Mathematik

The isomorphism problem for almost split Kac–Moody groups

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Abstract

The isomorphism problem for a given class of groups $\mathcal{C}$ asks to determine whether two groups $G, G' \in \mathcal{C}$ are isomorphic or not. In this thesis, we consider the class $\mathcal{C}$ of almost split Kac–Moody groups. These groups have been constructed by Rémy via Galois descent from split Kac–Moody groups as defined by Tits. We show that under certain technical assumptions, any isomorphism between groups in this class must preserve the canonical subgroup structure, i.e. the twin root datum associated to these groups, which generalizes results of Caprace in the split case.
This is achieved via the construction of maximal split subgroups inside almost split Kac–Moody groups and a detailed study of bounded subgroups, which generalizes results of Borel–Tits and Caprace.

Zusammenfassung

Das Isomorphie-Problem für eine gegebene Klasse von Gruppen $\mathcal{C}$ besteht darin zu entscheiden, ob zwei Gruppen $G, G' \in \mathcal{C}$ isomorph oder nicht isomorph sind. In der vorliegenden Arbeit betrachten wir die Klasse $\mathcal{C}$ der fast zerfallenden Kac–Moody-Gruppen. Diese Gruppen wurden von Rémy mittels Galois-Abstieg von zerfallenden Kac–Moody-Gruppen im Sinne von Tits konstruiert. Wir zeigen, dass unter gewissen technischen Voraussetzungen ein Isomorphismus zwischen zwei Gruppen dieser Klasse die kanonische Untergruppenstruktur, d.h. das zugehörige Zwillingswurzeldatum, erhält. Dieses verallgemeinert Resultate von Caprace im Fall von zerfallenden Kac–Moody-Gruppen.
Zu diesem Zweck konstruieren wir maximal zerfallende Untergruppen von fast zerfallenden Kac–Moody-Gruppen und untersuchen im Detail beschränkte Untergruppen. Dies verallgemeinert Resultate von Borel–Tits und von Caprace.

Introduction

There is a machine mathematicians call PSL which has two levers: with the first, one selects a natural number $n \geq 2$, and with the second a field k. For each choice of n and k, PSL produces a group $\mathrm{PSL}_n(k)$, and two different positions of the levers will usually give rise to two non-isomorphic groups. To be precise, for two natural numbers $n, n' \geq 2$ and two fields k, k', $\mathrm{PSL}_n(k)$ is isomorphic to $\mathrm{PSL}_{n'}(k')$ if and only if $n = n'$ and $k \cong k'$, except the two "accidental" isomorphims $\mathrm{PSL}_2(\mathbb{F}_4) \cong \mathrm{PSL}_2(\mathbb{F}_5)$ and $\mathrm{PSL}_2(\mathbb{F}_7) \cong \mathrm{PSL}_3(\mathbb{F}_2)$, cf. [Wil09, Chapter 3.3.5].
For a prime number p, the group $\mathrm{PSL}_n(\mathbb{F}_p)$ was constructed by Galois in 1830, while Jordan in 1870 constructed the groups $\mathrm{PSL}_n(\mathbb{F}_q)$, where q is a prime power. In modern language, these are finite groups of Lie type.
The following century witnessed spectacular developments in the emerging theories of Lie groups, their Lie algebras, and algebraic groups. An important theorem in this area is the classification of complex semisimple Lie algebras via their associated Cartan matrices. In 1968, Kac and Moody independently from one another generalized the notion of a complex semisimple Lie algebra to arrive at certain infinite-dimensional complex Lie algebras, which came to be known as Kac–Moody algebras.
These algebras can be integrated over fields of characteristic 0 to Kac–Moody groups. In 1987, Tits solved the difficult problem of defining Kac–Moody groups in arbitrary characteristic. In the seminal paper [Tit87] he associates to each Kac–Moody root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i \in I}, (h_i)_{i \in I})$ a functor $\mathcal{G}_\mathcal{D}$ from the category of commutative rings with 1 to the category of groups. When A is a classical Cartan matrix and k is a field, $\mathcal{G}_\mathcal{D}(k)$ coincides with the k-rational points of the split reductive k-group which has Λ as the character group of a maximal torus and the c_i resp. h_i as the associated roots resp. coroots. When A is no longer classical, $\mathcal{G}_\mathcal{D}(k)$ can be thought of as an "infinite-dimensional split reductive group".
The following classical fact led Rémy [Rém02] to the construction of almost split Kac–Moody groups: An arbitrary reductive algebraic group G defined over a field k splits over a finite Galois extension E of k, and G can be recovered from the split form via Galois descent.
Very roughly, in Rémy's theory the k-rational points of an almost split Kac–Moody group $G(k)$ are obtained by taking the fixed points of a suitable action ρ of $\mathrm{Gal}(E|k)$ on a split Kac–Moody group $\mathcal{G}_\mathcal{D}(E)$, where E is a separable extension of k.

In analogy with PSL the following natural question arises:

> Let G, G' be two almost split Kac–Moody groups which arise from parameter sets $\mathcal{P} = (\mathcal{D}, E|k, \rho)$ and $\mathcal{P}' = (\mathcal{D}', E'|k', \rho')$ and suppose that

G, G' are isomorphic as abstract groups. Do the parameter sets $\mathcal{P}$ and $\mathcal{P}'$ necessarily coincide?

The corresponding question for split Kac–Moody groups over arbitrary fields was settled by Caprace in 2005 ([Cap09]) after previous work by Kac-Peterson ([KP85]) and Caprace-Mühlherr ([CM05], [CM06]).
In the setting of almost split Kac–Moody groups, we develop some new tools which together with methods used in the split case lead to the answer of the question, i.e. the solution of the isomorphism problem for 2-spherical almost split Kac–Moody groups over fields of characteristic 0.

Main results
The key for all developments in this work is to compare (almost) split Kac–Moody groups with isotropic reductive algebraic groups. A first example in this direction is the following observation.

Proposition 1 (Restriction of scalars for Kac–Moody groups)**.** *Let k be a field and let E be a finite Galois extension of k. Let $\mathcal{G}_\mathcal{D}$ be a constructive Tits functor. Then there is a quasi-split Kac–Moody group $\mathcal{G}'$ such that $\mathcal{G}_\mathcal{D}(E) \cong \mathcal{G}'(k)$.*

This already shows that a group isomorphism in general will not preserve the full parameter set used to define an almost split Kac–Moody group.

Let k be a field and let E be a separable extension of k of degree 2. Let $h\colon E^3 \to E$ be a Hermitian form of Witt index 1 with associated unitary group SU_3, which can be thought of as an algebraic group defined over k. An important observation is that there is an inclusion of groups

$$\mathrm{SL}_2(k[t,t^{-1}]) \leq \mathrm{SU}_3(k[t,t^{-1}]) \leq \mathrm{SL}_3(k[t,t^{-1}])$$

where the groups on the left and on the right are split Kac–Moody groups, while the group in the middle is an almost split Kac–Moody group. In the setting of reductive algebraic groups, it is a classical theorem of Borel–Tits [BT65] that for a connected reductive algebraic k-group G with maximal k-split torus S, there is a connected reductive algebraic k-group $F \leq G$ which is split over k, contains S as a maximal torus and has the same Weyl group as G.
We can generalize this to groups endowed with a 2-spherical twin root datum (see Theorem 4.5 for a precise statement). As a corollary, we get the following generalization of the Borel–Tits theorem for almost split Kac–Moody groups.

Theorem 1. *Let k be an infinite field and let $G(k)$ be a 2-spherical almost split Kac–Moody group with maximal k-split torus $T_d(k)$. Then there is a subgroup $F \leq G(k)$ endowed with a locally split twin root datum which contains $T_d(k)$ and intersects each root group V_α non-trivially.*

Actually, a more precise statement holds (see Theorem 4.13) which shows the abundance of locally split subgroups sharing $T_d(k)$ as a maximal torus. It is this refined version that is needed in the solution of the isomorphism problem for almost split Kac–Moody groups.

An almost split Kac–Moody group $G(k)$ is generated by its anisotropic kernel $Z(k)$ and root subgroups V_α relative to a maximal split torus $T_d(k)$. For each $u \in V_\alpha$, there is a homomorphism $\psi_u\colon \operatorname{SL}_2(k) \to G(k)$ such that $u \in \operatorname{im}\psi_u$. If $\varphi\colon G(k) \to G'(k')$ is an isomorphism of two almost split Kac–Moody groups, this gives a representation

$$\varphi \circ \psi_u\colon \operatorname{SL}_2(k) \to G'(k').$$

A subgroup $H \leq G'(k')$ is called bounded if its action on both halves of the twin building associated to $G'(k')$ has bounded orbits. The importance of the notion of bounded subgroups comes from the fact that these are (central extensions of) rational points of algebraic groups.
An essential step in Caprace's solution of the isomorphism problem for split Kac–Moody groups was that for $k = \mathbb{Q}$, every homomorphism $\psi\colon \operatorname{SL}_2(k) \to G'(k')$ has bounded image. A natural and important question is to determine for which other fields k any representation $\psi\colon \operatorname{SL}_2(k) \to G'(k')$ has bounded image, since by the above remarks an isomorphism $\varphi\colon G(k) \to G'(k')$ induces lots of these representations, and whenever $\operatorname{im}\psi$ is bounded, the well-developed theory of algebraic groups can be used to study the isomorphism problem.
In this direction, we have the following theorem.

Theorem 2. *Let k, k' be two fields of characteristic 0, let $G'(k')$ be an almost split Kac–Moody group and let $\psi\colon \operatorname{SL}_2(k) \to G'(k')$ be an abstract homomorphism.*

(i) If k is a (possibly infinite) algebraic extension of $\mathbb{Q}$, i.e. $\operatorname{tr.deg}(k|\mathbb{Q}) = 0$, then $\operatorname{im}\psi$ is bounded.

(ii) If k is arbitrary and there is a twin apartment $\mathcal{A}'$ stabilized by $\psi(T)$, where T denotes the diagonal matrices, and such that $\psi(\operatorname{SL}_2(\mathbb{Q}))$ fixes two opposite points of $\mathcal{A}'$, then $\operatorname{im}\psi$ is bounded.

The second assertion can be used to give a different proof of Caprace's results for split Kac–Moody groups over fields of characteristic 0.

On the other hand we construct representations of algebraic groups with unbounded image, see Theorem 5.11:

Theorem 3. *Let k be a field with infinite transcendence degree over its prime field. Let $\mathbf{K}$ be a k-isotropic reductive k-group and let $n := \dim \mathbf{K}$. Let G be a split Kac–Moody group which has a Levi factor isomorphic to $\operatorname{SL}_{n+1}(k[t, t^{-1}])$. Then there is a homomorphism $\varphi\colon \mathbf{K}(k) \to G(k)$ with unbounded image.*

The knowledge about maximal split subgroups and the study of bounded subgroups is used to prove the main theorem:

Theorem 4 (Main theorem)**.** *Let k, k' be two fields of characteristic 0 and let $G(k), G'(k')$ be two 2-spherical almost split Kac–Moody groups obtained by Galois descent. Let $(Z(k), (U_\alpha(k))_{\alpha\in\Phi(W,S)})$ and $(Z'(k'), (V_\beta(k'))_{\beta\in\Phi(W',S')})$ denote the canonical twin root data associated to $G(k)$ and $G'(k')$.*
Then any isomorphism $\varphi\colon G(k) \to G'(k')$ is **standard***, i.e. there is some $x \in G'(k')$ and a bijection $i\colon \Phi(W,S) \to \Phi(W',S')$ such that $\varphi' := \operatorname{int} x \circ \varphi$ satisfies*

(i) $\varphi'(Z(k)) = Z'(k')$

(ii) $\varphi'(U_\alpha(k)) = V_{i(\alpha)}(k')$.

We discuss how this theorem can be used to describe the automorphism group of an almost split Kac–Moody group $G(k)$.
We end with indications how the methods used here can be used to tackle the isomorphism problem in positive characteristic. In particular, we expect the same conclusion to hold whenever $\operatorname{char} k = \operatorname{char} k' \geq 5$.

Organisation of the text
In the first two chapters which are strictly of expository nature, we review the theory of split Kac–Moody groups in the sense of Tits and Rémy's construction of almost split Kac–Moody groups. Chapter three contains some observations about split and almost split Kac–Moody groups. Chapter four studies maximal split subgroups of almost split Kac–Moody groups. In chapter five we discuss conditions under which an abstract representation of an algebraic group into a Kac–Moody group has bounded image. In chapter six, we give the solution of the isomorphism problem for 2-spherical almost split Kac–Moody groups in characteristic 0.

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Contents

1 Review of split Kac–Moody groups

In this chapter, we recall the fundamental notions associated to Tits's construction of Kac–Moody groups.

References. A good introduction to the subject are the two surveys by Rémy [Rém04] and Caprace–Rémy [CR09a]. Tits's original papers [Tit85], [Tit87], [Tit89] and [Tit92] remain *the* reference.
Nice expositions of the basic material covered here can be found in Abramenko's book [Abr96], Caprace's thesis [Cap09] and Chosson's thesis [Cho00]. For general building theory, we refer to [AB08]. Finally, Rémy's thesis [Rém02] gives a very detailed account of the theory.

1.1 Kac–Moody algebras

Let I be a finite index set, $n := |I|$ and let $A = (a_{ij})_{i,j\in I} \in \mathbb{Z}^{n\times n}$ be a **generalized Cartan matrix**, i.e. $a_{ii} = 2$ for all $i \in I$, $a_{ij} \leq 0$ for $i \neq j$ and $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$.
Let Λ be a free $\mathbb{Z}$-module of finite rank and denote by $\Lambda^\vee := \mathrm{Hom}(\Lambda, \mathbb{Z})$ its dual. For $i \in I$, let $c_i \in \Lambda$ and $h_i \in \Lambda^\vee$ be such that $h_i(c_j) = a_{ij}$. Then $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ is called a **Kac–Moody root datum.**

The set $\Pi := \{c_i : i \in I\}$ is called the **base** and the set $\Pi^\vee := \{h_i : i \in I\}$ the **cobase** of the root datum $\mathcal{D}$.

Let A be a generalized Cartan matrix. Two Kac–Moody root data involving A are given by the following two examples.
The **simply connected root datum** $\mathcal{D}^A_{sc}$ associated to A is given by $\Lambda := \bigoplus_{i\in I} \mathbb{Z}e_i$, $c_i := \sum_{j\in I} a_{ji}e_j$ and $h_i := e_i^\vee$, where $(e_i^\vee)_{i\in I}$ is the dual basis of $(e_i)_{i\in I}$.
The **minimal adjoint root datum** $\mathcal{D}^A_{min}$ is given by $\Lambda := \bigoplus_{i\in I} \mathbb{Z}e_i$, $c_i := e_i$ and $h_i := \sum_{j\in I} a_{ij}e_i^\vee$.
In general, though, neither will the family $(c_i)_{i\in I}$ be free nor generate Λ. Since for a root datum $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ its **dual** $\mathcal{D}^t := (I, A^t, \Lambda^\vee, (h_i)_{i\in I}, (c_i)_{i\in I})$ is again a root datum, a similar statement holds for the family $(h_i)_{i\in I}$.

Let K be a field of characteristic 0 and let $\mathcal{D}$ be a Kac–Moody root datum. The **Kac–Moody algebra** $\mathfrak{g} = \mathfrak{g}_\mathcal{D}$ of type $\mathcal{D}$ over K is the Lie algebra generated by $\mathfrak{g}_0 := \Lambda^\vee \otimes_\mathbb{Z} K$ and the symbols e_i, f_i $(i = 1, \ldots, n)$ subject to the following relations:

$$[h, e_i] = h(c_i)e_i,\ [h, f_i] = -h(c_i)f_i \ \text{ for } h \in \mathfrak{g}_0, \quad [\mathfrak{g}_0, \mathfrak{g}_0] = 0,$$

$$[e_i, f_i] = -h_i \otimes 1, \; [e_i, f_j] = 0 \;\text{ for } i \neq j,$$
$$(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = (\operatorname{ad} f_i)^{-a_{ij}+1} f_j = 0.$$

The universal enveloping algebra. Let $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$ denote the universal enveloping algebra of $\mathfrak{g}_{\mathcal{D}}$. Let $Q := \mathbb{Z}^n$ with standard basis vectors v_i. Then there is a well-defined Q-grading of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$ by setting $\deg h := 0$ for all $h \in \mathfrak{g}_0$, $\deg e_i := -\deg f_i := v_i$ and extending this. This means that there is a family of subspaces $(V_a)_{a \in Q}$ of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$ such that $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}} = \bigoplus_{a \in Q} V_a$ and for $v_a \in V_a, v_b \in V_b$, $[v_a, v_b] \in V_{a+b}$. As $\mathfrak{g}_{\mathcal{D}}$ can be identified with a subalgebra of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$, there is an induced grading $\mathfrak{g}_{\mathcal{D}} = \bigoplus_{a \in Q} \mathfrak{g}_a$. If a is such that $\mathfrak{g}_a \neq 0$, a is called a **root** and $\mathfrak{g}_a$ a nontrivial **root space**.

For $u \in \mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$ let $u^{[n]} := \frac{1}{n!} u^n$ and $\binom{u}{n} := \frac{1}{n!} u(u-1)\cdots(u-n+1)$.
Let $\mathcal{U}_0$ denote the subring of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$ generated by all elements $\binom{h}{n}$, where $h \in \Lambda^\vee$ and $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$ let $\mathcal{U}_i$ resp. $\mathcal{U}_{-i}$ be the subring $\sum_{n \in \mathbb{N}} \mathbb{Z} e_i^{[n]}$ resp. $\sum_{n \in \mathbb{N}} \mathbb{Z} f_i^{[n]}$.
Let $\mathcal{U}_{\mathcal{D}}$ be the subring generated by $\mathcal{U}_0$ and $\mathcal{U}_i, \mathcal{U}_{-i}$ $(i = 1, \ldots, n)$.
It can be shown that $\mathcal{U}_{\mathcal{D}}$ is a $\mathbb{Z}$-form of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$, i.e. the canonical map

$$\mathcal{U}_{\mathcal{D}} \otimes_{\mathbb{Z}} K \to \mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$$

is bijective.
For a subring A of $\mathcal{U}_{\mathfrak{g}_{\mathcal{D}}}$ and a ring R, we set $A_R := A \otimes_{\mathbb{Z}} R$. Then A_R inherits a grading. For $M \subseteq (\mathcal{U}_{\mathcal{D}})_R$, the **support** of M is the set of degrees which appear when decomposing elements of M into their homogeneous components.

The Weyl group. From the last two sets of defining relations of $\mathfrak{g}_{\mathcal{D}}$ it follows that $\operatorname{ad} e_i, \operatorname{ad} f_i$ are locally nilpotent derivations of $\mathfrak{g}$. Then $\exp \operatorname{ad} e_i, \exp \operatorname{ad} f_i$ are well-defined automorphisms of $\mathfrak{g}$. Let

$$s_i^* := \exp \operatorname{ad} e_i \cdot \exp \operatorname{ad} f_i \cdot \exp \operatorname{ad} e_i$$

and let $W^* := \langle s_i^* : i \in I \rangle \leq \operatorname{Aut}(\mathfrak{g})$.
The **Weyl group** of the generalized Cartan matrix A is defined as

$$W := W_A := \langle (s_i)_{i \in I} : (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $m_{ii} := 1$ and for $i \neq j$, $m_{ij} := 2, 3, 4, 6$ or ∞ according to whether $a_{ij} a_{ji} = 0, 1, 2, 3$ or ≥ 4. The group W_A acts on $Q = \mathbb{Z}^n$ via $s_i(v_j) := v_j - a_{ij} v_i$.
The connection between W^* and W is as follows: It can be shown that the assignment $s_i^* \mapsto s_i$ extends to a well-defined surjective homomorphism $\pi : W^* \to W$. The action of W^* permutes the root spaces of $\mathfrak{g}_{\mathcal{D}}$, more precisely, we have $w^* \mathfrak{g}_a = \mathfrak{g}_{\pi(w^*)a}$.
A root a such that $\mathfrak{g}_a = w^* \mathfrak{g}_{\pm v_i}$ is called a **real root**. The set of all real roots is denoted by Δ^{re}.

The roots of a Coxeter group. The real roots can be identified with the set of roots $\Phi(W, S)$ of the Coxeter group W. Recall that for a Coxeter system (W, S) with associated length function l, the set of roots $\Phi(W, S)$ is defined as

$$\Phi(W, S) := \{ w \alpha_s : w \in W, s \in S \}$$

where $\alpha_s := \{w \in W : l(sw) > l(w)\}$ is a **simple root**. For a root $\alpha = w\alpha_s$ set $-\alpha := ws\alpha_s$, then $\alpha \cup -\alpha = W$.
With these definitions, $\Phi(W,S)$ is the disjoint union of positive and negative roots, $\Phi(W,S) = \Phi^+ \cup \Phi^-$, where $\Phi^+ = \{\alpha \in \Phi(W,S) : 1 \in \alpha\}$ and $\Phi^- = \{-\alpha : \alpha \in \Phi^+\}$. Write $\alpha > 0$ (resp. $\alpha < 0$) if α is a positive (resp. negative) root.

Let $\Psi \subseteq \Phi(W,S)$ be a set of roots. Ψ is called **prenilpotent** if there are elements $w, w' \in W$ such that $w \cdot \Psi \subseteq \Phi^+$ and $w' \cdot \Psi \subseteq \Phi^-$. For a prenilpotent pair of roots $\{\alpha, \beta\}$ the **closed root interval** $[\alpha, \beta]$ is defined as

$$[\alpha, \beta] := \{\gamma \in \Phi : \gamma \supset \alpha \cap \beta \text{ and } -\gamma \supset (-\alpha) \cap (-\beta)\}.$$

One sets $[\alpha, \beta) := [\alpha, \beta] \backslash \{\beta\}, (\alpha, \beta] := [\alpha, \beta] \backslash \{\alpha\}$ and $(\alpha, \beta) := [\alpha, \beta] \backslash \{\alpha, \beta\}$.
The set Ψ is called **closed** if for each prenilpotent pair of roots $\{\alpha, \beta\} \subseteq \Psi$, the closed interval $[\alpha, \beta]$ is contained in Ψ. A prenilpotent set of roots Ψ is called **nilpotent** if it is prenilpotent and closed.

Note that the terminology stems from the following fact: If Ψ is closed, the subspace $\mathfrak{g}_\Psi := \bigoplus_{a \in \Psi} \mathfrak{g}_a$ actually is a Lie subalgebra, which is furthermore nilpotent if Ψ is nilpotent.

1.2 The constructive Tits functor

To each Kac–Moody root datum $\mathcal{D}$, Tits associates a functor $\mathcal{G}_\mathcal{D}$ from the category of commutative rings with 1 to the category of groups. The value $\mathcal{G}_\mathcal{D}(K)$ of $\mathcal{G}_\mathcal{D}$ on a field K is called a **split Kac–Moody group**. We recall here the definition of this so-called constructive Tits functor.

The commutator formula. For $K = \mathbb{C}$ and $\Psi \subseteq \Delta^{re} = \Phi(W,S)$ a nilpotent set of roots, let $U_\Psi(\mathbb{C})$ denote the complex unipotent algebraic group with $\operatorname{Lie} U_\Psi = \mathfrak{g}_\Psi$. It can be shown that inside this group there is a generalization of the classical Chevalley commutator formula. In particular, let $\{a, b\}$ be a prenilpotent pair of roots and let $[a, b]$ be the associated closed root interval. Then any root $c \in [a, b]$, viewed as a real root, is a linear combination of a and b. For a single root c, $U_{\{c\}}$ is isomorphic to $(\mathbb{C}, +)$. Let $x_c : \mathbb{C} \to U_c$ be a fixed isomorphism. There are certain $r_{abc} \in \mathbb{Z}$ such that for all $r, s \in \mathbb{C}$ there is a commutator formula

$$(R_{ab}) \qquad [x_a(r), x_b(s)] = \prod_{c = ia + jb} x_c(r_{abc} r^i s^j).$$

In [Tit87] this is proved by appealing to the $\mathbb{Z}$-form of the universal enveloping algebra. By the work of Morita [Mor88], these constants can be computed in a more down-to-earth manner.

In the following, let $\mathcal{D}$ be a Kac–Moody root datum and let $\mathfrak{g}_\mathcal{D}$ be the associated Kac–Moody algebra. Let R be a commutative ring with 1.

The Steinberg functor. Let $\mathrm{St}_{\mathcal{D}}(R)$ be the quotient of the free group generated by the symbols $x_a(r)$ (where a is a real root and $r \in R$) by the relations $x_a(r)x_a(s) = x_a(r+s)$ and for each prenilpotent pair of roots $\{a, b\}$ the commutator relation (R_{ab}).

The split torus scheme. Let $T(R) := \mathrm{Hom}(\Lambda, R^\times)$ where $R^\times$ is the group of units of R. Clearly, $T(R) \cong (R^\times)^k$ as an abstract group, where k is the rank of Λ.
Let $r \in R^\times$ and $h \in \Lambda^\vee$. Then $r^h \colon \lambda \mapsto r^{h(\lambda)}$ is a well-defined element of $T(R)$. (This is why h is called a coroot: to each $r \in R^\times, h$ associates the element $r^h \in T(R)$.)

Construction 1.1. The **constructive Tits functor** $\mathcal{G}_{\mathcal{D}}$ associated to $\mathcal{D}$ is the functor which assigns to each commutative ring R with 1 the group $\mathcal{G}_{\mathcal{D}}(R)$ which is the quotient of $\mathrm{St}_{\mathcal{D}}(R) * T(R)$ by the following sets of relations (R_1) to (R_4):

For each simple root $a_i \in \Phi$, for each $r \in R$ and for each $t \in T = T(R)$,

$$(R_1) \qquad t x_{a_i}(r) t^{-1} = x_{a_i}(t(c_i) \cdot r)$$

This says that the torus normalizes each simple root group and acts on it via the character c_i.
For $r \in R^\times$ let $\tilde{s}_i(r) := x_{a_i}(r) x_{-a_i}(r^{-1}) x_{a_i}(r)$ and let $\tilde{s}_i := \tilde{s}_i(1)$. The action of the Weyl group W on Λ given by $s_i(t) := \lambda - h_i(\lambda) c_i$ gives rise to a W-action on T via $s_i(t)(\lambda) := t(s_i \cdot \lambda)$.
For each $i \in I$, each $r \in R^\times$ and each $t \in T$,

$$(R_2) \qquad \tilde{s}_i(r) t \tilde{s}_i(r)^{-1} = s_i(t)$$

This implies that reflections normalize the torus and act on it as elements of the Weyl group would.
For each $i \in I$ and each $r \in R^\times$,

$$(R_3) \qquad \tilde{s}_i(r^{-1}) = \tilde{s}_i r^{h_i}$$

This means that two lifts of the same reflection differ by a value of the coroot h_i.
For each $i \in I$, each $a \in \Phi$ and each $r \in R$,

$$(R_4) \qquad \tilde{s}_i x_a(r) \tilde{s}_i^{-1} = x_{s_i \cdot a}(r)$$

This says that the standard reflections will permute the root groups just as the corresponding lifts in the Weyl group W would permute the roots. □

The whole point of defining this group functor is that $G(\mathbb{C}) := \mathcal{G}_{\mathcal{D}}(\mathbb{C})$ can be thought of (via the adjoint representation) as (a central extension of) a group of automorphisms of the Kac–Moody algebra $\mathfrak{g}_{\mathcal{D}}$ over $\mathbb{C}$.
In particular, there is a presentation of $G(\mathbb{C})$ in terms of generators and relations.

Moreover, this definition is functorial and defined over $\mathbb{Z}$, which e.g. allows to evaluate $\mathcal{G}_\mathcal{D}$ on fields of positive characteristic.

A submatrix A_J of a generalized Cartan matrix A gives rise to a subfunctor of a constructive Tits functor.

Definition 1.2. *Let $A = (a_{ij})_{i,j\in I}$ be a generalized Cartan matrix and let $\mathcal{D}^{sc}_A$ denote the simply connected Kac–Moody root datum. For $J \subseteq I$ let $A_J := (a_{ij})_{i,j\in J}$ and let $\mathcal{D}^{sc}_{A_J}$ denote the corresponding simply connected root datum. The corresponding Tits functor $\mathcal{G}_{\mathcal{D}^{sc}_{A_J}}$ is called a* **subfunctor** *of $\mathcal{G}_{\mathcal{D}^{sc}_A}$.*

It can be checked that $\mathcal{G}_{\mathcal{D}_{A^{sc}_J}}$ actually is a subfunctor in the sense of category theory, i.e. for each ring there is an inclusion $\mathcal{G}_{\mathcal{D}^{sc}_{A_J}}(R) \hookrightarrow \mathcal{G}_{\mathcal{D}^{sc}_A}(R)$, cf. [CER08, Section 5]. Sometimes $\mathcal{G}_{\mathcal{D}^{sc}_{A_J}}(k)$ is called a **standard Kac–Moody subgroup** of $\mathcal{G}_{\mathcal{D}^{sc}_A}(k)$.

1.3 The adjoint representation

From the definition it is not obvious that $\mathcal{G}_\mathcal{D}(k)$ is not trivial. However, for each ring R there is an adjoint representation of $\mathcal{G}_\mathcal{D}(R)$ which generalizes the adjoint representation of $G(\mathbb{C})$, from which the non-triviality follows.
For each ring R, let $\mathrm{Aut}_{filt}(\mathcal{U}_\mathcal{D})_R$ denote the group of R-automorphisms of the R-algebra $\mathcal{U}_\mathcal{D} \otimes_\mathbb{Z} R$ which preserve the filtration (or grading) of $(\mathcal{U}_\mathcal{D})_R$ inherited from $\mathcal{U}_\mathcal{D}$ and the ideal $\mathcal{U}^+_\mathcal{D} \otimes_\mathbb{Z} R$. Here $\mathcal{U}^+_\mathcal{D}$ is the ideal of $\mathcal{U}_\mathcal{D}$ generated by $\mathfrak{g}_\mathcal{D}\mathcal{U}_{\mathfrak{g}_\mathcal{D}} \cap \mathcal{U}_\mathcal{D}$.

Theorem 1.3. *Let R be a ring. Then there is a homomorphism*

$$\mathrm{Ad}\colon \mathcal{G}_\mathcal{D}(R) \to \mathrm{Aut}_{filt}(\mathcal{U}_\mathcal{D})_R$$

characterised by the conditions

$$\mathrm{Ad}(u_a(r)) = \exp(\mathrm{ad}\, e_a \otimes r) = \sum_{n\geq 0} \frac{(\mathrm{ad}\, e_a)^n}{n!} \otimes r^n,$$

$$\mathrm{Ad}(T(R)) \text{ fixes } (\mathcal{U}_0)_R \text{ and } \mathrm{Ad}(h)(e_a \otimes r) = h(c_a) \cdot e_a \otimes r$$

for all $h \in T(R)$, $a \in \Phi$ and $r \in R$.

Proof. This is Theorem 9.5.3 in [Rém02]. (More precisely, there is a natural transformation $\mathrm{Ad}\colon \mathcal{G}_\mathcal{D} \to \mathrm{Aut}_{filt}(\mathcal{U}_\mathcal{D})$ between these two group functors.) ☐

The homomorphism Ad is called the **adjoint representation**.

Let K be a field and let $G := \mathcal{G}_\mathcal{D}(K)$ be a split Kac–Moody group. A subgroup $H \leq G$ is called **Ad-locally finite** if each $v \in (\mathcal{U}_\mathcal{D})_K$ is contained in a finite-dimensional $\mathrm{Ad}\, H$-invariant subspace.
A subgroup $H \leq G$ is called **Ad-diagonalizable** if there is a basis of $(\mathcal{U}_\mathcal{D})_K$ in which the H-action is diagonal.
From the explicit description of Ad it follows that $T(K)$ is Ad-diagonalizable.

1.4 Group combinatorics

Here we define the standard subgroups of a Kac–Moody group. Let K be a field and let $\mathcal{G}_{\mathcal{D}}$ be a Tits functor. Let $G(K) := \mathcal{G}_{\mathcal{D}}(K)$.

Characters and cocharacters. Let $X^*(T)_{\text{abs}} := \text{Hom}(T, K^\times)$ denote the group of (abstract) characters of T and $X_*(T)_{\text{abs}} := \text{Hom}(K^\times, T)$. Then Λ injects into $X^*(T)_{\text{abs}}$, while $\Lambda^\vee$ injects into $X_*(T)_{\text{abs}}$.
The group Λ is called the **group of algebraic characters** of T, while the group $\Lambda^\vee$ is called the **group of algebraic cocharacters** of T.

Cartan subgroups. Let $T = T(K) \leq G(K)$. Then T is called the **standard Cartan subgroup**, while any conjugate gTg^{-1} is called a **Cartan subgroup.**

Root groups. For a real root $a \in \Phi$, let $U_\alpha := \{x_a(t) : t \in K\}$. Then U_α is called a **root group** (relative to T).

It can be shown that the torus $T(K)$ and the root groups U_α embed in $G(K)$, i.e. $T(K)$ is isomorphic to $(K^\times)^k$, where k is the rank of Λ, while U_α is isomorphic to the additive group of K.
Note that $G(K)$ is generated by T and the root groups $U_\alpha, \alpha \in \Phi$. More precisely, $G(K)$ is already generated by T and the root groups $U_{\pm\alpha}$, where α runs through the positive simple roots.

Borel groups. Let $U_+ := \langle U_\alpha : \alpha > 0\rangle$ and $U_- := \langle U_\alpha : \alpha < 0\rangle$. Let $B_+ := TU_+$, $B_- := TU_-$. Then B_+ (resp. B_-) is called the **standard positive** (resp. negative) **Borel subgroup**, while any conjugate of B_+ resp. B_- is called a positive resp. negative Borel group. For $\epsilon \in \{\pm 1\}$, the group U_ϵ is called the **unipotent radical** of B_ϵ.
A positive Borel group B_1 and a negative Borel subgroup B_2 are called **opposite** if their intersection is a Cartan subgroup.
In contrast to the theory of algebraic groups, a (positive or negative) Borel subgroup B of a Kac–Moody group in general is not solvable. Indeed, B is solvable if and only if W is finite.

We recall the definition of a group G endowed with a twin root datum. Such a group is sometimes called a **group of Kac–Moody type**.

Definition 1.4. *Let (W, S) be a Coxeter system and let $\Phi = \Phi(W, S)$ be the set of its roots. Let G be a group and let $(U_\alpha)_{\alpha\in\Phi}$ be a family of non-trivial subgroups. Let $H \leq \cap_{\alpha\in\Phi} N_G(U_\alpha)$ and set $U_+ := \langle U_\alpha : \alpha > 0\rangle$, $U_- := \langle U_\alpha : \alpha < 0\rangle$. Then $(H, (U_\alpha)_{\alpha\in\Phi})$ is said to be a* **twin root datum** *for G (of type (W, S)) if the following conditions are satisfied:*

(TRD 1) $G = H\langle U_\alpha : \alpha \in \Phi\rangle$.

(TRD 2) *For each prenilpotent pair of roots* $\{\alpha, \beta\}$*, the commutator subgroup* $[U_\alpha, U_\beta]$ *is contained in* $U_{(\alpha,\beta)} := \langle U_\gamma : \gamma \in (\alpha, \beta)\rangle$.

(TRD 3) *For each* $s \in S$ *and each* $u \in U_{\alpha_s}\backslash\{1\}$*, there exist* $u', u'' \in U_{-\alpha_s}$ *such that* $m(u) := u'uu''$ *conjugates* U_β *onto* $U_{s\beta}$ *for all* $\beta \in \Phi$.
Moreover, for all $u, v \in U_{\alpha_s}\backslash\{1\}, m(u)H = m(v)H$.

(TRD 4) *For all* $s \in S$*,* $U_{\alpha_s} \not\subseteq U_-$ *and* $U_{-\alpha_s} \not\subseteq U_+$.

The definition of a twin root datum was made to capture the subgroup structure of a split Kac–Moody group $G(K)$. For $H = T(K)$, the defining relations $(R_{ab}), (R_1)-(R_4)$ are made such as to satisfy (TRD 1)-(TRD 3). The non-degeneracy conditions $U_\alpha \neq 1$ and (TRD 4) can be checked with the help of the adjoint representation.

Let G be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha\in\Phi(W,S)})$ of type (W, S). The group $G^\dagger := \langle U_\alpha : \alpha \in \Phi\rangle$ is called the **little projective group**; it is endowed with the twin root datum $(H \cap G^\dagger, (U_\alpha)_{\alpha\in\Phi})$.
For $B_\pm := HU_\pm$ and $N := H\langle m(u) : u \in U_{\alpha_s}\rangle$ and S a set of representatives for the reflections with respect to the simple roots, (B_+, B_-, N, S) is a **twin BN-pair** (see [AB08, Definition 6.78]) for G.
In particular, let $\Delta_\pm := G/B_\pm$ and $\Delta := (\Delta_+, \Delta_-)$. Then Δ is a **twin building** of type (W, S) (see [AB08, Definition 5.133]) and there is a **Bruhat decomposition** of G

$$G = \bigcup_{w\in W} B_+WB_+ = \bigcup_{w\in W} B_-WB_-$$

and a **Birkhoff decomposition**

$$G = \bigcup_{w\in W} B_+WB_- = \bigcup_{w\in W} B_-WB_+.$$

A **twin apartment** $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ is a subset of Δ which is isometric to the thin twin building of type (W, S) (see e.g. [AB08, Definition 5.171]).

If (W, S) is the Coxeter system with $|W| = 2$ and G is a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha\in\Phi(W,S)})$ (i.e. there are only two root groups altogether), then the associated twin building $\Delta(G)$ is called a **Moufang set**.
Recall that a subgroup P containing a conjugate of B_ϵ is called a **parabolic subgroup** of sign ϵ. If P contains B_ϵ, there is a set $J \subseteq S$ such that $P = B_\epsilon W_J B_\epsilon$, where $W_J := \langle s_i : i \in J\rangle \leq W$. If W_J is finite, W_J (or J) is called **spherical.**
Let $J \subseteq S$. Then $L^J := H\langle U_\alpha : \alpha \in \Phi(W_J, J)\rangle$ is called a **Levi factor**.

1.5 Geometric realizations

One of the equivalent ways to define a building is to view it as a simplicial complex covered by subcomplexes (the apartments) which are isomorphic to the standard

Coxeter complex. We briefly recall two important geometric realizations of this simplicial complex. A very good exposition of the interplay of these two constructions can be found in [Kra09, Appendix B.4].

The standard linear representation. Let $W = \langle (s_i)_{i\in I} : (s_is_j)^{m_{ij}} = 1\rangle$ be a Coxeter group. Let $A := (-\cos(\frac{\pi}{m_{ij}}))_{i,j}$. Let $V := \bigoplus_{i\in I} \mathbb{R}e_i$ and let B_I denote the bilinear form induced by A, i.e. $B_I(e_i, e_j) := a_{ij}$. Then the representation

$$\rho\colon W \to \mathrm{GL}(V), \rho(s_i)(e_j) := e_j - 2B(e_i, e_j)e_i$$

is called the **standard linear representation** of W, which can be shown to be faithful.
For a subset $J \subseteq I$, let $V_J := \bigoplus_{i\in J} \mathbb{R}e_i$ and write B_J for the restriction of B_I to V_J.

The CAT(0) realization. With the notation from above, for each $J \subseteq S$ such that W_J is spherical let $S_J := \{x \in V_J : x_i \geq 0, B_J(x, x) = 1\}$. Let C be the intersection of the cone generated by these spherical cells with the half spaces $B_I(e_i, -) \leq 1$. Then C serves as the model of a chamber.
For a building Δ of type (W, S), this gives a geometric realization of Δ via the mirror construction (see e.g. [Rém02, Section 4.2.1]). Moussong proved that the realization of an apartment in this realization has a natural metric which makes it a CAT(0) space. More precisely, the realization is a CAT(0) polyhedral complex with finitely many shapes of cells. By using retractions, Davis proved that the geometric realization of the entire building is CAT(0).
A point in the CAT(0) realization corresponds to a spherical residue of Δ. If $\Delta = \Delta(G)$ is the building associated to a group G endowed with a BN-pair, then G acts on the CAT(0) realization of Δ via isometries.

Example 1.5. Let (W, S) be a Coxeter system of type $\bullet \overset{\infty}{—} \bullet \overset{\infty}{—} \bullet \quad\quad \bullet$.
Then W is a so-called right-angled Coxeter group.
Part of the CAT(0) realization X of (W, S) is drawn below, with a chamber singled out. A point of X is contained in either 1,2,4 or 8 chambers.

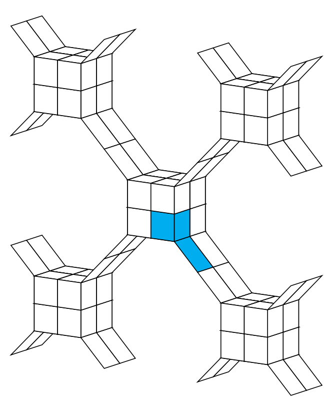

Figure 1.1: The CAT(0) realization of (W, S)

The cone realization. Again let (W, S) be a Coxeter group and let $\rho\colon W \to \mathrm{GL}(V)$ denote the standard linear representation. A **root** is a vector of the form $a = we_i$ for some $w \in W$ and some standard basis vector e_i; let $\Phi = \Phi_+ \cup \Phi_-$ denote the set of all roots. A root a is often identified with the **half-space**

$$D_a := \{f \in V^* : f(a) \geq 0\} \subseteq V^*$$

it determines.
Let $\overline{C} := \{f \in V^* : f(e_i) \geq 0 \text{ for all } i \in I\}$ be the so-called **fundamental chamber** and let $\overline{F}_{s_i} := \{f \in V^* : f(e_i) = 0\}$ denote the **wall** associated to the simple root e_i.
For an arbitrary root a let $\partial a := \{f \in V^* : f(a) = 0\}$ denote the wall of a.
Let W act on V^* in the contragredient way, i.e. $(w \cdot f)(v) := f(w^{-1}v)$. Then

$$\mathcal{C} := W \cdot \overline{C}$$

is called the **Tits cone** of W. It serves as a geometric realization of the Coxeter complex of W.
Let Δ be a building of type (W, S), viewed as a discrete set with a W-valued metric δ. Consider the topological space $\Delta_{cone} := \Delta \times \overline{C}/\sim$, where two points $(c, x), (d, y)$ are identified if and only if $x = y$ and $\delta(c, d) \in W_{J(x)}$. Here $J(x) := \{s_i \in S : x \in \overline{F}_{s_i}\}$ is the **type** of x.
For a twin building $\Delta = (\Delta_+, \Delta_-)$ the **cone realization** of Δ is defined as the link of Δ_+ and Δ_- with the origin of both realizations identified:

$$\Delta_{cone} := \Delta_+ * \Delta_- / \sim .$$

If $\mathcal{A}$ is a twin apartment of Δ, it turns out that its geometric realization in Δ_{cone} is homeomorphic to the realization $\mathcal{A}'$ of the thin twin building of type (W, S), which can be viewed as two copies of the Tits cone: $\mathcal{A}' \cong \mathcal{C} \cup -\mathcal{C} \subsetneq V^*$.
Note that if W is spherical, $\mathcal{C} = V^*$, while if W is infinite the Tits cone $\mathcal{C}$ is contained in a half-space. In both cases $\mathcal{A} = \mathcal{C} \cup -\mathcal{C}$ makes good sense.

Let $\mathcal{A}$ be a twin apartment of Δ and let $\Omega \subseteq \Delta$ be a set which is contained in $\mathcal{A}$. Identifying $\mathcal{A}$ with $\mathcal{C} \cup -\mathcal{C}$, the **convex hull** of Ω, $\mathrm{conv}_{\mathcal{A}}(\Omega)$ is defined as the convex hull of Ω in $\mathcal{A}$, and its **vectorial extension**, $\mathrm{vect}_{\mathcal{A}}(\Omega)$ as the vector subspace spanned by Ω. The set Ω is said to be **generic** if it is, viewed as a subset of $\mathcal{C} \cup -\mathcal{C}$, the intersection of $\mathcal{C} \cup -\mathcal{C}$ with a subspace L of V^* which meets the interior of $\mathcal{C}$: $\Omega = L \cap (\mathcal{C} \cup -\mathcal{C})$.
A subset $\Omega \subseteq \Delta_{cone}$ which is contained in a twin apartment $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$ is called **balanced** if $\Omega \cap \mathcal{A}_+ \neq \emptyset \neq \Omega \cap \mathcal{A}_-$ and Ω is contained in the union of a finite number of spherical facets. Here a **spherical facet** F is defined as

$$F = w \cdot \left(\bigcap_{i \in J} \partial e_i \cap \bigcap_{i \in I \setminus J} D_{e_i} \right)$$

for some $w \in W$ and some spherical subset $J \subseteq I$.
Two points x, y of the cone realization of a twin building are **(geometrically) opposite** if there is a twin apartment $\mathcal{A} \cong \mathcal{C} \cup -\mathcal{C} \subseteq V^*$ containing x and y such that in this identification, $x = -y$.

Example 1.6. In the cone realization of the thin twin building of type D_∞, the generic subspaces are precisely those different from $\{y = 0\}$. $\Omega = \{0\}$ is not a balanced subset. The spherical facets are the open half-rays (corresponding to panels) and the interior of the cones bounded by two consecutive open half-rays (corresponding to chambers).
The set of positive simple roots $\{\alpha, \beta\}$ which bound the positive fundamental chamber $\overline{C}$ is not prenilpotent.

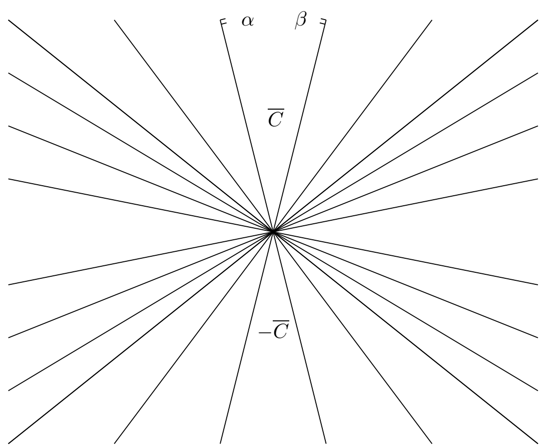

Figure 1.2: The cone realization of the thin twin building of type D_∞.

1.6 Bounded subgroups

In this section we recall the close connection between Ad-locally finite groups and fixators of balanced subsets.

Let G be a group endowed with a twin root datum $(T, (U_\alpha)_{\alpha \in \Phi(W,S)})$. A subgroup $H \leq G$ is called **bounded** if there exists $n \in \mathbb{N}$ and $w_1, \ldots, w_n \in W$ such that

$$H \subseteq B_+\{w_1, \ldots, w_n\}B_+ \cap B_-\{w_1, \ldots, w_n\}B_-,$$

i.e. for $\epsilon \in \{+, -\}$, H is contained in a finite number of double B_ϵ-cosets.

The following is Theorem 10.2.2 in [Rém02].

Theorem 1.7. *Let $G = \mathcal{G}_\mathcal{D}(K)$ be a split Kac–Moody group. For a subgroup $H \leq G$, the following conditions are equivalent.*

(i) H is bounded.

(ii) H fixes a point in the CAT(0) realization of both Δ_+ and Δ_-.

(iii) H is Ad-locally finite.

Sketch of proof. $(i) \Rightarrow (ii)$ is clear, as the orbits of the standard chambers C_+, C_- corresponding to B_+ and B_- under H are bounded, which allows to apply the Bruhat-Tits fixed point theorem [AB08, Theorem 11.23].
$(ii) \Rightarrow (iii)$ H is contained in the full stabilizer of two points $x \in \Delta_+, y \in \Delta_-$, so without loss of generality suppose that $H = \mathrm{Fix}_G(\{x, y\})$. Up to conjugation, we can assume that x, y are contained in the standard twin apartment. Then H has the form $H = L^J \ltimes U$, where L^J is a Levi factor of finite type, and $U = U_{[\alpha,\beta]}$ for a prenilpotent pair of roots α, β, see [Rém02, Theorem 6.3.4] For each $v \in (\mathcal{U}_{\mathcal{D}})_{\bar{K}}$, $\mathrm{Ad}\, U \cdot v$ spans a finite-dimensional subspace since $\mathrm{Ad}\, u$ is locally nilpotent for each $u \in U_\gamma$ and U is boundedly generated by $U_\gamma, \gamma \in [\alpha, \beta]$. Since the Levi factor is of finite type, the Bruhat decomposition for L^J, $L^J = \cup_{i=1}^n B' w_i B'$ where B' is a Borel subgroup of L^J, shows similarly that L is locally finite. As H is boundedly generated by L^J and U, the claim follows.
$(iii) \Rightarrow (i)$: Assume H is not bounded. Then there is a sequence $(g_i)_{i \in \mathbb{N}} \subseteq H$ such that $\{B_+ g_i B_+\}$ is infinite (replace '+' by '-' if necessary). Write $g_i = b_i w_i b_i'$. Let $v \in V_a$ be a non-trivial weight vector. Then $b_i' \cdot v$ has cv as a homogeneous component for some $c \neq 0$, from which it follows that $\mathrm{Ad}\, g_i \cdot v$ has a non-trivial component in the root space $w_i \cdot a$, proving that the support of $\mathrm{Ad}\, H \cdot v$ is infinite-dimensional. □

Now let $\Omega \subseteq \Delta_{cone}$ be a balanced subset which is contained in the standard apartment $\mathcal{A}$. By the previous theorem, $H := \mathrm{Fix}\, \Omega$ is Ad-locally finite. Rémy attaches to H a certain finite-dimensional $\mathrm{Ad}\, H$-invariant subspace whose construction we recall.
Let $\bar{K}$ be an algebraic closure of K. Let $\mathcal{L}_{\mathcal{D}} := \mathfrak{g}_{\mathcal{D}} \cap \mathcal{U}_{\mathcal{D}}$, where $\mathcal{U}_{\mathcal{D}}$ is the $\mathbb{Z}$-form of the universal enveloping algebra. Then $\mathcal{L}$ has a grading: $\mathcal{L}_{\mathcal{D}} = \mathcal{L}_0 \oplus \bigoplus_{a \in \Phi} \mathcal{L}_a$.
Let $\Delta(\Omega) := \{a \in \Phi : \Omega \subseteq a\}$, $\Delta^u(\Omega) := \{a \in \Phi : \Omega \subseteq a, \Omega \subsetneq \partial a\}$ and let $\Delta^m(\Omega) := \{a \in \Phi : \Omega \subseteq \partial a\}$. Here the roots are viewed as half-spaces in the cone realization. Write $L := T\langle U_\alpha : \alpha \in \Delta^m(\Omega)\rangle$ and $U := \langle U_\alpha : \alpha \in \Delta^u(\Omega)\rangle$.

Proposition 1.8. *Let $W = W_\Omega$ be the smallest Q-graded subspace of $(\mathcal{L}_{\mathcal{D}})_{\bar{K}}$ with the following properties:*

(i) W contains $(\mathcal{L}_0)_{\bar{K}}$ and $(\mathcal{L}_a)_{\bar{K}}$ for all $a \in \Delta(\Omega)$.

(ii) The Q-support of W contains $-\Delta^u(\Omega)$.

(iii) W is stable under $H := \mathrm{Fix}\, \Omega$.

Then the following properties hold:

(i) W is finite-dimensional and the kernel of $\mathrm{Ad}\colon H \to \mathrm{Ad}\, H|_W$ *is precisely the center of H.*

(ii) Let $\bar{H}$ *(resp.* $\bar{T}, \bar{L}, \bar{U}$*) denote the Zariski-closure of* $\operatorname{Ad} H|_W$ *(resp.* $\operatorname{Ad} T|_W$, $\operatorname{Ad} L|_W$, $\operatorname{Ad} U|_W$*). Then* $\bar{L}$ *is a connected reductive* K*-group,* $\bar{T}$ *is a maximal torus of* $\bar{L}$*,* $\bar{U}$ *is unipotent and* $\bar{H} = \bar{L} \ltimes \bar{U}$ *is a Levi decomposition.*

Proof. This is [Rém02, Lemma 10.3.1, Proposition 10.3.6]. □

2 Review of almost split Kac-Moody groups

In this chapter, we recall Rémy's construction of almost split Kac–Moody groups, cf. [Rém02], [Rém04]. These groups can be obtained via Galois descent, i.e. by taking the fixed points of a certain Galois group action on a split Kac–Moody group. One of the main features of an almost split Kac–Moody group is that it is again endowed with a twin root datum.
A result of Borel–Tits states (in modern language) that a connected K-isotropic algebraic K-group G has the property that $G(K)$ is endowed with a twin root datum. This justifies regarding almost split Kac–Moody groups as infinite-dimensional isotropic reductive groups.

2.1 The definition of almost split groups

Let K be a field, $\bar{K}$ an algebraic closure of K and K_s the separable closure of K in $\bar{K}$. Let $\mathcal{D}$ be a Kac–Moody root datum and let $\mathcal{G}_{\mathcal{D}}$ be a constructive Tits functor.
A **prealgebraic K-form** of $\mathcal{G}_{\mathcal{D}}$ is a couple $(\mathcal{G},\mathcal{U}_K)$, where $\mathcal{G}$ is a group functor on the category of field extensions of K which coincides with $\mathcal{G}_{\mathcal{D}}$ over extensions of $\bar{K}$, and $\mathcal{U}_K$ a K-form of the filtered algebra $(\mathcal{U}_{\mathcal{D}})_{\bar{K}}$ satisfying

(PA 1) The adjoint representation Ad is Galois-equivariant, i.e. for each K-algebra R and each $\sigma \in \Gamma := \mathrm{Gal}(K_s|K)$, the following diagram commutes, where $R_{\bar{K}} := \bar{K} \otimes_K R$:

$$\begin{array}{ccc} \mathcal{G}(R_{\bar{K}}) & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}_{filt}(\mathcal{U}_{\mathcal{D}})(R_{\bar{K}}) \\ \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\ \mathcal{G}(R_{\bar{K}}) & \xrightarrow{\mathrm{Ad}} & \mathrm{Aut}_{filt}(\mathcal{U}_{\mathcal{D}})(R_{\bar{K}}) \end{array}$$

(PA 2) If $\iota\colon K \to L$ is an injection of fields, then $\mathcal{G}(\iota)\colon \mathcal{G}(K) \to \mathcal{G}(L)$ is injective, too.

Let E be a field satisfying $K \subseteq E \subseteq \bar{K}$. Then a prealgebraic form $(\mathcal{G},\mathcal{U}_K)$ is said to **split** over E if it is E-isomorphic to the split form $(\mathcal{G}_{\mathcal{D}}, (\mathcal{U}_{\mathcal{D}})_E)$ over E (see [Rém02, 11.1.5] for a precise definition).

Convention. In this section, let $(\mathcal{G},\mathcal{U}_K)$ always be a prealgebraic K-form of $\mathcal{G}_{\mathcal{D}}$ which is assumed to split over an infinite field E such that $E|K$ is a normal field extension.

Let $\Gamma := \mathrm{Gal}(K^{sep}|K)$ be the absolute Galois group. Then for each field $L \subseteq \bar{K}$ and each $\gamma \in \Gamma$, there is an action of Γ on $\mathcal{G}$ given by $(\gamma \cdot \mathcal{G})(L) := \mathcal{G}(\gamma \cdot L)$. Since $E|K$ is assumed to be normal, Γ acts on $\mathcal{G}(E)$, and since $\mathcal{G}$ is assumed to split over E, each element of $\mathrm{Gal}(K^{\mathrm{sep}}|E)$ acts trivially on $\mathcal{G}(E)$, i.e. the Γ-action factors through $\mathrm{Gal}(E|K)$.
Fix an isomorphism $\Psi\colon \mathcal{G}(E) \to \mathcal{G}_{\mathcal{D}}(E)$. By abuse of notation, let $T(E) \leq \mathcal{G}(E)$ again denote the subgroup of $\mathcal{G}(E)$ which is mapped to the group $T(E) \leq \mathcal{G}_{\mathcal{D}}(E)$. Then Γ preserves the conjugacy class of $T(E)$ ([Rém02, 11.2.2]). For $\sigma \in \Gamma$, choose $g \in \mathcal{G}(E)$ such that the so-called **rectification** $\bar{\sigma} := \mathrm{int}\, g^{-1} \circ \sigma$ stabilizes $T(E)$. Then $\bar{\sigma}$ induces an automorphism of $W = N(T(E))/T(E)$.

Let $(\mathcal{G}, \mathcal{U}_K)$ be a prealgebraic K-form of G which splits over E. Then G is said to satisfy (SGR) if for each $\sigma \in \Gamma$, each rectified automorphism $\bar{\sigma}$ of $G(E)$ induces a permutation of the root groups relative to $T(E)$.
Remark 2.1. By the explicit description of $\mathrm{Aut}(\mathcal{G}_{\mathcal{D}}(E))$ by Caprace ([Cap09, Theorem A]) this condition is empty: $\bar{\sigma}$ automatically preserves root groups. Indeed, by the quoted result any automorphism φ can be written as a product $\varphi = \varphi_2 \circ \varphi_1$ of an inner automorphism φ_1 (which can be chosen to be trivial if $\varphi(T) = T$) and an automorphism φ_2 which permutes the root groups: $\varphi_2(x_\alpha(r)) = x_{\iota(\alpha)}(c_\alpha \sigma_\alpha(r))$, where $\iota\colon \Phi \to \Phi$ is a bijection, $c_\alpha \in E^\times$ and $\sigma_\alpha \in \mathrm{Aut}(E)$.
It follows that $\bar{\sigma}$ induces a permutation of the roots Φ of W. Moreover, $\bar{\sigma}$ induces an action on the groups $X^*(T(E))_{\mathrm{abs}}$ resp. $X_*(T(E))_{\mathrm{abs}}$ of abstract characters resp. cocharacters.

In this situation, $G = (\mathcal{G}, \mathcal{U})$ is called a **Kac–Moody K-group** if for each $\bar{\sigma}$,

(ALG 1) $\bar{\sigma}$ respects the Q-grading of $(\mathcal{U}_{\mathcal{D}})_E$ and the induced permutation of Q satisfies $\bar{\sigma}(na) = n(\bar{\sigma}(a))$ for all $n \in \mathbb{N}$.

(ALG 2) $\bar{\sigma}$ stabilizes the algebraic characters $\Lambda \leq X^*(T(E))_{\mathrm{abs}}$ resp. the algebraic cocharacters $\Lambda^\vee \leq X_*(T(E))_{\mathrm{abs}}$.

Let $G = (\mathcal{G}, \mathcal{U})$ be a Kac–Moody K-group. Then G is called **almost split** if the action of Γ on $\mathcal{G}(E)$ stabilizes the conjugacy classes of the standard Borel subgroups $B_+(E)$ and $B_-(E)$. The group G is called **quasi-split** if there are two opposite Borel groups B_1, B_2 which are stable under the Γ-action.
Note that a quasi-split Kac–Moody group is automatically almost split.

Remark 2.2. The terminology "almost split" stems from the following fact: although an almost split Kac–Moody group has an anisotropic kernel $Z(k)$, this group is *finite-dimensional*.

Galois descent. Let $G = (\mathcal{G}, \mathcal{U})$ be a Kac–Moody K-group. Then G is said to be obtained via **Galois descent** if G splits over the separable closure K_s of K in $\bar{K}$ and for each separable field sub-extension $E|K$, the group $\mathcal{G}(E)$ is precisely the fixed point set of $\mathrm{Gal}(K^{\mathrm{sep}}|E)$ in $\mathcal{G}(K^{\mathrm{sep}})$. In this case, $\mathcal{G}$ is said to satisfy the condition (DCS).

2.2 An explicit construction

In this section we recall the explicit construction of quasi-split Kac–Moody groups due to Rémy [Rém02, Ch. 13.2.3].

Construction 2.3. Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum. Then the **Dynkin diagram** $D = D_A$ of the generalized Cartan matrix A is a graph with vertices I and edges defined as follows. If $a_{ij}a_{ji} \leq 4$ and $|a_{ij}| \geq |a_{ji}|$ the vertices i, j are connected by $|a_{ij}|$ lines, with an arrow pointing toward i if $|a_{ij}| > 1$. If $a_{ij}a_{ji} > 4$, the vertices are connected by a line labelled $(|a_{ij}|, |a_{ji}|)$.
A **diagram automorphism** of A is a permutation of the index set I which induces an automorphism of the Dynkin diagram D_A.
A **diagram automorphism** φ of $\mathcal{D}$ is a $\mathbb{Z}$-linear automorphism of Λ which stabilizes the base $\{c_i : i \in I\}$ and such that the induced automorphism $\varphi^\vee$ of $\Lambda^\vee$ stabilizes the cobase $\{h_i : i \in I\}$. Moreover, let $\pi \in \mathrm{Sym}(I)$ denote the permutation such that $\varphi(c_i) = c_{\pi(i)}$. Then π is required to satisfy $\varphi^\vee(h_i) = h_{\pi(i)}$ and to induce an automorphism of the Dynkin diagram of A.
Let $\mathrm{Diag}(\mathcal{D})$ denote the group of diagram automorphisms of $\mathcal{D}$.

Let K be a field and let E be a finite Galois extension of K. Let $\Gamma := \mathrm{Gal}(E|K)$ be the corresponding Galois group. Let $\mathcal{D}$ be a Kac–Moody root datum and suppose that there is a homomorphism

$$*\colon \Gamma \to \mathrm{Diag}(\mathcal{D}).$$

We denote $\sigma^* := *(\sigma)$. Let $\mathcal{G}_\mathcal{D}$ be the constructive Tits functor associated to $\mathcal{D}$. For a field F, let $G(F) := \mathcal{G}_\mathcal{D}(F)$ denote the F-rational points of $\mathcal{G}$.
Let $x_{i,+}$ (resp. $x_{i,-}$) be the one-parameter group corresponding to the positive (resp. negative) simple root s_i. For $\sigma \in \Gamma$ and $r \in E$ set

$$\sigma^*(x_{i,\varepsilon}(r)) := x_{\sigma^*(i),\varepsilon}(\sigma(r))$$

for $\varepsilon \in \{+, -\}$ and for a torus element $t \in T(E)$ set $\sigma^*(t) := \{x \mapsto \sigma^{-1}(t(\sigma^*(x))\}$. By inspecting the defining relations of $G(E)$, it can be checked that σ^* extends to a well-defined automorphism of $G(E)$. Moreover, $*\colon \Gamma \to \mathrm{Aut}(G(E)), \sigma \mapsto \sigma^*$ defines a Γ-action on $G(E)$.
Let

$$G'(K) := \mathrm{Fix}_\Gamma(G(E))$$

denote the group of fixed points under this action. Then $G'(K)$ is a quasi-split Kac–Moody K-group. □

Remark 2.4. In the above situation, the Γ-action on the diagram is not required to be faithful. Indeed, if the $*$-action is trivial, $G'(K)$ coincides with the split Kac–Moody group $\mathcal{G}_\mathcal{D}(K)$.

Moreover it is possible that the associated rank 1 groups are defined over different fields.

Example 2.5. The following example is given in [Rém04, 3.5.B]. Let $E|K$ be a separable quadratic field extension, $\mathrm{Gal}(E|K) = \langle\sigma\rangle$ and let $\mathcal{G}_\mathcal{D}$ be the affine Kac–Moody group $\mathcal{G}_\mathcal{D}(K) = \mathrm{SL}_3(K[t, t^{-1}])$. Let $\mathrm{SU}_3(K) \le \mathrm{SL}_3(E)$ denote the group of matrices which preserve a fixed three-dimensional σ–Hermitian form of Witt index 1. Then the group $\mathrm{SU}_3(K[t, t^{-1}])$ is a quasi-split Kac–Moody group obtained by the $*$-action where σ^* switches two nodes of the diagram associated to $\mathcal{G}_\mathcal{D}$.

More generally, there is the following class of examples of affine quasi-split Kac–Moody groups.

Proposition 2.6. *Let $\mathcal{G}$ be a connected simply connected almost simple algebraic group defined over $\mathbb{F}_q$ which is $\mathbb{F}_q$-isotropic. Then for any field K containing $\mathbb{F}_q$, the group $\mathcal{G}(K[t, t^{-1}])$ is an almost split Kac–Moody $\mathbb{F}_q$-group.*

Proof. This follows from [Rém02, Chapter 11]. A detailed proof is given in [BGW09, Proposition 10.2]. □

2.3 The Galois action on the building

Let K be a field, let $E|K$ be a normal field extension, where E is infinite, and let $\Gamma := \mathrm{Gal}(E|K)$. Let G denote an almost split Kac–Moody K-group obtained by Galois descent which splits over E.
Let $\Delta = (G(E)/B_+(E), G(E)/B_-(E))$ denote the twin building associated to the group $G(E) \cong \mathcal{G}_\mathcal{D}(E)$. The Γ-action on $G(E)$ then gives rise to an action on Δ since it preserves the respective conjugacy classes of B_+, B_-, cf. [Rém02, 11.3.2].
Moreover, there is a better rectification of automorphisms available, that is, for each $\sigma \in G$ there is a $g_\sigma \in G(E)$ (well-defined up to an element in $T(E)$) such that $\sigma^* := \mathrm{int}\ g_\sigma^{-1} \circ \sigma$ stabilizes both $B_+(E)$ and $B_-(E)$.
This gives a well-defined action of Γ on W, called the **$*$-action**. This action stabilizes the generating set S, i.e. the action is by diagram automorphisms ([Rém02, 11.3.2]).
It follows that Γ acts on the CAT(0) realization of the buildings Δ_+, Δ_-. Although Γ might be infinite (there is no assumption that $E|K$ is finite, i.e. that G splits over a finite extension of K), it can be shown that each orbit is bounded [Rém02, 11.3.4], so by the Bruhat-Tits fixed point theorem, there are fixed points in both halves of the twin building. By the dictionary relating the building to its CAT(0) realization, this is equivalent to saying that there are spherical residues R_+, R_- in both buildings which are stable under the Galois group. The residues R_+, R_- in general will not be chambers, though. Indeed, Γ will fix two opposite chambers if and only if G is quasi-split.

The action on the cone realization. Similarly, Γ acts on the cone realization Δ_{cone} of Δ. Let Δ^Γ_{cone} denote the set of fixed points, then it is clear that $G(K)$ acts on Δ^Γ_{cone}. In what follows, certain subsets of Δ^Γ_{cone} will be singled out, the stabilizers of which then will form the ingredients of a twin root datum for $G(K)$.

To start with, a maximal generic subspace (i.e. a sub-vectorspace of an apartment which meets the interior of the Tits cone) which is fixed by Γ is called a **K-apartment.** These can be shown to exist if G splits over the separable closure of K.
In the cone realization of the standard twin apartment, such a generic subspace L is given by

$$L = \{x \in V^* : e_i(x) = 0 \;\forall\, i : s_i \in S_0 \text{ and } e_i(x) = e_j(x) \text{ for } \Gamma^* s_i = \Gamma^* s_j\},$$

cf. [Rém02, Lemma 12.6.1]. Here S_0 is the type of the facet containing a maximal K-chamber F, see below. Note that the type of a chamber is $\emptyset$.

Remark 2.7. S_0, in Tits indices, denotes the type of the anisotropic kernel, so this makes good sense.

A **K-facet** is the set of Γ-fixed points of a Γ-stable facet. A maximal K-facet is a **K-chamber**. A **K-root** (resp. **K-half-apartment**, resp. **K-wall**, resp. **K-panel**) is an apartment (resp. half-apartment, resp. wall, resp. panel) relative to a K-apartment A_K, i.e. the trace of the corresponding object on A_K, which is assumed to be non-empty.
Two K-chambers of the same sign are called **adjacent** if they contain a common K-panel in their closure.
Two K-chambers of opposite sign are called (geometrically) **opposite** if there is a twin apartment which contains them and in which they are opposite.

For a given K-apartment A_K, $\Delta_K^{re}(A_K)$ is defined as the set of all *real* K-roots, i.e. those whose relative wall is again a generic subspace, and $\Phi_K(A_K)$ as the set of all K-half-apartments relative to A_K.
For a real K-root a, let $a^\natural$ denote its restriction to A_K. Then let

$$\Delta_a := \{b \in \Delta^{re} : \exists \lambda \geq 1 : b^\natural = \lambda a^\natural\}.$$

Note that Δ_a is a prenilpotent set of roots which is Γ-stable.

Finally, a **standardisation** of the cone realization Δ_{cone} of $G(E)$ is a triple $(A, C, -C)$ where A is a twin apartment which contains the two opposite chambers C and $-C$ (this corresponds to fixing a maximal torus T and two opposite Borel groups B_1, B_2 such that $B_1 \cap B_2 = T$). A **rational standardisation** is a triple $(A_K, F, -F)$ where A_K is a K-apartment and $F, -F$ are two opposite K-chambers which are contained in A_K. Two of these triples are called **compatible** if A contains A_K and $C, -C$ contain $F, -F$ respectively.

2.4 The twin root datum of an almost split group

Let $K \subseteq E \subseteq K^{\text{sep}}$ be an inclusion of fields and let G be an almost split Kac–Moody K-group which is obtained by Galois descent and splits over E. For a subgroup $U \le G(K^{sep})$, let $U(E) := G(E) \cap U$ denote the **group of E-rational points** of U.
A Γ-invariant parabolic subgroup P of G is called a **K-parabolic subgroup**. Such a K-parabolic group is precisely the stabiliser of a K-facet.

The anisotropic kernel. Let $(A_K, F, -F)$ be a rational standardisation. Then $Z := Z(A_K) := \text{Fix}_{G(K^{sep})}(A_K)$ is called the **anisotropic kernel** (with respect to A_K). Let $Z(K)$ denote the set of its K-rational points.
Let $\Omega := F \cup -F$. Then $\text{Ad}_\Omega(Z(A_K))$ is isomorphic to a semisimple algebraic K-group which is K-anisotropic. It follows that Z contains a **maximal K-split torus** $T_d(K)$, which can be identified with the connected component of the identity of its center ([Rém02, 12.5.2]). The set of all $G(K)$-conjugates of $T_d(K)$ is in bijection with the K-apartments.

Rational root groups. For a real K-root a, let $V_a := \langle U_b : b \in \Delta_a \rangle(K)$. By (DCS), V_a is just the fixed point group of Γ acting on the Γ-invariant group $U_{a^\natural} := \langle U_b : b \in \Delta_a \rangle$.
(This process sometimes is called "lumping together root groups", cf. [AB08, Section 7.9.3].)

Rank 1 groups. Let E be a K-panel, $\Omega := E \cup -E$ and denote by $M(\Omega)(K^{\text{sep}})$ its fixator in $G(K^{\text{sep}})$. Then $M(\Omega) = Z\langle V_\alpha, V_{-\alpha} \rangle$ for the K-root α with $E \subseteq \partial\alpha$.
The group $M(\Omega)$ is a reductive algebraic group defined over K of split semisimple rank 1, which can be seen by considering $\text{Ad}_\Omega(M_\Omega)$. It follows that a rational root group V_α is isomorphic to a root group of a semisimple K-group (cf. [Rém02, 12.5.4]).

Let $N(K)$ denote the stabilizer of A_K in $G(K)$. Then $W^\natural := N(K)/Z(K)$ is called the **relative Weyl group**.
It can be shown that $W^\natural$ is in fact a Coxeter group with generating set $S^\natural$ whose set of roots is in bijection with the half-apartments of A_K, see below.

With these notions, Rémy proved the following important and difficult theorem ([Rém02, Theorem 12.4.3]).

Theorem 2.8. *Let G be an almost split Kac–Moody K-group which is obtained by Galois descent. Let $(A_K, F, -F)$ be a rational standardisation. Then the group of rational points $G(K)$ is endowed with a twin root datum $(Z(K), (V_\alpha)_{\alpha \in \Phi(W^\natural, S^\natural)})$.*

Sketch of proof. The key fact is that the rational root groups V_α enjoy the Moufang property. For a simple root α defined by F and a K-panel E of F, this means that V_α acts transitively on the set of K-chambers C' which have E as a panel and are

distinct from C. This follows from the Moufang property of the groups U_α and then passing to fixed points.
The non-triviality of V_α follows from the fact that Γ acts semi-linearly on the center of $U_{\alpha^\natural}$, which is a vector space over K^{sep}, from which the existence of fixed points follows by a classical theorem.
Proving that $(W^\natural, S^\natural)$ is a Coxeter system is not obvious and is accomplished via the detour of showing first that the system of subgroups $(Z(K), (V_\alpha)_{\alpha \in \Phi(W^\natural, S^\natural)})$ gives rise to a refined symmetric BN-pair, see [Rém02, Definition 1.2.1].
One can then identify the relative roots of A_K with roots of $W^\natural$. The existence of μ-maps, i.e. the verification of (TRD 3), is then a direct consequence of the Moufang property. The fact that $G = Z(k)\langle V_\alpha : \alpha \in \Phi(W^\natural)\rangle$ is proved by looking at the action of G on the set of K-apartments: the little projective group acts transitively on this set, while $N(K)$, the stabilizer of A_K, acts transitively on the chambers of A_K. The commutation formula follows from the existence and uniqueness of the product decomposition

$$U_{\Delta_a} = \prod_{\gamma \in \Delta_a} U_\gamma$$

and taking fixed points. The non-degeneracy conditions are again an easy consequence of the action on the building. □

Geometric realization of the associated twin building. It can be checked [Rém02, 12.4.4] that the set of Γ-fixed points in $\Delta(G(E))$ gives a geometric realization of the twin building associated to $G(K)$ in the sense that adjacency and opposition can be checked by looking at the fixed points in $\Delta_{cone}(G(E))$.

Just like in the finite-dimensional case (cf. [TW02, Chapter 42]), we have the following fact:

Proposition 2.9. *Let $G(K)$ be a quasi-split Kac–Moody group obtained via Galois descent. Then the derived group of the anisotropic kernel Z is trivial, i.e. $Z(K)$ is abelian.*

Proof. By definition, the Galois group Γ stabilizes two opposite Borel groups of $G(E)$, where E is a splitting field of G. Without loss of generality, these can be assumed to be the standard Borel groups B_+, B_-. By the explicit description of the generic subspace A_K it follows that A_K is entirely contained in the cone of C_+ and C_-. So any element $g \in G(E)$ which fixes A_K will stabilize both B_+ and B_-, from which it follows that $g \in T(E)$. Thus $Z(K) \leq T(E)$, which is abelian. □

2.5 Review of non-split reductive algebraic groups

Let $G = G(k)$ be an almost split Kac–Moody group obtained via Galois descent. Let Ω a balanced subset of Δ_{cone} and let $M := \mathrm{Fix}_{G(k)}(\Omega)$. Then $\mathrm{Ad}_\Omega(M)$ can be identified with the k-points of an algebraic group defined over k, and M itself is a central extension of this group.

(The fact that $\operatorname{Ad} M$ is defined over k is implied by the axioms that the adjoint representation be Galois equivariant and that $G(k)$ is obtained by Galois descent; this is one of the main motivations of introducing these two axioms.)
This is why we recall here some facts about k-rational points of algebraic groups. Most of these results can be found in the classical reference [BT65]. A convenient summary of the results of this paper we need can also be found in [Deo78, Section 1.2].

Let k be a field, $\bar{k}$ an algebraic closure of k and G a connected reductive linear algebraic group defined over k. For our purposes, we can assume that G comes with a fixed embedding, i.e. G is a Zariski-closed subgroup of some $\mathrm{GL}_n(\bar{k})$.
Let $S \leq G$ be a maximal k-split torus and $X^*(S)$ its character group. Suppose that G is isotropic over k, i.e. S is non-trivial.
Let $\Phi \subseteq X^*(S)$ be the corresponding k-root system of G with respect to S, i.e. the set of weights of S acting on $\mathfrak{g} := \operatorname{Lie} G$ via the adjoint representation.
For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$ denote the corresponding root space, i.e.

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \operatorname{Ad} s(X) = \alpha(s) \cdot X \ \forall\, s \in S\}.$$

Let $\mathfrak{u}_\alpha := \sum_{k>0} \mathfrak{g}_{k\alpha}$ and let U_α be the connected unipotent subgroup of G with $\operatorname{Lie} U_\alpha = \mathfrak{u}_\alpha$. In fact, the only positive multiples of α which could possibly belong to Φ are α and 2α. These cases actually do occur, cf. the examples below.
The group U_α then is split over k, cf. [BT65, Cor. 3.18] and normalized by the centralizer $Z := C_G(S)$ of S in G.
If $\alpha \in \Phi$ is such that $2\alpha \notin \Phi$, then $U_2 := U_\alpha$ is k-isomorphic to a vectorspace $\mathbf{G}_a{}^n$.
If $\alpha \in \Phi$ is such that $2\alpha \in \Phi$, then $U_1 := U_\alpha / U_{2\alpha}$ again is isomorphic over k to a vectorspace.
In both cases, under this identification the action of S on U_1 resp. U_2 is given via the homothety induced by α. This means that for $s \in S(k)$ and $u \in U_1(k)$ or $u \in U_2(k)$, we have

$$s \cdot u \cdot s^{-1} = \alpha(s) \cdot u.$$

We recall the following two classes of examples of algebraic groups of relative rank 1 taken from [Bor91, 23.4] and [BT84, 4.1.9].

Example 2.10. Let K be a field of characteristic $\neq 2$ and $n \in \mathbb{N}$. Let V be an n-dimensional K-vectorspace and $q\colon V \to K$ a quadratic form of Witt index 1. An important case is when $K = \mathbb{R}$ and $q(x) := \sum_{i=1}^{n-1} x_i^2 - x_n^2$.
The bilinear form F associated to q may be assumed to have the form

$$F = \begin{pmatrix} 0_{1\times 1} & 0_{n-2\times 1} & 1_{1\times 1} \\ 0_{n-2\times 1} & D & 0_{n-2\times 1} \\ 1_{1\times 1} & 0_{n-2\times 1} & 0_{1\times 1} \end{pmatrix}$$

where $D \in K^{(n-2)\times(n-2)}$ is diagonal. Here $1_{a\times a}$ denotes the $a \times a$-identity matrix, while $0_{a\times b}$ denotes the $a \times b$-zero matrix.

Let

$$G := \mathrm{SO}(q) = \{g \in \mathrm{SL}_n(K) : Q(gv) = Q(v)\ \forall\, v \in V\}$$

denote the special orthogonal group associated to q. Then

$$T(K) = \{h(t) := \mathrm{diag}(t, 1, \ldots, 1, t^{-1}) : t \in K^\times\}$$

is the set of K-rational points of a maximal K-split torus T, i.e. G is an algebraic K-group of K-rank 1. The centralizer of T is given by $Z_G(T) = T \cdot \mathrm{SO}(q_D)$, where q_D is the quadratic form induced by D. Let $\lambda : h(t) \mapsto t$. Then λ is a root with weight space

$$\mathfrak{g}_\lambda = \left\{ \begin{pmatrix} 0_{1\times1} & -v^t & 0_{1\times1} \\ 0_{n-2\times1} & 0_{n-2\times n-2} & v \\ 0_{1\times1} & 0_{n-2\times1} & 0_{1\times1} \end{pmatrix} : v \in K^{n-2} \right\}.$$

$\mathfrak{g}_\lambda$ is abelian of dimension $n-2$; and since the exponential function exp induces a group isomorphism, so is $U_+ := \exp \mathfrak{g}_\lambda$. The group U_+ is explicitly given as

$$U_+(K) = \left\{ \begin{pmatrix} 1 & -v^t & \frac{-v^t v}{2} \\ & 1_{n-2\times n-2} & v \\ & & 1 \end{pmatrix} : v \in K^{n-2} \right\}.$$

T acts on U_+ via $h(t)\exp(v)h(t)^{-1} = \exp(tv)$. □

Example 2.11. Let k be a field and let $E|k$ be a separable field extension of degree 2 with $\mathrm{Gal}(E|k) = \langle\sigma\rangle$, and write $\bar{x} := \sigma(x)$. Let q denote the Hermitian form on $V = E^3$ given by $q(x, y, z) := x\bar{z} + y\bar{y} + z\bar{x}$ and let

$$\mathrm{SU}(q) := \{g \in \mathrm{SL}_3(E) : q(gv) = q(v)\ \forall\, v \in V\}.$$

Then $\mathrm{SU}(q)$ can be viewed as an algebraic group defined over k. More precisely, there is an algebraic group G which is defined over k such that $G(k) \cong \mathrm{SU}(q)$. The k-points of a maximal k-split torus of G are given by

$$T(k) := \{h(t) := \mathrm{diag}(t, 1, t^{-1}) : t \in k^\times\}.$$

The k-points of the centraliser of $T(k)$ are given by

$$Z(k) = \{\mathrm{diag}(t, \frac{\sigma(t)}{t}, \sigma(t)^{-1}) : t \in E^\times\}$$

(note that these are actually the k-points under the identification of $\mathrm{SU}(q)$ with a k-group).

For $a, b \in L$, let $u(a,b) := \begin{pmatrix} 1 & -\bar{a} & -b \\ & 1 & a \\ & & 1 \end{pmatrix}$. The corresponding groups are given by

$$U_\alpha(k) = \{u(a,b) : a, b \in L, b + \bar{b} = a\bar{a}\} \text{ and } U_{2\alpha}(k) = \{u(0,b) : b \in L, b + \bar{b} = 0\}.$$

Then $U_2 := U_{2\alpha}(k)$ is a 1-dimensional k-vectorspace, while $U_1 := U_\alpha(k)/U_{2\alpha}(k)$ is a 2-dimensional k-vectorspace. A torus element $h(t)$ acts on U_2 via multiplication by t^2 and on U_1 via multiplication by t. □

3 Further properties of Kac–Moody K-groups

Convention. For the rest of this thesis, any almost split Kac–Moody group is understood to be obtained via Galois descent.

After the review of the material needed for the definition of almost split Kac–Moody groups and their basic properties in the previous two chapters, we begin here to work in earnest with almost split Kac–Moody groups.
In the first part, we collect relevant facts about the interplay of the maximal split torus $T_d \leq Z$ with the root groups of an almost split Kac–Moody group G. This information is essential, as it allows to construct a subgroup $F \leq G$ which is locally split and of Kac–Moody type (cf. Chapter 4), which in turn allows for an adaption of the methods of [Cap09] to the isomorphism problem for these groups.
In the second part, we collect some observations on the concepts developed before. While this material is not needed for later chapters, it nevertheless responds to some natural questions which arise when viewing Kac–Moody groups as infinite-dimensional reductive groups. Besides some technical remarks, we develop the concept of restriction of scalars for Kac–Moody groups.

3.1 Properties of an almost split Kac–Moody K-group

We briefly recall the discussion of reductive k-subgroups of G as given in [Rém02, 12.5.2] to make the interplay of the maximal split torus and the relative root groups of an almost split Kac–Moody group explicit.

Let k be a field and let $G = G(k)$ be an almost split Kac–Moody k-group which splits over a separable extension $E \subseteq k^{sep}$. Let $(A_k, F, -F)$ denote a rational standardisation.
By definition, F and $-F$ are two minimal Galois-stable opposite spherical facets of the twin building associated to $G(E)$. The stabilizer of $\Omega := F \cup -F$ in $G(E)$ can then be identified with a Levi factor $L^J(E) := T\langle U_\alpha : \alpha \in \Phi(W_J)\rangle$ where $J \subseteq S$ is spherical. From the defining relations of the constructive Tits functor, it follows that $L^J(E)$ is abstractly isomorphic to the E-points of a connected reductive group split over E. Since L^J is invariant under the Γ-action, it follows that L^J is defined over k. Write Z for the algebraic group L^J endowed with this k-structure. So

$Z(E) \cong L^J(E)$, while $Z(k)$ of course is in general very different from $L^J(k)$.
For Ω as above, $\mathrm{Ad}_\Omega(Z)$ is a connected semisimple algebraic group defined over k which is anisotropic over k. It follows that there exists a unique maximal k-split torus T_d contained in Z. The torus T_d is central in Z and can be identified with a maximal k-split subtorus of T.

More generally, let $x \in A_k$ be a k-facet. Then for $\Omega := x \cup -x$, the fixator of Ω in $G(k)$ can be identified with the k-rational points of some Levi factor $L^{J'}$ of $G(E)$, where the k-structure on $L^{J'}$ again is given by the Γ-action. (We dealt above with the case when x is a k-chamber.)
The point here is that fixators of opposite points of a twin apartment carry an intrinsic structure of (the k-points of) an algebraic group. For bounded subgroups in general, though, one has to pass to the adjoint representation.

We combine this discussion with the review of rational points of algebraic groups in Chapter 2.5 to sum up the interplay between the maximal split torus $T_d(k)$ and the root groups $V_a(k)$.
Let $\mathbf{G}_a$ denote the algebraic group with $\mathbf{G}_a(k) = (k, +)$. For a group G, let $\mathscr{Z}(G)$ denote the center of G (which should not be confused with the anisotropic kernel Z of an almost split Kac-Moody group).

Proposition 3.1. *Let k be an infinite field and let G be an almost split Kac–Moody group obtained by Galois descent. Let Z be the anisotropic kernel of G, $T_d \leq Z$ a maximal k-split torus and W_k the Weyl group of $G(k)$ with S_k its set of canonical generators. Let $\Phi_k = \Phi(W_k, S_k)$ denote the set of k-roots and $(V_\alpha(k))_{\alpha\in\Phi_k}$ the set of root groups of $G(k)$ relative to $T_d(k)$.*
Let Δ_k denote the set of simple roots of Φ_k.

(i) Z is a connected reductive algebraic group defined over k. The torus T_d is a maximal k-split torus of Z which is central in Z; the derived group of Z is anisotropic over k.

(ii) Let $J \subseteq S_k$ be such that $(W_k)_J$ is finite. Then $L^J := Z\langle V_\alpha : \alpha \in \Phi((W_k)_J)\rangle$ is a connected reductive algebraic K-group, in which T_d is a maximal k-split torus. L^J has split-semisimple rank $|J|$.

(iii) Let $\alpha \in \Delta_k$. Then $X_\alpha := Z\langle V_\alpha, V_{-\alpha}\rangle$ is a connected reductive algebraic k-group of split-semisimple rank 1. V_α is a root group in X_α normalized by Z. There are two possibilities:

a) V_α is abelian and is k-isomorphic to $\mathbf{G}_a^n$ for $n := \dim V_\alpha$. In this case, V_α is normalized by Z, and T_d acts on V_α via a character α. This means there is some $\alpha \in X^*(T_d)$ defined over k such that $tut^{-1} = \alpha(t)\cdot u$ for $t \in T_d$ and $u \in V_\alpha$.

b) V_α is metabelian. Then $\mathscr{Z}(V_\alpha)$ is k-isomorphic to $\mathbf{G}_a^n$, where $n := \dim \mathscr{Z}(V_\alpha)$, and $V_\alpha/\mathscr{Z}(V_\alpha)$ is k-isomorphic to $\mathbf{G}_a^{n'}$, where $n' := \dim V_\alpha -$

$\dim \mathscr{Z}(V_\alpha)$.
The anisotropic kernel Z normalizes both V_α and $\mathscr{Z}(V_\alpha)$. There is a character $\alpha \in X^*(T_d)$ defined over k such that T_d acts on $\mathscr{Z}(V_\alpha)$ via 2α and on $V_\alpha/\mathscr{Z}(V_\alpha)$ via α.

(iv) *Let $u \in \mathscr{Z}(V_\alpha(k))\backslash\{1\}$ and $s_\alpha := m(u) = u'uu''$ the associated μ-map. Then s_α normalizes $T_d(k)$.*

(v) *Let $\alpha \in \Phi_k$. If $t \in T_d$ centralizes some $u \in V_\alpha\backslash\{1\}$, then t^2 already centralizes V_α.*

(vi) *If $\alpha, \beta \in \Phi_k, \alpha \neq \pm\beta$ are such that $o(s_\alpha s_\beta) < \infty$, then there is an element $t \in T_d(k)$ such that t centralizes V_α but not V_β.*

Proof. Part (i) is clear by the above discussion; similarly, as L^J is the fixator of two opposite points x, $-x$, for (*ii*) it is sufficient to check the statement about the semisimple rank of L^J, which follows from the fact that $\mathrm{Ad}_{x\cup -x}(L^J)$ is a semisimple group in which the $(V_\beta : \beta \in \Phi(W_k)_J)$ form a system of root groups in the algebraic sense.
Part (iii) follows from (ii) and the discussion of rational points of semisimple algebraic groups in Chapter 2.5.
For part (iv), note that by (iii) X_α is a reductive group with T_d a maximal split torus. Then the Zariski closure of $\{sus^{-1} : s \in T_d\}$ is a one-dimensional subgroup of V_α, and so is part of a maximal split reductive subgroup $F \leq X_\alpha$ which contains T_d, as follows from the Borel–Tits theorem (see Theorem 5.1). As $m(u)$, computed in F, leaves T_d invariant, so must $m(u)$, as computed in X_α.
Part (v) follows from part (iii) by noting that if V_α is abelian, then necessarily $\alpha(t) = 1$ (and so already t must centralize V_α). In case V_α is metabelian, if $u \in \mathscr{Z}(V_\alpha)$, then $2\alpha(t) = \alpha(t^2) = 1$ (so t^2 centralizes V_α), while if $u \notin \mathscr{Z}(V_\alpha)$, then $\alpha(t) = 1$, so t already centralizes V_α.
For part (vi) it follows from the assumption that V_α, V_β are contained in some Levi factor L^J with $|J| = 2$. Since the characters associated to α and β are not proportional, $C_{T_d}(V_\alpha) = \ker\alpha$ does not contain $C_{T_d}(V_\beta) = \ker\beta$. As $T_d(k)$ is Zariski dense in T_d, the claim follows. □

Remark 3.2. Let X be a (not necessarily finite) set and $U \leq \mathrm{Sym}(X)$ a doubly transitive permutation group which is not sharply doubly transitive. Then U is called a **Zassenhaus group** if the stabilizer of three points is trivial. With this terminology, Proposition 3.1 (v) can be roughly stated as follows: the split torus of an algebraic group of relative rank 1 one satisfies a weak Zassenhaus condition for its action on the associated Moufang set $\mathbb{M}$.
This is to say that if $h \in T_d$ and h fixes three chambers of the panel $\mathbb{M}_\alpha$ of C which is associated to V_α, then h^2 must fix the entire panel. Indeed, that h fixes three chambers is equivalent to saying that h normalizes $V_\alpha, V_{-\alpha}$ and $X := vV_{-\alpha}v^{-1}$ for some $v \in V_\alpha\backslash\{1\}$. Since V_α is sharply transitive on $\mathbb{M}_\alpha\backslash\{C\}$, v is uniquely determined. This implies that $hvV_{-\alpha}v^{-1}h^{-1} = vV_{-\alpha}v^{-1}$ if and only if h centralizes v. As

noted before, if h centralizes v, then h^2 must centralize the entire root group.

Note, however, that this condition is in general not satisfied for arbitrary elements $g \in Z(k)$. Indeed, let k be an infinite field, $n \in \mathbb{N}, n \geq 5$ and let q be an n-dimensional quadratic form over k of Witt index 1. Then the anisotropic kernel of $\mathrm{SO}(q)$ can be identified with $\mathrm{SO}(q')$, where q' is a $(n-2)$-dimensional anisotropic quadratic form over k, and the associated root group U_+ is isomorphic as a $\mathrm{SO}(q')$-module to the standard module of $\mathrm{SO}(q')$, an $(n-2)$-dimensional vectorspace (cf. Example 2.10). It is clearly possible to choose reflections s_1, s_2 in $\mathrm{SO}(q')$ such that $s_1 s_2$ has infinite order and its fixed point set H is a codimension-2-hyperplane. Since $n \geq 5$, H is not reduced to 0, i.e. $s_1 s_2$ fixes a proper subspace of U_+, but no nontrivial power of $s_1 s_2$ will fix U_+.

3.2 Restriction of scalars for Kac–Moody groups

We give a class of examples of quasi-split Kac–Moody groups obtained by the classical process of restriction of scalars, cf. [PR94, Section 2.1.2]. These examples show that an abstract isomorphism $\psi\colon G_1 \to G_2$ of two almost split Kac–Moody groups does not in general preserve the full parameter set $(\mathcal{D}_i, E_i|K_i, \rho_i)$ attached to these groups.

Proposition 3.3. *Let k be a field and let $E|k$ be a finite Galois extension. Let G be a split Kac–Moody group. Then there is a quasi-split Kac–Moody group G' such that $G(E)$ is isomorphic to $G'(k)$.*

Proof. Let $\Gamma := \mathrm{Gal}(E|k), n := |\Gamma|$ and let G_0 be the direct product of n copies of G, indexed by the elements of Γ. Define an action of Γ on $G_0(E)$ by setting

$$\gamma \cdot (g_{\sigma_1}, \dots, g_{\sigma_n}) := (g_{\gamma\sigma_1}, \dots, g_{\gamma\sigma_n}).$$

Let $G'(k)$ denote the fixed point set of Γ acting on $G_0(E)$. Then $G'(k)$ is precisely the diagonal subgroup of $G_0(E)$, which is isomorphic to $G(E)$.
It remains to be checked that this Γ-action is the $*$-action induced by a Γ-action on the Dynkin diagram of G_0, which allows to apply Construction 2.3. This is immediate, though, as the Dynkin diagram of G_0 is the disjoint union of n copies of the Dynkin diagram of G, and Γ permutes these copies. □

Remark 3.4. Let $E|k$ be a finite Galois extension and let G be a connected almost simple k-group which is split over k. Then the group $G'(k) \cong G(E)$ provided by Proposition 3.3 is the group classically obtained by restriction of scalars. The isomorphism $\varphi\colon G(E) \to G'(k)$ is not covered by Borel–Tits's theory [BT73] since $G'(k)$ is not absolutely almost simple. Indeed, in this theory one restricts to absolutely almost simple groups for precisely this reason.

3.3 Generalized Cartan matrices and the centralizer of a torus

Remark 3.5. Split Kac–Moody groups (over algebraically closed fields) should be thought of as *connected*, as they have no proper (normal) subgroups of finite index. Indeed, in this case for a finite index subgroup $N \leq G$, for all α we have $[U_\alpha : (U_\alpha \cap N)] < \infty$ and thus $U_\alpha \leq N$ since U_α is divisible. Similarly $T(K) \cong (K^\times)^n$ is divisible as K is algebraically closed, so $T \leq N$ and $G = T\langle U_\alpha : \alpha \in \Phi\rangle \leq N$.

Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum. Let $\overline{W} \leq \mathrm{Aut}(\Lambda)$ denote the group generated by all $\bar{s}_i \colon \lambda \mapsto \lambda - h_i(\lambda)c_i$, where $i \in I$. The group $\overline{W}$ is called the **Weyl group** of $\mathcal{D}$, cf. [Rém02, 7.1.3]. Let W denote the Weyl group associated to A. Then there is a canonical homomorphism $\phi \colon W \to \overline{W}$ given by $\phi(s_i) = \bar{s}_i$, cf. [Rém02, 7.1.5].

One of the difficulties when dealing with split Kac–Moody groups is the fact that the action of the torus on two distinct root groups U_α, U_β can be via the same character. An easy example is provided by the affine Kac–Moody group $G(k) := \mathrm{SL}_2(k[t, t^{-1}])$, where $T(k) = \{\left(\begin{smallmatrix} t & \\ & t^{-1}\end{smallmatrix}\right) : t \in k^\times\}$ acts on all root groups $U_{\alpha,i} := \left\{\left(\begin{smallmatrix} 1 & t^i r \\ & 1\end{smallmatrix}\right) : r \in k\right\}$ via multiplication by t^2.
This is related to the fact that the torus is in general not self-centralizing in contrast to the finite-dimensional situation.

Proposition 3.6. *Let $\mathcal{D} = (I, A, \Lambda, (c_i)_{i\in I}, (h_i)_{i\in I})$ be a Kac–Moody root datum and suppose that $\Lambda^\vee$ is generated by the cobase $\{h_i : i \in I\}$. Let K be an infinite field and let $G := \mathcal{G}_\mathcal{D}(K)$ be the split Kac–Moody group of type $\mathcal{D}$ over K.*

(i) The subgroup $Z_G(T)/T \leq W$ coincides with the kernel of the action of W on Λ.

(ii) If A is invertible, the W-action on Λ is faithful.

Proof. The first point follows from [Cap09, Lemma 7.10(iii)], the second one from [Cap09, Lemma 7.6.2]. □

Until the end of this chapter let $A \in \mathbb{Z}^{n\times n}$ be a generalized Cartan matrix.

Remark 3.7. The matrix A is called **hyperbolic** if it is of indefinite type and every principal submatrix is a direct sum of matrices of finite or affine type, cf. [Kac90, Chapter 4]. In this case A is invertible, from which it follows that the torus is self-centralizing.

Lemma 3.8. *Let $\mathcal{D}$ be a Kac–Moody root datum and let k be a field with $|k| \geq 4$. Let $G := \mathcal{G}_\mathcal{D}(k)$ with maximal torus $T(k)$. Then the center of G is given by*

$$Z(G) = \{t \in T(k) : t(c_i) = 1 \;\forall\, i \in I\}.$$

Proof. By [Rém02, Lemma 8.4.3], $Z(G) = \{t \in T : t(\lambda) = 1 \;\forall\; \lambda \in \Lambda\}$. For $t \in T$ to be central in G it is already sufficient that t centralizes $U_{\pm\alpha_i}$ for each simple root α_i, since $G = T\langle U_{\pm\alpha_i} : i \in I\rangle$ and T is abelian. Since T acts on $U_{\pm\alpha_i}$ via $\pm c_i$, the claim follows. □

Example 3.9. Let $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ and let $\mathcal{D} = \mathcal{D}^A_{sc}$ denote the simply connected root datum. Let k be an infinite field. Then the derived group $G^{(1)}$ of $G := \mathcal{G}_{\mathcal{D}}(k)$ contains the root groups U_α, so $S := (h_1 + h_2)(k^\times)$ is contained in $G^{(1)}$. By the previous lemma, S is central in G, in particular, the center of the derived group is infinite.
In contrast, let G be a reductive algebraic group. Then the derived group $G^{(1)}$ is a semisimple algebraic group, so its center is finite.

Remark 3.10. If A is invertible and indecomposable, then there are vectors $v, w \in \mathbb{Z}^n$ such that

$$A' := \begin{pmatrix} A & v \\ w^t & 2 \end{pmatrix}.$$

again is an indecomposable invertible generalized Cartan matrix. This is because the system of inequalities $w^t A^{-1} v \neq 2$ and $v, w \in \mathbb{Z}^n, v, w \leq 0, v_i = 0 \Leftrightarrow w_i = 0$ always has a solution, e.g. take $v = w$, each $v_i < 0$ and replace v by λv if necessary. If we start with A invertible of indefinite type, it is inductively clear that there exist invertible indecomposable matrices of indefinite type for any given dimension.
Conversely, it is easy to produce a matrix A of indefinite type which has arbitrarily large corank. Indeed, let

$$A := \begin{pmatrix} 2 & -4 & & \\ -4 & 2 & -6 & \\ & -6 & 2 & -4 \\ & & -4 & 2 \end{pmatrix}$$

Then $\ker A = \langle (2, 1, -1, -2)^t \rangle$, so A is of indefinite type and has corank 1. Starting from A, it is possible to produce generalized Cartan matrices of arbitrary corank. Indeed, let

$$B := \begin{pmatrix} 2 & -6 & \\ -6 & 2 & -4 \\ & -4 & 2 \end{pmatrix}$$

and let $B_k := \operatorname{diag}(2, B, \ldots, B)$ denote the $(3k+1) \times (3k+1)$ block diagonal matrix. Let E_k denote the symmetric $(3k+1) \times (3k+1)$ matrix with only nonzero entries above the diagonal $(E_k)_{1,2} = (E_k)_{1,5} = \ldots = (E_k)_{1,3k-1} = -4$. Let $C_k = B_k + E_k$.

This means that $C_1 = A$,

$$C_2 = \begin{pmatrix} 2 & -4 & & & -4 & & \\ -4 & 2 & -6 & & & & \\ & -6 & 2 & -4 & & & \\ & & -4 & 2 & & & \\ -4 & & & & 2 & -6 & \\ & & & & -6 & 2 & -4 \\ & & & & & -4 & 2 \end{pmatrix}$$

and so on. It is clear that the kernel of C_k has dimension k.

4 Maximal split subgroups

4.1 Split subgroups of groups of Kac–Moody type

An almost split Kac–Moody group $G(k)$ obtained via Galois descent is by definition a subgroup of a split Kac–Moody group $\mathcal{G}_{\mathcal{D}}(E)$. On the other hand, we show in this chapter that $G(k)$ possesses a maximal split subgroup $F(k)$ of Kac–Moody type, i.e. a subgroup endowed with a twin root datum which is locally split and intersects each root group $V_\alpha(k)$ of $G(k)$ non-trivially.

Example 4.1. Let k be a field and let $E|k$ be a separable extension of degree 2. Let $h : E^3 \to E$ be a Hermitian form of Witt index 1 with associated unitary group SU_3, which can be thought of as an algebraic group defined over k. Then $\mathrm{SU}_3(k[t,t^{-1}])$ is an almost split Kac–Moody group obtained from the split Kac–Moody group $\mathrm{SL}_3(k[t,t^{-1}])$ via Galois descent, cf. Example 2.5.
On the other hand, there is an inclusion $\mathrm{SL}_2(k[t,t^{-1}]) \leq \mathrm{SU}_3(k[t,t^{-1}])$, which follows by Example 2.11, as for the associated root groups $(V_\alpha(k)_{\alpha\in\Phi(W,S)})$ of $\mathrm{SU}_3([k[t,t^{-1}])$ it follows that $\langle \mathscr{Z}(V_\alpha(k)) : \alpha \in \Phi\rangle \cong \mathrm{SL}_2(k[t,t^{-1}])$.
The twin building associated to $\mathrm{SU}_3(\mathbb{F}_q[t,t^{-1}])$ is a semi-regular twin tree with valencies $(1+q, 1+q^3)$ in which the twin building associated to $\mathrm{SL}_2(\mathbb{F}_q[t,t^{-1}])$, a regular twin tree with valency $1+q$, embeds.

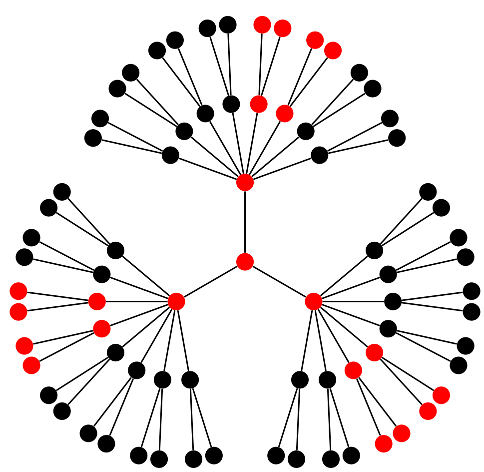

Figure 4.1: The twin trees associated to $\mathrm{SL}_2(\mathbb{F}_2[t,t^{-1}]) \leq \mathrm{SU}_3(\mathbb{F}_2[t,t^{-1}])$

Example 4.2. Let k be a field of characteristic $\neq 2$, $n \geq 2$ and let $q = \langle a_1, \dots, a_n\rangle$ be a quadratic form of Witt index 1 over k. We may assume that $\langle a_1, a_2\rangle = \langle 1, -1\rangle$

and that $\langle a_3, \ldots, a_n \rangle$ is anisotropic. Let $G := \mathrm{SO}(q)$ denote the associated special orthogonal group.
For $r = 2, \ldots, n$, let $q_r := \langle a_1, \ldots, a_r \rangle$ denote the truncated quadratic form and let $G_{q_r} := \mathrm{SO}(q_r)$ denote the associated special orthogonal group. Note in passing that $T := G_{q_2}(k) \cong k^\times$ and $G_{q_3}(k) \cong \mathrm{PGL}_2(k)$ – this follows from the fact that G_{q_3} is a split three-dimensional semisimple group, so it is either isomorphic to SL_2 or PGL_2, and these groups can be distinguished by the torus action on the root groups.
The point of this example is that there is a chain of reductive k-groups

$$T = G_{q_2} \leq G_{q_3} \leq \ldots \leq G_{q_n} = G$$

which share the same maximal torus $T = G_{q_2}$ of G. While G_{q_3} is split and contains the maximal split torus T, clearly it is not the only subgroup of G_{q_n} with this property – any $G'_i := \mathrm{SO}(\langle 1, -1, a_i \rangle)$ for some $i \in \{4, \ldots, n\}$ has the same property, and G_{q_3}, G'_i are not conjugate over k if $a_3 a_i^{-1} \notin k^2$.

The following is a classical result by Borel-Tits ([BT65, Theorem 7.2]).

Theorem 4.3. *Let G be a connected reductive k-group. Let S be a maximal k-split torus, $\Phi = \Phi(S, G)$ the system of k-roots of G and $\Phi' \subseteq \Phi$ the set of non-multipliable roots. Let Δ be a set of simple roots of Φ' and for each $a \in \Delta$ let $E_a \leq U_a$ be a k-subgroup which is normalized by S and is k-isomorphic to $\mathbf{G}_a$. Then there is a unique connected k-split reductive k-subgroup F which contains $S \cdot \langle E_a : a \in \Delta \rangle$.*

We prove a generalization of this result for a group G endowed with a 2-spherical root datum, which might be of independent interest as it provides "many" sub-twin buildings of the twin building associated to G. In our context, it will be used to construct a regular diagonalizable subgroup $H \leq G$ which is mapped under any isomorphism $\varphi \colon G \to G'$ again to a regular diagonalizable subgroup.

In a first step we define the necessary ingredients of a locally split subgroup and then go on to prove that these ingredients "integrate" to a locally split group of Kac–Moody type.
Recall that a Coxeter group $W = \langle (s_i)_{i \in I} : (s_i s_j)^{m_{ij}} = 1 \rangle$ is said to be **2-spherical** if $m_{ij} < \infty$ for all $i, j \in I$.
For elements $x, y \in G$ write ${}^y x := yxy^{-1}$. For a group G, let $G^* := G \backslash \{1\}$.

Definition 4.4. *Let $W = \langle (s_i)_{i \in I} : (s_i s_j)^{m_{ij}} = 1 \rangle$ be a 2-spherical Coxeter group and let G be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$.*
Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ denote the set of positive simple roots.
Let T_d be a subgroup of H and for each $\alpha \in \Delta$ let $E_\alpha \leq U_\alpha$ be a non-trivial subgroup.
For $\alpha \in \Delta$, let $s_\alpha := m(v)$ for some $v \in E^*_\alpha$.
Then $(T_d, (E_\alpha)_{\alpha \in \Delta})$ is called a **basis for a root subdatum** *if the following conditions are satisfied:*

(RSD 1) For all i, j, $(s_{\alpha_i} s_{\alpha_j})^{m_{ij}} \in T_d$.

(RSD 2) *For all* $r, t \in E_\alpha^*$, $m(r)m(t)^{-1} \in T_d$.

(RSD 3) *For all* $\alpha \in \Delta$, E_α *is normalized by* T_d *and each* s_α *normalizes* T_d.

(RSD 4) *For* $v \in E_\alpha^*$ *there exist* $v_1, v_2 \in E_\alpha^*$ *such that* $m(v)(:= v'vv'') = {}^{s_\alpha}v_1 \cdot v \cdot {}^{s_\alpha}v_2$.

(RSD 5) *If* $X \leq U_{(\alpha,\beta]}$ *is a subgroup normalized by* T_d *and* $x = u_1u_2 \in X$ *with* $u_1 \in U_{(\alpha,\beta)}, u_2 \in U_\beta$, *then* $u_1, u_2 \in X$.

As the name suggests, a basis for a root subdatum gives rise to a subgroup which has a root datum.

Theorem 4.5. *Let* (W, S) *be a 2-spherical Coxeter group, let* $\Phi = \Phi(W, S)$ *denote the set of its roots and let* Δ *be the set of simple roots.*
Let G *be a group endowed with a twin root datum* $(H, (U_\alpha)_{\alpha\in\Phi(W,S)})$. *Let* $(T_d, (E_\alpha)_{\alpha\in\Delta})$ *be a basis for a root subdatum.*
Let $M := T_d\langle s_\alpha : \alpha \in \Delta\rangle$, $V := \langle E_\alpha : \alpha \in \Delta\rangle$ *and* $F := \langle M, V\rangle$. *Set* $F_\gamma := F \cap U_\gamma$ *for* $\gamma \in \Phi$.
Then $(T_d, (F_\gamma)_{\gamma\in\Phi})$ *is a twin root datum for* F.

The proof, which will be given after a couple of preparatory lemmas, is very much inspired by [BT65, Proof of Theorem 7.2].

Lemma 4.6. *Let* G *be a group endowed with a twin root datum. Let* α, β *be two distinct positive simple roots. Then* $U_{-\alpha}$ *commutes with* U_β.

Proof. The set $\Psi := \{-\alpha, \beta\}$ is a prenilpotent set of roots since $s_\beta\Psi \subseteq \Phi^-$ and $s_\alpha\Psi \subseteq \Phi^+$. The open root interval $(-\alpha, \beta)$ is empty: Any positive root in $[-\alpha, \beta]$ must be mapped to a negative root by s_β and hence coincides with β, while any negative root in $[-\alpha, \beta]$ must be mapped to a positive one by s_α and hence coincides with $-\alpha$. By the commutator axiom, $[U_{-\alpha}, U_\beta] \leq U_{(-\alpha,\beta)} = 1$. □

We first analyze the structure of V.
Let $E_{-\alpha} := {}^{s_\alpha}E_\alpha$. Then $E_{-\alpha}$ is independent from the choice of $v \in E_\alpha^*$ in the definition of $s_\alpha = m(v)$ as for $v, v' \in E_\alpha^*$, $m(v)$ and $m(v')$ differ by an element of T_d by (RSD 2), and T_d normalizes E_α by (RSD 3).
For $\alpha, \beta \in \Delta$ let $E_{(\alpha,\beta)} := [E_\alpha, E_\beta]$ denote the commutator subgroup. Then $E_{(\alpha,\beta)}$ is normalized by E_β, since for $a \in E_\alpha, b, c \in E_\beta$,

$$c[a, b]c^{-1} = caba^{-1}b^{-1}c^{-1} = [c, a][a, cb].$$

Let $E_{(\alpha,\beta]} := E_{(\alpha,\beta)} \cdot E_\beta$.

Lemma 4.7. *Let* $\alpha, \beta \in \Delta$ *be two distinct positive roots.*

(i) $E_{(\alpha,\beta]} = \langle u_\alpha u_\beta u_\alpha^{-1} : u_\alpha \in E_\alpha, u_\beta \in E_\beta, u_\alpha \neq 1\rangle$.

(ii) $E_{(\alpha,\beta]}$ *is normalized by* s_α.

(iii) Let $E'_\alpha := \langle E_{(\alpha,\beta]} : \beta \in \Delta, \beta \neq \alpha \rangle$*. Then* $V = E_\alpha \ltimes E'_\alpha$.

Proof. (i) Let $X := \langle u_\alpha u_\beta u_\alpha^{-1} : u_\alpha \in E_\alpha, u_\beta \in E_\beta, u_\alpha \neq 1 \rangle$. Then $X \leq E_{(\alpha,\beta]}$ is clear. Conversely, since E_α and E_β are normalized by T_d, so is X. For $u_\alpha \in E^*_\alpha, u_\beta \in E_\beta$, note that $u_\alpha u_\beta u_\alpha^{-1} = [u_\alpha, u_\beta] u_\beta \in U_{(\alpha,\beta)} U_\beta$. By (RSD 5) it follows that u_β and $[u_\alpha, u_\beta]$ are contained in X, from which the claim follows.

(ii) By (i), it suffices to show that $s_\alpha (u_\alpha u_\beta u_\alpha^{-1}) s_\alpha^{-1} \in E_{(\alpha,\beta]}$, where $u_\alpha \neq 1$. Write $s_\alpha = u_1 u_2 u_\alpha^{-1}$ for some $u_1 \in E_\alpha, u_2 \in E_{-\alpha}$ – this is legitimate as $u_\alpha \neq 1$ and s_α is defined only up to elements of T_d.
Then

$$s_\alpha (u_\alpha u_\beta u_\alpha^{-1}) s_\alpha^{-1} = u_1 u_2 u_\beta u_2^{-1} u_1^{-1} = u_1 u_2 u_1^{-1}$$

since $u_2 \in E_{-\alpha}$ commutes with u_β by Lemma 4.6, from which the claim follows.

(iii) It is clear that E_α and E'_α are subgroups of V which generate V. From (ii) it is immediate that E_α normalizes E'_α. Let $v \in E_\alpha \cap E'_\alpha$. Then

$$^{s_\alpha}v \in {}^{s_\alpha}E_\alpha \cap {}^{s_\alpha}E'_\alpha = E_{-\alpha} \cap E'_\alpha \leq U_- \cap U_+ = 1.$$

□

Lemma 4.8. *(i) There is a canonical isomorphism* $\pi : M/T_d \to W$.

(ii) Let $\alpha \in \Delta$ *and* $w \in W$ *be such that* $w\alpha$ *is positive. Then* ${}^wE_\alpha \leq V$.

Proof. (i) Note that T_d is a normal subgroup of M by (RSD 3); by (RSD 1) it follows that $M/T_d \cong W$.

(ii) Since T_d normalizes E_α, ${}^wE_\alpha$ is well-defined. If $l(w) = 0$, there is nothing to prove, so suppose $l(w) \geq 1$. Since $w\alpha > 0$, we can write $w = s_\beta w'$, where β is a simple root distinct from α and w' is such that $w'\alpha > 0$. By induction, ${}^{w'}E_\alpha \leq V = E_\beta \ltimes E'_\beta$. Since ${}^{w'}E_\alpha \leq U_{w'\alpha}$ and $w'\alpha \neq \beta$, it follows that ${}^{w'}E_\alpha \leq E'_\beta$. Then ${}^wE_\alpha \leq {}^{s_\beta}E'_\beta = E'_\beta$.

□

The next step consists of exhibiting a Bruhat decomposition for F.

Lemma 4.9. *The group* F *can be written as* $F = VMV = \cup_{w \in W} VwV$.

Proof. The set $V \cdot M \cdot V$ contains V and M, is stable under inversion and closed under multiplication by elements in V or T_d from the right or left. To show that it coincides with F, it thus suffices to check that it is closed under multiplication from the right by $s_\alpha, \alpha \in \Delta$.

First step. For $\alpha \in \Delta$, $E_{-\alpha} \subseteq T_d E_\alpha \cup T_d E_\alpha s_\alpha E_\alpha$.
Indeed, $1 \in T_d E_\alpha$ while by definition, each $v \in E^*_{-\alpha}$ has the form $v = {}^{s_\alpha}v_0$ for some $v_0 \in E_\alpha$. By (RSD 4), there are $v_1, v_2 \in E_\alpha$ such that $m(v_0) = {}^{s_\alpha}v_1 v_0 {}^{s_\alpha}v_2$. Then

${}^{s_\alpha^{-1}}s_\alpha = v_1 v v_2$, i.e. $v \in E_\alpha s_\alpha E_\alpha$, from which the claim follows.

Second step. Since T_d normalizes V, we can write $VMV = \cup_{w\in W} VwV$ unambiguously. We will show that $VwVs_\alpha \subseteq Vws_\alpha V \cup VwV$, from which the claim will follow.

If $l(w) = 0$, i.e. $w = 1$, then by the first step and Lemma 4.7,

$$Vs_\alpha = E_\alpha s_\alpha {}^{s_\alpha}E'_\alpha \subseteq E_\alpha s_\alpha V.$$

Suppose $l(w) \geq 1$ and the claim is proven for all w' with $l(w') < l(w)$. Two cases can occur:
(1) $l(ws_\alpha) > l(w)$. This is the case if and only if $w\alpha > 0$. Then $wE_\alpha w^{-1} \leq V$ by Lemma 4.8 and we calculate

$$VwVs_\alpha = VwE_\alpha s_\alpha E'_\alpha = VE_\alpha ws_\alpha E'_\alpha \subseteq Vws_\alpha V.$$

(2) $l(ws_\alpha) < l(w)$, i.e. $w\alpha < 0$. Then we can write $w = w's_\alpha$ with $l(w') = l(w) - 1 \geq 0$. We calculate

$$VwVs_\alpha = Vw's_\alpha E_\alpha s_\alpha E'_\alpha = Vw'E_{-\alpha}E'_\alpha \subseteq Vw'V \cup Vw'Vs_\alpha V = Vw'V \cup Vw's_\alpha V.$$

Here the last equality follows because $w'\alpha > 0$, which allows us to apply the first case. □

We can turn to the proof of Theorem 4.5.

Proof. For $\gamma \in \Phi\backslash\Delta$ and $w \in W, \alpha \in \Delta$ such that $w\alpha = \gamma$ choose some lift $\tilde{w} \in M$ of w and set $E_\gamma := \tilde{w}E_\alpha\tilde{w}^{-1}$. Then for each $\gamma \in \Phi$, $E_\gamma \subseteq F_\gamma$. Assume for the moment that equality holds (in particular, E_γ will then not depend on the choice of α and $\tilde{w}$).
Then clearly for each $\gamma \in \Phi$, F_γ is nontrivial and normalized by T_d by (RSD 3). By (RSD 4), $s_\alpha \in \langle E_{-\alpha}, E_\alpha\rangle$, from which it follows that F is generated by T_d and $\langle E_\alpha, E_{-\alpha} : \alpha \in \Delta\rangle$, i.e. (TRD 1) holds. Set $V_- := \langle F_\gamma : \gamma < 0\rangle$. Then $V_- \cap V \leq U_- \cap U_+ = \{1\}$ and therefore (TRD 4) is satisfied. Similarly, (TRD 2) holds by the definition of F_γ and the corresponding property for G.
Axiom (TRD 3) holds for F_γ, $\gamma \in \Delta$ by (RSD 4).
It remains to prove that $F_\gamma = E_\gamma$ for $\gamma \in \Phi$, in particular $F_\alpha = E_\alpha$ for $\alpha \in \Delta$ which is not clear *a priori*.

First step. If $\gamma \in \Delta$, then $F \cap U_\gamma = E_\gamma$.
By the Bruhat decomposition $F = VMV$ it follows that $F \cap U_\gamma = V \cap U_\gamma$. Since $V = E_\gamma \ltimes E'_\gamma$ it follows that

$$s_\gamma(F \cap U_\gamma)s_\gamma^{-1} = s_\gamma V s_\gamma^{-1} \cap U_{-\gamma} = E_{-\gamma}E'_\gamma \cap U_{-\gamma} = E_{-\gamma} \cdot (E'_\gamma \cap U_{-\gamma}) = E_{-\gamma},$$

from which it follows that $F \cap U_\gamma = E_\gamma$.

Second step. If $\delta \in \Phi \backslash \Delta$ is arbitrary, then $F \cap U_\delta = E_\delta$.
Suppose first that $\delta \in \Phi^+$. Let $w = \tilde{w} \in M, \alpha \in \Delta$ as in the definition of E_δ. Then

$$w(F \cap U_\delta)w^{-1} = w(V \cap U_\delta)w^{-1} = wVw^{-1} \cap (U_+ \cap U_\alpha) \subseteq V \cap U_\alpha = E_\alpha.$$

By definition, $\tilde{w}E_\alpha\tilde{w}^{-1} \subseteq F_\delta$, and we have just shown the reverse inclusion, i.e. $F_\delta = E_\delta$.
Clearly the same reasoning works when $\delta \in \Phi^-$, which finishes the proof of the theorem. □

Remark 4.10. The statement of Lemma 4.9 that $F = \cup_{w \in W} VwV$ can be thought of as the fact that F is a graded subgroup of G. This means that whenever $f = b_1wb_2$ with $b_1, b_2 \in B$ and $w \in W$ is the Bruhat decomposition of an element $f \in F$, then b_1, b_2 and w can actually be chosen to be elements of F.

Remark 4.11. Let G be a group endowed with a 2-spherical twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$. Then $(H, (U_\alpha)_{\alpha \in \Delta})$ meets conditions (RSD 1)-(RSD 4), but not necessarily (RSD 5). Indeed, if (RSD 5) is met, it follows from the proof of the preceding theorem that $U_+ = \langle U_\alpha : \alpha \in \Delta \rangle$. This is satisfied for isotropic reductive k-groups with $|k| \geq 4$, but fails e.g. for $G_2(\mathbb{F}_2)$.

Remark 4.12. A geometric interpretation of the theorem is as follows: Let Δ be the twin building associated to G, $\mathcal{A}$ the twin apartment determined by H and C^+, C^- the two opposite chambers corresponding to B_+, B_-. On each panel F_α of C_+, fix chambers according to the action of E_α on F_α. Condition (RSD 4) ensures that these form a sub-Moufang set. The remaining conditions are the necessary compatibility conditions which ensure that these chambers give rise to a sub twin building with $\mathcal{A}$ as a twin apartment.
In particular, the twin building $\Delta(F)$ associated to F embeds in $\Delta(G)$ as a closed convex subcomplex. Methods from [Müh99] can be used to give a purely combinatorial argument of this fact.

4.2 The case of almost split Kac–Moody groups

We apply Theorem 4.5 to almost split Kac–Moody groups.
Let G be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$. Then the twin root datum is said to be **locally split** (over a family of fields $(k_\alpha)_{\alpha \in \Phi}$) if H is abelian and for each $\alpha \in \Phi$, $\langle U_\alpha, U_{-\alpha} \rangle$ is isomorphic to either $\mathrm{SL}_2(k_\alpha)$ or $\mathrm{PSL}_2(k_\alpha)$.

Theorem 4.13. *Let k be an infinite field and let $G(k)$ be a 2-spherical almost split Kac–Moody group obtained by Galois descent. Let $(Z(k), (V_\alpha)_{\alpha \in \Phi(W,S)})$ denote its canonical twin root datum and let $T_d(k) \leq Z(k)$ be a maximal k-split torus.*
For each simple root $\alpha \in \Delta$ let $E_\alpha \leq V_\alpha$ be a subgroup isomorphic to $(k, +)$ which is normalized by $T_d(k)$.
Then there is a subgroup $F(k) \leq G(k)$ which contains $T_d(k)\langle E_\alpha : \alpha \in \Delta \rangle$ and which is endowed with a locally split twin root datum.

Proof. Since G is assumed to be 2-spherical, for each pair of simple roots $\{\alpha, \beta\} \subseteq \Delta$ the group $X_{\alpha\beta} := Z(k)\langle V_{\pm\alpha}, V_{\pm\beta}\rangle$ can be identified with the k-points of a reductive algebraic k-group of relative rank 2. By Theorem 4.3, there is a split subgroup $Y_{\alpha\beta} \leq X_{\alpha\beta}$ which contains $T_d(k)$ and E_α, E_β. Now $Y_{\alpha\beta}$ is endowed with a spherical twin root datum, the properties of which imply that the axioms (RSD 1) to (RSD 4) of a root subdatum are satisfied, since these need to be checked only for rank 2 subgroups.
Since k is infinite, $T_d(k)$ is Zariski dense in T_d. For a subgroup $X \leq V_{(\alpha,\beta]}$ normalized by $T_d(k)$ it follows that $\overline{X}$ is normalized by T_d. By [BT65, Proposition 3.11] it follows that (RSD 5) is satisfied as well.

Theorem 4.13 gives the existence of F, and from the fact that the group $Y_{\alpha\beta}$ is a split reductive group it is immediate that the twin root datum for F is locally split. □

Definition 4.14. *Let k be an infinite field and let $G(k)$ be a 2-spherical almost split Kac–Moody group obtained by Galois descent. Any group F obtained from $G(k)$ in this way is called a* **maximal split subgroup** *of G.*

Remark 4.15. It is always poosible to find subgroups E_α as required in Theorem 4.13: just let E_α be a one-dimensional k-subspace of $\mathscr{Z}(V_\alpha)$. Then Proposition 3.1(iii) b) shows that E_α is normalized by $T_d(k)$.

In particular, any almost split 2-spherical Kac–Moody group is "sandwiched" between two split Kac–Moody groups: For a splitting field E of G, one has

$$F(k) \leq G(k) \leq G(E).$$

Here the Coxeter type of $F(k)$ is the same as the Coxeter type of $G(k)$, while the type of $G(k)$ equals the type of $G(E)$ if and only if G is already split split over k.

Remark 4.16. We used Theorem 4.5 to produce a locally split subgroup. The theorem is more general, though, as arbitrary sub-Moufang sets are allowed. In particular, we recover Example 4.2.

Remark 4.17. Another example of a basis for a root subdatum $(T_d, (E_\alpha)_{\alpha\in\Delta})$ as required in Theorem 4.5 comes from subfields: If $k \subseteq K$ and $\mathcal{G}_\mathcal{D}$ is a constructive Tits functor, then take $T_d := T(k)$ and $E_\alpha := U_\alpha(k)$ inside $\mathcal{G}_\mathcal{D}(K)$. Of course, the theorem can be applied more than once, i.e. pass first to a locally split subgroup and then to k-rational points.

Finally, just as Example 4.2 suggests, there is a chain condition on groups containing a maximal split subgroup.

Proposition 4.18. *Let k be an infinite field, char $k \neq 2$ and let $G(k)$ be an almost split Kac–Moody group over k obtained by Galois descent. Let $F \leq G$ be a maximal split subgroup. Let*

$$F = H_1 \leq H_2 \leq \ldots$$

be a chain of subgroups which are obtained by integrating a root subdatum and such that $H_i = H_i^\dagger$, i.e. H_i is generated by its root groups. Then the chain eventually becomes stationary.
More precisely, the length of a strictly increasing chain is bounded by $\sum_{\alpha\in\Delta} \dim_k V_\alpha$.

Proof. For each simple root $\alpha \in \Delta$ with corresponding root group $V_\alpha(k) \leq G(k)$ let $H_{i,\alpha} := H_i \cap V_\alpha$. Then $H_{i,\alpha} \cap \mathscr{Z}(V_\alpha)$ is a k-sub-vectorspace since it is invariant under $T_d(k)$, similarly for $(H_{i,\alpha} \cdot \mathscr{Z}(V_\alpha))/\mathscr{Z}(V_\alpha)$.
Recall from Proposition 3.1 that $V_\alpha(k)$ is an extension of two finite-dimensional k-vector spaces. This implies that $(H_{i,\alpha})$ eventually becomes stationary. Since the H_i are supposed to be generated by their root groups, the first claim follows.
Since in a strictly increasing chain of subgroups, in each step there is some $\alpha \in \Delta$ such that $H_{i,\alpha}$ is strictly contained in $H_{i+1,\alpha}$, the second claim follows. □

5 On bounded subgroups

In this chapter we are concerned with the following problem, which appears in [Cap09, Introduction].

Problem. *Let k, k' be two fields. Let $\mathcal{G}_{\mathcal{D}}$ be a constructive Tits functor, let $\mathbf{K}$ be a connected reductive k-isotropic k-group and let $\varphi\colon \mathbf{K}(k) \to \mathcal{G}_{\mathcal{D}}(k')$ be a homomorphism. Find conditions under which φ has bounded image.*

It will turn out that the transcendence degree of k over its prime field plays a central role in this problem.

Motivation. The fundamental rank 1 groups $X_\alpha(k)$ associated to an almost split Kac–Moody group are k-points of a connected reductive k-isotropic k-group. Then an isomorphism $\varphi\colon G(k) \to G'(k')$ of two almost split Kac–Moody groups will induce by restriction several representations $\varphi|_{X_\alpha}\colon X_\alpha(k) \to G'(k')$. If it can be shown that each $\varphi|_{X_\alpha}$ has bounded image, it will follow that φ maps bounded subgroups to bounded subgroups. By the results of [CM06], any isomorphism which preserves bounded subgroups will be standard:

Definition 5.1. *Let G, G' be two groups endowed with twin root data $(H, (U_\alpha)_{\alpha \in \Phi(W,S)})$ and $(H', (V_\beta)_{\beta \in \Phi(W',S')})$. Then a group isomorphism $\varphi\colon G \to G'$ is called* **standard** *if there is $x \in G'$ and a bijection $i\colon \Phi(W,S) \to \Phi'(W',S')$ such that $\varphi' := \operatorname{int} x \circ \varphi$ satisfies*

(i) $\varphi'(H) = H'$

(ii) $\varphi'(U_\alpha) = V_{i(\alpha)}$.

This approach then reduces the isomorphism problem for almost split Kac–Moody groups to the problem of showing that certain homomorphisms have bounded image, i.e. to the problem which appears above.

After some preliminary remarks in the first section, we show that a necessary condition for the field k appearing in the problem is that k have finite transcendence degree over its prime field k_0 – if $\operatorname{tr.deg}(k|k_0) = \infty$, we construct homomorphisms with unbounded image.
In a next step, we consider the case when $\operatorname{tr.deg}(k|k_0) = 0$, i.e. when k is an algebraic extension of its prime field. The main result in this direction is that if k is a (possibly infinite) algebraic extension of $\mathbb{Q}$, any homomorphism $\varphi\colon \mathrm{SL}_2(k) \to G'(k')$ has bounded image.

Finally, for arbitrary transcendence degree we give a geometric criterion which implies boundedness. This criterion can be used to recover Caprace's results for isomorphisms of split Kac–Moody groups over fields of characteristic 0.

5.1 Preliminaries

Remark 5.2. Let G be a group endowed with a twin root datum. Then the standard positive Borel group B_+ fixes the standard positive chamber C_+ in Δ_+, while it does not have a bounded orbit in Δ_- as soon as W is not spherical. Indeed, suppose that B_+ is bounded. Then there are $w_1, \dots, w_n \in W$ such that $B_+ \subseteq \cup_{i=1}^n B_- w_i B_-$. Then

$$\bigcup_{w\in W} B_- w B_- = G = B_- B_+ B_- \subseteq \bigcup_{i=1}^{n} B_- w_i B_-$$

from which it follows that W is finite.
More generally, let $\{\alpha, \beta\}$ be a pair of positive roots which is not prenilpotent and let $u_\alpha \in U_\alpha \backslash \{1\}$, $u_\beta \in U_\beta \backslash \{1\}$. Let $U' := \langle u_\alpha, u_\beta \rangle$. Then U' fixes the standard positive chamber C_+, while the U'-action on Δ_- does not have a bounded orbit. Indeed, since $\{\alpha, \beta\}$ is not prenilpotent, $s_\alpha s_\beta$ has infinite order. By induction

$$\begin{aligned} B_-(u_\alpha u_\beta)^{n+1} B_- &\subseteq (B_-(u_\alpha u_\beta)^n B_-)(B_- u_\alpha B_-)(B_- u_\beta B_-) \\ &= (B_-(s_\alpha s_\beta)^n B_-)(B_- s_\alpha B_-)(B_- s_\beta B_-) \\ &= B_-(s_\alpha s_\beta)^{n+1} B_- \end{aligned}$$

where the last equality follows from the BN-pair axioms and the fact that the length of the word increases. Then the first inclusion is an equality since $B_-(u_\alpha u_\beta)^{n+1} B_-$ is a double coset, so U' is not contained in a finite number of double B_--cosets.
Although the two halves of the twin building are related to each other via the codistance, this example emphasizes the fact that the codistance alone gives a rather weak connection between the halves of the twin building when studying the action of subgroups of G.

Let G be a group and $(U_i)_{i\in I}$ a family of subgroups. Then G is **boundedly generated** by $(U_i)_{i\in I}$ if there exists $n \in \mathbb{N}$ such each $g \in G$ is a product $g = g_1 \dots g_n$ of at most n elements $g_j \in \cup_{i\in I} U_i$.
Let X be a complete CAT(0) space and let G be a group which acts by isometries on X. Then G is called **bounded** if there is a bounded G-orbit, i.e. there is some $x \in X$ such that $\operatorname{diam} G \cdot x < \infty$. By the Bruhat-Tits fixed point theorem, this is equivalent to the fact that G fixes a point of X.
In our applications, X will be the CAT(0) realization of one half of the twin building associated to an almost split Kac–Moody group G on which a subgroup $U \leq G$ acts.

We will often use the following lemma.

Lemma 5.3. *Let X be a complete CAT(0) space and let $G \leq \operatorname{Isom}(X)$ be a group which is boundedly generated by a finite number of subgroups $U_1, \ldots, U_n$. Then G is bounded if and only if every U_i is bounded.*

Proof. This is Corollary 2.5 in [Cap09]. □

Lemma 5.4. *Let X be a complete CAT(0) space and $N, U \leq \operatorname{Isom}(X)$. If N normalizes U and both N and U are bounded, then N and U have a common fixed point.*

Proof. As N normalizes U, $N \cdot U$ is a group which is boundedly generated by N and U, from which the claim follows by the previous lemma. □

An important fact about Kac–Moody groups is that checking whether a subgroup is bounded or not can be done by looking at field extensions.

Lemma 5.5. *Let E be a field and let k be subfield of E. Let $\mathcal{G}_{\mathcal{D}}$ be a Tits functor and let $G := \mathcal{G}_{\mathcal{D}}(E)$. Let H be one of the following subgroups of G:*

(i) $H = G(k)$, where $G(k)$ is an almost split Kac–Moody k-group obtained from $\mathcal{G}_{\mathcal{D}}(E)$ by Galois descent.

(ii) $H = \mathcal{G}_{\mathcal{D}_{A_J}}(k)$ for some subfunctor $\mathcal{G}_{\mathcal{D}_{A_J}}$ (if $\mathcal{G}_{\mathcal{D}}$ is simply connected).

(iii) $H = L(k)$, where L is a Levi factor of $G(k)$.

Let $U \leq H$. Then U is bounded with respect to H, i.e. has a fixed point in both $\Delta_+(H), \Delta_-(H)$ if and only if U is bounded with respect to G, i.e. has a fixed point in both $\Delta_+(G), \Delta_-(G)$.

Proof. Note that for each H as above, H is endowed with a canonical twin root datum, so there is a twin building $\Delta(H) = (\Delta_+(H), \Delta_-(H))$ associated to H. Moreover, $\Delta(H)$ embeds as a closed convex subbuilding into the twin building $\Delta(G)$ associated to G. If U fixes points in both halves of $\Delta(H)$, the same points serve as fixed points in $\Delta(G)$. If $U \leq H$ fixes a point in $\Delta(G)$, it must also fix a point in $\Delta(H)$ since it leaves this closed convex set invariant, cf. [BH99, Proposition II.6.2 (4)]. □

The following lemma is a standard fact in the theory of algebraic groups, which has a geometric meaning in the context of twin buildings, as we will see shortly.

Lemma 5.6. *Let k be a field of characteristic 0 and let G be an algebraic group over k. Let G_u denote the unipotent radical of G. Let P be a linearly reductive subgroup of G such that $G = G_u P$ and let Q be any linearly reductive subgroup of G. Then there exists $t \in G_u(k)$ such that $tQ(k)t^{-1} \subseteq P(k)$.*

Proof. This is Proposition VIII.4.2 in [Hoc81]. □

The following proposition states that abstract representations of $\mathrm{SL}_2(\mathbb{Q})$ are in fact rational.

Proposition 5.7. *Let k be a field of characteristic 0 and let $n \in \mathbb{N}$. Let $\varphi\colon \mathrm{SL}_2(\mathbb{Q}) \to \mathrm{GL}_n(k)$ be a group homomorphism. Then there is a homomorphism of algebraic groups $\psi\colon \mathrm{SL}_2 \to \mathrm{GL}_n$ defined over k such that $\psi|_{\mathrm{SL}_2(\mathbb{Q})} = \varphi$.*

Proof. This is in [Ste85, p. 343]. Another proof is given in [Cap09, Lemma 5.9]. □

The following proposition is a refinement of an important result by Caprace.

Proposition 5.8. *Let k be a field and let $G(k)$ be an almost split Kac–Moody group obtained by Galois descent. Let $\varphi\colon \mathrm{SL}_2(\mathbb{Q}) \to G(k)$ be a homomorphism. Then* $\operatorname{im}\varphi$ *is bounded and fixes two opposite points of $\Delta(G)$.*

Proof. Let E be a field over which G splits and let $\iota\colon G(k) \to G(E)$ denote the canonical inclusion. By [Cap09, Corollary 5.8], $\iota \circ \varphi$ has bounded image and if φ is non-trivial (which we may assume), E has characteristic 0. By Lemma 5.5, $\operatorname{im}\varphi$ fixes points x, y in both halves of $\Delta(G(k))$. Then $\Omega := \{x, y\}$ is contained in a twin apartment, i.e. it is balanced. Let $\psi := \mathrm{Ad}_\Omega \circ \varphi\colon \mathrm{SL}_2(\mathbb{Q}) \to \mathrm{GL}(W_\Omega)$. Then ψ, although a priori only an abstract representation, is in fact rational by the previous proposition. It follows that the Zariski closure C of $\psi(\mathrm{SL}_2(\mathbb{Q}))$ is a reductive group. By Proposition 5.6, C can be conjugated inside a Levi factor of $\mathrm{Ad}_\Omega(\operatorname{Fix}\Omega)$ by an element of $G(k)$. This Levi factor is precisely the stabiliser of two opposite points, from which the claim follows. □

5.2 Unbounded algebraic subgroups

Let k be a field with infinite transcendence degree over its prime field and let $\mathbf{K}$ be a connected k-isotropic algebraic k-group. In this section, we will give a construction of a homomorphism $\varphi\colon \mathbf{K}(k) \to G(k)$ with unbounded image, where $G(k)$ is a certain Kac–Moody group.

In the case of affine Kac–Moody groups, there is an easy criterion to check whether a subgroup is bounded or not. Let k be a field and let $\deg_t$, $\deg_{t^{-1}}$ denote the valuations of $k(t)$ with t and t^{-1} as uniformizing parameter, respectively. For an element

$$f = \sum_{i=N_0}^{N_1} a_i t^i \in k[t, t^{-1}]$$

with $a_{N_0} \neq 0, a_{N_1} \neq 0$ this means that $\deg_t(f) = N_0$. Since $\deg_{t^{-1}}(f) = \deg_t f(\frac{1}{t})$ it follows that $\deg_{t^{-1}}(f) = -N_1$.
For a matrix $g = (g_{ij}) \in \mathrm{SL}_n(k[t, t^{-1}])$, let $\deg_t(g) := \max_{i,j} \deg_t g_{ij}$ and similarly let $\deg_{t^{-1}}(g) := \max_{i,j} \deg_{t^{-1}} g_{ij}$.
With this notation, we have the following:

Proposition 5.9. *A subgroup U of $\mathrm{SL}_n(k[t, t^{-1}])$ is bounded if and only if there exists $N \in \mathbb{N}$ such that $|\deg_t(g)| \leq N$ and $|\deg_{t^{-1}}(g)| \leq N$ for all $g \in U$.*

Proof. The positive (resp. negative) half of the twin building can be identified with the affine building of $\mathrm{SL}_n(k(t))$ with the discrete valuation $\deg_t$ (resp. $\deg_{t^{-1}}$). By [AB08, Ex. 11.40] U is bounded if and only if there is an upper bound on the absolute values of the matrix entries, which amounts to the claim. □

Let $\mathbf{K}$ be a reductive algebraic group defined over a field k which is isotropic over k.
Let $\mathfrak{k}$ denote the Lie algebra of $\mathbf{K}$. Then $\mathbf{K}(k)$ acts on $\mathfrak{k}(k)$, which is isomorphic to k^n as an abelian group, via the adjoint representation. Let $G := \mathrm{SL}_{n+1}(k[t, t^{-1}])$ and consider the following subgroup:

$$V = \left\{ \begin{pmatrix} & & \vdots \\ & \mathrm{Ad}\, g & v \\ & & \vdots \\ & & 1 \end{pmatrix} : g \in \mathbf{K}(k), v \in (k[t, t^{-1}])^n \right\}.$$

Note that $\det \mathrm{Ad}\, g = 1$ is automatic since $\mathbf{K}$ is reductive. Indeed, it suffices to check this over an algebraically closed field. Then $\mathbf{K}$ is the almost direct product of its center C and its derived group $[\mathbf{K}, \mathbf{K}]$. Now $\mathrm{Ad}\, C$ is trivial, whereas $\det \mathrm{Ad}\, g = 1$ for each element g of the derived group.

Lemma 5.10. *As an abstract group, V is isomorphic to*

$$\mathrm{Ad}\, \mathbf{K}(k) \ltimes \bigoplus_{i \in \mathbb{Z}} \mathfrak{k}(k),$$

where the action of $\mathrm{Ad}\, \mathbf{K}(k)$ on each summand of the direct sum is the natural one.

Proof. This is clear since $(k[t, t^{-1}], +)^n \cong \bigoplus_{i \in \mathbb{Z}} k^n$ by decomposing an element into its homogeneous components, and this decomposition is preserved under the action of $\mathrm{Ad}\, g$. The given matrix representation is just the standard one for a semidirect product of two linear groups. □

Until the end of this section, assume that the transcendence degree of k over its prime field k_0 is infinite. Let $T = (t_i)_{i \in \mathbb{Z}}$ be an infinite set of algebraically independent elements. Complete T by a set T' to a transcendence basis for k over k_0. For each $i \in \mathbb{Z}$, consider the derivation $\delta_i \colon k \to k$ obtained by extending the zero-derivation on $k_0(T' \cup T \backslash \{t_i\})$ to k by setting $\delta_i(t_i) = 1$, i.e. δ_i can be thought of as a partial derivative with respect to t_i. By the basic theory of derivations (cf. [Jac89]), this is clearly possible.
Since for each $x \in k$ there is a finite set I_x such that x is contained in an algebraic extension of $k_0(t_i : i \in I_x)$, it follows that for each $x \in k$, $\delta_i(x) \neq 0$ for only finitely many i.

The following observation is due to Borel-Tits [BT73, Example 8.18 b)] and is elaborated on in the paper of Lifschitz and Rapinchuk [LR01]. For a closed subgroup G of $\mathrm{GL}_n(k)$ and $\delta \colon k \to k$ a derivation, the matrix $g^{-1} \cdot \delta(g)$ is an element of $\mathfrak{g} = \mathrm{Lie}\, G$, where $\delta(g)$ is the matrix obtained by applying δ to all entries of g. Furthermore, the mapping $\varphi_\delta\colon G \to G \ltimes \mathfrak{g}, g \mapsto (g, g^{-1}\cdot \delta g)$ is a group homomorphism, where G acts on $\mathfrak{g}$ via the adjoint representation.

Following a suggestion by Caprace, we use this construction to describe the desired homomorphism with unbounded image.

Theorem 5.11. *Let k be a field with infinite transcendence degree over its prime field. Let $\mathbf{K}$ be a k-isotropic reductive k-group and let $n := \dim \mathbf{K}$. Let G be a Kac–Moody group whose diagram has a subdiagram of type $\widetilde{A_n}$ such that the derived group of the corresponding Levi factor $H(k)$ is isomorphic to $\mathrm{SL}_{n+1}(k[t,t^{-1}])$.*
Let $V = \mathrm{Ad}\,\mathbf{K}(k) \ltimes \bigoplus_{i\in\mathbb{Z}} \mathfrak{k}(k) \leq H(k) \leq G(k)$ be as above and let $(\delta_i)_{i\in\mathbb{Z}}$ be derivations as above.
Then the mapping

$$\varphi\colon \mathbf{K}(k) \to G(k), g \mapsto (\mathrm{Ad}\, g, (g^{-1}\cdot \delta_i(g))_{i\in\mathbb{Z}})$$

is a group homomorphism with unbounded image.

Proof. Due to the construction of the derivations, for each $g \in G$, $(g^{-1}\cdot\delta_i(g)) \neq 0$ for only finitely many $i \in \mathbb{Z}$, hence φ is well-defined. Since each φ_{δ_i} is a homomorphism, so is φ.
Let $T(k) \leq \mathbf{K}(k)$ be a k-split torus. Then for each i there is an element $g_i \in T(k) \cong (k^\times)^r$ such that $g^{-1}\cdot\delta_i(g) \neq 0$. This translates via Proposition 5.10 to the fact that $\varphi(g_i)$ has a matrix entry which has a homogeneous component of degree i. In particular, the degrees of the matrix entries of $\varphi(\mathbf{K}(k))$ are unbounded, which proves via Proposition 5.9 that φ has unbounded image in H. By Lemma 5.5, $\mathrm{im}\,\varphi$ is unbounded in G, too. □

Remark 5.12. If a field k is uncountable, it is of infinite transcendence degree over its prime field. In particular, a local field has infinite transcendence degree over its prime field, as follows from the classification of local fields.

This result is interesting in a different context, too. Following Farb [Far09], a group G is said to have property FA_n if every G-action by cellular isometries on an n-dimensional CAT(0) complex has a global fixed point.
Note that the geometric realization of $\mathrm{SL}_{n+1}(k[t,t^{-1}])$ has dimension n, since the apartments are tessellations of $\mathbb{R}^n$. This implies the following corollary.

Corollary 5.13. *Let k be a field with infinite transcendence degree over its prime field and let $\mathbf{K}$ be a reductive k-isotropic k-group of dimension n. Then $\mathbf{K}(k)$ does not have FA_n.*

The following discussion, leading up to Corollary 5.16, aims to make precise the informal statement that the space of quasi-morphisms of a Kac–Moody group G, although infinite-dimensional, cannot be used to check whether a subgroup is bounded or not. We quickly recall the relevant definitions and results in this direction.

For a group G, a map $\varphi\colon G \to \mathbb{R}$ is called a **quasi-morphism** if it satisfies

$$\sup_{g,h\in G} |\varphi(gh) - \varphi(g) - \varphi(h)| < \infty.$$

Let $\mathrm{QH}(G)$ denote the real vectorspace of all quasi-morphisms of G. Then $l^\infty(G)$, the space of bounded real-valued functions on G, and $\mathrm{Hom}(G,\mathbb{R})$ are subspaces of $\mathrm{QH}(G)$. Let $\widetilde{\mathrm{QH}}(G) := \mathrm{QH}(G)/(l^\infty(G)\oplus \mathrm{Hom}(G,\mathbb{R}))$ denote the space of non-trivial quasi-morphisms of G.

Theorem 5.14. *Let k be a field and let $G := \mathcal{G}_\mathcal{D}(k)$ be a split Kac–Moody group such that the Weyl group (W,S) of G is irreducible and neither spherical nor affine. Then $\widetilde{\mathrm{QH}}(G)$ is infinite-dimensional.*

Proof. This is [CF10, Theorem 1.1]. □

Let k be a local field and let G be a connected simply connected almost simple algebraic group defined over k of k-rank ≥ 2. By [BM99, Lemma 6.1], any *continuous* quasi-morphism $f\colon G(k) \to \mathbb{R}$ is trivial. When one restricts to $G = \mathrm{SL}_n(k)$ for $n \geq 3$, it is possible to drop the continuity assumption.

Proposition 5.15. *Let k be a field and let $n \geq 3$. Then any quasi-morphism $f\colon \mathrm{SL}_n(k) \to \mathbb{R}$ is bounded.*

Proof. The group $\mathrm{SL}_n(k)$ is boundedly generated by its root groups U_α, so it suffices to show that f is bounded on each U_α. Since $n \geq 3$, any two elements $a, b \in U_\alpha\backslash\{1\}$ are conjugate via some diagonal matrix. Indeed, without loss of generality let $U_\alpha = I_n + k\cdot e_{12}$ where I_n is the identity matrix and e_{12} is the matrix with the only non-zero entry $(e_{12})_{12} = 1$. Then $\mathrm{diag}(t,1,\dots,1,t^{-1})$ conjugates $I_n + e_{12}$ to $I_n + te_{12}$. Since f is bounded on conjugacy classes, the claim follows. □

For a group homomorphism $\varphi\colon H \to G$ there is a pull-back $\varphi^*\colon \mathrm{QH}(G) \to \mathrm{QH}(H)$ given by $\varphi^*(f)(x) := f(\varphi(x))$.

Corollary 5.16. *Let k be a field with infinite transcendence degree over its prime field. Let $G := \mathcal{G}_\mathcal{D}(k)$ be a split Kac–Moody group such that G contains $\mathrm{SL}_4(k[t,t^{-1}])$ as the derived group of a Levi factor and such that the Weyl group of G is irreducible and not affine. Then $\widetilde{\mathrm{QH}}(G)$ is infinite-dimensional.*
There is a homomorphism

$$\varphi\colon \mathrm{SL}_3(k) \to G$$

such that $\mathrm{im}\,\varphi$ is unbounded yet $|f(\mathrm{im}\,\varphi)| < K_f$ for each quasi-morphism $f \in \mathrm{QH}(G)$ and a constant K_f depending on f.

Proof. Note first that such a Kac–Moody group G exists, as it suffices to extend the root datum associated to $\mathrm{SL}_4(k[t,t^{-1}])$ to make the associated Cartan matrix to be of indefinite type. Then the first statement follows from Theorem 5.14.
Let $\varphi\colon \mathrm{SL}_3(k) \to G$ be the homomorphism with unbounded image constructed in Theorem 5.11. For $f \in \mathrm{QH}(G)$, the pull-back $\varphi^* f$ is bounded by Proposition 5.15, from which the claim follows. □

5.3 The case of number fields

While in the previous section we considered a field with infinite transcendence degree over its prime field, this section is concerned with the case where the field k is a finite algebraic extension of $\mathbb{Q}$.
We start with a variation on the classical primitive element theorem.

Lemma 5.17. *Let $L|\mathbb{Q}$ be a finite field extension and let $x \in L$ be a primitive element, i.e. $L = \mathbb{Q}(x)$. Then for $n \in \mathbb{N}$, there exists $y \in L$ such that $y, y^2, \dots, y^n$ all are primitive elements.*

Proof. Set $k := n!$ and observe that it is enough to find an element y such that y^k is primitive, since then for any p dividing k we clearly have $\mathbb{Q}(y^p) \supseteq \mathbb{Q}(y^k) = L$. Consider the sequence $y_i := x + i, i \in \mathbb{N}$ and set $z_i := {y_i}^k$. Since there are only finitely many intermediate fields between L and $\mathbb{Q}$, by the pigeonhole principle there is a subsequence (z_{i_l}) such that $\mathbb{Q}(z_{i_r}) = \mathbb{Q}(z_{i_s})$ for all i_r, i_s.
Note that in the polynomial ring $\mathbb{Q}[t]$ for pairwise different $a_1, \dots, a_{k+1} \in \mathbb{N}$ the polynomials $(t - a_i)^k$ form a basis of the subspace of polynomials of degree $\leq k$, since the coefficient vectors form a Vandermonde matrix (up to scaling). In particular, t lies in their $\mathbb{Q}$-span. Applying this analysis to $\mathbb{Q}(z_{i_r})$ then shows that this field contains x and hence equals L. ☐

Lemma 5.18. *Let L be a number field of degree n. Let $x \in L$ be such that x^2 is a primitive element. Then for $d := \operatorname{diag}(x, x^{-1})$, $\mathrm{SL}_2(L)$ is boundedly generated by d and $\mathrm{SL}_2(\mathbb{Q})$.*

Proof. Set $u_+(r) := \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right)$ and note that $d^i u_+(r) d^{-i} = u_+(rx^{2i})$. This implies that $U_L := \{u_+(l) : l \in L\}$ is generated by the subgroups $d^i U_{\mathbb{Q}} d^{-i}, i = 0 \dots, n-1$, where $n = [L : \mathbb{Q}]$.
Now $\mathrm{SL}_2(L)$ is boundedly generated by U_L and sU_Ls^{-1}, where $s := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, i.e. by $2n$ conjugates of $U_{\mathbb{Q}}$. ☐

Corollary 5.19. *Let K be a field and let $\mathcal{G}_{\mathcal{D}}$ be a Tits functor. Then for any number field L, every homomorphism $\varphi\colon \mathrm{SL}_2(L) \to \mathcal{G}_{\mathcal{D}}(K)$ has bounded image.*

Proof. Since $U_{\mathbb{Q}} \subseteq \mathrm{SL}_2(\mathbb{Q})$ has fixed points in both halves of the twin building by Proposition 5.8, so does every conjugate of $U_{\mathbb{Q}}$. By the preceding lemma and Lemma 5.3, the claim follows. ☐

Remark 5.20. (i) Similarly one can show: If $L|K$ is a finite extension, then $\mathrm{SL}_2(L)$ is boundedly generated by a finite number of conjugates of $\mathrm{SL}_2(K)$. Hence any homomorphism $\varphi\colon \mathrm{SL}_2(L) \to \mathcal{G}_{\mathcal{D}}(k')$ such that $\operatorname{im}\varphi(\mathrm{SL}_2(K))$ is bounded has bounded image.

(ii) As already remarked in [Cap09, Corollary 5.8], via bounded generation the above implies that any homomorphism of a split Chevalley group over a number field has bounded image.

The fixed point set of $\mathrm{SL}_2(L)$ might be smaller than that of its subgroup $\mathrm{SL}_2(\mathbb{Q})$, as the next example shows.

Example 5.21. Let L be a number field of degree $d > 1$ and let $\sigma_1, \sigma_2 \colon L \to \mathbb{C}$ be two different embeddings. Let $G = \mathrm{SL}_2$ and consider the homomorphism

$$\varphi \colon G(L) \to G(\mathbb{C}) \times G(\mathbb{C}), g \mapsto (\sigma_1(g), \sigma_2(g)).$$

Postcomposing with the standard inclusion $G(\mathbb{C}) \times G(\mathbb{C}) \to \mathrm{SL}_4(\mathbb{C})$, $G(L)$ then acts on the spherical building associated to $\mathrm{SL}_4(\mathbb{C})$. While $G(\mathbb{Q})$ fixes the residue R corresponding to the subspace $U := \langle(1,0,1,0),(0,1,0,1)\rangle$, the group $G(L)$ does not fix it.
In particular, $\varphi(G(\mathbb{Q}))$ is not Zariski dense in the closure of $\varphi(G(L))$: the closure of $\varphi(G(\mathbb{Q}))$ is the diagonal subgroup of the closure of $\varphi(G(L))$, which is $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. In particular, the dimension of the Zariski closure increases.
This observation leads to the proof of Theorem 5.31 in the next section.

5.4 The case of infinite algebraic extensions

In this section we address the question of infinite algebraic extensions. We use Margulis's rigidity result that any abstract representation of $\mathrm{SL}_2(L(S))$, where $L(S)$ is a certain subring of a number field, has semisimple Zariski closure.

Recall first that for a locally compact topological group G, a **lattice** Γ is a discrete subgroup of G such that G/Γ has a finite invariant measure. For locally compact topological groups $G_1, \ldots, G_n$, a lattice $\Gamma \leq G_1 \times \ldots \times G_n$ is said to be **irreducible** if the projection of Γ on each factor G_i is dense in G_i. If for each i, $G_i = \mathbf{G}_i(k_i)$, where k_i is a local field and $\mathbf{G}_i$ is a connected semisimple k_i-group without compact factors, then $\Gamma \leq G_1 \times \ldots \times G_n$ is irreducible if and only if no subgroup of finite index of Γ can be represented as the direct product of two infinite subgroups (cf. [Mar91, Introduction]).
The reference for the following paragraph is [Mar91, Introduction].
Let L be a number field and let R be the set of all (inequivalent) valuations of L. Let R_∞ denote the set of archimedean valuations of L, and for each $v \in R$ let L_v be the completion of L with respect to v. Let $|\cdot|_v$ denote the absolute value associated with v.
Let $S \subseteq R$ be a finite subset containing all archimedean valuations and suppose $|S| \geq 2$. Let $L(S) := \{x \in L : |x|_v \leq 1$ for all non-archimedean $v \in R \setminus S\}$ be the ring of S-integral elements of L.

Theorem 5.22 (Borel-Harish-Chandra-Behr-Harder reduction theorem)**.** *Let $G = \mathrm{SL}_2$. Then $\Gamma := G(L(S))$ is an irreducible lattice in $G_S := \prod_{v \in S} G(L_v)$.*

Proof. This is the special case $G = \mathrm{SL}_2$ of the result quoted in [Mar91, Page 1]. ☐

The following lemma combines several standard facts about lattices. For lack of a reference, we include a proof.

Lemma 5.23. *Let $\Gamma = G(L(S))$ be as above. Then there is a subgroup $\Gamma_0 \leq \Gamma$ of finite index such that Γ_0 has trivial center and is an irreducible lattice in G_S.*

Proof. Selberg's lemma for finitely generated linear groups implies that Γ is virtually torsion-free, i.e. there is a subgroup $\Gamma_0 \leq \Gamma$ of finite index which is torsion-free.
Since $\Gamma_0 \leq \Gamma$ and $\mathrm{vol}(G/\Gamma_0) = [\Gamma : \Gamma_0] \, \mathrm{vol}(G/\Gamma)$, Γ_0 is a lattice.
By the second criterion for irreducibility given above, it is clear that Γ_0 is irreducible, as a direct product of finite index in Γ_0 would be of finite index in Γ.
The Borel density theorem implies that the center of Γ is central in G_S. Since $\mathscr{Z}(G_S)$ is finite, any central element in Γ_0 has finite order, hence must be trivial. □

In this setting, Margulis's Rigidity Theorem takes the following form.

Theorem 5.24. *Let $G_S = \prod_{v \in S} \mathrm{SL}_2(L_v)$ be as above and let $\Gamma \leq G_S$ be an irreducible lattice. Let K be a field of characteristic 0, F an algebraic group defined over K and $\varphi \colon \Gamma \to F(K)$ a homomorphism. Then the Zariski closure of $\varphi(\Gamma)$ is a semisimple algebraic group defined over K.*

Proof. This is Theorem 3 of [Mar91, Introduction] adapted to the present situation. □

Remark 5.25. In general, Γ will admit non-trivial finite quotients. If Q is such a quotient, the natural map $\varphi \colon \Gamma \to Q \hookrightarrow \mathrm{GL}_n(F)$ for some $n \in \mathbb{N}$ and a field F shows that the Zariski closure of $\varphi(\Gamma)$ need not be connected.

To prove Theorem 5.31, we will apply Theorem 5.24 to the lattice Γ_0 provided by Lemma 5.23. Before proceeding to the proof of this theorem, we need some lemmas on bounded subgroups.

Let $U \leq \mathcal{G}(k)$ be a bounded subgroup of a split Kac–Moody group $\mathcal{G}(k)$. Let Δ^U denote the fixed point set of the U-action on the CAT(0) realization of the twin building. For $x \in \Delta^U_+$ and $y \in \Delta^U_-$ let $\Omega(x, y) := \{x, y\}$. Then there is a twin apartment $\mathcal{A}$ which contains x, y: choose a chamber C_+ which contains x and a chamber C_- which contains y, then there is a twin apartment which contains C_+ and C_-.

$\Omega(x, y)$ is a balanced subset of $\mathcal{A}$. Let $W := W_{\Omega(x,y),\mathcal{A}}$ denote the $\mathrm{Fix}(\Omega(x, y))$-subspace of $\mathcal{U}_{\mathcal{D}}$ provided by Proposition 1.8. Let $U_{x,y,\mathcal{A}}$ denote the Zariski closure of $\mathrm{Ad}\, U|_{W_{\Omega(x,y),\mathcal{A}}}$.

Remark 5.26. The group $U_{x,y,\mathcal{A}}$ depends in general on the choice of x and y. For instance, let $U = T$ be the standard torus which fixes the standard twin apartment $\mathcal{A} = (\mathcal{A}_+, \mathcal{A}_-)$. Let C_+ be the standard positive chamber and C_- the standard negative chamber.
Choose x_0 in the interior of C_+ and y_0 in the interior of C_-. Then $\Delta(x, y) = \emptyset$, i.e. there is no half-apartment containing both x and y. By the calculation of $\ker \mathrm{Ad}$ (see Proposition 1.8) it follows that $U_{x_0,y_0,\mathcal{A}} = 1$.
On the other hand, let α be a simple root and let $x_1 \in \mathcal{A}_+$ be in the interior of $\partial\alpha$ but not in the interior of any other $\partial\beta$, where β is a positive simple root distinct

from α. Let $y \in \mathcal{A}_-$ be chosen similarly. Then $\Delta(x_1, y_1) = \Delta^m(x_1, y_1) = \{\alpha, -\alpha\}$. By the calculation of $\ker \operatorname{Ad}$ it follows that $U_{x_1,y_1,\mathcal{A}}$ is a 1-dimensional torus.

The point of this remark is that one has to restrict to bounded subgroups which have trivial center.

Theorem 5.27. *Let k be a field and let $G := \mathcal{G}_\mathcal{D}(k)$ be a split Kac–Moody group. Let $U \leq G$ be a bounded subgroup with trivial center and let $x, x' \in \Delta_+, y, y' \in \Delta_-$ be U-fixed points. Let $\mathcal{A}$ be a twin apartment which contains x, y and $\mathcal{A}'$ a twin apartment which contains x', y'. Let $U_{x,y,\mathcal{A}}$ denote the Zariski closure of $\operatorname{Ad} U|_{W_{\Omega(x,y),\mathcal{A}}}$ and similarly $U_{x',y',\mathcal{A}'}$ for $x', y', \mathcal{A}'$.*
Then $U_{x,y,\mathcal{A}}$ and $U_{x',y',\mathcal{A}'}$ are isomorphic as algebraic groups.

Proof. Note first that if x_0, y_0 are two points fixed by U and $\mathcal{A}_0$ is a twin apartment containing them, then the restriction of $\operatorname{Ad} U$ to $W_{x_0,y_0,\mathcal{A}_0}$ is injective: the kernel of this restriction is central in $\operatorname{Fix}_G\{x_0, y_0\}$, so in particular its intersection with U must be central in U, but U has trivial center.

Step 1. If $\mathcal{A}''$ is another apartment containing x and y, then $U_{x,y,\mathcal{A}} \cong U_{x,y,\mathcal{A}''}$.
The group $\operatorname{Fix}_G\{x, y\}$ acts transitively on the set of apartments containing x and y by [Rém02, Proposition 10.4.4 (iii)]. Let $g \in \operatorname{Fix}(\{x, y\})$ be such that $g\mathcal{A} = \mathcal{A}''$. Then $\operatorname{Ad} g$ conjugates $W_{x,y,\mathcal{A}}$ to $W_{x,y,\mathcal{A}''}$ and hence $U_{x,y,\mathcal{A}}$ onto $U_{x,y,\mathcal{A}''}$.

Since $U_{x,y,\mathcal{A}}$ is independent from the twin apartment $\mathcal{A}$ containing x, y, we can write $U_{x,y} := U_{x,y,\mathcal{A}}$ unambiguously.

Step 2. It suffices to prove that $U_{x,y} \cong U_{x',y}$, where x, x', y are contained in a common apartment and x, x' are contained in a common chamber
Let $\rho\colon [0, r] \to \Delta_+, \rho(0) = x, \rho(r) = x'$ be a geodesic, where $r := d(x, x')$. Then $\operatorname{im} \rho$ is fixed by U.
Let $0 = i_0 < i_1 < \ldots < i_s = r$ be such that $\rho([i_j, i_{j+1}])$ is contained in a chamber C_j, $j = 0, \ldots, s-1$. Let $x_j := \rho(i_j)$. Suppose it is already proven that $U_{x_j,y} \cong U_{x_{j+1},y}$. Since $x_0 = x, x_s = x'$ it then follows that $U_{x,y} \cong U_{x',y}$, and arguing similarly for y it follows that $U_{x',y} \cong U_{x',y'}$.

Step 3. Conclusion. Let $F_I, F_{I'}$ be facets containing x, x' and maximal with this property. Then H fixes both F_I and $F_{I'}$ and hence $F_{I\cap I'}$. Replace the geodesic from x to x' by the union of a geodesic from x to a point z in $F_{I\cap I'}$ and a geodesic from z to x'. This allows to assume that $I \subseteq I'$ or $I' \subseteq I$, without loss of generality assume that $I \subseteq I'$.
With the notations from Proposition 1.8, let

$$\Delta_1 := \Delta^m(\{x, y\}) \cup \Delta^u(\{x, y\}) \cup -\Delta^u(\{x, y\})$$

and

$$\Delta_2 := \Delta^m(\{x', y\}) \cup \Delta^u(\{x', y\}) \cup -\Delta^u(\{x', y\}).$$

Then $\Delta_1 \subseteq \Delta_2$ as $I \subseteq I'$.
It follows that $W_{x,y,\mathcal{A}} \subseteq W_{x',y,\mathcal{A}}$. Since U is contained in $\mathrm{GL}(W_{x,y,\mathcal{A}}) \leq \mathrm{GL}(W_{x',y,\mathcal{A}})$, it follows that the Zariski closure of U, when computed in $\mathrm{GL}(W_{x,y,\mathcal{A}})$ is the same as when computed in $\mathrm{GL}(W_{x',y,\mathcal{A}})$, from which the claim follows. □

The independence of the fixed points x, y allows us to associate a canonical subgroup to a bounded subgroup with trivial center.

Definition 5.28. *Let k be a field and let $G := \mathcal{G}_{\mathcal{D}}(k)$ be a split Kac–Moody group. Let $U \leq G$ be a bounded subgroup with trivial center. Let $x \in \Delta_+, y \in \Delta_-$ be two points fixed by U. Then $\overline{\mathrm{Ad}\,U} := U_{x,y}$ is called the* **Zariski closure of U.**
Let $\mathrm{Addim}\,U := \dim \overline{\mathrm{Ad}\,U}$ denote the **Ad-dimension** *of U.*

We apply this to the study of the fixed point set.

Lemma 5.29. *Let U, U' be two bounded subgroups with trivial center such that $U' \leq U$. If $\mathrm{Fix}\,U \subsetneq \mathrm{Fix}\,U'$ then either there exists a finite index subgroup $U^* \leq U$ such that $\mathrm{Fix}\,U' \subseteq \mathrm{Fix}\,U^*$ or $\mathrm{Addim}\,U' < \mathrm{Addim}\,U$.*

Proof. Suppose without loss of generality that $\mathrm{Fix}_U(\Delta_+) \subsetneq \mathrm{Fix}_{U'}(\Delta_+)$. Consider the CAT(0) realization of Δ_+ and let $x \in \mathrm{Fix}_{U'}(\Delta_+) \setminus \mathrm{Fix}_U(\Delta_+)$ and $y \in \mathrm{Fix}_U(\Delta_+)$. Consider a geodesic segment $p\colon [0, r] \to \Delta_+$ such that $p(0) = x, p(r) = y$. Let $s \in [0, r]$ be minimal such that $p(s) \in \mathrm{Fix}_U(\Delta_+)$. Let $z \in \mathrm{Fix}_U(\Delta_-)$ and let $P := \mathrm{Fix}_G\{p(s), z\}$. Then U is contained in the bounded subgroup P.
Note that $s > 0$ and that for some $\varepsilon > 0$ the segment $[p(s - \varepsilon), p(s)]$ is contained in a residue R', which is a proper residue of the spherical building associated to $p(s)$. This in turn says that $\overline{\mathrm{Ad}\,U'}$ is contained in a proper parabolic P' of P, while $\overline{\mathrm{Ad}\,U}$ is not. If the connected component of the identity of $\overline{\mathrm{Ad}\,U}$ is not contained in P, we must have $\mathrm{Addim}\,U > \mathrm{Addim}\,U'$ by [Spr98, 1.8.2] applied to $\overline{\mathrm{Ad}\,U'}^0$ and $\overline{\mathrm{Ad}\,U}^0$. Otherwise let U^* be the preimage of $\mathrm{Ad}\,U \cap \overline{\mathrm{Ad}\,U}^0$ in U. Then U^* is of finite index in U and $\overline{\mathrm{Ad}\,U^*}$ is connected. By the same reasoningthe fixed point set of the group U^* must necessarily contain the fixed point set of U'. □

Remark 5.30. Clearly, the fixed point set of a finite index subgroup U' of U can be much larger. For example, $\mathrm{Sym}(n)$ operates on $V := k^n$ via permutation of the basis vectors and leaves the subspace generated by $v = (1, \ldots, 1)$ invariant, and the induced action on $V/\langle v\rangle \cong k^{n-1}$ is irreducible. It follows that $\mathrm{Sym}(n)$ is a subgroup of $\mathrm{SL}_{n-1}(k)$ which does not fix a proper residue of the spherical building associated to $\mathrm{SL}_{n-1}(k)$, while it virtually acts trivially.

Theorem 5.31. *Let L be an algebraic extension of $\mathbb{Q}$. Let k' be a field of characteristic 0 and let $G := \mathcal{G}_{\mathcal{D}}(k')$ a split Kac–Moody group. Then any homomorphism $\varphi\colon \mathrm{SL}_2(L) \to G$ has bounded image.*

Proof. If L is a finite extension, the result follows from Corollary 5.19. Otherwise let $(L_i)_{i\in\mathbb{N}} \subseteq L$ be an ascending sequence of numberfields such that $L = \bigcup L_i$. For each i, choose a set of valuations S_i of L_i such that $W_i := \mathrm{SL}_2(L_i(S_i))$ is an irreducible lattice (cf. Theorem 5.22) and such that $S_i \subseteq S_{i+1}$. Let $U_i \leq W_i$ be a finite index subgroup with trivial center, as provided by Lemma 5.23.

For each i, $\varphi(U_i)$ is bounded by Corollary 5.19 as $U_i \leq L_i$. By Theorem 5.24, the Zariski closure of $\varphi(U_i)$ is semisimple, hence by Lemma 5.6 contained in a Levi factor. In particular, the Ad-dimension of $\varphi(U_i)$ is bounded above since there are only finitely many conjugacy classes of Levi factors in G.
Pick i_0 such that $\operatorname{Addim} U_{i_0} = \max\{\operatorname{Addim} U_i : i \in \mathbb{N}\}$ and let x be a fixed point of U_{i_0}. Let U_i^* be the finite index subgroup of U_i provided by Proposition 5.29 which fixes x. Then $U^* := \langle U_i^* : i \in \mathbb{N}\rangle$ fixes x and for each i, the index of $U^* \cap U_i$ in U_i (and hence in W_i) is finite.
Let $\mathcal{O}_i$ denote the ring of integers of L_i. Then $\mathrm{SL}_2(\mathcal{O}_i) \leq W_i$ and hence $V_i := U^* \cap \mathrm{SL}_2(\mathcal{O}_i)$ has finite index in $\mathrm{SL}_2(\mathcal{O}_i)$.
We will show that U^* and $\mathrm{SL}_2(\mathbb{Q})$ boundedly generate $\mathrm{SL}_2(L)$, which will imply the claim by Lemma 5.3. Since every $g \in \mathrm{SL}_2(L)$ is contained in some $\mathrm{SL}_2(L_i)$, it suffices to prove that $\mathrm{SL}_2(\mathbb{Q})$ and V_i boundedly generate $\mathrm{SL}_2(L_i)$ and uniformly so. This is the content of the following lemma. □

Let N_0 denote the maximum number of elementary matrices needed to express a matrix $g \in G = \mathrm{SL}_2(K)$ as a product of elementary matrices. Note that each torus element t is the product of at most 6 elementary matrices, e.g. $t = m(u)m(1)$. By the Bruhat decomposition, $G = TU_+ \cup U_+TsU_+$, so $N_0 \leq 11$.

Lemma 5.32. *Let L be a number field and $\mathcal{O}$ its ring of integers. Let V be a subgroup of $\mathrm{SL}_2(\mathcal{O})$ of finite index. Then every element of $\mathrm{SL}_2(L)$ can be written as a product of at most $3N_0$ matrices from either $\mathrm{SL}_2(\mathbb{Q})$ or V.*

Proof. For x in L there is some $q \in \mathbb{N}$ such that $qx \in \mathcal{O}$. Since $\mathcal{O}$ is a ring containing $\mathbb{Z}$, $q^2x \in \mathcal{O}$. Since V has finite index in $\mathrm{SL}_2(\mathcal{O})$ there is some $a \in \mathbb{N}$ such that $u_+(a^2q^2x) \in V$.
We may write $u_+(x) = \operatorname{diag}((aq)^{-1}, aq)u_+(a^2q^2x)\operatorname{diag}(aq, (aq)^{-1})$, from which the claim follows. □

5.5 The general case

Let k, k' be two fields of characteristic 0, let $G(k')$ be a split Kac–Moody group and let $\varphi\colon \mathrm{SL}_2(k) \to G(k')$ be an abstract homomorphism. By Theorem 5.31, if $\operatorname{tr.deg}(k|\mathbb{Q}) = 0$, φ has bounded image, while if $\operatorname{tr.deg}(k|\mathbb{Q}) = \infty$ and G has a Levi factor isomorphic to $\mathrm{SL}_n(k[t, t^{-1}])$, φ might possibly have unbounded image by Theorem 5.11.
We still need to treat the case when $0 < \operatorname{tr.deg}(k|\mathbb{Q}) < \infty$. Here we propose a criterion that ensures that φ will have bounded image. This criterion can be checked to be satisfied when φ is induced by an abstract isomorphism $\psi\colon G(k) \to G(k')$ of two split Kac–Moody groups; in particular, ψ will preserve bounded subgroups.
This gives a variation of Caprace's proof of the isomorphism theorem for simply connected split Kac–Moody groups over fields of characteristic 0.

Theorem 5.33. *Let k, k' be two fields of characteristic 0, let $G := \mathcal{G}_{\mathcal{D}}(k')$ be a split Kac–Moody group with associated twin building Δ and let $\varphi\colon \mathrm{SL}_2(k) \to G(k')$ be a homomorphism.*
Suppose that there is a twin apartment $\mathcal{A}$ of Δ such that

$$T(k) := \{\mathrm{diag}(x, x^{-1}) : x \in k^\times\} \leq \mathrm{SL}_2(k)$$

stabilizes $\mathcal{A}$ and that $\varphi(\mathrm{SL}_2(\mathbb{Q}))$ fixes two opposite points x, y of $\mathcal{A}$.
Then φ has bounded image.

The proof of the theorem is achieved via some lengthy calculations. One has to rule out the case that a diagonal matrix t will act via a translation on $\mathcal{A}$. Intuitively, this is not possible since there is a large subgroup X containing $\mathrm{SL}_2(\mathbb{Q})$ which fixes x and y and such that X has a large intersection with ${}^tX$, while the intersection of $\varphi(X)$ and $\varphi({}^tX) = \varphi(t)\varphi(X)\varphi(t)^{-1}$ would be too small if $\varphi(t)$ acted as a translation.
We record some lemmas before proving the theorem.

Lemma 5.34. *Let (W, S) be a finitely generated Coxeter group and let $A \leq W$ be a solvable subgroup. Then A is finitely generated.*

Proof. Let X be the CAT(0) realization of W. Since W is finitely generated, X is finite-dimensional by construction and W acts on X properly and cocompactly. The conclusion follows now from [BH99, p. 439 Theorem I.1 (3),(4)]. ☐

Remark 5.35. Clearly, arbitrary subgroups of finitely generated Coxeter groups need not be finitely generated: By the strong Tits alternative for Coxeter groups [NV02], any finitely generated Coxeter group W which is not virtually abelian, i.e. neither spherical nor affine, contains a non-abelian free group F. Then $[F, F]$ is not finitely generated.

In the setting of Theorem 5.33, Lemma 5.34 will provide a subgroup $S \leq k^\times$ such that $k^\times/S$ is finitely generated. This is why we investigate such a group S more closely.

Let k be a field of characteristic 0 and S a subgroup of $k^\times$. Let

$$R_S := \{\sum_{i=1}^{n} a_i s_i^2 : n \in \mathbb{N}, a_i \in \mathbb{Q}, s_i \in S\}$$

Then R_S is a subring of k since $S \leq k^\times$ and coincides with $\mathbb{Q}[s^2 : s \in S]$.

The importance of R_S for our purposes will become clear later on. Here we record a first lemma where R_S appears.

For a field k and $s \in k^\times, r \in k$ let $h(s) := \left(\begin{smallmatrix} s & \\ & s^{-1} \end{smallmatrix}\right), u_+(r) := \left(\begin{smallmatrix} 1 & r \\ & 1 \end{smallmatrix}\right), u_-(r) := \left(\begin{smallmatrix} 1 & \\ r & 1 \end{smallmatrix}\right)$ denote the standard parametrization of diagonal resp. upper/lower unipotent matrices in $\mathrm{SL}_2(k)$.

Lemma 5.36. *Let k be a field of characteristic 0 and let $S \leq k^\times$ be a subgroup which contains $\mathbb{Q}^\times$. Let $r \in k$. Then $U_r := \langle h(S), u_+(r)\rangle = h(S) \ltimes u_+(rR_S)$ and $V_r := \langle h(S), u_-(r)\rangle = h(S) \ltimes u_-(rR_S)$.*

Proof. To show that $u_+(rR_S) \leq U_r$ it suffices to show that $u_+(ras^2) \in U_r$ where $a = \frac{b}{c} \in \mathbb{Q}_{>0}$ and $s \in S$. This follows from the fact that

$$u_+(ras^2) = (h(\tfrac{s}{c})u_+(r)h(\tfrac{s}{c})^{-1})^{bc}.$$

Clearly, $h(S)$ and $u_+(rR_S)$ generate U_r and have trivial intersection. Since $h(S)$ normalizes $u_+(rR_S)$, the claim follows. The proof for V_r is similar. □

The following theorem is a classical result by Albert Brandis ([Bra65]).

Theorem 5.37. *Let k_0 be an infinite field and let k be a field extension of k_0. If $k^\times/k_0^\times$ is finitely generated, then $k = k_0$.* □

Lemma 5.38. *Let $k = \mathbb{Q}(t)$ and let S be a subgroup of $k^\times$ such that $k^\times/S$ is finitely generated. Let $k_0 := \mathrm{Quot}(R_S)$ denote the field of fractions of R_S. Then $k_0 = k$.*

Proof. By Lüroth's theorem, $[k : k_0] < \infty$ since S contains an element x which is transcendental over $\mathbb{Q}$.
Let $S_1 := S \cap k_0^\times$. Note that for each $s \in S$, $s^2 \in R_S$, i.e. s is contained in a quadratic extension $k_0(s)$ of k_0. There are only finitely many of these since t is a primitive element for the field extension $k|k_0$.
Let $s_1, \ldots, s_r \in S$ be such that $\{k_0(s_1), \ldots, k_0(s_r)\} = \{k_0(s) : s \in S\}$. We claim that $S = S_1\langle s_i : i = 1, \ldots, r\rangle$. Indeed, for $s \in S$ there is some s_i and there are elements $a, b \in k_0$ such that $s = as_i + b$. If $a = 0$, $s \in S_1$; if $b = 0$, $s \in s_iS_1$. Otherwise note that $2abs_i = s^2 - (as_i)^2 - b^2 \in k_0$, whence $s_i \in k_0$, a contradiction.
This shows that $k^\times/S_1$ is finitely generated, so in particular $k^\times/k_0^\times$ is finitely generated. By the previous theorem, this implies that $k = k_0$. □

Remark 5.39. The requirement that $k = \mathbb{Q}(t)$ was used solely to prove the fact that $\mathrm{Quot}(R_S)$ admits only finitely many quadratic extensions inside k. It thus seems reasonable to suspect that the same conclusion holds whenever k is a field of characteristic 0. For our purposes, though, the lemma is sufficient.

The following lemma asserts that the intersection of some big subgroup of $\mathrm{SL}_2(\mathbb{Q}(t))$ with one of its conjugates still is rather big.

Lemma 5.40. *Let $k := \mathbb{Q}(t)$ and let $S \leq k^\times$ be a subgroup which contains $\mathbb{Q}^\times$ and such that $k^\times/S$ is finitely generated. Let $X := \langle h(S), \mathrm{SL}_2(\mathbb{Q})\rangle$ and set $I := X \cap h(t)Xh(t)^{-1}$.*
Then there is an element $r \in R_S$ such that $X \leq \langle u_-(\frac{1}{r^2}), I\rangle$.

Proof. By Lemma 5.38, $\mathrm{Quot}(R) = k$, so in particular there are $r, s \in R$ such that $r = t^2s$. Since X contains $u_+(r)$ and $h(t)Xh(t)^{-1}$ contains $u_+(t^2s)$, it follows that $u_+(r) \in I$. Let $W := \langle u_-(\frac{1}{r^2}), I\rangle$.

Since I contains S, it follows from Lemma 5.36 that W contains $u_-(\frac{1}{r^2}R)$. In particular, $u_-(\frac{-1}{r})$ and $u_-(\mathbb{Q})$ are contained in W. Then

$$s(r) := \begin{pmatrix} & r \\ \frac{-1}{r} & \end{pmatrix} = u_-(\tfrac{-1}{r})u_+(r)u_-(\tfrac{-1}{r})$$

is contained in W, and so is $s(r)u_-(\frac{-1}{r^2}R)s(r)^{-1} = u_+(R)$. It follows that W contains $u_+(\mathbb{Q})$, hence $\mathrm{SL}_2(\mathbb{Q})$ and X. □

Finally, we need a result on the intersection of two fixators of opposite points in $\mathcal{A}$. We freely use the notation of [Cap09, Section 3.1].

Lemma 5.41. *Let $L^J = P_+^J \cap P_-^J$ be a Levi factor of finite type and let $w \in W$. Then $L^J \cap wL^Jw^{-1} = L^{J\cap wJw^{-1}}$.*

Proof. We calculate

$$\begin{aligned} L^J \cap wL^Jw^{-1} &= (P_+^J \cap P_-^J) \cap (wP_+^J w^{-1} \cap wP_-^J w^{-1}) \\ &= (P_+^J \cap wP_-^J w^{-1}) \cap (P_-^J \cap wP_+^J w^{-1}) \\ &= (L^{J\cap wJw^{-1}} \ltimes U_+^{J,J,w}) \cap (L^{J\cap wJw^{-1}} \ltimes U_-^{J,J,w}) \\ &= L^{J\cap wJw^{-1}} \end{aligned}$$

Here the last equality follows since "$\supseteq$" is obvious, while $U_-^{J,J,w} \cap U_+^{J,J,w}$ is trivial. □

With these ingredients, we can turn to the proof of Theorem 5.33.

Proof. Let $T := T(k) = \{h(t) : t \in k^\times\} \leq \mathrm{SL}_2(k)$ denote the torus. From now on, we identify a diagonal matrix $h(t)$ with the field element t.
By assumption $\varphi(T)$ acts on $\mathcal{A}$. Let $S \leq k^\times$ be such that S is the kernel of this action. Then $\varphi(T)/\varphi(S)$ is a subgroup of the Coxeter group W of G. By Lemma 5.34, $k^\times/S$ is finitely generated. Let $s_1, \ldots, s_k$ be generators for T/S.
Assume first that each s_i has a fixed point in Δ. Since T is boundedly generated by the s_i and S, T has a fixed point, and then $\varphi(\mathrm{SL}_2(k))$ is bounded since T and $\mathrm{SL}_2(\mathbb{Q})$ boundedly generate $\mathrm{SL}_2(k)$. To arrive at a contradiction, suppose there is some $s_i =: t$ which acts as a hyperbolic element on $\mathcal{A}$. We may assume that t is transcendental over $\mathbb{Q}$, as otherwise $h(t)$ would be contained in some $\mathrm{SL}_2(L)$ for L a finite extension of $\mathbb{Q}$, which has bounded image by Corollary 5.19.

In this case, we might assume that $k = \mathbb{Q}(t)$. Let $T' := \mathrm{Fix}_G\,\mathcal{A}$ denote the corresponding torus of G and let $(U_\alpha)_{\alpha\in\Phi}$ denote the root groups relative to T'. Then $\varphi(S) \leq T'$.

By assumption, $\varphi(\mathrm{SL}_2(\mathbb{Q})) \leq L^J = T'\langle U_\alpha : \alpha \in \Phi(W_J)\rangle$ for some spherical subset J. Assume that J is minimal with respect to this property.
It follows that $X := \langle S, \mathrm{SL}_2(\mathbb{Q})\rangle$ satisfies $\varphi(X) \leq L^J$. Note that

$$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} t & \\ & \frac{1}{t} \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \frac{1}{t} & \\ & t \end{pmatrix} = \begin{pmatrix} t^2 & \\ & \frac{1}{t^2} \end{pmatrix},$$

so t^2 is contained in $\langle X, tXt^{-1}\rangle$. It follows that $\varphi(X)$ and $\varphi(tXt^{-1})$ do not have a common fixed point, since otherwise this point would be fixed by t^2, hence t would have a bounded orbit which contradicts the assumption that t acts a hyperbolic element.

As $\varphi(t)$ acts via a translation w on $\mathcal{A}$ and $\varphi(X) \leq L^J$, it follows that $\varphi(tXt^{-1})$ is contained in wL^Jw^{-1}. Let $I := X \cap tXt^{-1}$ denote the intersection of these two groups. By Lemma 5.41, $\varphi(I) \leq L^J \cap wL^Jw^{-1} = L^{J\cap wJw^{-1}}$, and $J \cap wJw^{-1}$ is a proper subset of J. By Lemma 5.40, there is an element $u = u_-(r) \in \mathrm{SL}_2(k)$ such that $\mathrm{SL}_2(\mathbb{Q})$ is contained in $\langle I, u\rangle$. Then $\varphi(u) \in L^J$ again is unipotent since u is conjugate to $\left(\begin{smallmatrix} 1 & \\ 1 & 1 \end{smallmatrix}\right)$.
Now u commutes with the lower diagonal matrices of $\mathrm{SL}_2(\mathbb{Q})$ and $\varphi(u)$ is unipotent. It follows from the representation theory of SL_2 that $\varphi(u)$ and $\varphi(u_-(\mathbb{Q}))$ are both contained in the unipotent radical V of some Borel subgroup. It follows that $\varphi(\mathrm{SL}_2(\mathbb{Q}))$ is contained in $L^{J\cap wJw^{-1}} \ltimes V$. This is a contradiction, as $\varphi(\mathrm{SL}_2(\mathbb{Q}))$ could then be conjugated into a smaller Levi factor. □

In the proof we used the fact that the element u commutes with the lower triangular matrices of $\mathrm{SL}_2(\mathbb{Q})$. This is essential, as the following example shows.

Example 5.42. Let k be a field of cardinality ≥ 4 and let $G := \mathrm{SL}_3(k)$. Then $L := \left\{\begin{pmatrix} A & \\ & \frac{1}{\det A} \end{pmatrix} : A \in \mathrm{GL}_2(k)\right\} \leq G$ is a Levi factor. Let u denote the unipotent element $u := \left(\begin{smallmatrix} 1 & & 1 \\ & 1 & \\ & 1 & 1 \end{smallmatrix}\right) = x_{13}(1)x_{32}(1)$. Since L contains the diagonal matrices T of G and there is an element $t \in T$ centralizing $x_{13}(1)$ but not $x_{32}(1)$ it follows that $K := \langle L, u\rangle$ contains $x_{13}(r)$ and $x_{32}(r)$ for arbitrary $r \in k$, hence $K = G$.
This example shows that if $L \leq G$ is a proper Levi factor and $u \in G$ is a unipotent element, then it is possible that $\langle L, u\rangle = G$.

Remark 5.43. The idea for the proof of Theorem 5.33 was inspired by the following example. Let (K, v) be a field of characteristic 0 with a discrete valuation and suppose that the residue field k is finite. Let $S = \ker v$ denote the group of units of the corresponding valuation ring $\mathcal{O}$. Then $K^\times/S \cong \mathbb{Z}$, in particular, it is finitely generated.
For $X := \langle S, \mathrm{SL}_2(\mathbb{Q})\rangle$, one calculates that $X = \mathrm{SL}_2(\mathcal{O})$. Let $t \in K^\times$. Then $I = X \cap tXt^{-1}$ is precisely the fixator of two points in the Bruhat-Tits tree associated to $\mathrm{SL}_2(K)$. Since the residue field is supposed to be finite, it follows that I has finite index in both X and tXt^{-1}, i.e. X and tXt^{-1} are commensurable.
This follows from the fact that the sphere D around x with radius $d(x, y)$, where x, y are the unique points fixed by X and tXt^{-1} respectively, is finite and $X \cap tXt^{-1}$ is a point stabilizer in the permutation action of X on D.
If k is infinite, X and tXt^{-1} are no longer commensurable, but their intersection

still is “large” in the sense made precise by Lemma 5.40.

We can combine Theorem 5.33 with the machinery of [Cap09] and the study of isomorphisms which preserve bounded subgroups [CM06] to recover as a corollary Caprace's result that any isomorphism of simply connected split Kac–Moody groups over fields of characteristic 0 is standard.

Corollary 5.44. *Let k, k' be two fields of characteristic 0 and let $G(k), G'(k')$ be two split simply connected Kac–Moody groups.*

(i) Any isomorphism $\Psi\colon G \to G'$ preserves bounded subgroups.

(ii) Ψ is standard, i.e. preserves the twin root data associated to G and G'.

Proof. (i) Using the machinery of [Cap09], one gets a particular regular diagonalizable subgroup $H \leq G$ such that $\varphi(H)$ again is regular diagonalizable. Since H is centralized by the diagonal matrices D_α of a fundamental rank 1 group X_α, which is isomorphic to either $\mathrm{SL}_2(k)$ or $\mathrm{PSL}_2(k)$, it follows that $\varphi(D_\alpha)$ stabilizes the fixed point set of $\varphi(H)$, which is reduced to a twin apartment $\mathcal{A}$. Since $\varphi(X_\alpha(\mathbb{Q}))$ fixes two opposite points in the building and H normalizes $X_\alpha(\mathbb{Q})$, these fixed points must be contained in $\mathcal{A}$. By Theorem 5.33, $\varphi(X_\alpha)$ is bounded; in particular, every root group U_α has bounded image.
If X is bounded, it is by definition contained in a conjugate of a group of the form $L^J \ltimes U_\Psi$. It is clear that U_Ψ is boundedly generated by root groups, so it suffices to check that L^J is boundedly generated by root groups. Since G is simply connected, it follows that L^J is simply connected. By Bruhat decomposition, it suffices to check that $B = TU$ is boundedly generated by root groups. This is clear for U, and for T it follows by a calculation in $\mathrm{SL}_2(k)$.

(ii) Given (i), this follows from the main theorem of [CM06].

□

6 The isomorphism problem

In this chapter we prove that every abstract isomorphism of two 2-spherical almost split Kac–Moody groups over fields of characteristic 0 is standard in the sense that it induces an isomorphism of the associated canonical twin root data.

Remark 6.1. To approach the isomorphism problem one has to look for a certain class of subgroups which is rigid enough to conclude that any isomorphism will preserve this class.
In the case of split Kac–Moody groups, for each connected component of the diagram the **Chevalley involution** is an automorphism which switches the positive and the negative Borel group of the corresponding Levi factor. In particular, any such rigid class of subgroups has to be symmetric with respect to the positive and negative half of the twin building.
This naturally leads us to look at the images of the torus T and the root groups U_α. As abstract groups, these are not rigid at all as they are abelian or 2-step nilpotent. Another caveat to bear in mind is that the restriction of an isomorphism φ to a fundamental rank 1 group $X_\alpha := Z\langle U_\alpha, U_{-\alpha}\rangle$ a priori might have unbounded image. However, it is the *interplay* of the maximal split torus T_d and the root groups U_α which accounts for the rigidity.

Remark 6.2. If G_1, G_2 are two groups endowed with twin root data, their direct product $G_1 \times G_2$ is endowed with at least three different root data corresponding to the action of $G_1 \times G_2$ on $\Delta(G_1)$ via the first factor, on $\Delta(G_2)$ via the second factor and on $\Delta(G_1) \times \Delta(G_2)$. In particular, an automorphism of $G_1 \times G_2$ will have no reason to preserve a twin root datum of this group if the anisotropic kernel H is large.
The canonical twin root datum associated to an almost split Kac–Moody group $G(k)$, however, is in some sense as fine as possible, which allows for a detailed investigation.

6.1 Preparatory lemmas

We need a couple of preparatory lemmas, of which the relevance for our purposes will become clear later on.

Lemma 6.3. *Let k be an infinite field and let T be a k-split torus. Let $S \leq T(k)$ be such that $T(k)/S$ is finitely generated. Then S is Zariski dense in T.*

Proof. Since T is split over k and k is infinite, $T(k)$ is Zariski dense in T. Assume that $\bar{S} \neq T$. Then $\bar{S}$ is defined over k and so is $\bar{S}^0$, which is a k-split subtorus of T by [BT65, Corollary 1.9 b)]. It follows that $\dim \bar{S}^0 < \dim T$. Passing to the rational points, we find that $X := T(k)/(S \cap \bar{S}^0(k))$ contains a copy of $k^\times$, which is not finitely generated. As $S \cap \bar{S}^0(k)$ has finite index in S, X is finitely generated. This is a contradiction since any subgroup of a finitely generated abelian group is finitely generated. □

Proposition 6.4. *Let G be a group endowed with a twin root datum $(H, (U_\alpha)_{\alpha\in\Phi(W,S)})$. Let L be a subgroup of G such that for each $l \in L\backslash\{1\}$ there is a root $\beta_l \in \Phi(W,S)$ such that $l \in U_{\beta_l}$. Then there is some $\beta \in \Phi(W,S)$ such that $L \leq U_\beta$.*

Proof. Note first that $\beta_x = \beta_{x^{-1}}$ as U_{β_x} is a subgroup and β_x is uniquely determined as $U_\alpha \cap U_\beta = 1$ for distinct roots $\alpha \neq \beta$.
Assume for a contradiction that there are $x, y \in L\backslash\{1\}$ such that $\beta_x \neq \beta_y$, in particular $xy \neq 1$. Let $\alpha := \beta_x, \beta := \beta_y$ and $\gamma := \beta_{xy}$.
This implies that $U_\alpha U_\beta \cap U_\gamma \neq \{1\}$. Moreover, for any permutation π of $\{\alpha, \beta, \gamma\}$, it follows that $U_{\pi(\alpha)}U_{\pi(\beta)} \cap U_{\pi(\gamma)} \neq \{1\}$: If $u_\alpha u_\beta = u_\gamma$, then $u_\beta^{-1}u_\alpha^{-1} = u_\gamma^{-1}$, $u_\beta u_\gamma^{-1} = u_\alpha^{-1}$, and the permutations $(\alpha\,\beta), (\alpha\,\gamma\,\beta)$ generate $\mathrm{Sym}(\{\alpha, \beta, \gamma\})$. This implies that if two of the three roots coincide, then all roots coincide. So we can suppose that all three roots are distinct.
If two of the three roots are positive and the remaining one is negative (or vice versa), we can assume that $\alpha > 0, \beta > 0$ and $\gamma < 0$ since the statement to be proved is invariant under permutations of the roots. But this is a contradiction as $U_+ \cap U_- = \{1\}$. If all three roots have the same sign, say $\alpha, \beta, \gamma \in \Phi^+$, then choose some $w \in W$ such that $w\gamma = \delta$ is a positive simple root. If $w\alpha$ or $w\beta$ is negative, this is a contradiction by the case just discussed. If $w\alpha, w\beta, w\gamma$ are all positive, then $s_\delta w\alpha > 0, s_\delta w\beta > 0, s_\delta w_\gamma < 0$, which is again a contradiction by the case just discussed. □

We need a version of the classical Jacobson-Morozov lemma on the level of algebraic groups. The following proposition is a folklore result. For a lack of a reference, we include a proof kindly pointed out by Brian Conrad.

Proposition 6.5. *Let k be a field of characteristic 0 and let G be a connected reductive algebraic group defined over k. Let $g \in G(k)\backslash\{1\}$ be a nontrivial unipotent element. Then there exists a morphism $\varphi : \mathrm{SL}_2 \to G$ defined over k such that $\varphi(u) = g$ for some unipotent element $u \in \mathrm{SL}_2(k)$.*

Proof. Let $U := \overline{\langle u \rangle}$. As k is of characteristic 0, U is a one-dimensional unipotent group which is defined over k since $u \in G(k)$. This implies that U is k-isomorphic to $\mathbf{G}_a$. Let $\mathfrak{u} := \mathrm{Lie}\, U$. By the Jacobson-Morozov lemma (usually stated for *semisimple* Lie algebras over a field of characteristic 0, but holding in fact for arbitrary completely reducible subalgebras $\mathfrak{g} \leq \mathfrak{gl}(V)$, see the original paper [Jac51, Theorem 3]), there is a three-dimensional Lie subalgebra x which is k-isomorphic to $\mathfrak{sl}_2$ and contains $\mathfrak{u}$. As $\mathrm{char}\, k = 0$, any perfect Lie subalgebra is the Lie algebra of a closed subgroup X ([Bor91, Corollary 7.9]). This translates into the fact that there is a closed subgroup $X \leq G$ defined over k which is k-isomorphic to either SL_2 or PGL_2. This implies the claim. □

Let G be a connected reductive group defined over k which splits over k. Let T be a maximal torus and U a unipotent group which is normalized by T. Then $\langle T, U\rangle$ is contained in a Borel group B, so there is an ordering on the set of roots $\Phi(T, G)$ of T in G such that $U \leq U_+$. It is then a classical fact (cf. [BT73, p.65 l.7]) that U is generated by the root groups U_α relative to T which are contained in U.
We need an analogue of this theorem in case that G is not necessarily split over k.

Proposition 6.6. *Let k be an infinite field. Let G be a connected reductive k-group which is k-isotropic and let S be a maximal k-split torus. Let $U \leq G$ be a unipotent subgroup defined over k which is normalized by S. Then U is contained in $\langle U_\alpha : \alpha > 0\rangle$ for some ordering $>$ of the set of roots $\Phi(S, G)$ of S in G.*

Proof. Let P be a minimal parabolic subgroup defined over k which contains U and S. Then P has a Levi decomposition $P = Z(S)P_u$, where $Z(S)$ is the centralizer of S and P_u is the unipotent radical of P. Since S is maximal k-split, $Z(S)(k)$ does not contain any unipotent elements. This implies that $U(k) \leq P_u(k)$ and since $U(k)$ is dense in U, it follows that $U \leq P_u$, which implies the claim. □

Recall that a subgroup S of an almost split Kac–Moody k-group $G(k)$ is called **diagonalizable** (over k) if there is $g \in G(k)$ such that $gSg^{-1} \leq T_d(k)$, where T_d is the standard maximal k-split torus of $G(k)$.
Furthermore, a diagonalizable subgroup S is called **regular** if the fixed point set of the S-action on the associated twin building consists of a single twin apartment.

Among all diagonalizable subgroups of G, regular subgroups can be characterized purely group-theoretically. The following characterization can be found in [Cap09, Proposition 5.13] for split Kac–Moody groups. We generalize this to almost split Kac–Moody groups, where care has to be taken of the anisotropic kernel.

Lemma 6.7. *Let k be a field of characteristic 0 and let G be an almost split Kac–Moody k-group. Let $S \leq G(k)$ be a diagonalizable subgroup. Then S is regular if and only if S does not centralize a subgroup $X \leq G(k)$ isomorphic to either* $\mathrm{SL}_2(\mathbb{Q})$ *or* $\mathrm{PSL}_2(\mathbb{Q})$.

Proof. Without loss of generality we may assume that S is contained in the standard maximal k-split torus $T_d(k)$. Suppose first that S is regular and centralizes X. As X has a fixed point in both Δ_+ and Δ_- and is normalized by S, these points can be assumed to lie in the standard twin apartment A_k. As X normalizes S, it must stabilize A_k. Hence there is a homomorphism $\Psi : X \to W = \mathrm{Stab}_G(A_k)/\mathrm{Fix}_G(A_k)$. $\Psi(X)$ then is a finite group as it is a subgroup of a point stabilizer, so a finite index subgroup $X' \leq X$ is contained in the anisotropic kernel $\mathrm{Fix}_G(A_k) = Z(k)$. Then $X' = X$ as $\mathrm{PSL}_2(\mathbb{Q})$ is simple and $\mathrm{SL}_2(\mathbb{Q})$ does not have a proper finite index subgroup either. (Indeed, since $U_+(\mathbb{Q})$ and $U_-(\mathbb{Q})$ are divisible, any finite index subgroup $N \leq \mathrm{SL}_2(\mathbb{Q})$ contains $U_+(\mathbb{Q})$ and $U_-(\mathbb{Q})$, hence is equal to $\mathrm{SL}_2(\mathbb{Q})$.)
Postcomposing with the adjoint representation Ad_Ω, where $(A_k, F, -F)$ is a rational standardisation and $\Omega := \{F, -F\}$, there is a homomorphism

$$X \to \mathrm{Ad}_\Omega(Z(k))$$

which induces a representation of $\mathrm{SL}_2(\mathbb{Q})$. This representation is rational and defined over k by Proposition 5.7. Since the target group is anisotropic over k and therefore does not contain k-rational unipotent elements, this homomorphism must be trivial. Then $X \leq \ker \mathrm{Ad}_\Omega$, which is a contradiction since the latter group is abelian.
Conversely, suppose that S fixes a point $x \notin A_k$, without loss of generality suppose that $x \in \Delta_+$. Then there is a panel E of A_k and a chamber $C_1 \not\subseteq A_k$ which has E as a panel and is fixed by S. Indeed, let $\mathcal{G} = (C_0, C_1, \ldots, C_n)$ be a gallery such that $C_0 \in A_k, C_n$ contains x and $\mathcal{G}$ is of minimal length among all such galleries. Then $n \geq 1$ since $x \notin A_k$, and C_1 is fixed by S since the $S-$action is type-preserving.
Let $E = C_0 \cap C_1$ and let α be the corresponding root of A_k determined by C_0 and E. The root group $V_\alpha \leq G(k)$ parametrizes the chambers which have E as a panel and which are different from C_0. Since S fixes A_k and $C_1 \not\subseteq A_k$, there are three chambers of the E-panel fixed by S. This means that there is some non-trivial $v \in V_\alpha$ centralized by S. If $v \in V_\alpha \backslash \mathscr{Z}(V_\alpha)$, this implies that S centralizes the entire group V_α, if $v \in \mathscr{Z}(V_\alpha)$, this implies at least that S centralizes $\mathscr{Z}(V_\alpha)$ (recall that the action of the split torus is via a character on both $V_\alpha/\mathscr{Z}(V_\alpha)$ and $\mathscr{Z}(V_\alpha)$).
In either case, S centralizes $\mathscr{Z}(V_\alpha)$ and also $\mathscr{Z}(V_{-\alpha})$. Hence S centralizes the group $T_d(k)\langle \mathscr{Z}(V_\alpha), \mathscr{Z}(V_{-\alpha})\rangle$. But this group contains a split semisimple group of rank 1 by Theorem 4.3, i.e. either $\mathrm{SL}_2(k)$ or $\mathrm{PGL}_2(k)$. In both cases the claim follows. □

6.2 Isomorphisms of almost split Kac–Moody groups in characteristic 0

Setting. Let k, k' be two fields of characteristic 0 and let G, G' be two 2-spherical almost split Kac–Moody groups over k, k', respectively. Let $G(k)$, $G'(k')$ denote their rational points and suppose that $\varphi \colon G(k) \to G'(k')$ is an abstract isomorphism.
Let

- $Z(k) \leq G(k), Z'(k') \leq G'(k')$ denote the respective anisotropic kernels of G, G'.
- $T_d(k) \leq Z(k), T'_d(k') \leq Z'(k')$ denote the respective maximal split tori.
- (W, S), (W', S') denote the respective Weyl groups, and Φ, Φ' the sets of roots.
- $(U_\alpha)_{\alpha \in \Phi}$ denote the rational root groups of G, and $(V_\beta)_{\beta \in \Phi'}$ the rational root groups of G'.
- Δ, Δ' denote the twin buildings associated to G and G'.
- $\mathcal{A}$ and $\mathcal{A}'$ denote the standard twin apartments fixed by $Z(k), Z'(k')$ respectively.

Strategy of proof. The proof strategy can be outlined as follows:
Step 1. Since $G(k)$ is assumed to be 2-spherical, $G(k)$ contains a maximal split

subgroup $F(k)$ containing $T_d(k)$. A generalization of arguments from [Cap09] can be used to exhibit a subgroup $S(\mathbb{Q}) \leq T_d(k)$ with the property that $S(\mathbb{Q})$ fixes precisely $\mathcal{A}$ and $\varphi(S(\mathbb{Q}))$ fixes precisely a twin apartment $\mathcal{A}''$ of Δ'. By postcomposing φ with an inner automorphism if necessary, we assume that $\mathcal{A}'' = \mathcal{A}'$.

Step 2. From the existence of $S(\mathbb{Q})$, which is in some sense a small subgroup of the split torus, we deduce the existence of two large subgroups $S_1 \leq T_d(k)$, $S_2 \leq T'_d(k')$ such that $\varphi(S_1) \leq Z'(k'), \varphi^{-1}(S_2) \leq Z(k)$. In particular, $\varphi(S_1)$ normalizes all root groups V_β and $\varphi^{-1}(S_2)$ normalizes all root groups U_α.

Step 3. We now focus on a root group U_α. Assume first that U_α is abelian (see Step 5 for the general case). Then for $u \in U_\alpha$, we show that $\varphi(u) \in L^J$ for some Levi factor L^J of finite type, which depends a priori on u.
Using the groups S_1 and S_2 we show that $\varphi(u)$ actually is a unipotent element which is contained in a group $V_{\beta_1} \cdots V_{\beta_r} \leq L^J$.

Step 4. Now root groups in a spherical Levi factor can be distinguished by the torus action. Again using the groups S_1 and S_2, it follows that with the above notation, $k = 1$, i.e. for each $u \in U_\alpha$ there is some $\beta_u \in \Phi'$ such that $\varphi(u) \in V_{\beta_u}$.
Since U_α is a group, it follows that $\varphi(U_\alpha) \leq V_{\beta(\alpha)}$ for some single $\beta(\alpha) \in \Phi'$.

Step 5. If U_α is not abelian, the analysis of steps 3 and 4 still applies to $\mathscr{Z}(U_\alpha)$. Let $u_1, \ldots, u_r$ be elements such that the canonical images of the u_i are a k-basis for $U_\alpha / \mathscr{Z}(U_\alpha)$. Arguing as in steps 3 and 4 for the groups $k \cdot u_i$, together with the knowledge about $\varphi(\mathscr{Z}(U_\alpha))$ allows to conclude that also in this case $\varphi(U_\alpha)$ is contained in a single root group $V_{\beta(\alpha)}$.

Step 6. By symmetry, each root group V_β satisfies $\varphi^{-1}(V_\beta) \leq U_{\alpha(\beta)}$, so actually equality holds. This allows to conclude that φ maps root groups to root groups and preserves the anisotropic kernel.

After these remarks, we now start the discussion which will lead to the proof of the main theorem.
The following lemma is a key step in comparing the twin root data of G and G'.

Lemma 6.8. *There exists a regular diagonalizable subgroup $S(\mathbb{Q}) \leq T_d(k)$ such that $\varphi(S(\mathbb{Q}))$ again is regular diagonalizable.*

Proof. Fix a maximal split subgroup $F(k)$ of $G(k)$ and let $(T_d(k), (F_\alpha(k))_{\alpha \in \Phi})$ denote the associated twin root datum. Then each rank 2 subgroup $F_{\alpha\beta}(k) := T_d(k)\langle F_{\pm\alpha}(k), F_{\pm\beta}(k)\rangle$ coincides with the k-points of a split reductive group of semisimple rank 2. Since these groups are defined over $\mathbb{Z}$, it is possible to consider $F(\mathbb{Q})$, the $\mathbb{Q}$-points of F. More precisely, for each root $\alpha \in \Phi$ let $f_\alpha : (k, +) \to F_\alpha(k)$ denote the corresponding isomorphism and $t : (k^\times)^n \to T_d(k)$ the canonical isomorphism. Then $F(\mathbb{Q}) := t((\mathbb{Q}^\times)^n) \cdot \langle f_\alpha(\mathbb{Q}) : \alpha \in \Phi\rangle$.

For each simple root α let $\psi_\alpha : \mathrm{SL}_2(\mathbb{Q}) \to \langle F_\alpha(\mathbb{Q}), F_{-\alpha}(\mathbb{Q})\rangle$ denote the canonical homomorphism. Let $D_\alpha := \langle \psi_\alpha(\mathrm{diag}(x, x^{-1})) : x \in \mathbb{Q}^\times\rangle$ and let $S(\mathbb{Q}) := \langle D_\alpha : \alpha \in \Delta\rangle$.

Claim 1. $S(\mathbb{Q})$ *is regular.* Note first that $S(\mathbb{Q})$ is invariant under the Weyl group. Indeed, it suffices to check that $s_\alpha(D_\beta) \le S(\mathbb{Q})$ for two simple roots α, β, and this can be verified in $F_{\alpha\beta}$ where it follows from the explicit description of the Weyl group action on the torus in a reductive group. Assume that $S(\mathbb{Q})$ is not regular. Then from the proof of Lemma 6.7 it follows that there is a root α such that $S(\mathbb{Q})$ centralizes $\mathscr{X}(V_\alpha)$, i.e. the character 2α vanishes on $S(\mathbb{Q})$. Write $\alpha = w\alpha_i$ for some $w \in W$ and a simple root α_i. Then $2\alpha_i$ vanishes on $S(\mathbb{Q})$ by the Weyl group invariance of $S(\mathbb{Q})$, but this is a contradiction since $S(\mathbb{Q})$ contains D_{α_i}.

Claim 2. $\varphi(S(\mathbb{Q}))$ *is diagonalizable over* k'. Since $S(\mathbb{Q})$ is boundedly generated by $(D_\alpha)_{\alpha\in\Delta}$ and $\varphi(D_\alpha) \le \varphi \circ \psi_\alpha(\mathrm{SL}_2(\mathbb{Q}))$, it follows that $\varphi(S(\mathbb{Q}))$ is bounded. Let $\Omega \subseteq \Delta'$ denote a balanced subset which is fixed by $\varphi(S(\mathbb{Q}))$.
Let $\overline{S(\mathbb{Q})}$ be the Zariski closure of $\mathrm{Ad}_\Omega(\varphi(S(\mathbb{Q})))$. As $S(\mathbb{Q})$ is commutative, so is $\overline{S(\mathbb{Q})}$. Note that $\overline{S(\mathbb{Q})}$ is connected as it is generated by connected subgroups.

By [Spr98, 3.1.1], $S := \overline{S(\mathbb{Q})}$ is the direct product of its semisimple and its unipotent elements: $S = S_s \times S_u$. Since the abstract representation $\rho := \mathrm{Ad}_\Omega \circ \varphi \circ \psi_\alpha$ actually is rational, it follows that the image of each $S_\alpha(\mathbb{Q})$ consists of semisimple elements only, i.e. is contained in S_s.
In particular, S is a torus since it is connected and contains semisimple elements only. Clearly, S is defined over k'. It remains to be checked that S is split over k'. Let $g \in S(\mathbb{Q})$ be of infinite order. Since g is contained in a k-split torus, the Zariski closure S_g of $\langle g\rangle$ is again a k-split torus by [BT65, Proposition 1.9 b)]. By induction, S/S_g is a k-split torus, from which the result again follows by [BT65, Proposition 1.9 b)]. This implies the claim.

Claim 3. $\varphi(S(\mathbb{Q}))$ *is regular diagonalizable.* This is a direct consequence of the group theoretic characterization of regular diagonalizable subgroups, Lemma 6.7. □

Remark 6.9. If K is algebraically closed and G is a split Kac–Moody group over K, it is even possible to exhibit *finite* regular diagonalizable groups which are mapped to regular diagonalizable subgroups, see [CM05]. Still in the split case over arbitrary fields, $T' := \ker(\alpha - \beta)$ for suitably chosen roots α, β is regular. In particular, the dimension of a regular diagonalizable subgroup can vary arbitrarily.

Remark 6.10. The assumption that $G(k), G'(k')$ be 2-spherical is essentially only used to produce a regular subgroup $S(\mathbb{Q}) \le G(k)$ which is again mapped to a regular diagonalizable subgroup in $G'(k')$.

Since maximal k'-split tori are conjugate under $G'(k')$ [Rém02, Theorem 12.5.3], there exists some $x \in G'(k')$ such that $(\mathrm{int}\, x \circ \varphi)(S(\mathbb{Q})) \le T'_d(k')$. Replacing φ by $\mathrm{int}\, x \circ \varphi$ if necessary, we assume from now on that $\varphi(S(\mathbb{Q})) \le T'_d(k')$.

Proposition 6.11. *There are subgroups $S_1 \leq T_d(k)$ and $S_2 \leq T'_d(k')$ with the property that $T_d(k)/S_1$, $T'_d(k')/S_2$ both are finitely generated and such that $\varphi(S_1) \leq Z'(k')$, $\varphi^{-1}(S_2) \leq Z(k)$.*

Proof. As $T_d(k)$ normalizes $S(\mathbb{Q})$, $T_d(k)$ acts via φ on the fixed point set of $\varphi(S(\mathbb{Q}))$, which is reduced to $\mathcal{A}'$. Let S_1 denote the kernel of this action, then $\varphi(S_1) \leq Z'(k')$ by definition of the anisotropic kernel as the fixator of $\mathcal{A}'$. As $\varphi(T_d(k))/\varphi(S_1)$ is an abelian subgroup of W', it is finitely generated by Lemma 5.34.
Similarly, as $T'_d(k')$ normalizes $\varphi(S(\mathbb{Q}))$, $T'_d(k')$ acts via φ^{-1} on the fixed point set of $S(\mathbb{Q})$, which is reduced to $\mathcal{A}$. Define S_2 as the kernel of this action, then S_2 is as required by similar arguments. □

The subgroups S_1 and S_2 should be thought of as "large" as they are Zariski dense in T_d and T'_d, respectively, by Proposition 6.3. Moreover, S_1 and $\varphi^{-1}(S_2)$ both normalize each root group $U_\alpha \leq G$, while $\varphi(S_1)$ and S_2 both normalize each root group $V_\beta \leq G'$.

The next step consists of showing that for certain unipotent elements $u \in U_\alpha \backslash \{1\}$, $\varphi(u) \leq L^J$ for a Levi factor of spherical type containing $Z'(k')$.

Definition 6.12. *Let $U_\alpha \leq G$ be a root group and let $\mathfrak{u} := \operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$. For an element $u \in U_\alpha(k)$ let $\log u \in \mathfrak{u}$ denote the unique element such that $\exp(\log u) = u$. Then $u \in U_\alpha$ is called* **pure** *if $\log u \in \mathfrak{g}_\alpha$ or $\log u \in \mathfrak{g}_{2\alpha}$.*

Note that if U_α is abelian, each element $u \in U_\alpha \backslash \{1\}$ is pure.

Lemma 6.13. *Let $u \in U_\alpha(k) \backslash \{1\}$ be a pure element. Then there exists a homomorphism $\psi_u : \mathrm{SL}_2(\mathbb{Q}) \to G(k)$ such that*

(i) $\psi_u(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)) = u$

(ii) $\operatorname{im} \psi_u$ *is normalized by* $S(\mathbb{Q})$.

Proof. This follows from the proof of the Proposition 6.5 or from Theorem 4.3. More precisely, since u is pure, the subalgebra $k \log u$ is invariant under $\operatorname{Ad} T_d(k)$, i.e. there is a subgroup $Y_u \leq U_\alpha$ which contains u and is isomorphic to $\mathbf{G}_a$. This isomorphism can be chosen to send u to 1. By Theorem 4.3, u is contained in a split group which contains $T_d \cdot Y_u$. Since $\mathbb{Q}u$ is is invariant under $S(\mathbb{Q})$, the claim follows. □

Proposition 6.14. *Let $u \in U_\alpha \backslash \{1\}$ be a pure element. Then $\varphi(u)$ fixes two opposite points $x, y \in \mathcal{A}'$, i.e. $\varphi(u) \in L^J$ for a Levi factor of finite type of G' relative to T'_d.*

Proof. Let $\psi_u : \mathrm{SL}_2(\mathbb{Q}) \to G(k)$ be as in the previous lemma. Then

$$\varphi \circ \psi_u : \mathrm{SL}_2(\mathbb{Q}) \to G'(k')$$

is a homomorphism whose images fixes two opposite points $x, y \in \Delta'$ by Proposition 5.8. As $\operatorname{im} \psi_u$ is normalized by $S(\mathbb{Q})$, both x and y must actually be contained in $\mathcal{A}'$ by Lemma 5.4 and the fact that $\varphi(S(\mathbb{Q}))$ fixes only $\mathcal{A}'$. □

It remains to prove that not only $\varphi(u) \in L^J$ but actually $\varphi(u) \in V_{\beta(u)}$ for some root $\beta(u)$ depending on u.

The following proposition uses the trick that a unipotent element u is an element of the derived group of a solvable group B_u, a property which is clearly preserved by a group isomorphism. This idea goes back to [BT73].

Proposition 6.15. *Let $u \in U_\alpha \backslash \{1\}$ be pure and let $J \subseteq S'$ be such that $\varphi(u) \in L^J$. Then*

$$cl(u) := \overline{\langle c\varphi(u)c^{-1} : c \in S_2 \rangle} \leq \overline{L^J}$$

is a unipotent group defined over k' and normalized by T'_d.

Proof. Let $Y_u := \langle \varphi^{-1}(c)u\varphi^{-1}(c^{-1}) : c \in S_2 \rangle$. Then $Y_u \leq U_\alpha$ since $\varphi^{-1}(S_2)$ normalizes U_α. Moreover Y_u is contained in $Y'_u := \langle sY_us^{-1} : s \in S(\mathbb{Q}) \rangle$.
By Proposition 6.15 there is a subset $J \subseteq S'$ such that $\varphi(u) \in L^J$. Since $\varphi(S(\mathbb{Q}))$ and S_2 are subgroups of $T'_d(k')$ it follows that $\varphi(Y'_u) \leq L^J$.
Let $\mathbb{Q}u := \exp(\mathbb{Q} \cdot \log u)$. Then $\mathbb{Q}u$ is a group normalized by $S(\mathbb{Q})$ by Proposition 6.13. The group $B_u := S(\mathbb{Q}) \cdot \mathbb{Q}u$ is solvable and u is contained in every finite index subgroup of the derived group of B_u. Indeed, since $S(\mathbb{Q})$ acts on $\mathbb{Q} \cdot u$ via a non-trivial character, for each $n \in \mathbb{N}$ there is some $s \in S(\mathbb{Q})$ such that $\frac{1}{n} \cdot u \in \langle [s, u] \rangle$.
As $B_u \leq Y'_u, \varphi(B_u) \leq L^J$. Since B_u is solvable, so is $\overline{\varphi(B_u)}$. By the Lie-Kolchin theorem, $\overline{\varphi(B_u)}$ has a finite index subgroup which is triagonalizable, and since u is in the derived group, it follows that $\varphi(u)$ is unipotent.
Note that $S(\mathbb{Q})$ and $\varphi^{-1}(S_2)$ commute, i.e. Y'_u is normalized by both $S(\mathbb{Q})$ and $\varphi^{-1}(S_2)$. Arguing similarly as for u, it follows that $\varphi(Y'_u)$ is unipotent since it is (up to finite index) contained in the derived group of a solvable group.
Since $\varphi(Y'_u) \leq G'(k')$, the Zariski closure $cl(u)$ is defined over k', and $cl(u)$ again is unipotent (cf. [Spr98]). By definition, $cl(u)$ is normalized by S_2 and hence by the Zariski closure of S_2, which is T'_d by Lemma 6.3. □

The following step is inspired by the proof of [CR09b, Proposition 23], which in turn is inspired by classical results.
We recall first the definition of a nibbling sequence of roots.

Definition 6.16. *Let (W, S) be a Coxeter group and let $\alpha_1 \ldots, \alpha_n \in \Phi(W, S)$ be such that $\{\alpha_i, \alpha_j\}$ is prenilpotent for all $i, j \in \{1, \ldots, n\}$. Then $(\alpha_1, \ldots, \alpha_n)$ is called a* **nibbling sequence** *of roots if for all $i < j$, $(\alpha_i, \alpha_j) \subseteq \{\alpha_{i+1}, \ldots, \alpha_{j-1}\}$.*

Proposition 6.17. *Let (W, S) be a spherical Coxeter group and let $\Psi \subseteq \Phi(W, S)$ be a nilpotent set of roots. Then the elements of Ψ can be arranged to form a nibbling sequence of roots.*

Proof. See [Rém02, Section 9.1.2]. □

Theorem 6.18. *Let $u \in U_\alpha \backslash \{1\}$ be pure and $J \subseteq S'$ spherical such that $\varphi(u) \in L^J$. Then $\varphi(u) \in V_\beta$ for some $\beta \in \Phi'$.*

Proof. By Proposition 6.15 and Proposition 6.6, it follows that

$$\psi(u) \in V_J^+ := \langle V_\beta : \beta \in \Phi(W_J'), \beta > 0\rangle$$

for a suitable ordering '>' of the roots of $\Phi(W_J')$. Since $\Phi(W_J')$ is finite it follows from Proposition 6.17 that there is an ordering on the positive roots β_i such that $(\beta_1, \dots, \beta_k)$ is a nibbling sequence. Then

$$\varphi(u) = v_{i_1} \cdots v_{i_r}$$

for certain $v_{i_j} \in V_{\beta_{i_j}}, v_{i_j} \neq 1$.
Assume for a contradiction that $r > 1$.
Claim. In this case, there are indices $i \neq j$ and elements $u_i, u_j \in U_\alpha\backslash\{1\}$ such that $\varphi(u_i) \in V_{\beta_i}$ and $\varphi(u_j) \in V_{\beta_j}$.
Since W_J' is spherical, for any two roots $\beta_i, \beta_j \in \Phi(W_J')$ such that $\beta_i \neq \pm\beta_j$, there is an element $t_{ij} \in \langle V_\alpha : \alpha \in \Phi(W_J')\rangle \cap S_2$ such t_{ij} centralizes V_{β_i} but not V_{β_j}. This follows from the fact that such an element exists in $T_d'(k')$ and the fact that S_2 is Zariski dense in T_d'.
Consider $v_1 := [t_{1,r}, \varphi(u)]$ and $v_2 := [t_{r,1}, \varphi(u)]$
Then $v_1, v_2 \in \varphi(U_\alpha)$ since $\varphi^{-1}(t_{ij})$ normalizes U_α. Furthermore, the support of v_1 contains β_1 but not β_r. Likewise, the support of v_2 does not contain β_1 but contains β_r.
By repeating the process for v_1 and v_2 inductively if necessary, the claim is proven.

Now take an element $s \in S_2$ of infinite order such that s centralizes V_{β_i} but not V_{β_j} and such that $\varphi^{-1}(s) \in T_d(k)$. The existence of such an element can be proven by appealing to the $\mathbb{Q}$-points of a split subgroup of $G'(k')$ and the fact that β_i, β_j are roots in a *spherical* Coxeter group. Then $\varphi^{-1}(s)$ centralizes u_i, since $\varphi(u_i) \in V_{\beta_i}$, so $\varphi^{-1}(s^2)$ centralizes U_α since $\varphi^{-1}(s) \in T_d(k)$. But this is a contradiction, since $\varphi^{-1}(s^2)$ does not centralize u_j, since $\varphi(u_j) \in V_{\beta_j}$. □

Corollary 6.19. *Let U_α be a root group. Then $\varphi(\mathscr{Z}(U_\alpha)) \leq V_\beta$ for some $\beta \in \Phi'$.*

Proof. By the preceding theorem, $\varphi(\mathscr{Z}(U_\alpha))$ is a group which satisfies the assumptions of Lemma 6.4, since each element $u \in \mathscr{Z}(U_\alpha)\backslash\{1\}$ is pure, so the conclusion follows. □

This corollary finishes the case where all root groups are abelian. Some more effort is required when there are metabelian root groups present. These technical problems are always present when one deals with metabelian root groups, see e.g. [Deo78] or [BT73].

The following lemma is inspired by the proof of [CM05, Theorem 2.2].

Lemma 6.20. *Let $u \in U_\alpha\backslash\{1\}$ be a pure element and let $\beta \in \Phi'$ be such that $\varphi(u) \in V_\beta$. Then the elements $u', u'' \in U_{-\alpha}$ such that $m(u) = u'uu''$ satisfy $\varphi(u'), \varphi(u'') \in V_{-\beta}$.*

Proof. From Lemma 6.13 it follows that $u' = u''$ and that u' is pure. Let $\gamma \in \Phi'$ be such that $\varphi(u') \in V_\gamma$. It is clear that $\varphi(\mathbb{Q}u) \leq V_\beta$ and that $\varphi(\mathbb{Q}u') \leq V_\gamma$. This induces a homomorphism $\psi : \mathrm{SL}_2(\mathbb{Q}) \to V_{\beta\gamma} := \langle V_\beta, V_\gamma \rangle$. Suppose that $\beta \neq -\gamma$. If $\{\beta, \gamma\}$ is a prenilpotent set of roots, $V_{\beta\gamma}$ is nilpotent since each root group V_α is nilpotent, which is a contradiction since ψ is nontrivial. If $\{\beta, \gamma\}$ is not prenilpotent, the free product $V_\beta * V_\gamma$ embeds in $G'(k')$, which is a contradiction since a conjugate of u in $\mathrm{SL}_2(\mathbb{Q})$ commutes with u', while this is not the case for $\varphi(u) \in V_\beta$ and $\varphi(u') \in V_\gamma$. □

Proposition 6.21. *Suppose that U_α is metabelian. Then $\varphi(U_\alpha) \leq V_\beta$ for some $\beta \in \Phi'$.*

Proof. Let $u_1, \ldots, u_r \in U_\alpha$ be pure such that $\log u_1, \ldots, \log u_r$ form a basis for $\mathfrak{g}_\alpha$. Let $U_i := k \cdot u_i$ and let $U_0 := \mathscr{Z}(U_\alpha)$. Let $\gamma_0, \ldots, \gamma_r \in \Phi'$ be such that $\varphi(U_i) \leq V_{\gamma_i}$. These clearly exist, as each U_i is a subgroup of U_α consisting of pure elements.
Suppose that there are i, j such that $\gamma_i \neq \gamma_j$. If $w := s_{\gamma_i} s_{\gamma_j}$ has finite order, γ_i and γ_j are roots in a Levi factor L^J. Then $U_{ij} := \langle U_i, U_j \rangle$ is mapped to a unipotent subgroup of L^J by arguments similar to those in the proof of Proposition 6.15. Arguing as in the proof of Theorem 6.18, this yields a contradiction as then there would exist a torus element $t \in T'_d(k')$ such that $\varphi^{-1}(t)$ centralizes U_i but not U_j.
It follows that w has infinite order. Note that $\varphi(U_\alpha)$ is contained in the set $V' := V_{\gamma_1} \cdots V_{\gamma_r} \cdot V_{\gamma_0}$, in particular, $\varphi(U_\alpha)$ is bounded.

Let $m_i, m_j \in G'(k')$ be such that m_i, m_j stabilize $\mathcal{A}'$, act on it via $s_{\gamma_i}, s_{\gamma_j}$ and such that $\varphi^{-1}(m_i), \varphi^{-1}(m_j)$ stabilize $\mathcal{A}$. These elements can be shown to exist via e.g. invoking a split subgroup of $G'(k')$.
From the previous proposition it follows that $\varphi^{-1}(m_i), \varphi^{-1}(m_j)$ map U_α to $U_{-\alpha}$.
For $t := m_i m_j$ it follows that $\varphi^{-1}(t)$ normalizes U_α.
Then for each $r \in \mathbb{Z}$ there exists some $u_r \in U_\alpha$ such that $\varphi(u_r) \in V_{w^r \gamma_i}$. This is the desired contradiction, as this implies that $\varphi(U_\alpha)$ is unbounded. □

To sum up: For each $\alpha \in \Phi$ there is a root $i(\alpha) \in \Phi'$ such that $\varphi(U_\alpha) \leq V_{i(\alpha)}$.
Arguing likewise for φ^{-1} (note that the corresponding twin apartments $\mathcal{A}, \mathcal{A}'$ are already aligned in the right fashion) we find that for each $\beta \in \Phi'$ there is a $j(\beta) \in \Phi$ such that $\varphi^{-1}(V_\beta) \leq U_{j(\beta)}$. From the inclusion

$$U_\alpha = \varphi^{-1}(\varphi(U_\alpha)) \leq \varphi^{-1}(V_{i(\alpha)}) \leq U_{j(i(\alpha))}$$

and the fact that $U_\alpha \neq 1, U_\alpha \cap U_\beta = 1$ for $\alpha \neq \beta$, it finally follows that i and j are inverse bijections and that equality holds all along.

This discussion can be succinctly summed up by saying that any isomorphism $\varphi \colon G(k) \to G'(k')$ is **standard**, cf. Definition 5.1. We have shown:

Theorem 6.22. *Let $G = G(k), G' = G'(k')$ be two 2-spherical almost split Kac–Moody groups over fields k, k' of characteristic 0. Let $(Z(k), (U_\alpha)_{\alpha \in \Phi(W,S)})$ and $(Z'(k'), (V_\beta)_{\beta \in \Phi(W',S')})$ denote the associated canonical twin root data. Suppose that $\varphi : G(k) \to G'(k')$ is an abstract isomorphism. Then φ is standard.*

Proof. By the previous discussion, there exists $x \in G'(k')$ such that $\varphi' := \operatorname{int} x \circ \varphi$ induces a bijection of the root groups. Note that $\operatorname{int} x$ can be chosen to be trivial if, with the notation from above, $\varphi(S(\mathbb{Q}))$ already fixes $\mathcal{A}'$. Since

$$Z(k) = \bigcap_{\alpha \in \Phi(W,S)} N_{G(k)}(U_\alpha), \; Z'(k') = \bigcap_{\beta \in \Phi(W',S')} N_{G'(k')}(V_\beta)$$

(there is actually an equality, not just an inclusion, see [Rém02, Proposition 1.5.3]), it follows that $\varphi'(Z(k)) = Z'(k')$. □

Remark 6.23. Let $\varphi : G \to G'$ be a standard isomorphism of two groups endowed with twin root data of type (W, S) resp. (W', S') with associated bijection i. Suppose that S and S' are finite.
It can be shown (cf. [CM05, Section 2.2]) that $i(-\alpha) = -i(\alpha)$ and that φ induces an isomorphism $\bar{i} : (W, S) \to (W', S')$ of Coxeter systems, i.e. an isomorphism $\bar{i} : W \to W'$ which maps S to S'. In particular, $\bar{i}$ interchanges the connected components of S and S', and for each such connected component J there is $\epsilon_J \in \{+, -\}$ such that $i(\Phi(W_J)^+) = \Phi(W'_{i(J)})^{\epsilon_J}$.

Remark 6.24. Since clearly $\mathscr{Z}(U_\alpha)$ is mapped to $\mathscr{Z}(V_\beta)$, it also follows that φ induces an automorphism of the refined root datum as given by Rémy [Rém02, Theorem 12.6.3].

Remark 6.25. In the setting of Borel-Tits's classical paper, one can proceed even further to show that an abstract isomorphism is induced from a field isomorphism and a rational map. In the case of split Kac–Moody groups, the knowledge of the fact that root groups are permuted can be used to describe possible isomorphisms in terms of the $\mathrm{SL}_2(k)$'s which determine the Kac–Moody group.
In the present generality of dealing with almost split Kac–Moody groups, a more explicit description of the induced isomorphisms of rank 1 groups can only be obtained when making assumptions on the rank 1 groups in question.
Moreover, no statement is made about the anisotropic kernel.

Proposition 6.26. *Let k, k' be two fields of characteristic 0 and let $G(k), G'(k')$ be two 2-spherical almost split Kac–Moody groups. Let $\varphi : G(k) \to G'(k')$ be an isomorphism rectified in such a fashion that φ maps root groups with respect to $T_d(k)$ to root groups with respect to $T'_d(k')$.*
Let $X_\alpha(k) := Z(k)\langle U_\alpha, U_{-\alpha}\rangle$ and $Y_\beta(k') := Z'(k')\langle V_\beta, V_{-\beta}\rangle$ be two rank 1 groups such that $\varphi(X_\alpha) = Y_\beta$.
Suppose that the derived groups of both X_α and Y_β are absolutely almost simple and that either X'_α is simply connected or that Y'_β is adjoint. Then there is a field isomorphism $\sigma : k \to k'$, a rational map $r : X'_\alpha \to Y'_\beta$ and a map $c : X'_\alpha(k) \to \mathscr{Z}(Y'_\beta(k'))$ such that for $x \in X'_\alpha$, $\varphi(x) = c(x) \cdot (r \circ \sigma(x))$.

Proof. The assumption are made as to conform to the assumptions of Borel-Tits's classical theorem [BT73, Theorem A], from which the claim follows. □

6.3 Outlook

We end with a discussion of extending these results to positive characteristic.
Let k, k' be two fields of cardinality ≥ 4 and let $G(k), G'(k')$ be two almost split Kac–Moody groups. Let $\varphi\colon G(k) \to G'(k')$ be an isomorphism. Reasoning as in [Cap09, Proposition 4.16], it follows that k and k' have the same characteristic p; we assume here that $p > 0$.
In characteristic 0, we exhibited a split Kac–Moody subgroup $F(\mathbb{Q}) \leq G(k)$ to produce a regular diagonalizable subgroup. A split semisimple k-group of rank 1 is isomorphic to either $\mathrm{SL}_2(k)$ or $\mathrm{PGL}_2(k)$. In particular, in positive characteristic p a rank 1 group X_α contains a split subgroup Y_α isomorphic to either $\mathrm{SL}_2(\mathbb{F}_p)$ or $\mathrm{PGL}_2(\mathbb{F}_p)$, the torus of which has cardinality $p-1$. If $p \geq 5$ it is possible to produce a regular diagonalizable subgroup by means of the diagonal matrices of the Y_α, $\alpha \in \Delta$: It is automatic that $\varphi(Y_\alpha(\mathbb{F}_p))$ is bounded. By Jordan decomposition in characteristic p, it follows that the image of a diagonal matrix in $Y_\alpha(\mathbb{F}_p)$ is again diagonalizable.
Note that if $X = \mathrm{SL}_2(\mathbb{F}_q)$ where $q = p^e$ with $e > 1$, there are representations of X which do not map diagonalizable elements to diagonalizable elements: The easiest example is provided by the restriction of scalars $\mathrm{SL}_2(\mathbb{F}_4) \to \mathrm{SL}_4(\mathbb{F}_2)$.
This allows us to produce a regular diagonalizable subgroup S of $G(k)$ such that $\varphi(S)$ is again diagonalizable in $G'(k')$. It remains to check that $\varphi(S)$ is regular.
The remainder of the proof of Theorem 6.22 used the fact that the characteristic is 0 in some arguments involving algebraic groups. It thus seems reasonable to expect that the main theorem remains correct when the assumption of characteristic 0 is replaced by the requirement that k, k' be two infinite perfect fields of characteristic ≥ 5.
Since in the case of finite ground fields the bounded subgroups are precisely the finite subgroups, methods from [CM06] should yield the same result in the case that k, k' are finite fields of sufficiently large cardinality.

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