Münster J. of Math. **13** (2020), 425–444
DOI 10.17879/90169651061
urn:nbn:de:hbz:6-90169651373

Münster Journal of Mathematics

Aspects of p-adic operator algebras

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(Communicated by Peter Schneider)

Dedicated to Christopher on the occasion of his 60th birthday

Abstract. In this article, we propose a p-adic analog of complex Hilbert space and consider generalizations of some well-known theorems from functional analysis and the basic study of operators on Hilbert spaces. We compute the K-theory of the analog of the algebra of compact operators and the algebra of all bounded operators. This article contains a survey on results from the thesis of the first author.

1. Introduction

While there exists a rich literature on p-adic functional analysis in general (cp. Schneider's book [14] as a comprehensive source), it seems that only few publications treat p-adic operator algebras, their K-theory and their application to group rings. In this article, the authors want to give their contribution to the subject with focus on an p-adic analog of the classical Hilbert space featuring phenomena such as self-duality etc. This p-adic *Hilbert space* $\mathbb{Q}_p(X)$ (sometimes called the restricted product of $\mathbb{Q}_p$ indexed by X) is defined as the set of all maps $\xi\colon X \to \mathbb{Q}_p$ such that $|\xi(x)|_p > 1$ holds for only finitely many elements $x \in X$. The space $\mathbb{Q}_p(X)$ is not a $\mathbb{Q}_p$-vector space, but, equipped with the canonical addition, scalar multiplication with scalars in $\mathbb{Z}_p$ and an appropriate topology τ, a locally compact topological $\mathbb{Z}_p$-module. We will introduce a *scalar product* $\langle\ ,\ \rangle\colon \mathbb{Q}_p(X) \times \mathbb{Q}_p(X) \to S^1$ on $\mathbb{Q}_p(X)$. It turns out that the Pontryagin dual of $\mathbb{Q}_p(X)$ is isomorphic to $\mathbb{Q}_p(X)$ as a topological group and all characters can be uniquely represented by a scalar product with an element of $\mathbb{Q}_p(X)$, this correspondence yielding the isomorphism of $\mathbb{Q}_p(X)$ with its dual. As in the usual Archimedean case, one can define the algebra $\mathcal{B}(\mathbb{Q}_p(X))$ of continuous $\mathbb{Z}_p$-linear operators on $\mathbb{Q}_p(X)$. Using the notion of adjoint operators (cp. Section 2.10) and of the operator norm (cp. Section 2.14), $\mathcal{B}(\mathbb{Q}_p(X))$ can be given the structure of a complete normed $*$-algebra over $\mathbb{Z}_p$,

This research was supported in part by the ERC Consolidator Grant No. 681207.

i.e., a Banach-$*$-algebra over $\mathbb{Z}_p$. In analogy with the Archimedean case, it is possible to define a continuous functional calculus for certain operators in $\mathcal{B}(\mathbb{Q}_p(X))$, the so-called *normal contractions* (cp. Section 2.21). The definition is based on Mahler's representation theorem of the continuous functions $\mathbb{Z}_p \to \mathbb{Z}_p$ as infinite $\mathbb{Z}_p$-linear combinations of binomial coefficients.

It is possible to define an analog $\mathcal{K}(\mathbb{Q}_p(X))$ for the ideal of compact operators in a Hilbert space (cp. Section 3.1). Furthermore, we will introduce and study a matrix representation of operators in $\mathcal{B}(\mathbb{Q}_p(X))$ (Section 2.28).

In Section 3.4, we will be interested in idempotents of the ring $\mathcal{B}(\mathbb{Q}_p(X))$ and its K-theory, namely, its K_0-group. Compared to the usual case of projections in the complex Hilbert space, idempotents and projections in $\mathcal{B}(\mathbb{Q}_p(X))$ are much harder to study. For example, there is, in general, no projection onto the intersection of images of two given projections etc. But at least, we will show that, as in the Archimedean case, we have the isomorphisms $K_0(\mathcal{K}(\mathbb{Q}_p(X))) \cong \mathbb{Z}$ and $K_0(\mathcal{B}(\mathbb{Q}_p(X))) = 0$ (cp. Section 3.8 and Section 3.14). Interestingly, also the fact that each idempotent in the quotient algebra $\mathcal{B}(\mathbb{Q}_p(X))/\mathcal{K}(\mathbb{Q}_p(X))$ can be lifted to an idempotent in $\mathcal{B}(\mathbb{Q}_p(X))$ remains true, but the proof is different from the analog Archimedean theorem (cp. Section 3.21).

This article is a short version of the first three chapters in the thesis of one of the authors (cp. [4]), and most parts are taken from there. In the last two chapters of [4], the reader can find additional considerations, e.g., on the definition of the tensor product of operator algebras acting on $\mathbb{Q}_p(X)$, the application of our approach to the case that $X = \Gamma$ is a countable group, the p-adic analog of the group von Neumann algebra etc.

2. The p-adic analog of a Hilbert space

2.1. **The space $\mathbb{Q}_p(X)$ and the topology τ.** Let X be a countable set. Consider the set

$$\mathbb{Q}_p(X) := \{\xi\colon X \to \mathbb{Q}_p; |\xi(i)|_p \leq 1 \text{ for all but finitely many } i \in X\}.$$

On this set, we define a topology τ by saying that a set $A \subseteq \mathbb{Q}_p(X)$ is open if for all $P \subseteq X$ with $|P| < \infty$, the set

$$\Big(\prod_{i\in P} \mathbb{Q}_p \times \prod_{j\in X\setminus P} \mathbb{Z}_p\Big) \cap A$$

is open in $\prod_{i\in P} \mathbb{Q}_p \times \prod_{j\in X\setminus P} \mathbb{Z}_p$ with respect to the product topology. Note that τ is the largest topology on $\mathbb{Q}_p(X)$ such that all the inclusions of the form

$$\prod_{i\in P} \mathbb{Q}_p \times \prod_{j\in X\setminus P} \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p(X)$$

with finite $P \subseteq X$ are continuous.

Using the terminology of [11, Def. I.1.1.12], the space $\mathbb{Q}_p(X)$ is called the *restricted product* of countably many copies of $\mathbb{Q}_p$ with respect to the open subgroups $\mathbb{Z}_p \subseteq \mathbb{Q}_p$.

Furthermore, notice the similarity of this construction with the construction of the *adele-rings* in [9, Section 4.3.7].

For $x \in X$, we define the element $\delta_x \in \mathbb{Q}_p(X)$ by $\delta_x(x) = 1$ and $\delta_x(y) = 0$ for $y \in X \setminus \{x\}$.

The following lemma is easy to prove.

Lemma 2.2. *With respect to τ, a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $\mathbb{Q}_p(X)$ converges to $\xi \in \mathbb{Q}_p(X)$ if and only if it converges entrywise to ξ and if the set $\{x \in X;$ there exists $n \in \mathbb{N}\colon |\xi_n(x)|_p > 1\}$ is finite.*

Equipped with the natural coordinate-wise addition and the topology τ, the set $\mathbb{Q}_p(X)$ becomes a locally compact and σ-compact Hausdorff topological abelian group, where the subset

$$\mathbb{Z}_p(X) := \prod_{i\in X} \mathbb{Z}_p$$

is (according to Tychonoff's theorem) a compact open subgroup. The group $\mathbb{Q}_p(X)$ additionally carries a natural structure of a $\mathbb{Z}_p$-module, but it is not a $\mathbb{Q}_p$-vector space if X is infinite.

The topological groups $\mathbb{Q}_p(X)$ have already been considered in [13], where the authors show that all self-dual (in Pontryagin's sense) metrizable locally compact torsion-free abelian groups are either of this form or of the form $\mathbb{R}^n$, of the form $D \oplus \widehat{D}$, where D is a countable torsion-free divisible discrete group, or a (local) direct sum of groups of these types.

Also the following lemma is easy to verify.

Lemma 2.3. *The abelian group $\mathbb{Q}_p(X)$ is a Polish group.*

Definition 2.4. The set of all $\mathbb{Z}_p$-linear τ-continuous operators on $\mathbb{Q}_p(X)$ is denoted by $\mathcal{B}(\mathbb{Q}_p(X))$.

Note that a τ-continuous group homomorphism $A\colon \mathbb{Q}_p(X) \to \mathbb{Q}_p(X)$ is already in $\mathcal{B}(\mathbb{Q}_p(X))$. The set $\mathcal{B}(\mathbb{Q}_p(X))$ forms a $\mathbb{Z}_p$-module with the canonical operations.

The following two lemmas are special cases of well-known versions of the open mapping and closed graph theorems for certain topological groups (cp. [7, Thm. 1.5] and [8, p. 213]). For these useful lemmas, the assumption on X to be countable becomes relevant.

Lemma 2.5. *Let $A\colon \mathbb{Q}_p(X) \to \mathbb{Q}_p(X)$ be a group homomorphism. The following statements are equivalent:*

(a) *A is τ-continuous,*
(b) *the graph $\mathcal{G}(A) := \{(\xi, A\xi) \in \mathbb{Q}_p(X) \times \mathbb{Q}_p(X); \xi \in \mathbb{Q}_p(X)\}$ of the map A is closed in $\mathbb{Q}_p(X) \times \mathbb{Q}_p(X)$,*
(c) *for every sequence $(\xi_n)_{n\in\mathbb{N}}$ with $\tau\text{-}\lim_{n\to\infty} \xi_n = 0$, $\tau\text{-}\lim_{n\to\infty} A\xi_n = \eta$, we have $\eta = 0$.*

Recall that a map $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is called *open* if it maps open sets onto open sets. As a consequence of Lemma 2.5, one easily sees that any surjective $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is open.

2.6. **The pairing on $\mathbb{Q}_p(X)$ and duality aspects.** We want to introduce a natural pairing on our space $\mathbb{Q}_p(X)$ that can be compared to a scalar product on a usual Hilbert space. Define

$$\langle\ ,\ \rangle\colon \mathbb{Q}_p(X)\times\mathbb{Q}_p(X)\to S^1$$

by

$$\langle\xi,\eta\rangle := \iota\Big(\sum_{i\in X}(\xi(i)\eta(i)+\mathbb{Z}_p)\Big),$$

where $\iota\colon \mathbb{Q}_p/\mathbb{Z}_p = \mathbb{Z}[1/p]/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z} \cong S^1$ is the canonical map (here, the identification $\mathbb{R}/\mathbb{Z}\cong S^1$ is given as usual by $t\mapsto e^{2\pi i t}, t\in\mathbb{R}/\mathbb{Z}$, and the identification $\mathbb{Z}[1/p]/\mathbb{Z}$ with $\mathbb{Q}_p/\mathbb{Z}_p$ is given by the composited map $\mathbb{Z}[1/p]\hookrightarrow \mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p$ that factors through $\mathbb{Z}[1/p]/\mathbb{Z}$).

The pairing is symmetric and jointly continuous because it is the composition of two continuous maps (where $\mathbb{Q}_p/\mathbb{Z}_p$ is equipped with the discrete topology). Furthermore, it induces a $\mathbb{Z}_p$-linear identification of $\mathbb{Q}_p(X)$ with its Pontryagin dual (cp. [13] or [11, Prop. I.1.1.13]). As a topological group, $\mathbb{Q}_p(X)$ is isomorphic to its Pontryagin dual. This is an analogy to Riesz' theorem on the self-duality for Hilbert spaces.[1]

Remark. In the definition of the pairing $\langle\ ,\ \rangle$, it would have been possible to take other embeddings j of $\mathbb{Q}_p/\mathbb{Z}_p\cong\mathbb{Z}[1/p]/\mathbb{Z}$ into S^1 instead of ι. Each such embedding differs from ι by a unique $\alpha_j\in\mathbb{Z}_p^\times$, the unit group of $\mathbb{Z}_p$, in the way that $j=\iota\circ M_{\alpha_j}$, where $M_{\alpha_j}\colon \mathbb{Q}_p/\mathbb{Z}_p\to\mathbb{Q}_p/\mathbb{Z}_p$ denotes the multiplication by α_j.

Let us pursue the analogy between $\mathbb{Q}_p(X)$ and Hilbert spaces:

Definition 2.7. Let K be a subset of $\mathbb{Q}_p(X)$. Define

$$K^\perp := \{\xi\in\mathbb{Q}_p(X);\ \langle\xi,\eta\rangle = 0 \text{ for all } \eta\in K\}.$$

For subsets $K,L\subseteq\mathbb{Q}_p(X)$, we write $K\perp L$ if $\langle\xi,\eta\rangle=0$ for all $\xi\in K$ and $\eta\in L$.

The set $K^\perp$ is a closed sub-$\mathbb{Z}_p$-module of $\mathbb{Q}_p(X)$. It may happen that $K\cap K^\perp\neq\{0\}$. For example, we have $\mathbb{Z}_p(X)^\perp=\mathbb{Z}_p(X)$.

Lemma 2.8. *Let $H\subseteq\mathbb{Q}_p(X)$ be a closed subgroup. The Pontryagin dual $\widehat{H}$ of H is topologically isomorphic to $\mathbb{Q}_p(X)/H^\perp$.*

Proof. According to [6, Cor. 3.6.2], each character φ on H extends to $\mathbb{Q}_p(X)$. Because of the self-duality of $\mathbb{Q}_p(X)$, it can be represented by some vector $\xi\in\mathbb{Q}_p(X)$, i.e.,

$$\varphi(\eta)=\langle\eta,\xi\rangle \quad \text{for all } \eta\in H,$$

[1]Notice, however, that τ is *not* equal to the weak topology with respect to this pairing, i.e., the initial topology with respect to the maps of the form $\mathbb{Q}_p(X)\to S^1, \xi\mapsto\langle\xi,\eta\rangle$ (where $\eta\in\mathbb{Q}_p(X)$), cp. the remark following Theorem 1.8.2 in W. Rudin's book "Fourier analysis on groups", p. 30.

where ξ is determined uniquely up to elements in $H^\perp$. We obtain a bijective group homomorphism Φ from $\widehat{H}$ to $\mathbb{Q}_p(X)/H^\perp$. Note that both groups are Polish: the second as a quotient of the Polish group $\mathbb{Q}_p(X)$, the first as a quotient of the Polish group $\widehat{\mathbb{Q}}_p(X) \cong \mathbb{Q}_p(X)$ (as H is a closed subgroup of $\mathbb{Q}_p(X)$, use [6, Prop. 3.6.1]). As Φ is a bijective continuous group homomorphism between Polish groups, the map Φ^{-1} must be an isomorphism of topological groups (use again [7, Thm. 1.5]). □

Lemma 2.9. *Let $K, L \subseteq \mathbb{Q}_p(X)$ be closed sub-$\mathbb{Z}_p$-modules. Then, the following properties hold:*

(a) $K = K^{\perp\perp}$,
(b) $K \subseteq L \Rightarrow L^\perp \subseteq K^\perp$,
(c) $(K + L)^\perp = K^\perp \cap L^\perp$,
(d) $(K \cap L)^\perp = \mathrm{cl}(K^\perp + L^\perp)$, *where the closure is taken in the τ-topology.*

Proof. First, we prove the property (a). Lemma 2.8 yields the following exact sequence:

$$0 \to K^\perp \to \mathbb{Q}_p(X) \to \widehat{K} \to 0.$$

Now, Pontryagin duality shows (cp. [6, Cor. 3.6.2]) that the dual sequence

$$0 \to K \to \mathbb{Q}_p(X) \to \widehat{K}^\perp \to 0$$

is also exact. Replacing K by $K^\perp$ in the first sequence, we obtain the exact sequence

$$0 \to K^{\perp\perp} \to \mathbb{Q}_p(X) \to \widehat{K}^\perp \to 0.$$

The maps $\mathbb{Q}_p(X) \to \widehat{K}^\perp$ in the second and the third sequence coincide, i.e., their kernels K and $K^{\perp\perp}$ coincide as well. The statements (b) and (c) are obvious. The statement (d) follows from statement (c) using statement (a). □

2.10. **The adjoint of an operator.** We obtain a further analogy of $\mathbb{Q}_p(X)$ and ordinary Hilbert spaces:

Lemma 2.11. *For every $A \in \mathcal{B}(\mathbb{Q}_p(X))$, there is a unique operator $A^* \in \mathcal{B}(\mathbb{Q}_p(X))$ satisfying $\langle A\xi, \eta\rangle = \langle \xi, M\eta\rangle$ for all $\xi, \eta \in \mathbb{Q}_p(X)$. We will call it the* adjoint operator *of A. For $A, B \in \mathcal{B}(\mathbb{Q}_p(X))$, $\lambda \in \mathbb{Z}_p$, we have*

$$A^{**} = A, \quad (A + \lambda B)^* = A^* + \lambda B^*, \quad (AB)^* = B^* A^*.$$

Proof. The uniqueness and existence of a group homomorphism A^* with the above property can be proved as in the usual Hilbert space case, and to prove the continuity of the homomorphism M, one then applies the third characterization of τ-continuity in Lemma 2.5 (or one simply uses the fact that Pontryagin duality is a functor together with the self-duality of $\mathbb{Q}_p(X)$). The formulas for the adjoint operator are clear. □

Lemma 2.12. *For every $A \in \mathcal{B}(\mathbb{Q}_p(X))$, we have* $\ker(A) = \mathrm{im}(A^*)^\perp$.

Proof. The direction $\ker(A) \subseteq \operatorname{im}(A^*)^\perp$ is clear. For the other direction, suppose $\eta \in \operatorname{im}(A^*)^\perp$. For all $\xi \in \mathbb{Q}_p(X)$, we see that

$$\langle A\eta, \xi\rangle = \langle \eta, A^*\xi\rangle = 0.$$

Hence, since our natural pairing is non-degenerate, we obtain that $A\eta = 0$ or $\eta \in \ker(A)$. □

For reasons of completeness, we finally want to state a more general version of Lemma 2.11.

Theorem 2.13. *Let $\sigma\colon \mathbb{Q}_p(X) \times \mathbb{Q}_p(X) \to S^1$ be a biadditive form that is separately continuous. Then, there exists a unique $A \in \mathcal{B}(\mathbb{Q}_p(X))$ such that*[2]

$$\langle A\xi, \eta\rangle = \sigma(\xi, \eta) \quad \text{for all } \xi, \eta \in \mathbb{Q}_p(X).$$

The proof can be found in [4, Thm. 4.4].

2.14. **The norm topology on $\mathbb{Q}_p(X)$ and $\mathcal{B}(\mathbb{Q}_p(X))$.** For an element $\xi \in \mathbb{Q}_p(X)$, we define $\|\xi\| := \max_{i\in X}|\xi(i)|_p$. It is clear that we have defined an ultra-norm on the $\mathbb{Z}_p$-module $\mathbb{Q}_p(X)$ in this way: $\|\xi+\eta\| \le \max\{\|\xi\|, \|\eta\|\}$ for all $\xi, \eta \in \mathbb{Q}_p(X)$. Note that all norm-convergent sequences also converge with respect to τ, but not the other way around. The norm topology is therefore stronger than the τ-topology (strictly stronger if X is infinite). The following lemma is easy to verify.

Lemma 2.15. *A subset $K \subseteq \mathbb{Q}_p(X)$ is τ-compact if and only if it is norm-bounded, τ-closed and there is a finite subset $S \subseteq X$ such that*

$$K \subseteq \prod_{x\in S} \mathbb{Q}_p \times \prod_{x \in X\setminus S} \mathbb{Z}_p.$$

We want to investigate some further properties of the norm and of norm-continuous operators. The following two lemmas are easy to verify.

Lemma 2.16. *The space $\mathbb{Q}_p(X)$ is complete with respect to the norm.*

Lemma 2.17. *Let $A\colon \mathbb{Q}_p(X) \to \mathbb{Q}_p(X)$ be a $\mathbb{Z}_p$-linear map. Then A is norm-continuous if and only if A is bounded, i.e., there is $C > 0$ such that*

$$\|A\xi\| \le C\|\xi\| \quad \text{for all } \xi \in \mathbb{Q}_p(X).$$

Lemma 2.18. *A τ-continuous $\mathbb{Z}_p$-linear map on $\mathbb{Q}_p(X)$ is also norm-continuous.*

Proof. Suppose that A is a τ-continuous $\mathbb{Z}_p$-linear map, i.e., $A \in \mathcal{B}(\mathbb{Q}_p(X))$. As $\mathbb{Z}_p(X)$ is τ-compact, also its image under A is τ-compact, and therefore norm-bounded by Lemma 2.15. This fact implies the boundedness and hence the continuity of A. □

Unfortunately, the converse does not hold (this is a consequence, for example, of [4, Thm. 2.4.1]).

[2]Note that the seemingly nontrivial part lies in showing that already *separate* continuity of σ is sufficient.

Definition 2.19. For each $A \in \mathcal{B}(\mathbb{Q}_p(X))$, we define its *operator norm* in the usual way by

$$\|A\| := \sup_{\xi \in \mathbb{Q}_p(X), \|\xi\| \leq 1} \|A\xi\|.$$

By Lemma 2.18, this is a real number and it is clear that it makes $\mathcal{B}(\mathbb{Q}_p(X))$ an ultra-normed $\mathbb{Z}_p$-module. For $A, B \in \mathcal{B}(\mathbb{Q}_p(X))$ and $\xi \in \mathbb{Q}_p(X)$, we have

$$\|A + B\| \leq \max\{\|A\|, \|B\|\}, \quad \|AB\| \leq \|A\|\|B\|, \quad \|A\xi\| \leq \|A\|\|\xi\|.$$

Lemma 2.20. *The $\mathbb{Z}_p$-module $\mathcal{B}(\mathbb{Q}_p(X))$ is norm-complete.*

Once we will have established the matrix representation of the operators in $\mathcal{B}(\mathbb{Q}_p(X))$ (Theorem 2.30), this lemma will be easy to show, and therefore we skip the proof for the moment.

2.21. **Mahler's algebra and continuous functional calculus.** For $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, we will need the binomial coefficient

$$\binom{x}{k} := \frac{x(x-1)\dots(x-(n-1))}{n!} \in \mathbb{Z}_p.$$

The next lemma has a nice combinatorial proof.

Lemma 2.22. (a) *For $x \in \mathbb{Z}_p$ and $m, n \in \mathbb{N}$, the following identity holds:*

$$\binom{x}{m}\binom{x}{n} = \sum_{l=m\vee n}^{m+n} \frac{l!}{(m+n-l)!(l-m)!(l-n)!}\binom{x}{l}.$$

(b) *For $x \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, the following identity holds:*

$$x\binom{x}{n} = n\binom{x}{n} + (n+1)\binom{x}{n+1}.$$

Proof. (a) It is sufficient to show the formula for the case $x \in \mathbb{N}$, $x > m + n$. We assume this.

Then consider a finite set X with cardinality $|X| = x$. The left side of the above equation is exactly the number of pairs (M, N) of subsets $M, N \subseteq X$ such that $|M| = m$ and $|N| = n$. Each such pair is uniquely characterized by the set $M \cup N$ and the subdivision of $M \cup N$ into the subsets $M \setminus N$, $N \setminus M$ and $M \cap N$, and this is precisely what the right side corresponds to. Indeed, the number l corresponds to $|M \cup N|$, the binomial coefficient on the right corresponds to the choices of the set $M \cup N$ and the fraction to the number of subdivisions. Hence, the two sides of the equation coincide.

(b) This is just a consequence of the first part of the lemma (set $m = 1$). □

The following theorem is due to Mahler, see [2] for an elementary proof.

Theorem 2.23 (Mahler's theorem)**.** *Every element $f \in C(\mathbb{Z}_p, \mathbb{Z}_p)$ has a unique representation of the form*

$$f(x) = \sum_{n=0}^{\infty} T_n(f)\binom{x}{n}$$

such that $T_n(f) \in \mathbb{Z}_p$ *and* $\lim_{n\to\infty} T_n(f) = 0$. *The convergence of this series is uniform and the equality*

$$\|f\|_{\sup} = \max_{n\in\mathbb{N}} |T_n(f)|_p$$

holds. In other words, there is an isometric isomorphism $\sigma\colon C(\mathbb{Z}_p,\mathbb{Z}_p) \to c_0(\mathbb{N},\mathbb{Z}_p)$ *of* $\mathbb{Z}_p$*-modules given by* $f \mapsto (T_n(f))_{n\in\mathbb{N}}$.

Definition 2.24. An operator $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is called a *normal contraction* if the quotient

$$\binom{A}{n} := \frac{A(A-1)\cdots(A-(n-1))}{n!} \in \mathcal{B}(\mathbb{Q}_p(X))$$

is defined and is a contraction, i.e., its norm is not greater than one.

It is not difficult to show that $|n!|_p = p^{-\frac{n-s_p(n)}{p-1}}$ for $n \in \mathbb{N}$, where $s_p(n)$ denotes the digit sum in the p-adic decomposition

$$n = \sum_{k=0}^{\infty} n_k p^k$$

of n (with $n_k \in \{0,\dots,p-1\}$), i.e.,

$$s_p(n) = \sum_{k=0}^{\infty} n_k.$$

Therefore, we obtain that A is a normal contraction if and only if

$$\|A(A-1)\dots(A-(n-1))\| \le p^{-\frac{n-s_p(n)}{p-1}} \quad \text{for all } n \in \mathbb{N}.$$

For example, a contractive diagonal operator on $\mathbb{Q}_p(X)$ is always a normal contraction. Note that the formulas in Lemma 2.22 remain true if one replaces x by a normal contraction A. If A is a normal contraction, we obtain a natural functional calculus using Mahler's theorem:

Theorem 2.25. *If* $A \in \mathcal{B}(\mathbb{Q}_p(X))$ *is a normal contraction, then there is a natural contractive homomorphism of* $\mathbb{Z}_p$*-algebras*

$$\pi_A\colon C(\mathbb{Z}_p,\mathbb{Z}_p) \to \mathcal{B}(\mathbb{Q}_p(X)),$$

with $\pi_A(\mathrm{id}_{\mathbb{Z}_p}) = A$.

As usual, we write $f(A)$ instead of $\pi_A(f)$. Note that for a normal contraction A and $f \in C(\mathbb{Z}_p,\mathbb{Z}_p)$, also the operator $f(A)$ is a normal contraction because as $f(A)$ can be represented by a function in $C(\mathbb{Z}_p,\mathbb{Z}_p)$, also the binomial coefficients $\binom{f(A)}{n}$ can and are therefore well-defined contractions.

Proof. By Theorem 2.23, there is a natural isometric isomorphism of $\mathbb{Z}_p$-modules $\sigma\colon C(\mathbb{Z}_p,\mathbb{Z}_p) \to c_0(\mathbb{N},\mathbb{Z}_p)$ satisfying $\sigma(\mathrm{id}_{\mathbb{Z}_p}) = \delta_1$. For $f \in C(\mathbb{Z}_p,\mathbb{Z}_p)$, define

$$\pi_A(f) := \sum_{n=0}^{\infty} \sigma(f)(n)\binom{A}{n}.$$

This definition yields a contractive homomorphism of $\mathbb{Z}_p$-algebras and the proof is finished. □

For example, if $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is a normal contraction and $z \in p\mathbb{Z}_p$, the operator

$$F_z(A) := \sum_{n=0}^{\infty} z^n \binom{A-1}{n}$$

is well defined.

Example. An example of a normal contraction A acting on the space $\mathbb{Q}_p(\mathbb{N})$ is given by the operator defined by $A(\delta_n) = n\delta_n + (n+1)\delta_{n+1}$. Indeed, one can show by induction that the n-th row of the matrix representing the operator $A(A-1)\cdots(A-k)$ (cp. Theorem 2.30) is given by

$$\left(\binom{k+1}{n-i}\binom{n}{k+1}(k+1)!\right)_{i\in\mathbb{N}}$$

for $n, k \in \mathbb{N}$.

Let us recall the following lemma.

Lemma 2.26. *The sequence (f_n) of functions $\mathbb{Z}_p \to \mathbb{Z}_p$ that is defined by*

$$f_n(x) = x^{p^n}$$

for all $x \in \mathbb{Z}_p$ converges uniformly to a function that is constant on each equivalence class for the equivalence relation of having distance less than 1.

The result is well known and the limit $\lim_{n\to\infty} f_n(x)$ is called the *Teichmüller representative* of x (cp. [9, Section 4.3.4]). The proof is also repeated in [4, Lem. 1.6.5].

Now, it is possible to define a polynomial with coefficients in $\mathbb{Z}_p$ mapping all the nonzero Teichmüller representatives to 0 and 0 to 1, namely, the polynomial $P_{\mathbb{Q}_p}(X) := \frac{(X-\lambda_1)\cdots(X-\lambda_{p-1})}{(-1)^{p-1}\lambda_1\cdots\lambda_{p-1}}$.

Corollary 2.27. *Suppose that $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is a normal contraction. Then the sequence $P_{\mathbb{Q}_p}(A^{p^n})$ converges to an idempotent in the operator norm.*

Remark. It is also possible to formulate the above functional calculus for finite field extensions K of $\mathbb{Q}_p$ (cp. [4, Section 1.6]), but we prefer working with $\mathbb{Q}_p$ for now.

2.28. **The matrix representation of operators.** Let A be an operator in $\mathcal{B}(\mathbb{Q}_p(X))$. Associate the matrix $M_A := (A_{ij})_{i,j\in X}$ to A whose coefficients are given by $A_{ij} = (A(\delta_j))(i)$. Note that $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is uniquely determined by M_A. Furthermore, for continuity reasons, we have $A(\xi)(i) = \sum_{j\in X} A_{ij}\xi(j)$ for all $\xi \in \mathbb{Q}_p(X)$.

First, we will state a lemma and second, we will characterize all matrices that can be written in the form M_A for an operator $A \in \mathcal{B}(\mathbb{Q}_p(X))$.

Lemma 2.29. *Let A be in $\mathcal{B}(\mathbb{Q}_p(X))$, then we have $M_{A^*} = M_A^T$, where $M_A^T = (A_{ji})_{i,j\in X}$ is just the transposed matrix of M_A.*

Proof. Let λ be a number in $\mathbb{Q}_p$ and let $i, j \in X$. Observe that

$$\iota(\lambda A_{ij} + \mathbb{Z}_p) = \langle A\delta_j, \lambda\delta_i\rangle = \langle \lambda\delta_j, A^*\delta_i\rangle = \iota(\lambda A^*_{ji} + \mathbb{Z}_p).$$

This can only hold for every $\lambda \in \mathbb{Q}_p$ if $A^*_{ji} = A_{ij}$ for all $i, j \in X$. Therefore, M_{A^*} is exactly the transpose of M_A. □

Theorem 2.30. *A necessary and sufficient condition for a matrix $M = (a_{ij})_{i,j\in X}$ to be of the form $M = M_A$ for an operator $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is that* (a) *M admits only finitely many entries in $\mathbb{Q}_p \setminus \mathbb{Z}_p$ and* (b) *for $k \in X$, one always has $\lim_{i\to\infty} a_{ik} = 0$ and $\lim_{j\to\infty} a_{kj} = 0$.*

Proof. To see why (a) is necessary, suppose that M has infinitely many entries in $\mathbb{Q}_p \setminus \mathbb{Z}_p$. As in each row and in each column, there are clearly only finitely many entries in $\mathbb{Q}_p \setminus \mathbb{Z}_p$, it is possible to choose an infinite subset $Y \subseteq X$ such that for each $y \in Y$, one has $\{i \in X; a_{iy} \in \mathbb{Q}_p \setminus \mathbb{Z}_p\} \neq \varnothing$ and $\{i \in X; a_{iy} \in \mathbb{Q}_p \setminus \mathbb{Z}_p\} \cap \{j \in X; a_{jz} \in \mathbb{Q}_p \setminus \mathbb{Z}_p\} = \varnothing$ for $y, z \in Y$, $y \neq z$. Consider the element $\chi_Y \in \mathbb{Q}_p(X)$, the characteristic function of the set Y. One has $\chi_Y = \lim_{n\to\infty} \chi_{Y_n}$ (convergence with respect to τ), where $(Y_n)_{n\in\mathbb{N}}$ is an increasing sequence of finite subsets of Y with the property that $Y = \bigcup_n Y_n$. If there existed $A \in \mathcal{B}(\mathbb{Q}_p(X))$ such that $M = M_A$, the sequence $(A\chi_{Y_n})_n$ would by continuity converge in $\mathbb{Q}_p(X)$. The choice of the set Y shows that this is not the case. Therefore, condition (a) is necessary for the existence of such an operator A.

On the other hand, suppose that there is an element $x \in X$ and $\varepsilon > 0$ such that $\{j \in X; |a_{jx}| > \varepsilon\}$ is infinite. For $\lambda \in \mathbb{Q}_p$ with $\varepsilon|\lambda| > 1$, the element $\lambda\chi_{\{x\}}$ lies in $\mathbb{Q}_p(X)$, but as $(\lambda a_{ix})_{i\in X}$ does not lie in $\mathbb{Q}_p(X)$, the matrix M is not of the form $M = M_A$ for $A \in \mathcal{B}(\mathbb{Q}_p(X))$. The same holds for the case that $\{j \in X; |a_{xj}| > \varepsilon\}$ is infinite (considering the adjoint matrix M^* and using the lemma above). Therefore, condition (b) is equally necessary for the existence of such an $A \in \mathcal{B}(\mathbb{Q}_p(X))$.

In order to prove that (a) and (b) are sufficient for the existence of A, define A, being given a matrix M such that (a) and (b) hold, by $(A\xi)_i = \sum_{j\in X} a_{ij}\xi_j$, where $\xi = (\xi_j)_{j\in X} \in \mathbb{Q}_p(X)$, $i \in X$. One can easily verify that A lies indeed in $\mathcal{B}(\mathbb{Q}_p(X))$ and that $M = M_A$. □

The following lemma is easy to prove:

Lemma 2.31. *Let A be in $\mathcal{B}(\mathbb{Q}_p(X))$, then we have*

$$\|A\| = \max\{|A_{ij}|_p; i, j \in X\}.$$

By using Theorem 2.30 and Lemma 2.31, the completeness of $\mathcal{B}(\mathbb{Q}_p(X))$ with respect to the norm, i.e., Lemma 2.20, becomes obvious.

To finish the section, we want to give a link of our topic to Willis' notion of the scale of an operator. Recall that, for an endomorphism α on a totally disconnected locally compact group G (i.e., a continuous group homomorphism

$G \to G$), the *scale* $s(\alpha)$ is defined as the minimum of all possible values $[\alpha(U) : (U \cap \alpha(U))]$ for compact open subgroups U of G (the group G always has a base of neighborhoods of the identity that consists only of compact open subgroups, cp. [15, Thm. 2.1]).

For an arbitrary compact open subgroup U of G, the scale can be calculated as $s(\alpha) = \lim_{n\to\infty}[\alpha^n(U) : (U \cap \alpha^n(U))]^{1/n}$, cp. [15, Prop. 8.3].

Also for operators in $\mathcal{B}(\mathbb{Q}_p(X))$, we can ask how to calculate their scale. If X is finite, then we have $\mathbb{Q}_p(X) = \mathbb{Q}_p^n$ (for an appropriate $n \in \mathbb{N}$) and $s(\alpha)$ is the norm of the product of all eigenvalues of α with norm greater than 1 (in a finite field extension of $\mathbb{Q}_p$, where the characteristic polynomial of α decomposes in linear factors, use the Frobenius normal form to show this), i.e.,

$$s(\alpha) = \sup_n \Big\| \bigwedge^n \alpha \Big\|.$$

However, it seems to be a more difficult question how to determine the scale of an operator in $\mathcal{B}(\mathbb{Q}_p(X))$ for infinite X. It seems reasonable to expect that the scale of a general operator is the limit of the scales of the finite minors in its matrix representation and that a similar formula as above holds – but we were unable to show this.

For every operator $A \in \mathcal{B}(\mathbb{Q}_p(X))$, we have $s(A^*) = s(A)$. Even a more general statement can be proved: Let G be a totally disconnected locally compact *abelian* group and let A be an endomorphism on G; then, the adjoint endomorphism A^* acting on the Pontryagin dual G' of G has the same scale as A.

3. Various operator algebras and their K-theory

3.1. **Compact operators in $\mathcal{B}(\mathbb{Q}_p(X))$.** It is interesting to see that also the ideal of compact operators of usual Archimedean functional analysis have a natural analogy in our context.

Definition 3.2. Define $\mathcal{K}(\mathbb{Q}_p(X))$ to be the set of all operators in $\mathcal{B}(\mathbb{Q}_p(X))$ that map norm-bounded sets onto relatively τ-compact sets in $\mathbb{Q}_p(X)$. We want to call the elements of $\mathcal{K}(\mathbb{Q}_p(X))$ the *compact operators* on $\mathbb{Q}_p(X)$.

In the rest of this section, we will always assume $X = \mathbb{N}$ (without any restriction of generality).

Lemma 3.3. *For an operator $A \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$, the following three statements are equivalent:*

(a) *A is a compact operator,*
(b) *the matrix-entries of A converge to zero,*
(c) *it maps norm-bounded sets onto relatively norm-compact sets in $\mathbb{Q}_p(\mathbb{N})$.*

The operators with this property form a selfadjoint ideal in $\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$, i.e., an ideal that is closed under the adjoint operation.

Proof. (a) $\Rightarrow$ (b): Consider $N \in \mathbb{N}$. If A is compact, then the image of $M := \{p^{-N}\delta_n; n \in \mathbb{N}\}$ (as a norm-bounded set) must be relatively τ-compact.

According to Lemma 2.15, the entries of the elements of $A(M)$ have always to be in $\mathbb{Z}_p$ for sufficiently high indices. But the entries of $A(p^{-N}\delta_n)$ are exactly the matrix entries of the n-th column of A, multiplied by p^{-N}. This shows that the matrix entries of A must have norm at most p^{-N} for sufficiently high row-numbers. But in the (only finitely many) rows, where entry-norms greater than p^{-N} occur, A can only have finitely many entries with norm greater than p^{-N} because the row entries converge to zero in each row (cp. Theorem 2.30). Therefore, A has only finitely many matrix entries of norm greater than p^{-N}. As $N \in \mathbb{N}$ is arbitrary, the matrix entries of A must converge to zero.

(b) $\Rightarrow$ (c): Suppose that the matrix entries of A converge to zero. To prove (c), it is sufficient to show that all sets of the form $M_N := A(\{\xi \in \mathbb{Q}_p(\mathbb{N});\ \|\xi\| \leq p^N\})$, $N \in \mathbb{N}$, are relatively compact in $\mathbb{Q}_p(\mathbb{N})$. Suppose $N \in \mathbb{N}$. There exists, for each $k \in \mathbb{N}$, a number $m_k \in \mathbb{N}$ such that all matrix entries of A in a row with number $n \geq m_k$ have norm less than $p^{-N}p^{-k}$. We now obtain $|(A\xi)(n)| \leq p^{-k}$ for $\xi \in \mathbb{Q}_p(\mathbb{N})$ with $\|\xi\| \leq p^N$ and $n \geq m_k$. Therefore, we can construct a sequence $(p^{-a_l})_{l\in\mathbb{N}}$ ($a_l \in \mathbb{Z}$ for $l \in \mathbb{N}$) with $p^{-a_l} \xrightarrow{l\to\infty} 0$ (in $\mathbb{R}$) such that $|(A\xi)(n)| \leq p^{-a_n}$ for all $n \in \mathbb{N}$ and $\xi \in \mathbb{Q}_p(\mathbb{N})$, $\|\xi\| \leq p^N$. We see that

$$M_N \subseteq Q := \prod_{l\in\mathbb{N}} B_{p^{-a_l}},$$

where $B_\varepsilon := \{\lambda \in \mathbb{Q}_p; |\lambda| \leq \varepsilon\}$, $\varepsilon > 0$. To prove (c), it is sufficient to show the norm-compactness of Q. But this is a consequence of the Tychonoff theorem: Notice that the norm-topology on Q coincides exactly with the product topology because we assumed $p^{-a_l} \xrightarrow{l\to\infty} 0$ (in $\mathbb{R}$).

(c) $\Rightarrow$ (a): This is clear since every relatively norm-compact set in $\mathbb{Q}_p(\mathbb{N})$ is also relatively τ-compact (note that a norm-convergent sequence in $\mathbb{Q}_p(\mathbb{N})$ is also τ-convergent).

The fact that the compact operators form a selfadjoint ideal in $\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ follows easily if one uses the matrix representation for compact operators. ☐

3.4. **Some results on idempotents in $\mathcal{B}(\mathbb{Q}_p(X))$.** In the following sections, we want to analyze properties of idempotents in $\mathcal{B}(\mathbb{Q}_p(X))$ and calculate the K_0-groups of $\mathcal{K}(\mathbb{Q}_p(X))$ and $\mathcal{B}(\mathbb{Q}_p(X))$. As a good introduction into K-theory, we recommend [12].

The K-theory of nonarchimedean Banach rings (i.e., complete normed rings whose norm satisfies submultiplicativity and the strong triangle inequality) has been investigated by Adina Calvo in her thesis [3].

We want to collect first information on the idempotents in $\mathcal{B}(\mathbb{Q}_p(X))$. If $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is an *idempotent*, i.e., it satisfies the equation $A^2 = A$, then the operators $1 - A$, A^* and $1 - A^*$ are idempotents as well. Note that we have $\ker(1 - A) = \mathrm{im}(A)$ and that similar equations hold for A^*, $1 - A$ and $1 - A^*$ instead of A. A selfadjoint idempotent will be called a *projection*. The following lemma is easy to prove.

Lemma 3.5. *For an idempotent $A \in \mathcal{B}(\mathbb{Q}_p(X))$, we have the following identities:*

$$\operatorname{im}(A)^\perp = \operatorname{im}(1 - A^*) \quad \textit{and} \quad \operatorname{im}(A) = \operatorname{im}(1 - A^*)^\perp.$$

Note that $\operatorname{im} A = \ker(1 - A)$ is closed. Combining the preceding lemma with Lemma 2.8, we obtain the following.

Lemma 3.6. *If $A \in \mathcal{B}(\mathbb{Q}_p(X))$ is an idempotent, then the Pontryagin dual of* $\operatorname{im}(A)$ *is isomorphic to* $\operatorname{im}(A^*)$.

It would be interesting to know if one can define the usual operations (like supremum and infimum) on the set of idempotents (or projections) in our context.

Our first conjecture in this direction was that for a sequence $(e_n)_{n\in\mathbb{N}}$ of idempotents in $\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$, with

$$e_{n+1}\mathbb{Q}_p(\mathbb{N}) \subseteq e_n\mathbb{Q}_p(\mathbb{N}), \quad (1 - e_n)\mathbb{Q}_p(\mathbb{N}) \subseteq (1 - e_{n+1})\mathbb{Q}_p(\mathbb{N}), \quad \|e_n\| \leq 1$$

for all $n \in \mathbb{N}$, there always exists an idempotent $e \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ such that

$$e\mathbb{Q}_p(\mathbb{N}) = \bigcap_{n\in\mathbb{N}} e_n\mathbb{Q}_p(\mathbb{N}),$$

$$(1 - e)\mathbb{Q}_p(\mathbb{N}) = \tau\text{-cl}\Big(\bigcup_{n\in\mathbb{N}} (1 - e_n)\mathbb{Q}_p(\mathbb{N})\Big).$$

This conjecture, however, turns out to be false (even in the case where the e_n are required to be projections). Counter-examples can be found in [4, Section 3.1].

Second, we would like to know if for two projections $e, f \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$, there is always a projection (or at least an idempotent) $g \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ such that $\operatorname{im} g = \operatorname{im} e \cap \operatorname{im} f$. Unfortunately, also this conjecture is false (cp. [4, Section 3.1] for a counter-example).

Theorem 3.7. *There is a decreasing sequence of contractive projections $(e_n)_{n\in\mathbb{N}}$ in $\mathcal{B}(\mathbb{Q}_p(X))$ such that $\bigcap_{n\in\mathbb{N}} e_n\mathbb{Q}_p(\mathbb{N})$ is not the image of an idempotent in $\mathcal{B}(\mathbb{Q}_p(X))$. There are contractive projections $e, f \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ such that $\operatorname{im} e \cap \operatorname{im} f$ is not the image of an idempotent in $\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$.*

3.8. **The group $K_0(\mathcal{K}(\mathbb{Q}_p(X)))$.** In order to calculate $K_0(\mathcal{K}(\mathbb{Q}_p(X)))$, we will first establish some more general lemmas.

Two idempotents $e, f \in A$ in a unital Banach-$\mathbb{Z}_p$-algebra A are called *equivalent* with respect to A if there is an invertible element $g \in A$ such that $g^{-1}eg = f$. The following two lemmas should essentially be well-known and, in fact, hold for an arbitrary unital Banach-$\mathbb{Z}_p$-algebra.

Lemma 3.9. *Let A be a closed sub-$\mathbb{Z}_p$-algebra of $\mathcal{B}(\mathbb{Q}_p(X))$ that contains the identity. Let $e, f \in A$ be idempotents such that $e \neq 0$ and $\|e - f\| < 1/\|e\|$. Then, e and f are equivalent with respect to A.*

Proof. If e and f are as in the lemma, we obtain

$$\begin{aligned}\|f+e-2fe\| &= \|f-fe+e-fe\| \\ &\leq \max\{\|e\|\|e-f\|, \|f\|\|e-f\|\} < \|e\|\frac{1}{\|e\|} = 1\end{aligned}$$

because $\|e\| \geq 1 > \|e-f\|$ and therefore $\|f\| = \|e\|$. As in the Archimedean case, one can, since A is closed, use the Neumann series (geometric series) to show that the element $u = 1-f-e+2fe \in A$ is invertible in A. On the other hand, one has $fu = fe = ue$ and the lemma follows. □

Lemma 3.10. *Let $\mathcal{A}$ be an ultra-normed Banach algebra. Suppose that $a \in \mathcal{A}\setminus\{0\}$ satisfies $\|a^2-a\| < 1/\|a\|^2$. Then there is an idempotent element $e \in \mathcal{A}$ such that $\|a-e\| < \min\{1/\|a\|, 1\}$. The idempotent e is given as the limit of the sequence $P_m(a)$ as $m \to \infty$ for a certain sequence P_m of polynomials in $\mathbb{Z}[x]$.*

Proof. The result is clear if $\|a\| < 1$, so suppose $\|a\| \geq 1$. Then we have, in particular, $\|a^2 - a\| < 1$. First, we will have to establish that for each $m \in \mathbb{N}$, $m \geq 1$, there is exactly one polynomial $P_m \in \mathbb{Z}[x]$ such that

$$P_m(0) = 0, \quad P_m(1) = 1 \quad \text{and} \quad P_m^{(i)}(0) = P_m^{(i)}(1) = 0$$

for $i \in \{1, \dots, m-1\}$ and $\deg P_m \leq 2m-1$. The ansatz $P_m(x) = \sum_{i=0}^{2m-1} a_i x^i$ yields $P_m^{(k)}(x) = \sum_{i=k}^{2m-1} a_i(i(i-1)\cdots(i-k+1))x^{i-k}$, thus $f^{(k)}(0) = a_k k! = 0$ and $a_k = 0$ for $k \in \{1, \dots, m-1\}$. Furthermore, one gets

$$f^{(k)}(1) = \sum_{i=m}^{2m-1} a_i \binom{i}{k} = \delta_k, \quad k \in \{0, \dots, m-1\},$$

where δ_k denotes the value 1 for $k = 0$ and 0 else. The resulting system of linear equations has m equations and m variables. Using a result from [1], Chapter “Gitterwege und Determinanten”, one can easily see that the determinant of the coefficient matrix of this system is 1. Therefore, it admits a unique solution and the unique existence of the polynomial $P_m \in \mathbb{Z}[x]$ is proved.[3]

Consider now the sequence $P_m(a)$. We notice that

$$\|P_{m+1}(a) - P_m(a)\| = \|(a^2-a)^m(\alpha a + \beta)\| \leq \|a^2 - a\|^m \|a\| \to 0$$

for $m \to \infty$ (where $\alpha, \beta \in \mathbb{Z}$) and

$$\|P_m(a)^2 - P_m(a)\| = \|(a^2-a)^m g_m(a)\| \leq \|a^2-a\|^m \|a\|^{2m-2}$$

for a certain polynomial g_m of degree at most $2m-2$ over $\mathbb{Z}$. Choose $d > 2$ such that $\|a^2 - a\| < 1/\|a\|^d$ and define $c := 1 - 2/d > 0$, i.e., $d(1-c) = 2$.

[3] It is possible to prove an explicit formula for the polynomials P_m:

$$P_m(x) = \sum_{k=m}^{2m-1} x^k \sum_{i=k-m+1}^{m} (-1)^{i+1} \binom{m}{i} \binom{m-1+k-i}{k-i}.$$

Then we obtain

$$\begin{aligned}\|P_m(a)^2 - P_m(a)\| &\leq \|a^2 - a\|^m \|a\|^{2m-2} \\ &< \frac{1}{\|a\|^{d(1-c)m}} \|a^2 - a\|^{cm} \|a\|^{2m-2} \\ &= \frac{\|a^2 - a\|^{cm}}{\|a\|^2} \to 0\end{aligned}$$

for $m \to \infty$. Hence, the sequence $(P_m(a))$ converges to an idempotent $e \in \mathcal{A}$. The inequality $\|P_{m+1}(a) - P_m(a)\| \leq \|a^2 - a\|^m \|a\| < 1/\|a\|$ for $m \in \mathbb{N}$, $m \geq 1$, and the convergence $P_{m+1}(a) - P_m(a) \to 0$ imply that $\|e - a\| < 1/\|a\|$. □

Theorem 3.11. *Let $(A_n)_{n\in\mathbb{N}}$ be an increasing sequence of closed sub-$\mathbb{Z}_p$-algebras of $\mathcal{B}(\mathbb{Q}_p(X))$. Moreover, let A be the closed union of the A_n. Then $K_0(A)$ is isomorphic to the direct limit of the sequence of the $K_0(A_n)$ with the canonical homomorphisms.*

Proof. The proof is (as in the Archimedean case) a straight-forward application of the two preceding lemmas (cp. [10, pp. 234–240], cp. also [4, p. 46]). □

Again, a more general result is true: Let (A_n, φ_n) be a sequence of Banach-$\mathbb{Z}_p$-algebras A_n and contractive homomorphisms $\varphi_n \colon A_n \to A_{n+1}$, and let A be their direct limit as a $\mathbb{Z}_p$-Banach algebra. Then $K_0(A)$ is the direct limit of the sequence $(K_0(A_n), K_0(\varphi_n))$.

Observe that $\mathcal{K}(\mathbb{Q}_p(X))$ is the closure of the set of all operators whose matrices have only finitely many non-vanishing entries. As the finite-dimensional matrix algebras $\mathbb{Q}_p^{n\times n}$ (as well as $\mathbb{Z}_p^{n\times n}$) have K_0-group $\mathbb{Z}$, we can therefore state the following corollary as an application of the preceding theorem (here, we let $\mathcal{K}_{(1)}(\mathbb{Q}_p(X))$ denote the set of all operators in $\mathcal{K}(\mathbb{Q}_p(X))$ with norm not greater than 1).

Corollary 3.12. *We have $K_0(\mathcal{K}(\mathbb{Q}_p(X))) = \mathbb{Z}$. The canonical map*

$$K_0(\mathcal{K}_{(1)}(\mathbb{Q}_p(X))) \to K_0(\mathcal{K}(\mathbb{Q}_p(X)))$$

is an isomorphism.

Theorem 3.13. *Let e be an idempotent in $\mathcal{K}(\mathbb{Q}_p(X))$. Then the image of e is a finite-dimensional $\mathbb{Q}_p$-vector space.*

Proof. A compact operator e in $\mathcal{B}(\mathbb{Q}_p(X))$ has the property that it can be approximated in norm by an operator F in $\mathcal{B}(\mathbb{Q}_p(X))$ having only finitely many non-vanishing matrix-entries such that $\|e - F\| < 1/\|e\|^3$. If e is an idempotent, then we have $\|F^2 - F\| \leq \max\{\|F^2 - e^2\|, \|e - F\|\} \leq \max\{\|F - e\|\|F\|, \|F - e\|\|e\|, \|e - F\|\} < 1/\|e\|^2$. Therefore, there is an idempotent f with only finitely many matrix entries such that $\|f - F\| < 1/\|e\|$. We also obtain $\|f - e\| < 1/\|e\|$ and therefore the equivalence of e and f. As f has finite-dimensional image, also e must have finite-dimensional image. □

3.14. **The group $K_0(\mathcal{B}(\mathbb{Q}_p(X)))$.** Let $\mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ denote the set of all operators in $\mathcal{B}(\mathbb{Q}_p(X))$ with norm not greater than 1. Next, we want to show that $K_0(\mathcal{B}_{(1)}(\mathbb{Q}_p(X))) = 0$.

Lemma 3.15. *If X is countably infinite, the ring $\mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ is an infinite sum ring. In particular, $K_0(\mathcal{B}_{(1)}(\mathbb{Q}_p(X))) = 0$.*

Proof. First, we show that it is a sum ring:[4] Choose a decomposition of X into a countable number $X_0, X_1, X_2, \ldots$ of countably infinite subsets (i.e., their disjoint union is X). Now, choose four operators $\alpha_0, \beta_0, \alpha_1, \beta_1 \in \mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ such that the following properties hold: α_0 is a bijection of X_0 onto X (here, we identify the elements $x \in X$ with the corresponding elements $\delta_x \in \mathbb{Q}_p(X)$) and maps the elements of $X_1 \cup X_2 \cup \cdots$ to 0; β_0 maps X bijectively onto X_0; β_1 maps, for each $n \in \mathbb{N}$, the elements of X_n bijectively onto X_{n+1}; α_1 maps, for each $n \in \mathbb{N}, n \geq 1$, the elements of X_n bijectively onto X_{n-1} and those of X_0 to 0. Furthermore, one requires that $\alpha_0\beta_0|_X = \mathrm{id}_X$ and $\alpha_1\beta_1|_X = \mathrm{id}_X$.

Operators fulfilling these requirements are easily verified to satisfy the relations

$$\alpha_0\beta_0 = \alpha_1\beta_1 = 1, \beta_0\alpha_0 + \beta_1\alpha_1 = 1$$

that imply that $\mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ is a sum ring.

But $\mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ is even an infinite sum ring, that is, for each operator $a \in \mathcal{B}_{(1)}(\mathbb{Q}_p(X))$, define a^∞ to be the operator in $\mathcal{B}(\mathbb{Q}_p(X))$ that acts as a diagonal operator on each X_n as if it acted as a on X, i.e., more precisely, that maps $x \in X_n$ to $\beta_1^n\beta_0 a\alpha_0\alpha_1^n x$. The operator a^∞ lies in $\mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ because its matrix admits no entries in $\mathbb{Q}_p \setminus \mathbb{Z}_p$ (as does the matrix of $a \in \mathcal{B}_{(1)}(\mathbb{Q}_p(X))$). As one has $a^\infty = \sum_{n\in\mathbb{N}} \beta_1^n\beta_0 a\alpha_0\alpha_1^n$ (pointwise limit), it is easy to see that it always satisfies the equation

$$\beta_0 a\alpha_0 + \beta_1 a^\infty \alpha_1 = a^\infty.$$

This fact implies indeed that $\mathcal{B}_{(1)}(\mathbb{Q}_p(X))$ is an infinite sum ring (because the map $a \mapsto a^\infty$ is a unital ring homomorphism) and that $K_0(\mathcal{B}_{(1)}(\mathbb{Q}_p(X))) = 0$. The proof is complete. □

It remains to prove that $K_0(\mathcal{B}(\mathbb{Q}_p(X))) = 0$ for a countable set X. In the sequel, we will always (without loss of generality) assume $X = \mathbb{N}$ for simplicity. It is sufficient to show that each idempotent $e \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ is stably equivalent to zero because $M_m(\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))) \cong \mathcal{B}(\mathbb{Q}_p(\mathbb{N} \times \{1, \ldots, m\})) \cong \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ for all $m \in \mathbb{N}$, $m \geq 1$. Our strategy will be the following: First, we construct an idempotent $f \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ with the finite-dimensional image $\operatorname{im} f = \mathbb{Q}_p e_1 + \cdots + \mathbb{Q}_p e_n$ (where the columns $e_1, \ldots, e_n$ of e are chosen in such a way that they contain all entries of e in $\mathbb{Q}_p \setminus \mathbb{Z}_p$) and with the further

[4] A *sum ring* is a unital ring R with elements $a_0, b_0, a_1, b_1 \in R$ such that $a_0b_0 = a_1b_1 = 1$ and $b_0a_0 + b_1a_1 = 1$, cp. [5, p. 10]. In this case, $\boxplus\colon R\times R \to R$, $(x, y) \mapsto x\boxplus y = b_0xa_0 + b_1ya_1$, is a unital ring homomorphism. An *infinite sum ring* is a sum ring R with a unital ring homomorphism $R \to R$, $a \mapsto a^\infty$, such that $a \boxplus a^\infty = a^\infty$ holds for all $a \in R$. According to [5, Prop. 2.3.1], infinite sum rings always have vanishing K_0-group.

property that also $g = e - f$ is an idempotent with $\operatorname{im} g \subseteq \operatorname{im} e$ and $\|g\| \leq 1$. Second, we show the stable equivalence of g and e (which proves the result because of Lemma 3.15).

In the first step, we want to show that finite-dimensional subspaces have a complement.

Lemma 3.16. *Let $e \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ be an idempotent and define $U = e\mathbb{Q}_p(\mathbb{N})$. Furthermore, let $V \subseteq U \cap c_0(\mathbb{N}, \mathbb{Q}_p)$ be a finite-dimensional $\mathbb{Q}_p$-vector space. Then there exists a τ-continuous $\|\ \|$-contractive idempotent endomorphism $\tilde{f} \colon U \to U$ such that* $\operatorname{im} \tilde{f} = V$.

Proof. Choose a basis $(\tilde{v}_1, \dots, \tilde{v}_m)$ for V. Using certain operations (addition of a multiple of a basis vector to another, multiplication of a basis vector with a number), it is possible to transform this basis into a basis $(v_1, \dots, v_m)$ of V such that $\|v_1\| = \cdots = \|v_m\| = 1$ and with the property that for each $k = 1, \dots, m$, there is a number $a_k \in \mathbb{N}$ such that $v_i(a_k) = \delta_{ik}$ for $i = 1, \dots, m$ (where $v_i(a_k)$ is the a_k-th entry of v_i).

For an element $\xi \in U$, define now $\tilde{f}(\xi) = \xi(a_1)v_1 + \cdots + \xi(a_m)v_m$. The function $\tilde{f} \colon U \to U$ satisfies the required properties of the lemma. $\square$

Now, we can proceed to the announced decomposition of the idempotent e.

Lemma 3.17. *Let $e \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ be an idempotent and define $U = e\mathbb{Q}_p(\mathbb{N})$. Let e_i, $i \in \mathbb{N}$, be the columns of e (considered as a matrix) and choose $n \in \mathbb{N}$ such that e_i does not contain entries in $\mathbb{Q}_p \setminus \mathbb{Z}_p$ for $i > n$. Then there is an idempotent $f \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ with the properties that $fe = ef = f$, that* $\operatorname{im} f$ *is the finite-dimensional $\mathbb{Q}_p$-vector space $\mathbb{Q}_p e_1 + \cdots + \mathbb{Q}_p e_n$ and that $e - f$ is an idempotent with $\|e - f\| \leq 1$.*

Proof. Define V to be the space $\mathbb{Q}_p e_1 + \cdots + \mathbb{Q}_p e_n \subseteq U$ and apply the preceding lemma on it. Let $\tilde{f} \colon U \to U$ be the $\|\ \|$-contractive τ-continuous idempotent of the preceding lemma with $\operatorname{im} f = V$. Define $f = \tilde{f} \circ e$. It is clear that f is τ-continuous (and therefore in $\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$) and that it is an idempotent with $\operatorname{im} f = V$. The equation $ef = fe = f$ follows from $V \subseteq U$. Furthermore, we obtain $(e-f)^2 = e - fe - ef + f = e - f - f + f = e - f$, and $e - f$ is thus an idempotent.

We still have to show that $\|e - f\| \leq 1$. Let k be in $\{0, \dots, n\}$. A calculation yields

$$(e - f)\delta_k = e\delta_k - (\tilde{f} \circ e)\delta_k = e_k - \tilde{f}e_k = e_k - e_k = 0.$$

On the other hand, for $k \in \mathbb{N}$, $k > n$, we obtain

$$\|(e - f)\delta_k\| = \|e\delta_k - f\delta_k\| = \|e_k - \tilde{f}e_k\| \leq \|e_k\| \vee \|\tilde{f}e_k\| \leq 1,$$

because $\tilde{f}$ is $\|\ \|$-contractive and $\|e_k\| \leq 1$.

Hence, considered as a matrix, $e - f$ contains no entries in $\mathbb{Q}_p \setminus \mathbb{Z}_p$ and we have shown $\|e - f\| \leq 1$. $\square$

Lemma 3.18. *Let A be a closed sub-$\mathbb{Z}_p$-algebra of $\mathcal{B}(\mathbb{Q}_p(X))$ that contains the identity. Let $e \in A$ be an idempotent such that $e \neq 0$ and let $a \in A$ be such that $\|e - a\| < 1/\|e\|^3$. Then the sequence $(P_m(a))_{m\in\mathbb{N}}$ converges to an idempotent $e_a \in A$ that is equivalent to e.*

The polynomials $P_m \in \mathbb{Z}[x]$, $m \in \mathbb{N}$, $m \geq 1$, have been defined in the proof of Lemma 3.10.

Proof. First, we obtain $\|e\| \geq 1$ and $\|e - a\| < 1/\|e\|^3 \leq 1 \leq \|e\|$ and hence $\|a\| = \|e\| \geq 1$. Now, notice that $((e - a) + a)^2 = (e - a) + a$, i.e.,

$$(e-a)^2 + a^2 + (e-a)a + a(e-a) = (e-a) + a$$

or

$$\|a^2 - a\| = \|(e-a) - (e-a)^2 - (e-a)a - a(e-a)\| \leq \|e-a\|\|a\| < 1/\|a\|^2.$$

Recall from the proof of Lemma 3.10 that the sequence $P_m(a)$ converges to an idempotent element $e_a \in A$ such that $\|a - e_a\| < 1/\|a\| = 1/\|e\|$. We therefore obtain that also $\|e - e_a\| < 1/\|e\|$ holds. According to Lemma 3.9, the idempotents e and e_a are equivalent with respect to A. □

Lemma 3.19. *Let $f \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ be an idempotent whose image is a finite-dimensional $\mathbb{Q}_p$-vector space. Then f is stably equivalent to zero.*

Proof. The case $f = 0$ is obvious; assume therefore $f \neq 0$. Observe that, as f has a finite-dimensional image, it must be a compact operator. Therefore, its entries converge to zero.

Choose an element $a \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ that has (considered as a matrix) only finitely many non-vanishing entries and satisfies $\|a - f\| < 1/\|f\|^3$. Choose $n \in \mathbb{N}$ such that the entry a_{ij} of a is zero if $i > n$ or $j > n$, i.e., such that $a \in M_{\{0,\ldots,n\}}(\mathbb{Q}_p) \subseteq \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$. On the one hand, the polynomials $P_m(a)$ will, according to Lemma 3.18, converge in norm to an idempotent $e_a \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ that is equivalent to f. On the other hand, as the polynomials P_m have no constant term, $P_m(a)$ never leaves the set $M_{\{0,\ldots,n\}}(\mathbb{Q}_p) \subseteq \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$, and hence also e_a has only finitely many non-vanishing entries. It is a well-known fact from linear algebra that e_a is equivalent to a matrix of the form

$$D = \begin{bmatrix} E_N & 0 \\ 0 & 0 \end{bmatrix},$$

where E_N is the $N \times N$-unity matrix ($N \in \mathbb{N}$) and 0 means vanishing matrices of appropriate size. In the end, we obtain that D and e are equivalent matrices, and therefore e is stably equivalent to zero. □

Theorem 3.20. *We have $K_0(\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))) = 0$.*

Proof. Let e be an idempotent in $M_m(\mathcal{B}(\mathbb{Q}_p(\mathbb{N})))$. We want to show that e is stably equivalent to zero. Because of the isomorphism $M_m(\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))) \cong \mathcal{B}(\mathbb{Q}_p(\mathbb{N} \times \{1,\ldots,m\})) \cong \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ for all $m \in \mathbb{N}$, $m \geq 1$, it is sufficient to treat the case $e \in \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$. Consider the decomposition $e = f + (e - f)$ stemming from Lemma 3.17. As we have $f(e - f) = (e - f)f = 0$, we obtain that the stable equivalence class of e is exactly the sum of the stable equivalence classes of f and of $e-f$. But the stable equivalence class of f is zero according to Lemma 3.19 and the stable equivalence class of $e-f$ is zero as well because of Lemma 3.15 (since $\|e-f\| \leq 1$). Hence, the proof is finished. □

3.21. **Lifting of idempotents in $\mathcal{B}(\mathbb{Q}_p(X))/\mathcal{K}(\mathbb{Q}_p(X))$.**

Theorem 3.22. *Let $A \subseteq \mathcal{B}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$ be a norm-closed subalgebra containing the set $\mathcal{K}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$ of contractive compact operators. If E is an idempotent element in the quotient algebra $A/\mathcal{K}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$, then it has an idempotent lift e in A, i.e., $e^2 = e \in A$ and $e + \mathcal{K}_{(1)}(\mathbb{Q}_p(\mathbb{N})) = E$.*

Proof. Choose an arbitrary lift $a \in A$ of E. Then we get $a^2 - a \in \mathcal{K}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$ and also $a^n - a = (a^{n-2} + \cdots + a + 1)(a^2 - a) \in \mathcal{K}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$ for $n > 2$. Observe that there is a number $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, the entries of $a^n - a$ (considered as a matrix) at the positions $(i,j) \in \mathbb{N}^2 \setminus \{0,\ldots,N\}^2$ have absolute value smaller than 1 (because the entries of $a^2 - a$ converge to zero and one can write $a^n - a = (a^2 - a)b_n = b_n(a^2 - a)$ for an operator b_n with $\|b_n\| \leq 1$).

Therefore, there must be $m, n \in \mathbb{N}$ with $n < m$ such that $\|a^m - a^n\| < 1$. For $i, j \in \mathbb{N}$ such that $i > N$ or $j > N$, the entries of $a^n - a$ and $a^m - a$ (hence of $a^m - a^n$) at the position (i,j) have absolute value smaller than 1 anyway and for the finitely many positions in $\{1,\ldots,N\}^2$, the entries of $a^m - a$ and $a^n - a$ become arbitrarily close for certain m, n for compactness reasons (we used $a^m - a^n = (a^m - a) - (a^n - a)$). Now, choose $k \in \mathbb{N}$ such that $k(m-n) > n$. Then we also have $\|a^{(k+1)(m-n)} - a^{k(m-n)}\| < 1$ and thus $\|a^{2k(m-n)} - a^{k(m-n)}\| < 1$.

Finally, we apply our usual technique. As $b = a^{k(m-n)}$ and its square have distance less than 1, the sequence of polynomials $P_l(b)$ (defined in the proof of Lemma 3.10) converges to an idempotent e for $l \to \infty$ that has distance less than 1 from b. As all the operators $P_l(b) - b$ are of the form $(b^2 - b)Q(b)$ (where Q is a polynomial with coefficients in $\mathbb{Z}$), they are all compact, as well as $b - a$ and therefore $P_l(b) - a$. As the ideal of compact operators $\mathcal{K}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$ is norm-closed in $\mathcal{B}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$, we obtain that also $e - a$ is compact, i.e., e is an idempotent lift of E. Note that in the whole procedure, we did not leave the algebra A (even if it is nonunital) because we assumed it to be norm-closed, i.e., $e \in A$. That finishes the proof. □

As the operators in $\mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ have (considered as matrices) only finitely many entries not in $\mathbb{Z}_p$ and differ therefore only by a compact difference from operators in $\mathcal{B}_{(1)}(\mathbb{Q}_p(\mathbb{N}))$, we easily get the following corollary.

Corollary 3.23. *Let $A \subseteq \mathcal{B}(\mathbb{Q}_p(\mathbb{N}))$ be a norm-closed subalgebra containing the set $\mathcal{K}(\mathbb{Q}_p(\mathbb{N}))$ of compact operators. If E is an idempotent element in the quotient algebra $A/\mathcal{K}(\mathbb{Q}_p(\mathbb{N}))$, then it has an idempotent lift e in A, i.e., $e^2 = e \in A$ and $e + \mathcal{K}(\mathbb{Q}_p(\mathbb{N})) = E$.*

Acknowledgments. This research was supported in part by the ERC Consolidator Grant No. 681207. We thank the referee for useful comments that improved the exposition of the results. The results presented in this paper are part of the PhD project of the first author.

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Received April 24, 2019; accepted June 17, 2019

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