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Locally unitary principal series representations of $\mathbf{GL}_{d+1}(\boldsymbol{F})$

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On the occasion of the 60th birthday of Peter Schneider

Abstract. For a local field F we consider tamely ramified principal series representations V of $G = \mathrm{GL}_{d+1}(F)$ with coefficients in a finite extension K of $\mathbb{Q}_p$. Let I_0 be a pro-p-Iwahori subgroup in G, let $\mathcal{H}(G, I_0)$ denote the corresponding pro-p-Iwahori Hecke algebra. If V is locally unitary, i.e. if the $\mathcal{H}(G, I_0)$-module V^{I_0} admits an integral structure, then such an integral structure can be chosen in a particularly well organized manner, in particular its modular reduction can be made completely explicit.

Contents

1.	Introduction	115
2.	Functions on symmetric groups	117
3.	Hecke lattices in principal series representations I	123
4.	Hecke lattices in principal series representations II	128
5.	$\mathcal{H}(G, I_0)_k$-modules of W-type	131

1. Introduction

Let F be a local nonarchimedean field with finite residue field k_F of characteristic $p > 0$, let $G = \mathrm{GL}_{d+1}(F)$ for some $d \in \mathbb{N}$. Let K be another local field which is a finite extension of $\mathbb{Q}_p$, let $\mathfrak{o}$ denote its ring of integers, $\pi \in \mathfrak{o}$ a nonzero element in its maximal ideal and k its residue field.

The general problem of deciding whether a given smooth (or, more generally, locally algebraic) G-representation V over K admits a G-invariant norm—or equivalently: a G-stable free $\mathfrak{o}$-sub module containing a K-basis of V—is of great importance for the p-adic local Langlands program. It is not difficult to formulate a certain *necessary* condition for the existence of a G-invariant norm on V. This has been emphasized first by Vignéras, see also [2], [3], [6], [7]. If V is a tamely ramified smooth principal series representation and if $d = 1$ then this condition turns out to also be *sufficient*, see [8]. Unfortunately, if $d > 1$ it is unknown if this condition is sufficient. See however [4] for some recent progress.

In this note we consider tamely ramified smooth principal series representations V of G over K for general $d \in \mathbb{N}$. More precisely, we fix a maximal split torus T, a Borel subgroup P and a pro-p-Iwahori subgroup I_0 in G fixing a chamber in the apartment corresponding to T. We then consider a smooth K-valued character Θ of T which is trivial on $T \cap I_0$, view it as a character of P and form the smooth induction $V = \mathrm{Ind}_P^G \Theta$.

Let $\mathcal{H}(G, I_0)$ denote the pro-p-Iwahori Hecke algebra with coefficients in $\mathfrak{o}$ corresponding to I_0. The K-subspace V^{I_0} of I_0-invariants in V is naturally a module over $\mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} K$. The said necessary condition for the existence of a G-invariant norm on V is now equivalent with the condition that the $\mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} K$-module V^{I_0} admits an integral structure, i.e. an $\mathfrak{o}$-free $\mathcal{H}(G, I_0)$-sub module L containing a K-basis of V^{I_0}. One might phrase this as the condition that V be locally integral, or locally unitary.

It is not difficult to directly read off from Θ whether V is locally unitary. (Besides [2, Prop. 3.2] we mention the formulation in terms of Jacquet modules as propagated by Emerton ([3]), see also Section 4 below.) We rederive this relationship here. However, the proper purpose of this paper is to provide *explicit* and particularly *well structured* $\mathfrak{o}$-lattices L_∇ in V^{I_0} as above whenever V is locally unitary.

Our approach is completely elementary; for example, it does not make use of the integral Bernstein basis for $\mathcal{H}(G, I_0)$ (e.g. [7]). It is merely based on the investigation of certain $\mathbb{Z}$-valued functions ∇ on the finite Weyl group $W = N(T)/T$, and thus on combinatorics of W. We consider the canonical K-basis $\{f_w\}_{w \in W}$ of V^{I_0} where $f_w \in V^{I_0}$ has support PwI_0 and satisfies $f_w(w) = 1$ (we realize W as a subgroup in G). We then ask for functions $\nabla : W \to \mathbb{Z}$ such that $L_\nabla = \oplus_{w \in W} (\pi)^{\nabla(w)} f_w$ is an $\mathfrak{o}$-lattice as desired. We show (Theorem 4.2) that whenever V is locally unitary, then V^{I_0} admits an $\mathcal{H}(G, I_0)$-stable $\mathfrak{o}$-lattice of this particular shape.

The structure of the $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$-modules $L_\nabla \otimes_{\mathfrak{o}} k$ so obtained is then encoded in combinatorics of the (finite) Coxeter group W. Approaching them abstractly we suggest the notion of an $\mathcal{H}(G, I_0)_k$-module *of W-type* (or: a *reduced standard $\mathcal{H}(G, I_0)_k$-module*): This is an $\mathcal{H}(G, I_0)_k$-module $M[\theta, \sigma, \epsilon_\bullet]$ with k-basis parametrized by W and whose $\mathcal{H}(G, I_0)_k$-structure is characterized, by means of some explicit formulae, through a set of data $(\theta, \sigma, \epsilon_\bullet)$ as follows: θ is a character of $I/I_0 = (T \cap I)/(T \cap I_0)$ where $I \supset I_0$ is the corresponding Iwahori subgroup; σ is a function $\{w \in W \mid \ell(ws_d) > \ell(w)\} \to \{-1, 0, 1\}$ where s_d is the simple reflection corresponding to an end in the Dynkin diagram, and ℓ is the length function on W; finally, $\epsilon_\bullet = \{\epsilon_w \mid w \in W\}$ is a set of units in k. (But not any such set of data $(\theta, \sigma, \epsilon_\bullet)$ defines an $\mathcal{H}(G, I_0)_k$-module $M[\theta, \sigma, \epsilon_\bullet]$.)

The explicit nature of $L_\nabla \otimes_{\mathfrak{o}} k$, and more generally of an $\mathcal{H}(G, I_0)_k$-module of W-type, is particularly well suited for computing its value under a certain functor from finite dimensional $\mathcal{H}(G, I_0)_k$-modules to (φ, Γ)-modules (if $F = \mathbb{Q}_p$), see [5].

We intend to generalize the results of the present paper to other reductive groups in the future. Moreover, the relationship between $\mathcal{H}(G, I_0)_k$-modules of W-type (reduced standard $\mathcal{H}(G, I_0)_k$-modules) and standard $\mathcal{H}(G, I_0)_k$-modules should be clarified.

The outline is as follows. In Section 2 we first introduce the notion of a balanced weight of length $d + 1$: a $(d + 1)$-tuple of integers satisfying certain boundedness conditions which later on will turn out to precisely encode the condition (on Θ) for V to be locally unitary. Given such a balanced weight, we show the existence of certain functions $\nabla : W \to \mathbb{Z}$ "integrating" it. In Section 3 we introduce $V = \mathrm{Ind}_P^G \Theta$ and show that if a function ∇ "integrates" the "weight" associated with Θ, then L_∇ is an $\mathcal{H}(G, I_0)$-stable $\mathfrak{o}$-lattice as desired. In Section 4 we put the results of Sections 2 and 3 together. In Section 5 we introduce $\mathcal{H}(G, I_0)_k$-modules of W-type.

2. Functions on symmetric groups

For a finite subset I of $\mathbb{Z}_{\geq 0}$ we put

$$\Delta(I) = \sum_{i \in I} i - \frac{|I| \cdot (|I| - 1)}{2}.$$

Definition. Let $d, r \in \mathbb{N}$. We say that a sequence of integers $(n_i)_{0 \leq i \leq d} = (n_0, \dots, n_d)$ is a balanced weight of length $d+1$ and amplitude r if $\sum_{i=0}^{d} n_i = 0$ and if for each subset $I \subset \{0, \dots, d\}$ we have

$$r\Delta(I) \geq \sum_{i \in I} n_i \geq -r\Delta(\{0, \dots, d\} - I). \tag{1}$$

Lemma 2.1. *If $(n_i)_{0 \leq i \leq d}$ is a balanced weight of length $d + 1$ and amplitude r, then so is $(-n_{d-i})_{0 \leq i \leq d}$.*

Proof. For any $I \subset \{0, \dots, d\}$ we compute

$$\begin{aligned}
\Delta(I) &= \sum_{i \in I} i - \frac{|I| \cdot (|I| - 1)}{2} \\
&= \sum_{i=0}^{d} i - \sum_{i \notin I} i - d|I| - \frac{|I|^2}{2} + \frac{(d+1)|I| + d|I|}{2} \\
&= \frac{d(d+1)}{2} - \sum_{i \notin I} i - d|I| - \frac{|I|^2}{2} + \frac{(d+1)|I| + d|I|}{2} \\
&= d(d + 1 - |I|) - \sum_{i \notin I} i - \frac{(d + 1 - |I|)(d - |I|)}{2}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i \notin I} (d-i) - \frac{(d+1-|I|)(d-|I|)}{2} \\
&= \Delta(\{d-i \mid i \in \{0,\dots,d\} - I\}).
\end{aligned}
$$

Together with the assumption $\sum_{i=0}^{d} n_i = 0$ this shows that the set of inequalities (1) for $(n_i)_{0\le i\le d}$ is equivalent with the same set of inequalities for $(-n_{d-i})_{0\le i\le d}$. Namely, given $I \subset \{0,\dots,d\}$, the inequalities (1) for $(n_i)_{0\le i\le d}$ and I are equivalent with the inequalities (1) for $(-n_{d-i})_{0\le i\le d}$ and $\{d-i \mid i \in \{0,\dots,d\} - I\}$. $\square$

Lemma 2.2. *Let $(n_i)_{0\le i\le d}$ be a balanced weight of length $d+1$ and amplitude r.*

(a) There is a balanced weight $(\tilde{n}_i)_{0\le i\le d}$ of length $d+1$ and amplitude r such that $\tilde{n}_0 = 0$ and $0 \le n_i - \tilde{n}_i \le r$ for all $1 \le i \le d$.

(b) There is a balanced weight $(m_i)_{0\le i\le d-1}$ of length d and amplitude r such that $0 \le n_i - m_{i-1} \le r$ for each $i = 1,\dots,d$.

Proof. We first show that (b) follows from (a). Indeed, suppose we are given $(\tilde{n}_i)_{0\le i\le d}$ as in (a). Then put $m_{i-1} = \tilde{n}_i$ for $i = 1,\dots,d$. We clearly have $\sum_{i=0}^{d-1} m_i = 0$. Next, let $I \subset \{0,\dots,d-1\}$. Putting $I^+ = \{i+1 \mid i \in I\}$ and $I_0^+ = I^+ \cup \{0\}$ we then find

$$
\begin{aligned}
r\Delta(I) &= r\Big(\sum_{i\in I} i - \frac{|I|(|I|-1)}{2}\Big) \\
&= r\Big(\sum_{i\in I_0^+} i - |I| - \frac{|I|(|I|-1)}{2}\Big) \\
&= r\Big(\sum_{i\in I_0^+} i - \frac{|I_0^+|(|I_0^+|-1)}{2}\Big) \\
&= r\Delta(I_0^+) \\
&\overset{(i)}{\ge} \sum_{i\in I_0^+} \tilde{n}_i = \sum_{i\in I} m_i
\end{aligned}
$$

where (i) holds true by assumption. Similarly, we find

$$
\begin{aligned}
(2)\qquad & -r\Delta\Big(\{0,\dots,d-1\} - I\Big) \\
&= -r\Big(\sum_{i\in\{0,\dots,d-1\}-I} i - \frac{(d-|I|)(d-|I|-1)}{2}\Big) \\
&= -r\Big(\sum_{i\in\{0,\dots,d\}-I^+} i - (d-|I|) - \frac{(d-|I|)(d-|I|-1)}{2}\Big) \\
&= -r\Big(\sum_{i\in\{0,\dots,d\}-I^+} i - \frac{(d+1-|I^+|)(d-|I^+|)}{2}\Big)
\end{aligned}
$$

$$= -r\Delta(\{0,\dots,d\} - I^+)$$
$$\overset{(ii)}{\leq} \sum_{i\in I^+} \tilde{n}_i = \sum_{i\in I} m_i$$

where (ii) holds true by assumption.

Now we prove statement (a) in three steps.

Step 1: *For any sequence of integers $t_1,\dots,t_d$ satisfying*

$$r|I|(d-\frac{1}{2}(|I|-1)) \geq \sum_{i\in I} t_i \geq \frac{1}{2}r|I|(|I|-1) \tag{3}$$

for each subset $I\subset\{1,\dots,d\}$, there exists another sequence of integers $\tilde{t}_1,\dots,\tilde{t}_d$, again satisfying formula (3) for each $I\subset\{1,\dots,d\}$ and such that $\sum_{i=1}^d \tilde{t}_i = \frac{1}{2}rd(d-1)$ and $0\leq t_i-\tilde{t}_i\leq r$ for all $1\leq i\leq d$.

For a subset $I\subset\{1,\dots,d\}$ we write $I^c=\{1,\dots,d\}-I$. Put

$$\delta = \sum_{i=1}^d t_i - \frac{1}{2}rd(d-1).$$

To construct $\tilde{t}_1,\dots,\tilde{t}_d$ as desired, we put $s_i^{(0)}=t_i$ and define inductively sequences $s_1^{(m)},\dots,s_d^{(m)}$ for $1\leq m\leq\delta$ such that $0\leq t_i-s_i^{(m)}\leq r$, such that $0\leq s_i^{(m-1)}-s_i^{(m)}\leq 1$, such that $\delta-m=\sum_{i=1}^d s_i^{(m)}-\frac{1}{2}d(d-1)$ and such that for any fixed m the sequence $(s_i^{(m)})_i$ satisfies (3) for each subset $I\subset\{1,\dots,d\}$. Once all the $(s_i^{(m)})_i$ are constructed we may put $\tilde{t}_i=s_i^{(\delta)}$.

Suppose $(s_i^{(m)})_i$ have been constructed for some $m<\delta$. Let $I_0\subset\{1,\dots,d\}$ be maximal such that $\sum_{i\in I_0} s_i^{(m)}=\frac{1}{2}r|I_0|(|I_0|-1)$. We have

$$s_{i_0}^{(m)} < s_k^{(m)} \text{ for each } i_0\in I_0 \text{ and each } k\in I_0^c. \tag{4}$$

This follows from combining the three formulae

$$\sum_{i\in I_0\cup\{k\}} s_i^{(m)} \geq \frac{1}{2}r|I_0\cup\{k\}|(|I_0\cup\{k\}|-1) = \frac{1}{2}r|I_0|(|I_0|-1)+r|I_0|,$$
$$\sum_{i\in I_0} s_i^{(m)} = \frac{1}{2}r|I_0|(|I_0|-1),$$
$$\sum_{i\in I_0-\{i_0\}} s_i^{(m)} \geq \frac{1}{2}r|I_0-\{i_0\}|(|I_0-\{i_0\}|-1) = \frac{1}{2}r|I_0|(|I_0|-1)-r(|I_0|-1)$$

(the first one and the last one holding by hypothesis).

Claim: There is some $k\in I_0^c$ such that $s_k^{(m)}+r>t_k$.

Suppose that, on the contrary, $s_k^{(m)}+r=t_k$ for all $k\in I_0^c$. As $(t_i)_i$ satisfies (3) we then have

$$r|I_0^c|(d-\frac{1}{2}(|I_0^c|-1)) \geq \sum_{k\in I_0^c} s_k^{(m)}+r$$

or equivalently

$$r|I_0^c|(d-1-\frac{1}{2}(|I_0^c|-1)) \geq \sum_{k\in I_0^c} s_k^{(m)}.$$

On the other hand, as $m<\delta$ we find

$$\begin{aligned}\sum_{k\in I_0^c} s_k^{(m)} &= \Big(\sum_{k\in I_0} s_k^{(m)}\Big) - \sum_{k\in I_0} s_k^{(m)}\\ &> \frac{1}{2}rd(d-1) - \frac{1}{2}r|I_0|(|I_0|-1)\\ &= r\sum_{n=|I_0|}^{d-1} n\\ &= r|I_0^c|(d-1-\frac{1}{2}(|I_0^c|-1)).\end{aligned}$$

Taken together this is a contradiction. The claim is proven.

We choose some $k\in I_0^c$ such that $s_k^{(m)}+r>t_k$ and put $s_k^{(m+1)}=s_k^{(m)}-1$ and $s_i^{(m+1)}=s_i^{(m)}$ for $i\in\{1,\ldots,d\}-\{k\}$.

Claim: $(s_i^{(m+1)})_i$ *satisfies the inequality on the right hand side of (3) for each* $I\subset\{1,\ldots,d\}$.

If $k\notin I$ this follows from the inequality on the right hand side of (3) for I and $(s_i^{(m)})_i$. Similarly, if $\sum_{i\in I}s_i^{(m)}>\frac{1}{2}r|I|(|I|-1)$ the claim is obvious. Now assume that $k\in I$ and $\sum_{i\in I}s_i^{(m)}=\frac{1}{2}r|I|(|I|-1)$. We then find some $i_0\in I_0$ with $i_0\notin I$, because otherwise $I_0\subset I$ and hence (since $k\in I$ but $k\notin I_0$) even $I_0\subsetneq I$, which would contradict the maximality of I_0 as chosen above. Formula (4) gives $s_k^{(m+1)}\geq s_{i_0}^{(m)}$, hence the inequality on the right hand side of (3) for $(I-\{k\})\cup\{i_0\}$ and $(s_i^{(m)})_i$ implies the inequality on the right hand side of (3) for I and $(s_i^{(m+1)})_i$.

The claim is proven. All the other properties required of $(s_i^{(m+1)})_i$ are obvious from its construction.

Step 2: *The sequence* $t_1,\ldots,t_d$ *defined by* $t_i=n_i+r(d-i)$ *satisfies formula (3) for each subset* $I\subset\{1,\ldots,d\}$.

Indeed, for each $I\subset\{1,\ldots,d\}$ the formula (3) for $(t_i)_{1\leq i\leq d}$ is equivalently converted into the formula (1) for $(n_i)_{1\leq i\leq d}$ by means of the following equations:

$$r|I|(d-\frac{1}{2}(|I|-1)) = r\Delta(I)+\sum_{i\in I} r(d-i),$$

$$\frac{1}{2}r|I|(|I|-1) = -r\Delta(\{0,\ldots,d\}-I)+\sum_{i\in I} r(d-i).$$

Step 3: *If for the* t_i *as in Step 2 we choose* $\tilde{t}_i$ *as in Step 1, then the sequence* $(\tilde{n}_i)_{0\leq i\leq d}$ *defined by* $\tilde{n}_0=0$ *and* $\tilde{n}_i=\tilde{t}_i-r(d-i)$ *for* $1\leq i\leq d$ *satisfies the requirements of statement (a).*

It is clear that $\tilde{n}_0 = 0$ and $0 \leq n_i - \tilde{n}_i \leq r$ for all $1 \leq i \leq d$, as well as $\sum_{i=0}^{d} \tilde{n}_i = 0$. It remains to see that $(\tilde{n}_i)_{0 \leq i \leq d}$ satisfies the inequalities (1) for any $I \subset \{0, \dots, d\}$. If $0 \notin I$ then, using the same conversion formulae as in the proof of Step 2, this follows from the fact that $(\tilde{t}_i)_{1 \leq i \leq d}$ satisfies formula (1) for each $I \subset \{1, \dots, d\}$. If however $0 \in I$ then we use the property $\sum_{i=0}^{d} \tilde{n}_i = 0$: it implies that, for $(\tilde{n}_i)_{0 \leq i \leq d}$, the left hand (resp. right hand) side inequality of formula (1) for I is equivalent with the right hand (resp. left hand) side inequality of formula (1) for $\{0, \dots, d\} - I$, thus holds true because the latter holds true—as we just saw. $\square$

Let W denote the finite Coxeter group of type A_d. Thus, W contains a set $S_0 = \{s_1, \dots, s_d\}$ of Coxeter generators satisfying $\mathrm{ord}(s_i s_{i+1}) = 3$ for $1 \leq i \leq d-1$ and $\mathrm{ord}(s_i s_{j+1}) = 2$ for $1 \leq i < j \leq d-1$. Put $\overline{u} = s_d \cdots s_1$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the length function.

It is convenient to realize W as the symmetric group of the set $\{0, \dots, d\}$ such that $s_i = (i-1, i)$ (transposition) for $1 \leq i \leq d$. For $w \in W$ and $1 \leq i \leq d$ we then have

$$\ell(ws_i) > \ell(w) \text{ if and only if } w(i-1) < w(i), \tag{5}$$

see [1, Prop. 1.5.3].

Let W' denote the subgroup of W generated by $s_1, \dots, s_{d-1}$. Any element w in W can be uniquely written as $w = \overline{u}^i w'$ for some $w' \in W'$, some $0 \leq i \leq d$. We may thus define $\mu(w) = i$; equivalently, $\mu(w) \in \{0, \dots, d\}$ is defined by asking $\overline{u}^{-\mu(w)} w \in W'$.

Theorem 2.3. *Let $(n_i)_{0 \leq i \leq d}$ be a balanced weight of length $d+1$ and amplitude r. There exists a function $\nabla : W \to \mathbb{Z}$ such that for all $w \in W$ we have*

$$\nabla(w) - \nabla(w\overline{u}) = -n_{\mu(w)} \tag{6}$$

and such that for all $s \in S_0$ and $w \in W$ with $\ell(ws) > \ell(w)$ we have

$$\nabla(w) - r \leq \nabla(ws) \leq \nabla(w). \tag{7}$$

Proof. We argue by induction on d. The case $d = 1$ is trivial. Now assume that $d \geq 2$ and that we know the result for $d-1$. By Lemma 2.2 we find a balanced weight $(m_i)_{0 \leq i \leq d-1}$ of length d and amplitude r such that $0 \leq n_i - m_{i-1} \leq r$ for each $i = 1, \dots, d$. Put $\overline{u}' = s_{d-1} \cdots s_1$. Define

$$\mu' : W' \to \{0, \dots, d-1\}$$

by asking that for any $w \in W'$ the element $(\overline{u}')^{-\mu'(w)} w$ of W' belongs to the subgroup generated by $s_1, \dots, s_{d-2}$. By induction hypothesis there is a function $\nabla' : W' \to \mathbb{Z}$ with

$$\nabla'(w) - \nabla'(w\overline{u}') = -m_{\mu'(w)}$$

for all $w \in W'$ and

$$\nabla'(w) - r \leq \nabla'(ws) \leq \nabla'(w)$$

for all $w \in W', s \in \{s_1, \ldots, s_{d-1}\}$ with $\ell(ws) > \ell(w)$. Writing $w \in W$ uniquely as $w = w'\overline{u}^j$ with $w' \in W'$ and $0 \le j \le d$ we define

$$\nabla(w) = \nabla'(w') + \sum_{t=0}^{j-1} n_{\mu(w'\overline{u}^t)}.$$

That this function ∇ satisfies condition (6) for all $w \in W$ is obvious. We now show that it satisfies condition (7) for $s = s_d$ and all $w \in W$ with $\ell(ws_d) > \ell(w)$. Write $w = w'\overline{u}^j$ with $w' \in W'$ and $0 \le j \le d$.

If $j = d$ then $w = w'\overline{u}^d = w's_1 \cdots s_d$ so that $\ell(ws_d) < \ell(w)$ (since $w' \in W'$). Thus, for $j = d$ there is nothing to prove.

Now assume $1 \le j \le d-1$. We then have

$$ws_d = w\overline{u}^{-j}s_{d-j}\overline{u}^j = w's_{d-j}\overline{u}^j$$

with $w's_{d-j} \in W'$, and we claim that $\ell(ws_d) > \ell(w)$ implies $\ell(w's_{d-j}) > \ell(w')$. Indeed, $\ell(ws_d) > \ell(w)$ means $w(d-1) < w(d)$, by formula (5). As $\overline{u}^j(d) = d-j$ and $(\overline{u}')^j(d-1) = d-1-j$ this implies $w'(d-1-j) < w'(d-j)$, hence $\ell(w's_{d-j}) > \ell(w')$, again by formula (5). The claim is proven.

Moreover, for $0 \le t \le j-1$ we have

$$w's_{d-j}\overline{u}^t = w'\overline{u}^t s_{d-j+t}$$

with $s_{d-j+t} \in W'$. This implies $\mu(w's_{d-j}\overline{u}^t) = \mu(w'\overline{u}^t)$. Therefore the claim $\nabla(w) - r \le \nabla(ws_d) \le \nabla(w)$ is reduced to the assumption $\nabla'(w') - r \le \nabla'(w's_{d-j}) \le \nabla'(w')$.

Finally assume that $j = 0$, i.e. $w = w' \in W'$. Then $\nabla(w) = \nabla'(w)$ and

$$\begin{aligned} \nabla(ws_d) &= \nabla(w\overline{u}'\overline{u}^d) \\ &= \nabla'(w\overline{u}') + \sum_{t=0}^{d-1} n_{\mu(w\overline{u}'\overline{u}^t)}. \end{aligned} \tag{8}$$

Here $\nabla'(w\overline{u}') = \nabla'(w) + m_{\mu'(w)}$ by the assumption on ∇'. On the other hand $\sum_{t=0}^{d-1} n_{\mu(w\overline{u}'\overline{u}^t)} = -n_{\mu(ws_d)}$ as $\sum_{i=0}^{d} n_i = 0$. Now we claim that

$$\mu'(w) + 1 = \mu(ws_d).$$

Indeed, we have $w(d) = d - \mu(w)$ and hence also $ws_d(d) = d - \mu(ws_d)$ for $w \in W$. Similarly, we have $w(d-1) = d-1-\mu'(w)$ and hence also

$$ws_d(d) = w(d-1) = d-1-\mu'(w)$$

for $w \in W'$, and the claim is proven.

Inserting all this transforms the assumption $0 \le n_{\mu(ws_d)} - m_{\mu(ws_d)-1} \le r$ into the condition (7) (for $s = s_d$).

We have proven condition (7) for $s = s_d$ and all $w \in W$ with $\ell(ws_d) > \ell(w)$. Condition (7) for all $s \in S_0$ and all $w \in W$ with $\ell(ws) > \ell(w)$ can be checked directly as well. However, alternatively one can argue as follows.

In the setting of Section 3 (and in its notations) choose an arbitrary F with residue field $\mathbb{F}_q$ (for an arbitrary q), and choose $K/\mathbb{Q}_p$ and $\pi \in K$ such that our present r satisfies $\pi^r = q$. We use the elements $t_{\overline{u}^i}$ of T (explicitly given by formula (14)) to define the character $\Theta : T \to K^\times$ by asking that $\Theta(t_{\overline{u}^i}) = \pi^{-n_{i-1}}$ and that $\Theta|_{T\cap I} = \theta$ be the trivial character. (This is well defined as T is the direct product of $T \cap I$ and the free abelian group on the generators $t_{\overline{u}^i}$ for $0 \leq i \leq d$.) The implication (iii)⇒(ii) in Lemma 3.5, applied to this Θ, shows that what we have proven so far is enough. □

3. Hecke lattices in principal series representations I

Fix a prime number p. Let $K/\mathbb{Q}_p$ be a finite extension field, $\mathfrak{o}$ its ring of integers and k its residue field.

Let F be a nonarchimedean locally compact field, $\mathcal{O}_F$ its ring of integers, $p_F \in \mathcal{O}_F$ a fixed prime element and $k_F = \mathbb{F}_q$ its residue field with $q = p^{\log_p q} \in p^{\mathbb{N}}$ elements.

Let $G = \mathrm{GL}_{d+1}(F)$ for some $d \in \mathbb{N}$. Let T be a maximal split torus in G, let $N(T)$ be its normalizer. Let P be a Borel subgroup of G containing T, let N be its unipotent radical.

Let X be the Bruhat–Tits building of $\mathrm{PGL}_{d+1}(F)$, let $A \subset X$ be the apartment corresponding to T. Let I be an Iwahori subgroup of G fixing a chamber C in A, let I_0 denote its maximal pro-p-subgroup. The (affine) reflections in the codimension-1-faces of C form a set S of Coxeter generators for the affine Weyl group. We view the latter as a subgroup of the extended affine Weyl group $N(T)/T \cap I$. There is an $s_0 \in S$ such that the image of $S_0 = S - \{s_0\}$ in the finite Weyl group $W = N(T)/T$ is the set of simple reflections.

We find elements $u, s_d \in N(T)$ such that $uC = C$ (equivalently, $uI = Iu$, or also $uI_0 = I_0u$), such that $u^{d+1} \in \{p_F \cdot \mathrm{id}, p_F^{-1} \cdot \mathrm{id}\}$ and such that, setting

$$s_i = u^{d-i} s_d u^{i-d} \text{ for } 0 \leq i \leq d$$

the set $\{s_1, \ldots, s_d\}$ maps bijectively to S_0, while $\{s_0, s_1, \ldots, s_d\}$ maps bijectively to S; we henceforth regard these bijections as identifications. Let $\overline{u} = s_d \cdots s_1 \in W \subset G$. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function with respect to S_0.

For convenience one may realize all these data explicitly, e.g. according to the following choice: T consists of the diagonal matrices, P consists of the upper triangular matrices, N consists of the unipotent upper triangular matrices (i.e. the elements of P with all diagonal entries equal to 1). Then W can be identified with the subgroup of permutation matrices in G. Its Coxeter generators s_i for $i = 1, \ldots, d$ are the block diagonal matrices

$$s_i = \mathrm{diag}\left(I_{i-1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{d-i}\right)$$

while u is written in block form as

$$u = \begin{pmatrix} & I_d \\ p_F & \end{pmatrix}.$$

(Here I_m, for $m \geq 1$, always denotes the identity matrix in GL_m.) The Iwahori group I consists of the elements of $\mathrm{GL}_{d+1}(\mathcal{O}_F)$ mapping to upper triangular matrices in $\mathrm{GL}_{d+1}(k_F)$, while I_0 consists of the elements of I whose diagonal entries map to $1 \in k_F$.

For $s \in S_0$ let $\iota_s : \mathrm{GL}_2(F) \to G$ denote the corresponding embedding. For $a \in F^\times$, $b \in F$ put

$$h_s(a) = \iota_s\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right), \ \nu_s(b) = \iota_s\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right), \ \delta_s = \iota_s\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right).$$

We realize W as a subgroup of G in such a way that

$$\iota_s\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = s$$

for all $s \in S_0$. Notice that $\mathrm{Im}(\nu_s) \subset N$ for all $s \in S_0$.

Lemma 3.1. *(a) For $s \in S_0$ and $a \in F^\times$ we have*

$$s\nu_s(a)s = h_s(a^{-1})\nu_s(a)\delta_s s\nu_s(a^{-1}). \tag{9}$$

(b) For $w \in W$ and $s \in S_0$ with $\ell(ws) > \ell(w)$ and for $b \in F$ we have

$$w\nu_s(b)w^{-1} \in N. \tag{10}$$

Proof. Statement (a) is a straightforward computation inside $\mathrm{GL}_2(F)$. For statement (b) write $s = s_i$ for some $1 \leq i \leq d$. Then the matrix $w\nu_s(b)w^{-1}$ has entry b at the $(w(i-1), w(i))$-spot (and coincides with the identity matrix at all other spots). As $\ell(ws_i) > \ell(w)$ implies $w(i-1) < w(i)$ by formula (5), this implies $w\nu_s(b)w^{-1} \in N$. $\square$

Let $\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ denote the $\mathfrak{o}$-module of $\mathfrak{o}$-valued compactly supported functions f on G such that $f(ig) = f(g)$ for all $g \in G$, all $i \in I_0$. It is a G-representation by means of the formula $(g'f)(g) = f(gg')$ for $g, g' \in G$. Let

$$\mathcal{H}(G, I_0) = \mathrm{End}_{\mathfrak{o}[G]}(\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{\mathrm{op}}$$

denote the corresponding pro-p-Iwahori Hecke algebra with coefficients in $\mathfrak{o}$. Then $\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ is naturally a right $\mathcal{H}(G, I_0)$-module. For a subset H of G we let χ_H denote the characteristic function of H. For $g \in G$ let $T_g \in \mathcal{H}(G, I_0)$ denote the Hecke operator corresponding to the double coset $I_0 g I_0$. It sends $f : G \to \mathfrak{o}$ to

$$T_g(f) : G \longrightarrow \mathfrak{o}, \ h \mapsto \sum_{x \in I_0 \backslash G} \chi_{I_0 g I_0}(hx^{-1}) f(x).$$

In particular we have

$$T_g(\chi_{I_0}) = \chi_{I_0 g} = g^{-1}\chi_{I_0} \text{ if } gI_0 = I_0 g. \tag{11}$$

Let R be an $\mathfrak{o}$-algebra, let V be a representation of G on an R-module. The submodule of V^{I_0} of I_0-invariants in V carries a natural (left) action by the R-algebra $\mathcal{H}(G, I_0)_R = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} R$, resulting from the natural isomorphism $V^{I_0} \cong \mathrm{Hom}_{R[G]}((\mathrm{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}) \otimes_{\mathfrak{o}} R, V)$. Explicitly, for $g \in G$ and $v \in V^{I_0}$ the action of T_g is given as follows: If the collection $\{g_j\}_j$ in G is such that $I_0gI_0 = \coprod_j I_0g_j$, then

$$T_g(v) = \sum_j g_j^{-1}v. \tag{12}$$

Let $\overline{T} = (I \cap T)/(I_0 \cap T) = I/I_0$.

Suppose we are given a character $\Theta : T \to K^\times$ whose restriction $\theta = \Theta|_{I\cap T}$ to $I \cap T$ factors through $\overline{T}$. As $\overline{T}$ is finite, θ takes values in $\mathfrak{o}^\times$, hence induces a character (denoted by the same symbol) $\theta : \overline{T} \to k^\times$. For any $w \in W$ it defines a homomorphism

$$\theta(wh_s(.)w^{-1}) : k_F^\times \to k^\times,\ x \mapsto \theta(wh_s(x)w^{-1})$$

and it makes sense to compare it with the constant homomorphism $\mathbf{1}$ taking all elements of $k_F^\times$ to $1 \in k^\times$. Notice in the following that $\theta(wh_s(.)w^{-1}) = \mathbf{1}$ if and only if $\theta(wsh_s(.)sw^{-1}) = \mathbf{1}$. For $w \in W$ and $s \in S_0$ put

$$\kappa_{w,s} = \kappa_{w,s}(\theta) = \theta(w\delta_s w^{-1}) \in \{\pm 1\}.$$

Read Θ as a character of P by means of the natural projection $P \to T$ and consider the smooth principal series representation

$$\begin{aligned} V &= \mathrm{Ind}_P^G \Theta \\ &= \{f : G \to K \text{ locally constant } \mid f(pg) = \Theta(p)f(g) \text{ for } g \in G,\ p \in P\} \end{aligned}$$

with G-action $(gf)(x) = f(xg)$. For $w \in W$ let $f_w \in V$ denote the unique I_0-invariant function supported on PwI_0 and with $f_w(w) = 1$. It follows from the decomposition $G = \coprod_{w\in W} PwI_0$ that the set $\{f_w\}_{w\in W}$ is a K-basis of the $\mathcal{H}(G, I_0)_K$-module V^{I_0}.

Lemma 3.2. *Let $w \in W$ and $s \in S_0$, let $a \in \mathcal{O}_F$.*

(a) If $\ell(ws) > \ell(w)$ and $a \notin (p_F)$ then $ws\nu_s(a)s \notin PwI_0$.
(b) If $\ell(ws) > \ell(w)$ then $v\nu_s(a)s \notin PwI_0$ for all $v \in W - \{ws\}$.
(c) $v\nu_s(a)s \notin PwI_0$ for all $v \in W - \{w, ws\}$.

Proof. We have $\nu_s(\mathcal{O}_F) \subset I_0$. Therefore all statements will follow from standard properties of the decomposition $G = \coprod_{w\in W} PwI_0$, or rather the restriction of this decomposition to $\mathrm{GL}_{d+1}(\mathcal{O}_F)$; notice that this restriction projects to the usual Bruhat decomposition of $\mathrm{GL}_{d+1}(k_F)$.

(a) The assumption $a \notin (p_F)$, i.e. $a \in \mathcal{O}_F^\times$, implies that $ws\nu_s(a)s \in wIsI$, by formula (9). The assumption $\ell(ws) > \ell(w)$ implies $wIsI \subset PwsI = PwsI_0$ by standard properties of the Bruhat decomposition, hence $wIsI \cap PwI_0 = \varnothing$.

(b) Standard properties of the Bruhat decomposition imply $vI_0s \subset PvsI_0 \cup PvI_0$, as well as $vI_0s \subset PvsI_0$ if $\ell(vs) > \ell(v)$. As $\ell(ws) > \ell(w)$ and $v \neq ws$ statement (b) follows.

(c) The same argument as for (b). ☐

Lemma 3.3. *Let $w \in W$ and $s \in S_0$. We have*

$$T_s(f_w) = \begin{cases} f_{ws}, & \text{if } \ell(ws) > \ell(w), \\ q f_{ws}, & \text{if } \ell(ws) < \ell(w) \text{ and } \theta(wh_s(.)w^{-1}) \neq \mathbf{1}, \\ q f_{ws} + \kappa_{ws,s}(q-1) f_w, & \text{if } \ell(ws) < \ell(w) \text{ and } \theta(wh_s(.)w^{-1}) = \mathbf{1}. \end{cases}$$

Proof. We have $I_0 s I_0 = \coprod_a I_0 s \nu_s(a)$ where a runs through a set of representatives for k_F in $\mathcal{O}_F$. For $y \in G$ we therefore compute, using formula (12):

$$(T_s(f_w))(y) = \Big(\sum_a \nu_s(a) s f_w\Big)(y)$$

$$= \sum_a f_w(y \nu_s(a) s). \tag{13}$$

Suppose first that $\ell(ws) > \ell(w)$. For $a \notin (p_F)$ we then have $ws\nu_s(a)s \notin PwI_0$ by Lemma 3.2, hence $f_w(ws\nu_s(a)s) = 0$. On the other hand $f_w(ws\nu_s(0)s) = f_w(w) = 1$. Together we obtain $(T_s(f_w))(ws) = 1$. For $v \in W - \{ws\}$ and any $a \in \mathcal{O}_F$ we have $v\nu_s(a)s \notin PwI_0$ by Lemma 3.2, hence $(T_s(f_w))(v) = 0$. It follows that $T_s(f_w) = f_{ws}$.

Now suppose that $\ell(ws) < \ell(w)$. Then $ws\nu_s(a)sw^{-1} \in N$ for any a, by formula (10), hence

$$f_w(ws\nu_s(a)s) = \theta(ws\nu_s(a)sw^{-1}) f_w(w) = 1.$$

Summing up we get

$$(T_s(f_w))(ws) = \sum_a f_w(ws\nu_s(a)s) = |k_F| = q.$$

To compute $(T_s(f_w))(w)$ we first notice that $f_w(w\nu_s(0)s) = f_w(ws) = 0$. On the other hand, for $a \notin (p_F)$ we find

$$\begin{aligned} f_w(w\nu_s(a)s) &= f_w(wss\nu_s(a)s) \\ &\overset{(i)}{=} f_w(wsh_s(a^{-1})\nu_s(a)\delta_s s\nu_s(a^{-1})) \\ &= \theta(wsh_s(a^{-1})\nu_s(a)\delta_s sw^{-1}) f_w(w\nu_s(a^{-1})) \\ &\overset{(ii)}{=} \theta(wsh_s(a^{-1})\delta_s sw^{-1}) \\ &= \kappa_{ws,s}\theta(wsh_s(a^{-1})sw^{-1}). \end{aligned}$$

Here (i) uses formula (9) while (ii) uses $f_w(w\nu_s(a^{-1})) = f_w(w) = 1$ as well as

$$(wsh_s(a^{-1})\nu_s(a)\delta_s sw^{-1}) \cdot (wsh_s(a^{-1})\delta_s sw^{-1})^{-1} = ws\nu_s(a^{-1})sw^{-1} \in N,$$

formula (10). Now

$$\sum_{a \notin (p_F)} \theta(wsh_s(a)sw^{-1}) = \begin{cases} q-1, & \theta(wh_s(.)w^{-1}) = \mathbf{1}, \\ 0, & \theta(wh_s(.)w^{-1}) \neq \mathbf{1}. \end{cases}$$

Thus

$$\sum_{a \notin (p_F)} f_w(w\nu_s(a)s) = \begin{cases} \kappa_{ws,s}(q-1), & \theta(wh_s(.)w^{-1}) = \mathbf{1}, \\ 0, & \theta(wh_s(.)w^{-1}) \neq \mathbf{1}. \end{cases}$$

We have shown that

$$(T_s(f_w))(w) = \begin{cases} \kappa_{ws,s}(q-1), & \theta(wh_s(.)w^{-1}) = \mathbf{1}, \\ 0, & \theta(wh_s(.)w^{-1}) \neq \mathbf{1}. \end{cases}$$

Finally, for $v \in W - \{w, ws\}$ and $a \in \mathcal{O}_F$ we have $v\nu_s(a)s \notin PwI_0$ by Lemma 3.2, hence $(T_s(f_w))(v) = 0$. Summing up gives the formulae for $T_s(f_w)$ in the case $\ell(ws) < \ell(w)$. □

As $\overline{u}$ is the unique element in $W \subset G$ lifting the image of u in $W = N(T)/T$ we have $\overline{u}^{-1}u \in T$. For $w \in W$ we define

$$t_w = w\overline{u}^{-1}uw^{-1} \in T.$$

We record the formulae

$$\overline{u}^{-1}u = t_{\overline{u}^0} = \mathrm{diag}(p_F, I_d),$$

$$t_{\overline{u}^i} = \mathrm{diag}(I_{d-i+1}, p_F, I_{i-1}) \text{ for } 1 \leq i \leq d, \tag{14}$$

In particular we notice that $t_w = t_{ws_i}$ for $2 \leq i \leq d$.

Lemma 3.4. *For $w \in W$ we have*

$$T_{u^{-1}}(f_w) = \Theta(t_w)f_{w\overline{u}^{-1}} \text{ and } T_u(f_w) = \Theta(t_{w\overline{u}}^{-1})f_{w\overline{u}}. \tag{15}$$

For $w \in W$ and $t \in T \cap I$ we have

$$T_t(f_w) = \theta(wt^{-1}w^{-1})f_w. \tag{16}$$

Proof. We use formula (11) in both cases: First,

$$(T_{u^{-1}}(f_w))(w\overline{u}^{-1}) = (uf_w)(w\overline{u}^{-1}) = f_w(w\overline{u}^{-1}u) = \Theta(t_w)f_w(w) = \Theta(t_w)$$

but

$$(T_{u^{-1}}(f_w))(v) = (uf_w)(v) = f_w(vu) = \Theta(vu\overline{u}^{-1}v^{-1})f_w(v\overline{u}) = 0$$

for $v \in W - \{w\overline{u}^{-1}\}$, hence the first one of the formulae in (15); the other one is equivalent with it (or alternatively: proven in the same way). Next,

$$(T_t(f_w))(w) = (t^{-1}f_w)(w) = f_w(wt^{-1}) = \theta(wt^{-1}w^{-1})f_w(w) = \theta(wt^{-1}w^{-1}),$$

but

$$(T_t(f_w))(v) = (t^{-1}f_w)(v) = f_w(vt^{-1}) = \theta(vt^{-1}v^{-1})f_w(v) = 0$$

for $v \in W - \{w\}$, hence formula (16). □

We assume that there is some $r \in \mathbb{N}$ and some $\pi \in \mathfrak{o}$ such that $\pi^r = q$ and such that Θ takes values in the subgroup of $K^\times$ generated by π and $\mathfrak{o}^\times$. Notice that, given an arbitrary Θ, this can always be achieved after passing to a suitable finite extension of K. Let $\mathrm{ord}_K : K \to \mathbb{Q}$ denote the order function normalized such that $\mathrm{ord}_K(\pi) = 1$.

Suppose we are given a function $\nabla : W \to \mathbb{Z}$. For $w \in W$ we put $g_w = \pi^{\nabla(w)}f_w$ and consider the $\mathfrak{o}$-submodule

$$L_\nabla = L_\nabla(\Theta) = \bigoplus_{w \in W} \mathfrak{o}.g_w$$

of V^{I_0} which is $\mathfrak{o}$-free with basis $\{g_w \mid w \in W\}$. We ask under which conditions on ∇ it is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0}. Consider the formulae

$$\nabla(w) - \nabla(w\overline{u}) = \mathrm{ord}_K(\Theta(t_{w\overline{u}})), \tag{17}$$

$$\nabla(w) - r \leq \nabla(ws) \leq \nabla(w). \tag{18}$$

Lemma 3.5. *The following conditions (i), (ii), (iii) on ∇ are equivalent:*

(i) L_∇ is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0}.
(ii) ∇ satisfies formula (17) for any $w \in W$, and it satisfies formula (18) for any $s \in S_0$ and any $w \in W$ with $\ell(ws) > \ell(w)$.
(iii) ∇ satisfies formula (17) for any $w \in W$, and it satisfies formula (18) for $s = s_d$ and any $w \in W$ with $\ell(ws_d) > \ell(w)$.

Proof. For $t \in T \cap I$ and $w \in W$ it follows from Lemma 3.4 that 1

$$T_t(g_w) = \theta(wt^{-1}w^{-1})g_w, \tag{19}$$

$$T_{u^{-1}}(g_w) = \pi^{\nabla(w) - \nabla(w\overline{u}^{-1})}\Theta(t_w)g_{w\overline{u}^{-1}}, \tag{20}$$

$$T_u(g_w) = \pi^{\nabla(w) - \nabla(w\overline{u})}\Theta(t_{w\overline{u}}^{-1})g_{w\overline{u}}. \tag{21}$$

For $w \in W$ and $s \in S_0$ it follows from Lemma 3.3 that

$$T_s(g_w) = \begin{cases} \pi^{\nabla(w) - \nabla(ws)}g_{ws}, & \text{if } \ell(ws) > \ell(w), \\ \pi^{r + \nabla(w) - \nabla(ws)}g_{ws}, & \text{if } \ell(ws) < \ell(w) \\ & \text{and } \theta(wh_s(.)w^{-1}) \neq \mathbf{1}, \\ \pi^{r + \nabla(w) - \nabla(ws)}g_{ws} + \kappa_{ws,s}(\pi^r - 1)g_w, & \text{if } \ell(ws) < \ell(w) \\ & \text{and } \theta(wh_s(.)w^{-1}) = \mathbf{1}. \end{cases} \tag{22}$$

From these formulae we immediately deduce that condition (i) implies both condition (ii) and condition (iii) on ∇. Now it is known that $\mathcal{H}(G, I_0)$ is generated as an $\mathfrak{o}$-algebra by the Hecke operators T_t for $t \in T \cap I$ together with $T_{u^{-1}}$, T_u and T_{s_d}. Thus, to show stability of L_∇ under $\mathcal{H}(G, I_0)$ it is enough to show stability of L_∇ under these operators. The above formulae imply that this stability is ensured by condition (iii). Thus (i) is implied by (iii), and a fortiori by (ii). $\square$

4. Hecke lattices in principal series representations II

In Lemma 3.5 we saw that the (particularly nice) $\mathcal{H}(G, I_0)$ stable $\mathfrak{o}$-lattices L_∇ in the $\mathcal{H}(G, I_0)_K$-module V^{I_0} for $V = \mathrm{Ind}_P^G\Theta$ are obtained from functions $\nabla : W \to \mathbb{Z}$ satisfying the conditions stated there. We now want to explain that the existence of such a function ∇ can be directly read off from Θ. For $0 \leq i \leq d$ put

$$n_i = -\mathrm{ord}_K(\Theta(t_{\overline{u}^{i+1}})).$$

Corollary 4.1. *If $(n_i)_{0\leq i\leq d}$ is a balanced weight of length $d+1$ and amplitude r then there exists a function $\nabla : W \to \mathbb{Z}$ such that L_∇ is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0}.*

Proof. By Theorem 2.3 there exists a function $\nabla : W \to \mathbb{Z}$ satisfying condition (iii) of Lemma 3.5. Thus we may conclude with that Lemma. □

Thus we need to decide for which Θ the collection $(n_i)_{0\leq i\leq d}$ is a balanced weight of length $d+1$ and amplitude r.

We now assume that $F \subset K$. We normalize the absolute value $|.| : K^\times \to \mathbb{Q}^\times \subset K^\times$ on K (and hence its restriction to F) by requiring $|p_F| = q^{-1}$. Let $\delta : T \to F^\times$ denote the modulus character associated with P, i.e. $\delta = \prod_{\alpha\in\Phi^+} |\alpha|$ where Φ^+ is the set of positive roots. Let $N_0 = N \cap I$ and

$$T_+ = \{t \in T \mid t^{-1}N_0 t \subset N_0\}.$$

The group W acts on the group of characters $\mathrm{Hom}(T, K^\times)$ through its action on T.

Theorem 4.2. *Suppose that for all $w \in W$ and all $t \in T^+$ we have*

$$|((w\Theta)(w\delta^{\frac{-1}{2}})\delta^{\frac{1}{2}})(t)| \leq 1 \tag{23}$$

and that the restriction of Θ to the center of G is a unitary character. Then $(n_i)_{0\leq i\leq d}$ is a balanced weight of length $d+1$ and amplitude r, and L_∇ is stable under the action of $\mathcal{H}(G, I_0)$ on V^{I_0}.

As the center of G is generated by the element $\prod_{j=0}^{d} t_{\overline{u}^j} = p_F I_{d+1}$ (cp. formula (14)) together with $\mathcal{O}_F^\times \cdot I_{d+1}$, the condition that the restriction of Θ to the center of G be a unitary character is equivalent with the condition

$$\prod_{j=0}^{d} |\Theta(t_{\overline{u}^j})| = 1. \tag{24}$$

Proof of Theorem 4.2. Recall that, for convenience, we work with the following realization: T is the group of diagonal matrices, P is the group of upper triangular matrices, s_i (for $1 \leq i \leq d$) is the $(i-1, i)$-transposition matrix and $u = \overline{u} \cdot \mathrm{diag}(p_F, 1, \ldots, 1)$. Thus T_+ is the subgroup of T generated by all $t \in \overline{T}$ (viewed as a subgroup of T by means of the Teichmüller character), by the scalar diagonal matrices (the center of G), and by all the matrices of the form $\mathrm{diag}(1, \ldots, 1, p_F, \ldots, p_F)$. The modulus character is

$$\delta : T \longrightarrow F^\times,\ \mathrm{diag}(\alpha_0, \ldots, \alpha_d) \mapsto \prod_{i=0}^{d} |\alpha_i|^{d-2i}.$$

Write $\Theta = \mathrm{diag}(\Theta_0, \ldots, \Theta_d)$ with characters $\Theta_j : F^\times \to K^\times$. Reading W as the symmetric group of the set $\{0, \ldots, d\}$, formula (23) for $t = \mathrm{diag}(\alpha_0, \ldots, \alpha_d)$ reads

$$\left| \prod_{i=0}^{d} \Theta_{\tau(i)}(\alpha_i)|\alpha_i|^{\tau(i)-i} \right| \leq 1 \tag{25}$$

for all permutations τ of $\{0,\dots,d\}$. Asking formula (25) for all $\mathrm{diag}(\alpha_0,\dots,\alpha_d) \in T^+$ is certainly equivalent with asking it for all $\mathrm{diag}(p_F^{-1},\dots,p_F^{-1},1\dots,1)$ and for all $\mathrm{diag}(1\dots,1,p_F,\dots,p_F)$ (and all τ). This is equivalent with asking

$$|q|^{\Delta(I)} \leq \left|\prod_{j\in I}\Theta_j(p_F)\right| \leq |q|^{-\Delta(\{0,\dots,d\}-I)} \tag{26}$$

for all $I \subset \{0,\dots,d\}$. Indeed, the inequalities on the left hand side of (26) are the inequalities (25) for the $\mathrm{diag}(p_F^{-1},\dots,p_F^{-1},1\dots,1)$ and suitable τ. The inequalities on the right hand side of (26) are the inequalities (25) for the $\mathrm{diag}(1\dots,1,p_F,\dots,p_F)$ and suitable τ. Now observe that $\Theta_j(p_F) = \Theta(t_{\overline{u}^{d+1-j}})$ and hence

$$|\Theta_j(p_F)| = |\pi^{\mathrm{ord}(\Theta(t_{\overline{u}^{d+1-j}}))}| = |\pi^{-n_{d-j}}|$$

for $0\leq j\leq d$. We also have $|q| = |\pi^r|$. Together with Lemma 2.2 we recover formula (1). On the other hand, formula (24) is just the property $\sum_{i=0}^d n_i = 0$. We thus conclude with Corollary 4.1. $\square$

Remarks. (1) We (formally) put $\chi = \Theta\delta^{-\frac{1}{2}}$. Let $\overline{P} \subset G$ denote the Borel subgroup opposite to P. The same arguments as in [3, p. 10] show that (at least if χ is regular) for all $w \in W$ the action of T on the Jacquet module $J_{\overline{P}}(V)$ of V (formed with respect to $\overline{P}$) admits a nonzero eigenspace with character $(w\chi)\delta^{\frac{-1}{2}}$, i.e. with character $(w\Theta)(w\delta^{\frac{-1}{2}})\delta^{\frac{-1}{2}}$. From [3] we then deduce that the conditions in Theorem 4.2 are a necessary criterion for the existence of an integral structure in V.

(2) This necessary criterion has also been obtained in [2]. Moreover, in *loc.cit.* it is shown (in a much more general context) that it implies the existence of an integral structure in the $\mathcal{H}(G,I_0)$-module V^{I_0}. The point of Theorem 4.2 is that it explicitly describes a particularly nice such integral structure.

(3) Consider the smooth dual $\mathrm{Hom}_K(V,K)^{\mathrm{sm}}$ of V; it is isomorphic with $\mathrm{Ind}_P^G\Theta^{-1}\delta$. Our conditions (23) and (24) for Θ are equivalent with the same conditions for $\Theta^{-1}\delta$.

Remark. Suppose we are in the setting of Corollary 4.1 or Theorem 4.2. Let H denote a maximal compact open subgroup of G containing I. Abstractly, H is isomorphic with $\mathrm{GL}_{d+1}(\mathcal{O}_F)$. Let $\mathfrak{o}[H].L_\nabla$ denote the $\mathfrak{o}[H]$-sub module of V generated by L_∇, let $(\mathfrak{o}[H].L_\nabla)^{I_0}$ denote its $\mathfrak{o}$-sub module of I_0-invariants. Then one can show (we do not give the proof here) that the inclusion map $L_\nabla \to (\mathfrak{o}[H].L_\nabla)^{I_0}$ is surjective (and hence bijective). On the one hand this may be helpful for deciding whether V contains an integral structure, i.e. a G-stable free $\mathfrak{o}$-sub module containing a K-basis of V. On the other hand it implies (in fact: is equivalent with it) that the induced map

$$L_\nabla \otimes_{\mathfrak{o}} k \longrightarrow (\mathfrak{o}[H].L_\nabla)\otimes_{\mathfrak{o}} k$$

is injective. This might be a useful observation about the $\mathcal{H}(G,I_0)_k$-module $L_\nabla \otimes_{\mathfrak{o}} k$ (which we call an $\mathcal{H}(G,I_0)_k$-module of W-type in Section 5).

5. $\mathcal{H}(G, I_0)_k$-modules of W-type

We return to the setting of Section 3. For $w \in W$ we define

$$\epsilon_w = \epsilon_w(\Theta) = \pi^{-\mathrm{ord}_K(\Theta(t_w))}\Theta(t_w).$$

Let us write $W^{s_d} = \{w \in W \mid \ell(ws_d) > \ell(w)\}$. For a function $\sigma : W^{s_d} \to \{-1, 0, 1\}$, for $w \in W$ and $i \in \{-1, 0, 1\}$ we understand the condition $\sigma(w) = i$ as a shorthand for the condition

$$w \in W^{s_d} \text{ and } \sigma(w) = i.$$

For $w \in W$ we write $\kappa_w = \kappa_{ws_d, s_d}$.

Suppose that the function $\nabla : W \to \mathbb{Z}$ satisfies the equivalent conditions of Lemma 3.5. Define a function $\sigma : W^{s_d} \to \{-1, 0, 1\}$ by setting

$$\sigma(w) = \begin{cases} 1, & \text{if } \nabla(ws_d) = \nabla(w), \\ 0, & \text{if } \nabla(w) - r < \nabla(ws_d) < \nabla(w), \\ -1, & \text{if } \nabla(w) - r = \nabla(ws_d). \end{cases} \tag{27}$$

The action of $\mathcal{H}(G, I_0)$ on L_∇ induces an action of $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$ on $L_\nabla \otimes_{\mathfrak{o}} k$. The $\mathfrak{o}$-basis $\{g_w \mid w \in W\}$ of L_∇ induces a k-basis $\{g_w \mid w \in W\}$ of $L_\nabla \otimes_{\mathfrak{o}} k = L_\nabla(\Theta) \otimes_{\mathfrak{o}} k$ (we use the same symbols g_w).

Corollary 5.1. *The action of $\mathcal{H}(G, I_0)_k$ on $L_\nabla \otimes_{\mathfrak{o}} k$ is characterized through the following formulae: For $t \in T \cap I$ and $w \in W$ we have*

$$T_t(g_w) = \theta(wt^{-1}w^{-1})g_w, \tag{28}$$

$$T_{u^{-1}}(g_w) = \epsilon_w g_{w\overline{u}^{-1}} \text{ and } T_u(g_w) = \epsilon_{w\overline{u}}^{-1} g_{w\overline{u}}, \tag{29}$$

$$T_{s_d}(g_w) = \begin{cases} g_{ws_d}, & \text{if } [\sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}(.)w^{-1}) \neq \mathbf{1}] \\ & \text{or } \sigma(w) = 1, \\ -\kappa_w g_w, & \text{if } \sigma(ws_d) \in \{0, 1\} \text{ and } \theta(wh_{s_d}(.)w^{-1}) = \mathbf{1}, \\ g_{ws_d} - \kappa_w g_w, & \text{if } \sigma(ws_d) = -1 \text{ and } \theta(wh_{s_d}(.)w^{-1}) = \mathbf{1}, \\ 0, & \text{all other cases.} \end{cases} \tag{30}$$

Proof. Formula (28) follows from formula (19). The assumption $\nabla(w\overline{u}^{-1}) - \nabla(w) = \mathrm{ord}_K(\theta(t_w))$ implies that the formulae in (29) follow from formulae (20) and (21). Finally, formula (30) follows from formula (22) by a case by case checking. □

Forgetting their origin from some Θ and ∇, we formalize the structure of $\mathcal{H}(G, I_0)_k$-modules met in Corollary 5.1 in an independent definition.

Definition. We say that an $\mathcal{H}(G, I_0)_k$-module M is of W-type (or: a *reduced standard module*) if it is of the following form $M = M(\theta, \sigma, \epsilon_\bullet)$. First, a k-vector space basis of M is the set of formal symbols g_w for $w \in W$. The $\mathcal{H}(G, I_0)_k$-action on M is characterized by a character $\theta : \overline{T} \to k^\times$ (which we also read as a character of $T \cap I$ by inflation), a map $\sigma : W^{s_d} \to \{-1, 0, 1\}$

and a set $\epsilon_\bullet = \{\epsilon_w\}_{w\in W}$ of units $\epsilon_w \in k^\times$. Namely, for $w \in W$ we define $\kappa_w = \kappa_w(\theta) = \theta(ws_d\delta_{s_d}s_dw^{-1}) \in \{\pm 1\}$. Then it is required that for $t \in T \cap I$ and $w \in W$ formulae (28), (29) and (30) hold true.

Conversely we may begin with a character $\theta : \overline{T} \to k^\times$, a map $\sigma : W^{s_d} \to \{-1, 0, 1\}$ and a set $\epsilon_\bullet = \{\epsilon_w\}_{w\in W}$ of units $\epsilon_w \in k^\times$ and ask:

Question 1: For which set of data θ, σ, $\epsilon_\bullet$ do formulae (28), (29) and (30) define an action of $\mathcal{H}(G, I_0)_k$ on $\oplus_{w\in W} k.g_w$?

Question 2: For which set of data θ, σ, $\epsilon_\bullet$ does there exist some $\mathcal{H}(G, I_0)$-module $L_\nabla(\Theta)$ as in Corollary 5.1 such that $L_\nabla(\Theta) \otimes_{\mathfrak{o}} k \cong M(\theta, \sigma, \epsilon_\bullet)$ as an $\mathcal{H}(G, I_0)_k$-module ?

In Question 2 we regard θ as taking values in $\mathfrak{o}^\times \subset K^\times$ by means of the Teichmüller lifting. Clearly those θ, σ, $\epsilon_\bullet$ asked for in Question 2 belong to those θ, σ, $\epsilon_\bullet$ asked for in Question 1.

We do not consider Question 1 in general, but provide a criterion for a positive answer to Question 2. Suppose we are given a set of data θ, σ, $\epsilon_\bullet$ as above.

Proposition 5.2. *Suppose that $\epsilon_w = \epsilon_{ws_i}$ for all $2 \le i \le d$ and that there exists a function $\partial : W \to [-r, r] \cap \mathbb{Z}$ with the following properties:*

$$\sigma(w) = \begin{cases} 1, & \text{if } w \in W^{s_d} \text{ and } \partial(w) = 0, \\ 0, & \text{if } w \in W^{s_d} \text{ and } 0 < \partial(w) < r, \\ -1, & \text{if } w \in W^{s_d} \text{ and } \partial(w) = r, \end{cases} \tag{31}$$

$$\partial(ws_d) = -\partial(w), \tag{32}$$

$$\partial(w\overline{u}^{d-i}) + \partial(ws_i\overline{u}^{d-j}) = \partial(w\overline{u}^{d-j}) + \partial(ws_j\overline{u}^{d-i}) \tag{33}$$

for $1 \le i < j - 1 < d$,

$$\begin{aligned} &\partial(w\overline{u}^{d-i}) + \partial(ws_i\overline{u}^{d-i-1}) + \partial(ws_is_{i+1}\overline{u}^{d-i}) \\ &\quad = \partial(w\overline{u}^{d-i-1}) + \partial(ws_{i+1}\overline{u}^{d-i}) + \partial(ws_{i+1}s_i\overline{u}^{d-i-1}) \end{aligned} \tag{34}$$

for $1 \le i < d$.

Then there exists an extension $\Theta : T \to K^\times$ of θ and a function $\nabla : W \to \mathbb{Z}$ as before such that we have an isomorphism of $\mathcal{H}(G, I_0)_k$-modules $L_\nabla(\Theta) \otimes_{\mathfrak{o}} k \cong M(\theta, \sigma, \epsilon_\bullet)$.

Proof. Step 1: Let $w, v \in W$. Choose a (not necessarily reduced) expression $v = s_{i_1} \cdots s_{i_r}$ (with $i_m \in \{1, \ldots, d\}$) and put

$$\partial(w, v) = \sum_{m=1}^{r} \partial(ws_{i_1} \cdots s_{i_{m-1}}\overline{u}^{d-i_m}).$$

Claim: This definition does not depend on the chosen expression $s_{i_1} \cdots s_{i_r}$ for v.

Indeed, it follows from hypothesis (33) that for $1 \leq i < j-1 < d$ we have $\partial(w, s_i s_j) = \partial(w, s_j s_i)$ where on either side we use the expression of $s_i s_j = s_j s_i$ as indicated. Similarly, it follows from hypothesis (34) that for $1 \leq i < d$ we have $\partial(w, s_i s_{i+1} s_i) = \partial(w, s_{i+1} s_i s_{i+1})$ where on either side we use the expression of $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ as indicated. Finally, for $1 \leq i \leq d$ we have $\partial(w, s_i s_i) = 0$ where we use the expression $s_i s_i$ for the element $s_i s_i = s_i^2 = 1 \in W$: this follows from the definition of ∂ and from $s_i \overline{u}^{d-i} = \overline{u}^{d-i} s_d$. Thus we see that our definition of $\partial(w, v)$ (viewed as a function in $v \in W$, with fixed $w \in W$) respects the defining relations for the Coxeter group W. Iterated application implies the stated claim.

Step 2: The definition of $\partial(w, v)$ implies $\partial(w, v) + \partial(wv, x) = \partial(w, vx)$ for $v, w, x \in W$. Therefore there is a function $\nabla : W \to \mathbb{Z}$, uniquely determined up to addition of a constant function $W \to \mathbb{Z}$, such that

$$\nabla(w) - \nabla(wv) = \partial(w, v) \text{ for all } v, w \in W.$$

It has the following properties. First, it fulfils formula (27). Next, we have

$$\nabla(w) - \nabla(w\overline{u}) = \nabla(ws_i) - \nabla(ws_i\overline{u}) \text{ for } w \in W \text{ and } 1 \leq i \leq d-1. \tag{35}$$

$$\nabla(w\overline{u}^{-1}) - \nabla(w) = \nabla(w\overline{u}^{-1}s_i) - \nabla(ws_i) \text{ for } w \in W \text{ and } 2 \leq i \leq d. \tag{36}$$

These formulae are equivalent, as $s_i\overline{u} = \overline{u}s_{i+1}$ for $1 \leq i \leq d-1$. To see that they hold true we compute

$$\begin{aligned}
\nabla(w) - \nabla(ws_i) &= \partial(w, s_i) \\
&= \partial(w\overline{u}^{d-i}) \\
&= \partial(w\overline{u}, s_{i+1}) \\
&= \nabla(w\overline{u}) - \nabla(w\overline{u}s_{i+1}) \\
&= \nabla(w\overline{u}) - \nabla(ws_i\overline{u}) \qquad (37)
\end{aligned}$$

and formula (35) follows.

Step 3: For $w \in W$ we define

$$\Theta(t_w) = \pi^{\nabla(w\overline{u}^{-1}) - \nabla(w)} \epsilon_w \in K^\times.$$

Formula (36) together with our assumption on the ϵ_w implies that this is well defined, because for $w, w' \in W$ we have $t_w = t_{w'}$ if and only if $w^{-1}w'$ belongs to the subgroup of W generated by $s_2, \ldots, s_d$. As $T/T \cap I$ is freely generated by the t_w this defines a character $\Theta : T \to K^\times$ extending $T \cap I \to \overline{T} \xrightarrow{\theta} k^\times \subset K^\times$, as desired. ☐

Corollary 5.3. *Assume that $d \leq 2$. If we have $\epsilon_w = \epsilon_{ws_i}$ for all $2 \leq i \leq d$ then there exists an extension $\Theta : T \to K^\times$ of θ and a function $\nabla : W \to \mathbb{Z}$ such that we have an isomorphism of $\mathcal{H}(G, I_0)_k$-modules $L_\nabla(\Theta) \otimes_{\mathfrak{o}} k \cong M(\theta, \sigma, \epsilon_\bullet)$.*

Proof. Choose a function $\partial : W^{s_d} \to [0, r] \cap \mathbb{Z}$ such that

$$\partial(w) = 0 \text{ if } \sigma(w) = 1,$$

$$0 < \partial(w) < r \text{ if } \sigma(w) = 0,$$
$$\partial(w) = r \text{ if } \sigma(w) = -1.$$

Extend ∂ to a function $\partial : W \to [-r, r] \cap \mathbb{Z}$ by setting $\partial(ws_d) = -\partial(w)$ for $w \in W^{s_d}$. Then, as we assume $d \le 2$, properties (33) and (34) are empty resp. fulfilled for trivial reasons. Therefore we conclude with Proposition 5.2. □

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