Dispersive Effects in Quantum Kinetic Equations

The Wigner-Poisson-Fokker-Planck System

Inaugural-Dissertation
zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften

Elidon Dhamo

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Zusammenfassung

Die vorliegende Arbeit beschäftigt sich mit der mathematisch rigorosen Analyse von zeitabhängigen quantenmechanischen Modellen, die nichtlineare Phänomene einschließen. Im Speziellen wird dabei das nichtlineare dissipative Wigner-Poisson-Fokker-Planck (WPFP) System untersucht.

Die Quanten Wigner-Fokker-Planck (WFP) Gleichung im Ort-Geschwindigkeit-Phasenraum tritt als Modell für offene Quantensysteme auf, also in der Beschreibung von Teilchen, die sich in Interaktion mit ihrer "Umwelt" befinden. Ein typisches Beispiel für solche Systeme sind Halbleiter-Elektronen, gekoppelt an ein Wärmebad aus Phononen. In der zeitlichen Entwicklung des Elektronen-Subsystems werden im Allgemeinen dissipative und diffusive Effekte zu berücksichtigen sein. Zusätzlich werden Interaktionen zwischen den Teilchen, z.B. Coulombkräfte zwischen den Elektronen, durch ein selbstkonsistentes Potential vom Hartree-Typ einbezogen. Um die physikalische Wohlgestelltheit zu sichern, muss das dadurch modellierte System in der sogenannten Lindblad Form vorliegen.

Die Arbeit besteht aus zwei Teilen, untergliedert jeweils in zwei Kapitel:

I. Im ersten Kapitel wird Existenz, Eindeutigkeit und Regularität einer zeitglobalen Lösung des gleichmäßig elliptischen WPFP Systems in drei Dimensionen gezeigt. Die Analysis wird in einem geeignet gewichteten L^2-Phasenraum durchgeführt, so dass die makroskopische Teilchendichte wohldefiniert ist, und der lineare Fokker-Planck-Operator eine dissipative stark stetige Halbgruppe von beschränkten Operatoren erzeugt. Die parabolische Regularisierung der linearen WFP Gleichung kontrolliert die Nichtlinearität lokal in der Zeit, so dass Letztere als Störung der Halbgruppe behandelt werden kann. Die notwendige a-priori Abschätzung für die Lösung wird dann durch eine entsprechende a-priori Abschätzung für das elektrische Feld ermöglicht, eine Strategie, die die dispersiven Effekte des freien Transport-Operators zu Nutze macht, und in der Theorie des klassischen Gegenstücks, der Vlasov-Poisson-Fokker-Planck Gleichung, Anwendung gefunden hat.

Im zweiten Kapitel wird eine neue Strategie zur theoretischen Behandlung des dreidimensionalen WPFP Systems vorgestellt. Sie ermöglicht eine globale rein kinetische Existenz- und Eindeutigkeitsanalyse im L^2-Phasenraum, sowohl des elliptischen, als auch des physikalisch wichtigen hypoelliptischen Systems. Ausschlaggebend dabei ist, wiederum anhand der dispersiven Effekte des freien Transportes, eine zeitglobale Neudefinition des selbstkonsistenten Potentials und des elektrischen Feldes, und somit die Umgehung der Wohldefiniertheit der Teilchendichte, was ein zentrales Problem der quanten-kinetischen Theorie ist. Der parabolische Charakter der WFP

Gleichung führt schließlich zur C^∞–Regularität der Wigner-Funktion, der Teilchendichte sowie des elektrischen Feldes für positive Zeiten.

II. Des Weiteren befasst sich diese Arbeit mit der numerischen Approximation des eindimensionalen nichtlinearen WPFP Systems mit periodischen Randbedingungen in der Ortsvariable, mathematisch wohldefiniert in einem gewichteten L^2-Raum. Es wird eine Diskretisierung in der Zeit mittels eines Operator-Splitting-Verfahrens erster Ordnung vorgestellt, das auf der Produktformel für Halbgruppen von Operatoren beruht. Diese Splitting-Methode wird auf natürlicher Weise durch das im Phasenraum "orthogonale" Wirken der in der WFP Gleichung auftretenden Differentialoperatoren begünstigt. Im dritten Kapitel wird die nichtlineare Stabilität und die Konvergenz erster Ordnung dieser Semi-Diskretisierung gezeigt. Dabei wird die Nichtlinearität als lokale Lipschitz-stetige Störung der Produktformel betrachtet. Ferner macht die parabolische Regularisierung des Fokker-Planck Operators den Beweis einer Konvergenz "niedriger" Ordnung möglich, der nur mit einem zusätzlichen Moment in der Geschwindigkeitsrichtung und ohne Glattheitsvoraussetzungen an die Anfangsdaten auskommt.

Das vierte Kapitel ist der numerischen Realisierung des vorgestellten Operator-Splitting-Verfahrens gewidmet. Es wird ein gemischtes numerisches Schema vorgestellt, bestehend aus einer Finite-Differenzen-Methode in der Ortsrichtung mit Berücksichtigung der periodischen Randbedingungen, und einer Spektral-Kollokationsmethode in der Geschwindigkeitsrichtung, die die numerische Behandlung der nicht-lokalen Nichtlinearität möglich macht. Diverse numerische Simulationen zur Veranschaulichung der zeitlichen Evolution der approximierten Wigner-Funktion, sowohl unter der Wirkung eines gegebenen Stufen-Potentials, als auch unter der Wirkung des selbstkonsistenten Potentials schließen diese Arbeit ab.

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Contents

Introduction

In the present work the focus is on quantum mechanical multi-particle systems coupled to an external reservoir, i.e. so called *open quantum systems* [**Da, BrPe**]. The dynamics of such systems can often be approximately described by kinetic equations in the mean-field limit, i.e. by a Markovian approximation. Such self-consistent models appear in a wide range of physical applications, both classical and quantum mechanical, for example in gas dynamics, stellar dynamics, plasma physics, and electron transport. The corresponding nonlinear evolution equations are obtained as approximations to the underlying many-particle models, and there exists a vast body of literature on their mathematically rigorous derivation, cf. [**Sp**] and the references therein for an extended overview of such derivations for a variety of kinetic equations.

Before presenting the model to be considered, let us first give a short overview about its physical origin. A well known example (without external reservoir) from the classical kinetic theory is the coupled nonlinear Vlasov-Poisson (VP) system, i.e. the Vlasov equation

$$f_t + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = 0, \quad x, v \in \mathbb{R}^d,\ t > 0, \tag{1.0.1}$$

self-consistently coupled with the Poisson equation for the potential $V(x,t)$

$$-\Delta V(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\, dv, \quad x \in \mathbb{R}^d,\ t > 0. \tag{1.0.2}$$

It describes the collisionless evolution of the phase-space density $f(x,v,t)$ of a species of charged particles with Coulomb interaction (cf. [**BrHe, Ba**] for the derivation in the case of "smooth" and, resp. singular particle interaction potentials).

When including the interaction with an environment, one of the simplest model is the Fokker-Planck equation (cf. [**Ri, Sp**] for applications and its derivation as the Brownian motion limit of a Rayleigh gas). Specifically, we mention the Vlasov-Fokker-Planck (VFP) equation,

$$f_t + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f \;=\; \beta \mathrm{div}_v(vf) + \sigma \Delta_v f, \quad x, v \in \mathbb{R}^d,\ t > 0, \tag{1.0.3}$$

which is the classical counterpart of the model analyzed here. It describes the time-evolution of the phase-space density $f(x, v, t)$ under the action of the potential $V(x, t)$. Here, $\sigma > 0$ and $\beta > 0$ denote, respectively, the diffusion and friction constants.

On the quantum level, similarly to the Vlasov-Poisson equation (1.0.1)-(1.0.2), the Hartree equation describes the self-consistent transport of charged (spin-less) particles, e.g. ballistic electrons in a collisionless regime, cf. [**MRS**]. The Hartree equation can be obtained from the N-body Schrödinger equation in the mean field limit (cf. [**Sp**] for a derivation in the case of bounded particle interactions and [**ErYa**] for the Coulomb case). By using the Wigner transform (cf. [**Wi**]), the Hartree equation can be equivalently represented in phase-space, leading to the so called Wigner-Poisson (WP) system for the Wigner function $w = w(x, v, t)$, which is a real-valued quasi-distribution function in the position-velocity (x, v) space for the considered quantum system at time t. In three dimensions this system reads

$$\partial_t w + v \cdot \nabla_x w - \Theta[V]w \;=\; 0, \quad x, v \in \mathbb{R}^3,\ t > 0, \tag{1.0.4}$$

where the (real-valued) self-consistent Coulomb potential $V = V(x, t)$, defined as in (1.0.2), enters in the equation through the pseudo-differential operator $\Theta[V]$, defined as

$$\begin{aligned} (\Theta[V]w)(x, v, t) \;&=\; i\Big[V(x + \frac{1}{2i}\nabla_v, t) - V(x - \frac{1}{2i}\nabla_v, t)\Big] w(x, v, t) \\ &=\; \frac{i}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} \delta V(x, \eta, t) \mathcal{F}_{v\to\eta} w(x, \eta, t) e^{iv\cdot\eta}\, d\eta \end{aligned} \tag{1.0.5}$$

where

$$\delta V(x, \eta, t) \;=\; V\Big(x + \frac{\eta}{2}, t\Big) - V\Big(x - \frac{\eta}{2}, t\Big).$$

$\mathcal{F}_{v\to\eta} w$ denotes the Fourier transform of w with respect to v:

$$\mathcal{F}_{v\to\eta} w(t, x, \eta) \;=\; \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} w(t, x, v')e^{-iv'\cdot\eta}\, dv'.$$

Here, for simplicity of the notation, the Planck constant, particle mass and charge are set equal to unity. The classical limit of (1.0.4) is indeed the 3D Vlasov-Poisson system (cf. [**LPa, MM**]). Providing a kinetic description of quantum mechanics the Wigner formalism has attracted considerable attention of solid state physicists for simulating quantum effects in ultra–integrated devices, e.g. like resonant tunneling diodes ([**KKFR**]).

In addition to a self-consistent Coulomb field we shall here be interested in quantum systems which also have a dissipative interaction with their environment. In many (practical) applications of such open quantum systems the interaction with a reservoir is described in a rather simple phenomenological manner, often using diffusion operators, quantum-BGK or relaxation-type terms [**CL, DeRi, Ar96**] when considered in a kinetic formalism.

Thus, in order to describe this (non-reversible) interaction of a quantum system with its environment, e.g. the interaction of electrons with a phonon bath, a possible modification of (1.0.4) consists in introducing a Fokker-Planck type operator on the right hand side (cf. [**CL, CEFM**] for derivations from reversible quantum systems, and [**GGKS, Stro**] for applications in quantum transport):

$$(1.0.6)\quad w_t+v\cdot\nabla_x w-\Theta[V]w \;=\; \beta\mathrm{div}_v(vw)+\sigma\Delta_v w+2\gamma\mathrm{div}_v(\nabla_x w)+\alpha\Delta_x w,$$

for $x, v \in \mathbb{R}^3$, $t > 0$. The Cauchy problem for Wigner equations, like (1.0.4) and (1.0.6), is augmented by the initial condition $w(x, v, t = 0) = w_0(x, v)$, $(x, v) \in \mathbb{R}^6$. In (1.0.6), the friction parameter $\beta \geq 0$, the (classical) diffusion parameter $\sigma > 0$, and the quantum-diffusion coefficients $\alpha, \gamma \geq 0$ constitute the phase-space diffusion matrix of the system. In the Fokker-Planck equation of classical mechanics (cf. [**Ri, CSV**]) one would have $\alpha = \gamma = 0$. For the Wigner-Fokker-Planck (WFP) equation (1.0.6), the so-called Lindblad condition (cf. [**Li**])

$$\begin{pmatrix} \alpha & \gamma+\frac{i}{4}\beta \\ \gamma-\frac{i}{4}\beta & \sigma \end{pmatrix} \;\geq\; 0$$

has to hold: it guarantees that the evolution of the system is "quantum mechanically correct". More precisely, it guarantees that the corresponding von Neumann equation is in Lindblad form and that the density matrix of the quantum system stays a positive operator under temporal evolution (see [**Di, ALMS**] for details). For the mathematical analysis, however, it suffices that (1.0.6) is parabolic or degenerate parabolic. Thus, we shall only assume $\alpha\sigma \geq \gamma^2$ henceforth.

Both the Wigner (1.0.4) and the WFP (1.0.6) equations constitute valuable models for simulations in semiconductor device theory (cf. [**GGKS, MRS, RA**] and references therein), for quantum Brownian motion and quantum optics (cf. [**CL, Di, De, Va01, Va02**]).

This work is concerned with the Wigner-Poisson-Fokker-Planck (WPFP) system, i.e. the described the WFP equation (1.0.6) self-consistently coupled with the Poisson equation for the (real-valued) potential $V = V[w](x, t)$:

$$(1.0.7)\qquad -\Delta V \;=\; n[w], \quad x \in \mathbb{R}^3, \quad t > 0,$$

which models the repulsive Coulomb interaction within the considered particle system (in a mean-field description), e.g. the interaction between electrons. The particle density on the right hand side is formally defined by

$$n[w](x,t) \;:=\; \int_{\mathbb{R}^3} w(x,v,t)\,dv. \tag{1.0.8}$$

How to define rigorously the Hartree-potential in a quantum kinetic framework is indeed one of the crucial points in this work. This kinetic system is the quantum mechanical analogue of the classical Vlasov-Poisson-Fokker-Planck (VPFP) system, i.e. (1.0.3) coupled with (1.0.2). The solution of the Poisson equation (1.0.7) is formally given by

$$V \;=\; V(x,t) \;:=\; -\frac{1}{4\pi|x|} * n(x,t). \tag{1.0.9}$$

The corresponding electrical field is then defined by

$$E(x,t) \;:=\; \nabla_x V(x,t). \tag{1.0.10}$$

The work is split in two parts, consisting of two chapters, respectively. Here, a short abstract of each of them is presented.

I: The first part is concerned with the well-posedness analysis for the non-linear WPFP system in three dimensions. The state of the art of the mathematical treatment of this model, accompanied by a detailed description of the existing results, is presented at the beginning of each chapter.

In the first chapter the uniformly elliptic WPFP system is considered, i.e. assuming $\alpha\sigma \;\geq\; \gamma^2 + \frac{\beta^2}{16}$, $\alpha\sigma \;>\; \gamma^2$. Existence, uniqueness and regularity of global solutions to the Cauchy problem are established. The analysis is carried out in a weighted L^2–space, such that the particle density is properly defined, and the linear quantum Fokker-Planck operator generates a dissipative semigroup. The non-linear potential can be controlled by using the parabolic regularization of the system.
The main technical difficulty for establishing global-in-time solutions is to derive a-priori estimates on the electric field: Inspired by a strategy for the classical Vlasov-Fokker-Planck equation, the dispersive effects of the free transport operator have been exploited. As a "by-product" also a new a-priori estimate on the field in the Wigner-Poisson equation (1.0.4) is derived.

The purely kinetic L^2–analysis, presented in the second chapter, allows then a unified treatment of the elliptic and hypoelliptic cases. The crucial novel tool of the analysis is to exploit in the quantum framework the dispersive effects of the free transport equation. It yields an a-priori estimate on the electric field for all time which allows a new nonlocal-in-time definition of the self-consistent potential and field. Thus, one can circumvent the lacking

v-integrability of the Wigner function, which is a central problem in quantum kinetic theory. It is to be mentioned, that this new approach could be suitable for a broad range of quantum kinetic problems. Then, due to the (degenerate) parabolic character of this system, the C^∞–regularity of the Wigner function, its macroscopic density, and the field are established for positive times.

II: In the second part the one-dimensional nonlinear WPFP system is considered with periodic boundary conditions in the space variable, well-posed in a weighted L^2-space with respect to velocity. The third chapter is concerned with the analysis of a semi-discretization in time of this model through an operator splitting method. First-order convergence and nonlinear stability are established in the weighted L^2-framework, by handling the nonlinearity as perturbation of the product formula for linear semigroups. Further, due to the parabolic regularization of the Fokker-Planck operator, a low-order convergence is proved by increasing the velocity moments but without smoothness assumptions for the initial data.

In the last chapter the implementation of the proposed splitting algorithm is discussed, and a mixed finite-difference-spectral-collocation method is presented for the numerical realization. The finite difference method is applied in the t- and in x-direction, and allows various boundary conditions to be used. In accordance with our convergence analysis, we supplement the x-discretization with periodic boundary conditions.

The v-direction, in which the pseudo-differential operator Θ acts, is discretized by spectral collocation using trigonometric functions. Spectral methods are natural candidates for the discretization of Θ because of its nonlocal nature and its definition via the Fourier transforms.

Numerical examples are presented to illustrate these methods. Finally, simulations of the time-evolution of the approximated Wigner function under the effect of a given bandgap potential, as well as under the effect of the self-consistent Poisson potential conclude this work.

Part I

Analysis of the WPFP System

Chapter 2

Global-in-time analysis of the uniformly elliptic WPFP

Abstract

This chapter is concerned with the nonlinear uniformly elliptic Wigner-Poisson-Fokker-Planck system. Existence, uniqueness and regularity of global solutions to the Cauchy problem in 3 dimensions are established. The analysis is carried out in a weighted L^2–space, such that the particle density is properly defined, and the linear quantum Fokker-Planck operator generates a dissipative semigroup. The non-linear potential can be controled by using the parabolic regularization of the system.

The main technical difficulty for establishing global-in-time solutions is to derive a-priori estimates on the electric field: Inspired by a strategy for the classical Vlasov-Fokker-Planck equation, we exploit dispersive effects of the free transport operator. As a "by-product" we also derive a new a-priori estimate on the field in the Wigner-Poisson equation.

[1] The goal of this chapter is to prove the existence and uniqueness of global-in-time solutions to the coupled *uniformly elliptic* Wigner-Poisson-Fokker-Planck (WPFP) system (1.0.6)-(1.0.8) in three dimensions. Therefore, we shall assume

$$\alpha\sigma \;\geq\; \gamma^2 + \frac{\beta^2}{16} \quad \text{and} \quad \alpha\sigma \;>\; \gamma^2. \tag{2.0.1}$$

[1]The content of this chapter is a joint work with my Ph.D. adviser A. Arnold and C. Manzini (cf. [**ADM04**])

The main analytical challenge for tackling Wigner-Poisson systems is the proper definition of $n[w]$ in appropriate L^p spaces. Due to the definition of the operator Θ in Fourier space, $w \in L^2(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ is the natural set-up. Without further assumptions, of course, this does not justify to define $n[w]$. We shall now summarize the existing literature of this field and the typical strategies to overcome the above problem:

a) The standard approach for the *Wigner-Poisson* equation is to reformulate it as a Schrödinger-Poisson system, where the particle density then appears in L^1 (cf. [**BM, Ca97**] for the 3D–whole space case).

b) In *one spatial dimension with periodic boundary conditions* in x the Wigner-Poisson system (and WPFP) can be dealt with directly on the kinetic level. For w in a weighted L^2-space, the nonlinear term $\Theta[V]w$ is then bounded and locally Lipschitz [**AR96, ACD**]. The same strategy was also used in [**Ma**] for the Wigner-Poisson system on a bounded (spatial) domain in three dimensions (local-in-time solution).

c) By adapting L^1-techniques from the classical Vlasov-Fokker-Planck equation, the *3D Wigner-Poisson-Fokker-Planck* system was analyzed in [**ALMS**] (local-in-time solution for the friction-free problem) and [**CLN**] (global-in-time solution). The latter paper, however, is not a purely kinetic analysis as it requires to assume the positivity of the underlying density matrix. In both cases the dissipative structure of the system allows to control $n[w]$.

d) In [**Ar95, AS**] the 3D Wigner-Poisson and WPFP systems were reformulated as von-Neumann equations for the quantum mechanical *density matrix*. This implies $n \in L^1(\mathbb{R}^3)$. While this approach is the most natural, both physically and in its mathematical structure, it is restricted to whole space cases. Extensions to initial-boundary value problems (as needed for practical applications and numerical analysis) seem unfeasible.

e) For the classical *Vlasov-Poisson-Fokker-Planck* equation there exists a vast body of mathematical literatur from the 1990's (cf. [**Bo93, Bo95, CSV, Ca, Ca98**]), and many of those tools will be closely related to the present work.

In spite of the various existing well-posedness results for the WPFP problem, there is a need for a purely kinetic analysis, and this is our goal here. Such an approach could possibly allow for an extension to boundary-value problems in the Wigner framework (where the positivity of the related density matrix is a touchy question).

Mathematically we shall develop the following new tools and estimates that could be important also for other quantum kinetic applications: In all of the existing literature on Wigner-Poisson problems (except [**Ste**]) the potential V is bounded, which makes it easy to estimate the operator $\Theta[V]$ in L^2. Our framework for the local in time analysis does not yield a bounded potential. However, the operator Θ only involves δV, a potential difference, which has better decay properties at infinity. This observation gives rise to new estimates that are crucial for our local-in-time analysis.

In order to establish global-in-time solutions we shall extend *dispersive* tools of Lions, Perthame and Castella (cf. [**LP, Pe, CP**] for applications to classical kinetic equation) to the WP and WPFP systems. The fact that the Wigner function w also takes negative values gives rise to an important difference between classical and quantum kinetic problems: In the latter case, the conservation of mass and energy or pseudo-conformal laws do *not* provide useful a-priori estimates on w. We shall hence assume that the initial state lies in a weighted L^2-space, but we shall *not* require that our system has finite mass or finite kinetic energy. Since the energy balance will not be used, this also implies that the sign of the interaction potential does *not* play a role in our analysis.

This chapter is organized as follows: In Section 2.1 we introduce a weighted L^2-space for the Wigner function w that allows to define $n[w]$ and the non-linear term $\Theta[V]w$. In Section 2.2 we obtain a local-in-time, mild solution for WPFP using a fixed point argument and the parabolic regularization of the Fokker-Planck term. In Section 2.3 we establish a-priori estimates to obtain global-in-time solutions. The key point is to derive first L^p-bounds for the electric field ∇V by exploiting dispersive effects of the free kinetic transport. "Bootstraping" then yields estimates on the Wigner function in a weighted L^2-space. Finally, we give regularity results on the solution. The technical proofs of several lemmata are defered to the Appendix of this chapter.

2.1 The functional setting

In this section we shall discuss the functional analytic preliminaries for studying the non-linear problem (1.0.6)-(1.0.8). First we shall introduce an appropriate "state space" for the Wigner function w which allows to "control" the particle density $n[w]$ and the selfconsistent potential $V[w]$. Next, we shall discuss the linear Wigner-Fokker-Planck equation and the dissipativity of its (evolution) generator A.

2.1.1 State space and selfconsistent potential

Let us introduce the following weighted (real valued) L^2-space

$$X := L^2(\mathbb{R}^6; (1+|v|^2)^2\, dx\, dv), \tag{2.1.1}$$

endowed with the scalar product

$$(2.1.2) \qquad < u, w >_X \;=\; \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} u\, w(1+|v|^2)^2\, dx\, dv.$$

The following proposition motivates the choice of X as the state space for our analysis.

2.1 Proposition. *For all $w \in X$, the function $n[w]$, defined by*

$$n[w](x) := \int w(x,v)\, dv, \qquad x \in \mathbb{R}^3,$$

belongs to $L^2(\mathbb{R}^3)$ and satisfies

$$(2.1.3) \qquad \|n[w]\|_{L^2(\mathbb{R}^3)} \;\le\; C\|w\|_X,$$

with a constant C independent of w.

Here and in the sequel C shall denote generic, but not necessarily equal, constants.
Proof. By using Hölder inequality in the v-integral, we get

$$\begin{aligned}\|n[w]\|^2_{L^2(\mathbb{R}^3)} \;&\le\; \int\left(\int |w(x,v)|^2(1+|v|^2)^2\, dv\right)\left(\int\frac{dv}{(1+|v|^2)^2}\right) dx\\ &=\; C\|w\|^2_X.\end{aligned}$$

□

2.2 Remark. The choice of the $|v|^2$ weight was already seen to be convenient to control the L^2-norm of the density on a bounded domain of $\mathbb{R}^3_x$ (cf. [**Ma**]). The subsequent analysis would hold also by including a symmetric weight in the x-variable (i.e. for $w \in L^2(\mathbb{R}^6; (1+|x|^2+|v|^2)^2\, dx\, dv)$), which would yield a L^p-bound with $p \in (3/2, 2]$ for the density. On the other hand, Lemmata 2.8, 2.10 would prevent us from introducing a non-symmetric weight in x.

In this framework the following estimates for the self-consistent potential hold.

2.3 Proposition. *For all $w \in X$, the (Newton potential) solution $V = V[w]$ of the equation $-\Delta_x V[w] = n[w]$, $x \in \mathbb{R}^3$, satisfies*

$$(2.1.4) \qquad \|\nabla V[w]\|_{L^6(\mathbb{R}^3)} \;\le\; C\|n[w]\|_{L^2(\mathbb{R}^3)}.$$

Proof. Since $V = -\frac{1}{4\pi|x|} * n$, we have $\nabla V = \frac{x}{4\pi|x|^3} * n$, and the estimate follows from the generalized Young inequality. □

2.4 Remark. Note that $n \in L^2(\mathbb{R}^3)$ does not yield (via the generalized Young inequality) a control of V in any L^r-space (even $n \in L^p(\mathbb{R}^3)$ with $p \in (3/2, 2]$ would not "help"). However, the operator $\Theta[V]$ involves only the function δV, which is slightly "better behaved". We anticipate that we will later recover some information on the potential V via new *a-priori* estimates on the electric field $\nabla V[w]$ (see Corollary 2.33).

Omitting the time-dependence we have

$$
\begin{aligned}
\delta V(x,\eta) &= V(x+\frac{\eta}{2}) - V(x-\frac{\eta}{2}) \\
&= \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} \frac{n[w](x-\frac{\eta}{2}-\xi) - n[w](x+\frac{\eta}{2}-\xi)}{|\xi|}\,d\xi \\
&= \frac{1}{4\pi}\int\limits_{\mathbb{R}^3} f(y;\eta)\,n[w](x-y)\,dy,
\end{aligned}
$$

with the "dipole-kernel" $f(y;\eta) := \left(\frac{1}{|y-\frac{\eta}{2}|} - \frac{1}{|y+\frac{\eta}{2}|}\right)$.

2.5 Proposition. *For all $w \in X$ and fixed $\eta \in \mathbb{R}^3$, we have $\delta V[w](.,\eta) \in L^q(\mathbb{R}^3_x)$, $6 < q \le \infty$. Moreover*

$$
\|\delta V[w](.,\eta)\|_{L^\infty(\mathbb{R}^3_x)} \le C|\eta|^{1/2}\|n[w]\|_{L^2(\mathbb{R}^3)}. \tag{2.1.5}
$$

Proof. By using the triangle inequality,

$$
|f(y;\eta)| \le \frac{|\eta|}{|y-\frac{\eta}{2}||y+\frac{\eta}{2}|},
$$

and the transformation $y = |\eta|x$, we estimate for $3/2 < p < 3$

$$
\|f(.\,;\eta)\|^p_{L^p(\mathbb{R}^3)} = |\eta|^{3-p}\int\limits_{\mathbb{R}^3} \frac{dx}{\left(|x-\frac{e}{2}||x+\frac{e}{2}|\right)^p} < \infty,
$$

where $e \in \mathbb{R}^3$ is some unit vector (due to the rotational symmetry of $\|f(.;\eta)\|^p_{L^p(\mathbb{R}^3)}$ with respect to η). Young inequality then gives $\delta V(.,\eta) \in L^q(\mathbb{R}^3)$, $6 < q \le \infty$, and the assertion holds. □

In most of the literature the *Wigner operator* Θ is defined on $L^2(\mathbb{R}^d_v)$ for bounded potentials V, cf. [**MR, MB, ACD**]. For our nonlinear problem (1.0.6)-(1.0.8), however, $V \in L^\infty(\mathbb{R}^3)$ does *not* hold. As a compensation we shall hence exploit the additional regularity of the Wigner function to define the quadratic term $\Theta[V[w]]w$ (cf. Prop. 2.8 in [**Ma**] for a similar strategy).

2.6 Proposition. *Let $u \in X$ and $\nabla_v u \in X$ be given. Then, the linear operator*

$$z \longmapsto \Theta[V[z]]u,$$

with the function $V[z] = -\frac{1}{4\pi|x|} * n[z]$, is bounded from the space X into itself and satisfies

$$\|\Theta[V[z]]u\|_X \leq C\{\, \|u\|_X + \|\nabla_v u\|_X\} \, \|z\|_X, \quad \forall z \in X. \tag{2.1.6}$$

Proof. To estimate $\|\Theta[V[z]]u\|_X$ we shall consider separately the two terms of the equivalent norm

$$\|u\|_{\tilde{X}}^2 := \|u\|_2^2 + \sum_{i=1}^{3} \|v_i^2 u\|_2^2. \tag{2.1.7}$$

First, by denoting $\hat{u} := \mathcal{F}_{v\to\eta} u$, we get

$$\begin{aligned} \|\Theta[V[z]]u\|_2^2 &= \iint |\delta(V[z])(x,\eta)\hat{u}(x,\eta)|^2 dx\, d\eta \\ &\leq \iint \|\delta(V[z])(\,.\,,\eta)\|_\infty^2 |\hat{u}(x,\eta)|^2 d\eta\, dx \\ &\leq C\|z\|_X^2 \iint \left(|\eta|^{1/2}|\hat{u}(x,\eta)|\right)^2 d\eta\, dx \\ &\leq C\|z\|_X^2 \left(\|u\|_2^2 + \|\nabla_v u\|_2^2\right), \end{aligned} \tag{2.1.8}$$

by applying first the Plancherel Theorem, then Hölder's inequality in the x variable, the estimates (2.1.3), (2.1.5) for the function $\delta V[z]$, and finally, Young inequality and the Plancherel Theorem to the last integral.
For the second term of $\|\Theta[V[z]]u\|_{\tilde{X}}$ we shall use

$$v_i^2 \Theta[V] w(x,v) = \frac{1}{4}\Theta[\partial_i^2 V] w(x,v) + \Omega[\partial_i V](v_i w)(x,v) + \Theta[V] v_i^2 w(x,v), \tag{2.1.9}$$

with the pseudo-differential operator

$$\begin{aligned} \Omega[V] &:= i(\delta_+ V)\left(x, \frac{\nabla_v}{i}\right), \\ (\delta_+ V)(x,\eta) &:= V\left(x + \frac{\eta}{2}\right) + V\left(x - \frac{\eta}{2}\right). \end{aligned} \tag{2.1.10}$$

Here and in the sequel we use the abreviation $\partial_i := \partial_{x_i}$. (2.1.9) is now estimated:

$$\|v_i^2 \Theta[V[z]]u\|_2 \leq \frac{1}{4}\|\delta(\partial_i^2 V[z])\hat{u}\|_2 + \|\delta_+(\partial_i V[z])\partial_{\eta_i}\hat{u}\|_2 + \|\delta V[z]\partial_{\eta_i}^2 \hat{u}\|_2 \tag{2.1.11}$$

The first two terms of (2.1.11) can be estimated as follows:

$$\begin{aligned}\|\delta(\partial_i^2 V[z])\hat{u}\|_{L^2(\mathbb{R}^6)} &\leq 2\|\partial_i^2 V[z]\|_{L^2(\mathbb{R}^3_x)}\|\hat{u}\|_{L^2(\mathbb{R}^3_x;L^\infty(\mathbb{R}^3_\eta))} \\ &\leq C\|z\|_X\|(1+|v|^2)u\|_{L^2(\mathbb{R}^6)},\end{aligned}$$

by applying Hölder's inequality, (2.1.3) and the Sobolev imbedding $\hat{u}(x,.\,) \in H^2(\mathbb{R}^3_\eta) \hookrightarrow L^\infty(\mathbb{R}^3_\eta)$.

$$\text{(2.1.12)}\quad \begin{aligned}\|\delta_+(\partial_i V[z])\partial_{\eta_i}\hat{u}\|_{L^2(\mathbb{R}^6)} &\leq C\|\partial_i V[z]\|_{L^4(\mathbb{R}^3_x)}\|\partial_{\eta_i}\hat{u}\|_{L^2(\mathbb{R}^3_x;L^4(\mathbb{R}^3_\eta))} \\ &\leq C\|z\|_X\|(1+v_i^2)u\|_2,\end{aligned}$$

by the Sobolev imbedding and $\nabla_\eta\hat{u}(x,.\,) \in H^1(\mathbb{R}^3_\eta) \hookrightarrow L^4(\mathbb{R}^3_\eta)$, and by estimate (2.1.4) for $\nabla V[z]$ and (2.1.3). For the last term of (2.1.11) we estimate as in (2.1.8):

$$\begin{aligned}\|\delta V[z]\partial^2_{\eta_i}\hat{u}\|_2^2 &\leq \iint \|\delta V[z](.\,,\eta)\|_\infty^2|\partial^2_{\eta_i}\hat{u}(x,\eta)|^2 d\eta\, dx \\ &\leq C\|z\|_X^2\iint\left(|\eta|^{1/2}\partial^2_{\eta_i}\hat{u}(x,\eta)\right)^2 d\eta\, dx \\ &\leq C\|z\|_X^2\left(\|\partial^2_{\eta_i}\hat{u}\|_2^2+\|\eta\partial^2_{\eta_i}\hat{u}\|_2^2\right) \\ &\leq C\|z\|_X^2\left(\|\partial^2_{\eta_i}\hat{u}\|_2^2+\|\partial^2_{\eta_i}(\eta\hat{u})\|_2^2+\|\partial_{\eta_i}\hat{u}\|_2^2\right) \\ &\leq C\|z\|_X^2\left(\|(1+v_i^2)u\|_2^2+\|v_i^2\nabla_v u\|_2^2\right),\end{aligned}$$

by interpolation and integration by parts.
This concludes the proof of estimate (2.1.6). □

2.7 Remark. The previous proposition shows that the bilinear map

$$(z,u) \longmapsto \Theta[V[z]]u$$

is well-defined for all $z, u \in X$, subject to $\nabla_v u \in X$. The unusual feature of the above proposition is the boundedness of this map with respect to the function z appearing in the self-consistent potential $V[z]$. This is in contrast to most of the existing literature ([**ACD, MB, MR**]), where the boundedness of the pseudo-differential operator $\Theta[V[z]]$ (with z fixed) is used. However, this can only hold for bounded potentials V.

2.1.2 Dissipativity of the linear equation

In our subsequent analysis we shall first consider the linear Wigner-Fokker-Planck equation, i.e. equation (1.0.6) with $V \equiv 0$. The generator of this evolution problem is the unbounded linear operator $A : D(A) \longrightarrow X$,

$$\text{(2.1.13)}\quad Au := -v\cdot\nabla_x u + \beta\,\mathrm{div}_v(vu) + \sigma\Delta_v u + 2\gamma\,\mathrm{div}_v(\nabla_x u) + \alpha\Delta_x u,$$

defined on

$$D(A) = \{u \in X \mid v \cdot \nabla_x u, v \cdot \nabla_v u, \Delta_v u, \mathrm{div}_v \nabla_x u, \Delta_x u \in X\}. \tag{2.1.14}$$

Clearly, $C_0^\infty(\mathbb{R}^6) \subset D(A)$. Hence, $D(A)$ is dense in X. Next we study whether the operator A is dissipative on the (real) Hilbert space $\tilde{X}$, i.e. if

$$< Au, u >_{\tilde{X}} \leq 0, \quad \forall\, u \in D(A) \tag{2.1.15}$$

holds.

2.8 Lemma. *Let the coefficients of the operator A satisfy $\alpha\sigma \geq \gamma^2$. Then $A - \kappa I$ with*

$$\kappa := \frac{3}{2}\beta + 9\sigma \tag{2.1.16}$$

is dissipative in $\tilde{X}$.

The proof is lengthy but straigthforward and deferred to the Appendix.

By Theorem 1.4.5b of [**Pa**] its closure, $\overline{A - \kappa I} = \overline{A} - \kappa I$ is also dissipative. A straightforward calculation using integrations by parts yields

$$< Au, w >_{\tilde{X}} = < u, A_1^* w >_{\tilde{X}} + < u, A_2^* w >_{\tilde{X}}, \quad \forall\, u, w \in D(A),$$

with

$$A_1^* w = v \cdot \nabla_x w - \beta v \cdot \nabla_v w + \sigma \Delta_v w + 2\gamma \mathrm{div}_v(\nabla_x w) + \alpha \Delta_x w,$$

$$\begin{aligned} < u, A_2^* w >_{\tilde{X}} = \sum_{i=1}^{3} \Bigg(&- \frac{4}{3}\beta \iint v_i^4 w u + \frac{8}{3}\sigma \iint v_i^3 w_{v_i} u \\ &+ \frac{12}{3}\sigma \iint v_i^2 w u + \frac{8}{3}\gamma \iint v_i^3 w_{x_i} u \Bigg). \end{aligned}$$

Hence, $A^*|_{D(A)}$ – the restriction of the adjoint of the operator A to $D(A)$ – is given by $A^* = A_1^* + A_2^*$. A^* is densly defined on $D(A^*) \supseteq D(A)$, and hence A is a closable operator (cf. Theorem VIII.1.b of [**RS1**]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (cf. [**RS1**], Theorem VIII.1.c).
Since $< A^* u, u > = < Au, u >$ the following lemma on the dissipativity of the operator A^* restricted to $D(A)$ holds.

2.9 Lemma. *Let the coefficients of the operator A satisfy $\alpha\sigma \geq \gamma^2$. Then $A^*|_{D(A)} - \kappa I$ is dissipative (with κ as in (2.1.16)).*

Next we consider the dissipativity of this operator on its proper domain $D(A^*)$, which, however, is not known explicitly. To this end we shall use the following technical lemma, which, for a matter of generality, is stated in the space with the symmetric x, v-weight. The proof is defered to the appendix: the arguments are inspired by [**ACD**], [**AS**], but there are also similar results for FP-type operators in [**HelN, HerN**], e.g.

2.10 Lemma. *Let $P = p(x, v, \nabla_x, \nabla_v)$ where p is a quadratic polynomial and*

$$D(P) \;:=\; C_0^\infty(\mathbb{R}^6) \subset Z \;:=\; L^2(\mathbb{R}^6; (1 + |x|^2 + |v|^2)^2\, dx\, dv).$$

Then $\bar{P}$ is the maximum extension of P in the sense that

$$D(\bar{P}) \;:=\; \{u \in Z \mid \textit{the distribution } Pu \in Z\}\,.$$

We now apply Lemma 2.10 to $P = A^* - \kappa I$, which is dissipative on $D(P) \subset D(A)$. Since A^* is closed, we have $D(A^*) = D(\overline{P}) = \{u \in X \mid A^* u \in X\}$ and $A^* - \kappa I$ is dissipative on all of $D(A^*)$.

Applying Corollary 1.4.4 of [**Pa**] to $\overline{A} - \kappa I$ (with $(\overline{A})^* = A^*$), then implies that $\overline{A} - \kappa I$ generates a C_0 semigroup of contractions on X, and the C_0 semigroup generated by $\overline{A}$ satisfies

$$\|e^{t\overline{A}} u\|_{\tilde{X}} \;\leq\; e^{\kappa t}\|u\|_{\tilde{X}}, \quad u \in X, \;\; t \geq 0.$$

Since $\|.\|_X$ and $\|.\|_{\tilde{X}}$ are equivalent norms in X with

$$\|u\|_{\tilde{X}} \;\leq\; \|u\|_X \;\leq\; 4\|u\|_{\tilde{X}},$$

we have

$$\|e^{t\overline{A}} u\|_X \;\leq\; 4e^{\kappa t}\|u\|_X, \quad u \in X, \;\; t \geq 0. \tag{2.1.17}$$

2.2 Existence of the local-in-time solution

In this section we shall use a contractive fixed point map to establish a local solution of the WPFP system. To this end the parabolic regularization of the linear WFP equation will be crucial to define the self-consistent potential term.

2.2.1 The linear equation

First let us consider the linear equation

$$w_t \;=\; \overline{A} w(t), \quad t > 0, \qquad w(t = 0) \;=\; w_0 \in X. \tag{2.2.1}$$

By the discussion in Subsection 2.1.2, its unique solution $w(t) = e^{t\overline{A}}w_0$ satisfies

$$\|w(t)\|_X \;\leq\; 4e^{\kappa t}\|w_0\|_X, \quad \forall\, t \geq 0. \tag{2.2.2}$$

Actually, the solution of the equation can be expressed as

$$w(x,v,t) \;=\; \iint G(t, x-x_0, v, v_0)\, w_0(x_0,v_0)\, dx_0\, dv_0, \quad \forall\,(x,v) \in \mathbb{R}^6, \tag{2.2.3}$$

where the Green's function G satisfies (in a weak sense) the equation (2.2.1) and the initial condition

$$\lim_{t\to 0} G(t, x-x_0, v, v_0) = \delta(x-x_0, v-v_0), \quad \forall\,(x_0,v_0) \in \mathbb{R}^6.$$

for any fixed $(x_0,v_0) \in \mathbb{R}^6$ (cf. Def. 2.1 and Prop. 3.1 in [**SCDM**]).
The Green's function reads

$$G(t, x-x_0, v, v_0) \;=\; e^{3\beta t} g(t, X_{-t}(x,v) - x_0, \dot{X}_{-t}(x,v) - v_0), \tag{2.2.4}$$

with

$$\begin{aligned} g(t,x,v) \;=\;& \frac{1}{(2\pi)^3\,(4\lambda(t)\nu(t) - \mu^2(t))^{3/2}} \\ &\cdot \exp\left\{ -\frac{\nu(t)|x|^2 + \lambda(t)|v|^2 + \mu(t)(x\cdot v)}{4\lambda(t)\nu(t) - \mu^2(t)} \right\}. \end{aligned} \tag{2.2.5}$$

The characteristic flow $\Phi_t(x,v) = (X_t(x,v), \dot{X}_t(x,v))$ of the first order part of (2.2.1), is given for $\beta > 0$ by

$$X_t(x,v) = x + v\left(\tfrac{1-e^{-\beta t}}{\beta}\right), \qquad \dot{X}_t(x,v) = ve^{-\beta t},$$

and for $\beta = 0$ by

$$X_t(x,v) = x + vt, \qquad \dot{X}_t(x,v) = v.$$

The functions $\lambda(t)$, $\nu(t)$, $\mu(t)$ in (2.2.5) are given for $\beta > 0$ by

$$\begin{aligned} \lambda(t) &= \alpha t + \sigma\left(\tfrac{e^{2\beta t} - 4e^{\beta t} + 3}{2\beta^3} + \tfrac{1}{\beta^2}\,t\right) + \gamma\left(\tfrac{2}{\beta}t - \tfrac{2}{\beta^2}(e^{\beta t} - 1)\right), \\ \nu(t) &= \sigma\,\tfrac{e^{2\beta t} - 1}{2\beta}, \\ \mu(t) &= \sigma\left(\tfrac{1-e^{\beta t}}{\beta}\right)^2 + \gamma\,\tfrac{2(1-e^{\beta t})}{\beta}. \end{aligned}$$

In case $\beta = 0$ they respectively read

$$\lambda(t) \;=\; \alpha t + \sigma\frac{t^3}{3} - \gamma t^2, \qquad \nu(t) \;=\; \sigma t, \qquad \mu(t) \;=\; \sigma t^2 - 2\gamma t.$$

The asymptotic behaviour of these functions $\lambda(t)$, $\nu(t)$, $\mu(t)$ for small t (for $\beta \geq 0$) is described by

$$\lambda(t) \;\sim\; \alpha t, \qquad \nu(t) \;\sim\; \sigma t, \qquad \mu(t) \;\sim\; -2\gamma t.$$

With these preliminaries, the following parabolic reguralization result can be deduced.

2.11 Proposition. *For each parameter set $\{\alpha, \beta, \gamma, \sigma\}$, there exist two constants $B = B(\alpha, \beta, \gamma, \sigma)$ and $T_0 = T_0(\alpha, \beta, \gamma, \sigma)$, such that the solution of the linear equation (2.2.1) satisfies*

$$\|\nabla_v w(t)\|_X \;\leq\; Bt^{-1/2}e^{\kappa t}\|w_0\|_X, \quad \forall\, 0 < t \leq T_0, \tag{2.2.6}$$

$$\|\nabla_x w(t)\|_X \;\leq\; Bt^{-1/2}e^{\kappa t}\|w_0\|_X, \quad \forall\, 0 < t \leq T_0, \tag{2.2.7}$$

for all $w_0 \in X$.

The proof is similar to [**Ca**] and it will be defered to the Appendix.

2.12 Remark. (a) Observe that the functions $\nabla_x w, \nabla_v w \in \mathcal{C}((0,\infty); X)$. The local boundedness of $\nabla_x w, \nabla_v w$ on any interval $(\tau, \tau + T_0]$ follows from (2.2.2) and Prop. 2.11.
(b) Note that the strategy of the next section will *not* work in the degenerated parabolic case $\alpha\sigma - \gamma^2 = 0$, since the decay rates of Prop. 2.11 would then be $t^{-3/2}$, which is not integrable at $t = 0$. This and the hypoelliptic case ($\sigma > 0$, $\alpha = \gamma = 0$), with a respective singularity of $\mathcal{O}(t^{-3})$ at $t = 0$, are the motivation for the analysis of the next chapter.

2.2.2 The non-linear equation: local solution

Our aim is to solve the following non-linear initial value problem

$$\begin{aligned} w_t(t) \;&=\; \overline{A}w(t) + \Theta[V[w(t)]]w(t), \quad \forall\, t > 0, \\ w(t=0) \;&=\; w_0 \in X, \end{aligned} \tag{2.2.8}$$

where the pseudo-differential operator Θ is formally defined by (1.0.5) and the potential $V[w(t)]$ is the (Newton potential) solution of the Poisson equation

$$-\Delta_x V(t,x) \;=\; n[w(t)](x) \;=\; \int_{\mathbb{R}^3} w(t,x,v)\,dv, \qquad x \in \mathbb{R}^3,$$

for all $t > 0$. Actually, if we assume $w(t) \in X$ for all $t \geq 0$, then the function $n[w(t)]$ is well-defined for all $t \geq 0$ (cf. Prop. 2.1), and the solution $V[w(t)]$ satisfies the estimates of Propositions 2.3, 2.5 for all $t \geq 0$.

The Propositions 2.6 and 2.11 motivate the definition of the Banach space

$$Y_T \;:=\; \Big\{ z \in \mathcal{C}([0,T];X) \,|\, \nabla_v z \in \mathcal{C}((0,T];X) \\ \text{with } \|\nabla_v z(t)\|_X \leq C t^{-1/2} \;\text{ for } t \in (0,T) \Big\},$$

endowed with the norm

$$\|z\|_{Y_T} \;:=\; \sup_{t\in[0,T]} \|z(t)\|_X + \sup_{t\in[0,T]} \|t^{1/2}\nabla_v z(t)\|_X,$$

for every fixed $0 < T < \infty$. We shall obtain the (local-in-time) well-posedness result for the problem (2.2.8) by introducing a non-linear iteration in the space Y_T, with an appropriate (small enough) T.

For a given $w \in Y_T$ we shall now consider the linear Cauchy problem for the function z,

$$\text{(2.2.9)} \qquad \begin{aligned} z_t \;&=\; \overline{A}z(t) + \Theta[V[z(t)]]w(t), \quad \forall\, t \in (0,T], \\ z(t=0) \;&=\; w_0 \in X. \end{aligned}$$

with $0 < T \leq T_0$ and T_0 is defined in Prop. 2.11. According to Prop. 2.6 the (time-dependent) operator $\Theta[V[.\,]]w(t)$ is, for each fixed $t \in (0, T_0]$, a well-defined, linear and bounded map on X, which we shall consider as a perturbation of the operator $\overline{A}$.

2.13 Lemma. *For all $w_0 \in X$ and $w \in Y_T$, with $T \leq T_0$, the initial value problem*

$$z_t \;=\; \overline{A}z(t) + \Theta[V[z(t)]]w(t), \quad \forall\, t \in (0,T], \qquad z(t=0) \;=\; w_0,$$

has a unique mild solution $z \in \mathcal{C}([0,T];X)$, which satisfies

$$\text{(2.2.10)} \qquad z(t) \;=\; \mathrm{e}^{t\overline{A}}w_0 + \int_0^t \mathrm{e}^{(t-s)\overline{A}}\Theta[V[z(s)]]w(s)\,ds, \quad \forall\, t \in [0,T].$$

Moreover, the solution z belongs to the space Y_T.

Proof. The first assertion follows directly by applying (a trivial extension of) Thm. 6.1.2 in [**Pa**]:

For any fixed $w \in Y_T$, the function $g(t,.) := \Theta[V[.]]w(t)$ is a bounded linear operator on X for all $t \in (0,T)$, and it satisfies $g \in L^1((0,T);\mathcal{B}(X)) \cap$

$C((0,T];\mathcal{B}(X))$ (by Prop. 2.6). Moreover, by estimates (2.1.17), (2.1.6), the following inequalities hold

$$\begin{aligned} \|z(t)\|_X &\leq 4\mathrm{e}^{\kappa t}\|w_0\|_X \\ &\quad + 4\int_0^t \mathrm{e}^{\kappa(t-s)} C\{\, \|w(s)\|_X + \|\nabla_v w(s)\|_X\} \|z(s)\|_X \, ds \\ &\leq 4\mathrm{e}^{\kappa t}\|w_0\|_X + 4C\mathrm{e}^{\kappa T}\|w\|_{Y_T}\int_0^t (1+s^{-1/2})\|z(s)\|_X\, ds, \end{aligned}$$

for all $t \in [0,T]$. Then, by Gronwall's Lemma,

$$\begin{aligned} \|z(t)\|_X &\leq 4e^{\kappa T}\|w_0\|_X\Big[1 + 4C\|w\|_{Y_T}(t+2t^{1/2}) \\ &\qquad \cdot e^{\left(\kappa T + 4Ce^{\kappa T}\|w\|_{Y_T}(T+2T^{1/2})\right)}\Big], \end{aligned} \tag{2.2.11}$$

for all $t \in [0,T]$. By differentiating equation (2.2.10) in the v-direction, we obtain

$$\nabla_v z(t) = \nabla_v e^{t\overline{A}} w_0 + \int_0^t \nabla_v e^{(t-s)\overline{A}} g(s,z(s))\, ds, \quad \forall\, t \in (0,T]. \tag{2.2.12}$$

Using the estimates (2.2.6), (2.1.6), and (2.2.11) then yields

$$\begin{aligned} \|\nabla_v z(t)\|_X &\leq Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X \\ &\quad + B\|w\|_{Y_T}\int_0^t (t-s)^{-1/2}\mathrm{e}^{\kappa(t-s)} C\{\, 1+s^{-1/2}\}\, \|z(s)\|_X\, ds \\ &\leq Bt^{-1/2}\mathrm{e}^{\kappa t}\|w_0\|_X + 4BCe^{2\kappa T}\|w_0\|_X\|w\|_{Y_T}\Big[\, \pi + 2t^{1/2} \\ &\quad + 4C\|w\|_{Y_T} e^{\left(\kappa T + 4Ce^{\kappa T}\|w\|_{Y_T}(T+2T^{1/2})\right)} \left(4t^{1/2} + \frac{3}{2}\pi t + \frac{4}{3}t^{3/2}\right)\Big], \end{aligned} \tag{2.2.13}$$

for all $t \in (0,T]$. The continuity in time of $\nabla_v z$ can be derived from (2.2.12) by using Remark 2.12 and the fact that $g(t,z(t)) \in \mathcal{C}((0,T];X)$. Hence, the function z belongs to the space Y_T. □

We now define the (affine) linear map M on Y_T (for any fixed $0 < T \leq T_0$):

$$w \longmapsto Mw := z,$$

where z is the unique mild solution of the initial value problem (2.2.9). According to Lemma 2.13, $z \in Y_T$. Next we shall show that M is a strict contraction on a closed subset of Y_T, for T sufficiently small. This will yield the local-in-time solution of the non-linear equation (2.2.8).

2.14 Proposition. *For any fixed $w_0 \in X$, let $R > \max\{4, B\}e^{\kappa}\|w_0\|_X$, with the constant B defined in Prop. 2.11. Then there exists a $\tau := \tau(\|w_0\|_X, B) > 0$ such that the map M,*

$$(2.2.14)\quad (Mw)(t) \;=\; e^{t\overline{A}}w_0 + \int_0^t e^{(t-s)\overline{A}}\Theta[V[Mw(s)]]w(s)\,ds, \quad \forall\, t \in [0,\tau],$$

is a strict contraction from the ball of radius R of Y_τ into itself.

Proof. By (the proof of) Lemma 2.13, the function $z = Mw \in Y_\tau$ satisfies (2.2.11). Under the assumption $\|w\|_{Y_\tau} \le R$, this estimate reads

$$\|Mw(t)\|_X \;\le\; 4e^{\kappa\tau}\|w_0\|_X\left[1 + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}(t + 2t^{1/2})\right],$$

for all $t \in [0,\tau]$. If we assume

$$(2.2.15)\qquad 4e^{\kappa\tau}\|w_0\|_X\left[1 + 4CRe^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}(\tau + 2\tau^{1/2})\right] \;\le\; \frac{R}{3},$$

then $\|Mw(t)\|_X \le \frac{R}{3}$. Similar to (2.2.13) we have

$$\|\nabla_v Mw(t)\|_X \;\le\; Bt^{-1/2}e^{\kappa t}\|w_0\|_X + 4BCRe^{2\kappa\tau}\|w_0\|_X\Big[\pi + 2t^{1/2} + 4CRe^{\left(\kappa\tau+4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}\left(4t^{1/2} + \frac{3}{2}\pi t + \frac{4}{3}t^{3/2}\right)\Big].$$

If we assume

$$(2.2.16)\quad Be^{\kappa\tau}\|w_0\|_X + 4BCRe^{2\kappa\tau}\|w_0\|_X\Big[\pi\tau^{1/2} + 2\tau + {} + 4CRe^{\left(\kappa\tau+4CRe^{\kappa\tau}(\tau+2\tau^{1/2})\right)}\left(4\tau + \frac{3}{2}\pi\tau^{3/2} + \frac{4}{3}\tau^2\right)\Big] \;\le\; \frac{R}{3},$$

then

$$t^{1/2}\|\nabla_v Mw(t)\|_X \;\le\; \frac{R}{3}, \quad \forall\, t \in [0,\tau].$$

Let us now choose

$$(2.2.17)\quad \tau \;:=\; \min\left\{1, \left(\frac{R/3 - 4e^{\kappa}\|w_0\|_X}{48CR\|w_0\|_X e^{2\kappa+12CRe^{\kappa}}}\right)^2, \left(\frac{R/3 - Be^{\kappa}\|w_0\|_X}{4BCRe^{2\kappa}\|w_0\|_X\left[\pi + 2 + 4CRe^{(\kappa+12CRe^{\kappa})}(\frac{3}{2}\pi + \frac{16}{3})\right]}\right)^2\right\}$$

which is positive since $\max\{4, B\}e^{\kappa}\|w_0\|_X < R$. Then, the estimates (2.2.15) and (2.2.16) hold, and hence the operator M maps the ball of radius R of Y_τ into itself.

To prove contractivity we shall estimate $\|Mu - Mw\|_{Y_\tau}$ for all $u, w \in Y_\tau$ with $\|u\|_{Y_\tau}, \|w\|_{Y_\tau} \leq R$. Since

$$\begin{aligned} Mu(t) - Mw(t) &= \int_0^t \mathrm{e}^{(t-s)\overline{A}} \Theta[V[(Mu - Mw)(s)]]u(s)\, ds \\ &\quad + \int_0^t \mathrm{e}^{(t-s)\overline{A}} \Theta[V[Mw(s)]](u - w)(s)\, ds, \quad \forall\, t \in [0, \tau], \end{aligned}$$

by analogous estimates,

$$\begin{aligned} \|Mu(t) - Mw(t)\|_X &\leq 4CR\,\mathrm{e}^{\kappa\tau} \left\{ \int_0^t (1 + s^{-1/2})\, \|(Mu - Mw)(s)\|_X\, ds \right. \\ &\qquad \left. + \|u - w\|_{Y_\tau} \int_0^t (1 + s^{-1/2})\, ds \right\}, \end{aligned}$$

and, by applying Gronwall's Lemma,

$$\begin{aligned} \|Mu(t) - Mw(t)\|_X &\leq 4CR\mathrm{e}^{\kappa\tau} \left[t + 2t^{1/2} + \right. \\ &\quad \left. + 4CR\mathrm{e}^{\left(\kappa\tau + 4CRe^{\kappa\tau}(\tau + 2\tau^{1/2})\right)} \left(2t + 2t^{3/2} + \frac{1}{2}t^2 \right) \right] \|u - w\|_{Y_\tau} \end{aligned}$$

follows. By using $0 \leq t \leq \tau \leq 1$, we obtain

(2.2.18)
$$\|Mu(t) - Mw(t)\|_X \leq 4CR\,\mathrm{e}^{\kappa} \left[3 + 18CR\mathrm{e}^{(\kappa + 12CRe^{\kappa})} \right] \tau^{1/2} \|u - w\|_{Y_\tau}.$$

Similarly,

$$\begin{aligned} &\|\nabla_v Mu(t) - \nabla_v Mw(t)\|_X \leq \\ &\qquad \leq CBR\mathrm{e}^{k\tau} \left\{ \int_0^t (t-s)^{-1/2}(1 + s^{-1/2})\, \|(Mu - Mw)(s)\|_X\, ds \right. \\ &\qquad\qquad \left. + \int_0^t (t-s)^{-1/2}(1 + s^{-1/2})\, ds \|u - w\|_{Y_\tau} \right\}, \end{aligned}$$

and, by using estimate (2.2.18),

$$\begin{aligned} \|\nabla_v Mu(t) - \nabla_v Mw(t)\|_X &\leq CBR\mathrm{e}^{\kappa\tau} \left(\pi + 2t^{1/2} \right) \|u - w\|_{Y_\tau} \\ &\quad \cdot \left[1 + 4CR\,\mathrm{e}^{\kappa} \left(3 + 18CR\mathrm{e}^{(\kappa + 12CRe^{\kappa})} \right) \tau^{1/2} \right], \end{aligned}$$

for all $t \in [0, \tau]$, follows. Then, by exploiting $0 < \tau \leq 1$,

(2.2.19)
$$\begin{aligned} t^{1/2} \|\nabla_v Mu(t) - \nabla_v Mw(t)\|_X &\leq CBR\mathrm{e}^{\kappa}(\pi + 2)\tau^{1/2} \|u - w\|_{Y_\tau} \\ &\quad \cdot \left[1 + 4CR\,\mathrm{e}^{\kappa} \left(3 + 18CR\mathrm{e}^{(\kappa + 12CRe^{\kappa})} \right) \right]. \end{aligned}$$

When choosing $\tau > 0$ small enough, estimates (2.2.18), (2.2.19) imply

$$\|Mu - Mw\|_{\mathcal{C}([0,\tau];X)} \leq C\|u - w\|_{\mathcal{C}([0,\tau];X)},$$

for some $C < 1$, and the assertion is proved. □

2.15 Corollary. *There exists a* $t_{\max} \leq \infty$ *such that the initial value problem (2.2.8) has a unique* mild *solution* w *in* Y_T, $\forall T < t_{\max}$, *which satisfies*

$$w(t) = e^{t\overline{A}}w_0 + \int_0^t e^{(t-s)\overline{A}}\Theta[V[w(s)]]w(s)\,ds, \quad \forall t \in [0, T]. \tag{2.2.20}$$

Moreover, if $t_{\max} < \infty$, *then*

$$\lim_{t \nearrow t_{\max}} \|w(t)\|_X = \infty.$$

Proof. The solution of the problem is the fixed point of the map M previously introduced. By Prop. 2.14 this solution exists for a time interval of length τ (depending only on $\|w_0\|_X$) and it belongs to the space Y_τ. Since, in particular, $w(\tau) \in X$, the solution can be repeatedly continued up to the maximal time $t_{\max}$. It will then belong to Y_T, $\forall T < t_{\max}$.
If the second assertion of the corollary would not hold, there would be a sequence of times $t_n \uparrow t_{\max}$ such that $\|w(t_n)\|_X \leq C$ for all n. Then, by solving a problem with the initial value $w(t_n)$, with t_n sufficiently close to $t_{\max}$, we would extend the solution up to a certain time $t_n + \tau(\|w(t_n)\|_X) > t_{\max}$. This construction would contradict our definition of $t_{\max}$.
The uniqueness of the mild solution follows by arguments analogous to those in the proof of Thm. 6.1.4 in [**Pa**]. □

2.16 Remark. Note that the last statement in the thesis of the Corollary 2.15 differs from the standard setting (cf. Thm. 6.1.4 in [**Pa**]). For $t_{\max} < \infty$ we conclude the 'explosion' of $w(t)$, $t \to t_{max}$ in X and not only in Y_t. This is due to the parabolic regularization of the problem (2.2.8).

2.3 Global-in-time solution, a-priori estimates

In this section we shall exploit dispersive effects of the free transport equation to derive an a-priori estimate on the electric field. This is the key ingredient for proving the main result of the chapter, the global solution for the WPFP system:

2.17 Theorem. *Let $w_0 \in X$ satisfy for some $\omega \in [0,1)$ and $T > 0$*

$$(\mathbf{A}) \qquad \Big\| \int w_0(x - \vartheta(t)v, v)\, dv \Big\|_{L^{6/5}(\mathbb{R}^3_x)} \;\le\; C_T \vartheta(t)^{-\omega}, \quad \forall\, t \in (0,T],$$

with $\vartheta(t) := \frac{1-e^{-\beta t}}{\beta}$ for $\beta > 0$, and $\vartheta(t) = t$ for $\beta = 0$. Then the WPFP equation (2.2.8) admits a unique global-in-time mild solution $w \in Y_T$, $\forall\, 0 < T < \infty$.

In order to prove that $t_{\max} = \infty$, we have to show that $\|w(t)\|_X$ is finite for all $t \ge 0$ (cf. Corollary 2.15). To this end, we shall derive a-priori estimates for $\|w(t)\|_2$ and $\||v|^2 w(t)\|_2$. Thus, the proof of Thm. 2.17 will be a consequence of a series of Lemmata, in particular of Lemma 2.18 and Lemma 2.35. In the sequel, $w(t)$ denotes the unique mild solution for $0 \le t \le T$, for an arbitrary $0 < T < t_{\max}$.

2.18 Lemma. *For all $w_0 \in X$, the mild solution of the WPFP equation (2.2.8) satisfies*

$$(2.3.1) \qquad \|w(t)\|_2^2 \;\le\; e^{3\beta t}\|w_0\|_2^2, \quad \forall\, t \in [0,T].$$

Proof. Roughly speaking, this follows from the dissipativity of the operator $\overline{A} - \frac{3\beta}{2}$ in $L^2(\mathbb{R}^6)$ (cf. (2.4.2)) and the skew-symmetry of the pseudo-differential operator. However, since we are dealing only with the *mild* solution of the equation, the proof requires an approximation of w by classical solutions.
Since the solution satisfies $w \in Y_T$, $\forall T < t_{\max}$, Prop. 2.6 shows that the function $f(t) := \Theta[V[w(t)]]w(t)$, $t \in (0, t_{\max})$ is well defined and it is in $\mathcal{C}((0,t_{\max}); X) \cap L^1((0,T); X)$, $\forall\, 0 < T < t_{\max}$.
For $0 < T < t_{\max}$ fixed, let us consider the following linear inhomogeneous problem:

$$(2.3.2) \qquad \frac{d}{dt}y(t) \;=\; \overline{A}y(t) + f(t), \quad t \in [0,T], \qquad y(t=0) \;=\; w_0 \in X.$$

Its mild solution in $[0,T]$ is the function w, due to the uniqueness of the mild solution of problem (2.2.8). For this linear problem, we can apply Thm. 4.2.7 of [**Pa**]: The mild solution w is the uniform limit (on $[0,T]$) of *classical* solutions of problem (2.3.2). More precisely, there is a sequence $\{w_0^n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\overline{A})$, $w_0^n \to w_0$ in X, and a sequence $\{f_n(t)\} \subset \mathcal{C}^1([0,T]; X)$, $f_n(t) \to f(t)$ in $L^1((0,T); X)$. And the classical solutions $y_n \in \mathcal{C}^1([0,T]; X)$ of the corresponding problems

$$(2.3.3) \qquad \frac{d}{dt}y_n(t) \;=\; \overline{A}y_n(t) + f_n(t), \quad t \in [0,T], \qquad y_n(t=0) \;=\; w_0^n,$$

converge in $C([0,T];X)$ to the solution w of problem (2.3.2).

We shall need these approximating classical solutions y_n in order to justify the derivation of the a-priori estimate: Multiplying both sides of (2.3.3) by $y_n(t)$ and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|y_n(t)\|_2^2 \;\le\; \frac{3\beta}{2}\,\|y_n(t)\|_2^2 + \iint y_n(t) f_n(t)\,dx\,dv,$$

since the operator $\overline{A}-\frac{3\beta}{2}$ is dissipative in $L^2(\mathbb{R}^6)$ (cf. (2.4.2)). By integrating in t and letting $n\to\infty$, we have

$$\|w(t)\|_2^2 \le \|w_0\|_2^2 + 3\beta\int_0^t \|w(s)\|_2^2\,ds + 2\int_0^t\iint w(s)f(s)\,dx\,dv\,ds, \quad \forall\, t\in[0,T].$$

The second integral is equal to zero since the pseudo-differential operator Θ is skew-symmetric. Hence, applying Gronwall's Lemma yields

$$\|w(t)\|_2^2 \;\le\; e^{3\beta t}\|w_0\|_2^2, \quad \forall\, t\in[0,T]. \tag{2.3.4}$$

□

In order to recover similar estimates for $\||v|^2 w(t)\|_2$, we first need a-priori bounds for the self-consistent field $E=\nabla V$. To this end, we are going to exploit dispersive effects of the free streaming operator. We shall adapt to the Wigner-Poisson and Wigner-Poisson-Fokker-Planck problems the strategies introduced for the (classical) Vlasov-Poisson problem ([**LP, Pe**]), and for the Vlasov-Poisson-Fokker-Planck problem ([**Bo93, Bo95, Ca98**]).

2.3.1 A-priori estimates for the electric field: the Wigner-Poisson case

To explain the strategy, we first consider the (simpler) WP problem: Let us assume that w^{wp} is a "regular" solution of the WP problem (e.g., let $w^{\text{wp}}(t)\in L^2_x(H^1_v), \nabla_x V[w^{\text{wp}}](t)\in \mathcal{C}_B(\mathbb{R}^3)$, uniformly on bounded t-intervals) for which the Duhamel formula holds:

$$w^{\text{wp}}(x,v,t) \;=\; w_0^{\text{wp}}(x-tv,v) + \int_0^t \Big(\Theta[V[w^{\text{wp}}]]w^{\text{wp}}\Big)(x-sv,v,t-s)\,ds.$$

We formally integrate in v:

$$\begin{aligned} n[w^{\text{wp}}](x,t) \;&=\; \int_{\mathbb{R}^3} w_0^{\text{wp}}(x-tv,v)\,dv \\ &\quad + \int_0^t\int_{\mathbb{R}^3}\Big(\Theta[V[w^{\text{wp}}]]w^{\text{wp}}\Big)(x-sv,v,t-s)\,dv\,ds \\ &=: \; n_0^{\text{wp}}(x,t) + n_1^{\text{wp}}(x,t), \end{aligned}$$

and split the self-consistent field accordingly:

$$(2.3.5) \qquad \begin{aligned} E_0^{\mathrm{wp}}(x,t) &:= \lambda \frac{x}{|x|^3} *_x n_0^{\mathrm{wp}}(x,t) \\ &= \lambda \frac{x}{|x|^3} *_x \int w_0^{\mathrm{wp}}(x-tv,v)\,dv, \end{aligned}$$

$$(2.3.6) \quad E_1^{\mathrm{wp}}(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x \int_0^t \!\! \int \Big(\Theta[V[w^{\mathrm{wp}}]] w^{\mathrm{wp}} \Big)(x-sv, v, t-s)\,dv\,ds,$$

with $\lambda = \frac{1}{4\pi}$.
Then, we can estimate separately the two terms $E_0^{\mathrm{wp}}(t)$, $E_1^{\mathrm{wp}}(t)$ by exploting the properties of the convolution kernel $1/|x|$ (cf. [**LP, Pe**] for VP, [**Bo93, Bo95, Ca98**] for VPFP, [**ALMS**] for WPFP). To this end, we need an appropriate redefinition of the pseudo-differential operator $\Theta[V]$ in (1.0.5). It is inspired by the operator $\nabla_x V \cdot \nabla_v w$ in the VP equation that can be recovered from $\Theta[V]w$ in the semiclassical limit (cf. Remark 2.21).

Let us recall that

$$\Theta[V]w(x,v) = \mathcal{F}^{-1}_{\eta\to v}\Big(i\,\delta V(x,\eta) \mathcal{F}_{v\to\eta} w(x,\eta) \Big).$$

We can rewrite

$$(2.3.7) \quad \delta V(x,\eta) \;=\; \int_{x-\eta/2}^{x+\eta/2} \nabla_x V(z)\cdot dz \;=\; \int_{-1/2}^{1/2} \eta\cdot\nabla_x V(x-r\eta)\,dr \;=\; \eta\cdot W(x,\eta)\,,$$

with the vector-valued function

$$W(x,\eta) \;:=\; \int_{-1/2}^{1/2} \nabla_x V(x-r\eta)\,dr\,, \quad \forall\,(x,\eta)\in\mathbb{R}^6\,.$$

Then, we define the vector-valued operator

$$(2.3.8) \qquad \Gamma[\nabla_x V]u(x,v) \;:=\; \mathcal{F}^{-1}_{\eta\to v}\Big(W(x,\eta)\mathcal{F}_{v\to\eta} u(x,\eta) \Big).$$

It holds:

2.19 Lemma. *Let $\nabla_x V \in \mathcal{C}_B(\mathbb{R}^3)$. Then*

1. $W(x,\eta) \in \mathcal{C}_B(\mathbb{R}^6)$, $\|W\|_\infty \;\le\; \|\nabla_x V\|_\infty$;

2. $\Gamma[\nabla_x V] : L^2(\mathbb{R}^6) \to L^2(\mathbb{R}^6)$ *and, for all* $u \in L^2(\mathbb{R}^6)$,

$$\|\Gamma[\nabla_x V]u\|_{L^2(\mathbb{R}^6)} \;\le\; \|\nabla_x V\|_\infty \|u\|_{L^2(\mathbb{R}^6)};$$

3. $\Gamma[\nabla_x V] : L^2_x(H^1_v) \to L^2_x(H^1_v)$ *and, for all* $u \in L^2_x(H^1_v)$,

$$\|\Gamma[\nabla_x V]u\|_{L^2_x(H^1_v)} \;\le\; \|\nabla_x V\|_\infty \|u\|_{L^2_x(H^1_v)}. \tag{2.3.9}$$

Proof. The first and the second assertion are obvious. For (2.3.9) we use

$$\begin{aligned} \partial_{v_j}\Gamma[\nabla_x V]u(x,v) &= i\mathcal{F}^{-1}_{\eta\to v}\Big(\eta_j W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big) \\ &= \Gamma[\nabla_x V]\partial_{v_j}u; \qquad j = 1,2,3. \end{aligned} \tag{2.3.10}$$

□

2.20 Lemma. *Let* $\nabla_x V \in \mathcal{C}_B(\mathbb{R}^3)$ *and* $u \in L^2_x(H^1_v)$. *Then*

$$\Theta[V]u(x,v) \;=\; \mathrm{div}_v\left(\Gamma[\nabla_x V]u\right)(x,v) \tag{2.3.11}$$

Proof. By the definition (2.3.7) and Lemma 2.19,

$$\|\delta V(\,.\,,\eta)\|_\infty \;\le\; |\eta|\,\|W(\,.\,,\eta)\|_\infty \;\le\; |\eta|\,\|\nabla_x V\|_\infty.$$

Thus, $\|\Theta[V]u\|_{L^2(\mathbb{R}^6)} \le \|\nabla_x V\|_\infty \|u\|_{L^2_x(H^1_v)}$; the right hand side of equation (2.3.11) is also well-defined in $L^2(\mathbb{R}^6)$ by estimate (2.3.9). Equality then follows by equation (2.3.7) and

$$\begin{aligned} i\mathcal{F}^{-1}_{\eta\to v}\Big(\eta\cdot W(x,\eta)\mathcal{F}_{v\to\eta}u(x,\eta)\Big) &= \sum_{j=1}^{3}\partial_{v_j}\Gamma_j[\nabla_x V]u(x,v) \\ &= \mathrm{div}_v\left(\Gamma[\nabla_x V]u\right)(x,v). \end{aligned}$$

□

2.21 Remark (The semiclassical limit). The correctly scaled version of the pseudo-diffe-rential operator with the reduced Planck constant $\hbar = \frac{h}{2\pi}$ reads

$$\Theta_\hbar[V]w(x,v) \;=\; \frac{i}{(2\pi)^{3/2}}\int_{\mathbb{R}^3}\frac{V(x+\frac{\hbar}{2}\eta)-V(x-\frac{\hbar}{2}\eta)}{\hbar}\mathcal{F}_{v\to\eta}w(x,\eta)e^{iv\cdot\eta}\,d\eta.$$

Under the assumptions of Lemma 2.20, we thus have

$$\begin{aligned} \mathcal{F}_{v\to\eta}(\Theta_\hbar[V]w(x,v)) &= \frac{i}{\hbar}\,\delta V(x,\hbar\eta)\mathcal{F}_{v\to\eta}w(x,\eta) \\ &= i\,W(x,\hbar\eta)\,\cdot\,\eta\,\mathcal{F}_{v\to\eta}w(x,\eta). \end{aligned}$$

The limit $\hbar \to 0$ then yields:

$$\begin{aligned} i\, W(x, \hbar\eta) \cdot \eta\, \mathcal{F}_{v\to\eta} w(x,\eta) &\longrightarrow i\nabla_x V(x) \cdot \eta\, \mathcal{F}_{v\to\eta} w(x,\eta) \\ &= \mathcal{F}^{-1}_{\eta\to v}\Big(\nabla_x V(x) \cdot \nabla_v w(x,v)\Big); \end{aligned}$$

and hence

$$\Theta_{\hbar}[V]w(x,v) \longrightarrow \nabla_x V(x) \cdot \nabla_v w(x,v) \quad \text{in } L^2(\mathbb{R}^6),$$

which is the non-linear term in the VP equation.

Using the redefinition (2.3.11) of the pseudo-differential operator, and under the additional assumptions $w^{\text{wp}} \in H^1_x(L^2_v)$, $\Delta V[w^{\text{wp}}] \in \mathcal{C}_B(\mathbb{R}^3)$, we have for $s \in \mathbb{R}$

$$\begin{aligned} \Big(\Theta[V[w^{\text{wp}}]]w^{\text{wp}}\Big)(x - sv, v) &= \operatorname{div}_v\Big(\Gamma[\nabla_x V[w^{\text{wp}}]]w^{\text{wp}}(x - sv, v)\Big) \\ &\quad + s\operatorname{div}_x\Big(\Gamma[\nabla_x V[w^{\text{wp}}]]w^{\text{wp}}\Big)(x - sv, v). \end{aligned} \tag{2.3.12}$$

Thus, the field E_1^{wp} in (2.3.6) can be rewritten as $(j = 1,2,3)$

(2.3.13)

$$\begin{aligned} (E_1^{\text{wp}})_j(x,t) &:= \lambda \frac{x_j}{|x|^3} *_x \operatorname{div}_x \int_0^t s \int \Big(\Gamma[\nabla_x V[w^{\text{wp}}]]w^{\text{wp}}\Big)(x - sv, v, t-s)\, dv\, ds \\ &= \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} \\ &\quad *_x \int_0^t s \int \Big(\Gamma_k[\nabla_x V[w^{\text{wp}}]]w^{\text{wp}}\Big)(x - sv, v, t-s)\, dv\, ds. \end{aligned}$$

The following two lemmata are concerned with giving a meaning to the definition (2.3.13) of the field E_1, independently of the previous derivation.

2.22 Lemma. *For all $u \in L^2(\mathbb{R}^6)$ and $E \in L^2(\mathbb{R}^3)$ the following estimate holds*

$$\Big\| \int_{\mathbb{R}^3_v} (\Gamma[E]u)(x - sv, v)\, dv \Big\|_{L^2(\mathbb{R}^3_x)} \le C s^{-3/2} \|E\|_2 \|u\|_2\,, \quad \forall\, s > 0. \tag{2.3.14}$$

Proof. Since the operator $\Gamma[.]$ was originally defined for $E \in \mathcal{C}_B(\mathbb{R}^3)$, we shall first derive (2.3.14) for $E \in \mathcal{C}^\infty_0(\mathbb{R}^3)$ and conclude by a density argument. By the definition (2.3.8) and by several changes of variables, the following chain of equalities holds:

$$\begin{aligned}
(\Gamma[E]u)(x,v) &= (2\pi)^{3/2}\left[\mathcal{F}^{-1}_{\eta\to v}\left(W(x,\eta)\right) *_v u\right](x,v) \\
&= \iiint_{-1/2}^{1/2} E(x-r\eta)\, e^{i\eta\cdot z}\, dr\, d\eta\, u(x,v-z)\, dz \\
&= \iiint_{-1/2}^{1/2} \frac{1}{|r|^3} E(x-\widetilde{\eta})\, e^{i\tilde{\eta}\cdot\frac{z}{r}}\, dr\, d\tilde{\eta}\, u(x,v-z)\, dz \\
&= \iint E(x-\tilde{\eta})\, e^{i\tilde{\eta}\cdot\tilde{z}}\, d\tilde{\eta} \int_{-1/2}^{1/2} u(x,v-r\tilde{z})\, dr\ d\tilde{z} \\
&= \iint E(-\hat{\eta})\, e^{i\hat{\eta}\cdot\tilde{z}}\, d\hat{\eta}\, e^{ix\cdot\tilde{z}} \int_{-1/2}^{1/2} u(x,v-r\tilde{z})\, dr\ d\tilde{z} \\
&= (2\pi)^{3/2} \int \mathcal{F}_{\eta\to\tilde{z}} E(\tilde{z}) \int_{-1/2}^{1/2} u(x,v-r\tilde{z})\, dr\, e^{ix\cdot\tilde{z}}\, d\tilde{z}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int \left(\Gamma[E]u\right)(x-sv,v)\, dv &= \\
&= (2\pi)^{\frac{3}{2}} \int \mathcal{F}_{\eta\to\tilde{z}} E(\tilde{z}) \left(\iint_{-1/2}^{1/2} u(x-sv, v-r\tilde{z})\, dr\, e^{-isv\cdot\tilde{z}}\, dv \right) e^{ix\cdot\tilde{z}}\, d\tilde{z} \\
&= \frac{1}{(2\pi s)^3} \int \mathcal{F}_{\eta\to\tilde{z}} E(\tilde{z})\, \mathcal{F}_{v\to\tilde{z}} \left(\int_{-1/2}^{1/2} u(x-v, \frac{v}{s} - r\tilde{z})\, dr \right) e^{ix\cdot\tilde{z}}\, d\tilde{z}\,.
\end{aligned}$$

Then,

$$\begin{aligned}
\left\| \int \left(\Gamma[E]u\right)(x-sv,v)\, dv \right\|_{L^2(\mathbb{R}^3_x)} &\le \\
&\le \frac{\|E\|_2}{(2\pi s)^3} \left\| \int_{-1/2}^{1/2} \mathcal{F}_{v\to\tilde{z}} \Big(u(x-v, \frac{v}{s} - r\tilde{z}) \Big)\, dr \right\|_{L^2(\mathbb{R}^3_{\tilde{z}}\times\mathbb{R}^3_x)} \\
&\le \frac{\|E\|_2}{(2\pi s)^3} \left(\int_{-1/2}^{1/2} \left\| \mathcal{F}_{v\to\tilde{z}} \Big(u(x-v, \frac{v}{s} - r\tilde{z}) \Big) \right\|^2_{L^2(\mathbb{R}^3_{\tilde{z}}\times\mathbb{R}^3_x)} dr \right)^{\frac{1}{2}},
\end{aligned}$$

by applying Hölder's inequality first in the $\tilde{z}$ integral and then in the r integral. Finally, it remains to prove that

$$\int_{-1/2}^{1/2} \left\| \mathcal{F}_{v\to z} \Big(u(x-v, \frac{v}{s} - rz) \Big) \right\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_z)} dr \;=\; s^3 \|u(x,v)\|^2_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)}.$$

This is obtained by using repeatedly Plancherel's equality:

$$\begin{aligned}
\int_{-1/2}^{1/2} \Big\| \mathcal{F}_{v\to z}\Big(u(x-v,\frac{v}{s}-rz)\Big)\Big\|^2_{L^2_{x,z}} dr \;=& \\
=& \int_{-1/2}^{1/2} \Big\| \mathcal{F}_{x\to\xi}\Big[\mathcal{F}_{v\to z}\Big(e^{-iv\xi}u(x,\frac{v}{s}-rz)\Big)\Big]\Big\|^2_{L^2_{\xi,z}} dr \\
=& \int_{-1/2}^{1/2} \Big\| \mathcal{F}_{x\to\xi}\Big(s^3 e^{-is(\xi+z)rz}\mathcal{F}_{v\to s(\xi+z)}u(x,v)\Big)\Big\|^2_{L^2_{\xi,z}} dr \;= \\
=& \; s^6\int_{-1/2}^{1/2} \Big\| \mathcal{F}_{x\to\xi}\Big(\mathcal{F}_{v\to s(\xi+z)}u(x,v)\Big)\Big\|^2_{L^2_{\xi,z}} dr \\
=& \; s^3\|u(x,v)\|^2_{L^2_{x,v}}.
\end{aligned}$$

□

2.23 Remark. Observe that the exponent of the variable s recovered in the Lemma is the same as obtained for the VP case (cf. [**Pe**], e.g.) in the L^2-estimate of $\int_{\mathbb{R}^3_v} Eu(x-sv,v)\,dv$. In the classical case, analogous L^p-estimates hold in addition. In the quantum counterpart, instead, the L^2-framework is the only possible, since the estimate had to be derived in Fourier space. Moreover, to derive a more refined version of this basic estimate (cf. [**LP, Bo93**]), the non-negativity of the classical distribution is a crucial ingredient. And this non-negativity does not hold for Wigner functions.

The following lemma is an immediate consequence of Lemma 2.22. We shall need the notation

$$V_{T,\omega} := \{E \in C((0,T];\; L^2_x(\mathbb{R}^3) \;\big|\; \|E\|_{V_{T,\omega}} < \infty\}$$

with

$$\|E\|_{V_{T,\omega}} := \sup_{0<t\le T} t^\omega \|E(t)\|_{L^2}.$$

2.24 Lemma. *For any fixed $T>0$, let $w \in \mathcal{C}([0,T];L^2_{x,v})$, and let w_0 satisfy for some $\omega \in [0,1)$:*

$$\text{(B)} \qquad \Big\|\int w_0(x-tv,v)\,dv\Big\|_{L^{6/5}(\mathbb{R}^3_x)} \;\le\; C_T t^{-\omega}, \quad \forall\, t\in(0,T].$$

Then, there exists a unique vector-field $E \in V_{T,\omega-\frac{1}{2}}$ which satisfies the linear equation for $j=1,2,3$:

$$\text{(2.3.15)} \qquad E_j(x,t) \;=\; \lambda\sum_{k=1}^{3}\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} *_x \int_0^t s\int \Big(\Gamma_k[E_0+E]w\Big)(x-sv,v,t-s)\,dv\,ds,$$

with E_0 defined by ($\lambda = \frac{1}{4\pi}$):

$$E_0(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x \int w_0(x - tv, v)\, dv.$$

PROOF. (2.3.15) has the structure of a Volterra integral equation of the second kind. Hence, we define the (affine) map $M : V_{T,\omega-\frac{1}{2}} \to V_{T,\omega-\frac{1}{2}}$ by

$$\begin{aligned}(ME)_j(x,t) \;&:=\; \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} \\ &\quad *_x \int_0^t s \int \Big(\Gamma_k[E_0 + E]w\Big)(x - sv, v, t-s)\, dv\, ds.\end{aligned}$$

Applying the generalized Young inequality to the definition of E_0 yields

$$\|E_0(t)\|_2 \;\le\; C \Big\| \int w_0(x - vt, v)\, dv \Big\|_{L^{6/5}(\mathbb{R}^3_x)}, \quad \forall\, t \in (0,T]\,. \tag{2.3.16}$$

Thus, by Lemma 2.22, the second convolution factor in (2.3.15) is well-defined and

$$\begin{aligned}&\Big\| \int_{\mathbb{R}^3_v} \Big(\Gamma_k[E_0 + E]w\Big)(x - sv, v, t-s)\, dv \Big\|_{L^2(\mathbb{R}^3_x)} \;\le \\ &\qquad \le\; C s^{-3/2} \|(E_0 + E)(t-s)\|_2 \|w(t-s)\|_2, \quad \forall\, s \in (0,t].\end{aligned}$$

By classical properties of the convolution with $\frac{1}{|x|}$ (cf. [**St**]) and the Young inequality, we get

$$\|(ME)_j(t)\|_2 \;\le\; C \int_0^t \frac{1}{\sqrt{s}} \Big(\|E_0(t-s)\|_2 + \|E(t-s)\|_2\Big) \|w(t-s)\|_2\, ds\,, \tag{2.3.17}$$

for all $t \in (0,T]$. Hence, the map M is well-defined from $V_{T,\omega-\frac{1}{2}}$ into itself and satisfies

$$\|ME(t)\|_2 \;\le\; C\Big(C_T + \sup_{s\in(0,T]} s^{\omega-\frac{1}{2}} \|E(s)\|_2\Big) \sup_{s\in[0,T]} \|w(s)\|_2 \,\Big(t^{1-\omega} + t^{\frac{1}{2}-\omega}\Big)\,,$$

for all $t \in (0,T]$. Since the map is affine, we have (by induction) for all $t \in (0,T]$

$$\begin{aligned}&\|M^n E(t) - M^n \widetilde{E}(t)\|_2 \;\le \\ &\quad \le\; C \sup_{s\in[0,T]} \|w(s)\|_2 \int_0^t \frac{1}{\sqrt{t-s}} \|M^{n-1}E(s) - M^{n-1}\widetilde{E}(s)\|_2\, ds \\ &\quad \le\; \Big(C \sup_{s\in[0,T]} \|w(s)\|_2\Big)^n C_{n-1} \int_0^t \frac{s^{\frac{n}{2}-\omega}}{\sqrt{t-s}}\, ds \sup_{s\in(0,T]} \Big(s^{\omega-\frac{1}{2}} \|E(s) - \widetilde{E}(s)\|_2\Big),\end{aligned}$$

with

$$\int_0^t \frac{s^{\frac{n}{2}-\omega}}{\sqrt{t-s}}\,ds \;=\; t^{\frac{n+1}{2}-\omega} B\left(\frac{1}{2}, \frac{n+2}{2}-\omega\right),$$

$$C_{n-1} \;=\; \prod_{j=1}^{n-1} B\left(\frac{1}{2}, \frac{j}{2}+1-\omega\right) \;=\; \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{3}{2}-\omega\right)}{\Gamma\left(\frac{n}{2}+1-\omega\right)},$$

where B denotes the Beta function and Γ the Gamma function. Thus, the map M^n is contractive for n large enough and admits a unique fixed point $E \in V_{T,\omega-\frac{1}{2}}$. □

With $E = E_1^{wp}$ the above lemma yields the regularity of the self-consistent field in the WP equation: It satisfies $\nabla_x V[w^{\rm wp}] = E_1^{wp} + E_0^{\rm wp} \in V_{T,\omega-\frac{1}{2}}$, under the assumptions that $w^{\rm wp} \in \mathcal{C}([0,T]; L^2_{x,v})$ and $w_0^{\rm wp}$ satisfies (**B**).

2.25 Proposition. *For any fixed $T > 0$, let $w^{wp} \in \mathcal{C}([0,T]; L^2_{x,v})$ be a mild solution of the WP equation with $\|w^{wp}(t)\|_2 = \|w_0^{wp}\|_2$, and with the initial value w_0^{wp} satisfying condition (**B**). Then, the self-consistent field satisfies the following estimates for all $t \in (0,T]$:*

$$\|E_0^{wp}(t)\|_2 \;\le\; C\left\|\int w_0^{wp}(x-vt,v)\,dv\right\|_{L^{6/5}(\mathbb{R}^3_x)} \;\le\; CC_T t^{-\omega}, \tag{2.3.18}$$

(2.3.19)

$$\|E_1^{wp}(t)\|_2 \;\le\; C\left(\|w_0^{wp}\|_2, \sup_{s\in(0,T]}\left\{s^\omega\left\|\int w_0^{wp}(x-sv,v)dv\right\|_{L^{6/5}}\right\}, T\right) t^{\frac{1}{2}-\omega}.$$

Here and in the sequel, the T-dependence of the constants C is continuous (on $T \in \mathbb{R}^+$).

Proof. The first estimate is (2.3.16) in Lemma 2.24. To derive the second one, we exploit eq. (2.3.17), the conservation of the L^2-norm of the solution and (2.3.18):

$$\begin{aligned}\|E_1^{\rm wp}(t)\|_2 \;&\le\; C\int_0^t s^{-1/2}(\|E_0^{\rm wp}(t-s)\|_2 + \|E_1^{\rm wp}(t-s)\|_2)\|w^{\rm wp}(t-s)\|_2\,ds\\ &\le\; C\|w_0^{\rm wp}\|_2 \sup_{s\in(0,T]}\left\{s^\omega\left\|\int w_0^{\rm wp}(x-sv,v)dv\right\|_{L^{6/5}}\right\} t^{\frac{1}{2}-\omega}\\ &\quad + C\|w_0^{\rm wp}\|_2\int_0^t (t-s)^{-1/2}\|E_1^{\rm wp}(s)\|_2\,ds.\end{aligned}$$

The thesis follows by Gronwall's Lemma. □

We shall now state a simple condition on w_0 that implies both conditions (**A**), (**B**). For $w_0 \in L^1_x(L^{6/5}_v)$ a Strichartz inequality for the free transport equation (cf. Th. 2 in [**CP**]) reads:

$$\left\| \int w_0(x - tv, v)\, dv \right\|_{L^{6/5}(\mathbb{R}^3_x)} \le t^{-\frac{1}{2}} \|w_0\|_{L^1_x(L^{6/5}_v)},\ \ t > 0, \tag{2.3.20}$$

and hence (**A**) and (**B**) hold.

2.26 Remark. Let us again compare the a-priori bounds (2.3.18), (2.3.19) with their classical counterparts. Using (2.3.20) we obtain the same $t^{-\frac{1}{2}}$–singularity of $\|E^{wp}(t)\|_2$ for the Wigner-Poisson system, as it was obtained in [**CP**] for the VP equation. In the latter case, similar L^p-estimates hold for p in a non-trivial interval. One crucial reason for this difference is the conservation of L^p-norm of the solution: while the WP equation only conserves the L^2-norm, all L^p-norms are constant in the VP case. A second reason is that we cannot exploit any pseudo-conformal law for the quantum case, since the Wigner functions are not non-negative (cf. [**Pe**] for the classical case).

As a by-product we obtain the following result for the self-consistent potential V, which follows directly from the splitting $V^{\mathrm{wp}} = V_0^{\mathrm{wp}} + V_1^{\mathrm{wp}}$

$$V_0^{\mathrm{wp}}(x,t) \ := \ \lambda \sum_{i=1}^{3} \frac{x_i}{|x|^3} *_x (E_0^{\mathrm{wp}})_i(x,t), \tag{2.3.21}$$

$$V_1^{\mathrm{wp}}(x,t) \ := \ \lambda \sum_{i=1}^{3} \frac{x_i}{|x|^3} *_x (E_1^{\mathrm{wp}})_i(x,t). \tag{2.3.22}$$

2.27 Corollary. *Under the assumptions of Proposition 2.25, the self-consistent potential $V^{wp} = V_0^{wp} + V_1^{wp}$ satisfies the following estimates for all $t \in (0,T]$:*

$$\|V_0^{wp}(t)\|_6 \ \le \ CC_T t^{-\omega}, \tag{2.3.23}$$

(2.3.24)

$$\|V_1^{wp}(t)\|_6 \ \le \ C\left(\|w_0^{wp}\|_2, \sup_{s\in(0,T]} \left\{ s^\omega \left\| \int w_0^{wp}(x - sv, v) dv \right\|_{L^{6/5}} \right\}, T \right) t^{\frac{1}{2}-\omega}.$$

2.3.2 A-priori estimates for the electric field: the WPFP case

According to Corollary 2.15, the mild solution of the WPFP problem satisfies for all $t \in [0,T]$ $(0 < T < t_{\max})$

$$\begin{aligned} w(x,v,t) \ &= \ \iint G(t, x - x_0, v, v_0) w_0(x_0, v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \iint G(s, x - x_0, v, v_0)(\Theta[V]w)(x_0, v_0, t - s)\, dx_0\, dv_0\, ds \end{aligned}$$

with the Green's function G from (2.2.4). According to [**SCDM**] we have

$$\int_{\mathbb{R}^3} G(t, x - x_0, v, v_0)\, dv \quad = \quad R(t)^{-3/2} \mathcal{N}\left(\frac{x - x_0 - \vartheta(t) v_0}{\sqrt{R(t)}}\right),$$

with

$$\mathcal{N}(x) \quad := \quad (2\pi)^{-3/2} \exp\left(-\frac{|x|^2}{2}\right), \tag{2.3.25}$$

$$\vartheta(t) \quad := \quad \frac{1 - e^{-\beta t}}{\beta} \quad = \quad \mathcal{O}(t), \quad \text{for } t \to 0; \tag{2.3.26}$$

$$\vartheta(t) \quad := \quad t, \quad \text{if } \beta = 0,$$

$$\begin{aligned} R(t) \quad &:= \quad 2\alpha t + \sigma \left[\frac{4e^{-\beta t} - e^{-2\beta t} + 2\beta t - 3}{\beta^3}\right] + 4\gamma \left[\frac{e^{-\beta t} + \beta t - 1}{\beta^2}\right] \\ &= \quad \mathcal{O}(t), \quad \text{for } t \to 0. \end{aligned} \tag{2.3.27}$$

By exploiting the redefinition (2.3.11) of the pseudo-differential operator, we obtain the following expression for the density $n[w]$

$$\begin{aligned} n[w](x,t) \quad &= \quad \int_{\mathbb{R}^3} w(x, v, t)\, dv \\ &= \quad \frac{1}{R(t)^{3/2}} \iint \mathcal{N}\left(\frac{x - x_0 - \vartheta(t) v_0}{\sqrt{R(t)}}\right) w_0(x_0, v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \frac{1}{R(s)^{3/2}} \iint \mathcal{N}\left(\frac{x - x_0 - \vartheta(s) v_0}{\sqrt{R(s)}}\right) \\ &\quad \cdot \operatorname{div}_{v_0} \left(\Gamma[\nabla_{x_0} V] w\right)(x_0, v_0, t - s)\, dx_0\, dv_0\, ds \\ &= \quad \frac{1}{R(t)^{3/2}} \iint \mathcal{N}\left(\frac{x - x_0}{\sqrt{R(t)}}\right) w_0(x_0 - \vartheta(t) v_0, v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \frac{\vartheta(s)}{R(s)^2} \iint (\nabla_x \mathcal{N}) \left(\frac{x - x_0 - \vartheta(s) v_0}{\sqrt{R(s)}}\right) \\ &\quad \cdot \left(\Gamma[\nabla_{x_0} V] w\right)(x_0, v_0, t - s)\, dx_0\, dv_0\, ds \\ &= \quad n_0(x,t) + n_1(x,t), \end{aligned}$$

where

$$\begin{aligned} n_0(x,t) \quad &:= \quad \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^{\vartheta}(x,t), \\ \text{with} \quad n_0^{\vartheta}(x,t) \quad &:= \quad \int w_0(x - \vartheta(t) v, v)\, dv, \end{aligned}$$

$$n_1(x,t) \;:=\; \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) \\ *_x \mathrm{div}_x \int \Big(\Gamma[\nabla_x V]w\Big)(x-\vartheta(s)v, v, t-s)\, dv\, ds.$$

Correspondingly, we can split the field (with $\lambda = \frac{1}{4\pi}$):

$$E_0(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x n_0(x,t) \\ = \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x E_0^\vartheta(x,t), \tag{2.3.28}$$

with

$$E_0^\vartheta(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x n_0^\vartheta(x,t),$$

and

$$E_1(x,t) \;:=\; \lambda \frac{x}{|x|^3} *_x n_1(x,t). \tag{2.3.29}$$

2.28 Remark. Note that the splitting of the density (and of the electric field) is the same as in [**Bo93, Bo95, Ca98**]: in the WPFP case the two components of the decomposition (n_0, n_1, as well as E_0, E_1) are smoothed versions (in fact, convolutions with a Gaussian) of those appearing in the WP case (namely $n_0^{\mathrm{wp}}, n_1^{\mathrm{wp}}, E_0^{\mathrm{wp}}, E_1^{\mathrm{wp}}$). Actually, the density $n_0^\vartheta(x,t)$ (which is convoluted with the Gaussian to give n_0) already differs from $n_0^{\mathrm{wp}}(x,t)$ in the WP case because the shift contains the function ϑ, which is due to friction (and analogously for $E_0^\vartheta(x,t)$ and $E_0^{\mathrm{wp}}(x,t)$).

From Lemma 2.22 we directly get

$$\left\| \int_{\mathbb{R}^3_v} \left(\Gamma[E]u\right)(x-\vartheta(s)v, v, t-s)\, dv \right\|_{L^2(\mathbb{R}^3_x)} \;\leq \\ \leq\; C\vartheta(s)^{-3/2} \|E(t-s)\|_2 \|u(t-s)\|_2, \quad \forall\, t \geq s > 0. \tag{2.3.30}$$

To derive an L^2-estimate on the field we shall proceed as in the WP case (Lemma 2.24, Proposition 2.25).

2.29 Lemma. *Let w be the mild solution of the WPFP equation (2.2.8) and let $w_0 \in X$ satisfy (**A**) for some $\omega \in [0,1)$. For any fixed $T > 0$ the electric field then satisfies $\nabla_x V \in V_{T,\omega-\frac{1}{2}}$ and the following estimates hold:*

1. *for $2 \leq p \leq 6$, $\theta = \frac{3(p-2)}{2p}$ and $t \in (0,T]$*

$$\|E_0(t)\|_p \;\leq\; C(T) \|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{L^{6/5}}^{1-\theta} = \mathcal{O}(t^{-\omega(1-\theta)}); \tag{2.3.31}$$

2. *for* $t \in (0,T]$

$$\|E_1(t)\|_2 \;\leq\; C\Big(T, \|w_0\|_2\,, \sup_{s\in(0,T]}\left\{\vartheta(s)^{\omega}\|n_0^{\vartheta}(s)\|_{L^{6/5}}\right\}\Big)t^{\frac{1}{2}-\omega}. \tag{2.3.32}$$

Proof. The estimate for $\|E_0(t)\|_p$, $p\in[2,6]$ is obtained by applying first the generalized Young inequality and then the Young inequality to the expression (2.3.28)

$$\begin{aligned}
\|E_0(t)\|_p &\leq C\left\|\frac{1}{R(t)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^{\vartheta}(x,t)\right\|_q \\
&\leq C\left\|\frac{1}{R(t)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right)\right\|_1 \|n_0^{\vartheta}(x,t)\|_q \\
&= C\|n_0^{\vartheta}(t)\|_q, \quad \text{with} \quad q=\frac{3p}{p+3}\in[6/5,2].
\end{aligned}$$

Next we interpolate n_0^{ϑ} between L^2 and $L^{6/5}$, use (2.1.3) and the dissipativity of the operator $-v\cdot\nabla_x-\frac{3}{2}$ in X (cf. Lemma 2.8):

$$\begin{aligned}
\|n_0^{\vartheta}(t)\|_q &\leq C\|w_0(x-\vartheta(t)v,v)\|_X^{\theta}\|n_0^{\vartheta}(t)\|_{6/5}^{1-\theta} \\
&\leq C\,e^{\frac{3}{2}\theta\vartheta(t)}\|w_0\|_X^{\theta}\|n_0^{\vartheta}(t)\|_{6/5}^{1-\theta},
\end{aligned}$$

with $\theta=\frac{5}{2}-\frac{3}{q}$. Hence

$$\|E_0(t)\|_p \;\leq\; C(T)\|w_0\|_X^{\theta}\|n_0^{\vartheta}(t)\|_{L^{6/5}}^{1-\theta}.$$

We rewrite the function $E_1(x,t)$ as

$$\begin{aligned}
(E_1)_j(x,t) &= \lambda\sum_{k=1}^{3}\frac{-3x_jx_k+\delta_{jk}|x|^2}{|x|^5} \\
&\quad *_x\int_0^t \frac{\vartheta(s)}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F_k(x,t,s)\,ds,
\end{aligned} \tag{2.3.33}$$

$$\text{with}\quad F_k(x,t,s) \;:=\; \int(\Gamma_k[E_0+E_1]w)\,(x-\vartheta(s)v,v,t-s)\,dv,$$

For estimating it we exploit classical properties of the convolution with the kernel $\frac{1}{|x|}$ and apply the Young inequality:

$$\begin{aligned}
\|E_1(t)\|_2 &\leq C\int_0^t\vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F(x,t,s)\right\|_2 ds \\
&\leq C\int_0^t\vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_1 \|F(x,t,s)\|_2\,ds \\
&\leq C(T)\|w_0\|_2\int_0^t\frac{\|E_0(t-s)\|_2+\|E_1(t-s)\|_2}{\sqrt{\vartheta(s)}}\,ds,
\end{aligned}$$

where the last inequality follows from (2.3.30) and the L^2–a-priori estimate on the solution w (cf. Lemma 2.18). By applying the estimate (2.3.31) to $\|E_0(t)\|_2$, we get

$$
\begin{aligned}
(2.3.34) \qquad \|E_1(t)\|_2 \;\le\; & C(T)\|w_0\|_2 \Bigg(\sup_{t\in(0,T]} \left\{ \vartheta(t)^\omega \|n_0^\vartheta(t)\|_{L^{6/5}} \right\} \\
& \cdot \int_0^t \vartheta(s)^{-\frac{1}{2}} \vartheta(t-s)^{-\omega}\, ds + \int_0^t \frac{\|E_1(t-s)\|_2}{\sqrt{\vartheta(s)}}\, ds \Bigg),
\end{aligned}
$$

where the function $\vartheta(s) = \mathcal{O}(s)$ as $s \to 0$. Thus the integrals are finite.
To establish a solution of (2.3.33) we introduce the fixed point map

$$
\begin{aligned}
(ME)_j(x,t) \;:=\; & \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} *_x *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}} \right) \\
& *_x \int \left(\Gamma_k[E_0 + E]w\right)(x - \vartheta(s)v, v, t-s)\, dv\, ds.
\end{aligned}
$$

By using $0 < \frac{\vartheta(T)}{T} t \le \vartheta(t)$, $\forall\, t \in (0,T]$ and (2.3.34), a simple fixed point argument as in the proof of Lemmma 2.24 with the contractivity estimate:

$$
\begin{aligned}
\|M^n E(t) - M^n \widetilde{E}(t)\|_2 \;\le\; & \left(C\sqrt{\frac{T}{\vartheta(T)}}\|w_0\|_2 \right)^n t^{\frac{n+1}{2}-\omega} \frac{\pi^{\frac{n}{2}}\Gamma\left(\frac{3}{2}-\omega\right)}{\Gamma\left(\frac{n+3}{2}-\omega\right)} \\
& \cdot \sup_{s\in(0,T]} \left(s^{\omega-\frac{1}{2}} \|E(s) - \widetilde{E}(s)\|_2 \right)
\end{aligned}
$$

shows that the linear equation (2.3.33) has a unique solution $E_1 \in V_{T,\omega-\frac{1}{2}}$. Hence $\nabla_x V = E_0 + E_1 \in V_{T,\omega-\frac{1}{2}}$ and Gronwall's Lemma then yields estimate (2.3.32). □

2.30 Remark. For the derivation of the a-priori bound on $\|E\|_2$, we did not use any moments of w (neither in x nor v), nor pseudo-conformal laws (cf. [**Bo93, Bo95, Pe, Ca98**] for the classical analogue, i.e. VPFP). In fact, the latter are not useful in the quantum case, since the Wigner function typically also takes negative values. Moreover, the convolution with the Gaussian did not play a role there; the estimate (2.3.32) relies just on the dispersive effect of the free-streaming operator. The parabolic regularization will be exploited in the "post-processing" Proposition 2.31.

The above lemma was the first crucial step towards proving global existence of the WPFP solution. Next we shall extend this estimates on the field E to a range of L^p-norms:

2.31 Proposition. *Let w be the mild solution of the WPFP equation (2.2.8) and let $w_0 \in X$ satisfy (**A**) for some $\omega \in [0,1)$. Then, we have for any fixed $T > 0$ and for all $p \in [2,6)$, $t \in (0,T]$:*

$$\|E_1(t)\|_{L^p} \;\le\; C\Big(T, \|w_0\|_2, \sup_{s\in(0,T]} \big\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\big\}\Big) t^{\frac{3}{2p}-\frac{1}{4}-\omega}. \tag{2.3.35}$$

PROOF. We shall estimate $E_1(t)$ (cf. (2.3.33)) by using classical properties of the convolution by the kernel $\frac{1}{|x|}$ and the following

$$\left\| \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_q \;=\; C R(s)^{\frac{3}{2q}}, \;\forall\, 1 \le q \le \infty\,. \tag{2.3.36}$$

Namely,

$$\begin{aligned}\|E_1[w](t)\|_{L^p(\mathbb{R}^3_x)} \;&\le\; C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F(x,t,s)\right\|_{L^p(\mathbb{R}^3_x)} ds \\ &\le\; C\int_0^t \frac{R(s)^{\frac{3}{2q}-\frac{3}{2}}}{\sqrt{\vartheta(s)}}\Big(\|E_0(t-s)\|_2 + \|E_1[w](t-s)\|_2\Big)\|w(t-s)\|_2\, ds,\end{aligned}$$

where we used the Young inequality with $1 + 1/p = 1/q + 1/2$ (thus, $p \ge 2$) and, for the L^2-norm, the Lemma 2.22. Then, by applying Lemma 2.29 we get

$$\begin{aligned}\|E_1[w](t)\|_{L^p(\mathbb{R}^3_x)} \;&\le\; C\Big(T, \sup_{s\in(0,T]}\big\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\big\}, \|w_0\|_{L^2(\mathbb{R}^6)}\Big)\cdot \\ &\qquad \cdot\int_0^t \frac{R(s)^{\frac{3}{2q}-\frac{3}{2}}}{\sqrt{\vartheta(s)}}\left(\vartheta(t-s)^{-\omega} + (t-s)^{\frac{1}{2}-\omega}\right) ds\end{aligned}$$

Since $\vartheta(t) = \mathcal{O}(t)\,, R(t) = \mathcal{O}(t)$ for $t \to 0$ (cf. (2.3.26), (2.3.27)), the last integral is finite for all $t > 0$ and for $3/(2q) - 2 > -1 \Leftrightarrow 3/(2p) - 5/4 > -1 \Leftrightarrow p < 6$. In fact the integral is $\mathcal{O}(t^{\frac{3}{2p}-\frac{1}{4}-\omega})$. □

2.32 Remark. Proposition 2.31 provides a non-trivial interval of L^p-estimates for the electric field in the WPFP case. This is due to the regularizing effect of the FP term. We remark that the corresponding Gaussian is "better behaved" than the classical one, since the quantum FP operator is uniformly elliptic in both x and v variables. On the other hand, exactly as in the WP case, the range of L^p-estimates for the WPFP equation is smaller in comparison to the counterpart VPFP and that depends again on the non-negativity of the classical distribution function.

As a further result, we obtain an a-posteriori information on the self-consistent potential V, which follows directly from the a-priori estimates on the field. Accordingly, we split the potential as $V = V_0 + V_1$, with

$$\begin{aligned} V_0(x,t) &:= \lambda \sum_{i=1}^{3} \frac{x_i}{|x|^3} *_x (E_0)_i(x,t), \\ V_1(x,t) &:= \lambda \sum_{i=1}^{3} \frac{x_i}{|x|^3} *_x (E_1)_i(x,t). \end{aligned}$$

2.33 Corollary. *Under the assumptions of Proposition 2.31, the self-consistent potential $V(t) = V_0(t) + V_1(t)$ belongs to $L^p(\mathbb{R}^3_x)$ with $6 \le p \le \infty$, and satisfies for all $t \in (0,T]$:*

$$\begin{aligned} \|V_0(t)\|_p &\le C(T)\|w_0\|_X^\theta \|n_0^\vartheta(t)\|_{L^{6/5}}^{1-\theta} = \mathcal{O}(t^{-\omega(1-\theta)}) \quad \textit{with } \theta = \frac{1}{2} - \frac{3}{p}, \\ \|V_1(t)\|_p &\le C\Big(T, \|w_0\|_2\,, \sup_{s\in(0,T]} \big\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\big\}\Big) t^{\frac{1}{2}-\omega}. \end{aligned}$$

2.3.3 A-priori estimates for the weighted L^2-norms

A first consequence of the a-priori estimates for the electric field is the following

2.34 Lemma. *For all $w_0 \in X$ such that (**A**) holds for some $\omega \in [0,1)$, the mild solution of the WPFP equation (2.2.8) satisfies for all $t \in [0,T]$:*

$$\|vw(t)\|_2^2 \le C\Big(T, \|w_0\|_X, \sup_{s\in(0,T]} \big\{\vartheta(s)^\omega \|n_0^\vartheta(s)\|_{L^{6/5}}\big\}\Big). \tag{2.3.37}$$

Proof. In order to justify the derivation of this a-priori estimate we need again the approximating classical solutions y_n introduced in the proof of Lemma 2.18. Mutiplying both sides of (2.3.3) by $v_i^2 y_n(t)$ and integrating yields

$$\frac{1}{2}\frac{d}{dt}\|v_i y_n(t)\|_2^2 = \iint v_i^2 y_n(t)\overline{A} y_n(t)\,dx\,dv + \iint v_i^2 y_n(t) f_n(t)\,dx\,dv.$$

By analogous calculations as in the proof of Lemma 2.8 (cf. also (2.4.3)) we get,

$$\iint |v|^2 y_n(t)\overline{A} y_n(t)\,dx\,dv \le 3\sigma\|y_n(t)\|_2^2 + \frac{\beta}{2}\|v y_n(t)\|_2^2,$$

and hence for $t \in [0,T]$:

$$\frac{1}{2}\frac{d}{dt}\|v y_n(t)\|_2^2 \le 3\sigma\|y_n(t)\|_2^2 + \frac{\beta}{2}\|v y_n(t)\|_2^2 + \iint |v|^2 y_n(t) f_n(t)\,dx\,dv.$$

By integrating in t, letting $n \to \infty$, and using (2.3.1), we have

$$\begin{aligned}\|vw(t)\|_2^2 \;\le\; & \|vw_0\|_2^2 + \frac{2\sigma}{\beta}(e^{3\beta t}-1)\|w_0\|_2^2 + \beta\int_0^t \|vw(s)\|_2^2\,ds \\ & + 2\int_0^t\iint |v|^2 w(s) f(s)\,dx\,dv\,ds, \quad \forall\, t\in[0,T].\end{aligned}$$

Using again the skew-symmetry of the pseudo-differential operator and the Hölder inequality yields

$$\begin{aligned}\int_0^t\iint v_i w(s) v_i f(s)\,dx\,dv\,ds \;=\; & \frac{1}{2}\int_0^t\iint v_i w(s)\Omega[\partial_i V[w(s)]]w(s)\,dx\,dv\,ds \\ \le\; & \frac{1}{2}\int_0^t \|v_i w(s)\|_2\,\|\Omega[\partial_i V[w(s)]]w(s)\|_2\,ds,\end{aligned}$$

with the operator Ω defined in (2.1.10). Estimating as in (2.1.12) and using the Sobolev inequality we obtain for $t\in[0,T]$:

$$\begin{aligned}\|\Omega[\partial_i V[w(t)]]w(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_v)} \;\le\; & C\|\partial_i V[w(t)]\hat{w}(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\eta)} \\ \le\; & C\|\partial_i V[w(t)]\|_3\|\hat{w}(t)\|_{L^2(\mathbb{R}^3_x;L^6(\mathbb{R}^3_\eta))} \\ \le\; & C\|\partial_i V[w(t)]\|_3\|\nabla_\eta\hat{w}(t)\|_{L^2(\mathbb{R}^3_x\times\mathbb{R}^3_\eta)} \\ \le\; & C\|\partial_i V[w(t)]\|_3\|vw(t)\|_2,\end{aligned} \tag{2.3.38}$$

where $\hat{w}(x,\eta,t) := \mathcal{F}_{v\to\eta}(w(x,v,t))$. Finally, using Prop. 2.31 (estimate (2.3.35) with $p=3$) yields

$$\begin{aligned}\|vw(t)\|_2^2 \;\le\; & C(T)\left(\|vw_0\|_2^2 + \|w_0\|_2^2\right) \\ & + C\Big(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\Big) \\ & \cdot\int_0^t \Big(s^{-\frac{\omega}{2}} + s^{-\omega+\frac{1}{4}} + \beta\Big)\,\|vw(s)\|_2^2\,ds, \quad t\in[0,T],\end{aligned} \tag{2.3.39}$$

and the Gronwall Lemma gives the result. □

With this result we can proceed to derive the a-priori estimate for $\||v|^2 w(t)\|_2$.

2.35 Lemma. *For all $w_0\in X$ such that (**A**) holds for some $\omega\in[0,1)$, the mild solution of the WPFP equation (2.2.8) satisfies for all $t\in[0,T]$:*

$$\||v|^2 w(t)\|_2^2 \;\le\; C\Big(T, \|w_0\|_X, \sup_{s\in(0,T]}\left\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\right\}\Big). \tag{2.3.40}$$

Proof. In order to control the term $\||v|^2 w(t)\|_2$, we shall use the same strategy as in the Lemmata 2.18 and 2.34. Multiplying both sides of (2.3.3) by $v_i^4 y_n(t)$ and integrating we get by using (2.4.3) and repeating the same limit procedure as in the previous lemma:

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{3}\iint v_i^4 w(t)^2\,dx\,dv \;\le\; 9\sigma\|w(t)\|_2^2+\left(3\sigma-\frac{\beta}{2}\right)\sum_{i=1}^{3}\iint v_i^4 w(t)^2\,dx\,dv + \sum_{i=1}^{3}\iint v_i^4 w(t) f(t)\,dx\,dv,\quad \forall\, t\in[0,T].$$

By integrating in t, using $C_1|v|^4 \le \sum v_i^4 \le C_2|v|^4$ and (2.3.1), we have

$$\||v|^2 w(t)\|_2^2 \;\le\; C\Bigg(\||v|^2 w_0\|_2^2+\frac{6\sigma}{\beta}(e^{3\beta t}-1)\|w_0\|_2^2 + (6\sigma-\beta)\int_0^t \||v|^2 w(s)\|_2^2\,ds + 2\sum_{i=1}^{3}\int_0^t\iint v_i^4 w(s) f(s)\,dx\,dv\,ds\Bigg). \tag{2.3.41}$$

Using again the skew-symmetry of the pseudo-differential operator Θ, the equation (2.1.9) and the Hölder inequality, we have

$$\int_0^t\iint v_i^2 w(s) v_i^2 f(s) dx\,dv\,ds \;\le\; \frac{1}{4}\int_0^t \|v_i^2 w(s)\|_2\,\|\Theta[\partial_i^2 V[w(s)]]w(s)\|_2\,ds + \int_0^t \|v_i^2 w(s)\|_2\,\|\Omega[\partial_i V[w(s)]]v_i w(s)\|_2\,ds. \tag{2.3.42}$$

Since $\hat{w}(x,.\,,t) \in H^2(\mathbb{R}^3_\eta)$, the Gagliardo-Nirenberg inequality (cf. [**Fr**], Chapter 1.9) yields for all $t\in[0,T]$

$$\begin{aligned}\|\hat{w}(x,.\,,t)\|_{L^\infty(\mathbb{R}^3_\eta)} &\le C\|\hat{w}(x,.\,,t)\|^{1/2}_{L^6(\mathbb{R}^3_\eta)}\|\widehat{|v|^2 w}(x,.\,,t)\|^{1/2}_{L^2(\mathbb{R}^3_\eta)}\\ &\le C\|\widehat{vw}(x,.\,,t)\|^{1/2}_{L^2(\mathbb{R}^3_\eta)}\|\widehat{|v|^2 w}(x,.\,,t)\|^{1/2}_{L^2(\mathbb{R}^3_\eta)}.\end{aligned} \tag{2.3.43}$$

Using

$$\begin{aligned}\|\Delta V[w(t)]\|_2 &= \|n[w(t)]\|_2\\ &= C\|\hat{w}(.\,,\eta=0,t)\|_{L^2(\mathbb{R}^3_x)}\\ &\le C\left(\int\|\hat{w}(x,.\,,t)\|^2_{L^\infty(\mathbb{R}^3_\eta)}\,dx\right)^{1/2},\end{aligned}$$

(2.3.43), the Hölder inequality, and (2.3.37) we can estimate:

$$\|\Theta[\partial_i^2 V[w(t)]]w(t)\|_2 \;\le\; C\|\Delta V[w(t)]\|_2\left(\int\|\hat{w}(x,.\,,t)\|^2_{L^\infty(\mathbb{R}^3_\eta)}\,dx\right)^{1/2} \tag{2.3.44}$$

$$
\begin{aligned}
&\le C\int \|\hat{w}(x,.\,,t)\|_\infty^2\,dx \\
&\le C\int \|\widehat{vw}(x,.\,,t)\|_{L^2(\mathbb{R}^3_\eta)}\,\|\widehat{|v|^2w}(x,.\,,t)\|_{L^2(\mathbb{R}^3_\eta)}\,dx \\
&\le C\Big(T,\|w_0\|_X,\sup_{s\in(0,T]}\big\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\big\}\Big)\||v|^2w(t)\|_2.
\end{aligned}
$$

For the second term of the r.h.s. of (2.3.42) we proceed as in (2.3.38) and use the estimate (2.3.35):

$$
\begin{aligned}
\|\Omega[\partial_i V[w(t)]]v_i w(t)\|_2 &\le C\|\partial_i V[w(t)]\|_3\|\widehat{v_iw}(t)\|_{L^2(\mathbb{R}^3_x;L^6(\mathbb{R}^3_\eta))} \\
&\le C\Big(T,\|w_0\|_X,\sup_{s\in(0,T]}\big\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\big\}\Big) \\
&\quad\cdot\Big(t^{-\frac{\omega}{2}}+t^{-\omega+\frac14}\Big)\||v|^2w(t)\|_2,
\end{aligned}
\tag{2.3.45}
$$

for all $t\in[0,T]$. Analogously to (2.3.39), combining the estimates (2.3.42), (2.3.44) and (2.3.45) the Gronwall Lemma gives the assertion. □

Proof of Theorem 2.17.
The Lemmata 2.18 and 2.35 show for all $0<T<t_{\max}$ that

$$
\|w(t)\|_X \le C\left(T,\|w_0\|_X,\sup_{s\in(0,T]}\big\{\vartheta(s)^\omega\|n_0^\vartheta(s)\|_{L^{6/5}}\big\}\right),\quad \forall\, t\in[0,T],
$$

with C being continuous in $T\in[0,t_{\max}]$. Then, Corollary 2.15 shows that the mild solution w exists on $[0,\infty)$. □

2.3.4 Regularity

The following result concerns the smoothness of the solution of WPFP, the macroscopic density and the force field, for positive times.

2.36 Corollary. *Under the assumptions of Theorem 2.17, the mild solution of the WPFP equation (2.2.8) satisfies*

$$
w\in\mathcal{C}((0,\infty);\mathcal{C}^\infty_{\mathcal{B}}(\mathbb{R}^6)),
$$

$$
n(t),E(t),V(t)\in\mathcal{C}((0,\infty);\mathcal{C}^\infty_{\mathcal{B}}(\mathbb{R}^3)).
$$

Proof. Obviously, $w(t)\in\mathcal{C}(\mathbb{R}^6)$ $\forall t>0$, because of the Green's function representation in (2.2.20), (2.2.3). If we differentiate equation (2.2.8) with respect to x_i and, resp., v_i, we obtain the following linear, inhomogeneous problems for any fixed $t_1 > 0$.

$$\begin{aligned} z_t(t) &= \overline{A}z(t) + \Theta[V[z(t)]]w(t) + \Theta[V[w(t)]]z(t), \quad \forall\, t > t_1, \\ z(t_1) &= \partial_{x_i} w(t_1) \in X \end{aligned}$$

and

$$\begin{aligned} y_t(t) &= \overline{A}y(t) + \beta y(t) - \partial_{x_i} w(t) + \Theta[V[w(t)]]y(t), \quad \forall\, t > t_1, \\ y(t_1) &= \partial_{v_i} w(t_1) \in X. \end{aligned}$$

By arguments analogous to Lemma 2.13, there exists a unique mild solution

$$z = \partial_{x_i} w \in \mathcal{C}([t_1, \infty); H^1(\mathbb{R}^6; (1 + |v|^2)^2\, dx\, dv)). \tag{2.3.46}$$

By an induction procedure, the derivatives $\nabla_x^\alpha \nabla_v^\beta w$, for $\alpha, \beta \in \mathbb{N}^3$, $|\alpha| + |\beta| = m > 1$ are also mild solutions of similar problems with additional well-defined inhomogeneities and with initial times $0 < t_1 < t_2 < ... < t_m$. This yields $\nabla_x^\alpha \nabla_v^\beta w \in \mathcal{C}([t_m, \infty); H^1(\mathbb{R}^6; (1 + |v|^2)^2\, dx\, dv))$, and thus $\nabla_x^\alpha \nabla_v^\beta w \in \mathcal{C}((0, \infty); X)$. Hence, the statement about smoothness of the density and the electric field is straightforward from Propositions 2.1 and 2.3 and Sobolev embeddings. □

2.4 Appendix

Proof of Lemma 2.8

For $u \in D(A)$ we have

$$< Au, u >_{\tilde{X}} \;=\; < Au, u >_{L^2(\mathbb{R}^6)} + \sum_{i=1}^{3} \iint v_i^4 u A u, \tag{2.4.1}$$

where $\iint f$ denotes the integral $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(x, v)\, dv\, dx$, and the norm $\|.\|_{\tilde{X}}$ is defined by (2.1.7). Using integrations by parts we shall calculate the three terms on the right hand side separately.

$$\begin{aligned} < Au, u >_{L^2(\mathbb{R}^6)} &= \sum_{i=1}^{3} \Big(-\iint v_i u_{x_i} u + \beta \iint (v_i u)_{v_i} u \\ &\quad + \sigma \iint u_{v_i v_i} u + 2\gamma \iint u_{x_i v_i} u + \alpha \iint u_{x_i x_i} u \Big) \end{aligned}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{3}\Bigg[3\beta\iint u^2+\beta\iint v_i u_{v_i}u-\sigma\iint u_{v_i}^2\\
&\qquad +\gamma\left(\epsilon\iint u_{x_i}^2+\frac{1}{\epsilon}\iint u_{v_i}^2\right)-\alpha\iint u_{x_i}^2\Bigg]\\
&= \frac{3}{2}\beta\|u\|_2^2+\left(\frac{\gamma}{\epsilon}-\sigma\right)\|\nabla_v u\|_2^2+(\epsilon\gamma-\alpha)\|\nabla_x u\|_2^2.
\end{aligned}
$$

With $\epsilon=\frac{\gamma}{\sigma}$ we obtain

$$
< Au,u >_{L^2(\mathbb{R}^6)} \;\le\; \frac{3}{2}\beta\|u\|_2^2. \tag{2.4.2}
$$

Next we estimate the second term of (2.4.1):

$$
\begin{aligned}
\sum_{i=1}^{3}\iint v_i^4 uAu &= \sum_{i,j=1}^{3}\Bigg(-\iint v_i^4 v_j u_{x_j}u+\beta\iint v_i^4 (v_j u)_{v_j}u\\
&\qquad +\sigma\iint v_i^4 u_{v_jv_j}u+2\gamma\iint v_i^4 u_{x_jv_j}u+\alpha\iint v_i^4 u_{x_jx_j}u\Bigg)\\
&\le \sum_{i,j=1}^{3}\Bigg[\beta\iint v_i^4u^2+\beta\iint v_i^4 v_j u_{v_j}u-\sigma\iint v_i^4 u_{v_j}^2\\
&\qquad -\frac{4}{3}\sigma\iint v_i^3 u_{v_i}u+\gamma\left(\epsilon\iint v_i^4u_{x_j}^2+\frac{1}{\epsilon}\iint v_i^4 u_{v_j}^2\right)-\alpha\iint v_i^4 u_{x_j}^2\Bigg]\\
&\le \sum_{i=1}^{3}\left(-\frac{1}{2}\beta\iint v_i^4u^2+6\sigma\iint v_i^2u^2\right)\\
&\le 9\sigma\|u\|_2^2+\left(-\frac{1}{2}\beta+3\sigma\right)\sum_{i=1}^{3}\iint v_i^4u^2,
\end{aligned}
$$

by choosing $\epsilon=\frac{\gamma}{\sigma}$ and by an interpolation.
Collecting the two estimates yields

$$
< Au,u >_{\tilde{X}} \;\le\; \left(\frac{3}{2}\beta+9\sigma\right)\|u\|_2^2+3\sigma\sum_{i=1}^{3}\iint v_i^4u^2 \;\le\; \left(\frac{3}{2}\beta+9\sigma\right)\|u\|_{\tilde{X}}^2.
$$

Thus, the operator $A-\kappa I$ is dissipative. $\square$

Proof of Lemma 2.10

To prove the assertion we shall construct for each $f\in D(\overline{P})\subset L^2(\mathbb{R}^6)$ a sequence $\{f_n\}\subset D(P)$ such that $f_n\to f$ in the graph norm

$$
\|f\|_P=\|f\|_{L^2}+\||x|^2f\|_{L^2}+\||v|^2f\|_{L^2}+\|Pf\|_{L^2}+\||x|^2Pf\|_{L^2}+\||v|^2Pf\|_{L^2}.
$$

To shorten the proof we shall consider here only the case

$$P = \theta + \nu v \cdot \nabla_x + \mu x \cdot \nabla_v + \beta v \cdot \nabla_v + \alpha \Delta_x + \sigma \Delta_v + \gamma \mathrm{div}_v \nabla_x$$

(cf. the definition of the operator A in (2.1.13)), but exactly the same strategy extends to the case, where P is a general quadratic polynomial.
First we define the mollifying delta sequence

$$\phi_n(x,v) := n^6 \phi(nx, nv), \qquad n \in \mathbb{N},\ x, v \in \mathbb{R}^3,$$

where

$$\begin{aligned}&\phi \in C_0^\infty(\mathbb{R}^6), \quad \phi(x,v) \geq 0,\\ &\iint \phi(x,v)dxdv = 1, \quad \text{and} \quad \mathrm{supp}\ \phi \subset \{|x|^2 + |v|^2 \leq 1\}.\end{aligned}$$

By definition we have the following properties:

(I) $\phi_n \to \delta$ in $\mathcal{D}'(\mathbb{R}^6)$,

(II) $\frac{1}{n}\partial_{x_i}\phi_n$, $\frac{1}{n}\partial_{v_i}\phi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$, $i = 1, 2, 3$,

(III) $(x,v)^\alpha \partial^\beta_{(x,v)} \left[(x,v)^\gamma \phi_n(x,v)\right] \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$, with $\alpha, \beta, \gamma \in \mathbb{N}_0^6$ multi-indexes
and $|\gamma| > 0$, since $(x,v)^\gamma \phi_n \to 0$ in $\mathcal{D}'(\mathbb{R}^6)$.

The cutoff sequence is

$$\psi_n(x,v) := \psi\left(\frac{|(x,v)|}{n}\right), \qquad n \in \mathbb{N},\ x, v \in \mathbb{R}^3,$$

where ψ satisfies

$$\psi \in C_0^\infty(\mathbb{R}), \quad 0 \leq \psi(z) \leq 1, \quad \mathrm{supp}\,\psi \subset [-1, 1], \quad \psi|_{[-\frac{1}{2},\frac{1}{2}]} \equiv 1,$$

and

$$|\psi^{(j)}(z)| \leq C_j, \quad \forall z \in \mathbb{R},\ j = 1, 2.$$

The sequence ψ_n has the following properties:

(IV) $\psi_n \to 1$ pointwise,

(V) $(x,v)^\alpha \partial^\beta_{(x,v)} \psi_n(x,v) = \frac{1}{n}\frac{(x,v)^\alpha (x,v)^\beta}{|(x,v)|}\psi'\left(\frac{|(x,v)|}{n}\right)$, with $\alpha, \beta \in \mathbb{N}_0^6$,
$|\alpha| = |\beta| = 1$, are supported in the annulus

$$\mathrm{supp}\left(\psi'\left(\frac{|(x,v)|}{n}\right)\right) \;=\; \{(x,v) \mid n/2 \leq |(x,v)| \leq n\} \;=:\; V_n,$$

and they are in $L^\infty(\mathbb{R}^6)$, uniformly in $n \in \mathbb{N}$.

(VI) $n\partial^\alpha_{(x,v)}\psi_n(x,v) = \frac{(x,v)^\alpha}{|(x,v)|}\psi'\left(\frac{|(x,v)|}{n}\right)$, with $\alpha \in \mathbb{N}_0^6$, $|\alpha| = 1$,

are uniformly bounded in $L^\infty(\mathbb{R}^6)$.

(VII) $\partial^\alpha_{(x,v)}\psi_n(x,v) = \frac{(x,v)^\alpha}{n^2|(x,v)|^2}\psi''\left(\frac{|(x,v)|}{n}\right) + \left(\frac{1}{n^2|(x,v)|} - \frac{(x,v)^\alpha}{n^3|(x,v)|^3}\right)\psi'\left(\frac{|(x,v)|}{n}\right)$,

with $|\alpha| = 2$ have support on V_n and converge uniformly to 0 in $L^\infty(\mathbb{R}^6)$.

We now define the approximating sequence

$$f_n(x,v) := (f * \phi_n)(x,v) \cdot \psi_n(x,v), \quad n \in \mathbb{N},$$

where '$*$' denotes the convolution in x and v.
By construction we have $f_n \in C_0^\infty(\mathbb{R}^6) = D(P)$.
Since we can split our operator as

$$\begin{aligned} P &= \sum_{i=1}^{3}\left[\frac{\theta}{3} + \nu v_i\partial_{x_i} + \mu x_i\partial_{v_i} + \beta v_i\partial_{v_i} + \alpha\partial^2_{x_i} + \sigma\partial^2_{v_i} + \gamma\partial_{v_i}\partial_{x_i}\right] \\ &= \sum_{i=1}^{3}\tilde{p}(x_i, v_i, \partial_{x_i}, \partial_{v_i}), \end{aligned}$$

we shall in the sequel only consider

$$\tilde{P} \;=\; \tilde{p}(y, z, \partial_y, \partial_z), \quad y, z \in \mathbb{R}$$

acting in one spatial direction $y = x_j$ and one velocity direction $z = v_j$.
We have to prove that $f_n(x,v) \to f(x,v)$ in the graph norm

$$\|f\|_{\tilde{P}} = \|f\|_{L^2} + \||x|^2 f\|_{L^2} + \||v|^2 f\|_{L^2} + \|\tilde{P}f\|_{L^2} + \||x|^2\tilde{P}f\|_{L^2} + \||v|^2\tilde{P}f\|_{L^2}.$$

According to the 6 terms of the graph norm we split the proof into 6 steps:
Step 1: By applying (P1) and (P4), we have

$$f_n \to f \quad \text{in} \quad L^2(\mathbb{R}^6).$$

Step 2: For the second term of the graph norm we write

$$x_i^2 f_n = (x_i^2 f * \phi_n)\psi_n + 2(x_i f * x_i\phi_n)\psi_n + (f * x_i^2\phi_n)\psi_n.$$

The first summand converges to $x_i^2 f$ in $L^2(\mathbb{R}^6)$ and both the second and the third terms converge to 0 by (III), since also $x_i f$ belongs to $L^2(\mathbb{R}^6)$ by interpolation.

Step 3: For the third term of the graph norm the same argument as in previous step can be used. Hence we have

$$f_n \to f \qquad \text{in} \quad Z.$$

Step 4: To prove that $\tilde{P} f_n \to \tilde{P} f$ in $L^2(\mathbb{R}^6)$ we write:

$$\begin{aligned}\tilde{P} f_n &= \frac{\theta}{3}(f * \phi_n)\psi_n + \nu(z f_y * \phi_n)\psi_n + \mu(y f_z * \phi_n)\psi_n + \beta(z f_z * \phi_n)\psi_n \\ &\quad + \alpha(f_{yy} * \phi_n)\psi_n + \sigma(f_{zz} * \phi_n)\psi_n + \gamma(f_{yz} * \phi_n)\psi_n + r_n^1(y,z) \\ &= (\tilde{P} f * \phi_n)\psi_n + r_n^1(y,z).\end{aligned}$$

As we shall show, all thirteen terms of the remainder

$$\begin{aligned} r_n^1 &= \nu(f * \partial_y(z\phi_n))\psi_n + \nu(f * \phi_n) z\partial_y \psi_n + \mu(f * y\partial_z \phi_n)\psi_n \\ &\quad + \mu(f * \phi_n) y \partial_z \psi_n + \beta(f * \partial_z(z\phi_n))\psi_n + \beta(f * \phi_n)) z\partial_z \psi_n \\ &\quad + 2\alpha(f * (\frac{1}{n}\partial_y \phi_n))(n\partial_y \psi_n) + \alpha(f * \phi_n))(\partial_y^2 \psi_n) + 2\sigma(f * \frac{1}{n}\partial_z \phi_n) n\partial_z \psi_n \\ &\quad + \sigma(f * \phi_n)\partial_z^2 \psi_n + \gamma(f * (\frac{1}{n}\partial_z \phi_n))(n\partial_y \psi_n) + \gamma(f * (\frac{1}{n}\partial_y \phi_n))(n\partial_z \psi_n) \\ &\quad + \gamma(f * \phi_n)\partial_y \partial_z \psi_n \end{aligned}$$

converge to 0 in $L^2(\mathbb{R}^6)$.
The first, the third and the fifth terms converge to 0 in $L^2(\mathbb{R}^6)$ by (III).
In the second, fourth and the sixth terms, exploiting (V) we have

$$\|(f * \phi_n)(z\partial_y \psi_n)\|_{L^2(R^6)} \leq C\|f * \phi_n - f\|_{L^2(V_n)} + \|f\|_{L^2(V_n)} \to 0, \tag{2.4.3}$$

because $\|f\|_{L^2(R^6)} = \|f\|_{L^2(B_{1/2}(0))} + \sum_{k=0}^{\infty} \|f\|_{L^2(V_{2^k})}$.
For what the seventh, ninth, eleventh and twelfth terms are concerned, we can exploit (VI) and then (II).
The remaining terms can be handled thanks to (VII).

Step 5: To prove that $|x|^2 \tilde{P} f_n \to |x|^2 P f$ in $L^2(\mathbb{R}^6)$ we write:

$$\begin{aligned} x_i^2 \tilde{P} f_n &= \frac{\theta}{3}(x_i^2 f * \phi_n)\psi_n + \nu(x_i^2 z f_y * \phi_n)\psi_n + \mu(x_i^2 y f_z * \phi_n)\psi_n \\ &\quad + \beta(x_i^2 z f_z * \phi_n)\psi_n + \alpha(x_i^2 f_{yy} * \phi_n)\psi_n + \sigma(x_i^2 f_{zz} * \phi_n)\psi_n \\ &\quad + \gamma(x_i^2 f_{yz} * \phi_n)\psi_n + r_n^2(y,z) \\ &= (x_i^2 \tilde{P} f * \phi_n)\psi_n + r_n^2(y,z). \end{aligned}$$

The remainder r_n^2 can be split in the following way ($y = x_j$, $z = v_j$):

$$
\begin{aligned}
r_{n,\theta}^2 &= \frac{2}{3}\theta(x_i f * y\phi_n)\psi_n + \frac{\theta}{3}(f * x_i^2\phi_n)\psi_n \\
r_{n,\nu}^2 &= 2\nu(zx_i f * \partial_y(x_i\phi_n))\psi_n - 2\nu\delta_{ij}(zf * x_i\phi_n)\psi_n + \nu(zf * \partial_y(x_i^2\phi_n))\psi_n \\
&\quad +\nu(x_i^2 f * z\partial_y\phi_n)\psi_n + 2\nu(x_i f * x_i z\partial_y\phi_n)\psi_n + \nu(f * x_i^2 z\partial_y\phi_n)\psi_n \\
&\quad +\nu(x_i^2 f * \phi_n)z\partial_y\psi_n + 2\nu(x_i f * x_i\phi_n)z\partial_y\psi_n + \nu(f * x_i^2\phi_n)z\partial_y\psi_n \\
r_{n,\mu}^2 &= 2\mu(x_i y f * \partial_z(x_i\phi_n))\psi_n + \mu(yf * \partial_z(x_i^2\phi_n))\psi_n + \mu(x_i^2 f * y\partial_z\phi_n)\psi_n \\
&\quad +2\mu(x_i f * x_i y\partial_z\phi_n)\psi_n + \mu(f * x_i^2 y\partial_z\phi_n)\psi_n + \mu(x_i^2 f * \phi_n)y\partial_z\psi_n \\
&\quad +2\mu(x_i f * x_i\phi_n)y\partial_z\psi_n + \mu(f * x_i^2\phi_n)y\partial_z\psi_n \\
r_{n,\beta}^2 &= 2\beta(x_i z f * x_i\partial_z\phi_n)\psi_n - 2\beta(x_i f * x_i\phi_n)\psi_n + \beta(zf * x_i^2\partial_z\phi_n)\psi_n \\
&\quad -\beta(f * x_i^2\phi_n)\psi_n + \beta(x_i^2 f * \partial_z(z\phi_n))\psi_n + 2\beta(x_i f * x_i\partial_z(z\phi_n))\psi_n \\
&\quad +\beta(f * x_i^2\partial_z(z\phi_n))\psi_n + \beta(x_i^2 f * \phi_n)z\partial_z\psi_n + 2\beta(x_i f * x_i\phi_n)z\partial_z\psi_n \\
&\quad +\beta(f * x_i^2\phi_n)z\partial_z\psi_n \\
r_{n,\alpha}^2 &= 2\alpha(x_i f * \partial_y^2(x_i\phi_n))\psi_n - 4\alpha\delta_{ij}(f * \partial_y(x_i\phi_n))\psi_n + \alpha(f * \partial_y^2(x_i^2\phi_n))\psi_n \\
&\quad +2\alpha(x_i^2 f * \frac{1}{n}\partial_y\phi_n)n\partial_y\psi_n + 4\alpha(x_i f * \frac{x_i}{n}\partial_y\phi_n)n\partial_y\psi_n \\
&\quad +2\alpha(f * \frac{x_i^2}{n}\partial_y\phi_n)n\partial_y\psi_n + \alpha(x_i^2 f * \phi_n)\partial_y^2\psi_n + 2\alpha(x_i f * x_i\phi_n)\partial_y^2\psi_n \\
&\quad +\alpha(f * x_i^2\phi_n)\partial_y^2\psi_n \\
r_{n,\sigma}^2 &= 2\sigma(x_i f * x_i\partial_z^2\phi_n)\psi_n + \sigma(f * x_i^2\partial_z^2\phi_n)\psi_n + 2\sigma(x_i^2 f * \frac{1}{n}\partial_z\phi_n)n\partial_z\psi_n \\
&\quad +4\sigma(x_i f * \frac{x_i}{n}\partial_z\phi_n)n\partial_z\psi_n + 2\sigma(f * \frac{x_i^2}{n}\partial_z\phi_n)n\partial_z\psi_n \\
&\quad +\sigma(x_i^2 f * \phi_n)\partial_z^2\psi_n + 2\sigma(x_i f * x_i\phi_n)\partial_z^2\psi_n + \sigma(f * x_i^2\phi_n)\partial_z^2\psi_n \\
r_{n,\gamma}^2 &= 2\gamma(x_i f * \partial_y(x_i\partial_z\phi_n))\psi_n - 2\gamma\delta_{ij}(f * x_i\partial_z\phi_n)\psi_n \\
&\quad +\gamma(f * \partial_y\partial_z(x_i^2\phi_n))\psi_n + \gamma(x_i^2 f * \frac{1}{n}\partial_z\phi_n)n\partial_y\psi_n \\
&\quad +2\gamma(x_i f * \frac{x_i}{n}\partial_z\phi_n)n\partial_y\psi_n + \gamma(f * \frac{x_i^2}{n}\partial_z\phi_n)n\partial_y\psi_n \\
&\quad +\gamma(x_i^2 f * \frac{1}{n}\partial_y\phi_n)n\partial_z\psi_n + 2\gamma(x_i f * \frac{x_i}{n}\partial_y\phi_n)n\partial_z\psi_n \\
&\quad +\gamma(f * \frac{x_i^2}{n}\partial_y\phi_n)n\partial_z\psi_n + \gamma(x_i^2 f * \phi_n)\partial_y\partial_z\psi_n \\
&\quad +2\gamma(x_i f * x_i\phi_n)\partial_y\partial_z\psi_n + \gamma(f * x_i^2\phi_n)\partial_y\partial_z\psi_n.
\end{aligned}
$$

By the properties (I)-(VII) and estimate like (2.4.3), it can be easily seen that each term converges to 0 in $L^2(\mathbb{R}^6)$.

Step 6: In analogy to $|x|^2\tilde{P}f_n$, the sequence $|v|^2\tilde{P}f$ can be split as

$$v_i^2\tilde{P}f_n = (v_i^2\tilde{P}f * \phi_n)\psi_n + r_n^3(y,z).$$

Due to the symmetry of the operator $\tilde{P}$ in x and v, the terms of the remainder r_n^3 can be obtained from r_n^2 by interchanging y and z (and changing the coefficients), except for the following term

$$v_i^2[\beta z \partial_z \left((f * \phi_n)\psi_n\right)] \quad = \quad \beta(v_i^2 z f_z * \phi_n)\psi_n + r_{n,\beta}^3,$$

where

$$\begin{aligned} r_{n,\beta}^3 \quad = \quad & 2\beta(v_i z f * \partial_z(v_i\phi_n))\psi_n - 2\beta(1+\delta_{ij})(v_i f * v_i\phi_n)\psi_n \\ & +\beta(z f * \partial_z(v_i^2\phi_n))\psi_n - \beta(f * v_i^2\phi_n)\psi_n + \beta(v_i^2 f * \partial_z(z\phi_n))\psi_n \\ & +2\beta(v_i f * v_i\partial_z(z\phi_n))\psi_n + \beta(f * v_i^2\partial_z(z\phi_n))\psi_n + \beta(v_i^2 f * \phi_n)z\partial_z\psi_n \\ & +2\beta(v_i f * v_i\phi_n)z\partial_z\psi_n + \beta(f * v_i^2\phi_n)z\partial_z\psi_n \end{aligned}$$

converges to 0 in $L^2(\mathbb{R}^6)$, since (I)-(VII) and (2.4.3) can be used. $\square$

Proof of Proposition 2.11

First, we shall prove the following estimates on the derivatives of the Green's function (2.2.4):

$$\text{(2.4.4)} \qquad |\nabla_v G(t, x-x_0, v, v_0)| \quad \leq \quad b\frac{G(t, \frac{x-x_0}{2}, \frac{v}{2}, \frac{v_0}{2})}{\sqrt{t}}, \quad \forall\, t \leq t_0,$$

$$\text{(2.4.5)} \qquad |\nabla_x G(t, x-x_0, v, v_0)| \quad \leq \quad b'\frac{G(t, \frac{x-x_0}{2}, \frac{v}{2}, \frac{v_0}{2})}{\sqrt{t}}, \quad \forall\, t \leq t_1.$$

with $b = b(\alpha,\gamma,\sigma), t_0 = t_0(\alpha,\beta,\sigma,\gamma), b' = b'(\alpha,\gamma,\sigma)$ and $t_1 = t_1(\alpha,\beta,\sigma,\gamma)$. The v-derivative of G is given by

(2.4.6)

$$\begin{aligned} \nabla_v G(t, x-x_0, v, v_0) \quad = \quad & -\frac{G(t, x-x_0, v, v_0)}{f(t)}\Bigg[\left(\mu(t)e^{\beta t} - 2\nu(t)\frac{e^{\beta t}-1}{\beta}\right) \\ & \cdot (x - \frac{e^{\beta t}-1}{\beta}v - x_0) + \left(2\lambda(t)e^{\beta t} - \mu(t)\frac{e^{\beta t}-1}{\beta}\right)(e^{\beta t}v - v_0)\Bigg] \end{aligned}$$

For all real $a, b, c > 0$ such that $c/\sqrt{a} \leq b\sqrt{2e}$, one easily verifies that

$$\text{(2.4.7)} \qquad c|x| \quad \leq \quad be^{a|x|^2}, \quad \forall\, x \in \mathbb{R}^3.$$

Since $\alpha, \sigma > 0$, we have for $t > 0$ small enough

$$\nu(t) - \frac{1}{2}\mu(t) \quad > \quad 0, \qquad \lambda(t) - \frac{1}{2}\mu(t) \quad > \quad 0.$$

In order to apply the estimate (2.4.7) to the two terms inside the squared bracket in (2.4.6) we shall use for t small:

$$\frac{c_1}{\sqrt{a_1}} \;:=\; \frac{\frac{\sqrt{t}}{f(t)}\left|\mu(t)e^{\beta t}-2\nu(t)\frac{e^{\beta t}-1}{\beta}\right|}{\sqrt{\frac{3}{4}\frac{\nu(t)-\frac{1}{2}\mu(t)}{f(t)}}} \;\sim\; \frac{2\gamma}{\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}} \;\leq\; b_1\sqrt{2e},$$

with $b_1=\gamma/\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}$. Similarly,

$$\frac{c_2}{\sqrt{a_2}} \;:=\; \frac{\frac{\sqrt{t}}{f(t)}\left|2\lambda(t)e^{\beta t}-\mu(t)\frac{e^{\beta t}-1}{\beta}\right|}{\sqrt{\frac{3}{4}\frac{\lambda(t)-\frac{1}{2}\mu(t)}{f(t)}}} \;\sim\; \frac{2\alpha}{\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}} \;\leq\; b_2\sqrt{2e},$$

with $b_2=\alpha/\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}$. Then, there exists some $t_0>0$ such that, for all $t\leq t_0$, the two inequalities can be combined with $b=\max\{b_1,b_2\}$ to give

$$\begin{aligned}
&\left|\frac{\left(\mu(t)e^{\beta t}-2\nu(t)\frac{e^{\beta t}-1}{\beta}\right)\left(x-\frac{e^{\beta t}-1}{\beta}v-x_0\right)}{f(t)}\right.\\
&\qquad\left.+\frac{\left(2\lambda(t)e^{\beta t}-\mu(t)\frac{e^{\beta t}-1}{\beta}\right)\left(e^{\beta t}v-v_0\right)}{f(t)}\right|\sqrt{t}\\
&\leq\; \frac{\sqrt{t}}{f(t)}\left\{\left|\mu(t)e^{\beta t}-2\nu(t)\frac{e^{\beta t}-1}{\beta}\right|\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|\right.\\
&\qquad\left.+\left|2\lambda(t)e^{\beta t}-\mu(t)\frac{e^{\beta t}-1}{\beta}\right|\left|e^{\beta t}v-v_0\right|\right\}\\
&\leq\; b\exp\left\{\frac{\left(\nu(t)-\frac{1}{2}\mu(t)\right)\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|^2+\left(\lambda(t)-\frac{1}{2}\mu(t)\right)\left|e^{\beta t}v-v_0\right|^2}{\frac{4}{3}f(t)}\right\}\\
&\leq\; b\ \exp\left\{\frac{\nu(t)\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|^2+\lambda(t)\left|e^{\beta t}v-v_0\right|^2}{\frac{4}{3}f(t)}\right.\\
&\qquad\left.+\frac{\mu(t)\left(x-\frac{e^{\beta t}-1}{\beta}v-x_0\right)\cdot\left(e^{\beta t}v-v_0\right)}{\frac{4}{3}f(t)}\right\}
\end{aligned}$$

Hence,

$$|\nabla_v G(t,x-x_0,v,v_0)| \;\leq\; b\,\frac{G(t,x-x_0,v,v_0)}{\sqrt{t}}$$

$$\cdot \exp\left\{\frac{\nu(t)\left|x-\frac{e^{\beta t}-1}{\beta}v-x_0\right|^2+\lambda(t)\left|e^{\beta t}v-v_0\right|^2}{\frac{4}{3}f(t)}\right.$$
$$\left.+\frac{\mu(t)\left(x-\frac{e^{\beta t}-1}{\beta}v-x_0\right)\cdot\left(e^{\beta t}v-v_0\right)}{\frac{4}{3}f(t)}\right\}$$

and the decay (2.4.4) follows by comparison with (2.2.4).
Next we consider the x-derivative of the Green's function,

$$\begin{aligned}\nabla_x G(t,x-x_0,v,v_0) &= G(t,x-x_0,v,v_0)\\ &\quad\cdot\left[-\frac{2\nu(t)(x-\frac{(e^{\beta t}-1)}{\beta}v-x_0)+\mu(t)(e^{\beta t}v-v_0)}{f(t)}\right].\end{aligned}$$

Analogously, the decay (2.4.5) follows by exploiting that for t small enough

$$\begin{aligned}\frac{\frac{\sqrt{t}}{f(t)}2\nu(t)}{\sqrt{\frac{3}{4}\frac{\nu(t)-\frac{1}{2}\mu(t)}{f(t)}}} &\sim \frac{2\sigma}{\sqrt{3(\alpha\sigma-\gamma^2)(\sigma+\gamma)}} \leq b_1'\sqrt{2e},\\ \frac{\frac{\sqrt{t}}{f(t)}|\mu(t)|}{\sqrt{\frac{3}{4}\frac{\lambda(t)-\frac{1}{2}\mu(t)}{f(t)}}} &\sim \frac{2\gamma}{\sqrt{3(\alpha\sigma-\gamma^2)(\alpha+\gamma)}} \leq b_2'\sqrt{2e},\end{aligned}$$

with appropriate $b_1'(\alpha,\gamma,\sigma), b_2'(\alpha,\gamma,\sigma)$.
Since

$$e^{t\overline{A}}w_0(x,v) = \iint G(t,x-x_0,v,v_0)w_0(x_0,v_0)\,dx_0\,dv_0,$$

we have

$$\begin{aligned}|\nabla_v e^{t\overline{A}}w_0(x,v)| &\leq \iint |\nabla_v G(t,x-x_0,v,v_0)|\,|w_0(x_0,v_0)|\,dx_0\,dv_0\\ &\leq bt^{-1/2}\iint G\left(t,\frac{x-x_0}{2},\frac{v}{2},\frac{v_0}{2}\right)|w_0(x_0,v_0)|\,dx_0\,dv_0\\ &= 64b\,t^{-1/2}\iint G(t,\tilde{x}-\tilde{x}_0,\tilde{v},\tilde{v}_0)|w_0(2\tilde{x}_0,2\tilde{v}_0)|\,d\tilde{x}_0\,d\tilde{v}_0\\ &= 64b\,t^{-1/2}e^{t\overline{A}}\tilde{w}_0(\tilde{x},\tilde{v}),\quad \forall\, t\leq t_0.\end{aligned} \tag{2.4.8}$$

Here we used the decay (2.4.4), and we put $\tilde{x}=\frac{x}{2}$, $\tilde{v}=\frac{v}{2}$ and $\tilde{w}_0(\tilde{x},\tilde{v}) = |w_0(2\tilde{x},2\tilde{v})|$. The assertion (2.2.6) follows directly by applying the estimate (2.2.2) to (2.4.8) and choosing $T_0=\min\{t_0,t_1\}$.
The estimate (2.2.7) can be obtained analogously. $\square$

Chapter 3

Dispersive effects and the hypoelliptic WPFP

Abstract

This chapter is concerned with a global-in-time well-posedness analysis for the Wigner-Poisson-Fokker-Planck system in three dimensions. The purely kinetic L^2–analysis here presented, allows a unified treatment of the elliptic and hypoelliptic cases. The crucial novel tool of the analysis is to exploit in the quantum framework the dispersive effects of the free transport equation. It yields an a-priori estimate on the electric field for all time which allows a new nonlocal-in-time definition of the self-consistent potential and field. Thus, one can circumvent the lacking v-integrability of the Wigner function, which is a central problem in quantum kinetic theory. Due to the (degenerate) parabolic character of this system, the C^∞–regularity of the Wigner function, its macroscopic density, and the field are established for positive times.

[1] In this chapter we present a new strategy for the well-posedness analysis of quantum kinetic problems that include a Hartree-type nonlinearity. We will focus here on the 3-dimensional Wigner-Poisson-Fokker-Planck (WPFP) system (1.0.6)-(1.0.8), but we expect this new approach to be suitable for a broad range of quantum kinetic problems. In particular, our analysis will cover the physically important hypoelliptic WPFP system as well, which has been *rigorously* derived from many-body quantum mechanics in [**CEFM, FMR**].

In classical kinetic theory the phase space density typically satisfies $f(.,.,t) \in L^1(\mathbb{R}^6)$ which yields a position density $n(\cdot,t) = \int f\,dv \in L^1(\mathbb{R}^3)$.

[1] The content of this chapter is a joint work with my Ph.D. adviser A. Arnold and C. Manzini (cf. [**ADM05**])

In quantum kinetic theory, however, the natural framework is $w(.,.,t) \in L^2(\mathbb{R}^6)$, which makes (1.0.8) meaningless and hence also the above definition of the mean-field potential. Solving or circumventing this problem is one of the key points for analyzing self-consistent quantum kinetic models. Accordingly, in order to establish well-posedness of the system (1.0.4)-(1.0.7) or (1.0.6)-(1.0.7), two strategies have been used so far. The first possibility is to reformulate the WP or WPFP systems either in terms of Schrödinger wave-function sequences (cf. [**BM, Ca97**]) or in terms of density matrices (cf. [**Ar95, AS**]). In such a framework, all physical quantities are well-defined, in particular $n(t) \in L^1_+(\mathbb{R}^3)$ and the physical conservation laws for mass and energy play a crucial role in the analysis for large time. Alternatively, one can keep to the kinetic formulation and to the use of kinetic tools, with the perspective of later tackling boundary-value problems, which are more reasonable models for real simulations.
The literature related to the latter approach can be split into two groups: in several articles (cf. [**AR96, ACD, Ma, ADM04**]), a L^2-setting is chosen for $w(t)$, such that $w(t)$ satisfies at least the necessary condition to describe a quantum system (cf. [**MRS, LPa**]). Then, v-weights are introduced in order to enforce integrability in the v-variable, so to give sense to (1.0.8). In other articles (cf. [**ALMS, CLN**]), instead, a L^1-setting is chosen with the same motivation. In order to prove global-in-time results for nonlinear quantum kinetic models, one might want to exploit the physical conservation laws. However, in neither of the two above approaches they can be exploited directly, since both the mass $\iint w\,dxdv$ and the kinetic energy $\frac{1}{2}\iint |v|^2 w\,dxdv$ are *not* positive functionals under the assumptions made at the kinetic level. This is the second crucial point in the quantum kinetic analysis.

A third aspect that differentiates quantum from classical kinetic theory, is the lack of a maximum principle for the Wigner function under time evolution. Indeed, $\|w(t)\|_{L^2(\mathbb{R}^6)}$ is the only conserved norm of (1.0.4). Due to the described differences, the analytic approach used for classical kinetic models like Vlasov-Poisson (VP) or Vlasov-Poisson-Fokker-Planck (VPFP) can *not* be adapted to quantum kinetic problems and a novel strategy is required.

In order to achieve a global-in-time result for the WPFP system (1.0.6)-(1.0.7), in Chapter 2 we exploit *dispersive* effects of the free-streaming operator jointly with the parabolic regularization of the Fokker-Planck term, since this yields a-priori estimates for the solution $w(t)$ in a weighted L^2-space. Such dispersive techniques for kinetic equations were first developed for the VP system (cf. [**LP, Pe**]) and then adapted to the VPFP equation (cf. [**Bo93, Ca98**]). In Chapter 2 these tools were extended to quantum kinetic theory.

In the present chapter, we will achieve as well a global-in-time well-posedness result for the WPFP system in the space $L^2(\mathbb{R}^6)$, *but without* introducing

weights. This is possible thanks to an alternative strategy that relies first of all on an a-priori estimate for the field $\nabla_x V(t)$ in terms of $\|w(t)\|_{L^2(\mathbb{R}^6)}$ only. This estimate was derived in Chapter 2 using dispersive effects of the free-streaming operator. It allows a *novel definition* of the macroscopic quantities (namely, the density, the self-consistent potential and the field), which, in contrast to their Definitions in (1.0.8), (1.0.9) and (1.0.10), is now non-local in time. This way, no v-integrability of w is needed, and hence no moments in v either. Secondly, we shall use the (degenerate) parabolic regularization of the Fokker-Planck term in order to construct (by a fixed point map) a global-in-time solution. These techniques allow to overcome the described analytical difficulties and they yield –a-posteriori– some L^p-estimates on the density.
In conclusion, our purely kinetic L^2-analysis solves both main problems of quantum kinetic theory, namely the definition of the density (due to the missing v-integrability of w) and the lack of usable a-priori estimates on w (due to its non-definite sign). Finally, we point out that we expect that this approach could also be a crucial step towards developing a kinetic analysis for the Wigner-Poisson system (1.0.4)-(1.0.7), which has been an open problem for 15 years.

This chapter is organized as follows: In Section 3.1 we motivate the new, non-local redefinition of the self-consistent field, and present the main results of this chapter. In Section 3.2 we derive a-priori estimates on the potential and the field which are the crucial ingredients for the global well-posedness analysis of Section 3.3. In the two different versions of the WPFP system, namely the elliptic ($\alpha > 0$) and hypoelliptic ($\alpha = 0$) cases, the solution exhibits a different asymptotic behaviour close to the initial time, and hence different analytical strategies will have to be applied. In Section 3.4 we establish –a-posteriori– the C^∞-regularity of the solution, and in Section 3.5 decay estimates on the particle density.

We anticipate that in the last two sections our technique will adequately highlight the (expected) different asymptotic behaviour close to the initial time of the regular solution of the different versions of WPFP system, namely the elliptic and hypoelliptic cases.

3.1 Strategy and main results

We shall prove existence and uniqueness of a mild solution $w(t) \in L^2(\mathbb{R}^6) = L^2(\mathbb{R}^6; dx\, dv)$ to the WPFP problem (1.0.6)-(1.0.8) on the time interval $[0, T]$, with $T > 0$ arbitrary, but fixed for the sequel. Accordingly, the solution has to satisfy, for all $t \in [0, T]$, the integral equation

(3.1.1)

$$\begin{aligned} w(x,v,t) &= \iint G(t, x-x_0, v, v_0)\, w_0(x_0, v_0)\, dx_0\, dv_0 \\ &\quad + \int_0^t \iint G(s, x-x_0, v, v_0)(\Theta[V]w)(x_0, v_0, t-s)\, dx_0\, dv_0\, ds, \end{aligned}$$

with the Green's function G given in Section 2.2.1.

The main difficulty in analyzing the WPFP system consists in defining the density $n(t)$ and the potential $V(t)$. As mentioned before, the standard, local-in-time definition (1.0.8) is unfeasible for a Wigner function $w(t) \in L^2(\mathbb{R}^6)$. We will show that it is possible to by-pass the definition of $n(t)$ by defining the potential $V[w]$ corresponding to a Wigner trajectory $w \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$. This *non-local* in time definition of $V[w]$ relies on dispersive effects of kinetic equations and it is inspired by a-priori estimates on the self-consistent field $\nabla_x V$ derived in Chapter 2.

To motivate our alternative definition of $V[w]$ let us first recall from Section 2.3.1 how the pseudo-differential operator $\Theta[V]$ has been reformulated in terms of $E = \nabla_x V$:

$$\Theta[V]u(x,v) = \operatorname{div}_v \left(\Gamma[E]u\right)(x,v), \tag{3.1.2}$$

with the vector-valued operator $\Gamma[E]$ defined in (2.3.7)-(2.3.8). The conditions, under which this redefinition of $\Theta[V]$ holds rigorously, are stated in Lemma 2.20.

Moreover, we shall exploit that

$$\int_{\mathbb{R}^3} G(t, x-x_0, v, v_0)\, dv = R(t)^{-3/2} \mathcal{N}\left(\frac{x - x_0 - \vartheta(t)v_0}{\sqrt{R(t)}}\right), \tag{3.1.3}$$

with the functions $\mathcal{N}(x)$, $\vartheta(t)$ and $R(t)$ defined in (2.3.25)-(2.3.27).

The parameter α (more precisely, $\alpha > 0$ or $\alpha = 0$) determines the asymptotic behaviour at $t = 0$ of the function $R(t)$, and hence the singularity of the convolution kernel (3.1.3) at $t = 0$. Since this convolution represents the parabolic regularization of the quantum Fokker-Planck operator, we have to distinguish the following two cases for the subsequent analysis,

the (degenerate) elliptic case

(I) $\quad \alpha\sigma \geq \gamma^2 \quad \text{and} \quad \alpha > 0 \quad \Rightarrow \quad R(t) = \mathcal{O}(t), \quad \text{for } t \to 0,$

and the hypoelliptic case

(II) $\quad \alpha = \gamma = 0, \quad \beta \geq 0 \quad \text{and} \quad \sigma > 0 \quad \Rightarrow \quad R(t) = \mathcal{O}(t^3), \quad \text{for } t \to 0.$

Following common practice for the VP (cf. [**LP**]) and VPFP systems (cf. [**Bo93**]), next we split the density, like in Section 2.3.2, into two terms: $n = n_0 + n_1$, where

$$\begin{aligned} n_0(x,t) &:= \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^\vartheta(x,t)\,, \\ \text{with} \quad n_0^\vartheta(x,t) &:= \int w_0(x - \vartheta(t)v, v)\,dv, \end{aligned} \tag{3.1.4}$$

$$\begin{aligned} n_1(x,t) &:= \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) \\ &\quad *_x \mathrm{div}_x \int \Big(\Gamma[E]w\Big)(x - \vartheta(s)v, v, t-s)\,dv\,ds. \end{aligned} \tag{3.1.5}$$

Analogously, we can split the self-consistent field, as in (2.3.28)-(2.3.29), into $E = E_0 + E_1$, where

$$E_0(x,t) := \lambda \frac{x}{|x|^3} *_x n_0(x,t), \tag{3.1.6}$$

$$E_1(x,t) := \lambda \frac{x}{|x|^3} *_x n_1(x,t), \tag{3.1.7}$$

with $\lambda = \frac{1}{4\pi}$. (3.1.5) now allows to rewrite E_1 as

$$\begin{aligned} (E_1)_j(x,t) &= \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} \\ &\quad *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F_k[w](x,t,s)\,ds, \end{aligned} \tag{3.1.8}$$

for $j = 1, 2, 3$ and

$$\text{with} \quad F_k[w](x,t,s) := \int \left(\Gamma_k[E_0 + E_1]w\right)(x - \vartheta(s)v, v, t-s)\,dv. \tag{3.1.9}$$

This is a linear Volterra integral equation of the second kind for the self-consistent field E_1. Note that all coefficients in the r.h.s. of (3.1.8) only depend on w_0 and w (and not on n). The advantage of the reformulation (3.1.2) of the pseudo-differential operator is precisely to obtain a closed equation for E_1, if w_0 and w are given. Starting with $w \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$, we shall prove that this integral equation has a unique solution. We remark that (3.1.5), instead, is not a closed equation for n_1 (for w_0 and w given); its r.h.s. also depends on the self-consistent field E. These motivations lead to our new definition of the Hartree-potential:

3.1 Definition (New definition of mean-field quantities). To a Wigner trajectory $w \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$ we associate

- the *field* $E[w] := E_0 + E_1[w]$, with E_0 given by (3.1.6), and $E_1[w]$ the unique solution of (3.1.8),
- the *potential* $V[w] := V_0 + V_1[w]$ with

$$
\begin{aligned}
&(3.1.10) \qquad & V_0(x,t) &:= \lambda \sum_{i=1}^{3} \frac{x_i}{|x|^3} *_x (E_0)_i(x,t), \\
&(3.1.11) \qquad & V_1[w](x,t) &:= \lambda \sum_{i=1}^{3} \frac{x_i}{|x|^3} *_x (E_1[w])_i(x,t),
\end{aligned}
$$

- and the *position density* $n[w] := -\mathrm{div} E[w]$ (at least in a distributional sense).

In contrast to the standard definitions (1.0.7)-(1.0.9), these new definitions are *non-local in time*. Also, the map $w \mapsto V[w]$ is now *non-linear*. For a given Wigner-trajectory these two definitions clearly differ in general. However, they coincide if w is the solution of the WPFP system. These new definitions of the self-consistent field and potential have the advantage that they only require $w \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$ and not $w(x,.,t) \in L^1(\mathbb{R}^3)$. If $w(t=0)$ only lies in $L^2(\mathbb{R}^6)$, the corresponding field and the potential will consequently only be defined for $t > 0$.

We shall now describe in detail our strategy to prove well-posedness of the WPFP system

$$
(3.1.12) \quad w_t \;=\; Aw + \Theta[V[w]]w, \quad t \in (0,T]; \quad w(t=0) \;=\; w_0 \in L^2(\mathbb{R}^6).
$$

(3.1.12) will be solved by a contractive fixed point map that is based on the linear equation

$$
(3.1.13) \qquad \tilde{u}_t \;=\; A\tilde{u} + \Theta[V[u]]\tilde{u}, \quad t \in (0,T]; \quad \tilde{u}(t=0) \;=\; w_0.
$$

While such an approach is standard for nonlinear PDEs, the key point is here the non-local definition of $V[u]$ via Definition 3.1. We will proceed as follows:

(i) the iteration will be considered in the set B_R, the ball of radius R in $\mathcal{C}([0,T]; L^2(\mathbb{R}^6))$, centered in the origin. Due to the a-priori estimate (3.3.4), we shall choose $R := e^{\frac{3}{2}\beta T}\|w_0\|_2$.

(ii) We will assume $w_0 \in L^2(\mathbb{R}^6)$ and satisfying (**C**) or (**D**) (see below). This will provide L^p-estimates on the field E_0, defined in (3.1.6). Accordingly, for $u \in B_R$, Definition 3.1 will yield a unique potential $V[u]$.

(iii) The estimates on $V[u]$ from (ii) will allow to prove existence and uniqueness of a mild solution for (3.1.13) that will satisfy $\tilde{u} \in B_R$.

(iv) We will finally define the non-linear map $M : B_R \longrightarrow B_R$ by $Mu := \tilde{u}$. Its unique fixed point will be the mild solution $w \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$ of the WPFP system (1.0.6)-(1.0.7), in the sense of (3.1.1).

We shall now specify the assumptions on the initial data w_0 that are mentioned in point (ii). We shall make two different assumptions on w_0, which will lead, however, to similar estimates on $E[u]$ and consequently on $V[u]$ (cf. Section 3.2).
In Section 3.3 we shall first prove existence and uniqueness of a mild solution of problem (1.0.6)-(1.0.7) under the assumption

$$(\mathbf{C}) \qquad w_0 \in L^2(\mathbb{R}^6) \quad \text{and} \quad \|n_0^\vartheta(t)\|_{L^\theta(\mathbb{R}^3_x)} \;\leq\; C_T t^{-\omega_\theta}, \quad \forall\, t \in (0,T],$$

for some $\omega_\theta \geq 0$.
For example, such an estimate for the "shifted" density n_0^ϑ (cf. (3.1.4)) can be concluded by the Strichartz estimate for the (free) kinetic equation [**CP**]

$$\|n_0^\vartheta(t)\|_{L^\theta(\mathbb{R}^3_x)} \;\leq\; C t^{-\omega_S(\theta)} \|w_0\|_{L^1_x(L^\theta_v)}, \quad \forall\, t \in (0,T], \tag{3.1.14}$$

with $\omega_S(\theta) := 3(1 - 1/\theta)$. At least in case (**I**) this typical example is always included in our main result, Theorem 3.8.
Here we introduce the constants determining the decay of n_0^ϑ that will be admitted in the subsequent analysis (cf. Thm. 4.1, e.g.).

$$\theta \in I_\theta \;:=\; \begin{cases} [1, \frac{6}{5}], & \text{case } (\mathbf{I}), \\ (\frac{9}{8}, \frac{6}{5}], & \text{case } (\mathbf{II}), \end{cases} \qquad \kappa(\theta) \;:=\; \begin{cases} 2 - \frac{3}{2\theta}, & \text{case } (\mathbf{I}), \\ 4 - \frac{9}{2\theta}, & \text{case } (\mathbf{II}). \end{cases}$$

Alternatively to assumption (**C**) we shall also consider initial data that satisfy:

$$(\mathbf{D}) \qquad w_0 \;\in\; L^2(\mathbb{R}^6) \cap L^1(\mathbb{R}^3_v; L^\theta(\mathbb{R}^3_x)), \quad \text{for some } \theta \in \left[1, \frac{6}{5}\right].$$

Considering estimate (3.1.14), this second assumption is complementary to the first one in the sense that the $x-$ and $v-$integrability of w_0 are interchanged. As we shall see, the well-posedness analysis performed under the assumption (**C**) will immediately extend to initial data satisfying (**D**).

The main result of this chapter is

Theorem 3.8 *Let either (**C**) hold for some $\theta \in I_\theta$ and $0 \le \omega_\theta < \kappa(\theta)$, or let (**D**) hold for some $\theta \in I_\theta$. Then, there exists a unique mild solution $w \in \mathcal{C}([0,\infty); L^2(\mathbb{R}^6))$ of the WPFP problem (1.0.6)-(1.0.7). In case (**I**), we also get $V[w] \in \mathcal{C}((0,\infty); L^\infty(\mathbb{R}^3))$.*

A-posteriori, we shall obtain the following regularity result for the solution:

Theorem 3.12 *Let (**C**) hold for some $\theta \in I_\theta$ and $0 \le \omega_\theta < \kappa(\theta)$, or let (**D**) hold for some $\theta \in I_\theta$ (in the latter case set $\omega_\theta := 0$). Then, the unique mild solution $w \in \mathcal{C}([0,\infty); L^2(\mathbb{R}^6))$ of the WPFP problem (1.0.6)-(1.0.7) satisfies*

$$w \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^6)),$$

with the estimate

$$\|D_x^l D_v^m w(t)\|_{L^2(\mathbb{R}^6)} \le C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{-\frac{L}{2}} t^{-\frac{M}{2}}, \quad \forall\, t \in (0,T],$$

for all $T > 0$, and all multiindices l, $m \in \mathbb{N}_0^3$, with $|l| = L$, $|m| = M \in \mathbb{N}_0$. Moreover, $E[w] \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$, satisfying for all $T > 0$, $t \in (0,T]$:

$$\|D_x^l E[w](t)\|_{L^2(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{L-1}{2}} t^{-\omega_\theta},$$

where $D_x^l E[w]$ represents the derivative with multiindex l of each component of the field $E[w]$. Accordingly (cf. Def. 3.1), the density

$$n[w] = -\mathrm{div} E[w] \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$$

satisfies in particular

$$\|n[w](t)\|_{L^2(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)} t^{-\omega_\theta}, \quad \forall\, t \in (0,T].$$

The self-consistent potential $V[w] \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$ and its Fourier transform $\widehat{V}[w](t)$ satisfy the estimates

$$\|V[w](t)\|_{L^\infty(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{1-\frac{3}{2\theta}} t^{-\omega_\theta}, \quad \forall\, t \in (0,T],$$

$$\|\widehat{V}[w](t)\|_{L^1(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, R(t)^{1-\frac{3}{2\theta}} t^{-\omega_\theta}, \quad \forall\, t \in (0,T].$$

Analogous regularity results on the classical VPFP equation were obtained in [**Bo95**] (Hölder regularity of the density and field) and rather recently in [**OS**] ($w, n, E \in C^\infty$ for positive time). Under the assumption (**D**), we shall also show for WPFP that $w \in C([0,\infty); L_v^1(L_x^\theta))$. Hence, the solution $w(t)$ remains in the space of the initial condition w_0 (cf. (**D**)). This allows to define the position density $n[w]$ in the standard sense (cf. Def. 1.1) and to derive an additional decay estimates for the density:

Theorem 3.13 *Let (**D**) hold for some $\theta \in I_\theta$. Then, the solution of the WPFP problem (1.0.6)-(1.0.7) satisfies*

(i) $w \in C([0,\infty); L^1_v(L^\theta_x))$,

(ii) the density $n(t)$ satisfies for all $T > 0$ and $\theta \le p \le 2$:

$$\|n(t)\|_{L^p(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\,, \|w_0\|_{L^1_v(L^\theta_x)}\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)},$$

for all $t \in (0,T]$.

3.2 A-priori estimates for the self-consistent potential

In this section we shall derive a-priori estimates for the previously defined fields $E_0, E_1[w]$ and the potentials $V_0, V_1[w]$. Such estimates rely on dispersive effects of the free-streaming operator *and* on the parabolic regularization of the quantum Fokker-Planck operator. Since the regularization is different in the two cases (**I**) and (**II**), the corresponding estimates will also differ. The following a-priori estimates generalize the results in Section 2.3.2 to the hypoelliptic case (**II**). To make this chapter self-contained we shall include a sketch of the proofs.

We start with an estimate on the field E_0, defined in (3.1.6) (cf. Lemma 2.29):

3.2 Proposition. *Let (**C**) hold for some $1 \le \theta \le 6/5$. Then, for all $p \ge 2$, the estimate*

$$\|E_0(t)\|_{L^p(\mathbb{R}^3_x)} \;\le\; C_T\, R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+\frac{1}{2}} t^{-\omega_\theta}, \quad \forall\, t \in (0,T] \tag{3.2.1}$$

holds.

Proof. The estimate is obtained by applying first the generalized Young inequality with $1/q = 1/p + 1/3$, and then the Young inequality with $1/q = 1/r + 1/\theta - 1$ to (3.1.6)

$$\begin{aligned}
\|E_0(t)\|_p &\le C \left\| \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) *_x n_0^\vartheta(x,t) \right\|_q \\
&\le C \left\| \frac{1}{R(t)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(t)}}\right) \right\|_r \|n_0^\vartheta(x,t)\|_\theta \\
&= C_T\, R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+\frac{1}{2}} \vartheta(t)^{-\omega_\theta}.
\end{aligned}$$

The assumed restriction on θ is necessary for the case $p = 2$. $\square$

Let us denote

$$N_\theta \equiv N_\theta(T) := \sup_{s\in(0,T]} \left\{ s^{\omega_\theta} \|n_0^\vartheta(s)\|_{L^\theta} \right\} < \infty ,$$

$$\mu(\theta) := \begin{cases} \frac{9}{4} - \frac{3}{2\theta}, & \text{case } (\mathbf{I}) , \\ \frac{19}{4} - \frac{9}{2\theta}, & \text{case } (\mathbf{II}) . \end{cases}$$

For a given Wigner-trajectory $u \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$ we now consider the inhomogeneous integral equation for the field $E_1 = E_1[u]$:

$$\text{(3.2.2)} \quad (E_1)_j(x,t) = \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F_k[u](x,t,s)\, ds,$$

$$\text{with} \quad F_k[u](x,t,s) := \int (\Gamma_k[E_0 + E_1]u)\,(x - \vartheta(s)v, v, t-s)\, dv,$$

and the vector valued operator $\Gamma[E_0 + E_1]$ defined in (2.3.7)-(2.3.8).

3.3 Proposition. *Let $u \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$ and $(\mathbf{C})$ hold for some $1 \le \theta \le 6/5$ and $0 \le \omega_\theta < \mu(\theta)$. Then, the integral equation (3.2.2) has a unique solution $E_1 = E_1[u] \in \mathcal{C}((0,T]; L^p(\mathbb{R}^3_x))$, which satisfies*

$$\text{(3.2.3)} \quad \|E_1[u](t)\|_{L^p(\mathbb{R}^3_x)} \le C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+\frac{1}{2}} t^{\frac{1}{2}-\omega_\theta},$$

for all $t \in (0,T]$ and for

$$\text{(3.2.4)} \quad 2 \le p < p_1 := \begin{cases} 6, & \text{case } (\mathbf{I}) , \\ \frac{18}{7}, & \text{case } (\mathbf{II}) . \end{cases}$$

Hence, $E[u] = E_0 + E_1[u]$ satisfies

$$\text{(3.2.5)} \quad \|E[u](t)\|_{L^p(\mathbb{R}^3_x)} \le C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+\frac{1}{2}} t^{-\omega_\theta},$$

for $2 \le p < p_1$ and $t \in (0,T]$.

Proof. By Lemma 2.22 it holds for $T \ge t \ge s > 0$:

(3.2.6)

$$\|F_k[u](.,t,s)\|_{L^2(\mathbb{R}^3_x)} \le C\vartheta(s)^{-3/2} \|(E_0 + E_1)(t-s)\|_{L^2(\mathbb{R}^3_x)} \|u(t-s)\|_{L^2(\mathbb{R}^3_x)}.$$

Proceeding as in the proof of Lemma 2.29 one can show the existence and uniqueness of the solution $E_1 = E_1[u]$ of (3.2.2) by a Banach fixed point argument in the space

$$\left\{ E \in \mathcal{C}((0,T];L^2(\mathbb{R}^3)) \,\middle|\, \sup_{0<t\le T} t^{\omega_\theta - \frac{1}{2}} R(t)^{\frac{3}{2\theta}-\frac{5}{4}} \|E(t)\|_{L^2} < \infty \right\}.$$

Moreover, we get as in Proposition 2.31 the following estimate for the solution of (3.2.2). Using classical properties of the convolution with $\frac{1}{|x|}$ (cf. [**St**]) yields:

(3.2.7)

$$\begin{aligned}
\|E_1[u](t)\|_{L^p(\mathbb{R}^3_x)} &\le C\int_0^t \vartheta(s) \left\| \frac{1}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x F[u](x,t,s)\right\|_{L^p(\mathbb{R}^3_x)} ds \\
&\le C\int_0^t \vartheta(s) \left\| \frac{1}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_{L^q(\mathbb{R}^3_x)} \|F[u](x,t,s)\|_{L^2(\mathbb{R}^3_x)}\, ds \\
&\le C\|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))} \int_0^t \frac{R(s)^{\frac{3}{2q}-\frac{3}{2}}}{\sqrt{\vartheta(s)}} \Big(\|E_0(t-s)\|_2 + \|E_1[u](t-s)\|_2\Big)\, ds,
\end{aligned}$$

with $1/2 + 1/p = 1/q$. For the integrability of the function $\|E_0(t)\|_{L^2(\mathbb{R}^3_x)}$ in $(0,T]$ (cf. Proposition 3.2), we need $\omega_\theta < \mu(\theta)$. The assertion for $p = 2$ follows from Gronwall's Lemma for (3.2.7). Then, we get (3.2.3) for $p > 2$ by using (3.2.7), provided condition (3.2.4) holds. □

Via (3.1.10)-(3.1.11), the above estimates on the field $E[u] = E_0 + E_1[u]$ immediately yield estimates for the potential $V[u] := V_0 + V_1[u]$:

3.4 Corollary. *Let $u \in \mathcal{C}([0,T];L^2(\mathbb{R}^6))$ and (**C**) hold for some $1 \le \theta \le 6/5$ and $0 \le \omega_\theta < \mu(\theta)$. Then, the potential $V[u] \in \mathcal{C}((0,T];L^p(\mathbb{R}^3_x))$ satisfies $\forall\, t \in (0,T]$:*

$$\begin{aligned}
&(3.2.8) & \|V_0(t)\|_p &\le C_T R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+1} t^{-\omega_\theta}, \\
&(3.2.9) & \|V_1[u](t)\|_p &\le C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+1} t^{\frac{1}{2}-\omega_\theta}, \\
&(3.2.10) & \|V[u](t)\|_p &\le C\left(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+1} t^{-\omega_\theta},
\end{aligned}$$

*for $6 \le p \le \infty$ in the case (**I**) or $6 \le p < 18$ in the case (**II**).*

PROOF. The admissible p-intervals follow immediately from condition (3.2.4) on E_1. □

3.5 Remark. We note that a-posteriori regularity of the solution $w(t)$ will imply for the self-consistent potential, $V[w](t) \in L^\infty(\mathbb{R}^3)$, $t > 0$ also in case (**II**) (cf. Theorem 3.12).

To close this section we shall now derive the analogous a-priori estimates under the assumption (**D**). For $T > 0$ fixed, we consider the density

$$(3.2.11) \qquad n_0(x,t) \;=\; \iint G(t,x,v,v_0) *_x w_0(x,v_0)\, dv_0\, dv, \quad \forall\, t \in [0,T];$$

cf. (3.1.4) for a different representation of n_0. The following decay estimate for the (classical) Vlasov-FP equation (cf. Lemma 2 in [**Ca**]) carries over to the Green's function G (cf. [**SCDM**]) for the WFP equation:

$$(3.2.12) \qquad \| \int G(t,x,v,v_0) *_x w_0(x,v_0)\, dv_0 \|_{L^p_x(L^1_v)} \;\le\; CR(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)} \|w_0\|_{L^1_v(L^\theta_x)},$$

for all $p \ge \theta$ and $t \in (0,T]$. Hence, if we assume (**D**) then (3.2.12) implies

$$(3.2.13) \qquad \|n_0(t)\|_{L^p_x} \;\le\; CR(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)} \|w_0\|_{L^1_v(L^\theta_x)}, \quad \forall\, p \ge \theta, \quad \forall\, t \in (0,T].$$

Thus, we can handle the affine term E_0 analogously to Proposition 3.2:

3.6 Corollary. *Let (*__D__*) hold for some* $1 \le \theta \le 6/5$. *Then, we have for all* $p \ge 2$ *and* $t \in (0,T]$*:*

$$(3.2.14) \qquad \|E_0(t)\|_{L^p_x} \;\le\; CR(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+\frac{1}{2}} \|w_0\|_{L^1_v(L^\theta_x)},$$

$$(3.2.15) \qquad \|V_0(t)\|_{L^p_x} \;\le\; CR(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)+1} \|w_0\|_{L^1_v(L^\theta_x)}.$$

Proof. This follows from (3.1.6), (3.1.10) with the generalized Young inequality and (3.2.13). □

3.7 Remark. Note that these decay rates of E_0 and V_0 correspond exactly to case (**C**) with $\omega_\theta = 0$ (cf. (3.2.1), (3.2.8)). Hence, all results of Proposition 3.3 and Corollary 3.4 carry over to case (**D**) when setting $\omega_\theta = 0$. In the subsequent well-posedness analysis for WPFP, the only relevant information on w_0 is the rate of singularity at $t = 0$ (and hence the integrability on $(0,T]$) of E_0 and V_0. In this respect, the analysis of the WPFP problem under assumption (**D**) appears just as a special case of the situation under assumption (**C**). Therefore, the existence and uniqueness result of Theorem 3.8 for case (**C**) directly implies an analogous result for case (**D**).

3.3 Existence and uniqueness of a global solution

The goal of this section is to prove the following

3.8 Theorem. *Let either* (**C**) *hold for some* $\theta \in I_\theta$ *and* $0 \le \omega_\theta < \kappa(\theta)$, *or let* (**D**) *hold for some* $\theta \in I_\theta$. *Then, there exists a unique mild solution* $w \in \mathcal{C}([0,\infty); L^2(\mathbb{R}^6))$ *of the WPFP problem (1.0.6)-(1.0.7). In case* (**I**), *we also get* $V \in \mathcal{C}((0,\infty); L^\infty(\mathbb{R}^3))$.

We will follow the strategy outlined in Section 3.1. Theorem 3.8 will follow from a sequence of auxiliary results that we derive first. In this section we shall only discuss the analysis according to the assumption (**C**). Due to Remark 3.7, however, all results of this section apply verbatim (with $\omega_\theta = 0$) to case (**D**).
Let $T > 0$ be arbitrary but fixed, $w_0 \in L^2(\mathbb{R}^6)$, and set $R := e^{\frac{3}{2}\beta T}\|w_0\|_2$. Let us denote with B_R the ball of radius R centered in the origin of $\mathcal{C}([0,T]; L^2(\mathbb{R}^6))$ and let u belong to B_R. Then, we consider the linear equation

$$\tilde{u}_t = A\tilde{u} + \Theta[V[u]]\tilde{u}, \quad t \in (0,T], \tag{3.3.1}$$

with the initial value

$$\tilde{u}(t=0) = w_0.$$

In case (**I**), we have $V[u](t) \in L^\infty(\mathbb{R}^3)$ for $t > 0$ (cf. (3.2.10)) and hence $\Theta[V[u](t)]$ is a bounded linear operator on $L^2(\mathbb{R}^6)$, which satisfies

$$\|\Theta[V[u](t)]\|_{\mathcal{B}(L^2(\mathbb{R}^6))} \le C\big(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)\, t^{1-\frac{3}{2\theta}-\omega_\theta}, \tag{3.3.2}$$

for all $t \in (0,T]$.
Moreover, if $\omega_\theta < \kappa(\theta)$, then $\Theta[V[u](\cdot)] \in L^1((0,T); \mathcal{B}(L^2(\mathbb{R}^6)))$.
In case (**II**), however, we lack an a-priori bound for $\|V[u](t)\|_\infty$. Thus, $\Theta[V[u(t)]]$ will *not* be a bounded operator on $L^2(\mathbb{R}^6)$. Instead, we shall exploit the a-priori bound for $\|V[u](t)\|_6$, jointly with the regularization of the semigroup e^{tA}. This is the key-idea of the following

3.9 Proposition. *Let* (**C**) *hold for some* $\theta \in I_\theta$ *and* $0 \le \omega_\theta < \kappa(\theta)$. *Also assume that* $u \in B_R$. *Then, the equation (3.3.1) has a unique mild solution* $\tilde{u} \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$, *which satisfies*

$$\begin{aligned} \tilde{u}(x,v,t) &= \int G(t,x,v,v_0) *_x w_0(x,v_0)\, dv_0 \\ &\quad + \int_0^t \int G(s,x,v,v_0) *_x (\Theta[V[u]]\tilde{u})(x,v_0,t-s)\, dv_0\, ds \end{aligned} \tag{3.3.3}$$

and

$$\|\tilde{u}(t)\|_2 \le e^{\frac{3}{2}\beta t}\|w_0\|_2, \quad \forall\, t \in [0,T]. \tag{3.3.4}$$

Proof. In case (**I**), $A - \frac{3}{2}\beta I$ generates a C_0 semigroup of contractions on $L^2(\mathbb{R}^6)$ (see Section 2.1.2 for the details) and $\Theta[V[u](.)]$ is a bounded perturbation, integrable in time. The assertion then follows from standard semigroup theory (cf. Thm. 6.1.2 in [**Pa**]).

In case (**II**), we define the affine map P for all $\tilde{z} \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$:

$$
\begin{aligned}
(3.3.5)\quad P\tilde{z}(x,v,t) \;:=&\; \int G(t,x,v,v_0) *_x w_0(x,v_0)\,dv_0 \\
&+ \int_0^t \int G(s,x,v,v_0) *_x (\Theta[V[u]]\tilde{z})(x,v_0,t-s)\,dv_0\,ds.
\end{aligned}
$$

We will show that it has a unique fixed point $\tilde{u} \in \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$. The crucial step is to prove that P maps into $\mathcal{C}([0,T]; L^2(\mathbb{R}^6))$.

To this end we first state the following estimate on the x-derivatives (with multiindex l) of the Green's function g (cf. (2.2.5)) that can be proved directly by calculating the integral. It reflects the regularization of the semigroup e^{At}.

$$
(3.3.6)\qquad \|D_x^l g(t)\|_{L^1_{x,v}} \;\le\; C_T R(t)^{-\frac{L}{2}}, \quad \forall\, t \le T,\ |l| = L,\ L \ge 0.
$$

Thus, by the Sobolev embedding $W^{\frac{1}{2},1}(\mathbb{R}^3_x) \hookrightarrow L^{6/5}(\mathbb{R}^3_x)$ and interpolation in (3.3.6) between $L = 0$ and $L = 1$, we get

$$
(3.3.7)\qquad \|g(t)\|_{L^1_v(L_x^{6/5})} \;\le\; C_T R(t)^{-\frac{1}{4}}, \quad \forall\, t \le T.
$$

Next we note that G does *not* act in (3.3.5) as a convolution in the v-variable. However, it *is* a convolution in the characteristic variables $\bar{x} := x + v\left(\frac{1-e^{\beta t}}{\beta}\right)$, $\bar{v} := ve^{\beta t}$ (cf. (2.2.4)):

$$
(3.3.8)\quad \iint G(t,x-x_0,v,v_0)\phi(x_0,v_0)\,dx_0 dv_0 \;=\; e^{3\beta t}\Big(g(t) *_{x,v} \phi\Big)(\bar{x},\bar{v}).
$$

By using the Jacobian of the transformation, $\left|\frac{d(x,v)}{d(\bar{x},\bar{v})}\right| = e^{-\frac{3}{2}\beta t}$, it follows for all p, q such that $\frac{1}{p} + \frac{1}{q} = \frac{3}{2}$:

$$
\begin{aligned}
(3.3.9)\quad \left\|\int G(t,x,v,v_0) *_x \phi(x,v_0)\,dv_0\right\|_{L^2_{x,v}} \;&=\; e^{\frac{3}{2}\beta t}\|g(t) *_{x,v} \phi\|_{L^2_{x,v}} \\
&\le\; e^{\frac{3}{2}\beta t}\|g(t)\|_{L^1_v(L^p_x)}\|\phi\|_{L^2_v(L^q_x)},
\end{aligned}
$$

for all $\phi \in L^2_v(L^q_x)$. In addition, it can be checked by duality that

$$
(3.3.10)\quad \|(\Theta[V[u]]\tilde{z})(t-s)\|_{L^2_v(L_x^{3/2})} \;\le\; 2\|V[u](t-s)\|_{L^6_x}\|\tilde{z}(t-s)\|_{L^2_{x,v}}.
$$

Now we estimate the L^2-norm of (3.3.5) by applying the results (3.3.6)-(3.3.10) and the a-priori bound (3.2.10) with $p = 6$. We finally get

$$
\begin{aligned}
(3.3.11)\quad \|P\tilde{z}(t)\|_{L^2_{x,v}} &\leq e^{\frac{3}{2}\beta t}\|w_0\|_{L^2_{x,v}} \\
&\quad + 2e^{\frac{3}{2}\beta t}\int_0^t \|g(s)\|_{L^1_v(L^{6/5}_x)}\|V[u](t-s)\|_{L^6_x}\|\tilde{z}(t-s)\|_{L^2_{x,v}}ds \\
&\leq e^{\frac{3}{2}\beta t}\|w_0\|_{L^2_{x,v}} + C\big(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \\
&\quad \cdot \|\tilde{z}\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\int_0^t s^{-\frac{3}{4}}\,(t-s)^{\frac{9}{2}\left(\frac{1}{6}-\frac{1}{\theta}\right)+3-\omega_\theta}\,ds.
\end{aligned}
$$

The condition $\omega_\theta < \kappa(\theta)$ guarantees that the last integral is in $\mathcal{C}[0,T]$.
Concerning the contractivity of P, we obtain analogously for all $\tilde{z}_1, \tilde{z}_2 \in \mathcal{C}([0,T];L^2(\mathbb{R}^6))$ by induction:

$$
\begin{aligned}
\|(P^n\tilde{z}_1 - P^n\tilde{z}_2)(t)\|_{L^2_{x,v}} &\leq C\big(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \\
&\quad \cdot \int_0^t s^{-\frac{3}{4}}\,(t-s)^{\frac{9}{2}\left(\frac{1}{6}-\frac{1}{\theta}\right)+3-\omega_\theta}\|(P^{n-1}\tilde{z}_1 - P^{n-1}\tilde{z}_2)(t-s)\|_{L^2_{x,v}}\,ds \\
&\leq C\big(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)^n C_{n-1}\|\tilde{z}_1 - \tilde{z}_2\|_{\mathcal{C}([0,T];L^2_{x,v})} \\
&\quad \cdot \int_0^t \frac{s^{(n-2)(1-a-b)-a}}{(t-s)^b}\,ds,
\end{aligned}
$$

for $n \in \mathbb{N}$ and some $a, b \geq 0$ with $a + b < 1$. Further,

$$
\int_0^t \frac{s^{(n-2)(1-a-b)-a}}{(t-s)^b}\,ds = t^{(n-1)(1-a-b)}B\Big(1-b, (n-2)(1-a-b)+1-a\Big),
$$

and

$$
C_{n-1} = \prod_{j=1}^{n-1} B\Big(1-b, j(1-a-b)+1-a\Big) = \frac{\Gamma(1-b)^{n-1}\,\Gamma(1-a)}{\Gamma((n-1)(1-a-b)+1)},
$$

where B denotes the Beta function and Γ the Gamma function. Clearly,

$$
C\big(T, \|u\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big)^n C_{n-1}\int_0^t \frac{s^{(n-2)(1-a-b)-a}}{(t-s)^b}\,ds < 1
$$

for n large enough. Thus, the map P^n is contractive and admits a unique fixed point in B_R.
Formally, the L^2-bound (3.3.4) follows from the dissipativity of the operator $A - \frac{3}{2}\beta I$ in $L^2(\mathbb{R}^6)$ and the skew-symmetry of $\Theta[V[u]]$. This can be justified as follows:

Applying $|\nabla_x|^{1/2}$ to (3.3.3) yields by using (3.3.6):

(3.3.12)

$$\begin{aligned}
\||\nabla_x|^{1/2}\tilde{u}(t)\|_{L^2_{x,v}} &\leq C(T)\left\||\nabla_x|^{1/2}g(t)\right\|_{L^1_{x,v}}\|w_0\|_{L^2_{x,v}} \\
&\quad + C(T)\int_0^t \left\||\nabla_x|^{1/2}g(s)\right\|_{L^1_{x,v}}\|(\Theta[V[u]]\tilde{u})(t-s)\|_{L^2_{x,v}}ds \\
&\leq C(T)t^{-\frac{3}{4}}\|w_0\|_{L^2_{x,v}} \\
&\quad + C(T)\int_0^t s^{-\frac{3}{4}}\|V[u](t-s)\|_{L^6_x}\|\tilde{u}(t-s)\|_{L^2_v(\dot{H}^{1/2}_x)}ds,
\end{aligned}$$

where $\|.\|_{\dot{H}^{1/2}} = \||\nabla|^{1/2}.\|_{L^2}$ denotes the $H^{1/2}$-seminorm on $\mathbb{R}^3$. Applying the Gronwall Lemma to (3.3.12) then yields $t^{\frac{3}{4}}\tilde{u} \in \mathcal{C}([0,T]; L^2_v(H^{1/2}_x))$ and hence (using (3.2.10) with $p = 6$) $f := \Theta[V[u]]\tilde{u} \in \mathcal{C}((0,T]; L^2(\mathbb{R}^6)) \cap L^1((0,T); L^2(\mathbb{R}^6))$. Now (3.3.1) can be written as

$$\tilde{u}_t = A\tilde{u} + f(t), \quad t \in (0,T], \quad \tilde{u}(t=0) = w_0, \tag{3.3.13}$$

and the L^2-estimate on $\tilde{u}$ can finally be obtained by a standard approximation of $\tilde{u}$ by classical solutions of (3.3.13) (cf. Thm. 4.2.7 in [**Pa**], Lemma 2.18 for the details). ☐

We now consider the non-linear map $M : B_R \longrightarrow B_R$ defined as

$$Mu := \tilde{u}, \tag{3.3.14}$$

which is well-defined by the previous proposition. The next goal is to prove that the map M admits a unique fixed point in $B_R \subset \mathcal{C}([0,T]; L^2(\mathbb{R}^6))$, which will be the mild solution of our WPFP problem. To this end we need the following result.

3.10 Lemma. *Let* (**C**) *hold for some* $1 \leq \theta \leq 6/5$ *and* $0 \leq \omega_\theta < \mu(\theta)$. *Then, for* $u_1, u_2 \in B_R$,

$$\begin{aligned}
\|V_1[u_1](t) - V_1[u_2](t)\|_{L^6(\mathbb{R}^3_x)} &\leq \\
&\leq C(T,R,N_\theta)\,R(t)^{\frac{5}{4}-\frac{3}{2\theta}}t^{\frac{1}{2}-\omega_\theta}\|u_1-u_2\|_{\mathcal{C}([0,t];L^2(\mathbb{R}^6))}
\end{aligned}\tag{3.3.15}$$

holds for all $t \in (0,T]$. *In the case* (**I**), *it also holds*

$$\begin{aligned}
\|V_1[u_1](t) - V_1[u_2](t)\|_{L^\infty(\mathbb{R}^3_x)} &\leq \\
&\leq C(T,R,N_\theta)\,t^{\frac{3}{2}-\frac{3}{2\theta}-\omega_\theta}\|u_1-u_2\|_{\mathcal{C}([0,t];L^2(\mathbb{R}^6))},
\end{aligned}\tag{3.3.16}$$

for all $t \in (0,T]$.

Proof.

$$
\begin{aligned}
(E_1[u_1] - E_1[u_2])_j\,(x,t) &= \lambda \sum_{k=1}^{3} \frac{-3x_j x_k + \delta_{jk}|x|^2}{|x|^5} \\
&\quad *_x \int_0^t \frac{\vartheta(s)}{R(s)^{3/2}} \mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right) *_x \left(F_k[u_1,u_1] - F_k[u_2,u_2]\right)(x,t,s)\,ds,
\end{aligned}
$$

with

$$
F_k[u,\tilde{u}](x,t,s) := \int \left(\Gamma_k[E_0 + E_1[u]]\tilde{u}\right)(x - \vartheta(s)v, v, t-s)\,dv.
$$

By classical properties of the convolution with $\frac{1}{|x|}$ (cf. [**St**]) and the Young inequality, we get

$$
\|E_1[u_1](t) - E_1[u_2](t)\|_2 \leq C\int_0^t \vartheta(s)\|\left(F_k[u_1,u_1] - F_k[u_2,u_2]\right)(t,s)\|_2\,ds.
$$

We write

$$
F_k[u_1,u_1] - F_k[u_2,u_2] = F_k[u_1,u_1] - F_k[u_1,u_2] + F_k[u_1,u_2] - F_k[u_2,u_2].
$$

By (3.2.6) (cf. Lemma 2.22) we have for $t \geq s > 0$:

$$
\begin{aligned}
\|F_k[u_1,u_2](t,s)\|_{L^2(\mathbb{R}^3_x)} &\leq C\vartheta(s)^{-3/2}\|u_2(t-s)\|_{L^2(\mathbb{R}^6)} \\
&\quad \cdot \|(E_0 + E_1[u_1])(t-s)\|_{L^2(\mathbb{R}^3_x)}.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&\|E_1[u_1](t) - E_1[u_2](t)\|_2 \leq \\
&\quad \leq C\int_0^t \frac{1}{\sqrt{\vartheta(s)}} \Big(\|(E_0 + E_1[u_1])(t-s)\|_2 \|(u_1 - u_2)(t-s)\|_2 \\
&\qquad + \|(E_1[u_1] - E_1[u_2])(t-s)\|_2 \|u_2(t-s)\|_2 \Big)\,ds.
\end{aligned} \tag{3.3.17}
$$

By (3.2.5) we obtain

$$
\begin{aligned}
&\|E_1[u_1](t) - E_1[u_2](t)\|_2 \leq \\
&\quad \leq C\,(T,R,N_\theta)\int_0^t \frac{1}{\sqrt{\vartheta(s)}} R(t-s)^{\frac{5}{4}-\frac{3}{2\theta}}(t-s)^{-\omega_\theta}\|(u_1 - u_2)(t-s)\|_2\,ds \\
&\quad + CR\int_0^t \frac{1}{\sqrt{\vartheta(s)}} \|(E_1[u_1] - E_1[u_2])(t-s)\|_2\,ds.
\end{aligned}
$$

By Gronwall's Lemma we get for $t \in (0,T]$:

(3.3.18)

$$\begin{aligned}
\|E_1[u_1](t) - E_1[u_2](t)\|_2 \;&\leq \\
\leq\; & C\,(T,R,N_\theta)\,\|u_1-u_2\|_{\mathcal{C}([0,t];L^2(\mathbb{R}^6))}\Bigg(\int_0^t \frac{1}{\sqrt{\vartheta(s)}}\,R(t-s)^{\frac{5}{4}-\frac{3}{2\theta}}(t-s)^{-\omega_\theta}\,ds \\
&+\; e^{C_T R t^{1/2}}\int_0^t\!\!\int_0^s \frac{1}{\sqrt{\vartheta(s)\vartheta(\tau)}}\,R(s-\tau)^{\frac{5}{4}-\frac{3}{2\theta}}(s-\tau)^{-\omega_\theta}\,d\tau\,ds\Bigg) \\
\leq\; & C\,(T,R,N_\theta)\,\|u_1-u_2\|_{\mathcal{C}([0,t];L^2(\mathbb{R}^6))}R(t)^{\frac{5}{4}-\frac{3}{2\theta}}t^{\frac{1}{2}-\omega_\theta}.
\end{aligned}$$

With

$$\|V_1[u_1](t) - V_1[u_2](t)\|_6 \;\leq\; C\|E_1[u_1](t) - E_1[u_2](t)\|_2$$

the assertion (3.3.15) follows.

To prove (3.3.16) in case (**I**) we proceed analogously and obtain by using Young's inequality

$$\begin{aligned}
\|E_1[u_1](t) - E_1[u_2](t)\|_4 \;&\leq \\
\leq\; & C\int_0^t \vartheta(s)\left\|\frac{1}{R(s)^{3/2}}\mathcal{N}\left(\frac{x}{\sqrt{R(s)}}\right)\right\|_{4/3}\|\left(F_k[u_1,u_1]-F_k[u_2,u_2]\right)(t,s)\|_2\,ds \\
\leq\; & C\int_0^t \frac{R(s)^{-3/8}}{\sqrt{\vartheta(s)}}\Big(\|(E_0+E_1[u_1])(t-s)\|_2\|(u_1-u_2)(t-s)\|_2 \\
&+\|(E_1[u_1]-E_1[u_2])(t-s)\|_2\|u_2(t-s)\|_2\Big)\,ds.
\end{aligned}$$

By applying the estimates (3.2.5) and (3.3.18), we then get

$$\begin{aligned}
(3.3.19) \qquad \|E_1[u_1](t) - E_1[u_2](t)\|_{L^4(\mathbb{R}^3_x)} \;&\leq \\
\leq\; & C\,(T,N_\theta,R)\,t^{\frac{11}{8}-\frac{3}{2\theta}-\omega_\theta}\|u_1-u_2\|_{\mathcal{C}([0,t];L^2(\mathbb{R}^6))}.
\end{aligned}$$

The second assertion (3.3.16) then follows from the Gagliardo-Nirenberg inequality (cf. [**Fr**], Chapter 1.9) using estimates (3.3.15) and (3.3.19). ☐

3.11 Proposition. *Let (*$\mathbf{C}$*) hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$. Then, the map M, defined by (3.3.14), has a unique fixed point in B_R.*

Proof. We give the proof only for the case (**II**); case (**I**) is easier due to the boundedness of $\Theta[V[u]]$ (cf. (3.3.2) and (3.3.16)).

For $u_1, u_2 \in B_R$ we start from equation (3.3.3). Estimating like in the proof of Proposition 3.9 and with (3.3.15), we obtain

$$
\begin{aligned}
&\|Mu_1(t) - Mu_2(t)\|_{L^2_{x,v}} \leq \\
&\quad \leq \int_0^t \left\| \int G(s,x,v,v_0) *_x \left(\Theta[V[u_1] - V[u_2]]Mu_2\right)(x,v_0,t-s)\, dv_0 \right\|_{L^2_{x,v}} ds \\
&\qquad + \int_0^t \left\| \int G(s,x,v,v_0) *_x \left(\Theta[V[u_1]]\left(Mu_1 - Mu_2\right)\right)(x,v_0,t-s)\, dv_0 \right\|_{L^2_{x,v}} ds \\
&\quad \leq C\left(T,R,N_\theta\right) \int_0^t s^{-\frac{3}{4}} (t-s)^{\frac{17}{4} - \frac{9}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,t-s];L^2(\mathbb{R}^6))}\, ds \\
&\qquad + C\left(T,R,N_\theta\right) \int_0^t s^{-\frac{3}{4}} (t-s)^{\frac{15}{4} - \frac{9}{2\theta} - \omega_\theta} \|(Mu_1 - Mu_2)(t-s)\|_{L^2_{x,v}}\, ds.
\end{aligned}
$$

By applying Gronwall's Lemma we get

$$
\begin{aligned}
&\|Mu_1(t) - Mu_2(t)\|_{L^2_{x,v}} \leq \\
&\quad \leq C\left(T,R,N_\theta\right) \int_0^t s^{-\frac{3}{4}} (t-s)^{\frac{17}{4} - \frac{9}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,t-s];L^2(\mathbb{R}^6))}\, ds \\
&\qquad + C\left(T,R,N_\theta\right) \int_0^t s^{\left(\frac{9}{2} - \frac{9}{2\theta} - \omega_\theta\right) - \frac{3}{4}} (t-s)^{\frac{15}{4} - \frac{9}{2\theta} - \omega_\theta} \|u_1 - u_2\|_{C([0,s];L^2(\mathbb{R}^6))} ds
\end{aligned}
$$

Then, the result follows by a contraction argument like in Proposition 3.9: The map M^n is contractive for n large enough. Thus, M admits a unique fixed point in B_R. □

The above auxiliary results directly yield the

Proof of Theorem 3.8.
The fixed point of the map M satisfies the equation (3.1.1) for all $T > 0$. Thus, it is the unique *global-in-time* mild solution of problem (1.0.6)-(1.0.7), in the sense of (3.1.1). □

3.4 Regularity of the solution

In this section we shall establish the C^∞-regularity of the unique, global, mild solution $(w, n, V, E = E_0 + E_1)$ of the WPFP-system (1.0.6)–(1.0.7). Based on a bootstrapping argument, our main result is

3.12 Theorem. *Let* (**C**) *hold for some $\theta \in I_\theta$ and $0 \leq \omega_\theta < \kappa(\theta)$, or let* (**D**) *hold for some $\theta \in I_\theta$ (in the latter case set $\omega_\theta := 0$). Then, the unique mild solution $w \in \mathcal{C}([0,\infty); L^2(\mathbb{R}^6))$ of the WPFP problem (1.0.6)-(1.0.7) satisfies*

$$
w \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^6)),
$$

with the estimate

$$\|D_x^l D_v^m w(t)\|_{L^2(\mathbb{R}^6)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\,, N_\theta\big)\, R(t)^{-\frac{L}{2}} t^{-\frac{M}{2}}, \tag{3.4.1}$$

for all $t \in (0,T],\ T > 0,$ *and all multiindices* $l,\, m \in \mathbb{N}_0^3$*, with* $|l| = L,\ |m| = M \in \mathbb{N}_0$*.*
Moreover, $E, V \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$*, satisfying for all* $t \in (0,T],\ T > 0$*:*

$$\|D_x^l E(t)\|_{L^2(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\,, N_\theta\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{L-1}{2}} t^{-\omega_\theta}, \tag{3.4.2}$$

where $D_x^l E$ *represents the derivative with multiindex* l *of each component of the field* E*.*
Accordingly, the density $n = -\mathrm{div} E \in \mathcal{C}((0,\infty); \mathcal{C}_B^\infty(\mathbb{R}^3))$ *satisfies*

$$\|n(t)\|_{L^2(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\,, N_\theta\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)} t^{-\omega_\theta}, \tag{3.4.3}$$

for all $t \in (0,T]$*.*
The self-consistent potential $V(t)$ *and its Fourier transform* $\widehat{V}(t)$ *satisfy for all* $t \in (0,T],\ T > 0,$ *the estimates*

$$\|V(t)\|_{L^\infty(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\,, N_\theta\big)\, R(t)^{1-\frac{3}{2\theta}} t^{-\omega_\theta}, \tag{3.4.4}$$

$$\|\widehat{V}(t)\|_{L^1(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}\,, N_\theta\big)\, R(t)^{1-\frac{3}{2\theta}} t^{-\omega_\theta}. \tag{3.4.5}$$

Proof. A calculation as in Proposition 3.2 gives

$$\|D_x^l E_0(t)\|_{L^2(\mathbb{R}^3)} \;\le\; C_T R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{L-1}{2}} t^{-\omega_\theta}, \quad \forall\, t \in (0,T]. \tag{3.4.6}$$

In the sequel we shall use the estimate (3.3.6) for the x-derivatives of the Green's function g and

$$\|D_v^m g(t)\|_{L^1_{x,v}} \;\le\; C_T t^{-\frac{M}{2}}, \quad \forall\, t \le T,\ |m| = M,\ M \ge 0, \tag{3.4.7}$$

which again can be proved directly by calculating the integrals.

Step 1:
By induction on $L = |l|$ we shall first prove the estimates for the x-derivatives, i.e. (3.4.1) for $m = 0$ and (3.4.2). We fix some multiindex $l = (l_1, l_2, l_3)$ with $L = |l| \ge 1$ and suppose that the inequalities (3.4.1), (3.4.2) hold for all multiindices $\tilde{l}$ with $0 \le |\tilde{l}| < L$.
We apply D_x^l to (3.1.1) and use (3.3.9) with $p = 1$ for the first summand and $p = \frac{6}{5}$ for the second one:

$$\begin{aligned}\|D_x^l w(t)\|_2 \;\le\;& e^{\frac{3}{2}\beta t}\|D_x^l g(t)\|_{L^1_{x,v}}\|w_0\|_2 \\ &+\int_{t/2}^{t} e^{\frac{3}{2}\beta s}\|D_x^l g(s)\|_{L^1_v(L^{6/5}_x)}\|(\Theta[V]w)(t-s)\|_{L^2_v(L^{3/2}_x)}\,ds \\ &+\sum_{\substack{0\le l_j^k\le l_j\\ j=1,2,3}}\binom{l_1}{l_1^k}\binom{l_2}{l_2^k}\binom{l_3}{l_3^k}\int_0^{t/2} e^{\frac{3}{2}\beta s}\|g(s)\|_{L^1_v(L^{6/5}_x)} \\ &\cdot\|(\Theta[D_x^{l-l^k}V]D_x^{l^k}w)(t-s)\|_{L^2_v(L^{3/2}_x)}\,ds,\end{aligned}$$

where $l^k=(l_1^k,l_2^k,l_3^k)$. Here we had to split the time integral and apply D_x^l, respectively, to the first and second convolution factor. Without this procedure the resulting integrals would diverge either at $s=0$ or at $s=t$. Using the estimates (3.3.6), (3.3.7), (3.3.10), and (3.2.10) we obtain:

$$\begin{aligned}\|D_x^l w(t)\|_2 \;\le\;& C_T R(t)^{-\frac{L}{2}}\|w_0\|_2 \\ &+C_T\int_{t/2}^{t}R(s)^{-\frac{2L+1}{4}}\|V(t-s)\|_6\|w(t-s)\|_2\,ds \\ &+C_T\sum_{\substack{0\le l_j^k\le l_j\\ j=1,2,3}}\int_0^{t/2}R(s)^{-\frac14}\|D_x^{l-l^k}V(t-s)\|_6\|D_x^{l^k}w(t-s)\|_2\,ds \\ \le\;& C\big(T,\|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))},N_\theta\big)\Big[R(t)^{-\frac{L}{2}}+R(t)^{-\frac{L}{2}+\frac32\left(\frac12-\frac1\theta\right)+\frac14}t^{1-\omega_\theta}\Big] \\ &+C_T\sum_{\substack{0\le l_j^k\le l_j,\ j=1,2,3\\ l^k\neq 0,\ l^k\neq l}}\int_0^{t/2}R(s)^{-\frac14}\|D_x^{l-l^k}E(t-s)\|_2\|D_x^{l^k}w(t-s)\|_2\,ds \\ &+C_T\int_0^{t/2}R(s)^{-\frac14}\|D_x^l E(t-s)\|_2\|w(t-s)\|_2\,ds \\ &+C_T\int_0^{t/2}R(s)^{-\frac14}\|E(t-s)\|_{L^2_x}\|D_x^l w(t-s)\|_2\,ds.\end{aligned}$$

By considering the range of ω_θ, using (3.2.5), (3.4.6), (3.4.1), and (3.4.2) for $|l|<L$ we obtain

(3.4.8)

$$\begin{aligned}\|D_x^l w(t)\|_2 \;\le\;& C\big(T,\|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))},N_\theta\big) \\ \cdot&\left[R(t)^{-\frac{L}{2}}+\sum_{k=1}^{L-1}R(t)^{-\frac14+\frac32\left(\frac12-\frac1\theta\right)-\frac{L-k-1}{2}}R(t)^{-\frac{k}{2}}t^{1-\omega_\theta}+R(t)^{\frac32\left(\frac12-\frac1\theta\right)-\frac{2L-1}{4}}t^{1-\omega_\theta}\right]\end{aligned}$$

$$
\begin{aligned}
&+ C_T \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))} \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^l E_1(t-s)\|_2 \, ds \\
&+ C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \\
&\cdot \int_0^{t/2} R(s)^{-\frac{1}{4}} R(t-s)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)+\frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 \, ds \\
\leq\ & C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \Bigg[R(t)^{-\frac{L}{2}} + \int_0^{t/2} R(s)^{-\frac{1}{4}} \|D_x^l E_1(t-s)\|_2 \, ds \\
&+ \int_0^{t/2} R(s)^{-\frac{1}{4}} R(t-s)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)+\frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 \, ds \Bigg].
\end{aligned}
$$

Since this inequality for $\|D_x^l w(t)\|_2$ is not closed, we have to consider in parallel the derivatives of the field. As before, we apply D_x^l to (3.2.2) and use (3.2.5), (3.2.6),(3.2.7) together with (3.4.6), (3.4.1) and (3.4.2) for $|l| < L$:

$$
\begin{aligned}
\text{(3.4.9)} \qquad \|D_x^l E_1(t)\|_2 \ \leq\ & C \int_{t/2}^t \vartheta(s) R(s)^{-\frac{L}{2}} \|F[w](t,s)\|_2 \, ds \\
&+ C \int_0^{t/2} \vartheta(s) \|D_x^l F[w](t,s)\|_2 \, ds \\
\leq\ & C \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))} \int_{t/2}^t \vartheta(s)^{-\frac{1}{2}} R(s)^{-\frac{L}{2}} \|E(t-s)\|_2 \, ds \\
&+ C \sum_{\substack{0 \leq l_j^k \leq l_j \\ j=1,2,3}} \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} \|D_x^{l-l^k} E(t-s)\|_2 \|D_x^{l^k} w(t-s)\|_2 \, ds \\
\leq\ & C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \Bigg(R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{L-1}{2}} t^{\frac{1}{2}-\omega_\theta} \\
&+ \sum_{k=1}^{L-1} R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{L-k-1}{2}} R(t)^{-\frac{k}{2}} t^{\frac{1}{2}-\omega_\theta} + \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} \|D_x^l E_1(t-s)\|_2 \, ds \\
&+ \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} R(t-s)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)+\frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 \, ds \Bigg) \\
\leq\ & C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \Bigg(R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{L-1}{2}} t^{\frac{1}{2}-\omega_\theta} \\
&+ \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} \|D_x^l E_1(t-s)\|_2 \, ds \\
&+ \int_0^{t/2} \vartheta(s)^{-\frac{1}{2}} R(t-s)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)+\frac{1}{2}} (t-s)^{-\omega_\theta} \|D_x^l w(t-s)\|_2 \, ds \Bigg).
\end{aligned}
$$

By applying Gronwall's Lemma to the coupled system (3.4.8)-(3.4.9), the estimates (3.4.1) for $m = 0$ and (3.4.2) follow.

Step 2:
Next we consider the v-derivatives of the solution $w(t)$ for some fixed multi-index $m \in \mathbb{N}_0^3$ with $|m| = M \geq 1$. As in Step 1, we assume that (3.4.1) holds for all multiindices $\tilde{m}$ with $0 \leq |\tilde{m}| \leq M-1$ and $l = 0$. By interpolation with the result of Step 1, the estimate (3.4.1) then also holds for all mixed derivatives $D_x^l D_v^{\tilde{m}} w(t)$ with $0 \leq |\tilde{m}| \leq M-2$ and $l \in \mathbb{N}_0^3$.
We apply D_v^m to (3.1.1), introduce the characteristic coordinates $\bar{x}_t := x + v\left(\frac{1-e^{\beta t}}{\beta}\right)$, $\bar{v}_t := ve^{\beta t}$ (and, analogously, $\bar{x}_s$, $\bar{v}_s$), and use (3.3.8). Here and in the sequel, $\frac{1-e^{\beta t}}{\beta}$ has to be replaced by its limit $-t$ if $\beta = 0$. This yields

$$\begin{aligned}
D_v^m w(x,v,t) \; &= \\
= \; & \int [D_v^m G(t,x,v,v_0)] *_x \; w_0(x,v_0)\, dv_0 \\
& + \int_0^t \int [D_v^m G(s,x,v,v_0)] *_x (\Theta[V]w)(x,v_0,t-s)\, dv_0\, ds \\
= \; & e^{3\beta t}\left[e^{\beta t} D_v + \frac{1-e^{\beta t}}{\beta} D_x\right]^m \Big(g(t) *_{x,v} \; w_0\Big)(\bar{x}_t,\bar{v}_t) \\
& + \int_{t/2}^t e^{3\beta s}\left(\left[e^{\beta s} D_v + \frac{1-e^{\beta s}}{\beta} D_x\right]^m g(s) *_{x,v} \; (\Theta[V]w)(t-s)\right)(\bar{x}_s,\bar{v}_s)\, ds \\
& + \int_0^{t/2} e^{3\beta s}\left(g(s) *_{x,v} \left[e^{\beta s} D_v + \frac{1-e^{\beta s}}{\beta} D_x\right]^m (\Theta[V]w)(t-s)\right)(\bar{x}_s,\bar{v}_s)\, ds.
\end{aligned}$$

Since $\|.\|_{L^2_{\bar{x},\bar{v}}} = e^{\frac{3}{2}\beta t}\|.\|_{L^2_{x,v}}$, we then obtain by estimating like in (3.3.10)-(3.3.11), using $D_v^{\tilde{m}}\Theta[V]w = \Theta[V]D_v^{\tilde{m}}w$ and (3.4.1):

(3.4.10)

$$\begin{aligned}
\|D_v^m w(t)\|_2 \; \leq \; & e^{\frac{3}{2}\beta t}\left\|\left[e^{\beta t} D_v + \frac{1-e^{\beta t}}{\beta} D_x\right]^m g(t)\right\|_{L^1_{x,v}} \|w_0\|_2 \\
& + \int_{t/2}^t e^{\frac{3}{2}\beta s}\left\|\left[e^{\beta s} D_v + \frac{1-e^{\beta s}}{\beta} D_x\right]^m g(s)\right\|_{L^1_v(L_x^{6/5})} \\
& \qquad \cdot \|(\Theta[V]w)(t-s)\|_{L^2_v(L_x^{3/2})}\, ds \\
& + C_T \sum_{\substack{|\tilde{m}|\leq 1 \\ \tilde{m},m-\tilde{m}\in\mathbb{N}_0^3}} \int_0^{t/2}\left(\frac{e^{\beta s}-1}{\beta}\right)^{|\tilde{m}|} \|g(s)\|_{L^1_v(L_x^{6/5})} \\
& \qquad \cdot \|(\Theta[V] D_x^{\tilde{m}} D_v^{m-\tilde{m}} w)(t-s)\|_{L^2_v(L_x^{3/2})}\, ds
\end{aligned}$$

$$+ C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\big) \cdot \sum_{k=2}^{M} \left(\frac{e^{\beta t}-1}{\beta}\right)^k R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{k-\frac{1}{2}}{2}} t^{-\omega_\theta+1} t^{-\frac{M-k}{2}}.$$

In the last integral we only kept the v-derivatives of the order M and $M-1$, as the estimates of mixed lower order v-derivatives of w are already known. For the second factor of the last integral we use (3.3.10) and the following interpolation (if $|\tilde{m}|=1$):

$$\left(\frac{e^{\beta s}-1}{\beta}\right)^{|\tilde{m}|} \|(\Theta[V] D_x^{\tilde{m}} D_v^{m-\tilde{m}} w)(t-s)\|_{L_v^2(L_x^{3/2})} \le \tag{3.4.11}$$
$$\le C\|V(t-s)\|_{L_x^6} \left(\left(\frac{e^{\beta s}-1}{\beta}\right)^{|M\tilde{m}|} \|D_x^{M\tilde{m}} w(t-s)\|_{L_{x,v}^2} + \sum_{j=1}^{3} \|D_v^{Me_j} w(t-s)\|_{L_{x,v}^2}\right),$$

where e_j denote the unit vectors in $\mathbb{N}^3$. Since (3.4.10), (3.4.11) would not yield a closed inequality for $\|D_v^m w(t)\|_2$, we sum (3.4.10) over all multiindices with $|m|=M$. Using the estimates (3.4.7), (3.3.6), (3.3.7), and (3.2.10) we get

$$\begin{aligned}
\sum_{|m|=M} \|D_v^m w(t)\|_2 \le{}& C_T t^{-\frac{M}{2}} \|w_0\|_2 \\
&+ C_T \int_{t/2}^{t} R(s)^{-\frac{1}{4}} s^{-\frac{M}{2}} \|V(t-s)\|_{L_x^6} \|w(t-s)\|_2 \, ds \\
&+ C_T \sum_{|m|=M} \int_0^{t/2} R(s)^{-\frac{1}{4}} \|V(t-s)\|_6 \\
&\cdot \left(\left(\frac{e^{\beta s}-1}{\beta}\right)^M \|D_x^m w(t-s)\|_2 + \|D_v^m w(t-s)\|_2\right) ds \\
&+ C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \\
&\cdot \sum_{k=2}^{M} \left(\frac{e^{\beta t}-1}{\beta}\right)^k R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{k-\frac{1}{2}}{2}} t^{-\omega_\theta+1} t^{-\frac{M-k}{2}} \\
\le{}& C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, N_\theta\right) \Bigg[t^{-\frac{M}{2}} + R(t)^{\frac{3}{2}\left(\frac{1}{6}-\frac{1}{\theta}\right)+\frac{3}{4}} t^{-\frac{M-2}{2}-\omega_\theta} \\
&+ \sum_{k=2}^{M} \left(\frac{e^{\beta t}-1}{\beta}\right)^k R(t)^{\frac{3}{2}\left(\frac{1}{2}-\frac{1}{\theta}\right)-\frac{k-\frac{1}{2}}{2}} t^{-\omega_\theta+1} t^{-\frac{M-k}{2}}
\end{aligned}$$

$$+ \sum_{|m|=M} \int_0^{t/2} R(s)^{-\frac{1}{4}} R(t-s)^{\frac{3}{2}\left(\frac{1}{6}-\frac{1}{\theta}\right)+1} (t-s)^{-\omega_\theta} \|D_v^m w(t-s)\|_2 \, ds \Bigg].$$

By applying Gronwall's Lemma and considering the range of ω_θ we finally obtain the estimate (3.4.1) for $l = 0$.

Further, (3.4.4) follows by using the Gagliardo-Nirenberg inequality (cf. [**Fr**], Chapter 1.9)

$$\|V(t)\|_{L^\infty(\mathbb{R}^3)} \;\leq\; C\|V(t)\|^{1/2}_{L^6(\mathbb{R}^3)} \|D_x^l E(t)\|^{1/2}_{L^2(\mathbb{R}^3)}, \quad \text{with } |l| = 1,$$

and the estimates (3.2.10), (3.4.2).

For (3.4.5) we use the Fourier transformed version of (3.1.10), (3.1.11), i.e.

$$\widehat{V}(t,\xi) \;=\; -i\,\frac{\xi \cdot \widehat{E}(\xi)}{|\xi|^2},$$

the estimate (3.4.2) with $L = 0$, $L = 1$, and the Hölder inequality. □

3.5 A-posteriori estimates on the particle density

In this section we present some additional decay results for the particle density that hold only under assumption (**D**). They are complementary to §5, since we recover estimates for $\|n(t)\|_{L^p(\mathbb{R}^3)}$ with $p \leq 2$. Since we have $\widehat{V}(t) \in L^1(\mathbb{R}^3)$ for $t > 0$ (cf. (3.4.5)), the following (rigorous) reformulation of the pseudo-differential operator $\Theta[V]$ holds (cf. [**ALMS**]):

$$\text{(3.5.1)} \qquad \Theta[V(t)]u(x,v) \;=\; -\frac{16}{(2\pi)^{3/2}} \operatorname{Re}\left(ie^{2iv\cdot x}\widehat{V}(t,2v)\right) *_v u(x,v).$$

Hence, by Young's inequality and (3.4.5),

$$\text{(3.5.2)} \qquad \begin{aligned} \|\Theta[V(t)](t)\|_{\mathcal{B}(L^1_v(L^\theta_x))} &\leq \frac{16}{(2\pi)^{3/2}} \|\widehat{V}(t)\|_{L^1(\mathbb{R}^3)} \\ &\leq C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, \|w_0\|_{L^1_v(L^\theta_x)}\big)\, R(t)^{1-\frac{3}{2\theta}} \end{aligned}$$

for all $t \in (0, T]$ follows.

The main result is

3.13 Theorem. *Let (***D***) hold for some $\theta \in I_\theta$. Then, the solution of the WPFP problem (1.0.6)-(1.0.7) satisfies:*

(i) $w \in C([0,\infty); L^1_v(L^\theta_x))$.

(ii) The density $n(t)$ *satisfies for all* $T > 0$ *and* $\theta \le p \le 2$*:*
(3.5.3)
$$\|n(t)\|_{L^p(\mathbb{R}^3)} \;\le\; C\big(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, \|w_0\|_{L^1_v(L^\theta_x)}\big)\, R(t)^{\frac{3}{2}\left(\frac{1}{p}-\frac{1}{\theta}\right)},$$

for all $t \in (0,T]$.

Proof.
(i) We estimate (3.1.1) by using (3.3.8) and (3.5.2):

$$\begin{aligned}
\|w(t)\|_{L^1_v(L^\theta_x)} \;\le\;& C_T \|g(t)\|_{L^1_{x,v}} \|w_0\|_{L^1_v(L^\theta_x)} \\
&+ C_T \int_0^t \|g(s)\|_{L^1_{x,v}} \|(\Theta[V]w)(t-s)\|_{L^1_v(L^\theta_x)}\, ds \\
\le\;& C_T \|w_0\|_{L^1_v(L^\theta_x)} + C\left(T, \|w\|_{\mathcal{C}([0,T];L^2(\mathbb{R}^6))}, \|w_0\|_{L^1_v(L^\theta_x)}\right) \\
&\times \int_0^t \|w(t-s)\|_{L^1_v(L^\theta_x)} R(t-s)^{1-\frac{3}{2\theta}}\, ds.
\end{aligned}$$

Then, Gronwall's Lemma yields the assertion.
(ii) The estimate (3.5.3) follows by interpolation between (3.4.3) and the estimate
$$\|n(t)\|_{L^\theta(\mathbb{R}^3)} \;\le\; \|w(t)\|_{L^1_v(L^\theta_x)}.$$

□

Part II

Numerical Approximation

Chapter 4

An operator splitting method on the periodic WPFP

Abstract

We consider the one-dimensional nonlinear Wigner-Poisson-Fokker-Planck system with periodic boundary conditions in the space variable, well-posed in a weighted L^2-space with respect to velocity. This chapter is concerned with the analysis of a semi-discretization in time of this model through an operator splitting method. First-order convergence and nonlinear stability are established in the weighted L^2-framework, by handling the nonlinearity as perturbation of the product formula for linear semigroups. Further, due to the parabolic regularization of the Fokker-Planck operator, a low-order convergence is proved by increasing the velocity moments but without smoothness assumptions for the initial data.

In this chapter we present and study an operator splitting method of first order for the time-discretization of the one-dimensional nonlinear Wigner-Poisson-Fokker-Planck system (WPFP) with periodic boundary conditions in the space-direction. Our aim is to give rigorous convergence and stability results of this semi-discretization.

The Wigner-Fokker-Planck (WFP) equation in one-dimension is given by

$$\partial_t w + v w_x + \Theta[V]w \;=\; \beta(vw)_v + \sigma w_{vv} + \alpha w_{xx}, \qquad t > 0, \tag{4.0.1}$$

on the phase space slab $x \in (0, 2\pi),\ v \in \mathbb{R}$ with periodic boundary conditions in x:

$$w(0, v, t) \;=\; w(2\pi, v, t),$$

and the initial condition

$$w(x, v, t = 0) \;=\; w^I(x, v).$$

A general description of this model is given in the Introduction (Chapter 1). We are considering here only the case with no mixed derivatives, i.e. $\gamma = 0$.

The WFP equation (4.0.1) is self-consistently coupled with the Poisson equation with periodic boundary condition for the (real-valued) potential $V = V[w](x, t)$:

$$V_{xx}(x, t) \;=\; n[w](x, t) - D(x), \qquad x \in (0, 2\pi),\; t > 0, \tag{4.0.2}$$
$$V(0, t) \;=\; V(2\pi, t), \tag{4.0.3}$$

with the given doping profile D and the particle density

$$n[w](x, t) \;=\; \int_{\mathbb{R}} w(x, v, t)\, dv. \tag{4.0.4}$$

Coupled to the WFP equation (4.0.1) via the pseudo-differential operator Θ,

$$(\Theta[V]w)(x, v, t) \;=\; \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} \delta V(x, \eta, t) \mathcal{F}_{v\to\eta} w(x, \eta, t) e^{iv\eta}\, d\eta \tag{4.0.5}$$

the self-consistent potential $V[u]$ has to be 2π-periodically extended.
$\mathcal{F}_{v\to\eta} w$ denotes here the one-dimensional Fourier transform of w with respect to v:

$$\mathcal{F}_{v\to\eta} w(x, \eta, t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} w(x, v', t) e^{-iv'\eta}\, dv'.$$

Also, for notational simplicity we have set the Planck constant, particle mass and charge equal to unity. In (4.0.1), $\beta \geq 0$ is the friction parameter and the parameters $\alpha \geq 0$, $\sigma > 0$ constitute the phase-space diffusion matrix of the system.

For the quantum-mechanical correctness, i.e. Lindblad condition (cf. Chapter 1 and [**Li**]), we have to assume

$$\begin{pmatrix} \alpha & \frac{i}{4}\beta \\ -\frac{i}{4}\beta & \sigma \end{pmatrix} \;\geq\; 0,$$

In the subsequent analysis, as well as in the next Chapter 5 about the numerical simulations, we require additionally $\alpha, \beta \;>\; 0$.

In [**ACD**] a rigorous well-posedness analysis of the x-periodic 1D WPFP-system is carried out in a weighted L^2-space with one momentum in the velocity direction, $L^2((0, 2\pi)\times\mathbb{R};\ (1+v^2)\, dx\, dv)$, which is a natural framework to properly define the particle density by (4.0.4). Note that the choice of a finite intervall in x is important for the well-posedness of this one-dimensional model. A whole-space analysis, similar to what is done for the 3D WPFP in the first two chapters, is here not possible, since the Poisson equation has no non-trivial whole-space fundamental solution.

In the context of simulating quantum effects in semiconductor devices, various numerical methods have been used for approximating the solution of the Wigner(-Poisson) equation: finite-difference-schemes ([**Fre, RKPKF**]), spectral collocation methods ([**Ri90, Ri91, Ri92**]) and a deterministic particle method ([**ArNi**]). An operator splitting method for the Vlasov equation, and later for the Wigner–Poisson system ([**AR95, AR96**]) has first been used by plasma physicists to study oscillations in one–dimensional plasmas ([**CK, SFB**]).

In our approach, the product formula for semigroups is the motivation for considering an operator splitting method to approximate the Wigner function. The "orthogonal" action of the operators in (4.0.1) in the (x, v)-phase space (cf. Section 4.1 for details), makes possible a suitable split of the equation (4.0.1) in time. We focus on a splitting scheme of first order in time. This approach was first introduced in [**AR96**] for the coupled nonlinear Wigner-Poisson equation with periodic boundary conditions in x. Here, we shall extend these techniques to include the Fokker-Planck terms by using their parabolic regularization. In order to obtain the order of convergence, we shall deal with the nonlinear coupling with the Poisson equation, via the pseudo-differential operator Θ, as a perturbation of the product formula for linear semigroups. In the sequel analysis, an explicit stability and first-order convergence proof is carried out. Also, due to the parabolic regularization of the Fokker-Planck operator, we shall present a low-order convergence result, i.e. a convergence result of order $1 - \epsilon$, $\forall\, \epsilon > 0$, which dispenses with smoothness assumptions on initial data (cf. Section 4.4). The numerical realization of this approach and numerical simulations are postponed to the next chapter.

4.1 Preliminaries

In this section we present the numerical approximation scheme for the WPFP-system with fixing necessary existence analysis results, and we sketch the strategy of the convergence analysis.

First we define the weighted L^2-spaces

$$L^2_{\mu_k} \;=\; L^2_{\mu_k}((0, 2\pi) \times \mathbb{R}) \;:=\; L^2((0, 2\pi) \times \mathbb{R};\, \mu_k\, dx\, dv),$$

with the weight $\mu_k = (1 + v^{2k})$ for $k \in \mathbb{N}_0$. Similarly, $H^s_{\mu_k}$, $s \in \mathbb{N}_0$ denotes the weighted Sobolev space $H^s((0,2\pi)\times\mathbb{R};\, \mu_k\, dx\, dv)$.

In [**ACD**] the well-posedness analysis of the x-periodic 1D WPFP-system was carried out in L^2_μ with $\mu := \mu_1$. Since $L^2_\mu \hookrightarrow L^2((0,2\pi), L^1(\mathbb{R}_v))$, this framework allows to properly define the particle density by (4.0.4). In the sequel we shall use the same approach.

We define the differential operators:

$$\begin{aligned} Au &:= -v\partial_x u + \alpha\partial_x^2 u; \\ Bu &:= \sigma\partial_v^2 u + \beta u + \beta v\partial_v u, \end{aligned}$$

which act on their respective domains

$$\begin{aligned} D_p(A) &:= \{u \in L^2_\mu \mid vu_x, u_{xx} \in L^2_\mu;\ u(0,v) = u(2\pi,v), \\ &\qquad\qquad u_x(0,v) = u_x(2\pi,v), \forall\, v \in \mathbb{R}\}; \\ D(B) &:= \{u \in L^2_\mu \mid u_{vv}, vu_v \in L^2_\mu\}; \\ D_p(A+B) &:= \{u \in L^2_\mu \mid vu_x, u_{vv}, vu_v, u_{xx} \in L^2_\mu;\ u(0,v) = u(2\pi,v), \\ &\qquad\qquad u_x(0,v) = u_x(2\pi,v), \forall\, v \in \mathbb{R}\}. \end{aligned}$$

Also, we define the corresponding norms

$$\begin{aligned} \|u\|_{D(A)} &:= \|u\|_{L^2_\mu} + \|vu_x\|_{L^2_\mu} + \|u_{xx}\|_{L^2_\mu}; \\ \|u\|_{D(B)} &:= \|u\|_{L^2_\mu} + \|vu_v\|_{L^2_\mu} + \|u_{vv}\|_{L^2_\mu}; \\ \|u\|_{D(A+B)} &:= \|u\|_{L^2_\mu} + \|vu_x\|_{L^2_\mu} + \|u_{xx}\|_{L^2_\mu} + \|vu_v\|_{L^2_\mu} + \|u_{vv}\|_{L^2_\mu}. \end{aligned}$$

In the sequel the subscript "p" will denote spaces of periodic functions on the domain $(0,2\pi)$. E.g. $W^{s,q}_p(0,2\pi)$, $s \in \mathbb{N}_0$, $1 \le q \le \infty$ denotes "periodic" Sobolev spaces, which are defined as the closure of $C^\infty_p([0,2\pi])$-functions with respect to the $W^{s,q}(0,2\pi)$-norm, as in [**Ad**].

The existence and uniqueness of a global-in-time solution of the nonlinear WPFP-model (4.0.1)-(4.0.4) has been established in Theorem 2.6 of [**ACD**]. Let us recall here the main result.

4.1 Theorem. *Let $D \in L^1(0,2\pi)$.*

a) For every $w^I \in L^2_\mu$, the WPFP-problem (4.0.1)-(4.0.4) has a unique mild solution $w \in C([0,\infty), L^2_\mu)$.

b) If $w^I \in D_p(\overline{A+B})$, w is a classical solution, i.e. $w \in C^1([0,\infty), L^2_\mu)$, and $w(t) \in D_p(\overline{A+B})$ for $t \ge 0$.

The operator splitting method we are going to establish and analyze consists in solving successively for each time step of length Δt the evolution equations:

Step I:

$$\text{(4.1.1)}\qquad \begin{cases} \partial_t u &= Bu - \Theta[V[u]]u, \qquad t_n < t \le t_{n+1} \\ u(t_n) &= w_n, \\ w_{n+\frac{1}{2}} &:= u(t_{n+1}), \end{cases}$$

self-consistently coupled with the Poisson equation

$$\text{(4.1.2)}\qquad \begin{aligned} V_{xx}(x,t) &= \int_{\mathbb{R}^3} u(x,v,t)\,dv - D(x), \qquad t_n \le t \le t_{n+1} \\ V(0,t) &= V(2\pi,t). \end{aligned}$$

A simple calculation shows that $\int u(t)\,dv$ is constant in time over Step I. Hence, $V[u(t)] = V[w_n]$ is just evaluated at the beginning of this time step.

Step II:

$$\text{(4.1.3)}\qquad \begin{cases} \partial_t u &= Au, \qquad t_n < t \le t_{n+1} \\ u(0,v,t) &= u(2\pi,v,t), \\ u(t_n) &= w_{n+\frac{1}{2}}, \\ w_{n+1} &:= u(t_{n+1}), \end{cases}$$

where $(x,v) \in (0,2\pi)\times\mathbb{R}$, and w_n and w_{n+1} denote the approximations of w at t_n and $t_{n+1} = t_n + \Delta t$ ($w_0 := w^I$ at $t_0 = 0$), respectively.

This splitting method is particularly suitable for the WFP equation since the operators A and $B+\Theta[V[.]]$ act in "orthogonal" directions of the (x,v)-phase space. Moreover, the two split evolution equations can be solved explicitly. From the existence analysis of the WPFP-model (cf. [**ACD**]) we know that the operators $\overline{A}$, $\overline{B}$, and $\overline{A+B}$ generate (quasi-) contractive C_0-semigroups on L^2 and L^2_μ which can be given explicitly.

For notational simplicity we denote these semigroups in the sequel by e^{tA}, e^{tB} and $e^{t(A+B)}$, respectively.

4.2 Proposition. *For the C_0-semigroups generated by $\overline{A}$, $\overline{B}$, and $\overline{A+B}$ the following statements hold for all $t \ge 0$:*

$$\begin{aligned} \|e^{tA}\|_{\mathcal{B}(L^2)} &\le 1; & \|e^{tA}\|_{\mathcal{B}(L^2_\mu)} &\le 1; \\ \|e^{tB}\|_{\mathcal{B}(L^2)} &\le e^{\frac{\beta}{2}t}; & \|e^{tB}\|_{\mathcal{B}(L^2_\mu)} &\le e^{(\sigma+\frac{\beta}{2})t}; \\ \|e^{t(A+B)}\|_{\mathcal{B}(L^2)} &\le e^{\frac{\beta}{2}t}; & \|e^{t(A+B)}\|_{\mathcal{B}(L^2_\mu)} &\le e^{(\sigma+\frac{\beta}{2})t}. \end{aligned}$$

Let $u \in L^2_\mu$ and $D \in L^1(0,2\pi)$. In order for the Poisson equation (4.0.2)-(4.0.3) to yield a smooth periodic solution

$$\text{(4.1.4)}\qquad \int_0^{2\pi}\int_{\mathbb{R}} u(x,v)\,dv\,dx = \int_0^{2\pi} D(x)\,dx$$

has to hold. In this case, the potential will satisfy additionally

$$V_x(0) \;=\; V_x(2\pi).$$

Also, (4.0.2)-(4.0.3) determines the potential $V[u]$ only up to an additive constant, which, however, drops out of $\Theta[V]$. Nevertheless, we shall make V unique by requiring

(4.1.5)
$$\begin{aligned} V_{xx} &= n[u] - D, \quad x \in (0, 2\pi), \\ V(0) &= V(2\pi) \;=\; 0. \end{aligned}$$

The solution of this boundary value problem (4.1.5) is explicitly given by

(4.1.6)
$$V[u](x) \;=\; \int_0^{2\pi} \mathcal{G}(x,\xi)\Big(n[u](\xi) - D(\xi)\Big)d\xi, \quad x \in (0,\pi),$$

where $\mathcal{G}$ denotes the Green's function on $(0, 2\pi)$

(4.1.7)
$$\mathcal{G}(x,\xi) \;=\; \begin{cases} x(\frac{\xi}{2\pi} - 1), & x < \xi \\ \xi(\frac{x}{2\pi} - 1), & x > \xi. \end{cases}$$

Using $u \in L^2_\mu$ yields the following estimates for the density and potential (cf. [**AR95, AR96, ACD**]):

$$\|n[u]\|_{L^2(0,2\pi)} \;\le\; C\|u\|_{L^2_\mu}$$

and

(4.1.8)
$$\|V[u]\|_{W^{1,\infty}(0,2\pi)} \;\le\; C\|n[u] - D\|_{L^1(0,2\pi)} \;\le\; C\left(\|u\|_{L^2_\mu} + \|D\|_{L^1(0,2\pi)}\right).$$

Here and in the sequel C denotes generic, but not necessarily equal constants. An integration by parts (cf. Proposition 2.3 in [**ACD**]) gives

(4.1.9)
$$\Theta[V](vu) \;=\; v\Theta[V]u + \Omega[V_x]u$$

with

(4.1.10)
$$\begin{aligned} \Omega[V]u(x,v) &:= \frac{1}{2\sqrt{2\pi}} \int_{\mathbb{R}} \delta_+ V(x,\eta)\mathcal{F}_v u(x,\eta)e^{iv\eta}d\eta, \\ \delta_+ V(x,\eta) &:= V\left(x + \frac{\eta}{2}\right) + V\left(x - \frac{\eta}{2}\right). \end{aligned}$$

Therefore, the pseudo-differential operator $\Theta[V]$ satisfies the estimates

$$\begin{aligned} \|\Theta[V[u]]\|_{\mathcal{B}(L^2)} &\le 2\|V[u]\|_{L^\infty(0,2\pi)}, \\ \|\Theta[V[u]]\|_{\mathcal{B}(L^2_\mu)} &\le C\|V[u]\|_{W^{1,\infty}(0,2\pi)}. \end{aligned}$$

Moreover,

$$\begin{aligned} \|\Omega[V[u]]\|_{\mathcal{B}(L^2)} &\leq \|V[u]\|_{L^\infty(0,2\pi)}, \\ \|\Omega[V[u]]\|_{\mathcal{B}(L^2_\mu)} &\leq C\|V[u]\|_{W^{1,\infty}(0,2\pi)}. \end{aligned}$$

Further, the operator $\Theta[V]$ commutes with ∂_v^n, $n \in \mathbb{N}$.
We denote with $\Gamma(t)$ the non-linear semigroup on L^2_μ, generated by $\overline{B} - \Theta[V[.]]$. It furnishes the solution of Step I of the splitting scheme.
Let $u_0 \in L^2_\mu$ be the initial data for (4.1.1) at $t = 0$. In the v-Fourier space, (4.1.1) reads

$$\begin{aligned} \hat{u}_t &= -\sigma\eta^2\hat{u} - \beta\eta\partial_\eta\hat{u} - i\delta V[u_0](x,\eta)\hat{u}, \quad t > 0, \\ \hat{u}(t=0) &= \hat{u}_0 \ := \ \mathcal{F}_{v\to\eta}u_0, \end{aligned} \tag{4.1.11}$$

and it can be solved explicitly by using characteristics (details can be found in [**Dh**]). Hence, the solution to (4.1.1) is given by

$$\Gamma(t)u_0(x,v) := \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} \mathcal{E}[u_0](x,\eta,t)\mathcal{F}_{v\to\eta}u_0(x,\eta e^{-\beta t})e^{iv\eta}d\eta, \quad \forall\, t \geq 0, \tag{4.1.12}$$

with

$$\mathcal{E}[u_0](x,\eta,t) := \exp\left\{\frac{\sigma}{2\beta}\eta^2(e^{-2\beta t}-1) - i\int\limits_0^t \delta V[u_0](x,\eta e^{\beta(\tau-t)})\,d\tau\right\}. \tag{4.1.13}$$

The step-forward operator describing our splitting scheme is now given by

$$F_{\Delta t} := e^{\Delta t A}\Gamma(\Delta t), \quad \forall\, \Delta t \geq 0. \tag{4.1.14}$$

Thus, the splitting scheme (4.1.1)-(4.1.3) reads

$$w_{n+1} = F_{\Delta t}w_n = e^{\Delta t A}\Gamma(\Delta t)w_n, \tag{4.1.15}$$

with $w_0 = w^I$ and the potential $V = V(t_n)$ being evaluated at the beginning of each time step $[t_n, t_n + \Delta t]$.

Let us remark that in the linear Wigner-Fokker-Planck equation (WFP), which consists of the equation (4.0.1) with a given constant-in-time potential $V = V(x)$, the strong L^2-convergence of the splitting scheme (4.1.1)-(4.1.3) is an immediate consequence of Trotter's product formula (cf. [**CHMM**]). In the nonlinear case an analogous convergence result has to be derived explicitly.

The splitting method (4.1.15) is a nonlinear product formula with first-order convergence. The main difficulty in proving this lies in handling the nonlinear coupling of (4.0.1) with the Poisson equation for the self-consistent potential. In order to carry out the convergence proof with "natural" assumptions on the initial data we shall need to deal with the nonlinearity in (4.1.14) as a perturbation of Trotter's product formula for linear semigroups, i.e. $e^{tA}e^{tB}$. For the consistency estimates on the error $e^{t(A+B)} - e^{tA}e^{tB}$ we will use a formal Taylor expansion of the product formula $e^{tA}e^{tB}$, where the derivative operator is replaced by the commutators of the linear generators of the semigroups, given in (4.3.2). Moreover, this Taylor expansion clearly underlies the maximal order of convergence to be expected in the approximation, namely first-order convergence. An adequate decomposing of the nonlinear splitting (4.1.15) based on Duhamel's formula shall then provide the first-order convergence by using the mentioned Taylor expansion.

We remark that a "Strang splitting" version of (4.1.15) is of second order, but we shall not pursue this here.

In the next two Sections 4.2 and 4.3 we prove stability and first-order convergence of the splitting scheme for $w^I \in D_p(A+B)$. Section 4.4 is devoted to a low-order (precisely, $1-\epsilon$) convergence result which holds for more general initial data, with just an additional momentum in v, i.e. $w^I \in L^2_{\mu_2}$. This will be a $(1-\epsilon)$-order convergence with no additional smoothness assumptions for w^I. A summary of convergence results for the linear WFP, with a given constant-in-time potential $V = V(x)$, is presented in Section 4.5. The detailed description of the numerical algorithm of the full discretization and some numerical simulations for the linear and nonlinear model to complete this work are presented in next Chapter 5.

4.2 Stability of the splitting scheme

To establish the convergence of the splitting method we will need nonlinear stability properties of the introduced splitting scheme (4.1.15) which are stated in the Propositions 4.5 and 4.6. The following proposition summarizes important a-priori estimates on the nonlinear semigroup $\Gamma(t)$.

4.3 Proposition. *Let $u_0 \in L^2_\mu$ and $D \in L^1(0,2\pi)$. Then,*

(i)

$$\|\Gamma(t)u_0\|_{L^2_\mu} \;\leq\; e^{tK_1\left(t,\|u_0\|_{L^2}\right)}\|u_0\|_{L^2_\mu}, \quad \forall\, t \geq 0, \tag{4.2.1}$$

with

$$K_1\left(t, \|u_0\|_{L^2}\right) \;:=\; \sigma + \frac{\beta}{2} + C\left(\|D\|_{L^1(0,2\pi)} + e^{\frac{\beta}{2}t}\|u_0\|_{L^2}\right).$$

(ii)

$$\|\Gamma(t)u_0\|_{L^2_\mu} \;\leq\; e^{tK_2\left(\|u_0\|_{L^2_\mu}\right)}\|u_0\|_{L^2_\mu}, \quad \forall\, t \geq 0, \tag{4.2.2}$$

with

$$K_2\left(\|u_0\|_{L^2_\mu}\right) \;:=\; \sigma + \frac{\beta}{2} + C\left(\|D\|_{L^1(0,2\pi)} + \|u_0\|_{L^2_\mu}\right).$$

PROOF. By considering the classical solution $u(t)$ of equation (4.1.1) in Step I, the idea of the proof is based on using the dissipativity of the operator $B - \left(\sigma + \frac{\beta}{2}\right) I$ in L^2_μ, which has been implicitly proved in Lemma 2.1 in [**ACD**]. However, since we are dealing only with the *mild* solution of this equation, an approximation of u by classical solutions is needed. More precisely, let $\{u_0^n\}_{n\in\mathbb{N}} \subset \mathcal{D}(B)$ be a sequence with $u_0^n \to u_0$ in L^2_μ. Then, for the classical solutions (cf. [**Pa**], Thm. 6.1.5) u^n we get

$$\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2_{L^2_\mu} \;\leq\; \left(\sigma + \frac{\beta}{2}\right)\|u^n(t)\|^2_{L^2_\mu} - <\Theta[V[u^n(t)]]u^n(t), u^n(t)>_{L^2_\mu}.$$

The skew-symmetry of the operator $\Theta[V]$ (cf. [**MR**]) implies

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2_{L^2_\mu} - \left(\sigma + \frac{\beta}{2}\right)\|u^n(t)\|^2_{L^2_\mu} \;&\leq \\
&\leq \int_0^{2\pi}\!\!\int_{\mathbb{R}} v u^n(t)\Omega[V_x[u^n(t)]]u^n(t)\,dv\,dx \\
&\leq \|V_x[u^n(t)]\|_{L^\infty(0,2\pi)}\|vu^n(t)\|_{L^2}\|u^n(t)\|_{L^2} &(4.2.3)\\
&\leq C\left(\|u^n(t)\|_{L^2_\mu} + \|D\|_{L^1(0,2\pi)}\right)\|vu^n(t)\|_{L^2}\|u^n(t)\|_{L^2}. &(4.2.4)
\end{aligned}$$

On the other hand, by applying the dissipativity of $B - \frac{\beta}{2}I$ in L^2 (cf. [**ACD**]) and again the skew-symmetry of $\Theta[V]$, we have

$$\frac{1}{2}\frac{d}{dt}\|u^n(t)\|^2_{L^2} \;\leq\; \frac{\beta}{2}\|u^n(t)\|^2_{L^2}. \tag{4.2.5}$$

Letting $n \to \infty$, we obtain from (4.2.4)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_\mu} \;\leq\; &\left(\sigma + \frac{\beta}{2}\right)\|u(t)\|^2_{L^2_\mu} \\
&+ C\left(\|u(t)\|_{L^2_\mu} + \|D\|_{L^1(0,2\pi)}\right)\|u(t)\|_{L^2_\mu}\|u(t)\|_{L^2},
\end{aligned} \tag{4.2.6}$$

and from (4.2.5)

$$\|u(t)\|_{L^2} \;\leq\; e^{\frac{\beta}{2}t}\|u_0\|_{L^2}, \quad \forall\, t \geq 0. \tag{4.2.7}$$

The estimate (4.2.1) follows directly by applying Gronwall's Lemma.
For the second estimate (4.2.2) we use $V[u^n(t)] = V[u_0^n]$, and from (4.2.3) we get as $n \to \infty$:

$$\begin{aligned}(4.2.8) \qquad \frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_\mu} \;&\leq\; \left(\sigma + \frac{\beta}{2}\right)\|u(t)\|^2_{L^2_\mu} \\ &\quad + C\left(\|u_0\|_{L^2_\mu} + \|D\|_{L^1(0,2\pi)}\right)\|u(t)\|_{L^2_\mu}\|u(t)\|_{L^2}.\end{aligned}$$

Hence, applying Gronwall's Lemma yields (4.2.2). □

4.4 Corollary. *Let $u_0 \in L^2$ and $D \in L^1(0,2\pi)$. Then,*

$$(4.2.9) \qquad \|\Gamma(t)u_0\|_{L^2} \;\leq\; e^{\frac{\beta}{2}t}\|u_0\|_{L^2}, \quad \forall\, t \geq 0.$$

An a-priori estimate on the step-forward operator F_t is given in the following proposition.

4.5 Proposition. *Let $u \in L^2_\mu$ and $D \in L^1(0,2\pi)$. Then, the step forward operator F_t, defined in (4.1.14), satisfies*

$$(4.2.10) \qquad \|F_t^N u\|_{L^2_\mu} \;\leq\; \|u\|_{L^2_\mu} e^{tNK_1\left(tN, \|u\|_{L^2}\right)}, \quad \forall\, t \geq 0, \quad N \in \mathbb{N}.$$

PROOF. By using Proposition 4.2 and (4.2.1) we get

$$(4.2.11) \qquad \|F_t u\|_{L^2_\mu} \;\leq\; \|u\|_{L^2_\mu} e^{tK_1\left(t, \|u\|_{L^2}\right)}, \quad \forall\, t \geq 0.$$

Using the definition of K_1 in Proposition 4.3 and (4.2.9) we obtain by induction

$$\begin{aligned}\|F_t^N u\|_{L^2_\mu} \;&\leq\; \|F_t^{N-1}u\|_{L^2_\mu}\, e^{tK_1\left(t, \|F_t^{N-1}u\|_{L^2}\right)} \\ &\leq\; \|F_t^{N-1}u\|_{L^2_\mu}\, e^{tK_1\left(t, e^{\frac{\beta}{2}t(N-1)}\|u\|_{L^2}\right)} \\ &=\; \|F_t^{N-1}u\|_{L^2_\mu}\, e^{tK_1\left(tN, \|u\|_{L^2}\right)} \\ &\leq\; \|u\|_{L^2_\mu}\, e^{tNK_1\left(tN, \|u\|_{L^2}\right)}.\end{aligned}$$

□

The next proposition gives the nonlinear stability property of the splitting scheme (4.1.15).

4.6 Proposition. *Let $u_1, u_2 \in L^2_\mu$ and $D \in L^1(0,2\pi)$. Then, the step forward operator F_t satisfies*

$$\begin{aligned}(4.2.12) \qquad \|F_t^N u_1 - F_t^N u_2\|_{L^2_\mu} \;&\leq\; \|u_1 - u_2\|_{L^2_\mu} \\ &\quad \cdot C\left(Nt, \|D\|_{L^1(0,2\pi)}, \|u_1\|_{L^2_\mu}, \|u_2\|_{L^2_\mu}\right),\end{aligned}$$

for all $t \geq 0$, $N \in \mathbb{N}$.

PROOF. First we will derive the estimate

$$\begin{aligned}(4.2.13)\quad \|\Gamma(t)u_1-\Gamma(t)u_2\|_{L^2_\mu} &\le \|u_1-u_2\|_{L^2_\mu}\\ &\cdot \exp\left\{t\left(\sigma+\frac{\beta}{2}+C\left(\|D\|_{L^1(0,2\pi)}+e^{\beta t}[\|u_1\|_{L^2_\mu}+\|u_2\|_{L^2_\mu}]\right)\right)\right\},\end{aligned}$$

by using the mean value theorem

$$(4.2.14)\qquad \|\Gamma(t)u_1-\Gamma(t)u_2\|_{L^2_\mu} \le \sup_{\epsilon\in[0,1]}\|D_{u_\epsilon}\Gamma(t)(u_1-u_2)\|_{L^2_\mu},$$

with $u_\epsilon=\epsilon u_1+(1-\epsilon)u_2$, and $D_{u_\epsilon}\Gamma(t)$ denotes the *Frchet* derivative of $\Gamma(t)$ in u_ϵ.

Formally, for the *Frchet* derivative $D_u\Gamma(t)(\bar u)$ for $u,\bar u\in L^2_\mu$, we get by using the definition (4.1.12)

$$\begin{aligned}(4.2.15)\quad \mathcal{F}_{v\to\eta}\left(D_u\Gamma(t)(\bar u)\right)(x,\eta) &= D_u\left(\mathcal{F}_{v\to\eta}u(x,\eta e^{-\beta t})\mathcal{E}[u](x,\eta,t)\right)(\bar u)\\ &= \mathcal{F}_{v\to\eta}\bar u(x,\eta e^{-\beta t})\mathcal{E}[u](x,\eta,t)+\mathcal{F}_{v\to\eta}u(x,\eta e^{-\beta t})D_u(\mathcal{E}[.])(\bar u)(x,\eta,t)\\ &= \mathcal{E}[u](x,\eta,t)\Big[\mathcal{F}_{v\to\eta}\bar u(x,\eta e^{-\beta t})\\ &\quad +\mathcal{F}_{v\to\eta}u(x,\eta e^{-\beta t})\left(-i\int_0^t\delta V^0[\bar u](x,\eta e^{\beta(\tau-t)})\,d\tau\right)\Big]\\ &=: \mathcal{F}_{v\to\eta}\Big(\Gamma[u](t)\bar u+\Gamma[u](t)f(t)\Big)(x,\eta),\end{aligned}$$

with

$$f(x,v,t) := -\frac{1}{\sqrt{2\pi}}\,u *_v\left[i\int_0^t\mathcal{F}^{-1}_{\eta\to v}\Big(\delta V^0[\bar u]\Big)(x,ve^{-\beta\tau})e^{-\beta\tau}\,d\tau\right]$$

Here, we used

$$(4.2.16)\quad D_uV[.](\bar u)(x) = V^0[\bar u](x) := \int_0^{2\pi}\mathcal{G}(x,\xi)n[\bar u](\xi)\,d\xi,\quad x\in(0,\pi),$$

which satisfies

$$\|V^0[\bar u]\|_{W^{1,\infty}_p(0,2\pi)} \le C\|\bar u\|_{L^2_\mu}.$$

The two terms in (4.2.15) are the solutions of a linear version of equation (4.1.1) with the given potential $V[u]$. Using an estimate of type (4.2.2) therefore yields

$$(4.2.17)\qquad \|D_u\Gamma(t)(\bar u)\|_{L^2_\mu} \le e^{tK_2(\|u\|_{L^2_\mu})}\|\bar u\|_{L^2_\mu}+e^{tK_2(\|u\|_{L^2_\mu})}\|f(t)\|_{L^2_\mu}.$$

Further,

$$
\begin{aligned}
(4.2.18) \qquad \|f(t)\|_{L^2} &= \|\mathcal{F}_{v\to\eta} f(t)\|_{L^2} \\
&\leq C\|u\|_{L^2} \Big\| \int_0^t \delta V^0[\bar{u}](x, \eta e^{\beta\tau})\, d\tau \Big\|_{L^\infty((0,2\pi)\times\mathbb{R})} \\
&\leq Ct\|u\|_{L^2}\|\bar{u}\|_{L^2_\mu}
\end{aligned}
$$

and

$$
\begin{aligned}
(4.2.19) \quad \|vf(t)\|_{L^2} &\leq Ct\|vu\|_{L^2}\|\bar{u}\|_{L^2_\mu} \\
&\quad + C\|u\|_{L^2} \Big\| \int_0^t e^{\beta\tau}\delta_+ V_x^0[\bar{u}](x, \eta e^{\beta\tau})\, d\tau \Big\|_{L^\infty((0,2\pi)\times\mathbb{R})} \\
&\leq C\|\bar{u}\|_{L^2_\mu}\Big(t\|u\|_{L^2_\mu} + te^{\beta t}\|u\|_{L^2}\Big)
\end{aligned}
$$

hold.

By applying the estimates (4.2.17)-(4.2.19) to (4.2.14) we get

$$
\begin{aligned}
\|\Gamma(t)u_1 - \Gamma(t)u_2\|_{L^2_\mu} \leq \|u_1 - u_2\|_{L^2_\mu} \\
\cdot \sup_{\epsilon\in[0,1]} \Big[e^{tK_2(\|u_\epsilon\|_{L^2_\mu})} \Big(1 + Ct\|u_\epsilon\|_{L^2_\mu} + Cte^{\beta t}\|u_\epsilon\|_{L^2}\Big)\Big],
\end{aligned}
$$

and the assertion (4.2.13) follows.

Finally, by using Proposition 4.2 and (4.2.13) and by induction,

$$
\begin{aligned}
\|F_t^N u_1 - F_t^N u_2\|_{L^2_\mu} &\leq \|u_1 - u_2\|_{L^2_\mu} \exp\Big\{ Nt\Big(\sigma + \frac{\beta}{2} + C\Big(\|D\|_{L^1(0,2\pi)} \\
&\quad + e^{\beta t}\big[\|\Gamma^{N-1}(t)u_1\|_{L^2_\mu} + \|\Gamma^{N-1}(t)u_2\|_{L^2_\mu}\big]\Big)\Big)\Big\}
\end{aligned}
$$

follows. □

4.3 First order convergence

The aim of the following analysis is a consistency estimate for the splitting scheme (4.1.15). Then, the convergence result will be derived in Theorem 4.16 by using the nonlinear stability results of the previous section.

Let us mention that our splitting scheme is a nonlinear product formula. In order to treat the nonlinearity by a perturbation argument, a consistency estimate for the *linear* product formula $e^{tA}e^{tB}$ will be needed. In the sequel we shall give a formal Taylor expansion of the product formula $e^{tA}e^{tB}$ and estimate the consistency error to $e^{t(A+B)}$ by using the commutators of A and B. For this purpose we use the following identity for generators of linear semigroups which was proven in Lemma 2.2 of [**DS**].

4.7 Lemma. *For $n \geq 1$,*

(4.3.1)
$$\partial_{e^{tA}} B \;=\; -\sum_{j=1}^{n} \frac{t^j e^{tA}}{j!}(\partial_A^j B) - R_{n+1}(t),$$

where

$$R_{n+1}(t) \;=\; \int_0^t e^{(t-s)A}(\partial_A^{n+1} B) e^{sA} \frac{(t-s)^n}{n!} ds,$$

and $\partial_A B$ denotes the commutator of A and B

$$\partial_A B \;:=\; [A, B] \;=\; AB - BA.$$

Denoting $W(t) := e^{tA} e^{tB}$ we obtain after differentiating formally with respect to time:

$$\dot{W}(t) - (A + B)W(t) \;=\; \partial_{e^{tA}} B e^{tB}.$$

An immediate consequence of Duhamel's formula and (4.3.1) is the general error formula

(4.3.2)
$$W(t) - e^{t(A+B)} \;=\; \int_0^t e^{(t-s)(A+B)} \left[e^{sA} \sum_{j=1}^{n} \frac{s^j}{j!}(\partial_A^j B) + R_{n+1}(s) \right] e^{sB} ds.$$

Calculating the commutators we have

$$\partial_A B \;=\; 2\sigma\partial_x\partial_v + \beta v \partial_x, \qquad \partial_A^2 B \;=\; -2\sigma\partial_x^2.$$

Hence, $\partial_A^j B = 0$ for $j > 2$, and $R_3 \equiv 0$ follows. Thus, we can simplify (4.3.2) to

(4.3.3)
$$e^{tA} e^{tB} - e^{t(A+B)} \;=\; \int_0^t e^{(t-s)(A+B)} \left[e^{sA}(2s\sigma\partial_x\partial_v + s\beta v\partial_x - s^2\sigma\partial_x^2) e^{sB} \right] ds.$$

This Taylor expansion clearly emphasizes the order of convergence to be achieved by this product formula. It is obvious that at best first-order convergence can be obtained. For higher convergence other product formulae have to be considered, e.g. a Strang splitting formula for second-order convergence (cf. [**AR96**]).

In the next lemma we state some properties of the semigroup e^{tB}, which will be used in the next proofs.

4.8 Lemma. *Let $T > 0$ and $k \in \mathbb{N}_0$. Then, the C_0-semigroup e^{tB} satisfies:*

$$
\begin{array}{lrcl}
\text{(i)} & \|\partial_v e^{tB}\|_{\mathcal{B}(L^2_{\mu_k})} & \le & C_T t^{-1/2}, \quad \forall\, t \in (0, T]. \\
\text{(ii)} & \partial_v^n \big(e^{tB} u\big) & = & e^{n\beta t} e^{tB} \big(\partial_v^n u\big) \ \textit{for} \ u \in H^n,\ n \in \mathbb{N}; \\
\text{(iii)} & v e^{tB} u & = & e^{-\beta t} e^{tB}(vu) \ \textit{for} \ u, vu \in L^2,\ t \ge 0.
\end{array}
$$

Proof. The C_0-semigroup $e^{tB} \in \mathcal{B}(L^2_{\mu_k})$, $t \ge 0$ is given by

$$
\begin{aligned}
e^{tB} u(x,v) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp\left\{ \frac{\sigma}{2\beta} \eta^2 (e^{-2\beta t} - 1) \right\} \mathcal{F}_{v \to \eta} u(x, \eta e^{-\beta t}) e^{iv\eta}\, d\eta \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} u(x, v') \int_{\mathbb{R}} \exp\left\{ \frac{\sigma}{2\beta} \eta^2 (e^{-2\beta t} - 1) \right\} e^{i(v - v' e^{-\beta t})\eta}\, d\eta\, dv' \\
&= \frac{e^{\beta t}}{\sqrt{2\pi \frac{\sigma}{\beta}(1 - e^{-2\beta t})}} \int_{\mathbb{R}} u(x, \xi e^{\beta t}) \exp\left\{ \frac{\beta}{2\sigma(e^{-2\beta t} - 1)} (v - \xi)^2 \right\} d\xi,
\end{aligned}
$$

where

$$
\begin{aligned}
\int_{\mathbb{R}} \exp\left\{ \frac{\sigma}{2\beta} \eta^2 (e^{-2\beta t} - 1) \right\} e^{i(v - v' e^{-\beta t})\eta}\, d\eta &= \\
&= \sqrt{2\pi} \mathcal{F}^{-1}_{\eta \to v} \left(\exp\left\{ \frac{\sigma}{2\beta} \eta^2 (e^{-2\beta t} - 1) \right\} \right) (v - v' e^{-\beta t}).
\end{aligned}
$$

Hence,

$$
\partial_v e^{tB} u(x,v) = \frac{e^{\beta t}}{\frac{\sigma}{\beta}(e^{-2\beta t} - 1) \sqrt{2\pi \frac{\sigma}{\beta}(1 - e^{-2\beta t})}} \int_{\mathbb{R}} u(x, \xi e^{\beta t}) g(v - \xi, t)\, d\xi,
$$

with

$$
g(v,t) := v \exp\left\{ \frac{1}{2 \frac{\sigma}{\beta}(e^{-2\beta t} - 1)} v^2 \right\}.
$$

Estimating in $L^2_{\mu_k}$ we finally obtain (i).
The last two statements are straightforward (for details cf. [**Dh**]). □

With the error formula (4.3.3) we can now prove first-order consistency of the product formula $e^{tA} e^{tB}$ under natural assumptions on the initial data.

4.9 Proposition. *Let $T > 0$ and*

$$
u \in Y := \left\{ f \in L^2_\mu \mid f_{xv},\ v f_x \in L^2_\mu \right\}.
$$

Then, the consistency estimate

$$\|e^{tA}e^{tB}u - e^{t(A+B)}u\|_{L^2_\mu} \;\leq\; C_T t^2 \left(\|u\|_{L^2_\mu} + \|u_{xv}\|_{L^2_\mu} + \|vu_x\|_{L^2_\mu}\right) \tag{4.3.4}$$

holds for all $t \in [0, T]$.

Proof. Analogously to the parabolic regularization of the heat equation with periodic boundary conditions, it can be easily checked that for the C_0-semigroup e^{tA} the following regularization holds:

$$\|\partial_x^n e^{tA}\|_{\mathcal{B}(L^2_{\mu_k})} \;\leq\; C_T t^{-n/2}, \quad \forall\, t \in [0, T], \quad n \in \mathbb{N}, \quad k \in \mathbb{N}_0. \tag{4.3.5}$$

Due to the spectral theorem of normal operators, the semigroups e^{tA} and e^{tB} commute with ∂_x^j for $j \geq 1$. Thus, by using the properties (ii) and (iii) of Lemma 4.8 we rewrite (4.3.3) as

$$\begin{aligned} e^{tA}e^{tB} - e^{t(A+B)} &= \\ = \int_0^t e^{(t-s)(A+B)} &\left[2s\sigma e^{\beta s} e^{sA} e^{sB} \partial_v \partial_x + s\beta e^{-\beta s} e^{sA} e^{sB} v\partial_x - s^2 \sigma \partial_x^2 e^{sA} e^{sB}\right] ds. \end{aligned}$$

Then, the estimate (4.3.4) follows by using Proposition 4.2 and (4.3.5). □

It is known from Theorem 4.1 that for initial data $w^I \in L^2_\mu$ a unique mild solution $w \in C([0, \infty), L^2_\mu)$ of (4.0.1)-(4.0.4) exists. This nonlinear solution semigroup, which we denote here by $w(t) = \Phi(t)w^I$, $t \geq 0$, is given by Duhamel's formula

$$\Phi(t)w^I \;=\; e^{t(A+B)}w^I - \int_0^t e^{(t-s)(A+B)}\Theta[V[\Phi(s)w^I]]\Phi(s)w^I\, ds, \tag{4.3.6}$$

and it satisfies the following estimate (4.3.7).

4.10 Proposition. *Let* $w^I \in L^2_\mu$ *and* $D \in L^1(0, 2\pi)$. *Then,*

$$\|\Phi(t)w^I\|_{L^2_\mu} \;\leq\; e^{tK_1\left(t, \|w^I\|_{L^2}\right)}\|w^I\|_{L^2_\mu}, \quad \forall\, t \geq 0, \tag{4.3.7}$$

with $K_1 = K_1\left(t, \|w^I\|_{L^2}\right)$ *defined in Proposition 4.3.*

Proof. The proof follows the lines of Proposition 4.3, by using the dissipativity of the operator $A + B - \left(\sigma + \frac{\beta}{2}\right) I$ in L^2_μ, which was proved in Lemma 2.1 of [**ACD**]. The estimates (4.2.3)-(4.2.8) will carry over identically. □

For the solution of the Poisson equation we have the following result, which is an immediate consequence of the explicit formula (4.1.6).

4.11 Proposition. *Let*

$$u \in \Big\{ f \in L^2_\mu \quad | \quad f_x, f_{xx} \in L^2_\mu;\ u(0,v) = u(2\pi, v), \\ u_x(0,v) = u_x(2\pi, v), \forall\, v \in \mathbb{R} \Big\}$$

and $D \in W_p^{2,1}(0,2\pi)$, satisfying (4.1.4). Then, the solution of the Poisson equation (4.1.5) satisfies $V[u] \in W_p^{3,\infty}(0,2\pi)$ with

$$\|V[u]\|_{W^{3,\infty}(0,2\pi)} \leq C \left(\sum_{i=0}^{2} \left\| \frac{d^i}{dx^i} u \right\|_{L^2_\mu} + \|D\|_{W^{2,1}(0,2\pi)} \right) \tag{4.3.8}$$

holds. Moreover, for $D \in H_p^m(0,2\pi)$, $V[u] \in H_p^{m+2}(0,2\pi)$ with $m = 1,2$ follows.

4.12 Proposition. *Let $T > 0$, $w^I \in D_p(A+B)$ and $D \in H_p^1(0,2\pi)$, satisfying*

$$\int_{\mathbb{R}} \int_0^{2\pi} w^I(x,v)\, dx\, dv = \int_0^{2\pi} D(x)\, dx. \tag{4.3.9}$$

Then, for the classical solution $w \in C([0,T], D_p(A+B)) \cap C^1([0,T], L^2_\mu)$ of the system (4.0.1)-(4.0.4), the estimate

$$\|w(t)\|_{D(A+B)} \leq \|w^I\|_{D(A+B)} C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right), \tag{4.3.10}$$

for all $t \in [0,T]$ holds.

Proof. The proof is in the spirit of Lemma 3.19 in [**Dh**], and can be done by straightforward calculation. The idea is to consider the evolution equation for $y(t) = (A+B)w(t)$, starting with $y^I = (A+B)w^I$:

$$\begin{aligned} y &= (A + B - \Theta[V[y]])\, y + f(t,y), \qquad t > 0, \\ y(0) &= y^I, \end{aligned} \tag{4.3.11}$$

and estimating the respective Duhamel's formula for its mild solution

$$y(t) = \Phi(t)y^I + \int_0^t \Phi(t-s) f(s, y(s))\, ds.$$

By using the results on the potential stated in Proposition 4.11, Proposition 4.10 and Gronwall's lemma it is clear that $\|w^I\|_{D(A+B)}$ will be the leading term of the right hand side of (4.3.10).

Practically, since the evolution equation (4.3.11) is complicated to be dealt as the whole one, (4.3.10) can be proved sequently (like in [**Dh**]), by considering the respective evolution equation for each differential operator term of $A+B$. We refer to Table 3.1 in [**Dh**] for the details and shapes of the inhomogeneities $f(t,y)$ for each case. $\square$

Let $D \in L^1(0,2\pi)$ and $\Psi(t)$, $t \geq 0$ denote the nonlinear semigroup on L^2_μ, which furnishes the solution $z(t) = \Psi(t)u$ of

$$z_t \;=\; -\Theta[V[z]]z, \quad t>0, \qquad \text{with} \quad z(t=0) \;=\; u \in L^2_\mu,$$

where $V[z]$ is the solution of the (periodic) Poisson equation (4.1.5). Analogously to Proposition 4.3 we have

$$\|\Psi(t)u\|_{L^2_\mu} \;\leq\; e^{tC\left(\|u\|_{L^2}+\|D\|_{L^1(0,2\pi)}\right)}\|u\|_{L^2_\mu}, \quad \forall\, t\geq 0, \quad u \in L^2_\mu. \tag{4.3.12}$$

The following lemma presents the consistency error for splitting $\Gamma(t)u$ (cf. (4.1.12)) in $\Psi(t)u$ and e^{tB}. This result will be needed to prove the first-order consistency of the entire splitting scheme (4.1.15). Let us remark that for $\beta=0$ no splitting error for $\Gamma(t)u$ will occur, since ∂_v^n and $\Theta[V]$ commute.

4.13 Lemma. *Let $T>0$, $D \in L^2(0,2\pi)$ satisfying (4.3.9), and $u \in D(B)$ with*

$$u_x \in L^2_\mu \quad \textit{and} \quad u(0,v) = u(2\pi,v),\ v \in \mathbb{R}. \tag{4.3.13}$$

Then, the error estimate

$$\|\Gamma(t)u - e^{tB}\Psi(t)u\|_{L^2_\mu} \;\leq\; t^2 C\left(T, \|u\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right)\|u\|_{D(B)}, \tag{4.3.14}$$

for all $t\in[0,T]$ holds.

Proof. For the nonlinear semigroup $\Gamma(t)u$, defined in (4.1.12)-(4.1.13), Duhamel's formula gives

$$\Gamma(t)u \;=\; e^{tB}u - \int_0^t e^{(t-s)B}\Theta[V[\Gamma(s)u]]\Gamma(s)u\, ds. \tag{4.3.15}$$

On the other hand, the semigroup $\Psi(t)u$ satisfies

$$\Psi(t)u \;=\; u - \int_0^t \Theta[V[\Psi(s)u]]\Psi(s)u\, ds. \tag{4.3.16}$$

Combining (4.3.15) and (4.3.16) we obtain

$$\begin{aligned}\Gamma(t)u - e^{tB}\Psi(t)u &= \\ &= \int_0^t \left[e^{tB}\Theta[V[\Psi(s)u]]\Psi(s)u - e^{(t-s)B}\Theta[V[\Gamma(s)u]]\Gamma(s)u\right] ds,\end{aligned}$$

and hence

$$\|\Gamma(t)u - e^{tB}\Psi(t)u\|_{L^2_\mu} \le t \sup_{s\in[0,t]} \|g(t,s)\|_{L^2_\mu}, \tag{4.3.17}$$

with the function

$$\begin{aligned} g(t,s) &:= e^{tB}\Theta[V[\Psi(s)u]]\Psi(s)u - e^{(t-s)B}\Theta[V[\Gamma(s)u]]\Gamma(s)u \\ &= e^{tB}\Theta[V[u]]\Psi(s)u - e^{(t-s)B}\Theta[V[u]]\Gamma(s)u. \end{aligned} \tag{4.3.18}$$

Since $u \in D(B)$ and (4.3.13) holds, the function $g \in C^1([0,T]^2, L^2_\mu)$, and expanding $g(t,s)$ about $t = s = 0$ yields

$$\begin{aligned} \|g(t,s)\|_{L^2_\mu} &\le |t| \sup_{\epsilon\in[0,1]} \|\partial_t g(t_\epsilon, s_\epsilon)\|_{L^2_\mu} + |s| \sup_{\epsilon\in[0,1]} \|\partial_s g(t_\epsilon, s_\epsilon)\|_{L^2_\mu} \\ &= |t| \sup_{\epsilon\in[0,1]} \left\| e^{t_\epsilon B} B\Theta[V[u]]\Psi(s_\epsilon)u - e^{(t_\epsilon - s_\epsilon)B} B\Theta[V[u]]\Gamma(s_\epsilon)u \right\|_{L^2_\mu} \\ &\quad + |s| \sup_{\epsilon\in[0,1]} \left\| e^{t_\epsilon B}\Theta[V[u]]\Theta[V[u]]\Psi(s_\epsilon)u - e^{(t_\epsilon - s_\epsilon)B} B\Theta[V[u]]\Gamma(s_\epsilon)u \right. \\ &\quad \left. - e^{(t_\epsilon - s_\epsilon)B}\Theta[V[u]](B - \Theta[V[u]])\Gamma(s_\epsilon)u \right\|_{L^2_\mu}, \end{aligned} \tag{4.3.19}$$

with $t_\epsilon = \epsilon t$, $s_\epsilon = \epsilon s$. The assertion (4.3.14) follows by the properties of the operator $\Theta[.]$, discussed in Section 4.1, (4.3.12), Propositions 4.2 and 4.3, and the following estimate on the classical solution of (4.1.1)

$$\|B\Gamma(t)u\|_{L^2_\mu} \le C\left(T, \|u\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right) \|u\|_{D(B)}, \quad \forall\, t \in [0,T]. \tag{4.3.20}$$

The last estimate (4.3.20) can be proved analogously to the proof of Proposition 4.12, by straightforward calculation. The assumption (4.3.13) on u is only needed to ensure that $V[u] \in W_p^{2,\infty}(0,2\pi)$ (cf. Proposition 4.11). $\square$

The next auxiliary lemma is concerned with the consistency error of the approximation scheme $e^{t(A+B)}\Psi(t)u$ for the solution $\Phi(t)u$ of the WPFP-system. This scheme arises from splitting the WFP operator in $A + B$ and $\Theta[V]$.

4.14 Lemma. *Let $T > 0$, $u \in D_p(A + B)$, and $D \in H_p^1(0, 2\pi)$ satisfying (4.3.9). Then, the error estimate*

(4.3.21)
$$\|\Phi(t)u - e^{t(A+B)}\Psi(t)u\|_{L^2_\mu} \leq t^2 C\left(T, \|u\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right)\|u\|_{D(A+B)},$$

holds for all $t \in [0, T]$.

Proof. Combining (4.3.6) and (4.3.16) we obtain

$$\begin{aligned}\Phi(t)u - e^{t(A+B)}\Psi(t)u &= \\ &= \int_0^t \left[e^{t(A+B)}\Theta[V[\Psi(s)u]]\Psi(s)u - e^{(t-s)(A+B)}\Theta[V[\Phi(s)u]]\Phi(s)u\right] ds,\end{aligned}$$

and hence

$$\|\Phi(t)u - e^{t(A+B)}\Psi(t)u\|_{L^2_\mu} \leq t \sup_{s\in[0,t]} \|\tilde{g}(t, s)\|_{L^2_\mu}, \tag{4.3.22}$$

with the function

$$\tilde{g}(t, s) := e^{t(A+B)}\Theta[V[u]]\Psi(s)u - e^{(t-s)(A+B)}\Theta[V[\Phi(s)u]]\Phi(s)u. \tag{4.3.23}$$

Since $u \in D_p(A+B)$, the function $\tilde{g} \in C^1([0, T]^2, L^2_\mu)$, and expanding $\tilde{g}(t, s)$ about $t = s = 0$ yields

(4.3.24)
$$\begin{aligned}\|\tilde{g}(t, s)\|_{L^2_\mu} &\leq |t| \sup_{\epsilon\in[0,1]} \|\partial_t\tilde{g}(t_\epsilon, s_\epsilon)\|_{L^2_\mu} + |s| \sup_{\epsilon\in[0,1]} \|\partial_s\tilde{g}(t_\epsilon, s_\epsilon)\|_{L^2_\mu} \\ &= |t| \sup_{\epsilon\in[0,1]} \Big\| e^{t_\epsilon(A+B)}(A + B)\Theta[V[u]]\Psi(s_\epsilon)u \\ &\quad - e^{(t_\epsilon-s_\epsilon)(A+B)}(A + B)\Theta[V[\Phi(s_\epsilon)u]]\Phi(s_\epsilon)u\Big\|_{L^2_\mu} \\ &\quad + |s| \sup_{\epsilon\in[0,1]} \Big\| e^{t_\epsilon(A+B)}\Theta[V[u]]\Theta[V[u]]\Psi(s_\epsilon)u \\ &\quad - e^{(t_\epsilon-s_\epsilon)(A+B)}(A + B)\Theta[V[\Phi(s_\epsilon)u]]\Phi(s_\epsilon)u \\ &\quad - e^{(t_\epsilon-s_\epsilon)(A+B)}\Theta\left[V\left[\left(A + B - \Theta[V[\Phi(s_\epsilon)u]]\right)\Phi(s_\epsilon)u\right]\right]\Phi(s_\epsilon)u \\ &\quad - e^{(t_\epsilon-s_\epsilon)(A+B)}\Theta[V[\Phi(s_\epsilon)u]]\left(A + B - \Theta[V[\Phi(s_\epsilon)u]]\right)\Phi(s_\epsilon)u\Big\|_{L^2_\mu},\end{aligned}$$

with $t_\epsilon = \epsilon t$, $s_\epsilon = \epsilon s$. The assertion (4.3.21) follows by the properties of the operator $\Theta[.]$, discussed in Section 4.1, (4.3.12), and the Propositions 4.3

and 4.10-4.12. $\square$

Collecting all the results of this section we can finally prove the first-order consistency of the nonlinear splitting scheme (4.1.15).

4.15 Theorem. *Let $T > 0$, $w^I \in D_p(A+B)$ and $D \in H_p^1(0,2\pi)$ satisfying (4.3.9). Then, for the splitting scheme (4.1.15) the error estimate*

$$\|\Phi(t)w^I - F_t w^I\|_{L^2_\mu} \;\leq\; t^2 C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right)\|w^I\|_{D(A+B)}, \tag{4.3.25}$$

for all $t \in [0,T]$ holds.

Proof. Using Propositions 4.2 and 4.9, and Lemma 4.13, we obtain

(4.3.26)

$$\begin{aligned}
&\|e^{t(A+B)}\Psi(t)w^I - F_t w^I\|_{L^2_\mu} \;\leq \\
&\quad\leq\; \|e^{t(A+B)}\Psi(t)w^I - e^{tA}e^{tB}\Psi(t)w^I\|_{L^2_\mu} + \|e^{tA}e^{tB}\Psi(t)w^I - F_t w^I\|_{L^2_\mu} \\
&\quad\leq\; C_T t^2\left(\|\Psi(t)w^I\|_{L^2_\mu} + \|\partial^2_{xv}\Psi(t)w^I\|_{L^2_\mu} + \|v\partial_x\Psi(t)w^I\|_{L^2_\mu}\right) \\
&\qquad + \|e^{tB}\Psi(t)w^I - \Gamma(t)w^I\|_{L^2_\mu} \\
&\quad\leq\; t^2 C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right)\|w^I\|_{D(A+B)}.
\end{aligned}$$

Here we used that

$$\begin{aligned}
\|\Psi(t)w^I\|_{L^2_\mu} + \|\partial^2_{xv}\Psi(t)w^I\|_{L^2_\mu} + \|v\partial_x\Psi(t)w^I\|_{L^2_\mu} \;&\leq \\
\leq\; C\left(T, \|u\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right)\|u\|_{D(A+B)}&,
\end{aligned}$$

which can be easily proved in the spirit of Proposition 4.12.
Hence, by using (4.3.26) and Lemma 4.14, we finally get

$$\begin{aligned}
\|\Phi(t)w^I - F_t w^I\|_{L^2_\mu} \;&\leq\; \|\Phi(t)w^I - e^{t(A+B)}\Psi(t)w^I\|_{L^2_\mu} \\
&\quad + \|e^{t(A+B)}\Psi(t)w^I - F_t w^I\|_{L^2_\mu} \\
&\leq\; t^2 C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right)\|w^I\|_{D(A+B)},
\end{aligned}$$

for all $t \in [0,T]$. $\square$

Hence, by using the nonlinear stability results in Section 4.2 and the previous consistency error estimate we conclude the first-order convergence of the scheme (4.1.15).

4.16 Theorem. *Let $T > 0$ and N be the number of iterations and $\Delta t = t/N$ with $t \in [0, T]$. Then, for $w^I \in D_p(A + B)$ and $D \in H^1_p(0, 2\pi)$ satisfying (4.3.9), the splitting scheme (4.1.15) is first-order convergent:*

$$(4.3.27) \quad \|w(t) - F^N_{\Delta t} w^I\|_{L^2_\mu} \;\leq\; \Delta t \|w^I\|_{D(A+B)} C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right),$$

where $w(t) = \Phi(t)w^I$ is the classical solution of the nonlinear WPFP-system (4.0.1)-(4.0.4).

Proof. By using the telescoping sum

$$(4.3.28) \quad w(t) - F^N_{\Delta t} w^I \;=\; \sum_{j=1}^{N} \left[F^{N-j}_{\Delta t} w(j\Delta t) - F^{N-j+1}_{\Delta t} w((j-1)\Delta t) \right],$$

the Propositions 4.6 and 4.12, and Theorem 4.15, the assertion (4.3.27) follows. ☐

4.4 $1 - \epsilon$ order convergence

First-order is the best convergence order to be expected from our splitting scheme (4.1.15) (even for smooth initial data). The aim of the subsequent analysis is to derive a low-order convergence result which admits more general initial data. Indeed, we shall prove that starting with just one additional momentum in v, i.e. $w^I \in L^2_{\mu_2}$, we shall get the following error estimate for the splitting scheme

$$\|w(N\Delta t) - F^N_{\Delta t} w^I\|_{L^2_\mu} \;\leq\; M_\epsilon \left(T, \|w^I\|_{L^2_{\mu_2}}, \|D\|_{H^2(0,2\pi)}\right) \Delta t^{1-\epsilon}, \quad \forall\, \epsilon \in (0,1),$$

for Δt small enough and $N \in \mathbb{N}$ such that $0 \leq N\Delta t \leq T$.
This is a powerful result in the sense that we get "almost" first-order convergence without any smoothness assumption on the initial data.
Analogously to [**ACD**] we can prove that the operator $\overline{A + B}$ generates a C_0-semigroup of linear operators also in $L^2_{\mu_2}$. It satisfies

$$(4.4.1) \qquad \|e^{t(A+B)} u\|_{L^2_{\mu_2}} \;\leq\; e^{\left(\frac{\beta}{2} + 6\sigma\right)t} \|u\|_{L^2_{\mu_2}}, \quad \forall\, u \in L^2_{\mu_2}, \quad t \geq 0.$$

In the next proposition we derive a uniform-in-time estimate of the approximation error of the linear product formula $e^{tA}e^{tB}$. It does not present a consistency result, but it provides a better convergence in time than the Trotter's formula can ensure (cf. [**CHMM**]).

4.17 Proposition. *Let $T > 0$. Then, the following consistency error estimate for the approximation of $e^{t(A+B)}$ by the product formula $e^{tA}e^{tB}$ holds:*

$$\|e^{tA}e^{tB} - e^{t(A+B)}\|_{\mathcal{B}(L^2_{\mu_2},L^2_\mu)} \leq C_T t, \quad \forall\, t \in [0,T]. \tag{4.4.2}$$

Proof. Due to the fact that e^{tA} and e^{tB} commute with ∂_x^j for $j \geq 1$, and the properties of e^{tB}, discussed in Lemma 4.8, we rewrite (4.3.3) as

$$\begin{aligned} e^{tA}e^{tB} - e^{t(A+B)} &= \\ = \int_0^t e^{(t-s)(A+B)} &\left[2s\sigma\partial_x e^{sA}\partial_v e^{sB} + s\beta\partial_x e^{sA} v e^{sB} - s^2\sigma\partial_x^2 e^{sA}e^{sB}\right] ds. \end{aligned} \tag{4.4.3}$$

Then, by using Proposition 4.2 and the regularizing results (4.3.5) and (i) in Lemma 4.8, the estimate (4.4.2) follows. □

In the sequel we apply the same strategy as in Section 4.3 (cf. Lemmata 4.13, 4.14 and Theorem 4.15) to derive the error behaviour

$$\|\Phi(t)w^I - F_t w^I\|_{L^2_\mu} = \mathcal{O}(t)$$

on the nonlinear splitting error, by assuming $w^I \in L^2_{\mu_2}$ and $D \in L^2(0,2\pi)$ (this result is stated in Proposition 4.20). Therefore, we present the following three results in the same order as in Section 4.3. First we present the consistency error made by splitting $\Gamma(t)u$ (cf. (4.1.12)) in $\Psi(t)u$ and e^{tB}, without assuming regularity of initial data.

4.18 Lemma. *Let $T > 0$ and $D \in L^1(0,2\pi)$. Then, for all $u \in L^2_\mu$ the error estimate*

$$\|\Gamma(t)u - e^{tB}\Psi(t)u\|_{L^2_\mu} \leq tC\left(T, \|u\|_{L^2_\mu}, \|D\|_{L^1(0,2\pi)}\right), \tag{4.4.4}$$

for all $t \in [0,T]$ holds.

Proof. Like in the proof of Lemma 4.13 we have

$$\|\Gamma(t)u - e^{tB}\Psi(t)u\|_{L^2_\mu} \leq t \sup_{s\in[0,t]} \|g(t,s)\|_{L^2_\mu},$$

with the function g, defined in (4.3.18). Since g is continuous in t and s, (4.4.4) follows by estimating the L^2_μ norm of $g(t,s)$ (cf. estimates on $\Theta[V]$ in Section 4.1, (4.3.12) and Propositions 4.2, 4.3). □

The next Lemma provides consistency estimate on the approximation of $\Phi(t)$ by a scheme arising from splitting the WFP operator in $A + B$ and $\Theta[V]$.

4.19 Lemma. *Let $T > 0$ and $D \in L^1(0, 2\pi)$. Then, for all $u \in L^2_\mu$ the error estimate*

$$\|\Phi(t)u - e^{t(A+B)}\Psi(t)u\|_{L^2_\mu} \ \le \ tC\left(T, \|u\|_{L^2_\mu}, \|D\|_{L^1(0,2\pi)}\right), \tag{4.4.5}$$

for all $t \in [0, T]$ holds.

Proof. Like in the proof of Lemma 4.14 we have

$$\|\Phi(t)u - e^{t(A+B)}\Psi(t)u\|_{L^2_\mu} \ \le \ t \sup_{s\in[0,t]} \|\tilde{g}(t, s)\|_{L^2_\mu},$$

with the function $\tilde{g}$, defined in (4.3.23). Since $\tilde{g}$ is continuous in t and s, (4.4.5) follows by estimating the L^2_μ norm of $\tilde{g}(t, s)$ (cf. estimates on $\Theta[V]$ in Section 4.1, (4.3.12) and Propositions 4.2, 4.3 and 4.10). □

Again, collecting these results we can finally prove the following consistency.

4.20 Proposition. *Let $T > 0$, $w^I \in L^2_{\mu_2}$ and $D \in L^2(0, 2\pi)$ satisfying (4.3.9). Then, for the splitting scheme (4.1.15) the error estimate*

$$\|\Phi(t)w^I - F_t w^I\|_{L^2_\mu} \ \le \ tC\left(T, \|w^I\|_{L^2_{\mu_2}}, \|D\|_{L^2(0,2\pi)}\right), \tag{4.4.6}$$

for all $t \in [0, T]$ holds.

Proof. The proof yields in exactly the same way as in Theorem 4.15, by using the Lemmata 4.18, 4.19 and Proposition 4.17, respectively.
Here we use that $\Psi(t)$ is a nonlinear semigroup on $L^2_{\mu_2}$ as well. Similarly to the proof of Proposition 4.3 we can estimate for $u(t) := \Psi(t)w^I$ by using Hölder's inequality and Sobolev embeddings:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2_{\mu_2}} \ &\le \ \iint 2v^2 u(t)\Omega[V_x[u(t)]]vu(t) - v^2u(t)\Theta[V_{xx}[u(t)]]u(t) \\
&\le \ \left(4\|V_x[u(t)]\|_{L^\infty(0,2\pi)} + \|V_{xx}[u(t)]\|_{L^2(0,2\pi)}\right)\|v^2u(t)\|_{L^2}\|vu(t)\|_{L^2} \\
&\le \ C\left(\|u(t)\|_{L^2_\mu} + \|D\|_{L^2(0,2\pi)}\right)\|u(t)\|^2_{L^2_{\mu_2}}.
\end{aligned}$$

Hence,

$$\|u(t)\|_{L^2_{\mu_2}} \ \le \ e^{tC\left(\|u(t)\|_{L^2_\mu} + \|D\|_{L^2(0,2\pi)}\right)}\|w^I\|_{L^2_{\mu_2}}, \quad \forall\, t \ge 0 \tag{4.4.7}$$

follows. □

The next Propositions 4.21 and 4.22 present regularizing results of parabolic type of the linear semigroup $e^{t(A+B)}$ and the nonlinear semigroup $\Phi(t)$, which will be necessary for the low-order convergence proof in Theorem 4.23.

4.21 Proposition. *Let $T > 0$, $m \in \mathbb{N}$, $k \in \mathbb{N}_0$, and the initial data $u \in L^2_{\mu_k}(\mathbb{R}^2)$. Then, the solution $e^{t(A+B)}u$ of the linear parabolic equation (4.0.1) with $V \equiv 0$ satisfies*

$$\|e^{t(A+B)}u\|_{H^m_{\mu_k}} \;\leq\; C_T t^{-m/2}\|u\|_{L^2_{\mu_k}}, \quad \forall\, 0 < t \leq T. \tag{4.4.8}$$

Proof. This parabolic regularizing property is an expected result in the theory of uniformly parabolic equations, and it can be checked for general initial-boundary-value problems in [**Fr**]. However, the formula for the fundamental solution to the whole-space problem, i.e. the WFP-equation with $V \equiv 0$ for $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$, is given in Chapter 3.1 in [**SCDM**], and in (3.3.6)-(3.4.7) an analogous result to (4.4.8) has been proved by directly calculating the norms of the derivatives of this solution. This can be easily adapted to our periodic case. $\square$

4.22 Proposition. *Let $T > 0$, $m \in \mathbb{N}$, $w^I \in L^2_\mu$ and $D \in H^{m-2}_p(0,2\pi)$ for $m > 1$, satisfying (4.3.9) (for $m = 1$ is only $D \in L^1(0,2\pi)$ necessary). Then, the mild solution $w(t) = \Phi(t)w^I$ of the WPFP-model (4.0.1)-(4.0.4) satisfies*

$$\|w(t)\|_{H^m_\mu} \;\leq\; t^{-m/2} C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^{m-2}(0,2\pi)}\right), \quad \forall\, t \in (0,T]. \tag{4.4.9}$$

Proof. From Proposition 4.21 we have

$$\|e^{t(A+B)}\|_{\mathcal{B}(L^2_\mu, H^m_\mu)} \;\leq\; C_T t^{-m/2}, \quad \forall\, 0 < t \leq T. \tag{4.4.10}$$

Let $t_0 > 0$ be small enough such that $mt_0 \in (0,T]$. By using the Duhamel's formula (4.3.6) and the estimate (4.4.10), we get

(4.4.11)

$$\begin{aligned}
&\|\partial_x w(t)\|_{L^2_\mu} \;\leq \\
&\leq\; \|\partial_x e^{t(A+B)}w^I\|_{L^2_\mu} + \int_0^t \|\partial_x e^{(t-s)(A+B)}\Theta[V[w(s)]]w(s)\|_{L^2_\mu} ds \\
&\leq\; C_T t^{-1/2}\|w^I\|_{L^2_\mu} + C_T \int_0^t (t-s)^{-1/2}\|\Theta[V[w(s)]]w(s)\|_{L^2_\mu} ds \\
&\leq\; C_T t^{-1/2}\|w^I\|_{L^2_\mu} + C_T \int_0^t (t-s)^{-1/2}\|V[w(s)]\|_{W^{1,\infty}(0,2\pi)}\|w(s)\|_{L^2_\mu} ds \\
&\leq\; C_T t^{-1/2}\|w^I\|_{L^2_\mu} + C_T \int_0^t (t-s)^{-1/2}\left(\|w(s)\|_{L^2_\mu} + \|D\|_{L^1(0,2\pi)}\right)\|w(s)\|_{L^2_\mu} ds \\
&\leq\; C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^1(0,2\pi)}\right) t^{-1/2}\|w^I\|_{L^2_\mu}, \quad \forall\, 0 < t \leq T.
\end{aligned}$$

An analogous estimate with the same time decay we get for the derivative in v as well.

Differentiating the equation (4.0.1) with respect to x we obtain the following (new) linear, inhomogeneous problem

$$\begin{aligned} z_t &= -(A+B+\Theta[V[w]])z - \Theta[V_x[w]]w, \quad \forall\, t > t_0, \\ z(t_0) &= \partial_x w(t_0) \in L^2_\mu, \end{aligned}$$

where $z(t) = \partial_x w(t)$. Since by Sobolev embeddings

$$\begin{aligned} \|\Theta[V_x[w(t)]]w(t)\|_{L^2_\mu} &\leq C\|V_x[w(t)]\|_{L^\infty(0,2\pi)}\|w(t)\|_{L^2_\mu} \\ &\quad + C\|V_{xx}[w(t)]\|_{L^2(0,2\pi)}\|\mathcal{F}_{v\to\eta}w(t)\|_{L^2((0,2\pi);L^\infty(\mathbb{R}_\eta))} \\ &\leq C\|V_x[w(t)]\|_{L^\infty(0,2\pi)}\|w(t)\|_{L^2_\mu} \\ &\quad + C\|n[w(t)] - D\|_{L^2(0,2\pi)}\|vw(t)\|_{L^2} \\ &\leq C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right)\|w^I\|_{L^2_\mu} \end{aligned}$$

holds, for the mild solution $z(t)$ of this problem (cf. Theorem 6.1.2 in [**Pa**]) we get by using again the Duhamel's formula:

(4.4.12)

$$\begin{aligned} \|z(t)\|_{L^2_\mu} &\leq \|e^{(t-t_0)(A+B)}z(t_0)\|_{L^2_\mu} \\ &\quad + \int_{t_0}^t \|e^{(t-s)(A+B)}\left(\Theta[V[w(s)]]z(s) + \Theta[V_x[w(s)]]w(s)\right)\|_{L^2_\mu} ds \\ &\leq C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right)\left(\|z(t_0)\|_{L^2_\mu} + \|w^I\|_{L^2_\mu}\right) \\ &\quad + C(T)\int_{t_0}^t \left(\|w^I\|_{L^2_\mu} + \|D\|_{L^1(0,2\pi)}\right)\|z(s)\|_{L^2_\mu}\, ds. \end{aligned}$$

Applying the Gronwall's Lemma for $t_0 \leq t \leq T$, the estimate

$$\|z(t)\|_{L^2_\mu} \leq C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right)\left[\|z(t_0)\|_{L^2_\mu} + \|w^I\|_{L^2_\mu}\right] \tag{4.4.13}$$

follows. Therefore, for all $t_0 < t \leq T$:

$$\begin{aligned} \|\partial_x z(t)\|_{L^2_\mu} &\leq \|\partial_x e^{(t-t_0)(A+B)}z(t_0)\|_{L^2_\mu} \\ &\quad + \int_{t_0}^t \|\partial_x e^{(t-s)(A+B)}\left(\Theta[V[w(s)]]z(s) + \Theta[V_x[w(s)]]w(s)\right)\|_{L^2_\mu} ds \\ &\leq C(T)(t-t_0)^{-1/2}\|z(t_0)\|_{L^2_\mu} \\ &\quad + C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right)\left[\|z(t_0)\|_{L^2_\mu} + \|w^I\|_{L^2_\mu}\right]. \end{aligned}$$

Finally we get

$$\|\partial_x z(2t_0)\|_{L^2_\mu} \leq C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right) t_0^{-1}. \tag{4.4.14}$$

Differentiating the equation (4.0.1) with respect to v we obtain the following (new) linear, inhomogeneous problem

$$\begin{aligned} y_t &= -(A+B-\beta+\Theta[V[w]])y - z, \quad \forall\, t > t_0, \\ y(t_0) &= \partial_v w(t_0) \in L^2_\mu, \end{aligned}$$

where $y(t) = \partial_v w(t)$. For the mild solution $y(t)$ of this problem we get, by using (4.4.13) and the Duhamel's formula (cf. calculation in (4.4.11)),

$$\begin{aligned} \|y(t)\|_{L^2_\mu} &\le C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right) \Big[\|y(t_0)\|_{L^2_\mu} + \|z(t_0)\|_{L^2_\mu} \\ &\quad + \|w^I\|_{L^2_\mu}\Big], \quad \forall\, t_0 \le t \le T. \end{aligned}$$

Further, we obtain analogously

$$\begin{aligned} \|\partial_v y(t)\|_{L^2_\mu} &\le C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right) \Big[(t-t_0)^{-1/2}\|y(t_0)\|_{L^2_\mu} \\ &\quad + \|z(t_0)\|_{L^2_\mu} + \|w^I\|_{L^2_\mu}\Big], \quad \forall\, t_0 \le t \le T. \end{aligned}$$

Therefore,

$$\|\partial_v y(2t_0)\|_{L^2_\mu} \le C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right) t_0^{-1}. \tag{4.4.15}$$

Going on successively, by applying derivatives of order k ($k = 2, ..., m$) on the equation (4.0.1) and considering the resulting problems for $T \ge t \ge kt_0$, we get

$$\|w(mt_0)\|_{H^m_\mu} \le C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^{m-2}(0,2\pi)}\right) t_0^{-m/2} \tag{4.4.16}$$

(cf. Proposition 4.11), which gives (4.4.9), since t_0 we chose arbitrary but small enough. □

Finally, we present the main result of this section by adapting the strategy introduced in Proposition 4.12 in [**DS**].

4.23 Theorem. *Let $T > 0$, $w^I \in L^2_{\mu_2}$ and $D \in H^2_p(0, 2\pi)$ satisfying (4.3.9). Then, for all $\epsilon \in (0,1)$ there exists a constant $M_{\epsilon,T} \ge 0$ such that, for all $\Delta t \in (0, \min\{T, 1\}]$ and N with $0 \le N\Delta t \le T$, the approximating error of the scheme (4.1.15) to the solution $w(t) = \Phi(t)w^I$ of the nonlinear WPFP-system (4.0.1)-(4.0.4) satisfies*

$$\|w(N\Delta t) - F^N_{\Delta t} w^I\|_{L^2_\mu} \le \Delta t^{1-\epsilon} M_{\epsilon,T}\left(\|w^I\|_{L^2_{\mu_2}}, \|D\|_{H^2(0,2\pi)}\right). \tag{4.4.17}$$

PROOF. Let $\epsilon = \frac{s}{2} - 1 > 0$ with some $s \in (2,4)$.
The difference $F_{\Delta t}^N - \Phi(\Delta t)^N$ can be decomposed as follows

$$\begin{aligned} F_{\Delta t}^N - \Phi(\Delta t)^N &= \\ &= \sum_{j=1}^{N-1} \left(F_{\Delta t}^{N-j} - F_{\Delta t}^{N-j-1}\Phi(\Delta t) \right) \Phi(\Delta t)^j + (F_{\Delta t}^N - F_{\Delta t}^{N-1}\Phi(\Delta t)). \end{aligned}$$

By using the Propositions 4.6 we have the estimate

$$\begin{aligned} \|F_{\Delta t}^N w^I - F_{\Delta t}^{N-1}\Phi(\Delta t)w^I\|_{L^2_\mu} &\leq \|F_{\Delta t}w^I - \Phi(\Delta t)w^I\|_{L^2_\mu} \\ &\cdot C\left((N-1)\Delta t, \|D\|_{L^1(0,2\pi)}, \|F_{\Delta t}w^I\|_{L^2_\mu}, \|\Phi(\Delta t)w^I\|_{L^2_\mu}\right). \end{aligned}$$

Thus, by using Propositions 4.5, 4.10 and 4.20,

$$\|F_{\Delta t}^N w^I - F_{\Delta t}^{N-1}\Phi(\Delta t))w^I\|_{L^2_\mu} \leq \Delta t C\left(T, \|w^I\|_{L^2_{\mu_2}}, \|D\|_{L^2(0,2\pi)}\right)$$

follows. On the other hand, by using Propositions 4.6 and 4.10, and Theorem 4.15, we have

$$\begin{aligned} \left\| \left(F_{\Delta t}^{N-j} - F_{\Delta t}^{N-j-1}\Phi(\Delta t)\right) \Phi(\Delta t)^j w^I \right\|_{L^2_\mu} &\leq \\ &\leq \| \left(F_{\Delta t} - \Phi(\Delta t)\right) \Phi(\Delta t)^j w^I\|_{L^2_\mu} C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^1(0,2\pi)}\right) \\ &\leq \Delta t^2 \|\Phi(j\Delta t)w^I\|_{D(A+B)} C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{H^1(0,2\pi)}\right). \end{aligned}$$

Thanks to the regularizing effect (4.4.9) of $\Phi(t)$, we obtain by extending the $D_p(A+B)$-norm by complementary terms of the H^s_μ-norm:

$$\begin{aligned} \|\Phi(t)w^I\|_{D(A+B)} &\leq \|\Phi(t)w^I\|_{H^s_\mu} + \|v\partial_v\Phi(t)w^I\|_{L^2_\mu} + \|v\partial_x\Phi(t)w^I\|_{L^2_\mu} \\ &\leq t^{-s/2} C\left(T, \|w^I\|_{L^2_{\mu_2}}, \|D\|_{H^2(0,2\pi)}\right), \quad \forall\, t \in (0,T]. \end{aligned}$$

Here we need the assumption (4.3.9) on D because of

$$\|v\Phi(t)w^I\|_{L^2_\mu} \leq \|w^I\|_{L^2_{\mu_2}} C\left(T, \|w^I\|_{L^2_\mu}, \|D\|_{L^2(0,2\pi)}\right),$$

(cf. proof of Proposition 4.12). Putting these estimates together we finally get

$$\begin{aligned} \|F_{\Delta t}^N w^I - \Phi(\Delta t)^N w^I\|_{L^2_\mu} &\leq \\ &\leq C\left(T, \|w^I\|_{L^2_{\mu_2}}, \|D\|_{H^2(0,2\pi)}\right) \left(\sum_{j=1}^{N-1} \frac{\Delta t^2}{(j\Delta t)^{s/2}} + \Delta t \right). \end{aligned}$$

The estimate (4.4.17) follows by using the fact that $\sum_{j=1}^{N-1} \frac{1}{j^{s/2}} \leq \Gamma(s/2)$, where Γ denotes the Gamma function. $\square$

4.5 Remarks on the linear WFP

This section is devoted to the linear Wigner-Fokker-Planck equation given by

$$\begin{aligned} w_t &= \Big(A + B - \Theta[V]\Big)w, \quad (x,v) \in \mathbb{R}^2, \quad t > 0, \\ w(t=0) &= w^I, \end{aligned} \tag{4.5.1}$$

with a given constant-in-time potential $V = V(x)$, $V \in L^\infty(\mathbb{R})$.

In the sequel we shall give a summary of convergence results for approximating the solution of (4.5.1) by a linear product formula of semigroups. We shall prove first-order convergence by regularity assumptions on initial data and on the potential, and present a low-order convergence result for the splitting scheme by using more general data.

The following results are strongly related to the respective results for the nonlinear WPFP-system (cf. Sections 4.1-4.4). They are organized in the same way and proved by adaption of strategies used in the previous sections. The linear operators $\overline{A}$, $\overline{B} - \frac{\beta}{2}I$ and $\overline{A+B} - \frac{\beta}{2}I$ are dissipative in $L^2(\mathbb{R}^2)$, as well. Hence, they generate the (quasi-) contractive C_0-semigroups e^{tA}, e^{tB} and $e^{t(A+B)}$ in $L^2(\mathbb{R}^2)$, respectively. Their definition areas will then be composed of the same terms as in Section 4.1 by replacing the norm L^2_μ by $L^2(\mathbb{R}^2)$ (the periodicity in x is not needed):

$$\begin{aligned} \tilde{D}(A) &:= \{u \in L^2(\mathbb{R}^2) \mid vu_x, u_{xx} \in L^2(\mathbb{R}^2)\}; \\ \tilde{D}(B) &:= \{u \in L^2(\mathbb{R}^2) \mid u_{vv}, vu_v \in L^2(\mathbb{R}^2)\}; \\ \tilde{D}(A+B) &:= \{u \in L^2(\mathbb{R}^2) \mid vu_x, u_{vv}, vu_v, u_{xx} \in L^2(\mathbb{R}^2)\}. \end{aligned}$$

Since

$$\|\Theta[V]\|_{\mathcal{B}(L^2(\mathbb{R}^2))} \leq 2\|V\|_{L^\infty(\mathbb{R})},$$

the pseudo-differential operator $\Theta[V]$ can be seen as a bounded perturbation of the generators $\overline{A+B}$ and $\overline{B}$. Thus, in view of the foregoing analysis of WPFP (cf. [**ACD**]), it is obvious that the following (quasi-) contractive C_0-semigroups of linear operators are generated in $L^2(\mathbb{R}^2)$:

$$\tilde{\Gamma}(t)u \;:=\; e^{t(B-\Theta[V])}u \;:=\; e^{tB}u - \int_0^t e^{(t-s)B}\Theta[V]\tilde{\Gamma}(s)u\,ds$$

and

$$\tilde{\Phi}(t)u \;:=\; e^{t(A+B-\Theta[V])}u \;:=\; e^{t(A+B)}u - \int_0^t e^{(t-s)(A+B)}\Theta[V]\tilde{\Phi}(s)u\,ds,$$

for $u \in L^2(\mathbb{R}^2)$ (cf. semigroup theory in [**Pa**]). Therefore, the following existence and uniqueness result, analogous to Theorem 4.1, holds:

4.24 Theorem. *Let $V \in L^\infty(\mathbb{R})$.*

a) For every $w^I \in L^2(\mathbb{R}^2)$, the linear WFP-equation (4.5.1) has a unique mild solution $\tilde{w} \in C([0,\infty), L^2(\mathbb{R}^2))$.

b) If $w^I \in \tilde{D}(\overline{A+B})$, $\tilde{w}$ is a classical solution, i.e. $\tilde{w} \in C^1([0,\infty), L^2(\mathbb{R}^2)) \cap C([0,\infty), \tilde{D}(\overline{A+B}))$.

The explicit definition of $\tilde{\Gamma}(t)$ is given through (4.1.12)-(4.1.13) by putting the given potential V into (4.1.13). Due to skew-symmetry of $\Theta[V]$ we have the estimates

$$(4.5.2) \qquad \|\tilde{\Gamma}(t)\|_{\mathcal{B}(L^2(\mathbb{R}^2))} \;\le\; e^{\frac{\beta}{2}t}, \qquad \|\tilde{\Phi}(t)u\|_{\mathcal{B}(L^2(\mathbb{R}^2))} \;\le\; e^{\frac{\beta}{2}t}, \quad \forall\, t \ge 0$$

(cf. (4.2.5)).

The splitting scheme we use here is an adaption of (4.1.15) to the linear model (4.5.1):

$$(4.5.3) \qquad w_{n+1} \;=\; \tilde{F}_{\Delta t} w_n, \quad n \in \mathbb{N} \quad \text{with} \quad w_0 = w^I,$$

where the step-forward operator $\tilde{F}_{\Delta t}$ is defined by

$$(4.5.4) \qquad \tilde{F}_{\Delta t} \;:=\; e^{\Delta t A}\tilde{\Gamma}(\Delta t), \quad \forall\, \Delta t \ge 0.$$

Let us remark here that the L^2-convergence of the splitting scheme (4.5.3) is an immediate consequence of Trotter's product formula (cf. [**CHMM**]).

4.25 Theorem. *Let $V \in L^\infty(\mathbb{R})$ and the initial data $w^I \in L^2(\mathbb{R}^2)$. Then, the operator splitting method (4.5.3) converges in $L^2(\mathbb{R}^2)$ as $\Delta t \to 0$.*

Further, the following adaption of Proposition 4.9 can be easily checked:

4.26 Proposition. *Let $T > 0$ and*

$$u \in \tilde{Y} \;:=\; \left\{ f \in L^2(\mathbb{R}^2) \mid f_{xv},\, v f_x \in L^2(\mathbb{R}^2) \right\}.$$

Then, the consistency estimate

$$\|e^{tA}e^{tB}u - e^{t(A+B)}u\|_{L^2(\mathbb{R}^2)} \;\le\; C_T t^2 \left(\|u\|_{L^2(\mathbb{R}^2)} + \|u_{xv}\|_{L^2(\mathbb{R}^2)} + \|v u_x\|_{L^2(\mathbb{R}^2)} \right)$$

holds for all $t \in [0,T]$.

By a slightly modification of Lemma 3.19 in [**Dh**] concerning the term $\alpha\partial_x^2$ of the operator A, which indeed does not change its proof, it can be shown that for $w^I \in \tilde{D}(A+B)$ and $V \in W^{2,\infty}(\mathbb{R})$ the classical solution

$\tilde{\Phi}(.)w^I \in C([0,T],\tilde{D}(A+B)) \cap C^1([0,T],L^2(\mathbb{R}^2))$ of equation (4.5.1) satisfies the estimate

$$\|\tilde{\Phi}(t)w^I\|_{\tilde{D}(A+B)} \;\leq\; C\left(T,\|V\|_{W^{2,\infty}(\mathbb{R})}\right)\|w^I\|_{\tilde{D}(A+B)}, \tag{4.5.5}$$

for all $t \in [0,T],\; T>0$.

To prove the first-order consistency of the scheme (4.5.3) we apply the strategy used in the Lemmata 4.13, 4.14 and Theorem 4.15.

4.27 Theorem. *Let $T>0$, $w^I \in \tilde{D}(A+B)$ and $V \in W^{2,\infty}(\mathbb{R})$. Then, for the splitting scheme (4.5.3) the error estimate*

$$\|\tilde{\Phi}(t)w^I - \tilde{F}_t w^I\|_{L^2(\mathbb{R}^2)} \;\leq\; t^2 C\left(T,\|V\|_{W^{2,\infty}(\mathbb{R})}\right)\|w^I\|_{\tilde{D}(A+B)}, \tag{4.5.6}$$

for all $t \in [0,T]$ holds.

Proof. Let $\tilde{\Psi}(t)$ denote the semigroup given by

$$\tilde{\Psi}(t)u \;=\; u - \int_0^t \Theta[V]\Psi(s)u\,ds, \quad u \in L^2(\mathbb{R}^2),$$

which satisfies $\|\tilde{\Psi}(t)\|_{\mathcal{B}(L^2(\mathbb{R}^2))} \leq 1$. Then, we have as in the proof of Lemma 4.13 (cf. (4.3.17)-(4.3.19))

$$\begin{aligned}
&\|\tilde{\Gamma}(t)w^I - e^{tB}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \;\leq \\
&\leq\; t\sup_{s\in[0,t]}\|e^{tB}\Theta[V]\tilde{\Psi}(s)w^I - e^{(t-s)B}\Theta[V]\tilde{\Gamma}(s)w^I\|_{L^2(\mathbb{R}^2)} \qquad (4.5.7)\\
&\leq\; |t|\sup_{\epsilon\in[0,1]}\left\|e^{t_\epsilon B}B\Theta[V]\tilde{\Psi}(s_\epsilon)w^I - e^{(t_\epsilon-s_\epsilon)B}B\Theta[V]\tilde{\Gamma}(s_\epsilon)w^I\right\|_{L^2(\mathbb{R}^2)}\\
&\quad + |s|\sup_{\epsilon\in[0,1]}\Big\|e^{t_\epsilon B}\Theta[V]^2\tilde{\Psi}(s_\epsilon)w^I - e^{(t_\epsilon-s_\epsilon)B}B\Theta[V]\tilde{\Gamma}(s_\epsilon)w^I\\
&\quad - e^{(t_\epsilon-s_\epsilon)B}\Theta[V](B-\Theta[V])\Gamma(s_\epsilon)w^I\Big\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

with $t_\epsilon = \epsilon t$, $s_\epsilon = \epsilon s$. Applying (4.5.2), the estimate

$$\|B\tilde{\Gamma}(t)w^I\|_{L^2(\mathbb{R}^2)} \;\leq\; C\left(T,\|V\|_{W^{1,\infty}(\mathbb{R})}\right)\|w^I\|_{\tilde{D}(B)}. \tag{4.5.8}$$

and the properties of the operator $\Theta[.]$, discussed in Section 4.1 (in particular (4.1.9)) we obtain

$$\|\tilde{\Gamma}(t)w^I - e^{tB}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \;\leq\; t^2 C\left(T,\|V\|_{W^{1,\infty}(\mathbb{R})}\right)\|w^I\|_{\tilde{D}(B)}, \tag{4.5.9}$$

for all $t \in [0, T]$. The estimate (4.5.8) can be proved analogously to (4.5.5), by straightforward calculation.

One the other hand, dealing with the splitting formula $e^{t(A+B)}\tilde{\Psi}(t)$ as in Lemma 4.14 (cf. (4.3.22)-(4.3.24)) we have

$$\|\tilde{\Phi}(t)w^I - e^{t(A+B)}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \leq$$

$$\begin{aligned}
(4.5.10) \quad &\leq t \sup_{s\in[0,t]} \|e^{t(A+B)}\Theta[V]\tilde{\Psi}(s)w^I - e^{(t-s)(A+B)}\Theta[V]\tilde{\Phi}(s)w^I\|_{L^2(\mathbb{R}^2)} \\
&\leq |t| \sup_{\epsilon\in[0,1]} \Big\| e^{t_\epsilon(A+B)}(A+B)\Theta[V]\tilde{\Psi}(s_\epsilon)w^I \\
&\quad - e^{(t_\epsilon - s_\epsilon)(A+B)}(A+B)\Theta[V]\tilde{\Phi}(s_\epsilon)w^I \Big\|_{L^2(\mathbb{R}^2)} \\
&\quad + |s| \sup_{\epsilon\in[0,1]} \Big\| e^{t_\epsilon(A+B)}\Theta[V]^2\tilde{\Psi}(s_\epsilon)w^I \\
&\quad - e^{(t_\epsilon - s_\epsilon)(A+B)}(A+B)\Theta[V]\tilde{\Phi}(s_\epsilon)w^I \\
&\quad - e^{(t_\epsilon - s_\epsilon)(A+B)}\Theta[V]\Big(A+B-\Theta[V]\Big)\tilde{\Phi}(s_\epsilon)w^I \Big\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

with $t_\epsilon = \epsilon t$, $s_\epsilon = \epsilon s$. Applying (4.5.2), (4.5.5) and the properties of $\Theta[V]$ we obtain the error estimate

(4.5.11)

$$\|\tilde{\Phi}(t)w^I - e^{t(A+B)}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \leq t^2 C\left(T, \|V\|_{W^{2,\infty}(\mathbb{R})}\right) \|w^I\|_{\tilde{D}(A+B)},$$

for all $t \in [0, T]$.

Collecting the results in (4.5.9), (4.5.11),

$$\|\partial^2_{xv}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} + \|v\partial_x\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \leq C\left(T, \|V\|_{W^{2,\infty}(\mathbb{R})}\right) \|w^I\|_{\tilde{D}(A+B)},$$

and Proposition 4.26 we conclude this proof by arguing exactly as in Theorem 4.15 and splitting the error in

(4.5.12)

$$\begin{aligned}
\|\tilde{\Phi}(t)w^I - \tilde{F}_t w^I\|_{L^2(\mathbb{R}^2)} \leq\ &\|\tilde{\Phi}(t)w^I - e^{t(A+B)}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \\
&+ \|e^{t(A+B)}\tilde{\Psi}(t)w^I - e^{tA}e^{tB}\tilde{\Psi}(t)w^I\|_{L^2(\mathbb{R}^2)} \\
&+ \|e^{tA}e^{tB}\tilde{\Psi}(t)w^I - \tilde{F}_t w^I\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

□

As in Theorem 4.16 the first-order convergence follows by directly by using the telescoping sum (4.3.28) and the previous results.

4.28 Theorem. *Let $T > 0$ and N be the number of iterations and $\Delta t = t/N$ with $t \in [0, T]$. Then, for $w^I \in \tilde{D}(A + B)$ and $V \in W^{2,\infty}(\mathbb{R})$, the splitting scheme (4.5.3) is first-order convergent:*

$$\|\tilde{w}(t) - \tilde{F}^N_{\Delta t} w^I\|_{L^2(\mathbb{R}^2)} \leq \Delta t \|w^I\|_{\tilde{D}(A+B)} C\left(T, \|V\|_{W^{2,\infty}(\mathbb{R})}\right), \tag{4.5.13}$$

where $\tilde{w}(t) = \tilde{\Phi}(t) w^I$ is the classical solution of the linear WFP equation (4.5.1).

Our next aim is to derive a low-order convergence result for the linear product formula (4.5.3), analogous to Theorem 4.23 for the nonlinear case. By starting again with just an additional momentum in v of the initial data, i.e. $w^I \in L^2_\mu(\mathbb{R}^2)$, and $V \in W^{3,\infty}(\mathbb{R})$ we shall prove $(1 - \epsilon)$-order convergence, as well. The analogous consistency result to Proposition 4.17 is

4.29 Proposition. *Let $T > 0$. Then, the following consistency error estimate for the approximation of $e^{t(A+B)}$ by the product formula $e^{tA} e^{tB}$ holds:*

$$\|e^{tA} e^{tB} - e^{t(A+B)}\|_{\mathcal{B}(L^2_\mu(\mathbb{R}^2), L^2(\mathbb{R}^2))} \leq C_T t, \quad \forall\, t \in [0, T]. \tag{4.5.14}$$

Proof. The estimate (4.5.14) can be easily checked by using the Taylor expansion (4.4.3) and the regularizing properties (4.3.5) and (i) in Lemma 4.8 of the semigroups e^{tA} and e^{tB}, respectively, which hold in $L^2(\mathbb{R}^2)$ as well. □

4.30 Proposition. *Let $T > 0$, $w^I \in L^2_\mu(\mathbb{R}^2)$ and $V \in W^{1,\infty}(\mathbb{R})$. Then, for the splitting scheme (4.5.3) the error estimate*

$$\|\tilde{\Phi}(t) w^I - \tilde{F}_t w^I\|_{L^2(\mathbb{R}^2)} \leq t C\left(T, \|V\|_{W^{1,\infty}(\mathbb{R})}\right) \|w^I\|_{L^2_\mu(\mathbb{R}^2)}, \tag{4.5.15}$$

for all $t \in [0, T]$ holds.

Proof. Estimating (4.5.7) yields

$$\|\tilde{\Gamma}(t) w^I - e^{tB} \tilde{\Psi}(t) w^I\|_{L^2(\mathbb{R}^2)} \leq t C\left(T, \|V\|_{L^\infty(\mathbb{R})}\right) \|w^I\|_{L^2(\mathbb{R}^2)}, \tag{4.5.16}$$

for all $t \in [0, T]$. Analogously, by estimating (4.5.10),

$$\|\tilde{\Phi}(t) w^I - e^{t(A+B)} \tilde{\Psi}(t) w^I\|_{L^2(\mathbb{R}^2)} \leq t C\left(T, \|V\|_{L^\infty(\mathbb{R})}\right) \|w^I\|_{L^2(\mathbb{R}^2)}, \tag{4.5.17}$$

for all $t \in [0, T]$ follows. Using these two estimates, Proposition 4.29 and

$$\|\tilde{\Psi}(t) w^I\|_{L^2_\mu(\mathbb{R}^2)} \leq e^{t\|V_x\|_{L^\infty(\mathbb{R})}} \|w^I\|_{L^2_\mu(\mathbb{R}^2)}$$

(cf. (4.4.7)), and splitting of the error as in (4.5.12) we obtain the assertion (4.5.15). □

Since $\overline{A+B} - \left(\sigma + \frac{\beta}{2}\right) I$ is dissipative in $L^2_\mu(\mathbb{R}^2)$, for $V \in W^{1,\infty}(\mathbb{R})$ the estimate

$$\|\tilde{\Phi}(t)\|_{\mathcal{B}(L^2_\mu(\mathbb{R}^2))} \;\leq\; e^{t\left(\sigma+\frac{\beta}{2}+\|V\|_{W^{1,\infty}(\mathbb{R})}\right)}, \quad \forall\, t>0 \tag{4.5.18}$$

holds. This can be easily proved by arguing as in Proposition 4.3 (cf. (4.2.3)).

In the spirit of Proposition 4.22 one can prove by straightforward calculation the following parabolic regularizing property of the solution of (4.5.1):

4.31 Proposition. *Let $T>0$, $m \in \mathbb{N}$, $w^I \in L^2(\mathbb{R}^2)$ and $V \in W^{m-1,\infty}(\mathbb{R})$ for $m \geq 1$. Then, the mild solution $\tilde{w}(t) = \tilde{\Phi}(t) w^I$ of the linear WFP-equation (4.5.1) satisfies*

$$\|\tilde{w}(t)\|_{H^m(\mathbb{R}^2)} \;\leq\; t^{-m/2} C\left(T, \|V\|_{W^{m-1,\infty}(\mathbb{R})}\right) \|w^I\|_{L^2(\mathbb{R}^2)}, \tag{4.5.19}$$

for all $t \in (0,T]$.

Finally, we present the $(1-\epsilon)$-convergence of splitting scheme (4.5.3) by adapting the strategy of Theorem 4.23.

4.32 Theorem. *Let $T>0$, $w^I \in L^2_\mu(\mathbb{R}^2)$ and $V \in W^{3,\infty}(\mathbb{R})$. Then, for all $\epsilon \in (0,1)$ there exists a constant $M_{\epsilon,T} \geq 0$ such that, for all $\Delta t \in (0, \min\{T,1\}]$ and N with $0 \leq N\Delta t \leq T$, the approximating error of the scheme (4.5.3) to the solution $\tilde{w}(t) = \tilde{\Phi}(t) w^I$ of the linear WFP-equation (4.5.1) satisfies*

$$\|\tilde{w}(N\Delta t) - \tilde{F}^N_{\Delta t} w^I\|_{L^2(\mathbb{R}^2)} \;\leq\; \Delta t^{1-\epsilon} M_{\epsilon,T}\left(\|w^I\|_{L^2_\mu(\mathbb{R}^2)}, \|V\|_{W^{3,\infty}(\mathbb{R})}\right). \tag{4.5.20}$$

Proof. (4.5.20) can be proved straightforwardly as in Theorem 4.23, by using Theorem 4.27, the Propositions 4.30 and 4.31, and (4.5.18). □

4.33 Remark. All the analysis of this section can be done in $L^2(\mathbb{R}^{2d})$, $d \in \mathbb{N}$ as well. The linear WFP equation will then have the shape

$$w_t \;=\; -v \cdot \nabla_x w + \beta \mathrm{div}_v(vw) + \sigma \Delta_v w + \alpha \Delta_x w - \Theta[V] w,$$

for $(x,v) \in \mathbb{R}^{2d}$ and $t>0$, with a given constant-in-time potential $V = V(x)$, $V \in L^\infty(\mathbb{R}^d)$ and the initial value

$$w(t=0) \;=\; w^I \in L^2(\mathbb{R}^{2d}).$$

We have to consider the following domains of the differential operators A, B and $A+B$:

$$\begin{aligned}
\tilde{D}(A) &:= \{u \in L^2(\mathbb{R}^{2d}) \mid v_j u_{x_j}, u_{x_j x_j} \in L^2(\mathbb{R}^{2d}),\ j = 1,...,d\};\\
\tilde{D}(B) &:= \{u \in L^2(\mathbb{R}^{2d}) \mid u_{v_j v_j}, v_j u_{v_j} \in L^2(\mathbb{R}^{2d}),\ j = 1,...,d\};\\
\tilde{D}(A+B) &:= \{u \in L^2(\mathbb{R}^{2d}) \mid v_j u_{x_j}, u_{v_j v_j}, v_j u_{v_j}, u_{x_j x_j} \in L^2(\mathbb{R}^{2d}),\ j = 1,...,d\},
\end{aligned}$$

and use, wherever is necessary in the proofs, the component-wise writing and the weight $\mu = (1+|v|^2)$.

Chapter 5

Implementation and simulations

In this section we briefly discuss the implementation of the proposed splitting algorithm (4.1.1)-(4.1.3) and present a numerical test examples. Although, theoretically, both steps of the splitting method can be carried out exactly, we will incur discretization errors at each time step since a finite dimensional representation of the solution has to be chosen.

We shall discuss a mixed finite-difference-spectral-collocation method for the numerical realization of the splitting scheme. A finite difference method is used in the x- and t-direction. The v-direction, in which the pseudo-differential operator Θ acts, is discretized by spectral collocation using trigonometric functions. Spectral methods are natural candidates for the discretization of Θ because of its nonlocal nature and its definition via the Fourier transforms.

The reason for choosing a difference method in the x-direction is that, for practical applications, various and sometimes quite complicated boundary conditions must be used (cf. [**KFR88, KFR89, RFK**]). Here, in accordance with our convergence analysis of the previous section, we supplement the x-discretization with periodic boundary conditions. Difference methods can be adapted more easily to different types of boundary conditions.

On the other hand all evaluations in the v-direction, specially the evaluation of Θ-operator, can be performed using Fast Fourier Transform (FFT) methods, which significantly reduce the amount of work per time-step. This kind of discretization, for evaluating the pseudo-differential operator Θ, has been used and discussed in several works in the framework of numerical realization of the Wigner-Poisson model (cf. [**Ri90, Ri91, Ri92, AR95, AR96**]). A rigorous convergence proof for the spectral accuracy of the semi-discretization in v is presented in [**Ri91**], at least for the Wigner-Poisson case. In [**AR95**] first order convergence of the fully discretized splitting scheme for the Wigner-Poisson problem with periodic boundary conditions has been carried out. There, because of treating periodic boundary conditions in x, an expansion in trigonometric functions in spatial direction has been chosen as well.

In our approach the Poisson equation (4.1.2) will be subject to homogeneous Dirichlet boundary conditions on the bounded segment in the x-direction, as introduced in (4.1.5).

In the Sections 5.1 and 5.2 we shall present the discretization of the splitting steps (4.1.1)-(4.1.3) separately, and illustrate numerically the convergence to their steady states. In Section 5.3 the linear parabolic Wigner-Fokker-Planck (WFP) equation is considered, and numerical tests to illustrate the convergence to its steady state have been done. For quantifying the convergence error, the discrete L^2-norm as well as the discrete quadratic relative entropy norm have been used. The last one is a suitable norm in the context of Fokker-Planck-equations (cf. [**AMTU, SCDM, AU**]) to prove the exponential decay of the solution to its equilibrium. In the Section 5.4 the barrier effect of a given potential is simulated and the quantum effect of the tunneling of the wave packets, even for "very strong" potential, are illustrated. Finally, in Section 5.5 a simulation of the nonlinear WPFF system is presented.

5.1 The spectral discretization in velocity direction

Given an equidistant time discretization $t_n = n\Delta t$, $n \in \mathbb{N}$, let w_n denote the approximation in time of the solution w of the WPFP-system (4.0.1)-(4.0.4) at t_n ($w_0 := w^I$ at $t_0 = 0$). In the sequel we are considering as space interval the x-interval $[-L, L]$, $L > 0$ (instead of $[0, 2\pi]$!).

Since the first split step (4.1.1)-(4.1.2)

(5.1.1)

$$\begin{aligned}
&\text{a)} & u_t &= \sigma\partial_v^2 u + \beta\partial_v(vu) - \Theta[V]u, \qquad t > 0,\\
&\text{b)} & u(x,v,t=0) &= w_n(x,v),\\
&\text{c)} & V_{xx}(x,t) &= \int u(x,v,t)\,dv - D(x),\\
&\text{d)} & V(-L,t) &= V(L,t) \equiv 0,
\end{aligned}$$

can be carried out exactly even in the nonlinear case (cf. (4.1.6), (4.1.12)), it only remains to choose an appropriate function space to approximate the function u. Following [**Ri91, AR96**], we approximate the solution u of

(5.1.1) by trigonometric polynomials with a period $2\pi/\mathtt{a}$ in the v–direction

(5.1.2)

$$\text{a)} \qquad u(x,v,t) \;\sim\; u^d(x,v,t) \;=\; \sum_{k=-M+1}^{M} \hat{u}(x,k,t)e^{i\mathtt{a}kv},$$

$$\text{b)} \qquad n^d(x,t) \;=\; \int_{-\pi/\mathtt{a}}^{\pi/\mathtt{a}} u^d(x,v,t)dv \;=\; \frac{2\pi}{\mathtt{a}}\,\hat{u}(x,0,t) \;=\; \frac{2\pi}{\mathtt{a}}\,\hat{w}_n(x,0).$$

Since the exact solution is integrable in the v–direction, and because of the cutoff in the integral (5.1.2)b), the period $2\pi/\mathtt{a}$ will have to tend to infinity together with $M\mathtt{a}$ in order to achieve convergence. However, in general this approximation will be of spectral accuracy for smooth u, and so we can expect to achieve good results with modest values of M and $\mathtt{a}$ (see [**Ri91**] for details of the approximation properties of (5.1.2)).

Then, the split step (5.1.1) can be carried out in two intermediate steps. The first one is to solve

$$V_{xx} \;=\; \frac{2\pi}{\mathtt{a}}\,\hat{w}_n(x,0) - D(x) \tag{5.1.3}$$

together with the boundary conditions

$$V(-L) \;=\; V(L) \;=\; 0. \tag{5.1.4}$$

We shall solve (5.1.3)-(5.1.4) with a simple central finite difference scheme of second order in x, on an equidistant grid

$$x_j \;=\; j\Delta x - L, \qquad j \;=\; 0(1)N_x - 1 \quad \text{and} \quad \Delta x \;=\; \frac{2L}{N_x - 1}. \tag{5.1.5}$$

Denoting the approximating values of $V(x_j)$ by V_j, the difference scheme has the form

(5.1.6)

$$\frac{V_{j+1} - 2V_j + V_{j-1}}{(\Delta x)^2} \;=\; \frac{2\pi}{\mathtt{a}}\,\hat{w}_n(x_j,0) - D(x_j), \qquad j \;=\; j = 1(1)N_x - 2,$$

with the boundary conditions $V_0 \;=\; V_{N_x-1} \;=\; 0$.

Then, the next intermediate step is to apply the explicit formula for the semigroup $\Gamma(t)$, introduced in (4.1.12)-(4.1.13):

$$\mathcal{F}_{v\to\eta}\Big(\Gamma(t)w_n(x,v)\Big) \;=\; \mathcal{F}_{v\to\eta}w_n(x,\eta e^{-\beta t})\mathcal{E}(x,\eta,t) \tag{5.1.7}$$

with

$$\mathcal{E}(x,\eta,t) \;=\; \exp\left\{\frac{\sigma}{2\beta}\,\eta^2(e^{-2\beta t} - 1) - i\int_0^t \delta V(x,\eta e^{\beta(\tau-t)})\,d\tau\right\}. \tag{5.1.8}$$

In order to be able to carry out the evaluation for the time step Δt along the characteristics $\eta e^{-\beta\Delta t}$ we shall use here a standard interpolation with cubic splines. Let $N_v = 2M+1$ be the number of equidistant grid points in v, on the interval $[-\frac{\pi}{\mathtt{a}}, \frac{\pi}{\mathtt{a}}]$. Then, the discrete grid in v is given by

$$v_k \;=\; k\Delta v, \qquad k \;=\; -M(1)M, \qquad \Delta v \;=\; \frac{\pi}{\mathtt{a}M}, \tag{5.1.9}$$

and let $w_n(x,k)$ be the approximate value of $w_n(x,v_k)$. The corresponding grid points in the Fourier variable $\eta \in [\frac{\pi}{\Delta v}, \frac{\pi}{\Delta v}]$ are

$$\eta_k \;=\; k\Delta\eta, \qquad k \;=\; -M(1)M, \qquad \Delta\eta \;=\; \mathtt{a}\,, \tag{5.1.10}$$

and $\hat{w}_n(x,k)$ denotes the approximate value of $\mathcal{F}_{v\to\eta} w(x,\eta_k,t_n)$.

To use a cubic spline interpolation on the grid η_k with the function values $\hat{w}_n(x,k)$, $k = -M(1)M$ one usually needs conditions on the slope of the interpolating function at the end points of the interval $[\frac{\pi}{\Delta v}, \frac{\pi}{\Delta v}]$. For a general convergence analysis and implementation of cubic splines we refer for instance to the detailed study in [**Bo**]. Since no boundary information is available in our case, a cubic spline interpolation with *not-a-knot* end conditions is suitable (see [**Bo**] for details and convergence results).

Let us denote for fixed x the interpolated values of $\{(\eta_k, \hat{w}_n(x,k))\}_{k=-M(1)M}$ over the characteristic points $\eta_k e^{-\beta\Delta t}$ by $\hat{w}_n^{\mathrm{sp}}(x,k)$:

$$\hat{w}_n^{\mathrm{sp}}(x,k) \;:=\; \mathbf{spline}\Big(\{\eta_k\}, \{\hat{w}_n(x,k)\}, \eta_k e^{-\beta\Delta t}\Big) \tag{5.1.11}$$

On the other hand we have to evaluate the integral

$$\int_0^{\Delta t} \delta V(x_j, \eta_k e^{\beta(\tau-\Delta t)})\, d\tau, \quad j = 0(1)N_x - 1,\; k = -M(1)M, \tag{5.1.12}$$

in each time step. For that reason the potential V has to be extended periodically outside of the interval $(-L, L]$. Then, we shall apply the linear interpolation, introduced in [**Dh**], which gives a second order approximation of this integral in Δt as well as in $\Delta\eta$ (see in Chapter 5 of [**Dh**] for the convergence proof). Let us briefly recall here this interpolation.

Let $\lfloor . \rfloor$ denote the Gaussian brackets, i.e. $\lfloor y \rfloor$ is the largest integer less than or equal to $y \in \mathbb{R}$. Then, the approximation reads

$$\int_0^{\Delta t} \delta V(x_j, \eta_k e^{-\beta(\Delta t-\tau)})\, d\tau \;\approx\; \Big(\mathrm{INT}_{\mathrm{char}}\, \delta V\Big)(j,k) \tag{5.1.13}$$

$$
\begin{aligned}
:= \;& \Delta t\rho(k)\Big[V\Big(x_j + \Big\lfloor\frac{\eta_k e^{-\beta\Delta t}}{2\Delta x}\Big\rfloor\Delta x\Big) - V\Big(x_j - \Big\lfloor\frac{\eta_k e^{-\beta\Delta t}}{2\Delta x}\Big\rfloor\Delta x\Big)\Big] \\
& + \Delta t\Big(1-\rho(k)\Big)\Big[V\Big(x_j + \Big\lfloor\frac{\eta_k}{2\Delta x}\Big\rfloor\Delta x\Big) - V\Big(x_j - \Big\lfloor\frac{\eta_k}{2\Delta x}\Big\rfloor\Delta x\Big)\Big], \\
= \;& \Delta t\rho(k)\Big[V\Big(j + \Big\lfloor\frac{\eta_k e^{-\beta\Delta t}}{2\Delta x}\Big\rfloor\Big) - V\Big(j - \Big\lfloor\frac{\eta_k e^{-\beta\Delta t}}{2\Delta x}\Big\rfloor\Big)\Big] \\
& + \Delta t\Big(1-\rho(k)\Big)\Big[V\Big(j + \Big\lfloor\frac{\eta_k}{2\Delta x}\Big\rfloor\Big) - V\Big(j - \Big\lfloor\frac{\eta_k}{2\Delta x}\Big\rfloor\Big)\Big],
\end{aligned}
$$

with the interpolation weights

$$
(5.1.14) \qquad \rho(k) = \begin{cases} \frac{k}{k-\lfloor ke^{-\beta\Delta t}\rfloor}\Big[1 + \frac{1}{\beta\Delta t}(e^{-\beta\Delta t}-1)\Big], & k > \lfloor ke^{-\beta\Delta t}\rfloor \\ 1, & k = \lfloor ke^{-\beta\Delta t}\rfloor \end{cases}
$$

for all $k = -M(1)M$.

Thus, summarizing all the comments (5.1.2)-(5.1.13), starting with w_n we have for the time step $t_n \to t_n + \Delta t$ the following

5.1 Numerical algorithm.

1. for each point x_j of the x-grid, given in (5.1.5), compute the Fourier coefficients

$$
\hat{w}_n(x_j, k) = \frac{1}{2M}\sum_{l=-M+1}^{M} w_n(x_j, l)e^{-iakv_l}, \quad k = -M+1(1)M,
$$

   by using the FFT-algorithm. Here we should choose M equal to a power of 2 in order to take advantage of the FFT calculation. Then, set $\hat{w}_n(x_j, -M) = \hat{w}_n(x_j, M)$;

2. solve (5.1.6) with homogeneous boundary conditions on the x-grid (5.1.5);

3. apply the cubic spline interpolation, given in (5.1.11), for the characteristics tracing;

4. calculate the integral (5.1.12) via the linear interpolation given in (5.1.13)-(5.1.14);

5. evaluate the discretized version of (5.1.7)-(5.1.8):
   (5.1.15)

$$
\hat{w}_{n+\frac{1}{2}}(x_j, k) = \hat{w}_n^{\mathrm{sp}}(x_j, k)\exp\Big\{\frac{\sigma}{2\beta}\eta_k^2(e^{-2\beta\Delta t}-1) - i\Big(\mathrm{INT}_{\mathrm{char}}\,\delta V\Big)(j,k)\Big\}
$$

6. and finally, transform back the Fourier coefficients

$$w_{n+\frac{1}{2}}(x_j, k) = \sum_{l=-M+1}^{M} \hat{w}_{n+\frac{1}{2}}(x_j, l) e^{i\mathtt{a} l v_k}, \quad k = -M + 1(1)M,$$

   where $w_{n+\frac{1}{2}}(x_j, k)$ is an approximation of the value $u(x_j, v_k, t_n + \Delta t)$ of the solution of (5.1.1).

5.2 Remark. It should be noted that the Algorithm 5.1 preserves the conservation of the total mass in (5.1.1),

$$\mathcal{M}(t) := \int_{-L}^{L} \int_{-\pi/\mathtt{a}}^{\pi/\mathtt{a}} w(x, v, t)\, dv\, dx, \tag{5.1.16}$$

which is analytically preserved. This is due to the periodicity in x and v, and the skew-symmetry of Θ. After discretization, the total mass $\mathcal{M}$ at time t_n is defined as

$$\mathcal{M}_n = \Delta x \frac{2\pi}{\mathtt{a}} \sum_{j=0}^{N_x-1} \hat{w}_n(x_j, 0). \tag{5.1.17}$$

Therefore, since $\eta_0 = 0$, we have from (5.1.15) with $k = 0$:

$$\begin{aligned} \mathcal{M}_{n+\frac{1}{2}} &= \Delta x \frac{2\pi}{\mathtt{a}} \sum_{j=0}^{N_x-1} \hat{w}_{n+\frac{1}{2}}(x_j, 0) \\ &= \Delta x \frac{2\pi}{\mathtt{a}} \sum_{j=0}^{N_x-1} \hat{w}_n^{\mathrm{sp}}(x_j, 0) \exp\Big\{ - i\Big(\mathrm{INT}_{\mathrm{char}}\ \delta V\Big)(j, 0)\Big\} \\ &= \Delta x \frac{2\pi}{\mathtt{a}} \sum_{j=0}^{N_x-1} \hat{w}_n(x_j, 0) \\ &= \mathcal{M}_n. \end{aligned}$$

5.1.1 Numerical test of Algorithm 5.1 with $V \equiv 0$

In this subsection we test the Algorithm 5.1 without potential, i.e. we approximate the solution of the equation

$$\begin{aligned} u_t &= \sigma \partial_v^2 u + \beta \partial_v (v u), \qquad t > 0, \\ u(t = 0) &= w_n. \end{aligned} \tag{5.1.18}$$

In particular, we want to illustrate how the calculated solution behaves for different grid refinements in v and in the Fourier variable η. The reason to consider $V \equiv 0$ in (5.1.1) is, that we can exactly calculate the steady state of the equation (5.1.18) in the Fourier space with respect to v (cf. (5.1.19)). This leads to suitable numerical tests for the accuracy and stability of our algorithm.

Since the equation (5.1.18) is autonomous with respect to the x-variable, let x be fixed in the sequel. The discrete steady state of the Fourier transformed version of equation (5.1.18)

$$\hat{u}_t = -\sigma\eta^2\hat{u} - \beta\eta\partial_\eta\hat{u}, \qquad \hat{u}(\eta, t=0) = \hat{w}_n(\eta), \tag{5.1.19}$$

is given by

$$\hat{u}_\infty(k) = \hat{w}_n(0)\exp\Big\{-\frac{\sigma}{2\beta}\,\eta_k^2\Big\}, \qquad k = -M(1)M. \tag{5.1.20}$$

Starting with a Gaussian distribution function, centered at $v = 1$, the pictures in Figure 5.1 give a first impression about the time evolution of this equation in the v-variable, as well as in the Fourier space variable η, approximated by Algorithm 5.1 with $V \equiv 0$. The discretization and problem parameters are given on the top of the pictures.

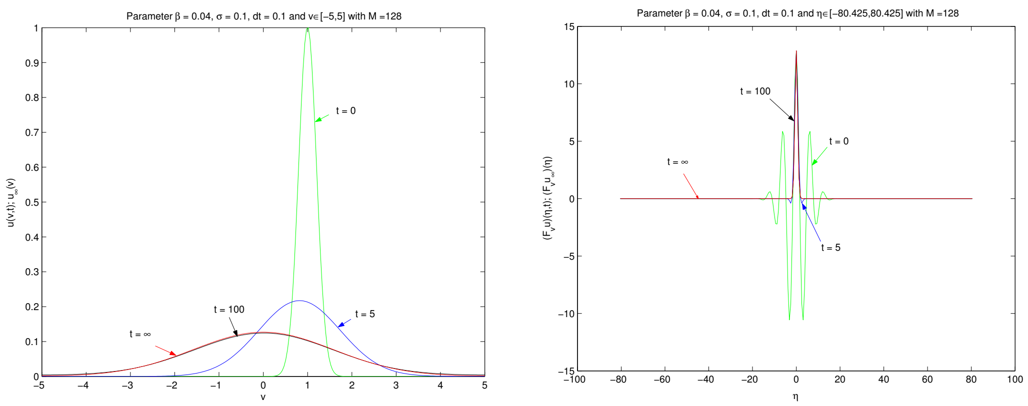

Figure 5.1: Time evolution of equation (5.1.19) in v, and in the Fourier variable η

u_∞ denotes the discrete Fourier transform of (5.1.20) from η back to v, which is a Gaussian distribution as well.

Starting with the same initial data as in Figure 5.1, the next Figure 5.2 illustrates the discrete L^2-difference of the approximated solution after "long" time to u_∞, when using different interval lengths $\frac{\pi}{\mathsf{a}}$ in v, and different number of grid points M. It can be seen that the length of the v-interval plays a crucial role in the accuracy of the numerical algorithm 5.1, since it affects the grid resolution in the Fourier variable η (see definition of $\Delta\eta$ in (5.1.10)). Even for a small number of grid points we obtain "good" accuracy for larger interval length, i.e. "bad" resolution in the v-space.

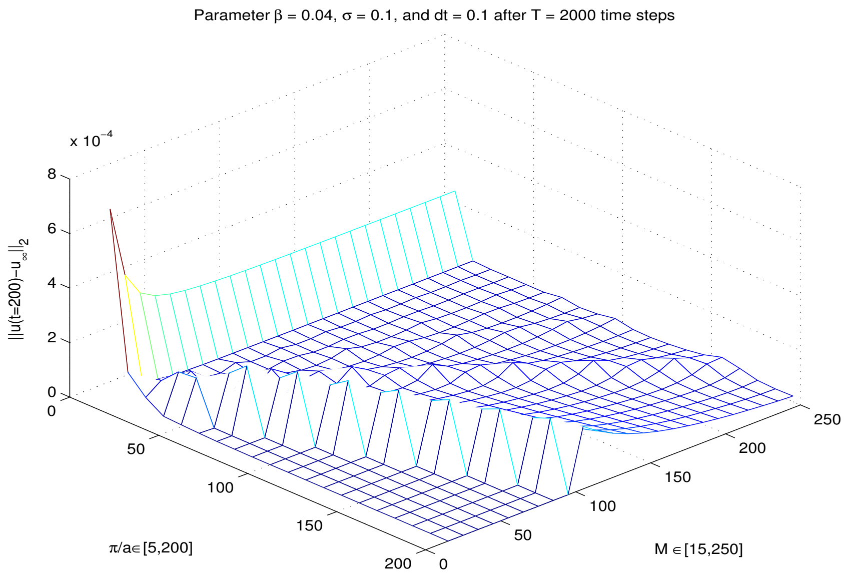

Figure 5.2: l^2-error to steady state at $t = 200$, depending on interval length $\pi/\mathtt{a}$ and number of grid points M

5.2 The convection-diffusion equation in space direction

In the split step (4.1.3) in the space direction $x \in [-L, L]$ we have to approximate the solution of the following convection-diffusion equation with periodic boundary conditions:

(5.2.1)

$$
\begin{array}{lrcll}
\text{a)} & u_t & = & -vu_x + \alpha u_{xx}, & t > 0, \\
\text{b)} & u(x, v, t = 0) & = & w_{n+\frac{1}{2}}(x, v), & \\
\text{c)} & u(-L, v, t) & = & u(L, v, t), & t > 0, \\
\text{d)} & u_x(-L, v, t) & = & u_x(L, v, t), & t > 0,
\end{array}
$$

where $v \in \mathbb{R}$ is fixed. This shall be solved using a finite difference scheme of Crank-Nicolson-type, which is unconditionally stable and second order accurate in time and space (cf. [**Stri**]). In the sequel we use the grids introduced in (5.1.5), (5.1.9), and the time grid $t_n = n\Delta t$, $n \in \mathbb{N}$. Let v_k be a fixed velocity for some $k = -M(1)M$, and $w_{n+1}(j, v_k)$ denote the approximate value of $w(x_j, v_k, t_n + \Delta t) = u(x_j, v_k, \Delta t)$.
Then, the difference scheme for solving the splitting step (5.2.1) reads

$$(5.2.2)\qquad \frac{w_{n+1}(j,v_k)-w_{n+\frac{1}{2}}(j,v_k)}{\Delta t} \;=\; \\ = \;\alpha\,\frac{w_{n+1}(j+1,v_k)-2w_{n+1}(j,v_k)+w_{n+1}(j-1,v_k)}{2(\Delta x)^2} \\ -\, v_k\frac{w_{n+1}(j+1,v_k)-w_{n+1}(j-1,v_k)}{4\Delta x} \\ +\,\alpha\,\frac{w_{n+\frac{1}{2}}(j+1,v_k)-2w_{n+\frac{1}{2}}(j,v_k)+w_{n+\frac{1}{2}}(j-1,v_k)}{2(\Delta x)^2} \\ -\, v_k\frac{w_{n+\frac{1}{2}}(j+1,v_k)-w_{n+\frac{1}{2}}(j-1,v_k)}{4\Delta x},$$

for $j = 1(1)N_x - 2$, with the periodic boundary conditions

$$(5.2.3)\qquad w_{n+1}(0,v_k) \;=\; w_{n+1}(N_x-2,v_k),$$

$$(5.2.4)\qquad w_{n+1}(N_x-1,v_k) \;=\; w_{n+1}(1,v_k).$$

Because of the periodicity in x the total mass is conserved,

$$\mathcal{M}_{n+1} \;=\; \mathcal{M}_{n+\frac{1}{2}} \;=\; \mathcal{M}_n.$$

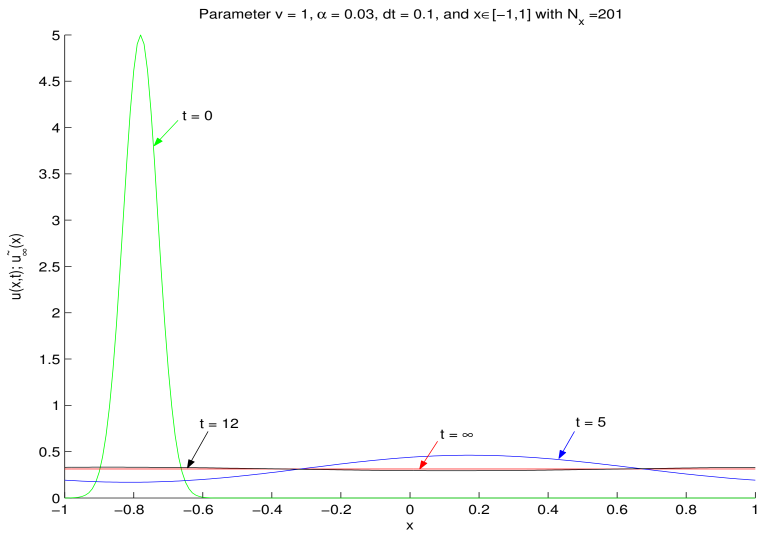

Figure 5.3: Initial data and its time evolution under (5.2.1)

Let us consider the x-interval $[-1,1]$ with $\Delta x = 0.01$. Starting with a Gaussian, centered at $x = -0.78$, Figure 5.3 shows its time evolution under (5.2.1), and the steady state (in red color), which is constant in x and given by

$$(5.2.5)\qquad \tilde{u}_\infty(x_j) \;\equiv\; \frac{\mathcal{M}}{2} \;=\; \frac{\Delta x}{2}\sum_{l=0}^{N_x-1} w_{n+\frac{1}{2}}(l,v_k), \qquad j \;=\; 0(1)N_x-1.$$

The discretization and problem parameters are given on the top of the pictures.

The accuracy with respect to the space step Δx of the above difference scheme (5.2.2) is well-known in the literature (cf. for instance [**Stri, GR**]). Hence, we want to illustrate the L^2-error behaviour according to the choice of the diffusion coefficient α. For "small" α we would have a singularly perturbed parabolic equation (cf. [**RST, GR**] for details on this class of problems). In this case a more involved difference scheme with *artificial diffusion* has to be applied, for instance an *Iljin*-scheme (cf. [**GR, Al, DMS**]). For instance, for a fixed α in (5.2.2) the maximum principle will be violated for large $|v|$. However, this will play an underpart since the cutoff-intervals in v will be chosen relatively small in our simulations.

The next Figure 5.4 illustrates exactly the above discussion. It shows the error after "long" time of the presented difference scheme (5.2.2) (on the interval $[-1,1]$) to the steady state in the discrete L^2-norm, for different diffusion parameters $\alpha \in [0.01, 0.25]$ and convection parameters $v \in [-10, 10]$. The relatively high error peaks close to $|v| = 10$, $\alpha = 0.01$, are caused by the violation of the maximum principle (cf. Chapter 6.4 of [**Stri**] for further details).

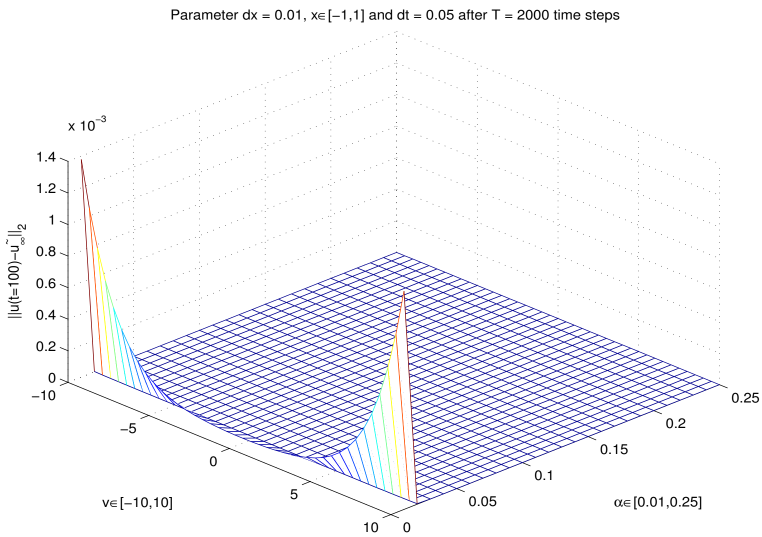

Figure 5.4: l^2-error to the steady state at $t = 100$, depending on convection coefficient v and diffusion coefficient α

5.3 The linear WFP equation with $V \equiv 0$

In this section we shall approximate the solution of the linear parabolic Wigner-Fokker-Planck equation

$$\partial_t w + v w_x = \beta(vw)_v + \sigma w_{vv} + \alpha w_{xx}, \qquad t > 0, \tag{5.3.1}$$

by the presented splitting scheme (5.1.1) and (5.2.1), consisting in applying the Algorithm 5.1 with $V \equiv 0$ and the difference scheme (5.2.2)-(5.2.4), successively, for each time step $t_n = n\Delta t$, $n \in \mathbb{N}$.

5.3 Numerical algorithm. Let x_j and v_k be the phase-space discretization given in (5.1.5), (5.1.9). Further, let $w_n(j,k)$ be the approximation of $w(x_j, v_k, t_n)$. Then, advancing in time from t_n to $t_{n+1} = t_n + \Delta t$ we have to apply:

I: Algorithm 5.1 to calculate numerically $w_{n+\frac{1}{2}}$. Since no potential is used, the items 2. and 4. of Algorithm 5.1) are not needed.

II: solve the linear system of equations arising from the finite difference scheme in (5.2.2) with the periodic boundary conditions (5.2.3)-(5.2.4), to calculate the new approximate solution w_{n+1}.

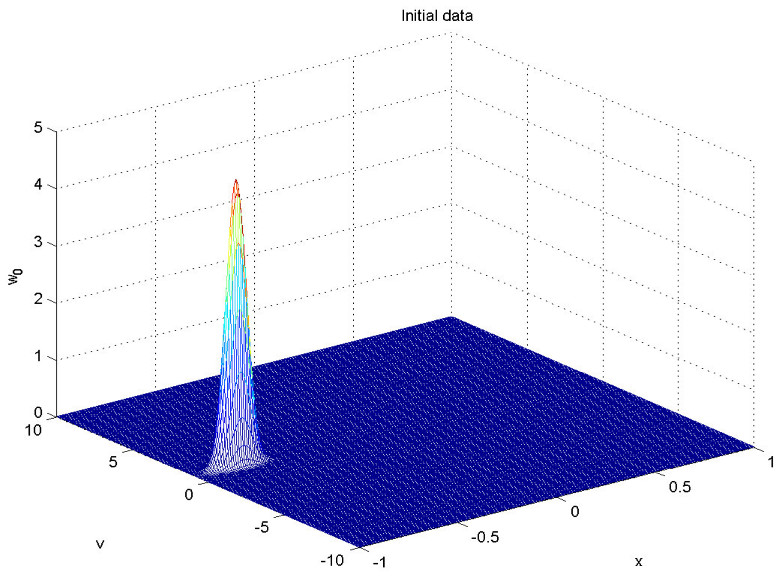

Figure 5.5: Initial Gaussian centered at $x = -0.78$ and $v = 1$

5.4 Numerical test. We consider a starting Gaussian, centered at $x = -0.78$ and $v = 1$ (Figure 5.5) and the following parameters:

- system parameters: $\beta = 0.04$, $\sigma = 0.1$, $\alpha = 0.03$.
- x-space: $x \in [-1, 1]$, $\Delta x = 0.01$, i.e. $N_x = 201$ grid points.
- v-space: $v \in [-10, 10]$, $\Delta v = 0.078125$, i.e. $M = 128$ grid points.
- time step $dt = 0.05$.

The Figures 5.6-5.7 give the time evolution of the initial data under the equation (5.3.1). The snapshots have been taken after 10, 30 and 50 time steps, respectively. Starting with a positive mean velocity, at $t = 0.5$ (Fig. 5.6) the initial Gaussian has been transported further into the considered domain. Moreover, the action of the diffusion operators in x and v are clearly illustrated.

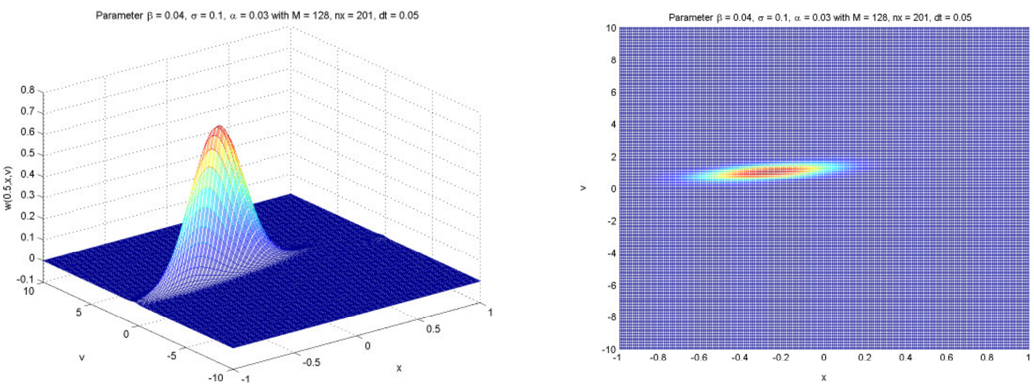

Figure 5.6: Time evolution of initial data in Fig. 5.5 under (5.3.1), at $t = 0.5$

The next pictures at $t = 1.5$ and $t = 2.5$ (Fig. 5.7) show the strong effect of diffusion, particularly in v, as well as the effect of using periodic boundary conditions in x. The outgoing parts of the wave at $x = 1$ reenter into the domain at $x = -1$.

The solution is converging (as $t \to \infty$) to the discrete steady state, whose Fourier coefficients are given by (cf. (5.1.20)-(5.2.5))

$$\hat{w}_\infty(j, k) = \frac{\Delta x}{2} \sum_{l=0}^{N_x - 1} \hat{w}_n(l, 0) \exp\left\{ -\frac{\sigma}{2\beta} \eta_k^2 \right\}, \qquad (5.3.2)$$

for all $k = -M(1)M$, $j = 0(1)N_x - 1$. The steady state is shown in Figure 5.8.

The time decay of the approximated solution to the steady state is illustrated in Figure 5.9a) in the discrete L^2-norm, and in Figure 5.9b) in terms of the

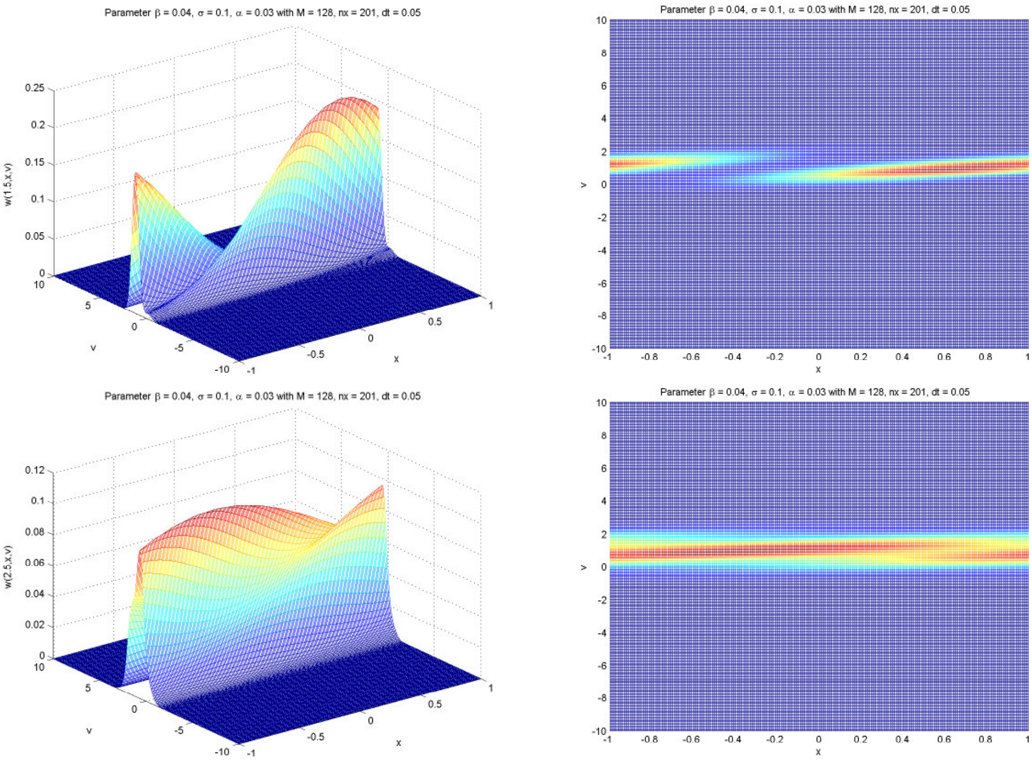

Figure 5.7: Time evolution of initial data in Fig. 5.5 under (5.3.1), at $t = 1.5$ and $t = 2.5$

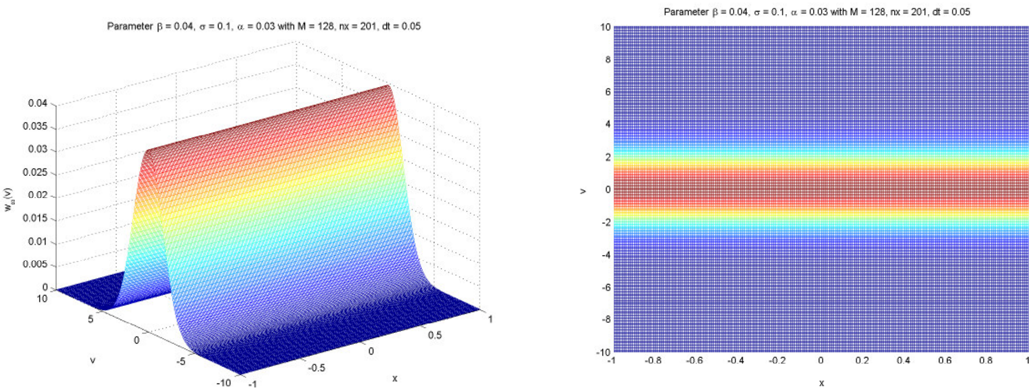

Figure 5.8: Steady state of the equation (5.3.1)

discrete quadratic relative entropy. The discrete quadratic relative entropy of w with respect to w_∞ is defined as the discrete version of (cf. [**AU**])

$$e\Big(\frac{w}{w_\infty}\Big) := \int_{-L}^{L}\int_{\mathbb{R}} \frac{\big(w(x,v,t) - w_\infty(x,v)\big)^2}{w_\infty(x,v)}\, dv\, dx. \tag{5.3.3}$$

The Figure 5.9b) clearly shows the expected exponential decay to the equilibrium of the approximated solution. After 3000 time steps effects of the phase-space discretization become visible.

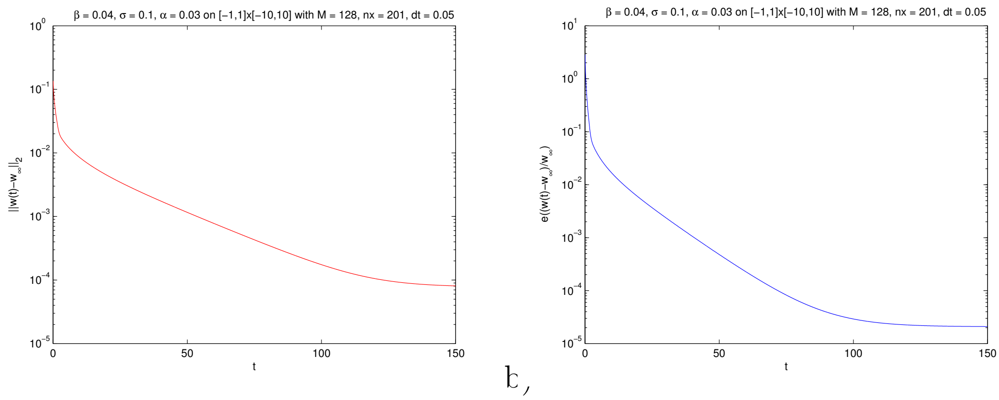

Figure 5.9: Time decay of the l^2-error (a), and the discrete quadratic relative entropy (b) w.r.t. to steady state (in logarithmic scale)

Starting again with the same initial data (cf. Fig. 5.5), Figure 5.10 gives an impression of the discrete L^2-error to the steady state after "long" time, and for different number of grid points M in v, as well as different refinements Δx of the x-discretization.

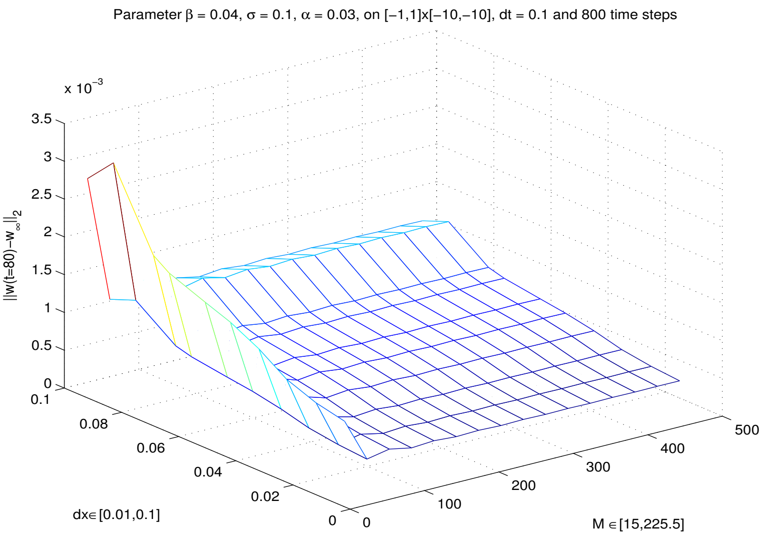

Figure 5.10: l^2-error to the steady state at $t = 80$, depending on M and Δx

5.4 The linear WFP equation with a bandgap potential

The aim of this section is to approximate the solution of the linear Wigner-Fokker-Planck equation

$$\partial_t w + v w_x \;=\; \beta(vw)_v + \sigma w_{vv} + \alpha w_{xx} - \Theta[V_B]w, \qquad t > 0, \tag{5.4.1}$$

with a given constant-in-time potential V_B. Convergence results of the semi-discretization in time of this equation via the splitting scheme (4.1.1)-(4.1.3) have been presented in Section 4.5.

The used algorithm consists in applying the two splitting steps (5.1.1) and (5.2.1), successively. Summarizing the detailed description of this steps, given in the Sections 5.1 and 5.2, yields the following algorithm for each time step $t_n = n\Delta t$, $n \in \mathbb{N}$.

5.5 Numerical algorithm. Let x_j and v_k be the phase-space discretization given in (5.1.5), (5.1.9). Further, let $w_n(j,k)$ be the approximated value of $w(x_j, v_k, t_n)$. Then, moving in time from t_n to $t_{n+1} = t_n + \Delta t$ we have to apply:

I: Algorithm 5.1 to calculate numerically $w_{n+\frac{1}{2}}$. Since the potential V_B is given, no Poisson-coupling (item 2. in Algorithm 5.1) is needed.

II: solve the linear system of equations arising from the finite difference scheme in (5.2.2) with the periodic boundary conditions (5.2.3)-(5.2.4), to calculate the new approximate solution w_{n+1}.

5.4.1 Single barrier potential

In the following test example we consider the tunneling of a wave packet, shown in Figure 5.11, again as a Gaussian distribution (centered at $x = -0.78$ and $v = 1$, and "mainly" localized in the $(x < 0, v > 0)$-quadrant), through a simple potential barrier. The barrier can be thought of as the difference in the bandgap for different materials (see [**Ri91**]). It is modeled here by a bandgap potential V_B. The ion–background D is chosen constant equal to zero. The Figures 5.12-5.14 show the time evolution of the initial wave packet with the following test parameters.

5.6 Numerical test.

- system parameters: $\beta = 0.04$, $\sigma = 0.1$, $\alpha = 0.03$.
- x-space: $x \in [-1, 1]$, $\Delta x = 0.01$, i.e. $N_x = 201$ grid points.

- v-space: $v \in [-5, 5]$, $\Delta v = 0.078125$, i.e. $M = 64$ grid points.
- time step $dt = 0.05$.
- single step, constant-in-time potential V_B on $[-0.2, 0.2]$ with height $= 20$.

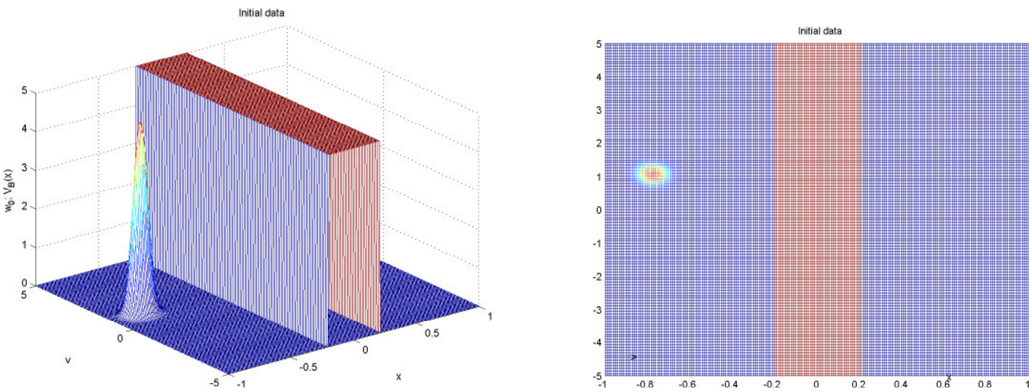

Figure 5.11: Initial Gaussian distribution and the single barrier potential with width = 0.4 and height = 20

Note that, for reasons of a better presentation, the height of the potential V_B is scaled to the maximum norm of the solution in each time step. Although the convergence analysis in Section 4.5 has been done for "smooth" potentials, we are using here a step potential for reasons of simplicity.

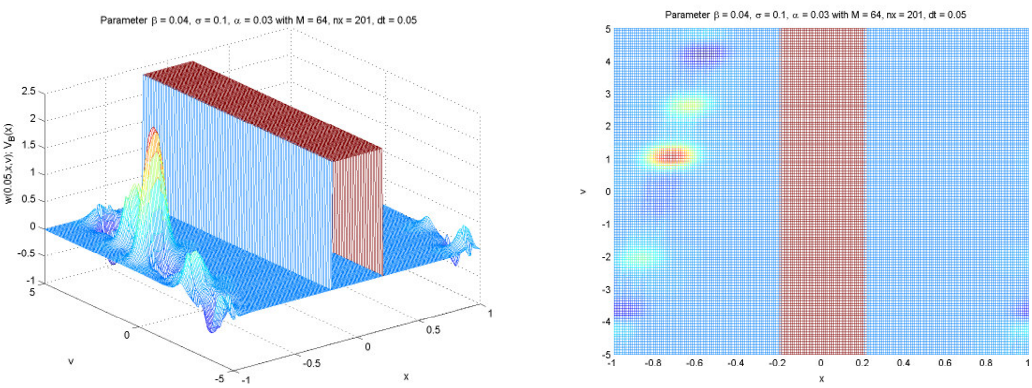

Figure 5.12: Time evolution of the Wigner function under the effect of a single barrier potential with width = 0.4 and height = 20, at $t = 0.05$

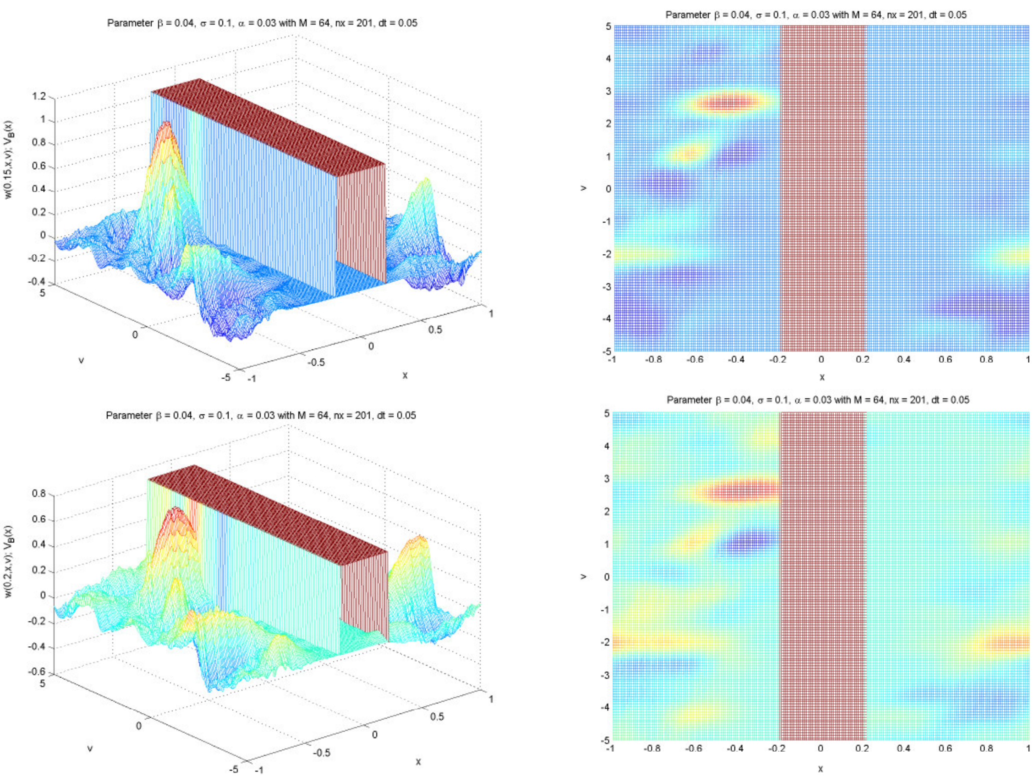

Figure 5.13: Time evolution of the Wigner function under the effect of a single barrier potential with width = 0.4 and height = 20, at $t = 0.15$, and $t = 0.2$

Due to the starting position ($v > 0$), the initial wave is subject to transport against the potential barrier. Already after one time step, at $t = 0.05$ (Fig. 5.12), it can be seen that parts of the wave with low kinetic energy (small v) have been reflected from the barrier and carried into the ($x < 0, v < 0$)-quadrant. In the next pictures at $t = 0.15$ and $t = 0.20$ (Fig. 5.13) we see that the parts of the wave with relatively high kinetic energy (big $v > 0$) are "struggling" to reach the barrier. The parts reflected into the ($v < 0$)-halfplane are transported out of the domain at $x = -1$. They reappear on the other side at $x = 1$, due to the periodic boundary conditions.

In the pictures of Figure 5.14 the part with the highest energy has tunneled through the potential barrier, loosing energy in the barrier area and gaining it back after the barrier, because of the conservation of the physical total energy. Due to periodic boundary conditions, wave packets on the ($x > 0$)-halfplane are moving against the barrier from the other side. At $t = 0.6$ one sees the onset of a concentration of mass around $v = 0$.

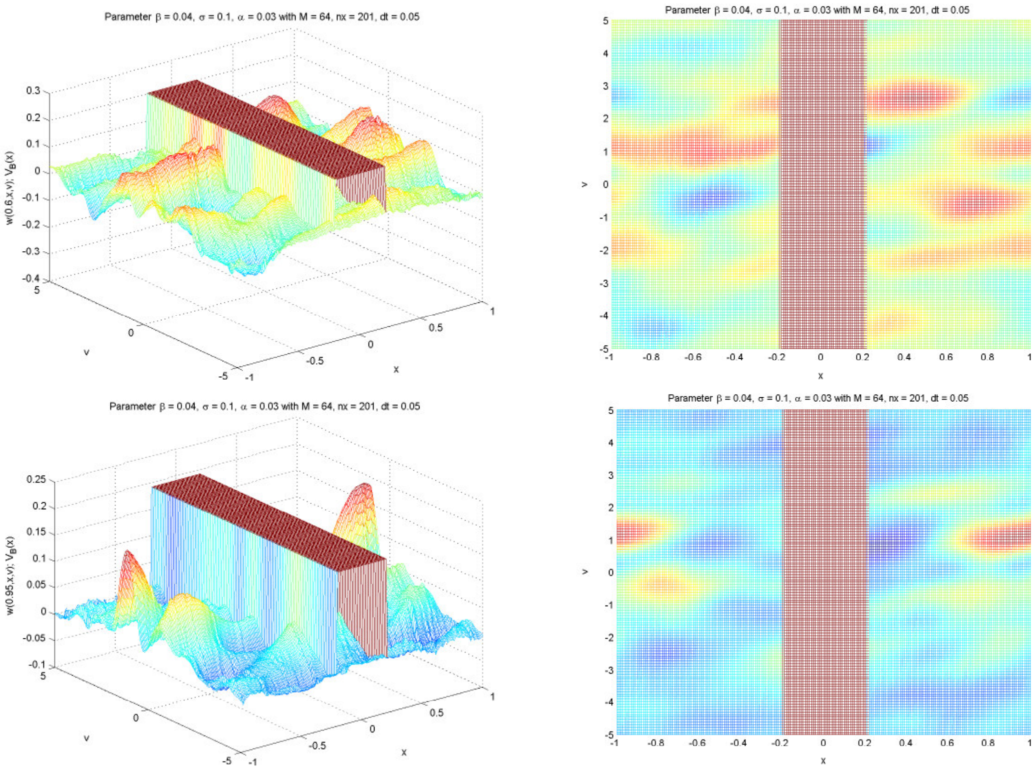

Figure 5.14: Time evolution of the Wigner function under the effect of a single barrier potential with width = 0.4 and height = 20, at $t = 0.6$ and $t = 0.95$

This evolution converges to the calculated numerical equilibrium, which is given approximately in Figure 5.15.

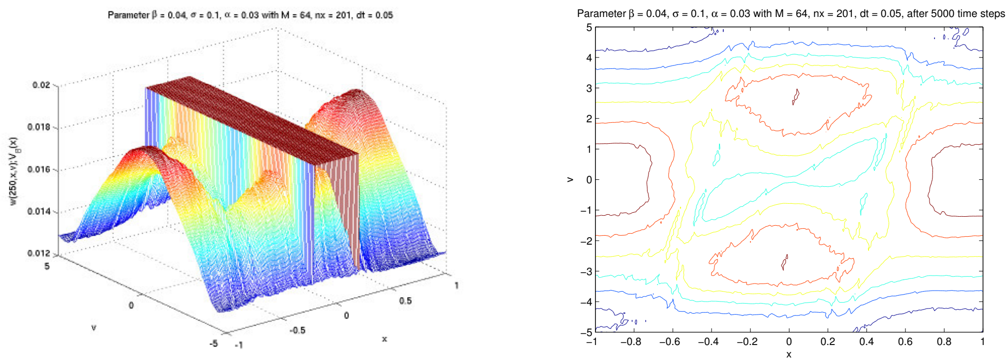

Figure 5.15: Wigner function under the effect of a single barrier potential with width = 0.4 and height = 20, after "long" time $t = 250$

For a better illustration of this process and, in particular of the numerical steady state, in Figure 5.16 the time evolution of the density $n[w]$, and its steady state are given.

Let us remark here that, although the L^2-convergence analysis in Section 4.5 does not ensure the definition of density of the Wigner function, the discrete definition over finite interval in v is possible.

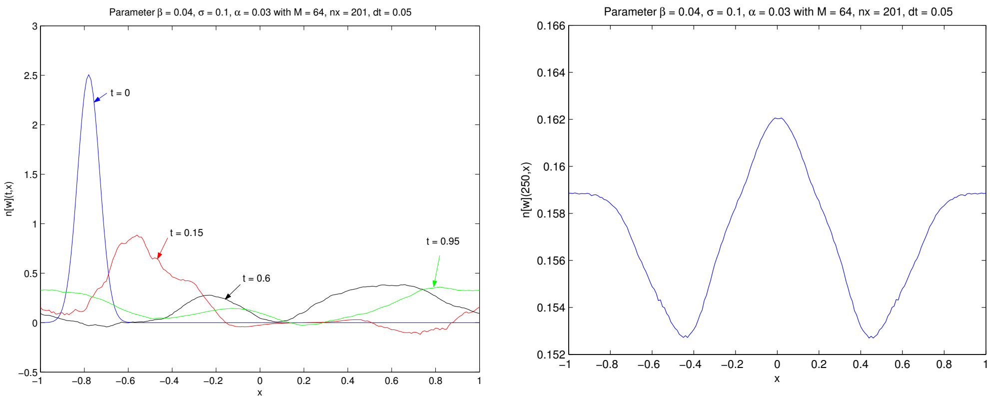

Figure 5.16: Time evolution of density of the Wigner function and its approximate steady state under the effect of a single barrier potential

5.4.2 The tunneling effect

In this subsection we consider the time evolution of the Wigner function under the barrier effect of a "very strong" potential. For the classical counterpart, the Vlasov(-Poisson) equation and/or Vlasov(-Poisson)-Fokker-Planck equation (cf. [**MRS, Sc**]), it is well-known that, when the potential height exceeds the kinetic energy of the wave packet, no tunneling is possible. Then, the wave packet is totally reflected (cf [**MRS**]) by the potential barrier.

The aim of the next numerical test is to illustrate, what happens when the potential is so strong that, in the classical case, we would not expect any barrier crossing of the wave packet. It shall turn out, that the wave parts with the highest kinetic energy will still tunnel through the barrier which is a typical quantum effect.

5.7 Numerical test.

- system parameters: $\beta = 0.04$, $\sigma = 0.1$, $\alpha = 0.03$.
- x-space: $x \in [-1, 1]$, $\Delta x = 0.01$, i.e. $N_x = 201$ grid points.
- v-space: $v \in [-5, 5]$, $\Delta v = 0.078125$, i.e. $M = 64$ grid points.

- time step $dt = 0.05$.
- single step, constant-in-time potential V_B on $[-0.2, 0.2]$ with height $= 200$.

The potential is chosen so high, that in the considered domain $v \in [-5, 5]$, the potential energy would be much higher then the kinetic energy. We use here the same initial data as in Figure 5.11. The next Figures 5.17-5.18 show the time evolution of the Wigner function under the effect of the given potential. Note here again that, for reasons of visualization, the height of the potential V_B is scaled to the maximum norm of the solution in each time step.

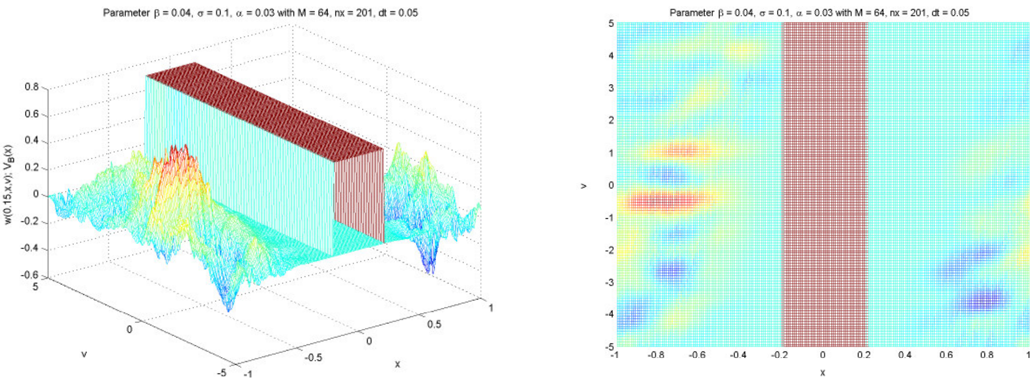

Figure 5.17: Wigner function under the effect of a very high (= 200) single barrier potential, at $t = 0.15$

Figure 5.17 shows, that already after three time steps (at $t = 0.15$), most of the initial wave has been reflected into the $(x < 0, v < 0)$-quadrant.

Three time steps later, at $t = 0.3$, we see that almost the entire wave has been reflected, carried out of the domain at $x = -1$ and entering back at $x = 1$, because of the periodic boundary conditions. But we also detect parts with big velocities which are moving to the potential barrier. The picture at $t = 0.95$ clearly shows that most of the wave packets are concentrated around $(-1, 0)$ and $(1, 0)$. This is due to the non-local repulsion of a quantum potential and the v-diffusion and -friction. The numerically calculated steady state is given in Figure 5.19, and its contour lines point up the quantum tunneling effect at big $|v|$-values.

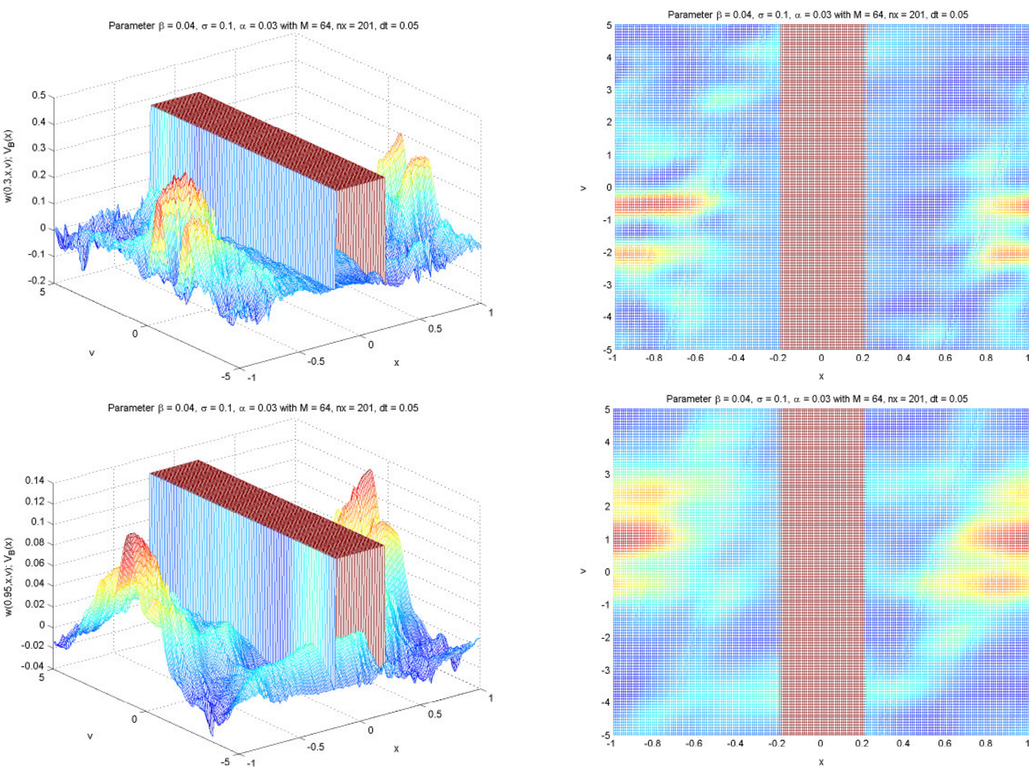

Figure 5.18: Wigner function under the effect of a very high (= 200) single barrier potential, at $t = 0.3$ and $t = 0.95$

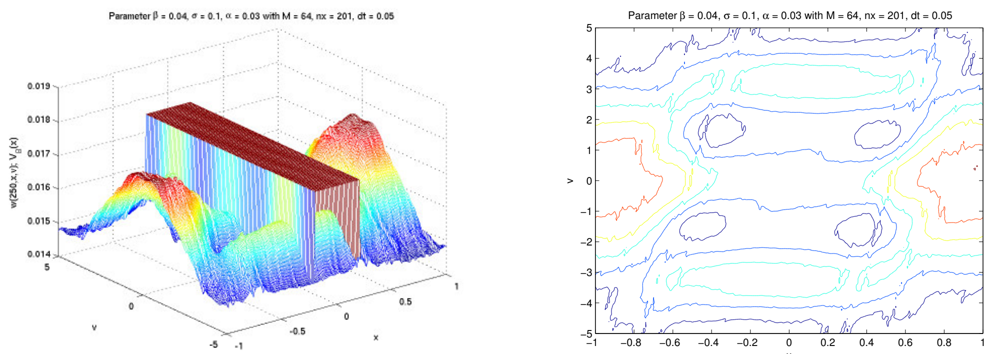

Figure 5.19: Wigner function and its contour lines after "long" time $t = 250$ (steady state)

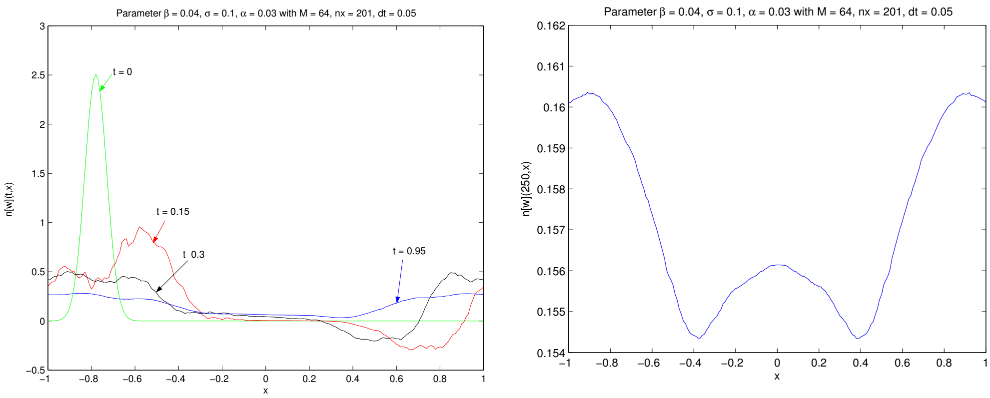

Figure 5.20: Time evolution of density of the Wigner function under the effect a very high (= 200) single barrier potential

5.5 A Wigner-Poisson-Fokker-Planck simulation

In this section we shall present the numerical approximation of the entire nonlinear WPFP system, coupled with the Poisson equation via the self-consistent potential. The used numerical method consists in applying the two splitting steps (5.1.1) and (5.2.1), successively. For each time step $t_n = n\Delta t$, $n \in \mathbb{N}$ the following algorithm yields.

5.8 Numerical algorithm. Let x_j and v_k be the phase-space discretization given in (5.1.5), (5.1.9). Further, let $w_n(j,k)$ be the approximated value of $w(x_j, v_k, t_n)$. Then, moving in time from t_n to $t_{n+1} = t_n + \Delta t$ we have to apply:

I: the complete Algorithm 5.1 to calculate $n[w_n]$, to solve the Poisson equation for $V[w_n]$, with homogeneous Dirichlet boundary conditions, by using the explicit difference scheme in (5.1.6), and then calculate numerically $w_{n+\frac{1}{2}}$ as done in detail in Section 5.1;

II: solve the linear system arising from the finite difference scheme in (5.2.2) with the periodic boundary conditions (5.2.3)-(5.2.4), to calculate the new approximate solution w_{n+1}.

The time evolution of the Wigner function under the effect of the self-consistent Poisson potential is simulated and shown in Figure 5.21.

As we can see from Figure 5.21 the effect of the self-consistent potential $V[w]$ is relatively modest, and the pictures are quite similar to those of Figures

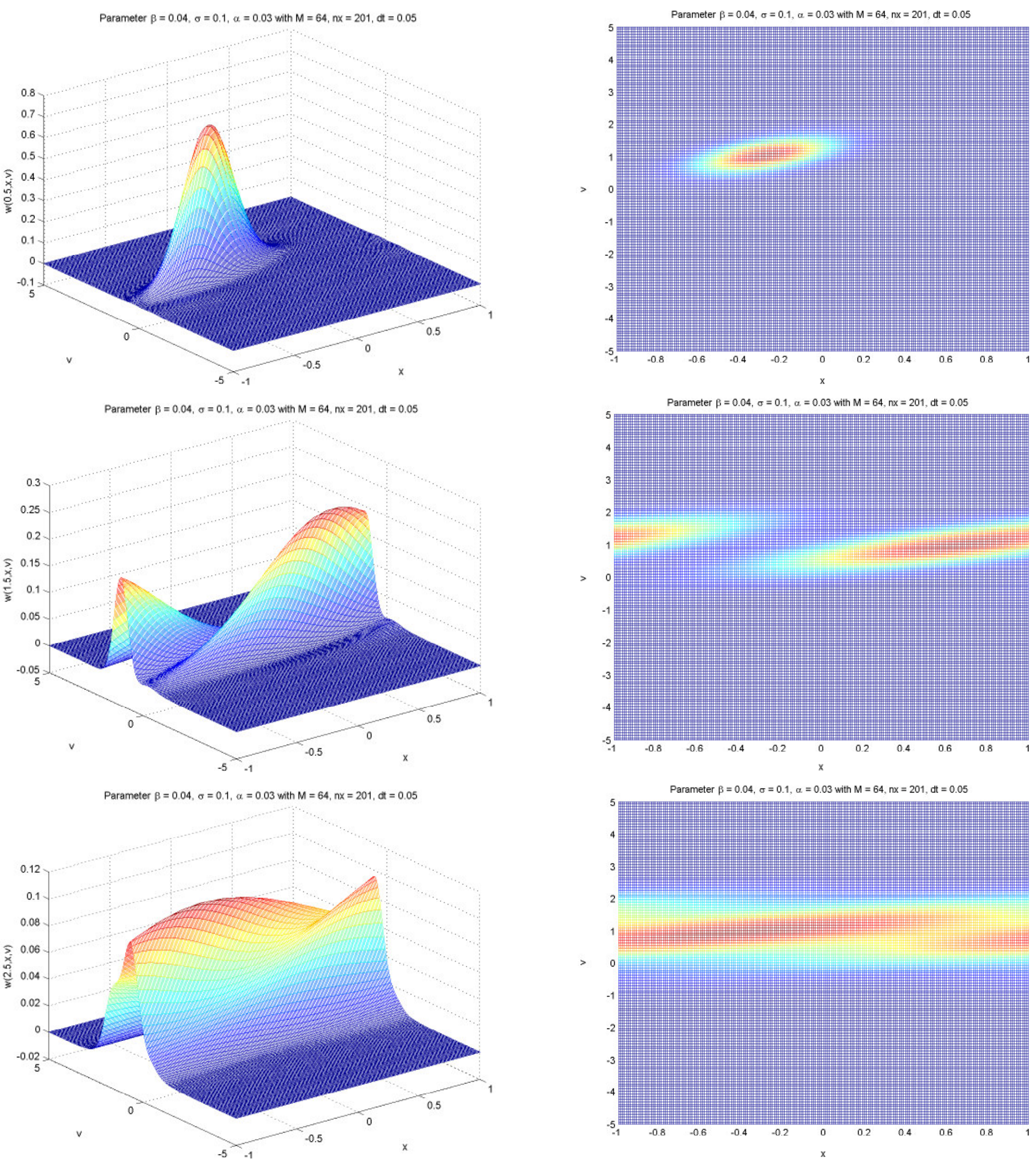

Figure 5.21: Time evolution of the Wigner function under the effect of Poisson potential, at $t = 0.5$, $t = 1.5$ and $t = 2.5$

5.6-5.7. In Figure 5.22 the numerically calculated steady state is presented. Its contour lines show a small detraction in v at around $x = 0$ in comparison with the steady state given in Figure 5.8.

The time evolution of the density $n[w]$ and the self-consistent potential $V[w]$ are illustrated in Figure 5.23, and their numerical steady states are given in Figure 5.24.

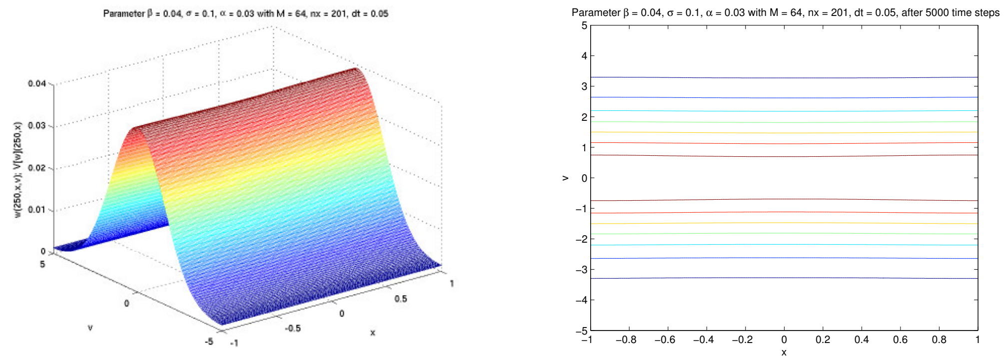

Figure 5.22: Wigner function and its contour lines after "long" time $t = 250$ (steady state)

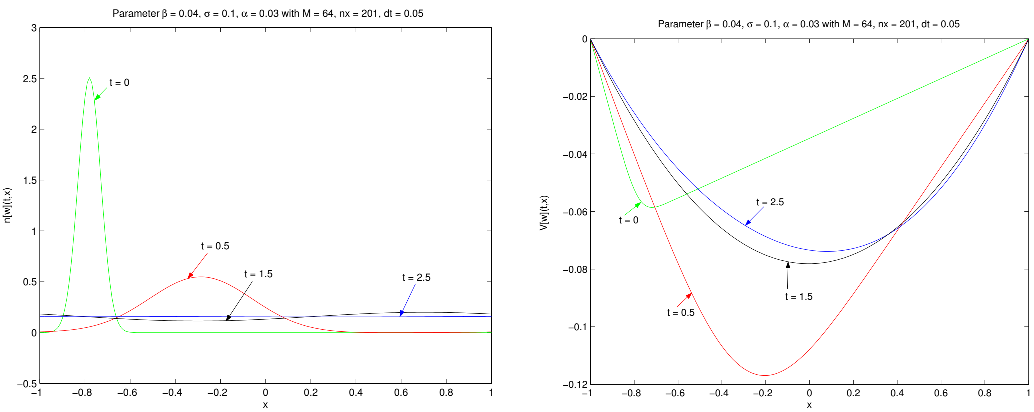

Figure 5.23: Time evolution of the density and self-consistent Poisson potential

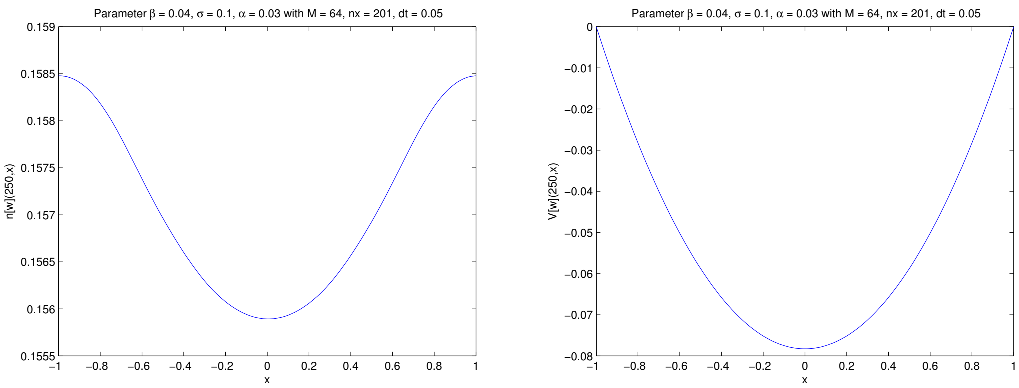

Figure 5.24: Density and self-consistent potential of the Wigner function after "long" time $t = 250$ (steady state)

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