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Certain actions of finite abelian groups on higher dimensional noncommutative tori

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Abstract. We study matrix type actions of finite abelian groups on simple higher dimensional noncommutative tori. For a given dimension d and a finite abelian group G, we apply a certain function to detect whether there is a simple noncommutative d-torus which admits such an action of G. For possible cases, we construct all such actions of G, and compute the K-theory of the resulting crossed products. We also give a necessary and sufficient condition of G under which the resulting crossed product is an AF algebra.

1. Introduction

The ordinary torus $\mathbb{T}^2 \cong \mathbb{R}^2/\mathbb{Z}^2$ admits an action of $\mathrm{SL}_2(\mathbb{Z})$ through matrix multiplication. This action can be generalized to one on an arbitrary rotation algebra [1, 13]. For a given real number θ, let $\mathcal{A}_\theta$ denote the rotation algebra associated with θ and u and v the unitaries that generate $\mathcal{A}_\theta$. Then the action $\alpha\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(\mathcal{A}_\theta)$ is defined by requiring

$$\alpha_A(u) = e^{\pi iac\theta}u^a v^c, \quad \alpha_A(v) = e^{\pi ibd\theta}u^b v^d,$$

where we assume

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The restriction of the action α to finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ that act on irrational rotation algebras have been systematically studied in [2]. It is known that a finite subgroup F of $\mathrm{SL}_2(\mathbb{Z})$ is necessarily isomorphic to $\mathbb{Z}_k$, with $k = 2, 3, 4$ or 6, see [8]. The main theorems of [2] show that each resulting crossed product $\mathcal{A}_\theta \rtimes \mathbb{Z}_k$ is an AF algebra for $k = 2, 3, 4$ and 6 and an irrational angle θ. Also, for each k, the K_0-group $K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_k)$ and its image under the induced map of the unique tracial state of $\mathcal{A}_\theta \rtimes \mathbb{Z}_k$ has been calculated, which indicates that $\mathcal{A}_\theta \rtimes \mathbb{Z}_k$ is isomorphic to $\mathcal{A}_{\theta'} \rtimes \mathbb{Z}_{k'}$ if and only if $k = k'$ and $\theta \equiv \pm\theta' \bmod \mathbb{Z}$ [2, Theorem 0.1]. Then they show that higher dimensional noncommutative tori admit flip actions of $\mathbb{Z}_2$, and for simple ones the crossed products are all AF algebras [2, Theorem 6.6].

As stated in the paper, the proof of the above results breaks into the following three steps.

(1) Computation of the K-theory of the crossed product.
(2) Proof that the crossed product satisfies the Universal Coefficient Theorem.
(3) Proof that the action has the tracial Rokhlin property.

The proof of step (2) and (3) is general enough so that it is also effective for higher dimensional cases. Hence, it is possible to generalize the above results to higher dimensional noncommutative tori if one obtains a proper realization of some finite groups, a proper generalization of the action α and a way to compute the K-theory of the crossed products. Jeong and Lee [3] manage to obtain new results in this way. For a given real $d \times d$ skew symmetric matrix $\Theta = (\theta_{ij})$, a higher dimensional noncommutative torus associated with Θ is the C^*-algebra generated by unitaries $u_1, u_2, \ldots, u_d$ such that

$$u_i u_j = \exp(2\pi i \theta_{ji}) u_j u_i.$$

Jeong and Lee [3] realize the higher dimensional noncommutative tori as twisted group C^*-algebras and consider a proper definition generalizing the action α, for which they use the terminology "canonical actions". Throughout this paper we are interested in actions of this type and call it "matrix type actions". They realize a given cyclic group as a subgroup of $\mathrm{GL}_d(\mathbb{Z})$ generated by the companion matrix of a properly chosen cyclotomic polynomial, and study matrix type actions of the cyclic group on simple higher dimensional noncommutative tori for certain dimensions. To compute the K-groups of the crossed product, Jeong and Lee [3] firstly apply [2, Lemma 2.1] and [9, Theorem 4.1] to write the crossed product in terms of another twisted group C^*-algebra. Then to compute its K-theory, it is the same to compute the K-theory of untwisted group C^*-algebra according to [2, Theorem 0.3]. Finally, by [6, Theorem 0.1] they manage to obtain the desired K-groups. They also show that if the dimension d and the order n satisfies some extra condition, there is a simple noncommutative d-torus on which $\mathbb{Z}_n$ acts but the K_1-group of the crossed product is not trivial [3, Theorem 3.6].

Based on these results, we study matrix type actions of finite abelian groups on simple higher dimensional noncommutative tori. By combining the tensor structure of higher dimensional noncommutative tori and the realization in [3] for cyclic groups and actions, we realize a class of matrix type actions of nontrivial finite cyclic groups. For a given dimension d, by known results on finite cyclic subgroups of $\mathrm{GL}_d(\mathbb{Z})$, we apply a certain function to detect whether we can realize the action of the cyclic group of order n on a simple noncommutative d-torus. This function, denoted by W, is defined by

$$W(n) := \begin{cases} \sum_{i=1}^{t} (p_i - 1) p_i^{e_i - 1} - 1, & p_1^{e_1} = 2, \\ \sum_{i=1}^{t} (p_i - 1) p_i^{e_i - 1}, & \text{otherwise}, \end{cases}$$

where primes p_i and powers e_i come from the prime factorization of n. Moreover, we describe the condition under which the crossed product is an AF algebra. The precise statement is as the following.

Theorem 1.1 (Theorems 3.1 and 3.3)**.** *For a given dimension d and an order $n > 2$, there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus $\mathcal{A}_\Theta$ if either $d - W(n) > 1$ or $d - W(n) = 0$. Considering the actions we realized, the resulting crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is an AT algebra. It is an AF algebra if and only if $d - W(n) = 0$, and n admits either of the following prime factorization forms:*

1. $2^k 3^j 5^i$ *with* $k \neq 1$, $0 \le j \le 2$ *and* $0 \le i \le 1$,
2. $2 3^j 5^i p_l^{e_l}$ *with* $0 \le j \le 2$ *and* $0 \le i \le 1$ *and* $p_l > 5$ *a prime,*
3. $2 3^j 5^i$ *with* $0 \le j \le 2$,
4. $2 3^j 5^i$ *with* $0 \le i \le 1$.

Throughout this paper we usually assume the dimension of a noncommutative torus is greater than 1, since there is no 1-dimensional noncommutative torus. Also, we usually suppose the order of the finite cyclic group is greater than 2, since the flip action, namely, the matrix type action of $\mathbb{Z}_2$ on a higher dimensional noncommutative torus has already been studied and is special in a sense we explain later in Section 3.

We subsequently generalize the function W for finite abelian groups by defining for a finite cyclic group $\mathbb{Z}_n$,

$$W(\mathbb{Z}_n) := \begin{cases} W(n), & n \neq 2, \\ 2, & n = 2, \end{cases}$$

and for a finite abelian group G,

$$W(G) := \min\left\{ \sum_{m=1}^{l} W(\mathbb{Z}_{n_m}) \;\middle|\; G \cong \prod_{m=1}^{l} \mathbb{Z}_{n_m} \text{ for some } l \right\}.$$

With the generalized function W, we detect and realize actions of finite abelian groups in a similar fashion as in Theorem 1.1.

Theorem 1.2 (Theorem 3.9)**.** *For a given dimension d and a finite abelian subgroup $G \le \mathrm{GL}_d(\mathbb{Z})$, with $d - W(G) > 1$ or $d - W(G) = 0$, there is a matrix type action of G on a simple noncommutative d-torus $\mathcal{A}_\Theta$. Considering the action we realized, the resulting crossed product $\mathcal{A}_\Theta \rtimes G$ is an AT algebra. It is an AF algebra if and only if $d - W(G) = 0$ and, if we write G as $G \cong \prod_{m=1}^{t} \mathbb{Z}_{n_m}$ so that*

$$W(G) = \sum_{m=1}^{t} W(\mathbb{Z}_{n_m}),$$

each n_m admits either of the following prime factorization forms:

1. $2^k 3^j 5^i$ *with* $k \neq 1$, $0 \le j \le 2$ *and* $0 \le i \le 1$,
2. $2 3^j 5^i p_l^{e_l}$ *with* $0 \le j \le 2$ *and* $0 \le i \le 1$ *and* $p_l > 5$ *a prime,*

(3) 23^j5^i *with* $0 \le j \le 2$,
(4) 23^j5^i *with* $0 \le i \le 1$.

As a corollary of Theorem 1.2, Corollary 3.10 states that the only possibility of a finite abelian group F acting on an irrational rotation algebra is when $F = \mathbb{Z}_k$, where $k = 2, 3, 4$ and 6. In Section 3 we prove the above theorems and discuss related results.

Then we find out that the function W contains more information as the following theorem indicates.

Theorem 1.3 (Theorem 4.5). *For a given dimension d and an order $n > 2$ with $d - W(n) = 1$, there is no matrix type action of $\mathbb{Z}_n$ on a simple noncommutative torus.*

In Section 4 we prove this theorem. A corollary shows that the only matrix type action on a simple 3-dimensional noncommutative torus is the flip action of $\mathbb{Z}_2$.

In Section 5 we deal with general matrix type actions of $\mathbb{Z}_n$ on simple noncommutative d-tori with $d = W(n)$. Under an additional assumption for the generator matrix of $\mathbb{Z}_n$, we show that the classification result of the crossed products of these matrix type actions is similar to that of the special actions we realized in Theorem 1.1.

2. Preliminaries

For a second-countable locally compact Hausdorff topological group G with its modular function $\Delta_G\colon G \to (0, +\infty)$ and a Borel 2-cocycle $\omega \in Z^2(G, \mathbb{T})$, we define the associated reduced (resp. full) twisted group C^*-algebra, denoted by $C^*_r(G, \omega)$ (resp. $C^*(G, \omega)$), in the following way. Regard $L^1(G)$ as a vector space, and equip it with the twisted convolution given by

$$f *_\omega g(t) = \int_G f(s) g(s^{-1}t) \omega(s, s^{-1}t)\, ds$$

and the involution given by

$$f^*(t) = \Delta_G(t^{-1}) \overline{\omega(t, t^{-1}) f(t^{-1})}.$$

Then $L^1(G, \omega) := (L^1(G), *_\omega, *)$ becomes a $*$-algebra which is called the *twisted convolution algebra*. To consider nondegenerate representations of $L^1(G, \omega)$, we turn to a twisted analog of unitary representations of G. An *ω-representation* of G on a Hilbert space $\mathcal{H}$ is a Borel map $V\colon G \to \mathcal{U}(\mathcal{H})$ such that

$$V(t)V(s) = \omega(t, s)V(ts),$$

where $\mathcal{U}(\mathcal{H})$ stands for the unitary group of $\mathcal{H}$ endowed with the strong operator topology. For example, we define $L_\omega\colon G \to \mathcal{U}(L^2(G))$, the *regular ω-representation* of G, by

$$L_\omega(s)\xi(t) = \omega(s, s^{-1}t)\xi(s^{-1}t)$$

for any $\xi \in L^2(G)$. Any ω-representation $V\colon G \to \mathcal{U}(\mathcal{H})$ induces a representation $\pi\colon L^1(G,\omega) \to B(\mathcal{H})$ given by

$$\pi(f) = \int_G f(s)V(s)\,ds$$

for any $f \in L^1(G,\omega)$. Then π is a contractive $*$-homomorphism. For simplicity of notation, let V still denote the representation π that V induces. Hence, we define the *reduced twisted group C^*-algebra* as

$$C_r^*(G,\omega) = \overline{L_\omega(L^1(G,\omega))}.$$

Moreover, every nondegenerate representation of $L^1(G,\omega)$ is induced by an ω-representation of G. We can see this by taking a norm one approximate identity in $L^1(G,\omega)$. Thus, we obtain the universal representation π_u of $L^1(G,\omega)$, and define the *full twisted group C^*-algebra* as

$$C^*(G,\omega) = \overline{\pi_u(L^1(G,\omega))}.$$

Note that $C^*(G,\omega)$ has the universal property for ω-representations and the $*$-homomorphism $L_\omega\colon C^*(G,\omega) \to C_r^*(G,\omega)$ is surjective. The reduced and full twisted group C^*-algebras coincide if G is amenable.

We realize higher dimensional noncommutative tori as twisted group C^*-algebras. For a given dimension d, take $G = \mathbb{Z}^d$. Throughout this paper we denote by $\mathcal{T}_d(\mathbb{R})$ the set of all $d\times d$ skew symmetric real matrices. For a matrix $\Theta = (\theta_{ij}) \in \mathcal{T}_d(\mathbb{R})$, let the induced 2-cocycle $\omega_\Theta\colon \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{T}$ be given by

$$\omega_\Theta(x,y) = \exp(\pi i\langle \Theta x, y\rangle)$$

for $x, y \in \mathbb{Z}^d$. Then $\mathcal{A}_\Theta := C^*(\mathbb{Z}^d, \omega_\Theta)$ is called a *noncommutative d-torus*. Since $\mathbb{Z}^d$ is discrete and amenable, there is no difference if we view $\mathcal{A}_\Theta$ as $C_r^*(\mathbb{Z}^d,\omega)$ up to isomorphism. Let e_i, with $i = 1,\dots,d$, denote the standard basis of $\mathbb{Z}^d$, and $l_\Theta\colon \mathbb{Z}^d \to \mathcal{U}(\ell^2(\mathbb{Z}^d))$ the regular ω_Θ-representation. Then we have

$$\mathcal{A}_\Theta := C^*(\mathbb{Z}^d,\omega_\Theta) = C^*\{l_\Theta(e_i) \mid i = 1,\dots,d\},$$

where each $l_\Theta(e_i)$ is a unitary and they satisfy the commuting relations

$$l_\Theta(e_i)l_\Theta(e_j) = \omega_\Theta(e_i,e_j)^2 l_\Theta(e_j)l_\Theta(e_i) = \exp(2\pi i\theta_{ji})l_\Theta(e_j)l_\Theta(e_i).$$

This is clearly a d-dimensional generalization of the rotation algebras.

By definition, $\mathcal{A}_\Theta$ is isomorphic to $C(\mathbb{T}^d)$, the ordinary d-torus, if Θ is the zero $d \times d$ matrix. But this is not of our interests since it is not simple. We list some notions and facts related to simplicity.

Definition 2.1. A matrix $\Theta \in \mathcal{T}_d(\mathbb{R})$ is said to be *nondegenerate* if whenever $x \in \mathbb{Z}^d$ satisfies $\exp(2\pi i\langle \Theta x, y\rangle) = 1$ for all $y \in \mathbb{Z}^d$, we have $x = 0$.

Theorem 2.2 ([10, Theorem 1.9]). *For a matrix $\Theta \in \mathcal{T}_d(\mathbb{R})$, the noncommutative d-torus $\mathcal{A}_\Theta$ is simple if and only if Θ is nondegenerate. Moreover, if $\mathcal{A}_\Theta$ is simple, then it has a unique tracial state.*

Theorem 2.3 ([10, Theorems 3.5 and 3.8]). *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be nondegenerate with $d \geq 2$. Then $\mathcal{A}_\Theta$ is a simple AT algebra with real rank zero, and has tracial rank zero.*

Proposition 2.4. *For matrices $\Theta_i \in \mathcal{T}_{d_i}(\mathbb{R})$, $i = 1, \ldots, m$, set $\Theta := \bigoplus_{i=1}^m \Theta_i \in \mathcal{T}_d(\mathbb{R})$, where $d := \sum_{i=1}^m d_i$. Then Θ is nondegenerate if and only if each Θ_i is nondegenerate.*

Proof. It suffices to show the case when $m = 2$. Suppose Θ_1 and Θ_2 are both nondegenerate. For $x = (x_1, x_2) \in \mathbb{Z}^d$ with $\exp(2\pi i\langle\Theta x, y\rangle) = 1$ for all $y \in \mathbb{Z}^d$, by taking $y_1 \in \mathbb{Z}^{d_1} \times \{0\}$ and $y_2 \in \{0\} \times \mathbb{Z}^{d_2}$, we obtain $\exp(2\pi i\langle\Theta_j x_j, y_j\rangle) = 1$ for all $y_j \in \mathbb{Z}^{d_j}$, where $j = 1, 2$. Then, by nondegeneracy of Θ_1 and Θ_2, we obtain $x = (x_1, x_2) = 0$, i.e., Θ is nondegenerate.

Conversely, suppose Θ is nondegenerate. Suppose $x_1 \in \mathbb{Z}^{d_1}$ is such that

$$\exp(2\pi i\langle\Theta_1 x_1, y_1\rangle) = 1$$

for all $y_1 \in \mathbb{Z}^{d_1}$. Consider $x = (x_1, 0) \in \mathbb{Z}^d$ and compute that

$$\exp(2\pi i\langle\Theta x, y\rangle) = \exp(2\pi i\langle\Theta_1 x_1, y_1\rangle) = 1$$

for all $y \in \mathbb{Z}^d$. Thus, $x_1 = 0$, by definition. Similarly, we also have $x_2 = 0$. Therefore, Θ_1 and Θ_2 are both nondegenerate. □

We now discuss a kind of actions on noncommutative tori. Let $\mathcal{A}_\Theta$ denote a noncommutative d-torus. The definition of these actions does not require simplicity of $\mathcal{A}_\Theta$. We view $\mathcal{A}_\Theta$ as a C^*-subalgebra of $B(\ell^2(\mathbb{Z}^d))$ through the regular ω_Θ-representation $l_\Theta \colon \ell^1(\mathbb{Z}^d, \omega_\Theta) \to B(\ell^2(\mathbb{Z}^d))$.

A matrix $A \in \mathrm{GL}_d(\mathbb{Z})$ defines a unitary u_A in $B(\ell^2(\mathbb{Z}^d))$ by

$$u_A \xi(x) = \xi(A^{-1}x)$$

for $\xi \in \ell^2(\mathbb{Z}^d)$ and $x \in \mathbb{Z}^d$. The unitary u_A defines an automorphism $\mathrm{Ad}\, u_A \in \mathrm{Aut}(B(\ell^2(\mathbb{Z}^d)))$. Then we have an action denoted by

$$\mathrm{Ad}\, u_\bullet \colon \mathrm{GL}_d(\mathbb{Z}) \to \mathrm{Aut}(B(\ell^2(\mathbb{Z}^d))).$$

For $\mathcal{A}_\Theta = C^*\{l_\Theta(e_i) \mid i = 1, \ldots, d\} = \overline{\mathrm{span}}\{l_\Theta(x) \mid x \in \mathbb{Z}^d\}$, we have the formula

$$\mathrm{Ad}\, u_A(l_\Theta(x))\xi(y) = l_{(A^{-1})^t \Theta A^{-1}}(Ax)\xi(y) \tag{1}$$

for $\xi \in \ell^2(\mathbb{Z}^d)$ and $y \in \mathbb{Z}^d$. Thus, for $A \in \mathrm{GL}_d(\mathbb{Z})$ such that $\Theta = (A^{-1})^t\Theta A^{-1}$, the restriction of $\mathrm{Ad}\, u_A$ is an automorphism of $\mathcal{A}_\Theta$. Putting

$$G_\Theta := \{A \in \mathrm{GL}_d(\mathbb{Z}) \mid \Theta = A^t\Theta A\},$$

we obtain an action

$$\mathrm{Ad}\, u_\bullet \colon G_\Theta \to \mathrm{Aut}(\mathcal{A}_\Theta).$$

This is a generalization of the action of $\mathrm{SL}_2(\mathbb{Z})$ on a rotation algebra $\mathcal{A}_\theta$. Particularly, we have $G_\Theta = \mathrm{SL}_2(\mathbb{Z})$ when $d = 2$. We are especially interested in such actions of some subgroup G of G_Θ on $\mathcal{A}_\Theta$, and in the remainder of this paper, we name these actions *matrix type actions*.

Theorem 2.5 ([2, Theorem 0.1]). *Let F be any of the finite subgroups $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6 \subseteq \mathrm{SL}_2(\mathbb{Z})$ with generators as the following:*

$$\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle = \mathbb{Z}_2, \quad \left\langle \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = \mathbb{Z}_3,$$

$$\left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle = \mathbb{Z}_4, \quad \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \right\rangle = \mathbb{Z}_6.$$

Let $\theta \in \mathbb{R}\backslash\mathbb{Q}$. Then the crossed product $\mathcal{A}_\theta \rtimes F$ is an AF algebra. For all $\theta \in \mathbb{R}$, we have

$$K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_2) \cong \mathbb{Z}^6, \quad K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_3) \cong \mathbb{Z}^8,$$

$$K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_4) \cong \mathbb{Z}^9, \quad K_0(\mathcal{A}_\theta \rtimes \mathbb{Z}_6) \cong \mathbb{Z}^{10}.$$

For $F = \mathbb{Z}_k$, $k = 2, 3, 4, 6$, the image of $K_0(\mathcal{A}_\theta \rtimes F)$ under the unique canonical tracial state of $\mathcal{A}_\theta \rtimes F$ is equal to $\frac{1}{k}(\mathbb{Z} + \mathbb{Z}\theta)$. As a consequence, $\mathcal{A}_\theta \rtimes \mathbb{Z}_k$ is isomorphic to $\mathcal{A}_{\theta'} \rtimes \mathbb{Z}_{k'}$ if and only if $k = k'$ and $\theta \equiv \pm\theta' \mod \mathbb{Z}$.

These are results on matrix type actions on irrational rotation algebras. Moreover, for the flip action of $\mathbb{Z}_2$ on higher dimensional noncommutative tori, the following result is known.

Theorem 2.6 ([2, Theorem 6.6]). *Let Θ be a nondegenerate real $d \times d$ skew symmetric matrix. Let $\varphi\colon \mathbb{Z}_2 \to \mathrm{Aut}(\mathcal{A}_\Theta)$ be the flip action. Then $\mathcal{A}_\Theta \rtimes \mathbb{Z}_2$ is an AF algebra.*

The tracial Rokhlin property for finite group actions on C^*-algebras is introduced by Phillips in [10]. It is an important property of actions on C^*-algebras.

Proposition 2.7. *Let $\Theta \in \mathcal{T}_d(\mathbb{R})$ be nondegenerate with $d \geq 2$. Then for any finite subgroup $G \leq G_\Theta$, the matrix type action*

$$\mathrm{Ad}\, u_\bullet \colon G \to \mathrm{Aut}(\mathcal{A}_\Theta)$$

has the tracial Rokhlin property.

Proof. Let τ denote the unique tracial state of $\mathcal{A}_\Theta$ according to Theorem 2.2, and $\pi_\tau\colon \mathcal{A}_\Theta \to B(\mathcal{H}_\tau)$ the GNS representation associated with τ. Write $(\mathrm{Ad}\, u_A)''$ for the automorphism of $\pi_\tau(\mathcal{A}_\Theta)''$ determined by $\mathrm{Ad}\, u_A$. Due to [2, Theorem 5.5], the matrix type action $\mathrm{Ad}\, u_\bullet\colon G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ has the tracial Rokhlin property if and only if $(\mathrm{Ad}\, u_A)''$is an outer automorphism of $\pi_\tau(\mathcal{A}_\Theta)''$ for any $A \in G\backslash\{I\}$, where I stands for the $d \times d$ identity matrix.

For any $A = (a_{ij}) \in G\backslash\{I\}$, by (1), we have

$$\mathrm{Ad}\, u_A(l_\Theta(e_i)) = l_\Theta(Ae_i) = \rho l_\Theta(e_1)^{a_{i1}} l_\Theta(e_2)^{a_{i2}} \cdots l_\Theta(e_d)^{a_{id}},$$

where $\rho \in \mathbb{T}$. There exists an integer k such that $(a_{k1}, a_{k2}, \ldots, a_{kd}) \neq e_k$, since $A \neq I$. Thus, by [2, Lemma 5.10], the automorphism $(\mathrm{Ad}\, u_A)''$ is outer and we complete the proof. □

The crossed products associated with actions having the tracial Rokhlin property preserve many C^*-algebraic properties. For example:

Proposition 2.8. *Let $\alpha\colon G \to \mathrm{Aut}(\mathcal{A}_\Theta)$ be an action of a finite group G on a simple noncommutative torus $\mathcal{A}_\Theta$ such that α has the tracial Rokhlin property. Then the crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ is also simple, with a unique tracial state and has tracial rank zero.*

Proof. It is straight-forward by Theorem 2.3, [11, Corollary 1.6] and [11, Theorem 2.6] that $\mathcal{A}_\Theta \rtimes_\alpha G$ is simple and has tracial rank zero.

The crossed product $\mathcal{A}_\Theta \rtimes_\alpha G$ has a unique tracial state due to [2, Proposition 5.7]. □

Under the assumption of Proposition 2.7, we can apply Proposition 2.8 to the resulting crossed product, denoted by $\mathcal{A}_\Theta \rtimes G$ for short. Moreover, it follows from [3, Proposition 3.1] that $\mathcal{A}_\Theta \rtimes G$ satisfies the Universal Coefficient Theorem. Hence, the resulting crossed products are classifiable and we may apply the following theorems.

Theorem 2.9 ([7, Theorem 5.2])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital separable simple nuclear C^*-algebras with tracial topological rank zero which satisfy the Universal Coefficient Theorem. Then $\mathcal{A} \cong \mathcal{B}$ if and only if they have isomorphic Elliott invariants, that is,*

$$(K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_\mathcal{A}], K_1(\mathcal{A})) \cong (K_0(\mathcal{B}), K_0(\mathcal{B})^+, [1_\mathcal{B}], K_1(\mathcal{B})).$$

Proposition 2.10 ([10, Proposition 3.7])**.** *Let $\mathcal{A}$ be a simple infinite-dimensional separable unital nuclear C^*-algebra with tracial rank zero which satisfies the Universal Coefficient Theorem. Then $\mathcal{A}$ is a simple AH algebra with real rank zero and no dimension growth. If $K_*(\mathcal{A})$ is torsion-free, $\mathcal{A}$ is an AT algebra. If, in addition, $K_1(\mathcal{A}) = 0$, then $\mathcal{A}$ is an AF algebra.*

We now review the results of Jeong and Lee in [3]. They apply companion matrices and cyclotomic polynomials to realize matrix type actions of finite cyclic groups on simple higher dimensional noncommutative tori.

To be precise, recall that for a given integer n, the value of Euler's totient function ϕ on n is defined by

$$\phi(n) = |\{k \in \mathbb{Z} \mid 1 \le k \le n,\ \gcd(k,n) = 1\}|.$$

Then let d be the dimension of some simple noncommutative torus, and n the order of some finite cyclic group with $d = \phi(n)$. The nth *cyclotomic polynomial* $\Phi_n(x)$ is defined by

$$\Phi_n(x) = \prod_{\substack{0<k\le n\\ \gcd(k,n)=1}} \Big(x - \exp\Big(2\pi i \frac{k}{n}\Big)\Big) = \sum_{i=0}^{d} a_i x^i.$$

$\Phi_n(x)$ is known to be a monic polynomial of degree $d = \phi(n)$ with integer coefficients. Thus, set

$$C_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & -a_{d-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{d-1} \end{pmatrix},$$

which is the companion matrix of $\Phi_n(x)$. Then C_n is in $\mathrm{GL}_d(\mathbb{Z})$ of order n.

Theorem 2.11 ([3, Theorem 4.2]). *Let $n \geq 3$ and $d := \phi(n)$. Then there exist simple d-dimensional tori on which the group $\mathbb{Z}_n = \langle C_n \rangle$ acts canonically.*

We remark that in this theorem the "canonical action" is what we called the matrix type action. We use their method of realization as a building block to obtain more matrix type actions. To compute the K-theory of the associated crossed product of such action, we need the following theorem.

Theorem 2.12 ([6, Theorem 0.1]). *Let $n, d \in \mathbb{N}$. Consider the extension of groups*

$$1 \to \mathbb{Z}^d \to \mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n \to \mathbb{Z}_n \to 1$$

such that the conjugation action α of $\mathbb{Z}_n$ on $\mathbb{Z}^d$ is free outside of the origin $0 \in \mathbb{Z}^d$. Then $K_0(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{s_0}$ for some $s_0 \in \mathbb{Z}$ and

$$K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \cong \mathbb{Z}^{s_1},$$

where $s_1 = \sum_{l \geq 0} \mathrm{rk}_{\mathbb{Z}}((\bigwedge^{2l+1} \mathbb{Z}^d)^{\mathbb{Z}_n})$. If n is even, $s_1 = 0$. If $n > 2$ is prime and $d = n-1$, then $s_1 = \frac{2^{n-1}-(n-1)^2}{2n}$.

Jeong and Lee [3] apply this method and obtain the following theorem which indicates that for certain dimension d and order n, the resulting crossed product is not an AF algebra.

Theorem 2.13 ([3, Theorem 3.6]). *Let $n \geq 7$ be an odd integer and $d := \phi(n)$. Consider the extension of groups $1 \to \mathbb{Z}^d \to \mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n \to \mathbb{Z}_n \to 1$ with $\mathbb{Z}_n = \langle C_n \rangle$. If $2d \geq n+5$, then*

$$K_1(C^*(\mathbb{Z}^d \rtimes_\alpha \mathbb{Z}_n)) \neq 0.$$

We denote by $\mathfrak{N}$ the smallest subcategory of the category of separable C^*-algebras, which contains separable Type I algebras and is closed under taking ideals, quotients, extensions, inductive limits, stable isomorphisms and crossed products by $\mathbb{Z}$ and $\mathbb{R}$. Then the following theorem holds.

Theorem 2.14 ([12, Künneth theorem]). *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with $\mathcal{A}$ in $\mathfrak{N}$. Then there is a natural short exact sequence*

$$0 \to K_*(\mathcal{A}) \otimes K_*(\mathcal{B}) \xrightarrow{\alpha} K_*(\mathcal{A} \otimes \mathcal{B}) \xrightarrow{\beta} \mathrm{Tor}(K_*(\mathcal{A}), K_*(\mathcal{B})) \to 0.$$

The sequence is $\mathbb{Z}_2$-graded with $\deg \alpha = 1$, $\deg \beta = 1$, where $K_p \otimes K_q$ and $\mathrm{Tor}(K_p \otimes K_q)$ are given degree $p+q$ for $p, q \in \mathbb{Z}_2$.

3. Realization of finite abelian groups acting on noncommutative d-tori

In this section we discuss a special realization of matrix type actions of finite abelian groups on noncommutative tori.

For given skew symmetric real matrices Θ_1 and Θ_2, there is an isomorphism between $\mathcal{A}_{\Theta_1} \otimes \mathcal{A}_{\Theta_2}$ and $\mathcal{A}_\Theta$, where $\Theta = \Theta_1 \oplus \Theta_2$ denotes the direct sum of Θ_1 and Θ_2. Suppose G_1 and G_2 are finite subgroups of G_{Θ_1} and G_{Θ_2} respectively, and consider matrix type actions of G_1 on $\mathcal{A}_{\Theta_1}$, and of G_2 on $\mathcal{A}_{\Theta_2}$. Since the product group $G = G_1 \times G_2$ is a subgroup of G_Θ, the product action provides us a matrix type action of G on $\mathcal{A}_\Theta$. The crossed product $\mathcal{A}_\Theta \rtimes G$ is isomorphic to $(\mathcal{A}_{\Theta_1} \rtimes G_1) \otimes (\mathcal{A}_{\Theta_2} \rtimes G_2)$ via the isomorphism defined by mapping $(a \otimes b, (g, h))$ to $(a, g) \otimes (b, h)$.

We combine this observation with Theorem 2.11 to realize more matrix type actions on higher dimensional noncommutative tori. To start, it is necessary to find out the possible finite cyclic subgroups of $\mathrm{GL}_d(\mathbb{Z})$, since G_Θ lives in it.

Let d denote the dimension of $\mathcal{A}_\Theta$, n the order of the finite cyclic group, and $n = \prod_{i=1}^t p_i^{e_i}$ the prime factorization of n, where the prime numbers p_i are subject to the condition $p_1 < \cdots < p_t$. Then, according to [5, Theorem 2.7], there is an element in $\mathrm{GL}_d(\mathbb{Z})$ of order n if and only if $W(n) \leq d$, where the function W is defined by

$$W(n) := \begin{cases} \sum_{i=1}^t (p_i - 1)p_i^{e_i - 1} - 1, & p_1^{e_1} = 2, \\ \sum_{i=1}^t (p_i - 1)p_i^{e_i - 1}, & \text{otherwise.} \end{cases}$$

Thus, there are finitely many candidates for n as a possible order of a finite cyclic subgroup of $\mathrm{GL}_d(\mathbb{Z})$. Let ϕ denote Euler's totient function. For n with $\phi(n) = d$, Theorem 2.11 enables us to find a simple noncommutative d-torus $\mathcal{A}_\Theta$, namely, to figure out a nondegenerate Θ, and to realize a matrix type action of $\mathbb{Z}_n$ on $\mathcal{A}_\Theta$. Moreover, it gives a formula for the K-theory $K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$, with which we can tell whether $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is AF or not. However, this argument does not work for n with $\phi(n) \neq d$, particularly when d is odd. We provide a way to realize matrix type actions of cyclic groups for these cases and study the resulting crossed products.

Theorem 3.1. *For a given dimension d and an order $n > 2$, there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus $\mathcal{A}_\Theta$ if either $d - W(n) > 1$ or $d - W(n) = 0$. Considering the matrix type action we realized, the resulting crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is an AT algebra.*

Proof. Let $n = \prod_{i=1}^t p_i^{e_i}$ still denote the prime factorization of n with $p_1 < \cdots < p_t$. According to [5, Theorem 2.7], we can assume $W(n) \leq d$. Then we split up our proof into two cases: $p_1^{e_1} \neq 2$ or $p_1^{e_1} = 2$, since the definition of W differs. If $p_1^{e_1} \neq 2$, using the companion matrix, we construct a $d_i \times d_i$ matrix A_i of order $p_i^{e_i}$, where $d_i = (p_i - 1)p_i^{e_i - 1}$. Note that in this case $\phi(p_i^{e_i}) = (p_i - 1)p_i^{e_i - 1}$ holds, which guarantees our construction. Then set $B := \bigoplus_{i=1}^t A_i$, which is a $W(n) \times W(n)$ matrix with $W(n) \leq d$.

By Theorem 2.11, for each i, we can find nondegenerate Θ_i in $\mathcal{T}_{d_i}(\mathbb{R})$ such that $\langle A_i\rangle = \mathbb{Z}_{p_i^{e_i}}$ acts on $\mathcal{A}_{\Theta_i}$.

If $d - W(n) > 1$, set $A := B \oplus I_{d-W(n)}$, which is a $d \times d$ matrix of order n. Find a nondegenerate Θ_0 and set

$$\Theta := \Big(\bigoplus_{i=1}^{t} \Theta_i\Big) \oplus \Theta_0,$$

which is nondegenerate as well. Then we have an action of $\langle A\rangle = \mathbb{Z}_n$ on $\mathcal{A}_\Theta$. Moreover, by our former discussion, one may check the following isomorphism:

$$\mathcal{A}_\Theta \rtimes \mathbb{Z}_n \cong \Big(\bigotimes_{i=1}^{t} (\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}})\Big) \otimes \mathcal{A}_{\Theta_0}.$$

To compute the K-groups of the crossed product, we apply the Künneth theorem in [12], as stated in Theorem 2.14.

Firstly, by [3, Proposition 3.4], each $\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}$ is an AT algebra, and so is $\mathcal{A}_{\Theta_0}$. Since $C(\mathbb{T}) \otimes F$ is separable and of Type I, where F stands for a finite-dimensional C^*-algebra, every $\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}$ and $\mathcal{A}_{\Theta_0}$ are in $\mathfrak{N}$. Secondly, by [3, Remark 3.2] and [6, Theorem 0.1], we know that for each i,

$$K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}) \cong K_*(C^*(\mathbb{Z}^{d_i} \rtimes \mathbb{Z}_{p_i^{e_i}}, \tilde{\omega}_{\Theta_i}) \cong K_*(C^*(\mathbb{Z}^{d_i} \rtimes \mathbb{Z}_{p_i^{e_i}})),$$

which is torsion-free.

Then, by applying the Künneth theorem, we obtain

$$K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) = \Big(\bigotimes_{i=1}^{t} K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}})\Big) \otimes K_*(\mathcal{A}_{\Theta_0}).$$

The crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ satisfies the Universal Coefficient Theorem [3, Proposition 3.1]. As we have already stated before, $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is then classifiable. Thus, under this realization, the crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is an AT algebra, since $K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$ is torsion-free.

If $d - W(n) = 0$, the situation is essentially the same. Instead we set $A := B = \bigoplus_{i=1}^{t} A_i$, which is a $d \times d$ matrix of order n. Then set

$$\Theta := \Big(\bigoplus_{i=1}^{t} \Theta_i\Big),$$

which is nondegenerate. Thus, we have an action of $\langle A\rangle = \mathbb{Z}_n$ on $\mathcal{A}_\Theta$. We also obtain

$$\mathcal{A}_\Theta \rtimes \mathbb{Z}_n \cong \bigotimes_{i=1}^{t} (\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}),$$

and again the Künneth theorem gives

$$K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) = \bigotimes_{i=1}^{t} K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}). \tag{2}$$

It follows from Proposition 2.10 that $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is an AT algebra.

If $p_1^{e_1} = 2$, we apply the argument for proving the first case to the order $\frac{n}{2}$. Thus, we construct a $d_i \times d_i$ matrix A_i of order $p_i^{e_i}$, where $d_i = (p_i - 1)p_i^{e_i - 1}$ for $i = 2, \ldots, t$. Since $W(n) = W(\frac{n}{2})$, the matrix $B := \bigoplus_{i=2}^{t} A_i$ is a $W(n) \times W(n)$ matrix with $W(n) \leq d$, and of order $\frac{n}{2}$.

Again we can find nondegenerate Θ_i in $\mathcal{T}_{d_i}(\mathbb{R})$ such that $\langle A_i \rangle$ acts on $\mathcal{A}_{\Theta_i}$ for $i = 2, \ldots, t$. If $d - W(n) = d - W(\frac{n}{2}) > 1$, set $A := B \oplus I_{d-W(n)}$, which is a $d \times d$ matrix of order $\frac{n}{2}$. It is possible to find a nondegenerate Θ_0, since $d - W(n) > 1$. Then set

$$\Theta := \Big(\bigoplus_{i=1}^{t} \Theta_i \Big) \oplus \Theta_0,$$

which is nondegenerate. Since

$$(-A)^t \Theta (-A) = A^t \Theta A = \Theta,$$

we have an action of $\langle -A \rangle = \mathbb{Z}_n$ on $\mathcal{A}_\Theta$. However, in this case, we write $\mathbb{Z}_n$ as

$$\mathbb{Z}_n \cong \Big(\prod_{\substack{2 \leq i \leq t \\ i \neq j}} \mathbb{Z}_{p_i^{e_i}} \Big) \times \mathbb{Z}_{2p_j^{e_j}},$$

where $2 \leq j \leq t$. Thus, instead of A, we consider $A_{(j)}$ as a generator of $\mathbb{Z}_n$, where $A_{(j)}$ is defined by

$$A_{(j)} = \begin{pmatrix} A_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -A_j & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & A_t & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & I_{d-W(n)} \end{pmatrix},$$

where $1 \leq j \leq t$. Then we consider actions of $\mathbb{Z}_n = \langle A_{(j)} \rangle$ on $\mathcal{A}_\Theta$.

Similarly, we have

$$\mathcal{A}_\Theta \rtimes \mathbb{Z}_n \cong \Big(\bigotimes_{\substack{2 \leq i \leq t \\ i \neq j}} (\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}) \Big) \otimes (\mathcal{A}_{\Theta_j} \rtimes \mathbb{Z}_{2p_j^{e_j}}) \otimes \mathcal{A}_{\Theta_0},$$

where $2 \leq j \leq t$. By the Künneth theorem, $K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$ is the tensor of K_* of each factor.

It is similar when $d - W(n) = d - W(\frac{n}{2}) = 0$, and the K-theory becomes

$$K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) \cong \Big(\bigotimes_{\substack{2 \leq i \leq t \\ i \neq j}} K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}) \Big) \otimes K_*(\mathcal{A}_{\Theta_j} \rtimes \mathbb{Z}_{2p_j^{e_j}}),$$

where $2 \leq j \leq t$. Proposition 2.10 still assures us that the crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is an AT algebra. □

Note that in the preceding proof, we factor $K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$ into "smaller blocks", by the Künneth formula. Thus, combining with known results, we can obtain more information of $K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$ and further classification results about $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$.

Definition 3.2. A natural number n is said to be of *form 3.2* if n admits either of the following prime factorization forms:

(1) $2^k 3^j 5^i$ with $k \neq 1$, $0 \leq j \leq 2$ and $0 \leq i \leq 1$,
(2) $2 3^j 5^i p_l^{e_l}$ with $0 \leq j \leq 2$ and $0 \leq i \leq 1$ and $p_l > 5$ a prime,
(3) $2 3^j 5^i$ with $0 \leq j \leq 2$,
(4) $2 3^j 5^i$ with $0 \leq i \leq 1$.

Theorem 3.3. *The crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ in the preceding proof is an AF algebra if and only if $d - W(n) = 0$, and n is of form 3.2.*

Proof. If $d - W(n) > 1$, the K_1-group $K_1(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) \neq 0$ does not vanish, since we have

$$K_0(\mathcal{A}_{\Theta_0}) \cong K_1(\mathcal{A}_{\Theta_0}) \cong \mathbb{Z}^{2^{d-W(n)}-1}.$$

Hence, $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is not an AF-algebra according to Proposition 2.10.

If $d - W(n) = 0$, by (2), we deduce that $K_1(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) = 0$ if and only if $K_1(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}) = 0$ for each i.

We now turn our attention to each $\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}$. The group $K_1(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}})$ is not trivial if p_i is an odd prime and subject to the condition

$$p_i^{e_i - 1}(p_i - 2) \geq 5,$$

according to [3, Theorem 3.6]. Therefore, if $K_1(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}}) = 0$, then $p_i^{e_i}$ is necessarily $3, 3^2, 5$ or 2^k, where $k > 1$. To compute the K_1-groups, we apply [6, Theorem 0.1], as stated in Theorem 2.12.

It follows from Theorem 2.12 that the K_1-group vanishes if $p_i^{e_i} = 3, 5$ or 2^k, where $k > 1$. In the following we check the case when $p_i^{e_i} = 3^2$. The theorem enables us to merely compute s_1 given by

$$s_1 = \mathrm{rk}_\mathbb{Z}\big(\big(\bigwedge^1 \mathbb{Z}^6\big)^{\mathbb{Z}_9}\big) + \mathrm{rk}_\mathbb{Z}\big(\big(\bigwedge^3 \mathbb{Z}^6\big)^{\mathbb{Z}_9}\big) + \mathrm{rk}_\mathbb{Z}\big(\big(\bigwedge^5 \mathbb{Z}^6\big)^{\mathbb{Z}_9}\big).$$

We obtain $\mathrm{rk}_\mathbb{Z}((\bigwedge^1 \mathbb{Z}^6)^{\mathbb{Z}_9}) = 0$, since the action is free outside the origin $0 \in \mathbb{Z}^6$. According to our realization, the cyclic group $\mathbb{Z}_9$ is realized by A in $\mathrm{GL}_6(\mathbb{Z})$, where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & 0 & 0 & -a_2 \\ 0 & 0 & 1 & 0 & 0 & -a_3 \\ 0 & 0 & 0 & 1 & 0 & -a_4 \\ 0 & 0 & 0 & 0 & 1 & -a_5 \end{pmatrix}$$

and

$$\Phi_9(x) = \prod_{\substack{0<k<9 \\ \gcd(k,9)=1}} (x - \zeta^k) = \sum_{i=0}^{5} a_i x^i, \zeta = \exp\Big(2\pi i \frac{1}{9}\Big).$$

Write $A = Q^{-1}\mathrm{diag}(\zeta, \zeta^2, \zeta^4, \zeta^5, \zeta^7, \zeta^8)Q$, for some $Q \in \mathrm{GL}_6(\mathbb{C})$. Notice that x is a fixed point of A acting on $\bigwedge^3 \mathbb{Z}$ if and only if Qx is a fixed point of $\mathrm{diag}(\zeta, \zeta^2, \zeta^4, \zeta^5, \zeta^7, \zeta^8)$ acting on $\bigwedge^3 Q\mathbb{Z}$. Then for $1 \leq i < j < k \leq 6$, suppose

$$\mathrm{diag}(\zeta, \zeta^2, \zeta^4, \zeta^5, \zeta^7, \zeta^8)e_i \wedge e_j \wedge e_k = e_i \wedge e_j \wedge e_k.$$

It is equivalent to the following statement: There is a way to pick three numbers in $1, 2, 4, 5, 7, 8$ such that the sum of them can be divided by 9, which is impossible. Thus, we obtain

$$\mathrm{rk}_{\mathbb{Z}}\big(\big(\bigwedge^3 \mathbb{Z}^6\big)^{\mathbb{Z}_9}\big) = 0.$$

Similarly, we also obtain

$$\mathrm{rk}_{\mathbb{Z}}\big(\big(\bigwedge^5 \mathbb{Z}^6\big)^{\mathbb{Z}_9}\big) = 0.$$

Hence, when $p_i^{e_i} = 3^2$, we obtain a matrix type action of $\mathbb{Z}_9$ on a simple noncommutative 6-torus $\mathcal{A}_{\Theta'}$ and

$$K_1(\mathcal{A}_{\Theta'} \rtimes \mathbb{Z}_9) = 0,$$

in other words the crossed product $\mathcal{A}_{\Theta'} \rtimes \mathbb{Z}_9$ is an AF algebra.

Let us leave the case when $d - W(n) = 1$ to the next section. To complete the proof, we discuss the case in which the given dimension d and order n are such that $W(n) \leq d$, and with $p_1^{e_1} = 2$ in the prime factorization of $n = \prod_{i=1}^t p_i^{e_i}$. Recall that by the proof of Theorem 3.1, we have

$$\mathcal{A}_\Theta \rtimes \mathbb{Z}_n \cong \Big(\bigotimes_{\substack{2 \leq i \leq t \\ i \neq j}} (\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}})\Big) \otimes (\mathcal{A}_{\Theta_j} \rtimes \mathbb{Z}_{2p_j^{e_j}}) \otimes \mathcal{A}_{\Theta_0},$$

where $2 \leq j \leq t$. By the Künneth theorem, $K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$ is the tensor of K_* of each factor. Again the part of $K_*(\mathcal{A}_{\Theta_0})$ forces $K_1(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) \neq 0$, which indicates that $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is not an AF algebra.

If $d - W(n) = d - W(\frac{n}{2}) = 0$, there is a similar isomorphism for the crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$, and its K-theory becomes

$$K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) \cong \Big(\bigotimes_{\substack{2 \leq i \leq t \\ i \neq j}} K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{p_i^{e_i}})\Big) \otimes K_*(\mathcal{A}_{\Theta_j} \rtimes \mathbb{Z}_{2p_j^{e_j}}),$$

where $2 \leq j \leq t$. Since $K_1(\mathcal{A}_{\Theta_j} \rtimes \mathbb{Z}_{2p_j^{e_j}})$ vanishes according to Theorem 2.12, the crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is then an AF algebra exactly when $\frac{n}{2}$ admits a form of $3^j 5^i p_l^{e_l}$, where j, i and e_l are all nonnegative integers with $j \leq 2$, $i \leq 1$. Note that if either j or i is 0, the prime number p_l could be 3 or 5, respectively. Otherwise p_l is prime number greater than 5. □

Corollary 3.4. *For an even dimension d and an order n, there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus if and only if*

$$W(n) \leq d.$$

Proof. Since $W(n)$ is always even when $n \geq 3$. □

Let us consider a similar realization of matrix type actions of finite abelian groups. The following two propositions enable us to deal with products of finitely many matrix type actions, and obtain classification results by computing the K_*-groups of the resulting crossed products.

Proposition 3.5. *For $i = 1, 2$, suppose the simple noncommutative torus $\mathcal{A}_{\Theta_i}$ admits a matrix type action $\alpha_i \colon G_i \to \mathrm{Aut}(\mathcal{A}_{\Theta_i})$. Then the product action*

$$\alpha_1 \otimes \alpha_2 \colon G_1 \times G_2 \to \mathrm{Aut}(\mathcal{A}_{\Theta_1} \otimes \mathcal{A}_{\Theta_2})$$

has the tracial Rokhlin property.

Proof. By [2, Theorem 5.5], it is equivalent to showing $(\alpha_1(g_1) \otimes \alpha_2(g_2))''$ is outer for any $(g_1, g_2) \neq (1_{G_1}, 1_{G_2})$.

Let τ_i denote the unique tracial state of $\mathcal{A}_{\Theta_i}$. Then $\tau_1 \otimes \tau_2$ is the unique tracial state on $\mathcal{A}_{\Theta_1} \otimes \mathcal{A}_{\Theta_2}$. Also we denote by

$$\pi_{\tau_i} \colon \mathcal{A}_{\Theta_i} \to B(\mathcal{H}_{\tau_i})$$

the GNS representation associated with τ_i. Then the GNS representation of $\mathcal{A}_{\Theta_1} \otimes \mathcal{A}_{\Theta_2}$ associated with $\tau_1 \otimes \tau_2$ is unitarily equivalent to $\pi_{\tau_i} \otimes \pi_{\tau_2}$. Since

$$(\pi_{\tau_1}(\mathcal{A}_{\Theta_1}) \otimes \pi_{\tau_2}(\mathcal{A}_{\Theta_2}))'' = \pi_{\tau_1}(\mathcal{A}_{\Theta_1})'' \otimes \pi_{\tau_2}(\mathcal{A}_{\Theta_2})'',$$

to show $(\alpha_1 \otimes \alpha_2)''$ is outer, it is then sufficient to show

$$\alpha_1(g_1)'' \otimes \alpha_2(g_2)'' \colon \pi_{\tau_1}(\mathcal{A}_{\Theta_1})'' \otimes \pi_{\tau_2}(\mathcal{A}_{\Theta_2})'' \to \pi_{\tau_1}(\mathcal{A}_{\Theta_1})'' \otimes \pi_{\tau_2}(\mathcal{A}_{\Theta_2})''$$

is an outer automorphism for any $(g_1, g_2) \neq (1_{G_1}, 1_{G_2})$.

Assume on the contrary $\alpha_1(g_1)'' \otimes \alpha_2(g_2)''$ is inner. Then, by [4, Theorem 13.1.16], it is equivalent to both of $\alpha_1(g_1)''$ and $\alpha_2(g_2)''$ are inner. Then, by Proposition 2.7 and [2, Theorem 5.5], it means $g_1 = 1_{G_1}$ and $g_2 = 1_{G_2}$. The contradiction shows that $\alpha_1(g_1)'' \otimes \alpha_2(g_2)''$ is outer. □

Proposition 3.6. *Under the assumption of Proposition 3.5, put $\alpha = \alpha_1 \otimes \alpha_2$, $\Theta = \Theta_1 \oplus \Theta_2$ and $G = G_1 \times G_2$. Then $\mathcal{A}_\Theta \rtimes_\alpha G$ satisfies the Universal Coefficient Theorem.*

Proof. Suppose that the dimension of $\mathcal{A}_\Theta$ is d, thus $\mathcal{A}_\Theta = C^*(\mathbb{Z}^d, \omega_\Theta)$. It is routine to verify that the following formula holds:

$$\omega_\Theta(g \cdot x, g \cdot y) = \omega_\Theta(x, y) \quad \text{for all } g \in G, \text{ and all } x, y \in \mathbb{Z}^d.$$

Hence, by [2, Lemma 2.1], there exists a cocycle $\widetilde{\omega}_\Theta \in Z^2(\mathbb{Z}^d \rtimes G, \mathbb{T})$ such that

$$\mathcal{A}_\Theta \rtimes_\alpha G \cong C^*(\mathbb{Z}^d \rtimes G, \widetilde{\omega}_\Theta).$$

Since $\mathbb{Z}^d \rtimes G$ is a closed subgroup of $\mathbb{R}^d \rtimes G$ and amenable, we complete the proof by [2, Corollary 6.2]. □

For a finite abelian group G, we may generalize our method to construct such actions of G on noncommutative tori. By the structure theorem for finitely generated abelian groups, we write G as a product group of several finite cyclic groups. Each one acts on a noncommutative torus via our construction of matrix type actions. Then the product action of them is an action of G on a noncommutative torus. We can also generalize the function W to detect the dimension cost for realizing the action. However, the definition of W should be modified for the following reason. Let G be the group $\mathbb{Z}_2$. Then G acts on an irrational rotation algebra though $W(2) = 0$. In other words, the dimension cost for realizing a matrix type action of $G = \mathbb{Z}_2$ on some noncommutative torus is 2 though $W(2) = 0$.

Definition 3.7. For a cyclic group $\mathbb{Z}_n$, the function W is given by

$$W(\mathbb{Z}_n) := \begin{cases} W(n), & n \neq 2, \\ 2, & n = 2. \end{cases}$$

Definition 3.8. Let G be a finite abelian group. By the structure theorem for finite abelian groups, we have

$$G \cong \prod_{i=1}^{s} \mathbb{Z}_{p_i^{e_i}},$$

where $p_1 \leq \cdots \leq p_s$ are primes and $e_i \leq e_{i+1}$ whenever $p_i = p_{i+1}$.

The function W of G is given by

$$W(G) := \min\biggl\{ \sum_{m=1}^{l} W(\mathbb{Z}_{n_m}) \biggm| G \cong \prod_{m=1}^{l} \mathbb{Z}_{n_m} \text{ for some } l \biggr\}.$$

Theorem 3.9. *For a given dimension d and a finite abelian subgroup G of $\mathrm{GL}_d(\mathbb{Z})$ with $d - W(G) > 1$ or $d - W(G) = 0$, there is a matrix type action of G on a simple noncommutative d-torus $\mathcal{A}_\Theta$. Considering the matrix type action we realized, the crossed product $\mathcal{A}_\Theta \rtimes G$ is an AT algebra. It is an AF algebra if and only if $d - W(G) = 0$ and, if we write G as $G \cong \prod_{m=1}^{t} \mathbb{Z}_{n_m}$ such that*

$$W(G) = \sum_{m=1}^{t} W(\mathbb{Z}_{n_m}),$$

each n_m is then of form 3.2.

Proof. By definition we can write G as $G \cong \prod_{m=1}^{t} \mathbb{Z}_{n_m}$ so that

$$W(G) = \sum_{m=1}^{t} W(\mathbb{Z}_{n_m}).$$

Since $d - W(\mathbb{Z}_{n_m}) > 1$ or $d - W(\mathbb{Z}_{n_m}) = 0$ for all m, it follows from Theorem 3.1 that for each m, there is a simple noncommutative torus $\mathcal{A}_{\Theta_m}$ that admits a matrix type action of $\mathbb{Z}_{n_m}$. Note that $\mathcal{A}_{\Theta_m}$ is necessarily chosen to be an irrational rotation algebra if $n_m = 2$. Then by the tensor product argument, similar in Section 3, we obtain a simple noncommutative d-torus $\mathcal{A}_\Theta$ and a matrix type action of G on it.

The rest of the proof is similar to that of Theorem 3.1. □

Corollary 3.10. *The only matrix type actions of finite abelian group on irrational rotation algebras are by cyclic groups $\mathbb{Z}_k$, where $k = 2, 3, 4$ and 6.*

Proof. Note that in the current case the dimension d is 2. Let G be a finite abelian group. Suppose that there is a matrix type action of G on an irrational rotation algebra. By definition, the value of the function W is always greater than or equal to 2. Thus, $W(G) = 2$ holds according to Theorem 3.9. Again, by the definition of the function, W we have $G = \mathbb{Z}_k$ where $k = 2, 3, 4$ and 6. □

4. The case $d - W(n) = 1$ and actions on odd-dimensional noncommutative tori

Let us study the remaining problem of the last section, that is, the case in which the given dimension d and the order $n = \prod_{i=1}^{t} p_i^{e_i} > 2$ with primes $p_1 < \cdots < p_t$ are such that $d - W(n) = 1$. We start our discussion again from a characterization of a matrix $A \in \mathrm{GL}_d(\mathbb{Z})$ that is of order n. Eventually we will show that there is a matrix $Q \in \mathrm{GL}_d(\mathbb{Q})$ such that

$$\Lambda := QAQ^{-1} = \begin{pmatrix} A_1 & 0 & \cdots & 0 & 0 \\ 0 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_t & 0 \\ 0 & 0 & \cdots & 0 & \pm 1 \end{pmatrix}, \tag{3}$$

for some matrices $A_1, A_2, \ldots, A_t$.

Proposition 4.1. *Let B be a matrix in $\mathrm{GL}_d(\mathbb{Z})$. Suppose $n = p^l$ with the prime power $p^l \neq 2$ or $n = 2p^l$ with the prime $p \neq 2$. If the minimal polynomial of B is $\Phi_n(x)$, then the size of the matrix B cannot be $\phi(n) + 1$.*

Proof. Suppose $n = p^l$ with $p \neq 2$. Since $\Phi_{p^l}(x)$ is an irreducible factor of $x^{p^l} - 1$ (over $\mathbb{Q}$), we have $B^{p^l} = I$ and this is true for no lower power of B. Hence, the order of B is p^l. There exist divisors n_i of p^l for $i = 1, 2, \ldots, s$, with $\mathrm{lcm}(n_1, n_2, \ldots, n_s) = p^l$, such that B is similar over $\mathbb{Q}$ to a matrix of the form

$$\begin{pmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_s \end{pmatrix},$$

where the minimal polynomial for B_i is $\Phi_{n_i}(x)$ for $i = 1, 2, \ldots, s$ (see the proof of [5, Theorem 2.7]). Suppose the size of B is $\phi(p^l) + 1$. Since $p^l \neq 2$, we have the inequalities

$$W(p^l) = \phi(p^l) \leq \sum_{i=1}^{s} \phi(n_i) \leq \phi(p^l) + 1,$$

according to [5, Lemma 2.8]. Because $\mathrm{lcm}(n_1, n_2, \ldots, n_s) = p^l$, there must exist some i_0 such that $p^l = n_{i_0}$ and hence $\phi(p^l) = \phi(n_{i_0})$. By definition, $\phi(p^m) > 2$

for any positive integer m, since $p \neq 2$. The inequalities then compel B to be similar over $\mathbb{Q}$ to a matrix of the form

$$\begin{pmatrix} B_1 & 0 \\ 0 & 1 \end{pmatrix}.$$

But this is impossible since the minimal polynomial of the above matrix is not $\Phi_{p^l}(x)$ but $\Phi_{p^l}(x)(x-1)$.

The rest of the proof follows by the same argument, considering the fact that $\phi(2) = 1$, $\Phi_2(x) = x+1$ and -1 is not a root of $\Phi_n(x)$, where $n = 2^l$ with $l \neq 1$ or $n = 2p^l$ with the prime $p \neq 2$. □

Now we are ready to prove the characterization (3) of the matrix A.

Proposition 4.2. *Let the positive integers d and $n = \prod_{i=1}^t p_i^{e_i} > 2$ with primes $p_1 < \cdots < p_t$ be such that $d - W(n) = 1$. Then a matrix $A \in \mathrm{GL}_d(\mathbb{Z})$ that is of order n is of the form* (3) *for some $Q \in \mathrm{GL}_d(\mathbb{Q})$.*

Proof. Similarly, to the argument in the proof of Proposition 4.1, there exist divisors n_i of n for $i = 1, 2, \ldots, s$, with $\mathrm{lcm}(n_1, n_2, \ldots, n_s) = n$, such that A is similar over $\mathbb{Q}$ to a matrix of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix},$$

where the minimal polynomial for A_i is $\Phi_{n_i}(x)$ for $i = 1, 2, \ldots, s$ (see the proof of [5, Theorem 2.7]). According to [5, Lemma 2.8], we have the inequalities

$$W(n) \leq \sum_{i=1}^{s} \phi(n_i) \leq d = W(n) + 1.$$

For each i, there exists j such that $p_i^{e_i}$ appears in the prime factorization of n_j and hence $\phi(p_i^{e_i}) \leq \phi(n_j)$. Hence, the inequalities force that:

(1) If $p_i \neq 2$, then $\phi(p_i^{e_i}) = \phi(n_j)$ and $n_j = p_i^{e_i}$ or $2p_i^{e_i}$.
(2) If $p_i = 2$ and $e_i \neq 1$, then $\phi(p_i^{e_i}) = \phi(n_j)$ and $n_j = p_i^{e_i} = 2^{e_i}$.

Then the proof is completed by the following case study:

(1) If $p_1 \neq 2$, then $s = t+1$. Renumbering if necessary, we have $n_i = p_i^{e_i}$ for $i = 1, 2, \ldots, t$ and $n_s = 1$.
(2) If $p_1 = 2$ with $e_1 \neq 1$, then $s = t+1$. Renumbering if necessary, we have $n_s = 1$ or 2 and $A_s = \pm 1$.
(3) If $p_1^{e_1} = 2$, then $s = t$. There are two possibilities. If there exists some i such that $n_i = 2p_i^{e_i}$, then we have $n_s = 1$ or 2 and $A_s = \pm 1$. Otherwise $n_s = 2$ and $A_s = -1$.

We draw the conclusion with the help of Proposition 4.1. □

Definition 4.3 ([3, Notation 4.1]). For a matrix $A \in \mathrm{GL}_d(\mathbb{Z})$, define

$$\mathcal{T}_{d,A}(\mathbb{R}) = \{\Theta \in \mathcal{T}_d(\mathbb{R}) \mid \Theta = A^t \Theta A\}.$$

$\mathcal{T}_{d,A}(\mathbb{R})$ is then the set of all skew symmetric matrices Θ such that the noncommutative torus $\mathcal{A}_\Theta$ admits the matrix type action of the group $\langle A \rangle$.

It is straight-forward to check that $\mathcal{T}_{d,A}(\mathbb{R}) = \mathcal{T}_{d,Q^{-1}\Lambda Q}(\mathbb{R}) = Q^t \mathcal{T}_{d,\Lambda}(\mathbb{R}) Q$. The following proposition characterizes the set $\mathcal{T}_{d,\Lambda}(\mathbb{R})$.

Proposition 4.4. *The matrix $\Theta \in \mathcal{T}_{d,\Lambda}(\mathbb{R})$ has the following form:*

$$\Theta = \begin{pmatrix} \Theta' & 0 \\ 0 & 0 \end{pmatrix}$$

for some $\Theta' \in \mathcal{T}_{d-1}(\mathbb{R})$.

Proof. Since Λ is of the form shown in (3), we write Θ as a $(t+1) \times (t+1)$ block matrix as follows:

$$\Theta = \begin{pmatrix} \Theta_{11} & \Theta_{12} & \cdots & \Theta_{1t} & \theta_1 \\ \Theta_{21} & \Theta_{22} & \cdots & \Theta_{2t} & \theta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Theta_{t1} & \Theta_{t2} & \cdots & \Theta_{tt} & \theta_t \\ -\theta_1 & -\theta_2 & \cdots & -\theta_t & 0 \end{pmatrix}.$$

Note that θ_i is a d_i-dimensional vector. By $\Lambda^t \Theta \Lambda = \Theta$, we obtain

$$A_i^t \theta_i = \pm \theta_i, \quad i = 1, 2, \ldots, t.$$

The construction of each A_i shows that A_i does not have an eigenvalue ± 1, hence $\theta_i = 0$. That is to say Θ is of the following form:

$$\Theta = \begin{pmatrix} \Theta' & 0 \\ 0 & 0 \end{pmatrix}$$

for some $\Theta' \in \mathcal{T}_{d-1}(\mathbb{R})$. □

We are now ready to prove the main theorem of this section.

Theorem 4.5. *For a given dimension d and an order $n > 2$ with $d - W(n) = 1$, there is no matrix type action of $\mathbb{Z}_n$ on a simple noncommutative torus.*

Proof. Let A denote a generator of $\mathbb{Z}_n$. Then there is a matrix $Q \in \mathrm{GL}_d(\mathbb{Q})$ such that (3) holds. For any matrix type action of $\mathbb{Z}_n$ on a noncommutative d-torus $\mathcal{A}_\Theta$, we have $\Theta \in \mathcal{T}_{d,A}(\mathbb{R}) = \mathcal{T}_{d,Q^{-1}\Lambda Q}(\mathbb{R}) = Q^t \mathcal{T}_{d,\Lambda}(\mathbb{R}) Q$. Therefore, according to Proposition 4.4, there exists some $\Theta' \in \mathcal{T}_{d-1}(\mathbb{R})$ such that

$$Q^t \Theta Q = \begin{pmatrix} \Theta' & 0 \\ 0 & 0 \end{pmatrix}.$$

The right-hand side is apparently degenerate. It then follows from [10, Lemma 1.8] that Θ is degenerate as well. Thus, the noncommutative torus $\mathcal{A}_\Theta$ is not simple. □

Corollary 4.6. *For a given dimension d and order n, there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus $\mathcal{A}_\Theta$ if and only if either $d - W(n) > 1$ or $d - W(n) = 0$.*

Corollary 4.7. *For a given dimension $d > 3$ which is odd and an order n, there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus if and only if there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative $(d-3)$-torus.*

Proof. Notice that $d - W(n)$ is an odd integer. Then, by Corollary 4.6 and Corollary 3.4, we draw the conclusion. □

Note that Theorem 4.5 is a generalization of [3, Theorem 5.2], shown by the following example.

Example 4.8. Consider a matrix type action of $\mathbb{Z}_n$ on a simple 3-dimensional noncommutative torus. It follows from Theorem 4.5 and the fact that $W(n)$ is always even that $d - W(n) = 3$. Thus, $n = 2$ necessarily holds. In other words, the only matrix type action of a nontrivial finite cyclic group on a simple 3-dimensional torus is the flip action by $\mathbb{Z}_2$ as stated in [3, Theorem 5.2].

5. General matrix type actions of cyclic groups on simple higher dimensional noncommutative tori

In this section we give some partial results on general matrix type actions. As discussed in Section 3, the function W surely describes our special realization of matrix type actions of finite abelian groups quite well. But for general ones, it is limited. For instance, suppose $n = 2^k p$ with $k \geq 3$ and the prime number $p > 5$. Let $d = \phi(n)$. It follows from the definition that $d = \phi(n) = 2^{k-1}(p-1) > W(n) + 1 = 2^{k-1} + (p-1) + 1$. According to Theorem 3.1 and Theorem 3.3, the crossed product of the matrix type action of $\mathbb{Z}_n$ on the simple noncommutative d-torus $\mathcal{A}_\Theta$ we realized is not an AF algebra. However, we can apply the method of Jeong and Lee directly and obtain another matrix type action of $\mathbb{Z}_n = \langle C_n \rangle$ on some simple noncommutative d-torus. As a consequence of Theorem 2.12, this time the resulting crossed product is an AF algebra. This is because the generator matrix of $\mathbb{Z}_n$ becomes different.

Although it is limited, the function W still indicates the least dimension to realize a matrix type action of some finite abelian group. Let us now remove the conditions for d and n in the last paragraph and suppose that there is a general matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus $\mathcal{A}_\Theta$ for some Θ with $d = W(n)$. Let A denote the matrix in $\mathrm{GL}_d(\mathbb{Z})$ that generates $\mathbb{Z}_n$. There exist divisors n_i of n for $i = 1, 2, \ldots, s$, with $\mathrm{lcm}(n_1, n_2, \ldots, n_s) = n$, and a matrix $Q \in \mathrm{GL}_d(\mathbb{Q})$ such that

$$QAQ^{-1} = \Lambda = \begin{pmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_s \end{pmatrix}, \tag{4}$$

where the minimal polynomial for A_i is $\Phi_{n_i}(x)$ for $i = 1, 2, \ldots, s$ (see the proof of [5, Theorem 2.7]). If we further assume that the matrix Q is in $\mathrm{GL}_d(\mathbb{Z})$, then the following theorem holds. It shows that the characterization of the crossed product of this general matrix type action is similar to that of the action we realized in Chapter 3.

Theorem 5.1. *Suppose there is a matrix type action of $\mathbb{Z}_n$ on a simple noncommutative d-torus $\mathcal{A}_\Theta$ with $d - W(n) = 0$. Assume that there exists a matrix $Q \in \mathrm{GL}_d(\mathbb{Z})$ such that* (4) *holds. Then the crossed product $\mathcal{A}_\Theta \rtimes \mathbb{Z}_n$ is an AT algebra. It is an AF algebra if and only if each n_i in* (4) *is either even,* 3*,* 5*, or* 9*.*

Proof. According to [5, Lemma 2.8], we have the inequalities

$$W(n) \le \sum_{i=1}^{s} \phi(n_i) \le d = W(n).$$

Let $n = \prod_{i=1}^{t} p_i^{e_i}$ be the prime factorization of n. The above equality forces that either of the following is true:

(1) if $p_1^{e_1} \neq 2$, renumbering if necessary, we have $s = t$ and $n_i = p_i^{e_i}$ or $2p_i^{e_i}$ for $i = 1, 2, \ldots, t$,
(2) if $p_1^{e_1} = 2$, renumbering if necessary, we have $s = t - 1$ and $n_i = p_{i+1}^{e_{i+1}}$ or $n_i = 2p_{i+1}^{e_{i+1}}$.

It also forces the size of A_i to be exactly $\phi(n_i)$ for each i, which indicates that the characteristic polynomial of A_i coincides its minimal polynomial $\Phi_{n_i}(x)$. The eigenvalues of each A_i and hence of A are roots of $\Phi_{n_i}(x)$, none of which are integers, since $n_i \neq 1$ or 2 for all i. This fact enables us to mimic the proof of [3, Theorem 4.2] to find for each i a nondegenerate Θ_i such that there is a matrix type action of $\langle A_i \rangle$ on the simple noncommutative torus $\mathcal{A}_{\Theta_i}$. Inspired by the discussion in Section 3, we consider the product action of $\mathbb{Z}_n = \langle \Lambda \rangle$ on $\bigotimes_{i=1}^{s} \mathcal{A}_{\Theta_i}$. Let $(\bigotimes_{i=1}^{s} \mathcal{A}_{\Theta_i}) \rtimes_\Lambda \mathbb{Z}_n$ denote the resulting crossed product. The Künneth formula gives

$$K_*\biggl(\biggl(\bigotimes_{i=1}^{s} \mathcal{A}_{\Theta_i}\biggr) \rtimes_\Lambda \mathbb{Z}_n\biggr) \cong \bigotimes_{i=1}^{s} K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{n_i}).$$

As mentioned in [3, Remark 3.2], it follows from [2, Theorem 0.3] that

$$K_i\biggl(\biggl(\bigotimes_{i=1}^{s} \mathcal{A}_{\Theta_i}\biggr) \rtimes_\Lambda \mathbb{Z}_n\biggr) \cong K_i(C^*(\mathbb{Z}^d \rtimes_\Lambda \mathbb{Z}_n)), \quad i = 0, 1.$$

Note that we denote by $\mathbb{Z}^d \rtimes_\Lambda \mathbb{Z}_n$ the semi-direct product of the conjugation action of $\mathbb{Z}_n$ on $\mathbb{Z}^d$ determined by Λ. Similarly, we also have

$$K_i(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) \cong K_i(C^*(\mathbb{Z}^d \rtimes_A \mathbb{Z}_n)), \quad i = 0, 1.$$

Since there exists a matrix $Q \in \mathrm{GL}_d(\mathbb{Z})$ such that (4) holds, the semi-direct product $\mathbb{Z}^d \rtimes_A \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}^d \rtimes_\Lambda \mathbb{Z}_n$. Therefore, we can apply the Künneth formula to compute $K_i(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n)$, that is, we have

$$K_*(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) \cong \bigotimes_{i=1}^{s} K_*(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{n_i}).$$

Note that the conjugation action of $\mathbb{Z}_{n_i}$ on $\mathbb{Z}^{\phi(n_i)}$ is free outside of origin. This guarantees us to apply Theorem 2.12. The fact that $K_1(\mathcal{A}_\Theta \rtimes \mathbb{Z}_n) = 0$ if and only if $K_1(\mathcal{A}_{\Theta_i} \rtimes \mathbb{Z}_{n_i}) = 0$ for each i then concludes the proof. □

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