \subseteq A'$. Note that this defines a partial order.

Definition 4.5. Let X be a geodesic metric space. A point $x \in X$ is called a **global cut-point** of X if the space $X \backslash \{x\}$ is not path connected.

Lemma 4.6 ([DS], Lemma 2.31)**.** *Let X be a complete geodesic space containing at least two points and let $\mathcal{C}$ be a non-empty set of global cut-points of X. There exists a smallest set of pieces $\mathcal{P}$ for X (with respect to $\prec$), such that X is tree-graded with respect to $\mathcal{P}$ and any piece in $\mathcal{P}$ is either a singleton or a set P with no global cut-point from $\mathcal{C}$. Moreover, the intersection of any two pieces from $\mathcal{P}$ is either empty or a point from $\mathcal{C}$.*

Definition 4.7 (cf. [DS], Definition 3.19)**.** Let X be a metric space with basepoint $e \in X$. Fix an ultrafilter μ on $\mathbb{N}$ and a scaling sequence α. Let $\mathcal{A}$ be a collection of subsets of X. Then for every sequence (A_n) of sets in $\mathcal{A}$, the set

$$\mathrm{Cone}_\mu\big((A_n), e, \alpha\big) := \{[x_n] \in \mathrm{Cone}_\mu(X, e, \alpha) : x_n \in A_n\}$$

is a (possibly empty) subset of the asymptotic cone of X. We say that X is **asymptotically tree-graded** with respect to $\mathcal{A}$ if $\mathrm{Cone}_\mu(X, e, \alpha)$ is tree-graded with respect to the set of pieces

$$\{\mathrm{Cone}_\mu\big((A_n), e, \alpha\big) : (A_n)_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}\}.$$

Here, we identify those pieces which coincide and disregard those which are empty.

Remark 4.8. Druţu and Sapir gave a characterization of being asymptotically tree-graded in terms of geometric properties of X with respect to $\mathcal{A}$ in Theorem 4.1 of [DS]. These do not depend on the choice of the ultrafilter μ and therefore if X is asymptotically tree-graded with respect to a collection of subsets $\mathcal{A}$ for some choice of μ, this will be true for every ultrafilter, cf. Corollary 4.30 in [DS].

Now we need an easy lemma, which I was unable to find in the literature. I include my own proof here, although the result is probably known to the experts.

Lemma 4.9. *Let X be a geodesic metric space, which is tree-graded with respect to a collection of pieces $\mathcal{P}$. Then X is also asymptotically tree-graded with respect to the same set of pieces.*

Proof. Fix an ultrafilter μ on $\mathbb{N}$ and a scaling sequence α as well as a base-point $e \in X$. Since the pieces from $\mathcal{P}$ cover X, it is clear that for every $[x_n] \in \mathrm{Cone}_\mu(X, e, \alpha)$ there is a sequence of pieces P_n such that $x_n \in P_n$ and therefore $[x_n] \in \mathrm{Cone}_\mu\big((P_n), e, \alpha\big)$, so the desired set of pieces covers the asymptotic cone of X.
We now have to verify the properties (T1) and (T2) for the asymptotic cone. For this, let (P_n) and (P'_n) be two sequences of pieces from $\mathcal{P}$, such that $\mathrm{Cone}_\mu\big((P_n), e, \alpha\big)$ is different from $\mathrm{Cone}_\mu\big((P'_n), e, \alpha\big)$; in particular $P_n \neq P'_n$ μ-almost surely. Suppose the intersection of the coned pieces is non-empty and choose $x = [x_n]$ and $y = [y_n]$ in this intersection. We take representatives such that $x_n \in P_n$ and also $y_n \in P_n$ for every $n \in \mathbb{N}$. We have to show that $x = y$.
By definition, we know that we can also find representatives $x = [x'_n]$ and $y = [y'_n]$ with $x'_n, y'_n \in P'_n$ for every $n \in \mathbb{N}$. For μ-almost every $n \in \mathbb{N}$, we know that the piece P_n intersects P'_n in at most one point, since X is tree-graded and the pieces are different. By Lemma 4.2, we find points $p'_n \in P'_n$ which are the unique projection from both, x_n and y_n, onto P'_n.
We know that

$$\lim_\mu \frac{d(x_n, y_n)}{\alpha_n} \leq \lim_\mu \frac{d(x_n, p'_n) + d(y_n, p'_n)}{\alpha_n} \leq \lim_\mu \frac{d(x_n, x'_n) + d(y_n, y'_n)}{\alpha_n} = 0.$$

This proves $x = y$ as desired. So any two pieces in the cone intersect in at most one point and this shows property (T1) for the cone.
For (T2), we use Corollary 4.18 in [D_2], which says that it is enough to check those simple geodesic triangles in the cone which are given as limits of simple geodesic triangles in the space X. Since X is tree-graded with respect to $\mathcal{P}$, we know that every such triangle lies in a single piece, and therefore the limit will lie in one of the sets we consider. This proves (T2). □

4.2 Infinite iteration for groups

We are now ready to state one of the main results of Druţu and Sapir in [DS]. They constructed a finitely generated group G which has $2^{\aleph_0}$ many pairwise non-homeomorphic asymptotic cones. Their construction will allow us to give an example of a group with infinitely many different iterated cones. We start with some technical definitions.

Definition 4.10. By a **metric graph** we mean a graph of finite valency where each edge has a positive weight. The metric on the set of vertices is then given by the length of the shortest path connecting two vertices, where the length of each edge is given by its weight.

Throughout the section, fix a number $\zeta \in (0,1)$. Consider a sequence (Γ_n) of finite metric graphs such that the following holds for $n \in \mathbb{N}$ sufficiently large:

- The weights of the edges of Γ_n are at least ζ^n and at most $\zeta^{\lfloor \frac{n}{2} \rfloor}$.
- The diameter of Γ_n is at most $10n$.

Let E_n be the number of edges in Γ_n and denote the vertex set of Γ_n by V_n. Fix a basepoint $O_n \in V_n$ for every $n \in \mathbb{N}$.

We will also need a certain function $\kappa \colon \mathbb{N} \to \mathbb{N}$ in the construction which is given by small cancellation properties of groups. For our purpose, we do not need to know the exact definition of κ; all we need is the fact that $\kappa(n)$ tends to infinity for $n \to \infty$. A precise definition of κ can be found in [DS, Proposition 7.13]. The reason why we need this function is because Theorem 4.12 below works with it.

Definition 4.11 ([DS], Definition 7.14)**.** An increasing sequence (α_n) of positive numbers is called **fast increasing** with respect to the sequence of graphs (Γ_n) if

i) For every $i \geq \lfloor \zeta^n \alpha_n \rfloor$ we have $\kappa(i) \geq E_n$.

ii) $\lim_{n\to\infty} \frac{\zeta^n \alpha_n}{\alpha_{n-1}} = \infty$.

iii) $\lim_{n\to\infty} \frac{E_n}{\zeta^n \alpha_n} = 0$.

Since $\kappa(n)$ tends to infinity, we know that such sequences exist. By ii), it is obvious that such a sequence will also be thin in the sense of Definition 3.15. We can now state the result of the construction in [DS].

Theorem 4.12 ([DS], Proposition 7.26 and Proposition 7.27)**.** *Let (Γ_n) be a sequence of graphs as above and let (α_n) be a fast increasing sequence with respect to these graphs. Then there exists a group G with 2 generators and for every $n \in \mathbb{N}$ a subset $\mathfrak{R}_n$ of G, such that for every ultrafilter μ, the asymptotic cone* $\mathrm{Cone}_\mu(G, e, \alpha)$ *is tree-graded with respect to the collection of pieces*

$$\mathcal{P} := \left\{ \mathrm{Cone}_\mu \left(g_n \mathfrak{R}_n, e, \alpha \right) : [g_n] \in \prod_\mu G \textit{ with } \lim_\mu \frac{d(e, g_n \mathfrak{R}_n)}{\alpha_n} < \infty \right\}$$

and different elements $[g_n]$ *correspond to different pieces of* $\mathcal{P}$*. Further consider the set*

$$\mathcal{X} := \left\{ \lim_{\mu}(V_n, x) : x \in \prod_{\mu} V_n \right\}.$$

Then every space in one of these collections is isometric with basepoint to a space in the other collection. Here the basepoint for each space in $\mathcal{P}$ *is the canonical basepoint* $\hat{e}$ *in the asymptotic cone of the group and the basepoint of* $\lim_\mu(V_n, x)$ *is just given by* x*. Moreover, every space in the second collection is isometric to* $2^{\aleph_0}$ *many spaces in the first collection.*

Some short remarks about the proof of the above theorem: The group G is constructed with a set of relations satisfying a small cancellation property. This means that different relators (written as words in the generators) can only have very small subwords in common, compared to their length, so when they are multiplied, only very “small cancellation” can happen, hence the name.

These properties give a considerable amount of freedom when “shaping” the Cayley graph of the group G. The graphs Γ_n are then prescribed in the Cayley graph of G, each rescaled (using the sequence α_n) to appear in the graph in such a way that they do not “interfere” with each other and such that the asymptotic cone of G carries the desired tree-graded structure with respect to ultralimits of the graphs with different observation points.

Also note that the isometries from the second part can be made explicit. In particular, when we consider the metric spaces up to isometry we can identify certain pieces, which will prove to be useful later.

We will now construct a suitable sequence of graphs for our purposes, again following [DS], Section 7.1.

Definition 4.13. Let X be a metric space and $\delta > 0$. A subset $A \subseteq X$ is called δ**-seperated** if for all $a, b \in A$ with $a \neq b$ we have $d(a, b) \geq \delta$. A subset $N \subseteq X$ is called a δ**-net** if for every $x \in X$, there is a point $y \in N$ with $d(x, y) < \delta$. A maximal δ-seperated set A is obviously a δ-net, which we will call a δ**-snet**.

Let X be a proper geodesic metric space, $e \in X$ a fixed basepoint and $\zeta \in (0, 1)$ as above. Let $B_n := \{x \in X : d(x, e) \leq n\}$ be the closed ball of radius n around e. Consider an increasing sequence of subsets

$$\{e\} \subseteq V_1 \subseteq V_2 \subseteq \dots$$

such that V_n is a ζ^n-snet in B_n. For every $n \in \mathbb{N}$, consider the finite graph Γ_n having vertex set V_n and an edge of length $d(x,y)$ between $x, y \in V_n$ if and only if

$$d(x,y) \leq \zeta^{\lfloor \frac{n}{2} \rfloor}.$$

Fix an ultrafilter μ on $\mathbb{N}$.

Lemma 4.14 ([DS], Lemma 7.5 (3))**.** *The spaces* $\lim_\mu(V_n, e)$ *(with the metric from* Γ_n*),* $\lim_\mu(\Gamma_n, e)$ *and* X *with basepoint* e *are isometric as pointed spaces.*

Now in order to use the Theorem 4.12 for our purpose, we consider a marked bullseye space X like the one constructed in Theorem 3.20 having infinitely many non-homeomorphic iterated cones. Fix again $\zeta \in (0,1)$ as above and define the sequence of graphs Γ_n to be an increasing sequence of nets like in the construction above. Then fix a sequence (α_n) which is fast increasing for these graphs. Note that it is possible to choose (α_n) independent of the underlying sequence of the bullseye space X, the number and position of bridges will not matter. Also note that X is indeed proper.

Theorem 4.15. *There exists a finitely generated group* G *and a sequence of scaling factors* α*, such that for every ultrafilter* μ *and every two numbers* $i, j \in \mathbb{N}$ *with* $i \neq j$ *we have*

$$\mathrm{Cone}^i_\mu(G, e, \alpha) \not\cong \mathrm{Cone}^j_\mu(G, e, \alpha)$$

Proof. Take X as above from Theorem 3.20, using the sequence α which increases fast to determine the underlying sequence. As was mentioned above, this will work since the α does not depend on the underlying sequence of X. Theorem 4.12 then gives a finitely generated group G which is asymptotically tree-graded. Fix an ultrafilter μ and set $C := \mathrm{Cone}_\mu(G, e, \alpha)$.
By construction, the space C is tree-graded with X as a piece. Theorem 4.12 gives a description of the pieces and X will be among them by Lemma 4.14. Since X is connected without any cut points, we can use Lemma 4.6 to recover X as a piece from the isometry class of C. In C we also have other pieces coming from cones of X with varying sequences of basepoints. To recover the underlying sequence of our bullseye space and its asymptotic density, we observe that for every possible sequence of observation points $x \in {}^*X$ we are in one of two cases: Either x has finite distance to the constant sequence e in the rescaled ${}^*\mathbb{R}$ metric (in which case the cone with basepoint x is isometric to X by Remark 1.15) or $d(x_n, e)/\alpha_n$ is μ-almost surely unbounded. But then the cone with respect to this x can only contain 2-dimensional or 3-dimensional parts (coming from the markings of X)

but not both. In particular such a piece cannot be isometric to any bullseye space, nor can any iterated cone be isometric to a bullseye space.
Therefore we are able to recover the asymptotic density of the underlying sequence of X. Now we iterate the process, taking the asymptotic cone of C, say C'. By Lemma 4.9, C' will again be tree-graded and the pieces will be cones of pieces, in particular the cone of X will occur as a piece. Since all iterated cones of X are pairwise non-homeomorphic, in particular the iterated cones of G will be pairwise non-homeomorphic as well. □

5 Slow ultrafilters

In this section, we want to answer a question asked by Druţu and Sapir in [DS]. In order to state this question and to explain its relevance, we need a definition.

Definition 5.1. Let $A = \{a_1 < a_2 < a_3 < ...\} \subseteq \mathbb{N}$. We call A **fast** if $\lim \frac{a_n}{n} = \infty$. Recall that A is called thin if $\lim \frac{a_n}{a_{n+1}} = 0$.

Remark 5.2. It is easy to see that every thin set is fast. The converse is not true, the set $A = \{2^n : n \in \mathbb{N}\}$ is an example of a fast set which is not thin.

Lemma 5.3. *The collection $\mathcal{S}$ of all cofinite sets together with complements of fast sets forms a filter.*

Proof. Since subsets of fast sets are clearly fast, it remains to show that the union of two fast sets is again fast, which is a simple calculation. □

Any ultrafilter extending the filter from the lemma will be called slow. So by construction, a slow ultrafilter can not contain any fast set, therefore it can't contain any thin set.

Question 5.4 (cf. [DS], Problem 1.20)**.** *Are there finitely generated groups with two bi-Lipschitz non-equivalent cones using slow ultrafilters?*

The relevance of this question lies in the fact that almost all examples of groups (or metric spaces in general) having different asymptotic cones rely on sequences of scaling factors which are thin when seen as subsets of $\mathbb{N}$. Equivalently, this means that these cones are formed using the standard scaling sequence ω and ultrafilters containing thin sets. This "thinness" is needed to have enough room for construction, as can be seen in the example in [TV] and also in the construction of the group in [DS]. So the hope is to be able to get rid of these unwanted examples by just using slow ultrafilters. If these ultrafilters yielded unique cones, one would just have to change the definition of asymptotic cones and then this space would not depend on the chosen ultrafilter.

Unfortunately this can't be done; in other words the answer to the above question is "Yes". In fact, the next theorem shows that any construction for cones that can be realised with an ultrafilter containing a thin set (like all examples considered above) can also be done with a slow ultrafilter.

Theorem 5.5. *Let A be a thin set and μ an ultrafilter containing A. Then there is a slow ultrafilter μ', such that for every pointed metric space (X, e), there is an isometry*

$$\varphi\colon \operatorname{Cone}_\mu(X, e, \omega) \to \operatorname{Cone}_{\mu'}(X, e, \omega).$$

Proof. Fix the thin set $A = \{a_1 < a_2 < a_3 < \ldots\}$. For every $L > 1$ and $n \in \mathbb{N}$, set

$$X_{L,a_n} := \left[\frac{1}{L}a_n, La_n\right] \cap \mathbb{N} \qquad \text{and for } I \subseteq A \text{, set} \qquad X_{L,I} := \bigcup_{a_n \in I} A_{L,a_n}$$

Since A is thin, these intervals will be disjoint for large n and fixed L, so it is no loss of generality to assume that this is always a disjoint union by getting rid of finitely many parts. We will first show that for any infinite $I \subseteq A$ the set $X_{L,I}$ is not fast and neither is its complement.

First note that an infinite set $X \subseteq \mathbb{N}$ is fast if and only if

$$\lim_{\substack{x \to \infty \\ x \in X}} \frac{|X \cap [1, x-1]|}{x} = 0. \tag{*}$$

In the set $X_{L,I}$, consider a subsequence of elements of the form La_n for $a_n \in I$. Then

$$\frac{|X_{L,I} \cap [1, La_n - 1]|}{La_n} \geq \frac{La_n - \frac{1}{L}a_n - 1}{La_n} = 1 - \frac{1}{L^2} - \frac{1}{La_n}.$$

Since $L > 1$, this will be bounded away from 0 for $n \to \infty$. Therefore, this subsequence doesn't satisfy (*) and this implies that the set $X_{L,I}$ is not fast.

For the complement $Y = \mathbb{N} \backslash X_{L,I}$, note that Y contains sets of the form

$$\left]La_{n-1}, \frac{1}{L}a_n\right[\cap \mathbb{N}.$$

It is no loss of generality to consider a subsequence in Y of elements of the form $\frac{1}{L}a_n$, instead of $\left(\frac{1}{L}a_n - 1\right)$. We see

$$\frac{|Y \cap [1, \frac{1}{L}a_n - 1]|}{\frac{1}{L}a_n} \geq \frac{\frac{1}{L}a_n - La_{n-1} - 1}{\frac{1}{L}a_n} = 1 - L^2\frac{a_{n-1}}{a_n} - \frac{L}{a_n}.$$

Since A is thin, the right hand side is bounded away from 0 as n goes to infinity, so again (*) is not satisfied for the complement.

Now consider the collection of sets $\{X_{L,I} : I \in \mu, I \subseteq A\}$ for some fixed $L > 1$.

Since μ is an ultrafilter, this collection is closed under finite intersections. And since all these sets are not fast and their complements are not fast either, we can find consider the filter $\mathcal{F}_L$ generated by $\mathcal{S}$ and this family.

Now fix a sequence $L_k > 1$ of numbers tending to 1 (strictly monoton). Then, for each $I \subseteq A, I \in \mu$ we have $X_{L_k,I} \subseteq X_{L_r,I}$ for $r > k$, which is equivalent to $L_k < L_r$. This means that each generating set of $\mathcal{F}_{L_r}$ contains a generating set of $\mathcal{F}_{L_k}$ and because filters are closed under taking supersets, it follows that $\mathcal{F}_{L_r} \subseteq \mathcal{F}_{L_k}$. This implies that we obtain an ascending sequence of filters

$$\mathcal{S} \subseteq \mathcal{F}_{L_1} \subseteq \mathcal{F}_{L_2} \subseteq \mathcal{F}_{L_3} \subseteq \dots$$

Their union is again a filter and so we find an ultrafilter μ' containing all these filters. In particular μ' is a slow ultrafilter since it contains $\mathcal{S}$. Define a map

$$\varphi\colon \operatorname{Cone}_\mu(X, e, \omega) \to \operatorname{Cone}_{\mu'}(X, e, \omega)$$

by setting $\varphi\big([x_m]\big) := [y_m]$, where y_m needs only be defined for $m \in X_{L_1,A}$, say. Set $y_m := x_{a_n}$ if $m \in X_{L_1,a_n}$. By construction, this map is well-defined: Consider another sequence $[x'_m]$ which agrees with $[x_m]$ on a set $I \in \mu$. Since $A \in \mu$, it is no loss of generality to assume $I \subseteq A$. The construction then implies that the image under φ of these sequences agrees on the set $X_{L_1,I} \in \mu'$.

Moreover, φ is a bi-Lipschitz homeomorphism with constant L_1. This is immediate since each $k \in X_{L_1,A}$ lies in exactly one interval X_{L_1,a_n} and we have

$$\frac{d(x_{a_n}, y_{a_n})}{L_1 a_n} \leq \frac{d\big(\varphi(x_k), \varphi(y_k)\big)}{k} \leq \frac{d(x_{a_n}, y_{a_n})}{\frac{1}{L_1} a_n}.$$

Consider now an arbitrary L_k. Since $L_k \leq L_1$ we know that $X_{L_k,A} \subseteq X_{L_1,A}$. Note that the actual definition of φ does not depend on the constant L_1, therefore the map φ can be defined for all L_k in the same way, it is actually the same map. It follows that φ is indeed a bi-Lipschitz map with constant L_k for all k. Since $L_k \to 1$, we find that φ is the desired isometry. □

This theorem shows that it is possible to "thicken" an ultrafilter containing a thin set in such a way that the same cone can be realised using a slow ultrafilter. Therefore if one has an example of a finitely generated group with two different asymptotic cones using different ultrafilters containing thin sets, one can modify the construction to obtain two slow ultrafilters yielding different cones. For instance this can be done in the example given by Thomas and Velickovic in [TV] or for the group in [DS] which has $2^{\aleph_0}$ many different non-homeomorphic asymptotic cones.

6 Concluding remarks and open questions

The paper [R] states on page 1295 (notation adapted): "In our applications we will find it most natural to fix the ultrafilter and then state results whose hypothesis is that some conditions hold in all $\mathrm{Cone}_\mu(X, e, \alpha)$ as e and α vary." A footnote there says: "Fixing α and varying e and μ would work similarly. In either case the point is not to lose information on some subsequence of $(X, d/\alpha_n)$."

Considering the sequence of metric spaces $(X, d/n)$ as n tends to infinity, the choice of a different scaling sequence α amounts to choosing a certain subsequence (if α has bounded accumulation). The choice of the ultrafilter however is a different matter: Whenever two disjoint subsequences are chosen, the ultrafilter states which of the two matters. My impression is that the choice of the ultrafilter is much stronger than just the choice of the α and it is not at all clear that an analogue of Proposition 1.18 can be stated and proven for a change of μ.

This is relevant in the light of Proposition 3.7, which states that the cone of scaling invariant spaces does not depend on the sequence of scaling factors. We would like to conclude from this that the cone is unique, but for this we would need such an analogue of Proposition 1.18.

Another question has been brought up by M. Sapir: Theorem 5.5 states that every cone that is formed using an ultrafilter containing a thin set can also be obtained as a cone using a slow ultrafilter. What about the converse? Given a slow ultrafilter, is it possible to construct an ultrafilter which is not slow (or maybe even thin) such that the cones are isometric? Clearly the same trick cannot work: In our proof of the theorem we took the crucial information in the sequences, which was by assumption very sparse, and "spread" it a little to get a slow ultrafilter. But maybe there is some other way to construct thin ultrafilters out of slow ones.

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