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Mahler measures and Fuglede-Kadison determinants

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Dedicated to Joachim Cuntz on the occasion of his 60th birthday

Abstract. The Mahler measure of a function on the real d-torus is its geometric mean over the torus. It appears in number theory, ergodic theory and other fields. The Fuglede–Kadison determinant is defined in the context of von Neumann algebra theory and can be seen as a noncommutative generalization of the Mahler measure. In the paper we discuss and compare theorems in both fields, especially approximation theorems by finite dimensional determinants. We also explain how to view Fuglede–Kadison determinants as continuous functions on the space of marked groups.

1. Introduction

For an essentially bounded complex valued measurable function P on the real d-torus $T^d = S^1 \times \ldots \times S^1$ the Mahler measure is defined by the formula $M(P) = \exp m(P) \geq 0$ where $m(P)$ is the integral

$$m(P) = \int_{T^d} \log |P| \, d\mu \quad \text{in} \quad \mathbb{R} \cup \{-\infty\}.$$

Here μ is the Haar probability measure on T^d. If P is a Laurent polynomial on T^d for example, it is known that $\log |P|$ is integrable on T^d unless $P = 0$, so that we have $M(P) > 0$ for $P \neq 0$ and $M(P) = 0$ for $P = 0$.

The Mahler measure appears in many branches of mathematics. It is especially interesting for polynomials with coefficients in $\mathbb{Z}$. If α is an algebraic integer with monic minimal polynomial P over $\mathbb{Q}$ then $m(P)$ is the normalized Weil height of α. This follows from an application of Jensen's formula. For polynomials in several variables there is no closed formula evaluating $m(P)$ but sometimes $m(P)$ can be expressed in terms of special values of L-functions, see e.g. [2], [14] and their references.

The logarithmic Mahler measure $m(P)$ of a Laurent polynomial P over $\mathbb{Z}$ also appears in ergodic theory as the entropy of a certain subshift defined by P

of the full shift for $\mathbb{Z}^d$ with values in the circle cp. [22] and [26]. For relations of $m(P)$ with hyperbolic volumes we refer to [4].

We now turn our attention to the determinants in the title.

Let $\mathcal{N}$ be a finite von Neumann algebra with a faithful normal finite trace τ. In this note we only need the von Neumann algebra $\mathcal{N}\Gamma$ of a discrete group Γ which is easy to define, cp. Section 2. For an invertible operator A in $\mathcal{N}$ the Fuglede-Kadison determinant [12] is defined by the formula

$$\det_{\mathcal{N}} A = \exp \tau(\log |A|).$$

Here $|A| = (A^* A)^{1/2}$ and $\log |A|$ are operators in $\mathcal{N}$ obtained by the functional calculus. For arbitrary operators A in $\mathcal{N}$ one sets

$$\det_{\mathcal{N}} A = \lim_{\varepsilon \to 0+} \det_{\mathcal{N}}(|A| + \varepsilon).$$

The main result in [12] asserts that $\det_{\mathcal{N}}$ is *multiplicative* on $\mathcal{N}$. This determinant has several interesting applications. It appears in the definitions of analytic and combinatorial L^2-torsion of Laplacians on covering spaces [19], [25] and [7], [21]. It was used in the work [13] on the invariant subspace problem in II_1-factors and it is related to the entropy of algebraic actions of discrete amenable groups [9], [10] and to Lyapunov exponents [8].

It was observed in [24, Example 3.13] that the Mahler measure has the following functional analytic interpretation. For the group $\Gamma = \mathbb{Z}^d$ there is a canonical isomorphism of $\mathcal{N}\Gamma$ with $L^\infty(T^d, \mu)$ which we write as $A \mapsto \hat{A}$. The relation of Mahler measures with Fuglede-Kadison determinants is then given by the formula:

$$\det_{\mathcal{N}\mathbb{Z}^d}(A) = M(\hat{A}) \text{ for all } A \in \mathcal{N}\mathbb{Z}^d. \tag{1}$$

In this note we review certain classical properties of Mahler measures and discuss their generalizations to Fuglede-Kadison determinants of group von Neumann algebras. In particular, this concerns approximation formulas e.g. by finite dimensional determinants. Usually the results for Mahler measures are stronger than the corresponding ones for general Fuglede-Kadison determinants and this raises interesting questions. In Section 2 we also extend part of the formalism of the theory of orthogonal polynomials on the unit circle to a noncommutative context. Moreover, in Section 3 we show that in a suitable sense $\det_{\mathcal{N}\Gamma}$ is continuous on the space of marked groups if the argument is invertible in L^1.

2. Approximation by finite dimensional determinants

In this section we discuss one way to approximate Mahler measures and more generally Fuglede-Kadison determinants of amenable groups by finite dimensional determinants. Another method which works for residually finite groups is explained in the next section as a special case of Theorem 17.

The Mahler measure aspect of this topic begins with Szegö's paper [29]. For an integrable function P on S^1 consider the Fourier coefficients

$$c_\nu = \int_{S^1} z^{-\nu} P(z)\, d\mu(z) \text{ for } \nu \in \mathbb{Z}$$

and define the following determinants for $n \geq 0$

$$D_n = \det \begin{pmatrix} c_0 & c_1 \cdots\cdots c_{n-1} \\ c_{-1} & c_0 \cdots\cdots c_{n-2} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ c_{-n+1} & c_{-n+2} \cdots\cdots c_0 \end{pmatrix}.$$

If P is real valued we have $\overline{c}_\nu = c_{-\nu}$ and if $P(z) \geq 0$ for all $z \in S^1$ we may view $c_{-\nu}$ as the ν-th moment of the measure $P(z)\, d\mu(z)$. In that case the D_n's are the associated Toeplitz determinants.

Theorem 1 (Szegö)**.** *If P is a continuous real valued function on S^1 with $P(z) > 0$ for all $z \in S^1$, then $D_n > 0$ for all $n \geq 0$ and we have the limit formula:*

$$M(P) = \lim_{n\to\infty} \sqrt[n]{D_n}.$$

Using the theory of orthogonal polynomials on the unit circle the conditions in Szegö's original theorem have been significantly relaxed, see [28] for the history:

Theorem 2. *The assertions in Szegö's theorem hold for every real-valued non-negative essentially bounded measurable function P on S^1 which is nonzero on a set of positive measure.*

Proof. The D_n's are determinants of Toeplitz matrices for the nontrivial measure $P\, d\mu$. These matrices are positive definite and in particular $D_n > 0$ for every $n \geq 0$, cp. [28, Section 1.3.2]. The limit formula $M(P) = \lim_{n\to\infty} \sqrt[n]{D_n}$ is a special case of [28, Theorem 2.7.14], equality of (i) with (vi) applied to the probability measure $P\|P\|_1^{-1}\, d\mu$ on S^1. (In following that proof, the shortcut in the remark on p. 139 of loc. cit. is useful.) □

Let us now explain the von Neumann aspect of these results. For a discrete group Γ we will view the elements of $L^p(\Gamma)$ as formal series $\sum_{\gamma\in\Gamma} x_\gamma \gamma$ with $\sum_\gamma |x_\gamma|^p < \infty$. It is then clear that Γ acts isometrically by left and right multiplication on $L^p(\Gamma)$. The von Neumann algebra $\mathcal{N}\Gamma$ of Γ may be defined as the algebra of bounded operators $A : L^2\Gamma \to L^2\Gamma$ which are left Γ-equivariant. For $\gamma \in \Gamma$ define the unitary operator $R_\gamma : L^2\Gamma \to L^2\Gamma$ by $R\gamma(x) = x\gamma$. The $\mathbb{C}$-algebra homorphism

$$r : \mathbb{C}\Gamma \to \mathcal{N}\Gamma \quad \text{with} \quad r\Big(\sum_\gamma f_\gamma \gamma\Big) = \sum_\gamma f_\gamma R_{\gamma^{-1}}$$

extends to a homomorphism $r : L^1(\Gamma) \to \mathcal{N}\Gamma$ with $\|r(f)\| \leq \|f\|_1$ for all $f \in L^1\Gamma$. By looking at $r(f)(e)$ where $e \in \Gamma \subset L^2\Gamma$ is the unit element of Γ, we see that r is injective. It will often be viewed as an inclusion in the following.

Setting $f^* = \sum \overline{f}_\gamma \gamma^{-1}$ for $f = \sum f_\gamma \gamma$ in $L^1\Gamma$ the equality $r(f^*) = r(f)^*$ holds. The canonical trace $\tau = \tau_{\mathcal{N}\Gamma}$ on $\mathcal{N}\Gamma$ is defined by the formula $\tau(A) = (Ae, e)$. It vanishes on commutators $[A, B] = AB - BA$ for A, B in $\mathcal{N}\Gamma$. For f in $L^1\Gamma$ we have $\tau(r(f)) = f_e$. Finally, $\det_{\mathcal{N}\Gamma} A$ is defined as in the introduction for every A in $\mathcal{N}\Gamma$.

For an abelian group Γ with (compact) Pontryagin dual $\hat{\Gamma} = \mathrm{Hom}_{\mathrm{cont}}(\Gamma, S^1)$ and Haar probability measure μ on $\hat{\Gamma}$, the Fourier transform provides an isometry of Hilbert spaces

$$\mathcal{F} : L^2\Gamma \xrightarrow{\sim} L^2(\hat{\Gamma}, \mu).$$

On the dense subspace $\mathbb{C}\Gamma$ it is given by $\mathcal{F}(f)(\chi) = \sum_\gamma f_\gamma \chi(\gamma)$ for $\chi \in \hat{\Gamma}$. One can show that under the induced isomorphism of algebras of bounded operators

$$\mathcal{B}(L^2(\Gamma)) \to \mathcal{B}(L^2(\hat{\Gamma}, \mu)), \quad A \mapsto \mathcal{F} \circ A \circ \mathcal{F}^{-1}$$

the von Neumann algebra $\mathcal{N}\Gamma$ maps isomorphically onto $L^\infty(\hat{\Gamma}, \mu)$ where the latter operates by multiplication on $L^2(\hat{\Gamma}, \mu)$. Denoting this isomorphism by $A \mapsto \hat{A}$ we have $\hat{A} = \mathcal{F}(A(e))$. Namely, $L^2(\Gamma)$ is a left $\mathbb{C}\Gamma$-algebra and for $f \in \mathbb{C}\Gamma$ we therefore have

$$\mathcal{F}(A(f)) = \mathcal{F}(fA(e)) = \mathcal{F}(f)\mathcal{F}(A(e)).$$

Now the assertion follows because $\mathcal{F}(\mathbb{C}\Gamma)$ is dense in $L^2(\hat{\Gamma}, \mu)$. It follows that we have

$$\tau(A) = (Ae, e) = (\mathcal{F}(A(e)), \mathcal{F}(e)) = (\hat{A}, 1) = \int_{\hat{\Gamma}} \hat{A} \, d\mu.$$

Hence there is a commutative diagram

$$\begin{array}{ccc} \mathcal{N}\Gamma & \xrightarrow{\hat{\ }\ \sim} & L^\infty(\hat{\Gamma}, \mu) \\ & {}_{\tau}\searrow \quad \swarrow_{\int_{\hat{\Gamma}}} & \\ & \mathbb{C} & \end{array}$$

where $\int_{\hat{\Gamma}}$ denotes integration against the measure μ. We conclude using the definition of $\det_{\mathcal{N}\Gamma}$ and Levi's theorem that we have:

$$\det_{\mathcal{N}\Gamma} A = \exp \int_{\hat{\Gamma}} \log |\hat{A}| \, d\mu \quad \text{for} \quad A \in \mathcal{N}\Gamma.$$

In particular, for $\Gamma = \mathbb{Z}^d$ we get formula (1) from the introduction. It also follows that the generalized Mahler measures studied in [17] can be expressed as Fuglede-Kadison determinants.

The noncommutative generalization of Szegö's theorem that we have in mind is valid for amenable groups. A Følner sequence (F_n) in Γ is a sequence of finite subsets $F_n \subset \Gamma$ which are almost invariant in the following sense: For any $\gamma \in \Gamma$ we have

$$\lim_{n \to \infty} \frac{|F_n \gamma \setminus F_n|}{|F_n|} = 0.$$

A countable discrete group Γ is said to be amenable if it has a Følner sequence. For example $\mathbb{Z}$ is amenable, the sets $F_n = \{0, 1, \dots, n-1\}$ forming a Følner sequence.

For a finite subset $F \subset \Gamma$ and an operator $A \in \mathcal{N}\Gamma$ consider the following endomorphism of $\mathbb{C}F$, the finite-dimensional $\mathbb{C}$-vector space over F:

$$A_F : \mathbb{C}F \xhookrightarrow{i_F} L^2\Gamma \xrightarrow{A} L^2\Gamma \xrightarrow{p_F} \mathbb{C}F.$$

Here i_F is the inclusion and p_F the orthogonal projection to $\mathbb{C}F$. We have $p_F^* = i_F$ for the L^2-adjoints and hence $(A_F)^* = (A^*)_F$.

Lemma 3. *If $A \in \mathcal{N}\Gamma$ is positive then A_F is positive as well and hence* $\det A_F \geq 0$*. If A is positive, and injective on $\mathbb{C}\Gamma$ then A_F is a positive automorphism of $\mathbb{C}F$ and hence* $\det A_F > 0$*.*

Proof. Set $B = \sqrt{A}$. For $v \in \mathbb{C}F$ we have $(A_F v, v) = (Av, v) = \|Bv\|^2$ and hence A_F is positive. Moreover $A_F v = 0$ implies $Bv = 0$ and hence $Av = B(Bv) = 0$. If A is injective on $\mathbb{C}\Gamma$ we get $v = 0$ and thus A_F is injective and hence an automorphism. □

The approximation result corresponding to Szegö's theorem is the following one which was proved in [9, Theorem 3.2]:

Theorem 4. *Let Γ be a finitely generated amenable group with a Følner sequence (F_n) and let A be a positive invertible operator in $\mathcal{N}\Gamma$. Then* $\det A_{F_n} > 0$ *for all n and we have:*

$$\det{}_{\mathcal{N}\Gamma} A = \lim_{n\to\infty} (\det A_{F_n})^{1/|F_n|}.$$

Positivity of $\det A_{F_n}$ follows from Lemma 3. The proof of the limit formula is based on an approximation result for traces of polynomials in A due to Schick [27] generalizing previous work of Lück. Theorem 4 follows by applying the Weierstraß approximation theorem to log and the fact that the spectrum of A and all A_{F_n} is uniformly bounded away from zero.

We would like to point out that another part of Szegö's theory which characterizes $M(P)$ by an extremal property has been generalized to the setting of von Neumann algebras in [1].

Example 5. Let us now show that Szegö's Theorem 1 is a special case of Theorem 4. Consider a measurable essentially bounded function $P : S^1 \to \mathbb{R}$ with $P(z) \geq 0$ for all $z \in S^1$. It defines a positive element P of the von Neumann algebra $L^\infty(S^1, \mu)$. Let A be the positive operator in $\mathcal{N}\mathbb{Z}$ with $\hat{A} = P$, i.e. with $\mathcal{F}(A(0)) = P$. For $\nu \in \mathbb{Z} \subset L^2(\mathbb{Z})$ write (ν) for its image in $L^2(\mathbb{Z})$. Then we have $\mathcal{F}(\nu) = z^\nu$ viewed as a character on S^1. Thus

$$c_\nu = \int_{S^1} z^{-\nu} P(z)\, d\mu(z) = (P, z^\nu) = (\mathcal{F}(A(0)), \mathcal{F}(\nu)) = (A(0), (\nu)).$$

Now consider the Følner sequence $F_n = \{0, 1, \dots, n-1\}$ of $\mathbb{Z}$. The matrix of A_{F_n} with respect to the basis $(0), (1), \dots, (n-1)$ of $\mathbb{C}F_n$ has (i, j)-th coefficient

$$(A_{F_n}(i), (j)) = (A(i), (j)) = (A(0), (j-i)) = c_{j-i}.$$

Thus we have $\det A_{F_n} = D_n$ and therefore Theorem 4 implies Theorem 1 (even with "continuous" replaced by "measurable essentially bounded").

Note that using other Følner sequences for $\mathbb{Z}$, Theorem 4 gives new limit formulas for the Mahler measure not covered by Szegö's theorem.

The analogue of Theorem 2 in our setting does not seem to be known. We formulate it as a question:

Question 6. *Let Γ be a finitely generated amenable group and A a positive operator in $\mathcal{N}\Gamma$. Does the limit formula*

$$\det_{\mathcal{N}\Gamma} A = \lim_{n\to\infty} (\det A_{F_n})^{1/|F_n|}$$

hold for every Følner sequence?

Remarks 1) In Theorem 2 the nonzero positive operators in $\mathcal{N}\mathbb{Z} \cong L^\infty(S^1, \mu)$ were considered. These are injective on $\mathbb{C}\mathbb{Z}$ because $\mathcal{F}(\mathbb{C}\mathbb{Z}) = \mathbb{C}[z, z^{-1}]$, and nonzero Laurent polynomials vanish only in a set of measure zero on S^1. Perhaps it is reasonable therefore to first consider only positive operators which are injective on $\mathbb{C}\Gamma$ so that by Lemma 3 all A_{F_n} are positive automorphisms. On the other hand, for $A = 0$ the limit formula is trivially true.
2) Because of the next proposition it would suffice to prove the inequality

$$\det_{\mathcal{N}\Gamma} A \le \varliminf_{n\to\infty} (\det A_{F_n})^{1/|F_n|}$$

in order to answer Question 5 affirmatively.

Proposition 7. *For a finitely generated group Γ and any positive operator A on $\mathcal{N}\Gamma$ we have*

$$\det_{\mathcal{N}\Gamma} A \ge \varlimsup_{n\to\infty} (\det A_{F_n})^{1/|F_n|}.$$

Proof. For A in $\mathbb{Z}\Gamma$ this is proved in [27]. In general we can argue as follows. For any endomorphism φ set $\varphi^{(\varepsilon)} = \varphi + \varepsilon \mathrm{id}$. Then we have $(A^{(\varepsilon)})_F = (A_F)^{(\varepsilon)}$ for finite $F \subset \Gamma$. The following relations hold:

$$\begin{aligned}\det_{\mathcal{N}\Gamma} A &\overset{(i)}{=} \lim_{\varepsilon\to 0+} \det_{\mathcal{N}\Gamma} A^{(\varepsilon)} \overset{(ii)}{=} \lim_{\varepsilon\to 0+} \lim_{n\to\infty} (\det (A^{(\varepsilon)})_{F_n})^{1/|F_n|} \\ &\overset{(iii)}{\ge} \varlimsup_{n\to\infty} (\det A_{F_n})^{1/|F_n|}.\end{aligned}$$

Here (i) is true by the definition of the Fuglede-Kadison determinant and (ii) follows from Theorem 4 applied to $A^{(\varepsilon)}$. Finally (iii) holds because $\det(A_{F_n})^{(\varepsilon)} \ge \det A_{F_n}$ for every $n \ge 1$ and $\varepsilon > 0$. □

In the rest of this section we develop a formalism for the determinants $\det A_F$ and $\det_{\mathcal{N}\Gamma} A$ which is suggested by the theory of orthogonal polynomials on the unit circle. We also point out the relation to Question 6.

We start with the following well known lemma:

Lemma 8. *For a block matrix over a field with A invertible the following formula holds:*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D - CA^{-1}B) \det A.$$

Proof. We have

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} I & B \\ CA^{-1} & D \end{pmatrix} \det\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \det\begin{pmatrix} I & B \\ 0 & D-CA^{-1}B \end{pmatrix} \det A.$$

□

Consider a countable discrete group Γ and finite subsets $F \subset F' \subset \Gamma$. Let $A \in \mathcal{N}\Gamma$ be positive, and injective on $\mathbb{C}\Gamma$, so that according to Lemma 3 the endomorphism A_F is positive and invertible. In terms of the decomposition

$$\mathbb{C}F' = \mathbb{C}F \oplus \mathbb{C}(F' \setminus F)$$

the endomorphism $A_{F'}$ is given by the block matrix

$$A_{F'} = \begin{pmatrix} A_F & p_F A i_{F'\setminus F} \\ p_{F'\setminus F} A i_F & p_{F'\setminus F} A i_{F'\setminus F} \end{pmatrix}.$$

Thus Lemma 8 gives the formula

$$\det A_{F'} = \det A_F \det(p_{F'\setminus F} A i_{F'\setminus F} - p_{F'\setminus F} A i_F A_F^{-1} p_F A i_{F'\setminus F}). \tag{2}$$

Now consider the endomorphism

$$\psi = i_F A_F^{-1} p_F A i_{F'} : \mathbb{C}F' \to \mathbb{C}F'$$

and the scalar product on $\mathbb{C}\Gamma$ defined by

$$(u, v)_A := (Au, v) = (u, Av).$$

It is positive since for $u, v \in \mathbb{C}\Gamma$ there is a finite subset $F \subset \Gamma$ with $u, v \in \mathbb{C}F$ and then we have $(u, v)_A = (A_F u, v)$ with the positive automorphism A_F.

Proposition 9. *The endomorphism ψ is the orthogonal projection of $\mathbb{C}F'$ to $\mathbb{C}F$ with respect to the scalar product $(,)_A$ on $\mathbb{C}F'$.*

Proof. For $u \in \mathbb{C}F$ we have $\psi(u) = A_F^{-1} A_F u = u$. This implies that $\psi^2 = \psi$ since ψ takes values in $\mathbb{C}F$. Moreover, $\operatorname{Im} \psi = \mathbb{C}F$. Next observe that

$$p_{F'} A \psi = p_{F'} A i_F A_F^{-1} p_F A i_{F'}$$

is selfadjoint since $i_F^* = p_F$ and A, A_F are selfadjoint. Hence we have $p_{F'} A\psi = \psi^* A i_{F'}$ and for $u, v \in \mathbb{C}F'$ therefore:

$$(\psi u, v)_A = (\psi u, Av) = (u, \psi^* Av) = (u, A\psi v) = (u, \psi v)_A.$$

□

By the proposition the endomorphism $\varphi = \mathrm{id} - \psi$ of $\mathbb{C}F'$ is the orthogonal projection to $\mathbb{C}F^{\perp_A}$ with respect to $(,)_A$. Formula (2) can be rewritten as

$$\det A_{F'} = \det A_F \det(p_{F'\setminus F} A \varphi i_{F'\setminus F}).$$

Corollary 10. *Assume that $F' = F \dot\cup \{\gamma\}$ and set $\Phi_\gamma = \varphi(\gamma)$. Then we have $\det A_{F'} = \|\Phi_\gamma\|_A^2 \det A_F$.*

Proof. Since $\mathbb{C}(F' \setminus F) = \mathbb{C}\gamma$ is one-dimensional and $\|\gamma\| = 1$, we have

$$\begin{aligned}\det(p_{F'\setminus F} A\varphi i_{F'\setminus F}) &= (p_{F'\setminus F} A\varphi(\gamma), \gamma) = (A\varphi(\gamma), \gamma) = (\varphi(\gamma), \gamma)_A \\ &= (\varphi^2(\gamma), \gamma)_A = (\varphi(\gamma), \varphi(\gamma))_A = \|\varphi(\gamma)\|_A^2.\end{aligned}$$

□

The corollary generalizes part of formula (1.5.78) of [28]. Using this orthogonalization process inductively we get the formula

$$\det A_F = \prod_{\gamma\in F} \|\Phi_\gamma\|_A^2.$$

Concerning the order of $\|\Phi_\gamma\|_A$ note the following equations

$$\begin{aligned}\|\Phi_\gamma\|_A^2 &= (\varphi(\gamma), \gamma)_A = (\gamma - i_F A_F^{-1} p_F A\gamma, \gamma)_A \\ &= \|\gamma\|_A^2 - (i_F A_F^{-1} p_F A\gamma, A\gamma) \\ &= \tau(A) - (A_F^{-1} s, s), \text{ where } s = p_F A\gamma \in \mathbb{C}F.\end{aligned}$$

In the situation of Corollary 10 we therefore obtain:

Corollary 11. *We have* $0 < \|\Phi_\gamma\|_A^2 \le \tau(A)$. *Moreover the following assertions are equivalent:*
1) $\|\Phi_\gamma\|_A^2 = \tau(A)$
2) $p_F A\gamma = 0$
3) $A_{F'} = \begin{pmatrix} A_F & 0 \\ 0 & c\end{pmatrix}$ *for some c (which must be* $c = \tau(A)$*).*

Now we generalize a calculation from the theory of orthogonal polynomials on S^1 which is used in one of the proofs of Theorem 2. Recall that for $\Phi \in \mathcal{N}\Gamma$ one sets $\|\Phi\|_2 = \tau(\Phi^*\Phi)^{1/2}$. It is known that we have

$$\det{}_{\mathcal{N}\Gamma}\Phi \le \|\Phi\|_2. \tag{3}$$

Namely, let E_λ be the spectral resolution of $|\Phi|$. Then we have by Jensen's inequality:

$$\begin{aligned}(\det{}_{\mathcal{N}\Gamma}\Phi)^2 &= \exp\int_0^\infty \log(|\lambda|^2) d\tau(E_\lambda) \le \int_0^\infty |\lambda|^2\, d\tau(E_\lambda) \\ &= \tau\Big(\int_0^\infty |\lambda|^2 dE_\lambda\Big) = \tau(\Phi^*\Phi) = \|\Phi\|_2^2.\end{aligned}$$

For positive $A \in \mathcal{N}\Gamma$ and any $\Phi \in \mathcal{N}\Gamma$ we find

$$\begin{aligned}(\det{}_{\mathcal{N}\Gamma} A)^{1/2}\det{}_{\mathcal{N}\Gamma}\Phi &= \det{}_{\mathcal{N}\Gamma}(\sqrt{A}\Phi) \le \|\sqrt{A}\Phi\|_2 \\ &= \tau(\Phi^* A\Phi)^{1/2} = (\Phi^* A\Phi e, e)^{1/2} \\ &= (A\Phi(e), \Phi(e))^{1/2} = \|\Phi(e)\|_A.\end{aligned}$$

Let $\sim: \mathbb{C}\Gamma \to \mathbb{C}\Gamma$ be defined by $\tilde{f} = \sum f_\gamma \gamma^{-1}$. Then for $f \in \mathbb{C}\Gamma$ the operator $r(f) \in \mathcal{N}\Gamma$ is right multiplication by $\tilde{f}$. For $f \in \mathbb{C}\Gamma \subset \mathcal{N}\Gamma$ where the inclusion is via r, we get

$$(\det{}_{\mathcal{N}\Gamma} A)^{1/2}\det{}_{\mathcal{N}\Gamma} f \le \|f(e)\|_A = \|\tilde{f}\|_A.$$

Applying this to $f = \tilde{\Phi}_\gamma$ we find

$$(\det_{\mathcal{N}\Gamma} A)^{1/2} \det_{\mathcal{N}\Gamma} \tilde{\Phi}_\gamma \leq \|\Phi_\gamma\|_A.$$

Combining this with Corollary 10 we obtain the following result:

Corollary 12. *Let $A \in \mathcal{N}\Gamma$ be positive, and injective on $\mathbb{C}\Gamma$. Assume that F and $F' = F \dot\cup \{\gamma\}$ are finite subsets of Γ. Then we have the inequality:*

$$(\det_{\mathcal{N}\Gamma} A)(\det_{\mathcal{N}\Gamma} \tilde{\Phi}_\gamma)^2 \leq \frac{\det A_{F'}}{\det A_F}.$$

Remark 13. Consider a function $P : S^1 \to \mathbb{R}$ as in Example 5 and assume that $\int_{S^1} P\, d\mu = 1$. Then $\mu_P = P\, d\mu$ is a probability measure on S^1 and we may consider the orthogonal projection φ_n of $\langle 1, z, \ldots, z^n \rangle$ onto $\langle 1, z, \ldots, z^{n-1} \rangle^\perp$ in $L^2(S^1, \mu_P)$. The monic polynomial of degree n given by $\Phi_n(z) = \varphi_n[z^n]$ is the n-th orthogonal polynomial with respect to μ_P. It is known that all zeroes of $\Phi_n(z)$ lie in the open unit disc. Following [28, p. 102], the shortest argument for this fundamental fact seems to be the following. We write $\| \; \|_P$ for the norm corresponding to the scalar product $(\alpha, \beta)_P := (P\alpha, \beta)_2 = (\alpha, P\beta)_2$ of $L^2(S^1, \mu_P)$. Let $z_0 \in \mathbb{C}$ be a zero of $\Phi_n(z)$. Then we have $zf = \Phi_n + z_0 f$ for a polynomial f of degree $n - 1$. This gives the equation

$$\|f\|_P^2 = \|zf\|_P^2 = \|\Phi_n\|_P^2 + |z_0|^2 \|f\|_P^2,$$

since $f \in \langle 1, z, \ldots, z^{n-1} \rangle$ implies $(\Phi_n, f)_P = 0$. It follows that $|z_0| < 1$.

As a consequence, note that Jensen's formula implies that $M(\Phi_n(z)) = 1$.

For $\Gamma = \mathbb{Z}$ and $F = \{0, \ldots, n-1\}$ and $F' = \{0, \ldots, n\}$ we have $\gamma = n$ in our above notation. Consider the operator $A \in \mathcal{N}\Gamma$ with $\hat{A} = P$ i.e. with $\mathcal{F}(A(0)) = P$. For Φ_γ defined as in Corollary 10, one checks that we have $\mathcal{F}(\Phi_\gamma) = \Phi_n(z)$.

Hence we get

$$\det_{\mathcal{N}\mathbb{Z}} \tilde{\Phi}_\gamma = M(\mathcal{F}(\Phi_\gamma)) = M(\Phi_n(z)) = 1.$$

Thus in this special case the inequality in Corollary 12 gives

$$\det_{\mathcal{N}\Gamma} A \leq \frac{\det A_{F_n}}{\det A_{F_{n-1}}}$$

where $F_n = \{0, \ldots, n\}$. This inequality is instrumental for the proof of Theorem 2. For general Γ, unfortunately we do not know whether $\det_{\mathcal{N}\Gamma} \tilde{\Phi}_\gamma \geq 1$ or even $\det_{\mathcal{N}\Gamma} \tilde{\Phi}_\gamma = 1$ holds under suitable conditions.

3. Approximation on the space of marked groups

According to a theorem of Lawton, Mahler measures of Laurent polynomials in several variables can be approximated by Mahler measures of one-variable Laurent polynomials. His result which we now recall resolved a conjecture of Boyd. For $r \in \mathbb{Z}^d$ set

$$q(r) = \min\{\|\nu\| \mid 0 \neq \nu \in \mathbb{Z}^d \text{ with } (\nu, r) = 0\}$$

where $\|\nu\| = \max |\nu_i|$ and $(\nu, r) = \sum_i \nu_i r_i$.

Theorem 14 (Lawton [15])**.** *For $r \in \mathbb{Z}^d$ and P in $\mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ set $P_r(X) = P(X^{r_1}, \ldots, X^{r_d})$. Then we have*

$$\lim_{q(r)\to\infty} M(P_r) = M(P).$$

If P does not vanish on T^d, so that $\log |P_r|$ is continuous on S^1 the theorem is much simpler to prove than in general. In the following we will generalize this easy case to a statement on the continuity of the Fuglede-Kadison determinant on the space of marked groups. A full generalization of Theorem 14 in this direction is a challenging problem.

For $d \geq 1$ the space X_d of marked groups on d-generators is the set of isomorphism classes $[\Gamma, S]$ of pairs (Γ, S) where Γ is a discrete group and $S = (s_1, \ldots, s_d)$ a family of d generators of Γ. Here repetitions are allowed. Two such pairs (Γ, S) and (Γ', S') are called isomorphic if there is an isomorphism $\alpha : \Gamma \xrightarrow{\sim} \Gamma'$ with $\alpha(S) = S'$. The set X_d becomes an ultra-metric space with the distance function

$$d([\Gamma_1, S_1], [\Gamma_2, S_2]) = 2^{-N}$$

where $N \leq \infty$ is the largest radius such that the balls of radius N around the origin in the Cayley graphs of (Γ_1, S_1) and (Γ_2, S_2) are isomorphic as oriented, labelled graphs with labels $1, \ldots, d$ corresponding to the generators. Thus intuitively two marked groups are close to each other if their Cayley graphs around the origin coincide on a big ball. An equivalent metric on X_d is obtained by setting

$$\delta([\Gamma_1, S_1], [\Gamma_2, S_2]) = 2^{-M}$$

if the bijection $S_1 \cong S_2$ induces a bijection of S_1- resp. S_2-relations of length less than M and if $M \leq \infty$ is maximal with this property. Here an S-relation in a group Γ is an S-word, i.e. a finite string of elements from S and their inverses, whose evalution in Γ is equal to e. The number of elements in the string defining a word is the length of the word e.g. $s_1^{-1} s_2 s_1 s_3^{-1} s_5$ has length 5.

Much more background on the space of marked groups can be found in [6, Section 2], for example.

Example 15. With notations as in Theorem 14 consider

$$(\Gamma, S) = (\mathbb{Z}^d, e_1, \ldots, e_d) \text{ and } (\Gamma_r, S_r) = (D(r)\mathbb{Z}, r_1, \ldots, r_d)$$

where $r \in \mathbb{Z}^d$ and $D(r)$ is the greatest common divisor of $r_1, \ldots, r_d$. Then we have

$$\lim_{q(r)\to\infty} [\Gamma_r, S_r] = [\Gamma, S] \text{ in } X_d.$$

Proof. As Γ_r is abelian, an S_r-word is a relation in Γ_r if and only if $\sum_{i=1}^d \nu_i r_i = 0$ where $\nu_i \in \mathbb{Z}$ is the sum of all exponents ± 1 of r_i in the word. The length of the relation is at least $\|\nu\|$. If a relation $\mathcal{R}$ has length less than $q(r)$ it follows that we have $\nu = 0$ and hence $\mathcal{R}$ is a relation of commutation. Hence for length less than $q(r)$ the relations in (Γ, S) and (Γ_r, S_r) are in canonical bijection. Thus we have

$$\delta([\Gamma, S], [\Gamma_r, S_r]) \leq 2^{-q(r)}$$

and the assertion follows. $\square$

Example 16. Let Γ be a countable group and (K_n) a sequence of normal subgroups of Γ. We write $K_n \to e$ if e is the only element of Γ which is contained in infinitely many K_n's. Equivalently, for any finite subset $Q \subset \Gamma$ we have $K_n \cap Q \subset \{e\}$ for n large enough.
Now assume that Γ is finitely generated and let S be a finite family of generators. Given epimorphisms $\varphi_n : \Gamma \twoheadrightarrow \Gamma_n$ we get finite families of generators S_n in Γ_n. Setting $d = |S|$, we claim that the limit formula

$$\lim_{n\to\infty} [\Gamma_n, S_n] = [\Gamma, S] \text{ in } X_d$$

is equivalent to $K_n \to e$, where $K_n = \operatorname{Ker} \varphi_n$.

Proof. Assume that $K_n \to e$. Let $\mathcal{R}_n$ be a relation of length l in Γ_n and let $\mathcal{R}$ be the corresponding S-word in Γ. The evaluation $\gamma = \operatorname{ev}(\mathcal{R})$ of $\mathcal{R}$ in Γ lies in K_n. Let $Q \subset \Gamma$ be the finite subset of at most l-fold products from $S \cup S^{-1}$. In particular $\gamma \in Q$. For $n \geq n(l)$, we have $K_n \cap Q \subset \{e\}$ since $K_n \to e$. Therefore the relations of length $\leq l$ in Γ and Γ_n are in canonical bijection if $n \geq n(l)$ and hence we have

$$\delta([\Gamma_n, S_n], [\Gamma, S]) \leq 2^{-l} \text{ for } n \geq n(l).$$

For the converse consider an element $\gamma \in \Gamma$ which is contained in infinitely many K_n's. Choose a word $\mathcal{W}$ in Γ with $\gamma = \operatorname{ev}(\mathcal{W})$ and let l be the length of $\mathcal{W}$. By assumption, there are arbitrarily large n's such that $\varphi_n(\mathcal{W})$ is a relation in Γ_n. But for $n \gg 0$ the relations of length l in Γ_n and Γ are in bijection. Hence $\mathcal{W}$ must be a relation i.e. $\gamma = \operatorname{ev}(\mathcal{W}) = e$. $\square$

In order to state the next result we introduce some notations.

For a homomorphism $\varphi : \Gamma \to \Gamma'$ of discrete groups denote by $\varphi_* : L^1(\Gamma) \to L^1(\Gamma')$ the map "integration along the fibres" defined by

$$\varphi_*\Big(\sum_{\gamma\in\Gamma} f_\gamma \gamma\Big) = \sum_{\gamma\in\Gamma} f_\gamma \varphi(\gamma) = \sum_{\gamma'\in\Gamma'} \Big(\sum_{\gamma\in\varphi^{-1}(\gamma')} f_\gamma\Big)\gamma'.$$

The map φ_* is a homomorphism of Banach $*$-algebras with units and it satisfies the estimate $\|\varphi_*(f)\|_1 \leq \|f\|_1$ for all $f \in L^1(\Gamma)$.

Recall that we view $L^1(\Gamma)$ as a subalgebra of $\mathcal{N}\Gamma$. Let $L^1(\Gamma)^\times$ be the group of invertible elements in $L^1(\Gamma)$. Then we have the following result

Theorem 17. *Consider a countable discrete group together with homomorphisms $\varphi_n : \Gamma \to \Gamma_n$. For $f \in L^1(\Gamma)$ set $f_n = \varphi_{n*}(f) \in L^1(\Gamma_n)$. Then we have:*

$$\det{}_{\mathcal{N}\Gamma} f \geq \overline{\lim_{n\to\infty}} \det{}_{\mathcal{N}\Gamma_n} f_n \textit{ if } K_n = \operatorname{Ker}\varphi_n \to e. \tag{4}$$

In case $f \in L^1(\Gamma)^\times$*, equality holds:*

$$\det_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \det_{\mathcal{N}\Gamma_n} f_n \text{ if } K_n \to e. \tag{5}$$

In particular, using Example 16 we get the following corollary:

Corollary 18. *For* $[\Gamma, S] \in X_d$*, epimorphisms* $\varphi_n : \Gamma \twoheadrightarrow \Gamma_n$ *and* $f \in L^1(\Gamma)^\times$ *we have*

$$\det_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \det_{\mathcal{N}\Gamma_n} f_n \quad \text{if} \quad [\Gamma_n, S_n] \to [\Gamma, S] \quad \text{in} \quad X_d.$$

Let us give two examples:

Example 19. For any countable residually finite group Γ there is a sequence of normal subgroups K_n with finite index such that $K_n \to e$. Set $\Gamma_n = K_n \setminus \Gamma$. Then we have:

$$\det_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} |\det r(f_n)|^{1/|\Gamma_n|} \quad \text{for any} \quad f \in L^1(\Gamma)^\times. \tag{6}$$

Note here that $r(f_n) \in \mathcal{N}\Gamma_n \subset \operatorname{End} \mathbb{C}\Gamma_n$. This formula follows immediately from Theorem 17 if we note that for a finite group G and an element $h \in \mathbb{C}G = L^1(G)$ we have:

$$\det_{\mathcal{N}G} h = |\det r(h)|^{1/|G|}.$$

Formula (6) was used in [10] to relate the growth rate of periodic points of certain algebraic Γ-actions to Fuglede-Kadison determinants. For f in $\mathbb{Z}\Gamma \cap L^1(\Gamma)^\times$ formula (6) is a special case of [23, Theorem 3.4, 3].

Example 20. Recall the situation of Example 15 and let P be a continuous function on T^d whose Fourier coefficients are absolutely summable. Thus we have $P = \mathcal{F}(f)$ for some $f \in L^1(\mathbb{Z}^d)$. If we assume that P does not vanish in any point of T^d it follows from a theorem of Wiener [30] that we have $f \in L^1(\mathbb{Z}^d)^\times$. Define $\varphi_r : \Gamma = \mathbb{Z}^d \to \Gamma_r$ by $\varphi_r(e_i) = r_i$ for $1 \le i \le r$. Corollary 18 now implies the formula

$$\det_{\mathcal{N}\Gamma} f = \lim_{q(r)\to\infty} \det_{\mathcal{N}\Gamma_r} f_r.$$

Since $\det_{\mathcal{N}\Gamma} f = M(P)$ and $\det_{\mathcal{N}\Gamma_r} f_r = M(P_r)$, we get the limit formula of Lawton's theorem in this (easy) case.

The theorem of Wiener mentioned above has been generalized to the non-commutative context. The ultimate result is due to Losert [18]. It asserts that $L^1(\Gamma)^\times = L^1(\Gamma) \cap C^*(\Gamma)^\times$ if and only if Γ is "symmetric". Thus for symmetric groups the question of invertibility in $L^1(\Gamma)$ is reduced to the easier question of invertibility in the C^*-algebra $C^*(\Gamma)$. Finitely generated virtually nilpotent discrete groups for example are known to be symmetric [20, Corollary 3], and hence we have the following equalities for them

$$L^1(\Gamma)^\times = L^1(\Gamma) \cap C_r^*(\Gamma)^\times = L^1(\Gamma) \cap (\mathcal{N}\Gamma)^\times. \tag{7}$$

Note that for amenable groups the C^*-algebra and the reduced C^*-algebra coincide. The classical Wiener theorem is a special case of (7):

$$L^1(\mathbb{Z}^d)^\times = L^1(\mathbb{Z}^d) \cap C^0(T^d)^\times = L^1(\mathbb{Z}^d) \cap L^\infty(T^d, \mu)^\times.$$

The assumptions in Corollary 18 are more restrictive than in Theorem 17. The advantage of its formulation lies in the intuition and results about X_d that one may use.

Proof of Theorem 17 First we need a simple result about traces. We claim that for any f in $L^1(\Gamma)$ and any complex polynomial $P(X)$ we have

$$\tau_{\mathcal{N}\Gamma}(P(f)) = \lim_{n\to\infty} \tau_{\mathcal{N}\Gamma_n}(P(f_n)) \text{ if } K_n \to e. \tag{8}$$

Since $P(f)$ lies in $L^1(\Gamma)$ as well, it suffices to prove (8) for $P(X) = X$. Writing $f = \sum_\gamma f_\gamma \gamma$ we have $\tau_{\mathcal{N}\Gamma}(f) = f_e$ and $\tau_{\mathcal{N}\Gamma_n}(f_n) = \sum_{\gamma\in K_n} f_\gamma$. Fix $\varepsilon > 0$. Since f is in $L^1(\Gamma)$, there is a finite subset $Q \subset \Gamma$ with $\sum_{\gamma\in\Gamma\setminus Q} |f_\gamma| < \varepsilon$. Because of the assumption $K_n \to e$, there is some $N \geq 1$ such that $K_n \cap Q \subset \{e\}$ for all $n \geq N$. For $n \geq N$ we therefore get the estimate

$$|\tau_{\mathcal{N}\Gamma}(f) - \tau_{\mathcal{N}\Gamma_n}(f_n)| = |f_e - \sum_{\gamma\in K_n} f_\gamma| \leq \sum_{\gamma\in K_n\setminus e} |f_\gamma| \leq \sum_{\gamma\in\Gamma\setminus Q} |f_\gamma| < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, formula (8) follows.

Next, for any $f \in L^1(\Gamma)$ we have

$$\|r(f_n)\| \leq \|f_n\|_1 \leq \|f\|_1 \text{ and } \|r(f)\| \leq \|f\|_1 \tag{9}$$

where $\| \ \|$ is the operator norm (between L^2-spaces).

Moreover, if $f \in L^1(\Gamma)^\times$, the relation $(f^{-1})_n = \varphi_n(f^{-1}) = \varphi_n(f)^{-1} = f_n^{-1}$ implies the estimates:

$$\|r(f_n^{-1})\| \leq \|f_n^{-1}\|_1 \leq \|f^{-1}\|_1. \tag{10}$$

Since $2 \det_{\mathcal{N}\Gamma} f = \det_{\mathcal{N}\Gamma} f^* f$ and

$$(f^*f)_n = \varphi_n(f^*f) = \varphi_n(f)^*\varphi_n(f) = f_n^* f_n$$

we may replace f by f^*f in the assertion of Theorem 17. Hence we may assume that $f \in L^1(\Gamma)$ and $f_n \in L^1(\Gamma_n)$ are positive in $\mathcal{N}\Gamma$ resp. $\mathcal{N}\Gamma_n$ i.e. that $r(f)$ and $r(f_n)$ are positive operators. If f is invertible it follows that the spectrum of $r(f)$ is contained in the interval $I = [\|f^{-1}\|_1^{-1}, \|f\|_1]$. Using the estimates (9) and (10) we see that the spectra of $r(f_n)$ lie in I as well for all n. Note here that for a positive bounded operator A on a Hilbert space we have $\|A\| = \max_{\lambda\in\sigma(A)} \lambda$. Fix $\varepsilon > 0$. Since I is a compact subinterval of $(0, \infty)$, it follows from the Weierstraß approximation theorem that there is a polynomial $P(X)$ with $\max_{x\in I} |P(x) - \log x| \leq \varepsilon$. Since $\sigma(r(f)), \sigma(r(f_n))$ lie in I it follows that we have:

$$\|\log r(f) - P(r(f))\| \leq \varepsilon \text{ and } \|\log r(f_n) - P(r(f_n))\| \leq \varepsilon.$$

Using the estimate $|\tau_{\mathcal{N}\Gamma} A| = |(Ae, e)| \leq \|A\|$ for any $A \in \mathcal{N}\Gamma$, we obtain:

$$|\tau_{\mathcal{N}\Gamma}(\log r(f)) - \tau_{\mathcal{N}\Gamma_n}(\log r(f_n))| \leq 2\varepsilon + |\tau_{\mathcal{N}\Gamma}(P(f)) - \tau_{\mathcal{N}\Gamma_n}(P(f_n))|.$$

Assertion (8) now implies formula (5) in Theorem 17. For the proof of (4) we can assume as above that $r(f)$ and the $r(f_n)$ are positive operators. The relations (9) imply that the spectra of $r(f)$ and of all $r(f_n)$ lie in $J = [0, \|f\|_1]$. Choose a sequence of polynomials $P_k(X) \in \mathbb{R}[X]$. converging pointwise to log in J and satisfying the inequalities $P_k > P_{k+1} > \log$ in J for all k. One may obtain such a sequence (P_k) as follows. The continuous functions φ_k on J defined by $\varphi_k(x) = 1/k + \log x$ for $x \geq 1/k$ and by $\varphi_k(x) = 1/k + \log 1/k$ for $0 \leq x \leq 1/k$ satisfy the inequalities $\varphi_k > \varphi_{k+1} > \log$ in J and converge pointwise to log in J. Setting

$$\psi_k = 2^{-1}(\varphi_k + \varphi_{k+1}) \text{ and } \varepsilon_k = \min_{x\in J}(\varphi_k(x) - \varphi_{k+1}(x)) > 0,$$

the Weierstraß approximation theorem provides us with polynomials P_k such that

$$\max_{x\in J} |\psi_k(x) - P_k(x)| \leq \frac{\varepsilon_k}{4}.$$

They satisfy the estimates $\varphi_k > P_k > \varphi_{k+1}$ for all k and hence have the desired properties. It follows that we have

$$\lim_{k\to\infty} \tau_{\mathcal{N}\Gamma}(P_k(f)) = \log \det{}_{\mathcal{N}\Gamma} f. \tag{11}$$

To see this, consider the spectral resolution E_λ of the operator $r(f)$. Then we have by the definition of $\det_{\mathcal{N}\Gamma} f$:

$$\begin{aligned}\log \det{}_{\mathcal{N}\Gamma} f &= \lim_{\varepsilon\to 0+} \tau_{\mathcal{N}\Gamma}(\log(r(f) + \varepsilon)) = \lim_{\varepsilon\to 0+} \int_J \log(\lambda + \varepsilon)\, d\tau_{\mathcal{N}\Gamma}(E_\lambda) \\ &\overset{(a)}{=} \int_J \log \lambda \, d\tau_{\mathcal{N}\Gamma}(E_\lambda) \\ &\overset{(b)}{=} \lim_{k\to\infty} \int_J P_k(\lambda)\, d\tau_{\mathcal{N}\Gamma}(E_\lambda) \\ &= \lim_{k\to\infty} \tau_{\mathcal{N}\Gamma}(P_k(f)).\end{aligned}$$

Here equations (a) and (b) hold because of Levi's theorem in integration theory (with respect to the finite measure $d\tau_{\mathcal{N}\Gamma}(E_\lambda)$ on J). Noting the estimate

$$\tau_{\mathcal{N}\Gamma_n}(P_k(f_n)) \geq \tau_{\mathcal{N}\Gamma_n}(\log(f_n)) \tag{12}$$

we obtain the relations:

$$\begin{aligned}\log \det{}_{\mathcal{N}\Gamma} f &\overset{(11)}{=} \lim_{k\to\infty} \tau_{\mathcal{N}\Gamma}(P_k(f)) \overset{(8)}{=} \lim_{k\to\infty} \lim_{n\to\infty} \tau_{\mathcal{N}\Gamma_n}(P_k(f_n)) \\ &\overset{(12)}{\geq} \overline{\lim_{n\to\infty}}\, \tau_{\mathcal{N}\Gamma_n}(\log f_n) = \overline{\lim_{n\to\infty}} \log \det{}_{\mathcal{N}\Gamma_n} f_n.\end{aligned}$$

□

Remark If $f \in L^1(\Gamma)$ is not invertible the question whether the equality (5) still holds becomes much more subtle. In the situation of Example 19, Lück has given a criterion in terms of the asymptotic behavior near zero of the spectral density function, which is hard to verify however, cp. [23, Theorem 3.4, 3]. Note that he discusses a slightly different version of the Fuglede-Kadison determinant where the zero-eigenspace is discarded. If $A \in \mathcal{N}\Gamma$ is injective on $L^2(\Gamma)$ the two versions of the FK-determinant agree. Incidentally, for a finitely generated amenable group Γ, a nonzero divisor $f \in \mathbb{C}\Gamma$ has the property that $r(f)$ is injective on $L^2(\Gamma)$, see [11].

For $\Gamma = \mathbb{Z}$ and the projections to $\Gamma_n = \mathbb{Z}/n$ the above question is related to the theory of diophantine approximation. This was first noted in ergodic theory because for $f \in \mathbb{Z}[\mathbb{Z}]$ the limit

$$\lim_{n\to\infty} \log \det_{\mathcal{N}\mathbb{Z}/n}(f_n) = \lim_{n\to\infty} n^{-1} \log \det(r(f_n)) \quad \text{(if it exists)}$$

is the logarithmic growth rate of the number of periodic points of a toral automorphism with characteristic polynomial $\hat{f} \in \mathbb{Z}[X^{\pm 1}]$. One wanted to know if it is equal to the topological entropy which turns out to be given by $m(\hat{f}) = \log \det_{\mathcal{N}\mathbb{Z}}(f)$. Using a theorem of Gelfond this was proved by Lind in [16, § 4]. See also [26, Lemma 13.53]. On the other hand there are examples of noninvertible $f \in L^1(\mathbb{Z})$ with $\hat{f} \in \mathbb{R}[X, X^{-1}]$ a linear polynomial for which formula (5) is false, see [24, Example 13.69].

On the other hand, for the sequence $\varphi_r : \Gamma = \mathbb{Z}^d \to \Gamma_r$ from Example 15 formula (5) holds for *all* $f \in \mathbb{C}[\mathbb{Z}^d]$ as follows from Lawton's Theorem 14 above. One may interpret his proof as an estimate for the spectral density function of $|f|$ near zero.

These cases suggest the following problem:

Question 21. *In the situation of Theorem 17 consider f in $\mathbb{Z}\Gamma$. Is it true that we have*

$$\det_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \det_{\mathcal{N}\Gamma_n} f_n \quad \textit{if} \;\; K_n \to e$$

even if f is not invertible in $L^1(\Gamma)$?

In the rest of this section we extend the previous theory somewhat by replacing the maps $\varphi_n : \Gamma \to \Gamma_n$ by a sequence of "correspondences". Thus, we consider discrete groups and homomorphisms

$$\Gamma \xleftarrow{\varphi} \tilde{\Gamma} \xrightarrow{\varphi_n} \Gamma_n \quad \text{with kernels} \quad K = \operatorname{Ker}\varphi \quad \text{and} \quad K_n = \operatorname{Ker}\varphi_n.$$

Given $\tilde{f} \in L^1(\tilde{\Gamma})$ write $f = \varphi_*(\tilde{f}) \in L^1(\Gamma)$ and $f_n = \varphi_{n*}(\tilde{f}) \in L^1(\Gamma_n)$. We will write $K_n \to K$ if one of the following equivalent conditions holds:

a No element $\tilde{\gamma} \in \tilde{\Gamma}$ is contained in $K \triangle K_n$ for infinitely many n.

b For any finite subset $Q \subset \tilde{\Gamma}$ we have $(K \triangle K_n) \cap Q = \varnothing$ if n is large enough.

Then Theorem 17 has the following generalization:

Theorem 22. *Consider diagrams of countable groups $\Gamma \xleftarrow{\varphi} \tilde{\Gamma} \xrightarrow{\varphi_n} \Gamma_n$ for $n \geq 1$ as above and fix $\tilde{f} \in L^1(\tilde{\Gamma})$. Then we have*

$$\det_{\mathcal{N}\Gamma} f \geq \overline{\lim_{n\to\infty}} \det_{\mathcal{N}\Gamma_n} f_n \quad \textit{if} \;\; K_n \to K. \tag{13}$$

For $\tilde{f} \in L^1(\tilde{\Gamma})^\times$, equality holds:

$$\det_{\mathcal{N}\Gamma} f = \lim_{n\to\infty} \det_{\mathcal{N}\Gamma_n} f_n \quad \textit{if} \;\; K_n \to K. \tag{14}$$

Proof. As before one first shows that:

$$\tau_{\mathcal{N}\Gamma}(P(f)) = \lim_{n\to\infty} \tau_{\mathcal{N}\Gamma_n}(P(f_n)) \quad \text{for} \;\; K_n \to K \tag{15}$$

whenever $\tilde{f} \in L^1(\tilde{\Gamma})$ and $P \in \mathbb{C}[X]$. Writing $\tilde{f} = \sum_{\tilde{\gamma}\in\tilde{\Gamma}} a_{\tilde{\gamma}}\tilde{\gamma}$ and using the inequality

$$|\tau_{\mathcal{N}\Gamma}(f) - \tau_{\mathcal{N}\Gamma_n}(f_n)| \le \sum_{\tilde{\gamma}\in K\triangle K_n} |a_{\tilde{\gamma}}|,$$

we can argue as in the proof of formula (8).

The rest of the proof is analogous to the one of Theorem 17 if we note that the spectra of $r(f)$ and $r(f_n)$ lie in $[0, \|\tilde{f}\|_1]$ for $\tilde{f} \in L^1(\tilde{\Gamma})$ and in $[\|\tilde{f}^{-1}\|_1^{-1}, \|\tilde{f}\|_1]$ if $\tilde{f}$ is invertible in $L^1(\tilde{\Gamma})$. This follows from the estimates

$$\|r(f)\| \le \|f\|_1 \le \|\tilde{f}\|_1 \text{ and } \|r(f_n)\| \le \|f_n\|_1 \le \|\tilde{f}\|_1 \text{ if } \tilde{f} \in L^1(\tilde{\Gamma})$$

and similar ones for the inverses of $\tilde{f}, f, f_n$ in case $\tilde{f} \in L^1(\Gamma)^\times$. □

Next, assume that $\tilde{\Gamma}$ is finitely generated and the maps φ and φ_n are surjective. A family of generators $\tilde{S}$ of $\tilde{\Gamma}$ gives families of generators S and S_n for Γ and Γ_n. If $d = |\tilde{S}|$ one can show as in Example 16 that the condition $K_n \to K$ is equivalent to $[\Gamma_n, S_n] \to [\Gamma, S]$ in X_d for $n \to \infty$.
For completeness let us give the argument for the implication needed in the following corollary. Assume that $\tilde{\gamma} \in \tilde{\Gamma}$ is contained in $K \triangle K_n$ for infinitely many n. Choose a word $\tilde{\mathcal{W}}$ in $\tilde{\Gamma}$ with $\tilde{\gamma} = \mathrm{ev}(\tilde{\mathcal{W}})$. Via φ_n, φ we obtain words $\mathcal{W}_n$ and $\mathcal{W}$ in Γ_n resp. Γ with $\gamma_n = \mathrm{ev}(\mathcal{W}_n)$ and $\gamma = \mathrm{ev}(\mathcal{W})$. By assumption there are infinitely many n, such that $\mathcal{W}$ is a relation in Γ but $\mathcal{W}_n$ is not a relation in Γ_n or vice versa. This is not possible however, since for large n the relations of length $\le l(\tilde{\mathcal{W}})$ in Γ and Γ_n are in canonical bijection if $[\Gamma_n, S_n] \to [\Gamma, S]$.

Corollary 23. **a** *In the situation above, we have for* $\tilde{f} \in L^1(\tilde{\Gamma})$

$$\lim_{n\to\infty} \tau_{\mathcal{N}\Gamma_n}(f_n) = \tau_{\mathcal{N}\Gamma}(f) \quad \textit{if} \quad [\Gamma_n, S_n] \to [\Gamma, S] \quad \textit{in} \quad X_d.$$

b *If* $\tilde{f}$ *is invertible in* $L^1(\tilde{\Gamma})$*, we have in addition*

$$\lim_{n\to\infty} \det{}_{\mathcal{N}\Gamma_n}(f_n) = \det{}_{\mathcal{N}\Gamma}(f) \quad \textit{if} \quad [\Gamma_n, S_n] \to [\Gamma, S] \quad \textit{in} \quad X_d.$$

Proof. The condition $[\Gamma_n, S_n] \to [\Gamma, S]$ implies that $K_n \to K$ and hence **a** follows from equation (15) and **b** from Theorem 22, (14). □

Corollary 24. *Consider the free group* F_d *on* d*-generators* $g_1, \ldots, g_d$*. For* $[\Gamma, S]$ *in* X_d *define an epimorphism* $\varphi : F_d \to \Gamma$ *by setting* $\varphi(g_i) = s_i$ *if* $S = (s_1, \ldots, s_d)$*.*
a *For every* $\tilde{f} \in L^1(F_d)$*, the following function is continuous:*

$$T(\tilde{f}) : X_d \to \mathbb{C} \quad \textit{defined by} \quad T(\tilde{f})[\Gamma, S] = \tau_{\mathcal{N}\Gamma}(\varphi(\tilde{f})).$$

b *For every* $\tilde{f} \in L^1(F_d)^\times$*, the function*

$$D(\tilde{f}) : X_d \to \mathbb{R}^{>0} \quad \textit{defined by} \quad D(\tilde{f})[\Gamma, S] = \det{}_{\mathcal{N}\Gamma}(\varphi(\tilde{f}))$$

is continuous.

Remarks The map $T(\tilde{f})$ depends only on the image of $\tilde{f}$ in the quotient of $L^1(F_d)$ by the subgroup generated by the commutators $[g,h] = gh - hg$. Moreover $D(\tilde{f})$ depends only on the image of $\tilde{f}$ in the abelianization of $L^1(F_d)^\times$. Note that assertion **b** is not a formal consequence of **a** since there is no functional calculus in $L^1(\Gamma)$ allowing us to define the logarithm on all invertible elements of the form $f^* f$.

4. Further problems

For a nonzero polynomial P in $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ it is well known that the Mahler measure satisfies the inequality $M(P) \geq 1$. In fact $m(P) = \log M(P)$ can be interpreted as the entropy of a suitable $\mathbb{Z}^d$-action and entropies are non-negative, cp. [22]. For discrete groups Γ the question whether $\det_{\mathcal{N}\Gamma} f \geq 1$ holds for $f \in \mathbb{Z}\Gamma$ has been much studied for the modified version of $\det_{\mathcal{N}\Gamma}$ where the zero eigenspace is discarded, cp. [24] for an overview. If $r(f)$ is injective on $L^2(\Gamma)$, these results apply to $\det_{\mathcal{N}\Gamma} f$ itself. It is known for example that for such f and all residually amenable groups Γ we have $\det_{\mathcal{N}\Gamma} f \geq 1$. For Mahler measures the polynomials P with $M(P) = 1$ are known by a theorem of Kronecker in the one-variable case and by a result of Schmidt in general, [26]. For them the above mentioned entropy is zero and this is significant for the dynamics. Apart from $\Gamma = \mathbb{Z}^d$ and finite groups Γ nothing seems to be known about the following problem:

Question 25. *Given a countable discrete group Γ, can one characterize the elements $f \in \mathbb{Z}\Gamma$ with $\det_{\mathcal{N}\Gamma} f = 1$?*

Even the case, where Γ is finitely generated and nilpotent would be interesting with the integral Heisenberg group as a starting point.

The polynomials $P \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ with $M(P) = 1$ are either units in $\mathbb{Z}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ or they have zeros on T^d and hence are not invertible in L^1. For $f \in \mathbb{Z}[\mathbb{Z}^d] \cap L^1(\mathbb{Z}^d)^\times$ we therefore have $\det_{\mathcal{N}\mathbb{Z}^d} f > 1$ unless f is a unit in $\mathbb{Z}[\mathbb{Z}^d]$. Is the same true in general?

Question 26. *Given a countable discrete group Γ and an element $f \in \mathbb{Z}\Gamma \cap L^1(\Gamma)^\times$ which does not have a left inverse in $\mathbb{Z}\Gamma$, is $\det_{\mathcal{N}\Gamma} f > 1$?*

Remark If Γ is residually finite and amenable, the answer is affirmative. This was shown in the proof of [10, Corollary 6.7] by interpreting $\log \det_{\mathcal{N}\Gamma} f$ as an entropy and proving that the latter was positive. Note that if f does have a left inverse in $\mathbb{Z}\Gamma$ i.e. $gf = 1$ for some $g \in \mathbb{Z}\Gamma$ we have $(\det_{\mathcal{N}\Gamma} g)(\det_{\mathcal{N}\Gamma} f) = 1$ which implies that $\det_{\mathcal{N}\Gamma} f = 1 = \det_{\mathcal{N}\Gamma} g$ if both determinants are ≥ 1. Incidentally, by a theorem of Kaplansky, $\mathcal{N}\Gamma$ and hence also the subrings $\mathbb{C}\Gamma$ and $L^1(\Gamma)$ are directly finite, i.e. left units are right units and vice versa.

The last topic we want to mention concerns a continuity property. Answering a question of Schinzel, Boyd proved the following result about the Mahler measure in [3]:

Theorem 27 (Boyd)**.** *For any Laurent polynomial $P \in \mathbb{C}[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$ the function $z \mapsto M(z - P)$ is continuous in $\mathbb{C}$.*

The proof is based on an estimate due to Mahler which in turn uses Jensen's formula.

Thus the question arises whether $\det_{\mathcal{N}\Gamma}(z - f)$ is a continuous function of $z \in \mathbb{C}$ for f in $\mathbb{C}\Gamma$. For A in $\mathcal{N}\Gamma$ the function $\varphi(z) = \log \det_{\mathcal{N}\Gamma}(z - A)$ is a subharmonic function on $\mathbb{C}$ cp. [5] and in particular it is upper semicontinuous. For $z \notin \sigma(A)$ (or even for z outside the support of the Brown measure) the function $\varphi(z)$ is easily seen to be continuous. If Γ is finite then $\det_{\mathcal{N}\Gamma}(z-f) = |\det(z-r(f))|^{1/|\Gamma|}$ is clearly continuous for $z \in \mathbb{C}$. For the discrete Heisenberg group Γ one may use formula (4) in [8] to get examples where $\det_{\mathcal{N}\Gamma}(z - f)$ can be expressed in terms of ordinary integrals. In all these cases one obtains a continuous function of z if f is in $\mathbb{C}\Gamma$.

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