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Honda-Tate-Theory for A-Motives and Global Shtukas

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Felix Rötting

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The result of Honda-Tate theory for abelian varieties is a bijective correspondence between simple abelian varieties over finite fields and certain algebraic integers, so-called Weil-numbers, induced by relating an abelian variety to its Frobenius endomorphism. Working on the function field side of algebraic number theory, we construct the analogue of Honda-Tate theory in this setting, generalizing an earlier result of Yu for Drinfeld-modules to the higher-dimensional case of A-motives and global shtukas.

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Dedicated to my father, †2015.

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Notations

We will use the following notational conventions throughout, unless explicitly stated otherwise:

C : a projective, geometrically irreducible, normal curve over $\mathbb{F}_q$.

∞ : a fixed $\mathbb{F}_q$-valued point of C.

A : the ring $\mathcal{O}(C \setminus \{\infty\})$ of rational functions of C outside ∞.

C' : the curve $C \setminus \{\infty\} = Spec\,(A)$

Q : the field of fractions of A, also the function field of C.

$\tilde{C}$: the curve corresponding to some CM-algebra E/Q (see below)

C_k, $\tilde{C}_k$: the curves $C \times Spec\,(k)$ and $\tilde{C} \times Spec\,(k)$, respectively

A_R, A_k : the rings $A \otimes_{\mathbb{F}_q} R$ and $A \otimes_{\mathbb{F}_q} k$.

k : a finite field extension of $\mathbb{F}_q$ of degree $e = [k : \mathbb{F}_q]$

γ : a fixed $\mathbb{F}_q$-homomorphism $A \to k$, making (k, γ) an A-field.

ε : the kernel of γ, the so-called A-characteristic of (k, γ).

$\mathbb{F}_\varepsilon$: residue field of ε, i. e. $\mathbb{F}_\varepsilon \cong A/\varepsilon$

$\underline{M}$: an A-motiv (M, τ_M) over some A-field k

$\underline{\mathcal{N}}$: a global shtuka $(\mathcal{N}, \tau_{\mathcal{N}}, \underline{c})$ over a base scheme S

$\mathrm{QEnd}_k(\underline{M})$: the Q-algebra of quasi-endomorphisms $\mathrm{End}_k(\underline{M}) \otimes Q$

E : a sub-Q-algebra in $\mathrm{QEnd}(\underline{M})$ of maximal rank $\mathrm{rk}_Q(\underline{M})$, called CM-algebra.

π : Frobenius endomorphism of an A-motive/global shtuka

α : an abstract (Drinfeld-)Weil-number in Q^{alg}

$\sigma : A_R \to A_R$: the map $a \otimes r \mapsto a \otimes r^q$.

$\sigma^* M$: the A_k-module $M \otimes_{\sigma,k} A_k$

$k\{\tau\}$: the skew-polynomial ring over k in the variable τ with $\tau.x = x^q.\tau$.

$u \mid v \mid w$: places of the CM-algebra E, of $F = Q(\pi) \subset E$ or $Q(\alpha) \subset W$, and of Q, respectively

z_u, z_v, z_w : uniformizing parameters at places u, v, w, often simply written as z

$\mathbb{F}_u, \mathbb{F}_v, \mathbb{F}_w$: residue fields at u, v, w

$\mathcal{O}_{C,v}$: the local ring of C at v.

A_v : the completion of $\mathcal{O}_{C,v}$ w. r. t. v.

Q_v : the field of fractions of A_v.

$A_{v,L}$: $A \,\hat{\otimes}_{\mathbb{F}_q} L$

$Q_{v,L}$: $A_{v,L}$ inverted at z_v.

Contents

Introduction

A simple abelian variety $\mathcal{A}$ over a finite field $\mathbb{F}_q$ may be associated with a specific set of complex numbers via its Frobenius endomorphism. More precisely, consider the minimal polynomial of the Frobenius, which will be a polynomial with integer coefficients, and form the set of its roots. We call the elements of this set the *Weil numbers* associated with $\mathcal{A}$. It was shown by John Tate that two isogenous abelian varieties have the same Frobenius minimal polynomial and therefore the same set of associated Weil numbers. Moreover, we can describe Weil numbers in an abstract, but very elementary way; they are complex algebraic numbers α which have absolute value $\sqrt{q}$ under every embedding $\mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$. One can therefore define an abstract Weil-q-number to be such an algebraic number, and having obtained this abstract definition of Weil numbers, it makes sense to ask if it is possible, given an abstract Weil number α, to associate an abelian variety $\mathcal{A}(\alpha)$ to the class of abstract Weil numbers represented by α, that is, to a conjugacy class of Weil-q-numbers under Galois operation, such that the Frobenius of $\mathcal{A}(\alpha)$ is conjugate to α. It was shown by Honda that this is indeed possible, and hence there is a one-to-one correspondence between the set of isogeny classes of abelian varieties over finite fields and certain finite subsets of the complex numbers. More precisely, if one denotes the set of isogeny classes of simple abelian varieties over $\mathbb{F}_q$ by $\mathrm{sabVar}^{\sim}_{\mathbb{F}_q}$ and the set of conjugagy classes of Weil-q-numbers by $\mathrm{W}^{\sim}_q$, one has a bijective map

$$\mathrm{sabVar}^{\sim}_{\mathbb{F}_q} \xleftrightarrow{1:1} \mathrm{W}^{\sim}_q \ .$$

The existence and description of this correspondence is usually referred to as *Honda-Tate-Theory* for abelian varieties. It is a part of number theory in characteristic zero, that is, the theory of finite field extensions of $\mathbb{Q}$ and its growth into arithmetic geometry.

Modern number theory exists in two parts, the first being the theory in characteristic zero, dealing with number fields, the aforementioned finite extensions of $\mathbb{Q}$, and the second the theory for characteristic $p > 0$, dealing with function fields – finite extensions of the fields $\mathbb{F}_p(t)$. The two sides are rich in analogues and similarities that have been researched and understood over the years. Elliptic curves and abelian varieties belong to the number field side; the endomorphism ring of elliptic curves includes the ring $\mathbb{Z}$ of rational integers. On the function field side, the ring $\mathbb{Z}$ is replaced by an (admissible, see Section. 1.1 for details) ring A of characteristic $p > 0$, the most simple example being $\mathbb{F}_q[t]$, and its field of fractions Q, in our example $\mathbb{F}_q(t)$, takes the place of the rational integers $\mathbb{Q}$. The uniquely determined, natural characteristic morphism $\mathbb{Z} \to k$ for any field k has to be replaced by a chosen morphism $\gamma : A \to k$, and the kernel $\varepsilon = \ker \gamma$ of this morphism is called the *A-characteristic* of k. In particular, it allows us to consider k as field extension of $\mathbb{F}_\varepsilon := A/ \ker \gamma$. The notion of *Drinfeld modules* plays the role that elliptic curves play for number fields, a Drinfeld module being some non-trivial ring homomorphism $\phi : A \ni a \mapsto \phi_a \in k\{\tau\}$, with $k\{\tau\}$ meaning the skew-polynomial ring over k in the variable τ, with action given by $\tau \cdot b = b^q \cdot \tau$ for $b \in k$. A Drinfeld-A-module ϕ has a rank, which is in the case of $A = \mathbb{F}_q[t]$ simply the τ-degree of the highest τ-power in ϕ_t with non-vanishing coefficient.

Every Drinfeld module ϕ defined over a finite field $k \cong \mathbb{F}_{q^e}$ possesses a Frobenius endomorphism π_ϕ, given by the q^e-th power map τ^e. Furthermore, it was proven by Yu ([Yu95], see also [Gos96, Thm. 4.12.15]) that the roots of its minimal polynomial $\min_{\pi,Q} \in Q[X]$ in Q^{alg} classify its isogeny class in a similar fashion as Weil-numbers do for elliptic curves and abelian varieties . In other words, if we write $A - \text{DM}^{\sim}_{k,r}$ for the set of isogeny classes of Drinfeld-A-modules of rank r over a field k of size (q^e) and $\text{DW}^{\sim}(A, q, e, r)$ for the (suitably defined, see Chapter 3) set of conjugacy classes of Drinfeld-Weil-numbers of rank r, there is a bijection

$$A - \text{DM}^{\sim}(k, r) \xleftrightarrow{1:1} \text{DW}^{\sim}(A, q, e, r)$$

given by mapping a Drinfeld-A-module ϕ of rank r with Frobenius endomorphism π_ϕ to the conjugacy class of roots of $\min_{\pi,Q}$. Thus, there is a Honda-Tate-Theory for Drinfeld modules.

Now, similar to how elliptic curves are just the 1-dimensional case of abelian varieties, Drinfeld modules can be understood as the 1-dimensional case of higher-dimensional objects, originally introduced in 1986 by Anderson in [And86] under the name *t-modules*, which can be defined as ring morphisms $\phi : A \to k\{\tau\}^{d\times d}$ for any dimension $d > 0$ satisfying some additional technical properties. The interested reader may find a quick overview in the Appendix, under the modern name *Anderson-A-modules*. In *loc. cit.* Anderson also introduced a category of *t-motives*, which is anti-equivalent to t-modules, and nowadays referred to in the literature as the category of *A-motives*. (Older works still use the terminology *Anderson-A-motive*.)

It is also possible to go one step further and construct a category of *Global Shtukas* which is even larger than the category of A-motives. Every A-motive lives over two places, the A-characteristic $\varepsilon = \ker \gamma$ and a point ∞, which is the place on the curve C corresponding to the field of fractions Q of A not in *Spec* (A). A global shtuka $\underline{\mathcal{N}} = (\mathcal{N}, \tau_{\mathcal{N}}, \underline{c})$ over a base scheme S is essentially an A-motive with more than two characteristic places given by a tuple of morphisms $\underline{c} = (c_i : S \to C : 1 \leq i \leq n)$. (A precise definition is included in later chapters.) The term *global* strongly implies the existence of something called *local shtuka*, which is indeed accurate; local shtukas are objects attached to A-motives and global shtukas at the places of C, which play are crucial role as the equivalents of p-divisible groups.

The theory of Drinfeld modules, Anderson-A-modules, A-motives and global/local shtukas was developed further by G. Anderson, V. Drinfeld, W. D. Brownawell, E.-U. Gekeler, D. Goss, G. Böckle, U. Hartl, M. Papanikolas, R. Pink, Y. Taguchi, D. Thakur, J.-K. Yu, amongst many others.

Our aim in these notes is to give a generalization of Honda-Tate correspondence to higher dimension, first and foremost to A-motives.

Let us briefly note some important similarities and differences between the theory of abelian varieties and the theory of A-motives.

Abelian varieties are always semisimple, that is, they may up to isogeny be written as product of simple subvarieties. In general, A-motives do not behave quite as well, in that one can construct non-semisimple A-motives. Over finite fields, however, A-motives are always semisimple after a suitably large field extension. Semisimple A-Motives show quite similar behaviour as abelian varieties.

An abelian variety A of dimension n over $\mathbb{C}$ can always be realized as a quotient V/Λ, where V is a n-dimensional $\mathbb{C}$-vector space and Λ a real lattice of $\mathbb{R}$-rank $2n$. Moreover, there is an explicit realization via its exponential function, i. e. $\mathrm{T}_0(A) \cong V \cong \mathbb{C}^n$ and $\Lambda \cong \ker \mathrm{Exp}_A$. Anderson-Motives on the other hand do not always allow a lattice realization; if they do, they are called *uniformizable*.

We already mentioned that the ring of rational integers $\mathbb{Z}$ can be found inside the endomorphism ring of an elliptic curve. More generally, this is true for abelian varieties of higher dimension as well, and if one considers the $\mathbb{Q}$-algebra of quasi-endomorphisms of an abelian variety $\mathcal{A}$, formed by tensoring the endomorphism ring with $\mathbb{Q}$, one can ask if this algebra includes a commutative semisimple Q-sub-algebra of maximal rank $2 \cdot \dim(\mathcal{A})$. If this is the case, the variety $\mathcal{A}$ is said to have *Complex Multiplication*, and one can describe abelian varieties with Complex Multiplication in a very abstract manner simply by defining the correct notion of *CM-type*. In this way, it is possible to construct abelian varieties with Complex Multiplication based only on information about their endomorphism ring. This is also a fundamental ingredient in the proof of Honda-Tate theory for abelian varieties, since knowledge of a Weil-number α associated to $\mathcal{A}$ can be translated into knowledge of the endomorphism algebra of $\mathcal{A}$.

Switching sides again, one can define the notion of Complex Multiplication and CM-types for A-motives (and global shtukas) in a manner quite similar to the above, and this will be a crucial step for us to develop Honda-Tate theory for A-motives (and global shtukas).

Finally, let us summarize our results. We give the definition of Weil-numbers and our main result for A-motives:

Definition 0.1 (see Chapter 3)**.** *Let A be an admissible ring, and let Q be its field of fractions. Let $\mathfrak{p} \subset A$ be a maximal ideal, and let $a > 0$ be a positive integer. A $\mathfrak{p}^a$-Weil-number is an element of Q^{alg} satisfying the following conditions:*

1. *The minimal polynomial $\min_{\alpha,Q}$ of α over Q has coefficients in A.*
2. *The element α does not lie over any place w outside $\mathfrak{p}$ and ∞, i. e.*

$$\mathrm{ord}_w(\alpha) = 0 \quad (\forall\, w \nmid \mathfrak{p} \cdot \infty).$$

Two Weil-numbers α, α' are called **conjugated**, written as $\alpha \sim \alpha'$, if they have the same minimal polynomial $\min_{\alpha,Q} = \min_{\alpha',Q}$ over Q. We denote the set of conjugacy classes of $\mathfrak{p}^a$-Weil-numbers by

$$\mathrm{W}^{\sim}(A, \mathfrak{p}, a)\,,$$

and the set of isogeny classes of simple A-motives over $k \cong \mathbb{F}_{q^a}$ of characteristic $\mathfrak{p}$ by

$$A\text{-}\mathrm{Mot}^{\sim}(\mathfrak{p}, a)\,.$$

Theorem 0.2. *Then there is a bijection between the set of isogeny classes of simple A-motives over a fixed finite A-field (k, γ) of size $\left(\#\,\mathbb{F}_{\mathfrak{p}}\right)^a$ and the set of conjugacy* *classes of Weil-numbers α in the sense of Definition 0.1 induced by $\underline{M} \mapsto \min_{\pi,Q}$, that is:*

$$A\text{-Mot}^{\sim}_{\mathrm{s}}(\mathfrak{p}, a) \xleftrightarrow[\Phi]{1:1} \mathrm{W}^{\sim}(A, \mathfrak{p}, a)$$

We also explain how to extract dimension and rank of the corresponding A-motive from a Weil-number, show that this correspondence can be restricted in a suitable manner to pure A-motives and give a definition of pure Weil-numbers. Furthermore, we prove an analogous result for Global Shtukas.

Our approach mirrors the works of Honda and Tate. Historically, injectivity and surjectivity were separate results by J. Tate and T. Honda [Hon68], respectively, with Hondas original proof having been written in 1967. Soon afterwards, in late 1968, a paper written by Tate alone appeared in Seminaire Bourbaki, crediting Honda, in which Tate gave a complete account of the bijection theorem, using the essence of Hondas approach, but avoiding some of Hondas elementary, but unwieldy ideal constructions and using instead an argument involving p-divisible groups. We will essentially follow this latter approach, employing the technique of local shtukas as analogue to Tates p-divisible group argument. The core steps of the proof are as follows:

- Choose a maximal ideal $\mathfrak{p}$, an order $a > 0$ and a $\mathfrak{p}$-Weil number α of order a.
- Deduce the correct endomorphism algebra from α.
- Obtain from it a CM-type and thereby an A-motive with (CM) over $\mathbb{C}_\infty$
- Move from a motive over $\mathbb{C}_\infty$ to a motive over some finite field $\tilde{k}/\mathbb{F}_q$.
- Show that some power α^m of α represents a Frobenius endomorpism.
- Finally, conclude that α is defined by a Frobenius endomorphism as well.

Some words about the organization of the work. The first chapter is primarily an overview of the theory of A-motives and global shtukas. The second chapter deals with the theory Complex Multiplication of A-motives, and includes a prove of the analogue to the formula of Shimura and Taniyama. The third chapter defines Weil-numbers for function field objects and states the Honda-Tate correspondence. The fourth, very short, chapter proves the injectivity of the correspondence. The fifth chapter deals with the surjectivity proof. The appendix includes short explanations of Anderson-A-modules, Complex Multiplication for abelian varieties, descent theory and Brauer groups. Each chapter starts with a more detailed overview of its contents.

During the work, we use **boldface** to indicate formal definitions, and *italics* for general emphasis. The numeration of Remarks, Propositions, Theorems and so on is continuous, and includes the chapter number first. Sections and subsections are labeled independently.

1. A-Motives and Global Shtukas

In this chapter we introduce all the fundamental objects of interest for our work, in particular A-rings, A-motives, global shtukas, (associated) local shtukas and the Frobenius endomorphism. We explain the notions of pure and (semi)simple A-motives, and we explain the structure of the endomorphism algebra of A-motives and global shtukas in terms of their Hasse invariants. The primary source are the two papers [BH11, BH09], as well as [Har16] and [HS16]. Note that the first two papers are older and written in the of so-called *abelian τ-sheaves*, which has not been adopted for general use, and we will therefore avoid entirely. Abelian τ-sheaves and A-motives are closely related (see [BH11, Thm. 3.1]), and results obtained for τ-sheaves are generally true for A-motives as well. A certain care, however, has to be taken when an argument including τ-sheaves touches the point ∞, which has to do with the definition of purity. More generally, the statements in [BH11, BH09] will usually also remain true if one replaces A-motives with global shtukas. This is true in particular whenever proofs work locally, i. e. with local shtukas.

1.1. The ring A and A-fields

As mentioned in the introduction, the theory of Drinfeld-modules and A-motives is situated above a ring A replacing the ring $\mathbb{Z}$ on the number-field side. Not every ring A may be used; instead, a ring A is called **admissible** if it arises in the following manner: Let C be a projective, geometrically irreducible and normal curve over $\mathbb{F}_q$ and ∞ be a fixed $\mathbb{F}_q$-rational point of C. Let $C' := C \setminus \{\infty\}$ be the curve outside this given point. Form the ring $A = \Gamma(C', \mathcal{O}_C)$ of functions of C regular outside ∞.

Let us now fix such a curve C, a point ∞ and a ring A as above. Let us write $Q := \mathrm{Frac}(A)$ for the ring of fractions of A.

Definition 1.1. *Let k be a field extension of $\mathbb{F}_q$. Let $\gamma : A \to k$ be an $\mathbb{F}_q$-homomorphism. The pair (k, γ) is then called an A-**field**. The A-**characteristic** ε of (k, γ) is defined as the kernel of γ. More generally, any (commutative, unitary) ring R together with a ring homomorphism $\gamma : A \to R$ is called an A-**ring**.*

Remark 1.2. *We are mostly interested in working with A-motives (and shtukas) over fields, but the theory can also be developed over A-rings, see for instance [Har16]. Since it will on occasion become necessary for us to consider objects over rings, we include the more general definition.*

Let R be a ring. Write $A_R := A \otimes_{\mathbb{F}_q} R$. We can define an endomorphism σ of A_R via

$$\sigma : A_R \ni a \otimes b \mapsto a \otimes b^q \in A_R.$$

We write $\sigma^*M := M \otimes_{A_R,\sigma} A_R$ (i.e. an element $a \in A_R$ acts as $a.m \otimes 1 = m \otimes \sigma(a)$) and $\sigma^*(m) := m \otimes 1$ for all $m \in M$. Since A_R acts canonically on the second component, σ^*M is an A_R-module.

1.2. Drinfeld-Modules

Definition 1.3. *Let $k/\mathbb{F}_q$ be a field extension. Let $k\{\tau\}$ denote the ring of skew-polynomials*

$$b_0 + b_1 \cdot \tau + \cdots + b_n \cdot \tau^n$$

in the variable τ with action given via the q-Frobenius of k. In concrete terms, this means $\tau^n \cdot b = b^{q^n} \cdot \tau^n$ for all $b \in k$ and $n \in \mathbb{N}$.

We have already developed all the necessary language to define one of the most important concepts in the arithmetic of finite fields, that plays a similar role as elliptic curves do for number fields.

Definition 1.4. *A **Drinfeld-A-Module** with coefficients in an A-field (k, γ) is a ring homomorphism $\phi : A \to k\{\tau\}$, such that the following two conditions are satisfied:*

(i) The image of ϕ is not contained in k, i.e.

$$\mathrm{Im}(\phi) \not\subset k.$$

(ii) The constant term of $\phi(a)$ is given by $\gamma(a)$, i.e. for all $a \in A$ we have

$$\phi(a) = \gamma(a) \cdot \tau^0 + a_1 \cdot \tau + \cdots + a_n \cdot \tau^n.$$

1.3. The Category of A-Motives

In this section we will define the objects of primary interest to us. They were also invented by Anderson [And86] under the name of *t-motives.*

1.3.1. τ-Modules

Let now M be an A_R-module and $\tau : M \to M$ be a σ-linear endomorphism of M, i.e. $\tau(a \cdot m) = \sigma(a) \cdot \tau(m)$ for all $a \in A_R$ and $m \in M$. Composition of τ with

$$\iota : M \ni m \mapsto m \otimes 1 \in \sigma^*M$$

yields $\tau := \tau_M \circ \iota : \sigma^*M \to M$, which is now a linear map of A_R-modules.

Definition 1.5. *A pair $\underline{M} := (M, \tau_M)$ is called a **τ-module** over Spec(A_R) of rank r, if M is a locally free A_R-module of finite rank r and $\tau_M : \sigma^*M \to M$ is a monomorphism of A_R-modules.*

A τ-module may also be given via the σ-linar map τ instead of τ_M.

1.3.2. A-Motives

Definition 1.6. *A τ-module $\underline{M} = (M, \tau_M)$ over A_R of rank r is called **effective A-motive** over an A-ring $(R, \gamma : A \to R)$ of rank r and dimension d, if the following conditions hold:*

(i) the cokernel $\operatorname{coker} \tau_M$ *of τ_M forms a d-dimensional R-module*

(ii) which is annihilated by the d-th power of the ideal $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a))_{A_R} \subset A_R$.

If in addition $R = k$ is a field and the following condition

(iii) M is finitely generated as $k\{\tau\}$-module

*holds, the A-motive $\underline{M}$ is called **abelian**.*

Remark 1.7. *Abelian A-motives are closely related to Drinfeld-modules and abelian Anderson-A-modules, as will be seen later. Under certain conditions (in particular* purity - *cf. Def. 1.35) Anderson A-motives are automatically abelian.*

Remark 1.8. *The ideal $\mathcal{J}$ is the kernel of $\gamma \otimes \mathrm{id}_k : A_R \to A_R$ and defines a locally free A_R-module of rank* 1*, or in other terminology, an invertible sheaf on Spec (A_R).*

The term *effective* refers to the fact that under the monomorphism τ_M the A_R-module $\sigma^* M$ is a subset of M. Dropping this condition, one is led to the more general notion of A-motive:

Definition 1.9. *An A-motive of rank r over an A-ring (R, γ) is a pair $\underline{M} = (M, \tau_M)$, where M is a locally free A_R-module of finite rank r, and τ_M is an isomorphism away from $\mathcal{J}$, that is,*

$$\tau_M : \sigma^* M_{|Spec\,(A_R) \setminus V(\mathcal{J})} \xrightarrow{\sim} M_{|Spec\,(A_R) \setminus V(\mathcal{J})}$$

As the following proposition shows, the two concepts are indeed closely related.

Proposition 1.10.

1. *Every effective A-motive $\underline{M}$ induces an A-motive in the natural way by restricting τ_M to Spec $(A_R) \setminus V(\mathcal{J})$.*
2. *For an A-motive (M, τ_M) to be effective it is sufficient and necessary to have*

$$\tau_M(\sigma^* M) \subset M.$$

In this case, the dimension $\dim(\underline{M})$ *is given by* $\dim_R(\operatorname{coker} \tau_M)$.

Proof. See [Har16, Prop. 2.3]. ⌟

Definition 1.11.

1. *A **morphism** $f : \underline{M} \to \underline{N}$ of A-motives is an A_R-module homomorphism f, such that $\tau_M \circ f = \sigma^* f \circ \tau_N$ where $\sigma^* f := f \otimes \mathrm{id} : \sigma^* M \to \sigma^* N$. As usual, we write* $\mathrm{Hom}_R(\underline{M}, \underline{N})$ *for the set of such morphisms, and we will leave out the index k when no confusion is possible.*

2. *If $f : \underline{M} \to \underline{N}$ is a surjective morphism of A-motives, then $\underline{N}$ is called a* ***quotient-motive*** *(also* ***factor-motive****) of $\underline{M}$.*

3. *A* ***quasi-morphism*** *$f : \underline{M} \to \underline{N}$ is an element of*

$$\mathrm{QHom}_R(\underline{M}, \underline{N}) := \mathrm{Hom}_R(\underline{M}, \underline{N}) \otimes_A Q.$$

We also write

$$\mathrm{QEnd}_R(\underline{M}) := \mathrm{QHom}_R(\underline{M}, \underline{M}).$$

So far, we have given the modern definition of A-motives over A-rings instead of A-fields, since it allows for a certain ease of notation in the context of reduction theory. However, we are for the most part interested in working with A-motives over fields k, and we will develop the rest of the theory of A-motives mostly for fields.

Remark 1.12. *There is a functor associating to a Drinfeld-module $\phi : A \to k\{\tau\}$ the pair*

$$\underline{M}(\phi) := (M(\phi) := k\{\tau\}, \tau_M : \sigma^* M(\phi) \to M(\phi), m \mapsto \tau \cdot m).$$

Then $\underline{M}(\phi)$ forms an abelian A-motive, and it can be shown that just as for Anderson-A-modules the effective, abelian A-motives of dimension 1 correspond exactly to the Drinfeld-A-modules.

A finite, surjective morphism of abelian varieties is called an **isogeny of abelian varieties**. The theory of Honda and Tate classifies abelian varieties over finite fields up to isogeny. We will now give the corresponding dual notion for A-motives.

Definition 1.13.

1. *An* ***isogeny*** *$f : \underline{M} \to \underline{N}$ of A-motives is a monomorphism $f \in \mathrm{Hom}(\underline{M}, \underline{N})$ with torsion cokernel. We denote the set of isogenies $\underline{M} \to \underline{N}$ by $\mathrm{Isog}(\underline{M}, \underline{N})$. If the set $\mathrm{Isog}(\underline{M}, \underline{N})$ is not empty, $\underline{M}$ and $\underline{N}$ are said to be* ***isogenous****, and we write $\underline{M} \sim \underline{N}$. If $\underline{M}$ and $\underline{N}$ are effective, there exists an induced map $\tau_{\mathrm{coker}\, f} : \sigma^* \mathrm{coker}\, f \to \mathrm{coker}\, f$, and if this map is (not) bijective, the isogeny f is said to be* ***(in)separable****.*

2. *A* ***quasi-isogeny*** *$f : \underline{M} \to \underline{N}$ of A-Motives is a quasi-morphism $f : \underline{M} \to \underline{N}$, which is invertible in $\mathrm{QHom}(\underline{M}, \underline{N})$. We denote the set of quasi-isogenies $\underline{M} \to \underline{N}$ as $\mathrm{QIsog}(\underline{M}, \underline{N})$.*

Proposition 1.14. *For any isogeny $f : \underline{M} \to \underline{N}$ between A-motives there exists an element $a \in A$ which annihilates the cokernel of f. Furthermore, there exists an isogeny $f_a^\vee : \underline{N} \to \underline{M}$ such that*

$$f \circ f_a^\vee = a.\mathrm{id}_{\underline{N}} \qquad \text{and} \qquad f_a^\vee \circ f = \mathrm{id}_{\underline{M}}.$$

Proof. See [BH11, Cor. 5.4] for the pure case, and [Har16, Cor. 5.15] for the general statement. ⌟

As an immediate consequence, we obtain the following Corollary:

Corollary 1.15. *A morphism $f : \underline{M} \to \underline{N}$ between A-motives is an isogeny if and only if it is a quasi-isogeny.*

Definition 1.16. *The isogeny $f_a^\vee$ is called the **dual isogeny** corresponding to (f, a).*

Corollary 1.17. *Let $\underline{M}$ and $\underline{M}'$ be A-motives and let $f \in \mathrm{QEnd}(\underline{M}, \underline{M}')$ be a quasi-isogeny $\underline{M} \to \underline{M}'$. Then*

$$\mathrm{QEnd}(\underline{M}) \cong \mathrm{QEnd}(\underline{M}').$$

Proof. Follows immediately from the existence of a dual isogeny. ⌟

Remark 1.18.

1. *Note that in general there is no canonical choice of a dualizing element a, and therefore no canonical choice of $f_a^\vee$. However, certain additional conditions guarantee a canonical choice; in particular, this is true for semisimple pure A-motives over finite fields.*

2. *The existence of dual isogenies implies that* equality up to isogeny *defines an equivalenve relation $\sim$ for A-motives. In particular, we can define the sets*

$$A\text{-}\mathrm{Mot}^\sim(k) := \{A\text{-motives } \underline{M} \text{ over } k\}/\sim$$

and

$$A\text{-}\mathrm{Mot}^\sim(k, d, r) := \{A\text{-motives } \underline{M} \text{ of dimension } d \text{ and rank } r \text{ over } k\}/\sim \ .$$

1.3.3. Global Shtukas

An A-motive, defined over some A-field (k, γ), lives over two special places, the characteristic ideal $\varepsilon = \ker \gamma$ and the fixed point ∞, with only the characteristic being allowed to vary from motive to motive. It is therefore natural to consider A-motives with "more characteristic places". We will call such a structure a *global shtuka*:

Definition 1.19. *A **global shtuka** of rank r with n paws over an $\mathbb{F}_q$-scheme S is a tuple*

$$\underline{\mathcal{N}} = (\mathcal{N}, \underline{c}, \tau_{\mathcal{N}}),$$

*where $\mathcal{N}$ is a locally free sheaf of rank r over $C_S = C \times S$ and $\underline{c}$ is a tuple $\underline{c} = (c_1, \ldots, c_n)$ of $\mathbb{F}_q$-morphisms Spec $(S) \to C$ such that $\tau_{\mathcal{N}}$ is an isomorphism on $\sigma^*\mathcal{N}_{|C_S \setminus \Gamma} \to N_{|C_S \setminus \Gamma}$ outside the graphs of the c_i, i. e. outside $\Gamma := \Gamma_{c_1} \cup \cdots \cup \Gamma_{c_n}$. We call the elements of $\underline{c}$ the **paws** of $\underline{\mathcal{N}}$.*

Remark 1.20. *The definition of a global shtuka $(\mathcal{N}, \underline{c}, \tau_{\mathcal{N}})$ does not put any restriction on the paws and their graphs of a global shtuka. In particular, the graphs don't have to be disjoint, which may cause problems when working locally. We are only interested in working with global shtukas over finite fields, in which case graphs are disjoint if any paw c_i given by a place on the curve C only appears once in $\underline{c}$.*

Definition 1.21. *Let $\underline{\mathcal{N}}$ and $\underline{\mathcal{N}'}$ be global shtukas over $S = Spec\,(k)$ with the same set of paws $c_1, \ldots, c_n$. Let η be the generic point of the curve $C_k = C \times Spec\,(k)$. Then we define the set of quasi-homomorphisms from $\underline{\mathcal{N}}$ to $\underline{\mathcal{N}'}$ as the set of morphisms between the generic fibres of $\underline{\mathcal{N}}$ and $\underline{\mathcal{N}'}$ compatible with $\tau_{\mathcal{N}} =$ and $\tau_{\mathcal{N}'}$, that is,*

$$\mathrm{QHom}(\underline{\mathcal{N}}, \underline{\mathcal{N}'}) := \{ f : \mathcal{N}_\eta \to \mathcal{N}'_\eta \mid f \circ \left(\tau_{\mathcal{N}}\right)_\eta = \left(\tau_{\mathcal{N}'}\right)_\eta \circ f \} \,.$$

From the definition it is quite immediately concluded that an A-motive is nothing but a global shtuka with two paws given by the characteristic point ε and the infinite point ∞.

Lemma 1.22. *Let $\underline{\mathcal{N}} = (\mathcal{N}, c_1, c_2, \tau_{\mathcal{N}})$ be a global shtuka of rank r over $Spec\,(k)$. The graph of c_2 consists of a single point, which we call ∞. Then the induced pair*

$$\underline{\mathrm{M}}(\underline{\mathcal{N}}) := \left(\Gamma(Spec\,(A_k), \mathcal{N}), \tau_{\mathcal{N}}\right)$$

forms a (not necessarily effective) A-motive of rank r over k. Vice versa, a (not necessarily effective) A-motive (M, τ_M) induces a global shtuka $\underline{\mathrm{N}}(\underline{M})$ by extending the associated sheaf $\widetilde{M}$ on $Spec\,(A_k)$ to C_S.

Proof. This follows immediately from the definitions; see also [HS16, Ex. 6.3]. ⌟

Remark 1.23. *For the special case of $n = 2$, Drinfeld [Dri87] introduced the concept of an F*-sheaf*, which in our terminology can be described as a global shtuka $(\mathcal{N}, c_1, c_2, \tau_{\mathcal{N}})$ such that c_1 and c_2 have disjoint graphs, $\tau_{\mathcal{N}}(\sigma^*\mathcal{N}) \subset \mathcal{N}$ outside Γ_{c_2} and $\tau_{\mathcal{N}}^{-1}(\mathcal{N}) \subset \sigma^*\mathcal{N}$ outside Γ_{c_1}, with the respective cokernel being locally free of rank 1 as $\mathcal{O}_S$-module. Global shtukas of this type are nowadays referred to as* Drinfeld shtukas*.*

Definition 1.24. *Let $\mathcal{N}$ and $\mathcal{N}'$ be two global shtukas over S with the same set of paws $(c_i : S \to C)_{1 \le i \le n}$. A **quasi-isogeny** between $\mathcal{N}$ and $\mathcal{N}'$ is an isomorphism*

$$f : \mathcal{N} \mid_{C_S \setminus D_S} \to \mathcal{N}' \mid_{C_S \setminus D_S}$$

such that $\tau_{\mathcal{N}'} \circ \sigma^* f = f \circ \tau_{\mathcal{N}}$, where D is some effective divisor on C.

Remark 1.25. *The definition of homomorphisms between global shtukas given above is consistent with the definition of homomorphism between A-motives, as can be seen from [BH11, Prop. 6.9]. Similarly, the definition of quasi-isogeny between global shtukas is consistent with the definition of quasi-isogenies between A-motives, see [BH11, Def. 6.1].*

1.3.4. The Frobenius endomorphism

Let us assume, that (k, γ) has characteristic $\varepsilon \neq 0$. In this case, $\mathbb{F}_\varepsilon := A/\varepsilon$ is a field. Let $e > 0$ be a positive multiple of $[\mathbb{F}_\varepsilon : \mathbb{F}_q]$. We call the morphism

$$\pi := \mathrm{Frob}_{q^e, \underline{M}} := \tau_M \circ \sigma^* \tau_M \circ \cdots \circ (\sigma^*)^{e-1} \tau_M \in \mathrm{Hom}(\sigma^{*e}\underline{M}, \underline{M})$$

the **q^e-Frobenius homomorphism** of $\underline{M}$. In particular, for the field $k = \mathbb{F}_{q^e}$, we get $(\sigma^*)^{[\mathbb{F}_\varepsilon : \mathbb{F}]} = (\sigma^*)^e = \mathrm{id}$, and therefore π is an endomorphism of $\underline{M}$, i. e. $\pi \in \mathrm{End}_k(\underline{M})$, simply called the **Frobenius endomorphism of $\underline{M}$**. Since τ_M is injective by definition and the rank of $\underline{M}$ is equal to the rank of $(\sigma^*)^e M$, the q-Frobenius π is an isogeny of $\underline{M}$.

Remark 1.26. *If one uses the non-linearized represention of* $\underline{M} = (M, \tau_{M,\sigma})$*, where* $\tau_{M,\sigma}$ *is a* σ*-linear injection* $M \hookrightarrow M$*, then the Frobenius is simply* $\tau_{M,\sigma}^{[k:\mathbb{F}_q]}$*.*

Remark 1.27. *Write* $\mathbb{F}_s = \mathbb{F}_{q^e}$ *and with* $S := Spec\,(\mathbb{F}_{q^e})$ *let* $\underline{N}$ *be a global shtuka over* S *with paws* $c_1, \ldots, c_n$ *and underlying morphism* $\tau_{\mathcal{N}} : \sigma^*\mathcal{N} \mid_{C_S \setminus \cup_i \Gamma_{c_i}} \to \mathcal{N} \mid_{C_S \setminus \cup_i \Gamma_{c_i}}$*. We call the quasi-morphism*

$$\pi_{\mathcal{N}} := \tau_{\mathcal{N}} \circ \sigma^*\tau_{\mathcal{N}} \circ \cdots \circ (\sigma^*)^{e-1}\tau_{\mathcal{N}} : \mathcal{N} \mid_{C_S \setminus \cup_i \Gamma_{c_i}} \to N \mid_{C_S \setminus \cup_i \Gamma_{c_i}}$$

the ***Frobenius of the global shtuka*** $\mathcal{N}$*. It is well-defined, since just like before* $\sigma^{*e} = \mathrm{id}_S$*, and an isogeny of global shtukas in the sense of Def. 1.24.*

Definition 1.28. *By Cor. 1.42, we have* $\min_{\pi,Q} \in A[X]$*, where* π *is the Frobenius endomorphism of an A-motive* $\underline{M}$*. We call* $\min_{\pi,Q}$ *the* ***Frobenius polynomial*** *of* $\underline{M}$*.*

Lemma 1.29. *Let* $\underline{M}$ *be an effective A-motive over the finite field* k *with* $q = p^e$ *elements with Frobenius endomorphism* π*, and let* k'/k *be a finite field extension. Let* $\underline{M}'$ *denote the A-motive* $\underline{M} \otimes_k k'$ *obtained by scalar extension. Let* π' *denote the Frobenius of* $\underline{M}'$*. Then*

$$\pi' = (\pi \otimes 1)^{[k':k]}\,.$$

Proof. The statement is most obvious when one uses the non-linearized representation $(M, \tau_{M,\sigma})$ of $\underline{M}$ and $\underline{M}'$, since then the Frobenius π is nothing but $\tau_{M,\sigma}^e$ and π' is given by

$$\left(\tau_{M',\sigma}\right)^{e\cdot[k':k]} = \left(\tau_{M,\sigma} \otimes 1\right)^{e\cdot[k':k]} = \left(\tau_{M,\sigma}^e \otimes 1\right)^{[k':k]} = (\pi_M \otimes 1)^{[k':k]}\,.$$

⌟

1.3.5. Simple and Semisimple A-Motives

An abelian variety A is always *semisimple up to isogeny*, that is, one can always find a finite number of simple abelian varieties A_i such that their product $\prod A_i$ is isogenous to A. Unfortunately, A-Motives (even pure ones, see next section) do not have this property.

Definition 1.30. *An A-Motive* $\underline{M} \neq 0$ *is called* ***simple****, if it has no non-trivial factor-motives. It is called* ***semisimple****, if, up to isogeny,* $\underline{M}$ *admits a decomposition into a finite direct sum of simple A-Motives.*

We use the notations

$$A\text{-Mot}_{\mathrm{ss}}^{\sim}(k) := \{\text{semisimple } A\text{-motives } \underline{M} \text{ over } k\}/\sim$$

and

$$A\text{-Mot}_{\mathrm{s}}^{\sim}(k) := \{\text{simple } A\text{-motives } \underline{M} \text{ over } k\}/\sim,$$

as well as $A\text{-Mot}_{\mathrm{ss}}^{\sim}(k, d, r)$ and $A\text{-Mot}_{\mathrm{s}}^{\sim}(k, d, r)$ of obvious meaning.

Remark 1.31. *In contrast to abelian varieties, A-motives do no need to be semisimple. For example, consider the motive $\underline{M} = (M, \tau_M)$ defined over $k \cong \mathbb{F}_q$ via $M := k[t]^2$ and $\tau_M = \left(\begin{smallmatrix} 1-t/\theta & t \\ 0 & 1-t/\theta \end{smallmatrix}\right)$ for some non-zero $\theta \in k^\times$. This motive has a canonical factor motive $\mathrm{pr}_2 : \underline{M} \twoheadrightarrow \underline{M}' = (k[t], 1-t/\theta)$, but it isn't semisimple, since then $\underline{M}'$ would have to be a direct summand of $\underline{M}$,i. e. there exists an $\iota : \underline{M}' \hookrightarrow \underline{M}$, and we obtain $\mathrm{pr}_2 \circ \iota = \mathrm{id}_{M'}$. Writing ι as matrix $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in k(t)^2$, one immediately deduces $y = 1$ and then compatibility of ι with τ_M and $\tau_{M'}$ yields the contradiction $0 = x - \sigma(x) = \frac{t}{1-t/\theta} \neq 0$, hence $\underline{M}$ cannot actually be semisimple. Since $\underline{M}$ is defined over a finite field, there exists a base-change to a suitable large finite field extension k'/k such that $\underline{M} \otimes k'$ will be semisimple, see Prop. 1.32.*

Proposition 1.32. *Let $\underline{M}$ be an A-motive of rank r and dimension d. If $\underline{M}$ is defined over a finite field $\mathbb{F}_s$, then there exists a finite field extension $\mathbb{F}_{s'} / \mathbb{F}_s$ such that $\underline{M} \otimes \mathbb{F}_{s'}$ is semisimple.*

Proof. This follows from [BH09, Thm. 6.15]. ⌟

Proposition 1.33. *If an A-motive $\underline{M}$ is semisimple over a finite field $\mathbb{F}_s$, then every scalar extension $\underline{M} \otimes \mathbb{F}_{s'}$ is semisimple.*

Proof. See [BH09, Cor. 6.16]. ⌟

Proposition 1.34. *Let $\underline{M}$ be a A-motive.*

1. *If $\underline{M}$ is simple, then $\mathrm{QEnd}(\underline{M})$ is a division algebra over Q.*
2. *If $\underline{M}$ is semisimple, then $\mathrm{QEnd}(\underline{M})$ is semisimple. More precisely, the finite decomposition $\underline{M} = \oplus_i \underline{M}_i$ of $\underline{M}$ into simple quotient-motives yields a decomposition of $\mathrm{QEnd}(\underline{M})$ into a finite direct sum of full matrix algebras over the division algebras $\mathrm{QEnd}(\underline{M}_i)$.*
3. *If $\underline{M}$ is defined over a finite field $\mathbb{F}_s$, then both statements above are equivalences.*

Proof. This is proven in [BH11, Thm. 7.8] and [BH09, Thm. 6.11] for abelian τ-sheaves and thereby for pure A-motives. The proof works just as well for not necessarily pure motives, since the behaviour at the place ∞ is not relevant to the argument. ⌟

1.3.6. Purity of A-Motives

We use the notation $A_{\infty,k}$ and $Q_{\infty,k}$ to indicate the tensor products of A_∞ and Q_∞ with k. Note that $A_{\infty,k}$ is a product of discrete valuation rings, while $Q_{\infty,k}$ if a product of fields.

Definition 1.35. *An A-Motive $\underline{M}$ of rank r and dimension d is called **pure**, if there exists a free $A_{\infty,k}$-submodule $W_M \subset M \otimes_{A_k} Q_{\infty,k}$ of rank r such that*

$$\tau_M^{n\cdot r}((\sigma^*)^{n\cdot r}(W_M)) = z_\infty^{-n\cdot d} W_M$$

for some $n \in \mathbb{N}$. In this case, the ratio

$$\mathrm{wt}(\underline{M}) := \frac{d}{r}$$

*is called the **weight** of $\underline{M}$. We use the notation*

$$A\text{-Mot}_{\mathrm{p}}^{\sim}(k, \mu)$$

to denote the set of isogeny classes of pure A-motives over k of weight μ.

There are obvious meanings attached to $A\text{-Mot}_{\mathrm{p}}^{\sim}(k, d, r)$, $A\text{-Mot}_{\mathrm{p,s}}^{\sim}(k, \mu), \dots,$ which we will not all list explicitly.

Remark 1.36. *By our definition, if an (effective) A-motive $\underline{M}$ is also finitely generated as $k\{\tau\}$-module, it is called* abelian. *However, if $\underline{M}$ satisfies the above definiton of purity and has dimension $d > 0$, it is automatically abelian according to [And86, §1.9].*

Remark 1.37. *Note that the purity condition is a local condition at ∞. Therefore, arguments for pure A-motives not depending on the behaviour at ∞ will generally work just fine for general (effective) A-motives. In order to work at the place ∞ more effectively, Hartl-Bornhofen [BH11, BH09] developed the notion of* abelian τ-sheaves. *This construction only works for pure motives; therefore, results from these two papers are only obtained for pure A-motives. However, keeping in mind that purity is only concerned with the point ∞, most of the results can be copied for non-pure A-motives.*

The name *purity* is justified by the following

Proposition 1.38. *Let $\underline{M}, \underline{M}'$ be two pure A-motives of different weights, i. e.*

$$\mathrm{wt}(\underline{M}) \neq (\underline{M}').$$

Then

$$\mathrm{Hom}(\underline{M}, \underline{M}') = \{0\}.$$

Proof. This is an immediate consequence of the isoshtuka decomposition of Thm. 1.59. It can also be argued directly, see [BH11, Cor. 3.5]. ⌟

Definition 1.39. *An A-motive is called **primitive**, if its rank r and dimension d are relatively prime, i. e. $(r, d) = 1$.*

In general, primitive motives do not need to be simple, since a direct sum of pure motives is never simple, but can be primitive; for instance,

$$\underline{M} := \big(A_k, \tau_1 = (t - \theta)\big) \oplus \big(A_k^2, \tau_2 = \left(\begin{smallmatrix} 0 & t-\theta \\ 1 & 1 \end{smallmatrix}\right)\big)$$

provides a specific counterexample, as $\underline{M}$ has rank 3 and dimension 2. However, for pure motives, we can conclude simplicity from primitivity:

Proposition 1.40. *A primitive pure A-motive $\underline{M}$ is automatically simple.*

Proof. (See [BH11, Prop. 7.4].) Let $\underline{M}$ be primitive pure, and let $\underline{M}'$ be a non-zero factor-A-motive with non-zero surjection morphism $p : \underline{M} \to \underline{M}'$. Then

$$\frac{d}{r} = \mathrm{wt}(\underline{M}) = \mathrm{wt}(\underline{M}') = \frac{d'}{r'}$$

by the last Proposition, and therefore $dr' = d'r$. Hence by the primitivity assumption we must have $d = d'$ and $r = r'$, and then p is an isomorphism. ⌟

Theorem 1.41. *Let $\underline{M}, \underline{N}$ be A-motives over some field k. Then the set of homomorphisms* $\mathrm{Hom}(\underline{M}, \underline{N})$ *is a projective A-module of rank* $\leq \mathrm{rk}(\underline{M}) \cdot \mathrm{rk}(\underline{N})$.

Proof. See [And86, Cor. 1.7.2] and [BH11, Thm. 9.5] for the original statement about effective (pure) A-motives and [Har16, Cor. 2.6] for the most general formulation. ⌟

As usual, we define the minimal polynomial of some homorphism $f : \underline{M} \to \underline{N}$ between A-motives to be the unique normalized generating element of the ideal

$$(\min_{f,Q}) = \ker(Q[X] \to \mathrm{QEnd}_k(\underline{M}), X \mapsto f)\,.$$

Corollary 1.42. *Let $\underline{M}$ be a pure A-motive over some field L and $f \in \mathrm{End}(\underline{M})$ be some endomorphism of $\underline{M}$. Then the minimal polynomial $\min_{f,Q}$ of f has coefficients in A.*

Proof. This is an immediate corollary of the preceding statement and can also be found as [BH11, Cor. 9.6]. ⌟

1.4. Reduction Theory

Let R be a Dedekind domain with field of fractions $\mathrm{Frac}(R) = k$.

Definition 1.43. *Let $\underline{M} = (M, \tau)$ be a τ-module over $Spec\,(A_R)$, and let $\mathfrak{p} \in Spec\,(R)$ with residue field $\kappa(\mathfrak{p})$. Let $\iota_\mathfrak{p} : Spec\,(\kappa(\mathfrak{p})) \hookrightarrow Spec\,(A_R)$ be the canonical inclusion induced by $A_R \twoheadrightarrow A_{\kappa(\mathfrak{p})}$. The **reduction $\underline{M}_\mathfrak{p}$ of $\underline{M}$ at $\mathfrak{p}$** is then defined as the pair consisting of the locally free $A_{\kappa(\mathfrak{p})}$-module*

$$M_\mathfrak{p} := \iota_\mathfrak{p}^* M = M \otimes_{A_R} A_{\kappa(\mathfrak{p})} = M \otimes_R \kappa(\mathfrak{p})$$

together with the induced homomorphism

$$\tau_\mathfrak{p} : \sigma^* M_\mathfrak{p} \to M_\mathfrak{p}.$$

Remark 1.44. *Note that the reduction defined above does not, in general, define a τ-module, since the induced morphism $\tau_\mathfrak{p}$ may not be injective, leading to the notion of* good *reduction defined in the next section.*

1.4.1. Good Reduction of τ-modules

We are ultimately interested in reduction theory of A-motives, which live over a field k, so we need to talk about reduction of τ-modules defined over k, where k is the field of fractions of a Dedekind domain $R = \mathcal{O}_k$. In order to reduce these to τ-modules over $R/\mathfrak{p} = \kappa(\mathfrak{p})$, we need to introduce the notion of a *model* $\underline{\mathcal{M}}$ of $\underline{M}$, i. e. a τ-module over R extending to $\underline{M}$ after base-change to k.

Definition 1.45. *Let $\underline{M}$ be a τ-module over k. A **model** of $\underline{M}$ over R is a τ-module $\underline{\mathcal{M}}$ over A_R with*

$$\underline{M} = \underline{\mathcal{M}}_k = \underline{\mathcal{M}} \otimes_R k.$$

We can now talk about a reduction of $\underline{M}$ at $\mathfrak{p} \in Spec\,(\mathcal{O}_k)$.

Definition 1.46. *A τ-module $\underline{M}$ over A_k has* ***good reduction at*** $\mathfrak{p} \in Spec\,(R)$*, if there exists a model $\underline{\mathcal{M}} = (\mathcal{M}, \tau)$ over A_R such that the induced map $\tau_\mathfrak{p}$ is an injection, i. e. the reduction of $\underline{\mathcal{M}}$ at $\mathfrak{p}$ is again a τ-module. Such a model is then called a* ***good model*** *of $\underline{M}$ at $\mathfrak{p}$.*

1.4.2. Potentially Good Reduction of τ-modules

As in the theory of abelian varieties, the property of having good reduction at a prime does depend on the base field. We are therefore led to introduce the notion of *potentially* good reduction:

Definition 1.47. *A τ-module $\underline{M}$ over k has* ***potentially good reduction at*** $\mathfrak{p} \in Spec\,(R)$*, if there exists a finite field extension k'/k such that $\underline{M}_{k'} = \underline{M} \otimes_k k'$ has good reduction at one place of k' above $\mathfrak{p}$.*

1.4.3. Reduction of A-motives

So far, we have only talked about τ-modules. However, as it turns out, the technique of reduction works well with the additional structural properties valid for A-motives.

Lemma 1.48. *Let $\underline{M}$ be an A-motive over k of rank r and dimension d. Let k'/k be a finite field extension. Then $\underline{M}_{k'} = \underline{M} \otimes_k k'$ is again an A-motive of rank r and dimension d.*

Let now R be a Dedekind domain such that $\gamma : A \to k$ factorizes through R.

Theorem 1.49. *Let $\underline{M}$ be a (pure) A-motive over k with good reduction at $\mathfrak{p} \in Spec\,(R)$. Let $\underline{\mathcal{M}}$ be a good model of $\underline{M}$ at $\mathfrak{p}$. Then the reduction $\underline{\mathcal{M}}_\mathfrak{p}$ is a (pure) A-motive over $\kappa(\mathfrak{p})$.*

Proof. This was proven in [Pel09, Thm. 2.3.9] for pure motives, and in [HH16, Thm. 4.7] for general A-motives. ⌟

1.5. Local Constructions

There are two important local objects attached to abelian varieties, namely their *Tate-Modules* and their *p-divisible groups*. For both classes of objects, analogies exist on the function field side. The former can be constructed quite easily; the latter, however, are obtained by a somewhat more involved process. As a reminder, a p-divisible group of height r over a ring R is given by a diagram

$$G_0 \xrightarrow{i_0} G_1 \xrightarrow{i_1} \dots \xrightarrow{i_{n-1}} G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{i_{n+1}} \dots$$

of finite commutative group schemes G_n of order $p^{n \cdot r}$ such that each sequence

$$0 \to G_n \xrightarrow{i_n} G_{n+1} \xrightarrow{p^n} G_{n+1}$$

is exact. (Some authors prefer to call the direct limit $\varinjlim G_n$ the p-divisible group defined by such a diagram above, while others refer to the diagram itself as a p-divisible group.) A direct analogy might therefore be expected to look similar in nature, involving a direct limit of sheafs of some kind. This can, in fact, be done, and one ends up with so-called *z-divisible local Anderson modules.*

Definition 1.50 ([HS17, HK18])**.** *Let A be a ring of the usual form, let $\varepsilon \subset A$ be some maximal ideal, and let $z \in Q$ be a local uniformizer at ε. Let S be some base scheme. A **z-divisible local Anderson module** over S is a $\mathbb{F}_\varepsilon[[z]]$-module-sheaf G on the big* fppf-*site of S such that*

(i) $G = \varinjlim G[z^n]$ *with* $G[z^n] := \ker(z^n : G \to G)$,

(ii) $z : G \twoheadrightarrow G$ *is an epimorphism,*

(iii) All $G[z^n]$ are representable by finite, locally free, strict $\mathbb{F}_\varepsilon$-module schemes over S (see [Fal02, Abr06]), and

(iv) locally on S there exists $d \in \mathbb{N}_0$ such that $(z-\zeta)^d = 0$ on ω_G, where

$$\omega_G := \varprojlim \omega_{G[z^n]} \quad \textit{with} \quad \omega_{G[z^n]} = (S \xrightarrow{0} G[z^n])^* \Omega^1_{G[z^n]/S}.$$

This definition is, unfortunately, somewhat unwieldy. Luckily, it can generally be avoided entirely, since the category of z-divisible local Anderson modules turns out to be anti-equivalent (see [HS16, Thm. 8.3]) to the much simpler category of (effective) *local shtukas*, which we will develop next.

1.5.1. Local Shtukas

There are two versions of local shtukas, depending on what power of Frobenius one is working with. We will give both, and explain their relationship.

Local $\hat{\sigma}$-shtukas

Let v be some place of Q away from ∞, let k_v be the completion of k at v and let $\mathcal{O}_v$ be the valuation ring of k_v. Let $z = z_v$ be a local uniformizer at v. In particular, $A_v = \mathbb{F}_v[\![z]\!]$ and $Q_v = \mathbb{F}_v((z))$. The image of z under the canonical extension of γ to A_v is labeled $\zeta := \zeta_v := \gamma(z)$. Write $q_v := \#\,\mathbb{F}_v$ and let $\hat{\sigma}$ be the endomorphism of $\mathcal{O}_v[\![z]\!]$ that is the identity on z and the q_v-th power map on elements of $\mathcal{O}_v$, so that $\hat{\sigma} : b \cdot z \mapsto b^{q_v} \cdot z$.

Definition 1.51. *Let $\hat{M}$ be a free $\mathcal{O}_v[\![z]\!]$-module of rank r and $\hat{\tau}_{\hat{M}}$ be an isomorphism*

$$\hat{\tau}_{\hat{M}} : \hat{\sigma}^* \hat{M}\left[\frac{1}{z-\zeta}\right] \to \hat{M}\left[\frac{1}{z-\zeta}\right].$$

*The pair $\underline{\hat{M}} := (\hat{M}, \hat{\tau}_{\hat{M}})$ is then called a **local $\hat{\sigma}$-shtuka of rank r over $\mathcal{O}_v$**. If additionally $\hat{\tau}_{\hat{M}}(\hat{\sigma}^* \hat{M}) \subset \hat{M}$ holds, then $\underline{\hat{M}}$ is called **effective**, and if we even have* *equality,* $\underline{\hat{M}}$ *is called* ***étale****. If the cokernel of* $\hat{\tau}_{\hat{M}}$ *is locally free of finite rank as* $\mathcal{O}_v$*-module, we call the* $\mathcal{O}_v$*-rank of* coker $\hat{\tau}_{\hat{M}}$ *the* ***dimension of*** $\underline{\hat{M}}$ *and write*

$$\dim \underline{\hat{M}} := \mathrm{rk}_{\mathcal{O}_v} \operatorname{coker} \hat{\tau}_{\hat{M}} \,.$$

Definition 1.52. *Let* $\underline{\hat{M}}$ *and* $\underline{\hat{N}}$ *be local* $\hat{\sigma}$*-shtukas over* $\mathcal{O}_v$*. A* ***morphism*** *of local* $\hat{\sigma}$*-shtukas* $\underline{\hat{M}} \to \underline{\hat{N}}$ *over* $\mathcal{O}_v$ *is a morphism of the underlying free* $\mathcal{O}_v[\![z]\!]$*-modules that is compatible with the underlying isomorphisms* $\hat{\tau}_{\hat{M}}$ *and* $\hat{\tau}_{\hat{N}}$*. A* ***quasi-morphism*** *of local* $\hat{\sigma}$*-shtukas is a morphism of* $\mathcal{O}_v[\![z]\!][\frac{1}{z}]$*-modules* $M[\frac{1}{z}] \to N[\frac{1}{z}]$ *compatible with the underlying isomorphisms, and we write* $\mathrm{QHom}_{\mathcal{O}_v}(\underline{\hat{M}}, \underline{\hat{N}})$ *for the set of quasi-morphims as well as* $\mathrm{QEnd}_{\mathcal{O}_v}(\underline{\hat{M}}) := \mathrm{QHom}_{\mathcal{O}_v}(\underline{\hat{M}}, \underline{\hat{M}})$*. A quasi-morphism is called a* ***quasi-isogeny*** *if it is an isomorphism of* $\mathcal{O}_V[\![z]\!][\frac{1}{z}]$*-modules.*

Let $\underline{M}$ be an A-motive over k of characteristic ε and $v \in \mathit{Spec}\,(A)$ a closed point. We want to construct a *local shtuka of* $\underline{M}$ *at* v. Assume that $\underline{M}$ has good reduction, and let $\underline{\mathcal{M}}$ be a good model of $\underline{M}$. Note that $\underline{\mathcal{M}}$ is an A-motive over $\kappa(v)$ of characteristic v. First, we discuss the case that $\mathbb{F}_v$ is not larger than $\mathbb{F}_q$, i. e., $[\mathbb{F}_v : \mathbb{F}_q] = 1$. In this situation,

$$\underline{\hat{M}}_v(\underline{M}) := (\mathcal{M} \otimes_{A_k} A_{v,k}, \tau_{\mathcal{M}} \otimes \mathrm{id})$$

forms a local $\hat{\sigma}$-shtuka at v.

If, however, $\mathbb{F}_v$ contains $\mathbb{F}_q$ only as a smaller subfield, more care has to be taken, since then $A_{v,\mathcal{O}_V}$ and $\mathbb{F}_v \otimes_{\mathbb{F}_q} \mathcal{O}_v$ are not integral domains. Write $f_v := [\mathbb{F}_v : \mathbb{F}_q]$. We now consider the sequence of ideals

$$\mathfrak{a}_i := (a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v) \subset A_{v,k}$$

for $0 \le i \le f_v - 1$. It is useful to write this interval as $\mathbb{Z}/(f_v)$ and consider the index i as an residue class $\mod f_v$. We now observe that the ideals $\mathfrak{a}_i$ have zero intersection $\prod_i \mathfrak{a}_i$ and are pairwise coprime. This can be seen from the fact that for each $a \in \mathbb{F}_v$ the equation

$$\mathrm{min}_{a,\mathbb{F}_q} \mid (X - a^{q^0}) \cdot (X - a^{q^1}) \cdots (X - a^{q^{f_v - 1}}) \in \mathbb{F}_q[X]$$

holds, since the right hand side has a as a root. The tensor product $\mathbb{F}_v \otimes_{\mathbb{F}_q} \mathcal{O}_v$ then splits as

$$\mathbb{F}_v \otimes_{\mathbb{F}_q} \mathcal{O}_v \cong \prod_{\mathrm{Aut}_{\mathbb{F}_q}(\mathbb{F}_v / \mathbb{F}_q)} \mathbb{F}_v \otimes_{\mathbb{F}_v} \mathcal{O}_v \cong \prod_{i \in \mathbb{Z}/(f_v)} \mathbb{F}_v \otimes_{\mathbb{F}_q} \mathcal{O}_v / \mathfrak{a}_i \,,$$

and we obtain as an immediate consequence the following decomposition:

Lemma 1.53. *The ring* $A_{v,k}$ *splits into a finite product*

$$A_{v,k} \cong \prod_{i \in \mathbb{Z}/(f_v)} A_{v,k}/\mathfrak{a}_i.$$

The factors are all canonically isomorphic to $\mathcal{O}_v[\![z]\!]$ *and are cyclically permutated by* σ *and left invariate under* $\hat{\sigma}$*. The ideals* $\mathfrak{a}_i$ *are in one-to-one-correspondence with the points* v_i *of* $C_{\mathbb{F}_v}$ *above* v*.*

We can now amend our definition of associated shtukas to be

$$\underline{\hat{M}}_v(\underline{M}) := (\mathcal{M} \otimes_{A_k} A_{v,k}/\mathfrak{a}_0, (\tau_{\mathcal{M}} \otimes \mathrm{id})^{f_v}) .$$

Note that this agrees with the definiton above whenever $f_v = 1$, that is, whenever $\mathbb{F}_v = \mathbb{F}_q$.

Definition 1.54. *The local shtuka $\underline{\hat{M}}_v(\underline{M})$ is called the **local $\hat{\sigma}$-shtuka at v associated to $\underline{M}$**, and the local isoshtuka $\underline{\hat{V}}_v(\underline{M})$ is called the **local $\hat{\sigma}$-isoshtuka at v associated to $\underline{M}$**.*

Note that forming the local $\hat{\sigma}$-shtuka $\underline{\hat{M}}$ even in the case of $f_v > 1$ does not amount to forgetting the structure of $\underline{\hat{M}} \otimes A_{v,\mathcal{O}_v}$, since the morphism $\tau_{\mathcal{M}} \otimes \mathrm{id}$ can be recovered from $\underline{\hat{M}}_v(\underline{M})$. This leads to the discussion of σ-shtukas in the next section.

Local σ-shtukas and their connection to $\hat{\sigma}$-shtukas

Definition 1.55. *Let k be an A-field.*

1. *A **local σ-shtuka** at $v \neq \infty$ over k of rank r is a pair $\underline{\hat{M}} = (\hat{M}, \hat{\tau})$, where $\hat{M}$ is a free $A_{v,k}$-module of rank r and $\hat{\tau} : \sigma^* \hat{M} \to \hat{M}$ is an injective $A_{v,k}$-homomorphism. The shtuka $\underline{\hat{M}}$ is called **étale**, if its underlying $A_{v,k}$-monomorphism is an isomorphism.*

2. *A **local σ-isoshtuka** at $v \neq \infty$ over k of rank r is a pair $\underline{\hat{V}} = (\hat{V}, \hat{\tau})$, where $\hat{V}$ is a free $Q_{v,k}$-module of rank r and $\hat{\tau}$ is a $Q_{v,k}$-isomorphism $\sigma^* \hat{V} \to \hat{V}$.*

3. *A **morphism of local σ-shtukas** $\hat{f} : \underline{\hat{M}} \to \underline{\hat{N}}$ is an $A_{v,k}$-homomorphism satisfying $\hat{\tau}_{\hat{N}} \circ \sigma^* \hat{f} = \hat{f} \circ \hat{\tau}_{\hat{M}}$, and similarly for σ-isoshtukas. We denote the set of such homomorphisms by $\mathrm{Hom}_{A_{v,k}[\tau]}(\underline{\hat{M}}, \underline{\hat{N}})$ and $\mathrm{Hom}_{Q_{v,k}[\tau]}(\underline{\hat{V}}, \underline{\hat{W}})$, respectively.*

Remark 1.56. *Local σ-isoshtukas are sometimes in the literature referred to as* z-isocrystals *or* Dieudonné-$\mathbb{F}_q ((z))$-modules.

Let $\underline{M}$ be an A-motive over k, and v be some place of Q away from ∞. We define the associated local σ-shtuka of $\underline{M}$ at v as

$$\underline{\hat{M}}_v(\underline{M}) := (M \otimes_{A_k} A_{v,k}, \tau_M \otimes \mathrm{id})$$

and the associated local σ-isoshtuka as

$$\underline{\hat{V}}_v(\underline{M}) := (M \otimes_{A_K} Q_{v,K}, \tau_M \otimes \mathrm{id}) .$$

Proposition 1.57. *Let i be some positive integer, and let $\mathfrak{a}_i$ be the ideal*

$$(a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v) \subset A_{v,k}$$

for all $0 \leq i < [\mathbb{F}_v : \mathbb{F}_q]$

(a) *There is an equivalence of categories between local σ-isoshtukas at v over k and local $\hat{\sigma}$-isoshtukas at v_i over k of the same rank, induced via reduction at $\mathfrak{a}_i$. More precisely, the functor is given by*

$$(\hat{V}, \tau) \mapsto (\hat{V}/\mathfrak{a}_i \cdot \hat{V}, \hat{\tau} = \tau^f : \sigma^{*f}\hat{V}/\mathfrak{a}_i \cdot V_i \to \hat{V}/\mathfrak{a}_i \cdot V_i)\,.$$

(b) *There is an equivalence of categories between étale local σ-shtukas at v over k and étale local $\hat{\sigma}$-shtukas at v_i over k of the same rank, induced via reduction at $\mathfrak{a}_i$ as above.*

(c) *At the characteristic place ε of $\underline{M}$, there exists an equivalence of categories between local σ-shtukas at ε and the category of $\hat{\sigma}$-shtukas at the point v_0 on $C_{\mathbb{F}_v}$ corresponding to the ideal $\mathfrak{a}_0$, induced by reduction at $\mathfrak{a}_0$.*

Proof. This is [BH11, Prop. 8.5] and [BH11, Prop. 8.8]. ⌟

Definition 1.58. *Let $v \in C$ be a closed point and let $z_v \in Q$ be a local uniformizer at v. Let l, m be integers without common divisors, i. e. $(m, l) = 1$, and let $l \geq 1$. The* ***standard-$\hat{\sigma}$-isoshtuka at** v **of slope** $\frac{m}{l}$ is defined as*

$$\underline{\hat{V}}_{m,l} := \left(k^{alg}(\!(z_v)\!)^{\oplus l}, \hat{\tau} = \begin{pmatrix} 0 & 0 & \dots & 0 & z_v^m \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 & 0 \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix} \right).$$

Theorem 1.59. *Every local isoshtuka $\underline{\hat{V}}$ at ∞ defined over an algebraically closed field $k = k^{alg}$ may be written as direct sum of standard-isoshtukas in the following way: There exists a unique set of pairs of coprime integers $(m_i, l_i) \in \mathbb{Z} \times \mathbb{N}_{>0}$ such that*

$$\underline{\hat{V}} \cong \bigoplus_i \underline{\hat{V}}_{m_i,l_i}\,.$$

Proof. This is the analogous statement to the classification of F-isocrystals by Dieudonné and Manin [Man63], and in our setting it was proven by Laumon [Lau96, Thm. 2.4.5]. See also [HJ16, §3.2]. ⌟

Local and Global Shtukas

Remark 1.60. *(See [HS16, Ex. 6.4] and [ARH14, Lemma 5.3].)*

(a) *As the name suggests, there is a close relationship between local and global shtuka. Let $\underline{\mathcal{N}} = (\mathcal{N}, c_1, \dots, c_n, \tau_{\mathcal{N}})$ be a global shtuka of rank r over k in the sense of (1.19). Pick a closed point v on the curve C with associated ideal sheaf $\mathfrak{p}_c \subset \mathcal{O}_C$ and let $z \in Q$ be a uniformizing parameter at v. Put $\zeta_i := c_i^*(z) \in k$. Then the formal completion M of $\mathcal{N}$ along the graph of c_i together with the morphism*

$$\hat{\tau}_M := \tau_{\mathcal{N}}^{[\mathbb{F}_v:\mathbb{F}_q]} : \sigma_q^{*f} M\left[\frac{1}{z-\zeta_i}\right] \xrightarrow{\sim} M\left[\frac{1}{z-\zeta_i}\right]$$

forms a local shtuka over k of rank r.

(b) *Note that the argument above may not always work if instead of working over a field k we allow the global shtuka $\mathcal{N}$ to live over a general ring R. In this situation, the element $\xi = c_i^*(z) \in R$ needs to be nilpotent. For $R = k$, this is automatic, since $c_i^*(z)$ will always be zero.*

(c) *Any A-motive $\underline{M}$ over k can be viewed as a global shtuka $\underline{\mathcal{N}} = (\mathcal{N}, \underline{c}, \tau_{\mathcal{N}})$ over Spec (k) according to Lemma 1.22, to which we just attached a local shtuka, leaving us with two apparently different notions of attached local shtukas for the same essential structure. However, the two of them coincide, since the completion of $\mathcal{O}_{C_k}$ at the graph of c_1 is naturally isomorphic to $A_{v,k}/\mathfrak{a}_0$ by [ARH14, Lemma 5.3].*

Purity and local shtukas

The notion of local shtuka allows us to give a characterization of purity by looking at the decomposition of the local isoshtuka at ∞.

Proposition 1.61. *Let $\underline{M}$ be an A-Motive of rank r and dimension d over an A-field k. Then the following conditions are equivalent:*

1. *$\underline{M}$ is pure.*
2. *There exists an $A_{\infty,k}$-lattice W_M in $\underline{\hat{V}}_\infty(\underline{M})$, such that $z^d\tau^r : (\sigma^*)W_M \to W_M$ gives an isomorphism of W_M.*
3. *We have*
$$\underline{\hat{V}}_\infty(\underline{M}) \otimes_{Q_{\infty,k}} Q_{\infty,k^{alg}} = \left(\underline{\hat{V}}_{m,l}\right)^{\oplus r/l},$$
where $\underline{\hat{V}}_{m,l}$ is the standard-σ-isoshtuka of slope $\frac{m}{l} = -\frac{\dim(M)}{\mathrm{rk}(M)}$.

Proof. What we have to show is the equivalence between the second and third condition. Assume the second condition holds for $\underline{M}$. Consider the unique decomposition

$$\underline{\hat{V}}_\infty(\underline{M}) \cong \bigoplus_i \underline{\hat{V}}_{m_i,l_i}$$

from Thm. 1.59. We define

$$W_i := W_M \otimes_{A_{\infty,k}} A_{\infty,k^{\mathrm{alg}}} \cap \underline{\hat{V}}_{m_i,l_i}.$$

In particular, each W_i is finitely generated over $A_{\infty,k^{\mathrm{alg}}}$ and $W_i \cdot Q_{\infty,k^{\mathrm{alg}}} = \underline{\hat{V}}_{m_i,l_i}$. But now we can conclude that $l = l_i$ and $m = m_i$, since $\tau_M^{l_i} = z^{m_i}$ on $\underline{\hat{V}}_{m_i,l_i}$.

Vice versa, assume that the unique decomposition into standard-isoshtukas looks like

$$\underline{\hat{V}}_\infty(\underline{M}) \otimes_{Q_{\infty,k}} Q_{\infty,k^{\mathrm{alg}}} \cong \underline{\hat{V}}_{d,r}^{\oplus r/l}.$$

Then a suitable lattice W_M can be defined as preimage of the canonical lattice $A_{\infty,k}^{\oplus r}$ found in the right hand side. ⌟

The last proposition deals with the relationship describes the local shtuka decomposition at ∞ for pure A-motives. Since we are interested in the more general case of not necessarily pure A-motives, we will have to discuss the situation in this more general case as well.

Proposition 1.62. *Let $\underline{M}$ be a semisimple A-motive over the finite field $k \cong \mathbb{F}_s \cong F_{q^e}$ with Frobenius endomorphism π. Write $F := Q(\pi)$. Pick some algebraic closure k^{alg} of k. Let w be some place of Q, and let*

$$\underline{\hat{V}}_w(\underline{M})_{k^{alg}} \quad \cong \quad \bigoplus_{i=1}^{s} \underline{\hat{V}}_{m_i,l_i}$$

be the decomposition of the local isoshtuka associated to $\underline{M}$ at w over k^{alg} with the slopes ordered by $m_i/l_i \leq m_{i+1}/l_{i+1}$. Then for every place v of F above w, we have

$$\frac{m_1}{l_1} \leq \frac{\mathrm{ord}_v(\pi_v)}{[k:\mathbb{F}_q]} \cdot \frac{[\mathbb{F}_w : \mathbb{F}_q]}{\mathrm{e}(v \mid w)} \leq \frac{m_s}{l_s}$$

In particular, for pure A-motives all the slopes m_i/l_i at $w = \infty$ are equal to negative of the weight of $\underline{M}$ and we get

$$-[k:\mathbb{F}_q] \cdot \mathrm{wt}(\underline{M}) = \frac{\mathrm{ord}_v(\pi)}{\mathrm{e}(v \mid w)}\,.$$

Proof. Pick an $A_{w,k}$-lattice $\Lambda \cong A_{w,k}^r \subset \underline{\hat{V}}_w(\underline{M})_{k^{\mathrm{alg}}}$ such that for some common multiple r of $e, l_1, \ldots, l_s$

$$\hat{\tau}_M^r = \tau_M^{[\mathbb{F}_w:\mathbb{F}_q]\cdot r} : \hat{\sigma}^{r*}\Lambda \hookrightarrow z_w^{\frac{m_1 \cdot r}{l_1}} \Lambda$$

and

$$(\hat{\tau}_M^r)^{-1} : \Lambda \hookrightarrow z_w^{-\frac{m_s \cdot r}{l_s}} \hat{\sigma}^{r*}\Lambda\,.$$

This is possible due to the form of the standard $\hat{\sigma}$-isoshtukas $\underline{\hat{V}}_{m_i,l_i}$ given in Def. 1.58; compare the proof of [HJ16, Prop. 3.10]. Writing

$$f := z_w^{-r\cdot m_1/l_1} \cdot \hat{\tau}_M^r \quad \text{and} \quad g := z_w^{r\cdot m_s/l_s} \cdot (\hat{\tau}_M^r)^{-1},$$

we obtain

$$\min\nolimits_{f,Q_w} \in A_{w,k}[X] \quad \text{and} \quad \min\nolimits_{g,Q_w} \in A_{w,k}[X]\,.$$

We now claim that $z_w^{r\cdot m_1/l_1} \cdot \pi^{[\mathbb{F}_w:\mathbb{F}_q]\cdot r/e}$ is a zero of $\min_{f,Q_w}$. Let F_0 denote the matrix representing the f-induced map $f_0 : \Lambda/\mathfrak{a}_0 \xrightarrow{\sim} \Lambda/\mathfrak{a}_0$, and let χ_0 be its characteristic polynomial with constant term $\pm \det F_0$. Since f_0 is an isomorphism, F_0 has to be invertible, i. e. $\det F_0$ has to be invertible in $A_{w,k}/\mathfrak{a}_0$, i. e. $|\det F_0|_w = 1$. In particular, all roots of χ_0 must be units as well. If now ρ is a root of $\min_{f,Q_w}$, it is an eigenvalue of π on $\underline{\hat{V}}_w(\underline{M})_{k^{\mathrm{alg}}}$, and therefore $z_w^{r\cdot m_1/l_1} \cdot \pi^{[\mathbb{F}_w:\mathbb{F}_q]\cdot r/e}$ is an an eigenvalue of f on $\underline{\hat{V}}_w(\underline{M})/\mathfrak{a}_0$.

Hence as a zero of $\min_{f,Q_w}$ the element $z_w^{r\cdot m_1/l_1} \cdot \pi^{[\mathbb{F}_w:\mathbb{F}_q]\cdot r/e}$ must be integral, i. e.

$$\mathrm{ord}_v\left(z_w^{r\cdot m_1/l_1} \cdot \pi^{[\mathbb{F}_w:\mathbb{F}_q]\cdot r/e}\right) \geq 0\,,$$

and thus

$$\mathrm{ord}_v(\pi) \geq \underbrace{\mathrm{ord}_v(z_w)}_{=\mathrm{e}(v|w)} \cdot \frac{m_1}{l_1} \cdot \frac{e}{[\mathbb{F}_w : \mathbb{F}_q]}\,,$$

thereby proving the first inequality. An analogous argument using $\min_{g,Q_w}$ yields the second inequality

$$-\operatorname{ord}_v(\pi)\cdot\frac{[\mathbb{F}_w:\mathbb{F}_q]}{e\cdot \mathrm{e}(v\mid w)}\geq\frac{m_s}{l_s}\,.$$

⌟

Corollary 1.63. *Under the assumptions of the last Proposition, we have*

$$q^{-[k:\mathbb{F}_q]\cdot\frac{m_s}{l_s}}\leq|\pi|_v\leq q^{-[k:\mathbb{F}_q]\cdot\frac{m_1}{l_1}}$$

and in particular

$$|\pi|_v=q^{\mathrm{wt}(\underline{M})\cdot e}$$

for all $v\mid\infty$ *if* $\underline{M}$ *is pure, where* $|\cdot|_v$ *extends the absolute value on* Q_w*.*

Proof. This follows immediately, since the valuation $|\cdot|_v=q_w^{-\frac{\operatorname{ord}_v(\cdot)}{\mathrm{e}(v|w)}}$ satisfies the equation

$$|\pi|_v=q_w^{-\frac{\operatorname{ord}_w(\cdot)}{\mathrm{e}(v|w)}}=q^{-\frac{[\mathbb{F}_w:\mathbb{F}_q]\cdot\operatorname{ord}_v(\pi)}{\mathrm{e}(v|w)}}$$

from general valuation theory. ⌟

Corollary 1.64. *Let* $\underline{M}$ *be a pure A-motive over a finite field* $k\cong\mathbb{F}_s\cong\mathbb{F}_{q^e}$ *of rank* r *and dimension* d*, and let* π *be its Frobenius endomorphism, and let* $\min_\pi\in A[X]$ *be its Frobenius polynomial. Then every root* $\rho\in Q_\infty^{alg}$ *of* $\min_\pi$ *has absolute value*

$$|\rho|_\infty=s^{\mathrm{wt}(\underline{M})}=s^{d/r}.$$

Proof. This is (rather briefly) proven in [BH09, Thm. 7.8], but it also follows directly from the preceeding corollary, since the Q_∞^{alg}-roots of $\min_{\pi,Q_\infty}$ are correspond bijectively to the set of Q-embeddings $F\hookrightarrow Q_\infty^{\mathrm{alg}}$. In particular, a valuations v_∞ extending ∞ to Q_∞^{alg} pulls back to some valuation v on $F=Q(\pi)$ above ∞, and we conclude

$$|\rho|_\infty=|\pi|_v=s^{\mathrm{wt}(\underline{M})}\,,$$

using the corollary above. ⌟

Corollary 1.65. *Let* $\underline{M}$ *be a semisimple A-motive over some finite field* $k\cong\mathbb{F}_{q^e}\cong\mathbb{F}_s$ *with Frobenius endomorphism* π*. Let* w *be a place of* Q *and let* v *be some place of* $F=Q(\pi)$ *above* w*. Assume that the local* $\sigma^{[\mathbb{F}_u:\mathbb{F}_q]}$*-isoshtuka of* $\underline{M}$ *at* v *with coefficients in an algebraic closure* k^{alg} *of* k *has the form*

$$\underline{\hat{V}}_v(\underline{M})_{k^{alg}}\quad\cong\quad\underline{\hat{V}}_{d_v,r_v}^{\oplus n}$$

with $\underline{\hat{V}}_{d_v,r_v}^{\oplus n}$ *defined as*

$$\left(F_{v,k}^{\oplus r_v},\hat{\tau}=\begin{pmatrix}0&0&\dots&0&z_v^{d_v}\\1&0&\dots&0&0\\\vdots&\ddots&\ddots&\vdots&\vdots\\0&\dots&1&0&0\\0&\dots&0&1&0\end{pmatrix}\right).$$

Then
$$q^{-\operatorname{ord}_v(\pi)\cdot[\mathbb{F}_w:\mathbb{F}_q]/\operatorname{e}(v|w)} = |\pi|_v = q^{\frac{d_v}{r_v}\cdot\frac{[k:\mathbb{F}_q]}{[F_v:Q_w]}} = (\#k)^{\frac{\operatorname{wt}_v(\underline{M})}{[F_v:Q_w]}}$$
for $\operatorname{wt}_v(\underline{M}) := \frac{d_v}{r_v}$.

Proof. Same as for Prop. 1.62, but with $\dim_{Q_{v,k}} \hat{\underline{V}}_v(\underline{M})_{k^{\mathrm{alg}}} = r_v$. ⌟

Remark 1.66. *Let* $E \subset \operatorname{QEnd}_k(\underline{M})$ *be a commutative, semisimple* Q*-subalgebra, and assume that* $\mathbb{F}_u$ *embeds into* k*. Then* $M \otimes_{A_k} W_{w,k}$ *is a module over the ring* $E \otimes_Q Q_w = \prod_{u|w} E_u$*. Let*
$$\mathfrak{b}_0 := (a \otimes 1 - 1 \otimes a \,:\, a \in \mathbb{F}_u \subset k) \subset E_u, k.$$
Then we define the local isoshtuka of $\underline{M}$ *at* u *as*
$$\hat{\underline{V}}_u(\underline{M}) := (\underline{M} \otimes_{A_k} Q_{w,k} \otimes_{Q_{w,k}} E_{u,k}/\mathfrak{b}_0, \tau_{\underline{M}}^{[\mathbb{F}_u:\mathbb{F}_q]})\,.$$

Corollary 1.67. *Let* $\underline{M}$ *be a semisimple* A*-motive over some finite field* $k \cong \mathbb{F}_{q^e} \cong \mathbb{F}_s$ *with Frobenius endomorphism* π *and (CM) via* $E \subset \operatorname{QEnd}(\underline{M})$*. Let* u *be some place of* E*. Assume that the local* $\sigma^{[\mathbb{F}_u:\mathbb{F}_q]}$*-isoshtuka of* $\underline{M}$ *at* u *with coefficients in an algebraic closure* k^{alg} *of* k *has the form*
$$\hat{\underline{V}}_u(\underline{M})_{k^{alg}} \quad\cong\quad \hat{\underline{V}}_{d_u,1} \cong (E_{u,k}/\mathfrak{b}_0, \hat{\tau} = z_u^{d_u})\,.$$
Then
$$\operatorname{ord}_u(\pi) = d_u \cdot \frac{[k:\mathbb{F}_q]}{[\mathbb{F}_u:\mathbb{F}_q]}\,.$$

Proof. Same as for Prop. 1.62 and Cor. 1.65, but with $\dim \hat{\underline{V}}_u(\underline{M}) = 1$. ⌟

1.5.2. Tate Modules

Definition 1.68. *Let* $\hat{\underline{M}}$ *be a local* $\hat{\sigma}$*-shtuka at* v*. The* ***Tate-module of*** $\hat{\underline{M}}$ ***at*** v *is defined as the* $\operatorname{Gal}(k^{sep}/k)$*-module of* $\tau_{\hat{M}}$*-invariants*
$$T_v\hat{\underline{M}} := (\hat{\underline{M}} \otimes_{A_{v,k}} A_{v,k^{sep}})^{\tau_{\hat{M}}}\,.$$
The ***rational Tate-module of*** $\hat{\underline{M}}$ ***at*** v *is defined as*
$$V_v\hat{\underline{M}} := T_v(\hat{\underline{M}}) \otimes_{A_v} Q_v.$$

We can associate a (rational) Tate module $T_v\underline{M}$ (respectively $V_v\underline{M}$) to an A-motive $\underline{M}$ at a place $v \in \mathit{Spec}\,(A)$ via the associated local $\hat{\sigma}$-shtuka, i. e.
$$T_v\underline{M} := T_v(\hat{\underline{M}}_v(\underline{M})) \qquad\text{and}\qquad V_v\underline{M} := V_v(\hat{\underline{M}}_v(\underline{M})).$$

1.5.3. The Tate Conjecture for A-motives

The following two important theorems are generally referred to as the *Tate Conjectures* for A-motives and go back to Taguchi and Tamagawa (c. f. [Tag95, Tam94, Tam95]) in the following formulation:

Theorem 1.69. *Let $\underline{M}$ and $\underline{M}'$ be two τ-modules over Spec (A_L) for a finitely generated field L and let G be the Galois group of the separable closure L^{sep} over L. Let $v \in$ Spec (A) be a place outside of* $\operatorname{supp}(\operatorname{coker}\tau')$*. Then*

$$\operatorname{Hom}(\underline{M}, \underline{M}') \otimes_A A_v \cong \operatorname{Hom}_{A_v[G]}(T_v\underline{M}, T_v\underline{M}').$$

Replacing Tate modules with shtukas, one ends up with the following statement for A-motives:

Theorem 1.70. *Let $\underline{M}, \underline{N}$ be A-motives over a finite field $\mathbb{F}_s$. Let v be a place of Q. Then there is a canonical isomorphism of Q_v-vector spaces*

$$\operatorname{QHom}(\underline{M}, \underline{N}) \otimes_Q Q_v \xrightarrow{\sim} \operatorname{Hom}_{Q_{v,k}[\hat{\tau}]}(\underline{\hat{V}}_v(\underline{M}), \underline{\hat{V}}_v(\underline{N}))\,.$$

In particular,

$$\operatorname{QEnd}(\underline{M}) \otimes_Q Q_v \xrightarrow{\sim} \operatorname{End}_{Q_{v,k}[\hat{\tau}]}(\underline{\hat{V}}_v(\underline{M}))$$

for all A-motives $\underline{M}$ over a finite field.

Proof. See [BH11, Thm. 8.6]. ⌟

Theorem 1.71. *Let $\underline{M}, \underline{N}$ be A-motives over a finite field $\mathbb{F}_s$. Let v be a maximal ideal of A. Then*

$$\operatorname{Hom}(\underline{M}, \underline{N}) \otimes_A A_v \xrightarrow{\sim} \operatorname{Hom}_{A_{v,k}[\hat{\tau}]}(\underline{\hat{M}}_v(\underline{M}))\,.$$

In particular,

$$\operatorname{End}(\underline{M}) \otimes_A A_v \xrightarrow{\sim} \operatorname{End}_{A_{v,k}[\hat{\tau}]}(\underline{\hat{M}}_v(\underline{M}), \underline{\hat{M}}_v(\underline{N}))$$

for all A-motives $\underline{M}$ over a finite field.

Proof. See [BH11, Thm. 8.7]. ⌟

Remark 1.72.
1. *Although [BH11, BH09] work only in the context of purity, both versions of the Tate-conjecture do not need purity assumptions. See [Sta10] for a complete argument; here the Tate conjectures are obtained as consequence of the semisimplicity conjecture.*
2. *Away from $\varepsilon \cdot \infty$, the assumption on L being finite can be relaxed to L being merely finitely generated, see [BH11, Thm. 9.9, Cor. 9.10].*

1.5.4. The Tate Conjecture for Global Shtukas

Theorem 1.73. *Let $k \cong \mathbb{F}_s \cong \mathbb{F}_{q^e}$ be some finite field and let $\underline{\mathcal{N}}, \underline{\mathcal{N}'}$ be global shtukas over $S = Spec\,(k)$ with the same set of paws $c_1, \ldots, c_n$. Let v be a place of Q. Then there is a canonical isomorphism of Q_v-vector spaces*

$$\mathrm{QHom}(\underline{\mathcal{N}}, \underline{\mathcal{N}'}) \otimes_Q Q_v \xrightarrow{\sim} \mathrm{Hom}_{Q_{v,k}[\hat{\tau}]}(\underline{\hat{V}}_v(\underline{\mathcal{N}}), \underline{\hat{V}}_v(\underline{\mathcal{N}'}))\,.$$

In particular,

$$\mathrm{QEnd}(\underline{\mathcal{N}}) \otimes_Q Q_v \xrightarrow{\sim} \mathrm{End}_{Q_{v,k}[\hat{\tau}]}(\underline{\hat{V}}_v(\underline{\mathcal{N}}))$$

for any global shtuka $\underline{\mathcal{N}}$ over a finite field.

Proof. We defined homomorphisms between global shtukas in Definition 1.21 as τ-compatible homomorphisms between their generic fibres

$$\mathrm{QHom}(\underline{\mathcal{N}}, \underline{\mathcal{N}'}) := \{ f : \mathcal{N}_\eta \to \mathcal{N}'_\eta \,|\, f \circ \left(\tau_{\mathcal{N}'}\right)_\eta = \left(\tau_{\mathcal{N}}\right)_\eta \circ f \}\,.$$

With this definition, the argument goes through in the same way as for A-motives in [BH11, Thm. 8.6]. ⌟

1.6. The Structure of the Endomorphism Algebra

Lemma 1.74. *Let $\underline{M}$ be an A-motive over some finite field $k \cong \mathbb{F}_{q^e}$ of characteristic ε. Assume that there exists some $v \in Spec\,(A)$ away from ε such that π_v is semisimple. Let χ_v be the characteristic polynomial of π_v, and write*

$$\chi_v = \prod_i f_i^{n_i}$$

for the prime factorization of χ_v. Then we have following isomorphism

$$\mathrm{End}_{Q_v[\tau]}(\underline{\hat{V}}_v \underline{M}) \cong \prod_i \left(Q_v[X]/(f_i)\right)^{n_i \times n_i} \cong \mathrm{QEnd}_k(\underline{M}) \otimes_Q Q_v$$

of Q_v-algebras.

Proof. See [BH09, Lemma 6.4]. ⌟

Proposition 1.75. *Let $\underline{M}$ be an A-motive over some A-field of characteristic ε k with Frobenius endomorphism π, and let $v \in Spec\,(A) \setminus \{\varepsilon\}$. Write $F = Q(\pi)$. Denote by E_v the endomorphism ring of the local isoshtuka of $\underline{M}$ at v and by π_v the image of π E_v. Write $F_v := Q_v[\pi_v]$. Then the following statements are equivalent:*

1. *π is semisimple.*
2. *$F = Q(\pi)$ is semisimple.*
3. *$F \otimes_Q Q_v \cong F_v$ is semisimple.*
4. *π_v is semisimple.*

5. $\mathrm{QEnd}_k(\underline{M}) \otimes_Q Q_v \cong \mathrm{QEnd}_k(\underline{M})_v$ *is semisimple.*

6. $\mathrm{QEnd}_k(\underline{M})$ *is semisimple.*

Proof. See [BH09, Prop. 6.8]. ⌟

Lemma 1.76. *Let $\underline{M}$ be an A-motive over a finite A-field $k \cong \mathbb{F}_{q^e}$. Then the following implications hold:*

1. *If* $\mathrm{QEnd}_k(\underline{M})$ *is simple, then $\underline{M}$ is simple.*

2. *If* $\mathrm{QEnd}_k(\underline{M})$ *is semisimple, then $\underline{M}$ is semisimple.*

Proof. See c[BH09, Theorem 6.11]. ⌟

In this section we collect some important structural information about the endomorphism algebra of A-motives. Since the results are important to us, and the proof is rather involved, but only exists in the literature (see [BH09, Thm. 9.1]) for pure A-motives in terms of τ-sheaves, we will give the argument for the relevant parts in detail, making certain no purity assumptions are needed at any step.

Theorem 1.77. *Let $\underline{M}$ be an effective, semi-simple A-motive of rank r defined over a finite field $k \cong \mathbb{F}_s = \mathbb{F}_{q^e}$ with Frobenius endomorphism π. Let $w \in Spec\,(A) \setminus \{\varepsilon\}$ be a place away from ε (and ∞), and let μ_w and χ_w be the minimal, respectively characteristic, polynomial of π_w. Then the following statements hold:*

1. *The Q-algebra* $\mathrm{QEnd}(\underline{M})$ *is semi-simple with center $F = Q(\pi)$.*

2. *The endomorphism algebra* $\mathrm{QEnd}(\underline{M})$ *is commutative if and only if it is equal to $F = Q(\pi)$, which is equivalent to* $\mathrm{QEnd}(\underline{M})$ *or F having rank r over Q.*

3. *The Q-rank of* $\mathrm{QEnd}(\underline{M})$ *is bounded by r from below and r^2 from above,*
$$\mathrm{rk}(\underline{M}) \leq [\mathrm{QEnd}(\underline{M}) : Q] \leq \mathrm{rk}(\underline{M})^2 .$$

4. *The center F of* $\mathrm{QEnd}(\underline{M})$ *is equal to Q if and only if* $\mathrm{QEnd}(\underline{M})$ *is already a central simple algebra over Q, which is equivalent to* $\mathrm{QEnd}(\underline{M})$ *having rank r^2 over Q.*

5. *The Hasse invariants of* $\mathrm{QEnd}(\underline{M})$ *at all places w of F are completely determined by*
$$\mathrm{inv}_w(\mathrm{QEnd}(\underline{M})) = -\frac{[\mathbb{F}_w : \mathbb{F}_q]}{[\mathbb{F}_s : \mathbb{F}_q]} \cdot \mathrm{ord}_w(\pi).$$
In particular, at all places $w \nmid \varepsilon \cdot \infty$ the Hasse invariant $\mathrm{inv}_w \mathrm{QEnd}(\underline{M})$ *vanishes.*

6. *If $\underline{M}$ is also pure, the local Hasse invariant at all places $w \mid \infty$ is given by*
$$\mathrm{inv}_w(\mathrm{QEnd}(\underline{M})) = \mathrm{wt}(\underline{M}) \cdot [F_w : Q_\infty] .$$
In the special case of $F = Q$, the only non-vanishing Hasse invariants are
$$\mathrm{inv}_\varepsilon(\mathrm{QEnd}(\underline{M})) = -\,\mathrm{wt}(\underline{M}) = -\,\mathrm{inv}_\infty(\mathrm{QEnd}(\underline{M})) .$$

Remark 1.78. *1. Note that by the last point of the structure theorem, the structure of the quaso-endomorphism algebra of an A-motive $\underline{M}$ over some finite field $\mathbb{F}_s$ is determined entirely by its Frobenius endomorphism $\pi_{\underline{M}}$, since all factors on the right hand side depend only on information encoded within the knowledge about $Q(\pi_{\underline{M}})$ and $\mathbb{F}_s$. We will make use of this information at a crucial point in our argument, allowing us to parse from a simple element α in some extension field of Q to an algebra E_α and further to an A-motive with endomorphism Q-algebra E_α.*

2. The statement as proven in [BH09, Thm. 9.1] includes some additional information concerning the characteristic polynomials χ_w, that have been left out of the formulation presented here, since they are not relevant to our purposes.

Proof. We prove the statement about the Hasse invariants, being the most crucial to our task ahead. Let w be some place of Q with resiue field $\mathbb{F}_w$ and let $\underline{V}_w$ be the associated σ-isoshtuka $\underline{V}_w(\underline{M})$ of $\underline{M}$ at w. Fix a $\mathbb{F}_q$-homomorphism $\mathbb{F}_l \to \mathbb{F}_s$, and write $l := q^f$ for the size of the intersection $\mathbb{F}_l = \{x \in \mathbb{F}_w : x^s = x\}$. Write

$$\mathfrak{a}_0 := (x \otimes 1 - 1 \otimes x : x \in \mathbb{F}_l) \subset Q_w \otimes_{\mathbb{F}_q} \mathbb{F}_s$$

and let

$$R := (Q_w \otimes_{\mathbb{F}_q} \mathbb{F}_s / \mathfrak{a}_0)[X] = Q_w \otimes_{\mathbb{F}_l} \mathbb{F}_s[X]$$

be the skew-polynomial ring with X-action given by l-Frobenius via

$$X \cdot (a \otimes b) = (a \otimes b^l) \cdot X\,.$$

Structually, R forms a non-commutative principal ideal domain with center $Q_w[X^{e/f}]$, since $Q_w \otimes_{\mathbb{F}_l} \mathbb{F}_s$ is a field. From Thm. 1.70, the Tate conjecture for isoshtukas, we obtain isomorphisms

$$\mathrm{QEnd}(\underline{M}) \otimes_Q Q_w \;\cong\; \mathrm{End}_{Q_w \otimes \mathbb{F}_s[\hat{\tau}]}(\underline{V}_w) \;\cong\; \mathrm{End}_R(\underline{V}_w/\mathfrak{a}_0 \cdot \underline{V}_w),$$

with X-action on $\underline{V}_w/\mathfrak{a}_0 \cdot \underline{V}_w$ given by $\hat{\tau}^f$. It follows from [Jac43, Thm. 3.19] that the R-module $\underline{V}_w/\mathfrak{a}_0 \cdot \underline{V}_w$ may be written as direct sum

$$\underline{V}_w/\mathfrak{a}_0 \cdot \underline{V}_w \cong \bigoplus_{v \in I} \underline{V}_v^{\oplus n_v},$$

where $\underline{V}_v$ are indecomposable. Without loss of generality, the $\underline{V}_v$ may be assumed pairwise non-isomorphic, i. e. the n_v are maximal. The annihilator of each $\underline{V}_v$ may then be written as (μ_v) for some central monic element $\mu_v \in Q_w[X^g]$ with $g = e/f$. In particular, the decomposition of $\underline{V}_w/\mathfrak{a}_0$ is compatible with the $Q_w[X^g]$-structure, and μ_v is the minimal polynomial of X^g acting on $\underline{V}_v$. Denote by μ the least common multiple of the μ_v, which is then the minimal polynomial of X^g on $\underline{V}_w/\mathfrak{a}_0$. Since the X^g-operation on $\underline{V}_w/\mathfrak{a}_0$ is given by Frobenius, the polynomial μ will be the minimal polynomial $\min_{\pi,Q_{\infty,k}}$ and therefore[1]

$$Q[X^g]/(\mu) \cong Q(\pi) = F.$$

[1]Here we need $\min_{\pi,Q_{\infty,k}} = \min_{\pi,Q}$. This is true, since base extension to Q_w is flat, and therefore the short exact sequence $0 \to Q[X] \xrightarrow{\cdot \min_{w,Q}} Q[X] \xrightarrow{X \mapsto \pi} \mathrm{QEnd}_k(\underline{M}) \to 0$ tensored with Q_w yields $\min_{w,Q} = \min_{pi,Q_{\infty,k}}$ as desired, using again Thm. 1.70.

Since π was assumed to be semisimple, μ is squarefree in $Q_w[X^g]$. We obtain the factorization

$$\mu = \prod_{v \in I} \mu_v \in Q_w[X^g].$$

Note that all the μ_v must be irreducible, since they have to be powers of irreducible polynomials according to [Jac43, Thm. 3.20], and no two of them can be equal, since the $\underline{V}_v$ are pairwise non-isomorphic (*loc. cit.*). The product decomposition of μ yields

$$F \otimes_Q Q_w \cong Q_w[X^g]/(\mu) \cong \prod_{v \in I} Q_w[X^g]/(\mu_v) \cong \prod_{v | w} F_v\,.$$

Write π_v for the image of π in F_v. Then π_v has minimal polynomial μ_v over Q_w, and we obtain

$$\mathrm{QEnd}(\underline{M}) \otimes_Q Q_w \cong \bigoplus_{v \in I} \underbrace{\mathrm{End}_R(\underline{V}_v^{n_v})}_{\cong \mathrm{QEnd}(\underline{M}) \otimes_F F_v} \cong \bigoplus_{v \in I} \mathrm{QEnd}(\underline{M}) \otimes_F F_v.$$

Let us now fix some place $v \mid w$ of F. With $h := [\mathbb{F}_v \cap \mathbb{F}_s : \mathbb{F}_{q^f}] = \gcd([\mathbb{F}_v : \mathbb{F}_{q^f}], g)$ and $i := [\mathbb{F}_w : \mathbb{F}_{q^f}]$, we get[2]

$$\begin{aligned}
[\mathbb{F}_w \cdot \mathbb{F}_s : \mathbb{F}_w] &= [\mathbb{F}_s : \mathbb{F}_{q^f}] = g,\\
[\mathbb{F}_w \cdot (\mathbb{F}_v \cap \mathbb{F}_s) : \mathbb{F}_w] &= [\mathbb{F}_v \cap \mathbb{F}_s : \mathbb{F}_{q^f}] = h,\\
[\mathbb{F}_w \cdot \mathbb{F}_s : (\mathbb{F}_w \cdot \mathbb{F}_s \cap \mathbb{F}_v)] &= [\mathbb{F}_v \cdot \mathbb{F}_s : \mathbb{F}_v] = [\mathbb{F}_s : \mathbb{F}_v \cap \mathbb{F}_s] = \frac{g}{h},\\
\mathbb{F}_w \cdot (\mathbb{F}_v \cap \mathbb{F}_s) &\subset \mathbb{F}_w \cdot \mathbb{F}_s \cap \mathbb{F}_v,\\
\mathbb{F}_w \cdot \mathbb{F}_s \cap \mathbb{F}_v &= \mathbb{F}_w \cdot (\mathbb{F}_v \cap \mathbb{F}_s) = \mathbb{F}_{q^{fhi}}\,.
\end{aligned}$$

Let $F_{v,k}$ be the compositum of $Q_w \otimes_{\mathbb{F}_{q^f}} \otimes \mathbb{F}_s$ and $\mathbb{F}_v$ inside an algebraic closure of Q_w. Note that $F_{v,k}$ is well defined since $\mathbb{F}_s / \mathbb{F}_{q^f}$ is a Galois extension. Now, let $F_{v,k}[Y]$ be the skew-polynomial ring with F_v inside the center and

$$Y \cdot (a \otimes b) := (a \otimes b^{q^{hif}}) \cdot Y$$

for all $a \otimes b \in Q_w \otimes \mathbb{F}_s$. This is well-defined, since $(Q_w \otimes_{\mathbb{F}_{q^f}} \mathbb{F}_s) \cap \mathbb{F}_v$ has residue field $\mathbb{F}_w \cdot \mathbb{F}_s \cap \mathbb{F}_v \cong \mathbb{F}_{q^{hfi}}$ and $F_{v,k}$ is unramified over Q_w, because $Q_w \otimes_{\mathbb{F}_{q^f}} \mathbb{F}_s$ is. Let

$$\Delta_v := F_{v,L}[Y]/(Y^{g/h} - \pi_v^i).$$

With $Z := Y^{[\mathbb{F}_v : \mathbb{F}_q]/fhi}$, the algebra Δ_v is just the cyclic algebra

$$\Delta_v \cong (F_{v,L}/F_v, Z, \pi_v^{[\mathbb{F}_v : \mathbb{F}_q]/fh}).$$

This is true, because the extension $F_{v,L}/F_v$ is unramified with degree $[\mathbb{F}_v\, \mathbb{F}_s : \mathbb{F}_v] = \frac{g}{h}$ and Z is its Frobenius automorphism and $Z^{g/h} = \pi_v^{[\mathbb{F}_v : \mathbb{F}_q]/fh}$ in Δ_v. Therefore, Δ_v has Hasse invariant

$$\mathrm{inv}_v(\Delta_v) = \frac{[\mathbb{F}_v : \mathbb{F}_q]}{[\mathbb{F}_s : \mathbb{F}_q]} \cdot \mathrm{ord}_v(\pi_v).$$

[2]The proof of [BH09, Thm. 9.1] includes a rather large diagram of all the following field extensions, which we haven't copied.

We now claim that
$$\operatorname{inv}_v(\operatorname{QEnd}(\underline{M})) = -\operatorname{inv}_v(\Delta_v),$$
which would prove the assertion of the theorem. Using [Jac43, Thm. 3.20] again, we choose some $m \in \mathbb{N}$ such that $\underline{V}_v^{\oplus m} \cong R/(\mu_v(X^g))$. Then
$$\begin{aligned}\operatorname{Mat}_m\big(\operatorname{QEnd}(\underline{M}) \otimes_F F_v\big) &\cong \operatorname{Mat}_m\big(\operatorname{End}_R(\underline{V}_v^{\oplus n_v})\big)\\ &\cong \operatorname{End}_R(\underline{V}_v^{\oplus m\cdot n_v})\\ &\cong \operatorname{Mat}_{n_v}\big((R/(\mu_v(X^g)))^{\mathrm{op}}\big).\end{aligned}$$
Now choose integers $a, b \in \mathbb{Z}$ such that $a > 1$ and $ai + bg = 1$. Let ρ denote the morphism
$$\begin{aligned}&R/(\mu_v(X^g)) \to \operatorname{Mat}_h(\Delta_v),\\ &\qquad x \otimes y \mapsto \operatorname{diag}(x \otimes y, x \otimes y^{q^f}, \cdots, x \otimes y^{q^{f\cdot(h-1)}}),\\ &\qquad X \mapsto \pi_v^b \cdot \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ Y^a & & & 0\end{pmatrix}.\end{aligned}$$
for all $x \in Q_w$ and $y \in \mathbb{F}_s$. This is compatible with the non-commutative ring structure, since both $X \cdot (x \otimes y)$ and $(x \otimes y^{q^f}) \cdot X$ are mapped to the same element (note that $Y^a = Y^{1/i}$ insice $\operatorname{Gal}(F_{v,L}/F_v)$). It is also compatible with the quotient on the left hand side, since $X^g \mapsto \pi_v^{ng} Y^{mg/h} \cdot \mathrm{id} = \pi_v \cdot \mathrm{id}$. Furthermore, ρ is an isomorphism of F_v-algebras: Injectivity follows from $R\mu_v(X^g) \subset R$ being a maximal (two-sided) ideal. For surjectivity, comparing dimensions as Q_w-vector spaces yields
$$\begin{aligned}\dim_{F_v}(\operatorname{Mat}_h(\Delta_v)) &= h^2 \cdot (\frac{g}{h})^2 = g^2,\\ \dim_{Q_w \otimes_{\mathbb{F}_{q^f}} \mathbb{F}_s}(R/(\mu_v(X^g))) &= g \cdot \deg \mu_v = g \cdot [F_v : Q_w],\\ \dim_{Q_w}(R/(\mu_v(X^g))) &= g^2 \cdot [F_v : Q_w] = \dim_{Q_w}(\operatorname{Mat}_h(\Delta_v)).\end{aligned}$$
Hence
$$\operatorname{Mat}_m(\operatorname{QEnd}(\underline{M}) \otimes_F F_v) \cong \operatorname{Mat}_{h\cdot n_v}(\Delta_v^{\mathrm{op}}),$$
thereby proving the assertion about the v-invariants. If now v is a place of F not dividing $\varepsilon \cdot \infty$, consider the local σ-shtuka of $\underline{M}$ at $w := v \cap Q$. Since this is an étale σ-shtuka, the constant coefficient of $\min_\pi$ is invertible in A_w, and hence $\operatorname{ord}_v(\pi_v) = 0$. It remains to prove the simplification in the pure case. Let $v \mid \infty$ and e_v be the ramification index of F_v/Q_∞. Since $\underline{M}$ is assumed to be pure and the residue field of Q_∞ is $\mathbb{F}_q$, we can calculate
$$(\#k)^{\mathrm{wt}(\underline{M})} = q^{e\cdot\mathrm{wt}(\underline{M})} = |\pi|_\infty = q^{-\operatorname{ord}_v(\pi_v)/\operatorname{e}(v|\infty)}$$
and therefore
$$\begin{aligned}\operatorname{inv}_v(\operatorname{QEnd}(\underline{M})) &= -\operatorname{ord}_v(\pi_v) \cdot \frac{[\mathbb{F}_v : \mathbb{F}_q]}{[k : \mathbb{F}_q]}\\ &= \frac{[\mathbb{F}_v : \mathbb{F}_q] \cdot \operatorname{e}(v \mid \infty) \cdot e \cdot \mathrm{wt}(\underline{M})}{e} = \mathrm{wt}(\underline{M}) \cdot [F_v : Q_\infty]\,.\end{aligned}$$

⌟

Corollary 1.79. *Let* $\underline{\mathcal{N}} = (\mathcal{N}, c_1, \ldots, c_n, \tau_{\mathcal{N}})$ *be a global shtuka over some finite field* $k \cong \mathbb{F}_s$ *with Frobenius endomorphism* π*. Then* $F = Q(\pi)$ *is the center of* $\mathrm{QEnd}(\underline{\mathcal{N}})$*, and it's Hasse invariants at all places* u *of* F *are completely determined by*

$$\mathrm{inv}_u(\mathrm{QEnd}(\underline{\mathcal{N}})) = -\frac{[\mathbb{F}_u : \mathbb{F}_q]}{[\mathbb{F}_s : \mathbb{F}_q]} \cdot \mathrm{ord}_u(\pi)\,.$$

In particular, all invariants at places not above $c_1 \cdot \ldots \cdot c_n$ *vanish.*

Proof. The proof is a calculation at shtuka-level, and therefore works for a global shtuka as well. ⌟

2. CM-types and the Taniyama-Shimura-formula

In this chapter we explain the notion of *Complex Multiplication* for A-motives and global shtukas, which closely resembles the analogous ideas in the theory of abelian varieties. Furthermore, we define *CM-types* for our objects, and we prove the equivalent statement to the formula of Taniyama and Shimura in its general form for unramified prime ideals, which is a central component in the surjectivity proof of Honda-Tate theory, both for abelian varieties and for function field objects.

2.1. Complex Multiplication

A Q-algebra E is called *semisimple*, if E is a finite product of fields.

Definition 2.1. *Let $\underline{M}$ be an A-motive of rank r, defined over some field k. Assume that there exists a commutative Q-algebra E inside the quasi-endomorphism algebra* $\mathrm{QEnd}_k(\underline{M})$ *of Q-dimension r. Then $\underline{M}$ is said to have* ***Complex Multiplication*** *(or simply* ***(CM)****) via E.*

Obviously, the notion of (CM) is compatible with the isogeny equivalence relation:

Proposition 2.2. *Let $\underline{M}$ and $\underline{M}'$ be isogenous A-motives. Then $\underline{M}$ has (CM) if and only if $\underline{M}'$ has (CM).*

Proof. By Cor. 1.17, we have $\mathrm{QEnd}(\underline{M}) \cong \mathrm{QEnd}(\underline{M}')$. ⌟

2.1.1. Complex Multiplication via $\mathcal{O}_E$

We use the notation $\mathcal{O}_E$ for the integral closure of A inside E.

Definition 2.3. *Let $\underline{M}$ be an A-motive over k with (CM) via a semisimple commutative Q-algebra E. We say that $\underline{M}$ has* ***Complex Multiplication via*** $\mathcal{O}_E$*, if the ring of integers $\mathcal{O}_E$ of E canonically embeds into the ring of endomorphism* $\mathrm{End}(\underline{M})$ *of $\underline{M}$, or equivalently, if*

$$\mathcal{O}_E = E \cap \mathrm{End}(\underline{M}).$$

While not every A-motive with (CM) necessarily has (CM) via $\mathcal{O}_E$, it turns out that its isogeny equivalence class will always include one representative that does:

Proposition 2.4. *Let $\underline{M}$ be an A-motive over an A-field k with (CM) via some semisimple Q-algebra E. Then there exists an abelian A-motive $\underline{M}'$ and an isogeny $f : \underline{M} \to \underline{M}'$ such that $\underline{M}'$ has (CM) via $\mathcal{O}_{E'}$, where $E' := f \cdot E \cdot f^{-1}$.*

Proof. This was proven in Corollary 3.3.3 in [Sch09]. We give a sketch. The proof consists of the following steps:

(i) The endomorphism ring $\mathrm{End}_k(\underline{M})$ forms an A-order in $\mathrm{QEnd}_k(\underline{M})$.

(ii) The integral closure of A is the only maximal order in E.

(iii) To every maximal order $\mathfrak{a}$ of $\mathrm{QEnd}_k(\underline{M})$ there exists an isogeny

$$f : \underline{M}' \to \underline{M}$$

such that $\mathfrak{a}$ has the form

$$\mathfrak{a} = f \cdot \mathrm{QEnd}_k(\underline{M}') \cdot f^{-1} \subset \mathrm{QEnd}_k(\underline{M}).$$

(iv) The ring of quasi-endomorphism of $\underline{M}$ which are integral over A form a maximal order in $\mathrm{QEnd}_k(\underline{M})$.

The first statement is immediately clear; the second one is first proven for fields and then generalized to semisimple algebras. The third was proven in [BH09][Thm 10.7]. The fourth one is only an application of standard techniques. Finally, the combination of the last two statements gives us the desired isogenous A-motive $\underline{M}'$. ⌟

Remark 2.5. *The result of the last Proposition can be rephrased to into: There exists an A-motive $\underline{M}'$ isogenous to $\underline{M}$ such that $\underline{M}'$ has Complex Multiplication via $\mathcal{O}_E$ under*

$$\mathcal{O}_E \hookrightarrow E \hookrightarrow \mathrm{QEnd}(\underline{M}) \xrightarrow[f\circ(\cdot)\circ f^{-1}]{\sim} \mathrm{QEnd}(\underline{M}')\,.$$

The following statement is very important, as it lays the foundation to understanding A-motives with (CM) as elements of the Picard group over a curve, which will be essential to constructing A-motives from a Weil-number.

Theorem 2.6. *Let $\underline{M}$ be an A-motive over an A-field k with (CM) via $\mathcal{O}_E$ with E semisimple. Then $\underline{M}$ is a τ-module of rank 1 over $\mathcal{O}_{E,k} = \mathcal{O}_E \otimes_{\mathbb{F}_q} k$, i. e. M is locally free of rank 1 as $\mathcal{O}_{E,k}$-module and*

$$\tau_M : \sigma^* M \hookrightarrow M$$

is an injective $\mathcal{O}_{E,k}$-module homomorphism.

Proof. This is Corollary 3.3.6 in [Sch09]. ⌟

2.1.2. Frobenius endomorphism of (CM)-motives

Theorem 2.7. *Let K be a finite field extension of Q, and let $\mathcal{O}_K$ be the integral closure of A in K. Let $\underline{M}$ be a pure A-Motive over K with (CM) via E. Let $\mathfrak{P}$ be a prime ideal of $\mathcal{O}_K$, such that $\underline{M}$ has good reduction at $\mathfrak{P}$. Let π be the Frobenius of the reduction $\underline{M}_{\mathfrak{P}}$ of $\underline{M}$ at $\mathfrak{P}$. Furthermore, assume $\underline{M}$ has (CM) via $\mathcal{O}_E$. Then π is in the image of* $\mathrm{End}\,\underline{M} \hookrightarrow \mathrm{End}\,\underline{M}_{\mathfrak{P}}$.

Proof. This was proven in Theorem 4.2.6 in [Pel09]. ⌟

Remark 2.8. *The preceeding theorem is valid only after making the technical assumption that $\underline{M}$ has (CM) via $\mathcal{O}_E$, i. e. that* $\operatorname{End}\underline{M} \cap E = \mathcal{O}_E$. *As we have just seen in Prop. 2.4, this is not a serious restriction, as one can always choose an A-motive $\underline{M}'$ in the same isogeny class satisfying this condition.*

2.1.3. Existence of Extension Sheaves

The underlying locally free A_k-module M of an A-motive $\underline{M}$ may of course always be understood as the locally free sheaf $\widetilde{M}$ of $Spec\,(A_k)$-modules. However, we want to work with the structure of M as an $\mathcal{O}_{E,k}$-module we just introduced. The argument we present is based on [Sch09, Section 3.4].

Lemma 2.9. *Let $\underline{M}$ be an A-motive over k with (CM) via $\mathcal{O}_E$. Then there exists a locally free sheaf M' on $Spec\,(O_{E,k})$ of rank 1 with global sections M.*

Proof. We can define M' as the associated coherent module sheaf to M. By construction, M' has global sections M. A standard localization argument shows that M' is also locally free of rank 1; see Lemma 3.4.1 in [Sch09] for details. ⌟

If the A-motive $\underline{M}$ is simple, its algebra $\operatorname{QEnd}(\underline{M})$ of quasi-endomorphisms forms a field. Let us denote by $\widetilde{C}$ the regular, projective curve over $\mathbb{F}_q$ defined by the function field $\operatorname{QEnd}(\underline{M})$. This curve has a projection map

$$p : \widetilde{C} \to C.$$

Let us write $\widetilde{C}_k$ for the base change to k, that is

$$\widetilde{C}_k := \widetilde{C} \times_{Spec\,(\mathbb{F}_q)} Spec\,(k).$$

If $\underline{M}$ is only semisimple, the Q-algebra $\operatorname{QEnd}(M)$ is in general not a field, but only a finite direct product

$$\operatorname{QEnd}(\underline{M}) = \prod E_i$$

of function fields. In this case, we can form curves $\widetilde{C}_i$ and $\widetilde{C}_{k,i}$ for every i; we then write $\widetilde{C}$ and $\widetilde{C}_k$ for the disjoint union of these curves.

Proposition 2.10. *Let $\underline{M}$ be a semisimple A-motive over k with (CM) via $\mathcal{O}_E$. Then there exists a locally free sheaf $\mathcal{M}$ of rank 1 on $\widetilde{C}_k$ such that M may be reconstructed from $\mathcal{M}$ by taking sections over $Spec\,(O_{E,k})$, i. e.*

$$M = \Gamma(Spec\,(O_{E,k}), \mathcal{M}).$$

Sketch of Proof. Use the last lemma to solve the problem locally and then glue these solutions together. Details can be be found in [Sch09, Lemma 3.4.3] ⌟

Remark 2.11. *One can also reverse this process to construct A-motives with (CM). We will make use of this reverse process when we build an A-motive out of a Weil number α.*

2.1.4. Definition over a finite extension

Let us assume an A-motive $\underline{M}$ of A-characteristic $\varepsilon = \ker\gamma$ is defined over an algebraically closed A-field $L = L^{\mathrm{alg}}$. Can we find a finite extension field K of $\mathbb{F}_\varepsilon$, the field of fractions of A/ε, such that $\underline{M}$ is already defined over K? Under the additional assumption of Complex Multiplication, the answer is generally yes, as the next Theorem shows:

Theorem 2.12. *Let $\underline{M}$ be an A-motive over an algebraically closed field L^{alg}, and assume that $\underline{M}$ has Complex Multiplication via $\mathcal{O}_E$, where E is some semisimple commutative Q-algebra. Then there exists a finite extension $K/\mathbb{F}_\varepsilon$ such that $\underline{M}$ is defined over K, that is, there exists an A-motive $\underline{M}'$ over K such that*

$$\underline{M} \cong \underline{M}' \otimes_K L\,.$$

Proof. This is [Sch09, Thm. 6.3.6]. ⌟

2.1.5. Reduction of A-Motives with (CM)

It is a natural to ask the question: What kinds of A-motives have potentially good reduction at what primes? As it turns out, the property of having (CM) is already enough to guarantee potentially good reduction everywhere, which will be useful to us later on:

Theorem 2.13. *Let $\underline{M}$ be an A-motive over a finite extension field K/Q with (CM) via E. Then $\underline{M}$ has potentially good reduction at every place of the integral closure $\mathcal{O}_K$ of A in K.*

Proof. This is a corollary to Thm. 2.6 together with [Gar03b, Prop. 2.11 9]. ⌟

2.1.6. CM-types and local CM-divisors

It is not in general enough to know the CM-algebra of an A-motive in order to know its isogeny class. The same is true for abelian varieties. In the theory of abelian varieties with Complex Multiplication, the so-called *CM-type* of an abelian variety is used to classify isogeny classes; there is a one-to-one correspondence between isogeny classes of abelian varieties on one side and isomorphism classes of CM-types on the other. In addition to the CM-algebra E, a CM-type knows about all the algebra homomorphisms $E \to \mathbb{C}$; more precisely, since morphisms $E \to \mathbb{C}$ exist in pairs under the action of complex conjugation, picking at random one element of every such pair defines a set $\Phi \subset \mathrm{Hom}(E, \mathbb{C})$ such that

$$H_E := \mathrm{Hom}(E, \mathbb{C}) = \Phi \cup \overline{\Phi}.$$

As it turns out, if one restricts oneself to algebras E with certain properties, so-called *CM-algebras*, a pair (E, Φ) uniquely defines an isogeny class of abelian varieties. For A-motives, we do not possess a complex conjugation. Furthermore, knowledge of H_E is not in general sufficient to fully describe isogeny classes. However, since an A-motive $\underline{M}$ lives over two places, namely the kernel ideal $\varepsilon := \ker\gamma$ and the fixed infinite place

∞, one needs to describe the behaviour of $\underline{M}$, that is the behaviour of its underlying morphism τ_M, at both places. Therefore, we will introduce the notion of a *local CM-divisor*, and define the total CM-divisor $\mathbb{D}_{\underline{M}}$ of $\underline{M}$ to consist of two *local* CM-divisors $\mathbb{D}_\varepsilon$ and $\mathbb{D}_\infty$, one for each characteristic place.

The CM-type

Pick[1] a (not necessarily effective) A-motive $\underline{M}$ of generic characteristic (that is, $\gamma : A \hookrightarrow k$ is injective) defined over a field k with (CM) via a semisimple commutative algebra E separable over Q. Furthermore, assume that E is separable over Q. Let us write

$$H_E := \mathrm{Hom}_Q(E, Q^{\mathrm{alg}}),$$

where Q^{alg} denotes a fixed algebraic closure of Q. We may assume $\psi(E) \subset k$ for all $\psi \in H_E$, since we can always base-change to a sufficiently large field $k' \supset k$. Let w be a place of Q and $z = z_v$ a local uniformizer at v. Write $\zeta = \zeta_v$ for the image of z under γ in K. Then Q embeds into $K[\![z-\zeta]\!]$ via the canonical map induced by $z \mapsto \zeta + (z-\zeta)$. Note that $K[\![z-\zeta]\!]$ is isomorphic to the completion of the local ring of C_K at $\mathrm{V}(\mathcal{J})$, which does not depend on v; hence our construction of $K[\![z-\zeta]\!]$ is independent of the choice of place v. Consider the $E_{[z-\zeta]} := E \otimes_Q k[\![z-\zeta]\!]$-module

$$\mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!]) = \sigma^* M \otimes_{A_k} k[\![z-\zeta]\!]$$

lying inside the $k(\!(z-\zeta)\!)$-vector space

$$\mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!])\left[\frac{1}{z-\zeta}\right],$$

which contains the **Hodge-Pink-lattice** $\mathfrak{q}_{\underline{M}}$ of $\underline{M}$,

$$\mathfrak{q}_{\underline{M}} := \tau_M^{-1}(M \otimes_{A_k} k[\![z-\zeta]\!]).$$

After we choose local uniformizers x_ψ (at some place of E) for all $\psi \in H_E$ such that $\psi(x_\psi) \neq 0$, the algebra $E_{[z-\zeta]}$ decomposes[2] as

$$E \otimes_Q k[\![z-\zeta]\!] = \prod_{H_E} k[\![x_\psi - \psi(x_\psi)]\!]\,.$$

We obtain a $E \otimes_Q k[\![z-\zeta]\!]$-decomposition of $\mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!])$ as

$$\begin{aligned}\mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!]) &= \mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!]) \otimes_{E_{[z-\zeta]}} E_{[z-\zeta]} \\ &= \prod_{H_E} \mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!]) \otimes_{E_{[z-\zeta]}} k[\![x_\psi - \psi(x_\psi)]\!],\end{aligned}$$

where each individual factor

$$\mathbf{H}^\psi(\underline{M}, k[\![x_\psi - \psi(x_\psi)]\!]) := \mathbf{H}^1_{\mathrm{dR}}(\underline{M}, k[\![z-\zeta]\!]) \otimes_{E_{[z-\zeta]}} k[\![x_\psi - \psi(x_\psi)]\!]$$

[1] Our discussion is based on [HS17, § 1].
[2] See [HS17, Lemma A.3].

is free of rank 1 as $k[\![x_\psi - \psi(x_\psi)]\!]$-module. Hence we can write the Hodge-Pink-lattice $\mathfrak{q}_{\underline{M}}$ of $\underline{M}$ as

$$\mathfrak{q}_{\underline{M}} = \prod_{H_E} \mathfrak{q}_{\underline{M}} \cap \mathbf{H}^\psi(\underline{M}, k[\![x_\psi - \psi(x_\psi)]\!]) \\ = \prod_{\psi \in H_E} (x_\psi - \psi(x_\psi))^{-d_\psi} \cdot \mathbf{H}^\psi(\underline{M}, k[\![x_\psi - \psi(x_\psi)]\!])$$

for suitable non-negative integers d_ψ. The tuple

$$\Psi_{\underline{M}} := (d_\psi)_{\psi \in H_E}$$

is then called the **CM-type of** $\underline{M}$. Note that $\mathfrak{q}_{\underline{M}}$ is a lattice of full rank $r = \mathrm{rk}(\underline{M})$ and that E as a semisimple, separable Q-algebra of rank r allows precisely $r = [E : Q]$ different embeddings $E \hookrightarrow \mathbb{C}_\infty$. Hence we get exactly r many integers d_ψ.

Remark 2.14. *If $\underline{M}$ is an effective A-motive with (CM) via E of CM-type Ψ, we obtain a decomposition of* $\operatorname{coker} \tau_M$ *in the following way: Define the tautological lattice* $\mathfrak{p}_{\underline{M}}$ *as*

$$\mathfrak{p}_{\underline{M}} := \mathbf{H}^1_{dR}(\underline{M}, K[\![z - \zeta]\!]) = \sigma^* \underline{M} \otimes_{A_K} K[\![z - \zeta]\!].$$

Since $\underline{M}$ is effective, we know that $\tau_M(\sigma^* M) \subset M$, and therefore $\mathfrak{p}_{\underline{M}} \subset \mathfrak{q}_{\underline{M}}$, with quotient being precisely the cokernel of τ_M, i. e.

$$\mathfrak{q}_{\underline{M}} / \mathfrak{p}_{\underline{M}} \xrightarrow[\tau_M]{\sim} \operatorname{coker} \tau_M.$$

Hence the decomposition of the Hodge-Pink lattice $\mathfrak{q}_{\underline{M}}$ yields

$$\operatorname{coker} \tau_M \cong \prod_{\psi \in H_E} K[\![x_\psi - \psi(x_\psi)]\!] / (x_\psi - \psi(x_\psi))^{d_\psi} \\ =: \prod_{\psi \in H_E} (\operatorname{coker} \tau_M)_\psi ,$$

where each $(\operatorname{coker} \tau_M)_\psi$ *is a K-module of rank d_ψ, adding up to the K-vector space* $\operatorname{coker} \tau_M$ *of dimension*

$$d = \dim(\underline{M}) = \dim_K(\operatorname{coker} \tau_M) = \sum_{\psi \in H_E} d_\psi.$$

The CM-divisor at the infinite place

We now define the local divisor of an A-motive $\underline{M}$ with CM at ∞ in a similar manner. As seen before, we can view $\underline{M}$ as $\mathcal{O}_{E,k}$-module of rank 1, with underlying morphism

$$\tau_M : \sigma^* M \to M$$

such that τ_M forms an isomorphism away from the two characteristic places ε and ∞.

As discussed before, τ_M has zeroes above ε, and the degrees of these zeroes determine the structure of the cokernel $\operatorname{coker} \tau_M$ of τ_M. In particular, the total degree of the divisor describing the zeroes above ε is precisely the k-vectorspace dimension of coker τ_M, i. e. the dimension d of $\underline{M}$. Choose some extension $(\mathcal{M}, \tau_{\tilde{M}})$ of (M, τ_M) to the whole curve $\widetilde{C}_k$. Above ∞, the morphism $\tau_{\tilde{M}}$ has poles, and their degrees also add up to dim $\underline{\mathcal{M}}$ (since the total degree of the divisor describing $\underline{M}$ has to be zero due to $\mathcal{M}$ forming an invertible sheaf on the curve $\widetilde{C}_k$). More specifically, let P be a point on the curve $\widetilde{C}_k$ aboce the point $\infty \in C$.The map $\tau_{\tilde{M}}$ has a pole of some degree $d^{\mathcal{M}}_{P,\infty}$ at P. Furthermore, we define

$$d_{u,\infty} := \sum_{P|u} d^{\mathcal{M}}_{u,P} \cdot [\kappa(P) : k]$$

at all places u of E above ∞. Note that the $d_{u,\infty}$ do not depend anymore on $\mathcal{M}$, since $d_{u,\infty} = \mathrm{ord}_P \left(\tau_{\mathcal{M}}^{[\mathbb{F}_u:\mathbb{F}_q]}\right)$. We write

$$\mathbb{D}_\infty := \{d_{u,\infty}\,;\, Spec\,(E) \ni u \mid \infty\}$$

for the resulting set of positive integers and call D_∞ the **local CM-divisor of M at ∞.**

Note that if $\underline{M}$ does not have CM via some algebra $E \subset \mathrm{QEnd}(\underline{M})$, then $\underline{M}$ cannot be seen as $\mathcal{O}_E$- or $\mathcal{O}_{E,k}$-module, and the discussion above does not make sense.

The CM-divisor at the characteristic place

Let $\underline{M} = (M, \tau_M)$ be an A-motive of characteristic ε over an A-field k. Assume that $\underline{M}$ has CM via $\mathcal{O}_E$ for some separable, commutative, Q-algebra E, and let Φ_ε be its CM-type. We define the local CM-divisor $\mathbb{D}_\varepsilon$ of $\underline{M}$ at the characteristic place in a similar manner as the local CM-divisor at the infinite place. The tuple $\mathbb{D}_\varepsilon := (d_{u,\varepsilon})_u$ indexed by the places u of $\mathcal{O}_E$ above ε is then called the **local CM-divisor of $\underline{M}$ at ε.**

Definition 2.15. *Let $\underline{M}$ be an A-motive with (CM) via $\mathcal{O}_E$, and let $\mathbb{D}_\varepsilon$ and $\mathbb{D}_\infty$ be the local CM-divisors of $\underline{M}$ as defined above. Then we call*

$$\mathbb{D}_{\underline{M}} := (\mathbb{D}_\varepsilon, \mathbb{D}_\infty)$$

the ***total CM-divisor of $\underline{M}$****.*

Remark 2.16. *The discussion above strongly suggests that for A-motives $\underline{M}$ with (CM) the structure of τ_M (and therefore $\underline{M}$) is determined by the total CM-divisor of $\underline{M}$, similar to how CM-types determine abelian varieties with complex multiplication. As we will later see, this is an accurate impression, and we will use this relationship to construct A-motives with pre-determined properties later on.*

Generalization to Global Shtukas

Let $\underline{\mathcal{N}} = (\mathcal{N}, c_1, \ldots, c_n, \tau_{\mathcal{N}})$ be some global shtukas of rank r over some finite field k, as defined in (1.19). In particular, $\tau_{\mathcal{N}}$ is an isomorphism on $\sigma^*\mathcal{N} \to \mathcal{N}$ outside the graphs of the c_i. The notion of Complex Multiplication carries over to global shtukas in a natural manner. We say that $\underline{\mathcal{N}}$ has Complex Multiplication (or short (CM)) via some commutative semisimple Q-algebra E of rank $\mathrm{rk}_Q E = \mathrm{rk}\underline{\mathcal{N}} = r$, if there is an embedding

$$E \hookrightarrow \mathrm{QEnd}(\underline{\mathcal{N}})$$

of commutative Q-algebras. Furthermore, such a global shtukas is said to have (CM) via $\mathcal{O}_E$ if $\mathcal{N}$ is invertible as $\mathcal{O}_{\tilde{C}_k}$-module. (Here $\tilde{C}$ is again the curve defined by E and $\tilde{C}_k = \tilde{C} \times \mathit{Spec}\,(k)$.)

We will now define a local CM-divisor $\mathbb{D}_i$ at each of the paws c_i by following the same approach as we used at the infinite place for A-motives. Namely, let P be some place on the curve $\tilde{C}_k$ with $\mathrm{ord}_P(\tau_{\mathcal{N}}) \neq 0$. In particular, P is lying above some some point $v_i \in C$ given by a paw c_i of $\underline{\mathcal{N}}$. For all $u \in \tilde{C}$ above v_i we let

$$d_{u,i} := \sum_{P|u} \mathrm{ord}_P(\tau_{\mathcal{N}}) \cdot [\kappa(P) : k] = \mathrm{ord}_P(\tau_{\mathcal{N}}^{[\mathbb{F}_u:\mathbb{F}_q]})\,.$$

Then the tuple $\mathbb{D}_i := (d_{u,i} : u \mid v_i)$ is called the local CM-divisor of $\underline{\mathcal{N}}$ at the paw c_i, and the tuple $\mathbb{D}_{\underline{\mathcal{N}}} = (D_i)$ is called the total CM-divisor of $\mathcal{N}$.

2.2. The Formula of Shimura and Taniyama

The question underlying the formula of Taniyama and Shimura may be stated as follows: Assume that we are given an abelian variety A over some number field K such that we can can reduce A at some prime $\mathfrak{P}$ in $\mathcal{O}_K$ to an abelian variety $A_{\mathfrak{P}}$ over $k := \mathcal{O}_K/\mathfrak{P}$. Now, since k is a finite field, $A_{\mathfrak{P}}$ has a Frobenius endomorphism π. What can we say about the algebraic nature of π? If A has Complex Multiplication via $\mathcal{O}_E$ with $\pi \in \mathcal{O}_E$, the formula of Taniyama and Shimura provides a complete and satisfying answer to this question. More specifically, it gives a full description of the ideal $(\pi)_{\mathcal{O}_E}$ generated by the Frobenius.

2.2.1. Unramified version

The Taniyama-Shimura formula goes back to a 1955 paper by Taniyama [Tan55] and was first proved by Shimura and Taniyama [Tan55, Shi55, Tan57], reducing to the case of $\mathfrak{P}$ having absolute degree 1. (As to the history of the proofs and some corrections, see also the discussion in [Hon68, §2].) It can also be found as [ST61, Thm. 1], however in a slightly weakened form; to be precise, it is assumed that $\mathfrak{p} := \mathfrak{P} \cap \mathbb{Z}$ is unramified. In order to prove the Honda-Tate-relationship between Weil numbers and Abelian Varieties, the stronger form is needed. However, since the proof of the weakened formula is rather elegant and was first used in [Pel09] to adapt the formula to the case of A-motives, we will first give and prove the weaker statement. This is, in essence, the proof given in [ST61], which may also be found in an updated form using modern language in Milne's expository article [Mil07] on the fundamental theorem of complex multiplication.

Theorem 2.17. *Let $\underline{M}$ be a pure A-motive over an A-field $(K, A \subset K)$ with (CM) given by a semisimple, separable Q-algebra E, such that K is finite over Q. Assume that K contains $\psi(E)$ for all $\psi \in \mathrm{Hom}(E, \mathbb{C}_\infty) = H_E$. Let $\mathfrak{P}$ be a prime ideal in the* *integral closure $\mathcal{O}_K$ of A in K, such that $\underline{M}$ has good reduction $\underline{M}_{\mathfrak{P}}$ at $\mathfrak{P}$. Assume that $\mathcal{O}_E = E \cap \operatorname{End}(\underline{M})$ and that $\mathfrak{p} := \mathfrak{P} \cap A$ is unramified in E. Let π be the Frobenius endomorphism of the reduction $\underline{M}_{\mathfrak{P}}$. Let $\Psi = (d_\psi \mid \psi \in H_E)$ be the CM-type of $\underline{M}$. Then the following equality of ideals in $\mathcal{O}_E$ holds:*

$$(\pi) = \prod_{\psi \in H_E} \psi^{-1}\big(\mathrm{N}_{K/\psi E)}\,\mathfrak{P}\big)^{d_\psi}. \tag{TSF}$$

Remark 2.18. *Note that the reduction $\underline{M}_{\mathfrak{P}}$ is an A-motive with characteristic ideal $\mathfrak{P} \cap A$. In order to avoid confusion, we will generally denote this characteristic ideal $\mathfrak{p} = \mathfrak{P} \cap A$ instead of ε during the course of this subchapter.*

Proof. Since $\underline{M}$ has good reduction at $\mathfrak{P}$ by assumption, we can choose a good model $\underline{\mathcal{M}}$ over the ring $\mathcal{O}_{K,\mathfrak{P}}$, with $\underline{M}_{\mathfrak{P}} = (M', \tau'_M)$ denoting the reduction at $\mathfrak{P}$. The cokernel $\operatorname{coker} \tau_{\mathcal{M}}$ of the model $\underline{\mathcal{M}}$ naturally relates to the cokernels of $\underline{M}$ and $\underline{M}_{\mathfrak{P}}$ in the following way: First, we have

$$\operatorname{coker} \tau_M \cong \operatorname{coker} \tau_{\mathcal{M}} \otimes_{\mathcal{O}_{K,\mathfrak{P}}} K$$

and

$$\operatorname{coker} \tau_{M'} \cong \operatorname{coker} \tau_{\mathcal{M}} / (\mathfrak{P} \cdot \operatorname{coker} \tau_{\mathcal{M}}).$$

Furthermore, from the decomposition

$$\operatorname{coker} \tau_M \cong \bigoplus_{\psi \in \Psi} (\operatorname{coker} \tau_M)_\psi$$

with

$$d_\psi = \dim_K (\operatorname{coker} \tau_M)_\psi \,,$$

and using that $\mathfrak{p}$ is unramified, we first obtain a corresponding decomposition

$$\operatorname{coker} \tau_{\mathcal{M}} \cong \bigoplus_{\psi \in \Psi} \underbrace{\big((\operatorname{coker} \tau_M)_\psi \cap \operatorname{coker} \tau_{\mathcal{M}}\big)}_{:=(\operatorname{coker} \tau_{\mathcal{M}})_\psi}$$

and then also

$$\operatorname{coker} \tau_{M'} \cong \bigoplus_{\psi \in \Psi} (\operatorname{coker} \tau_{M'})_\psi \,.$$

This decomposition argument breaks down for ramified $\mathfrak{p}$, which is why this proof cannot be adapted in the case of ramified characteristic. The ideal $(\pi) \subset \mathcal{O}_E$ generated by the Frobenius endomorphism $\pi \in \mathcal{O}_E$ may be written as a product of prime ideals

$$(\pi) = \prod_{\mathfrak{p}_v \mid \mathfrak{p}} \mathfrak{p}_v^{\operatorname{ord}_{\mathfrak{p}_v}(\pi)}.$$

Let us write $n_v := \operatorname{ord}_{\mathfrak{p}_v}(\pi)$. Note that the reduction $\underline{M}_{\mathfrak{P}}$ is an A-motive over the A-field $k := \mathcal{O}_K/\mathfrak{P}$, where $\gamma : A \to k$ is given by $A \hookrightarrow \mathcal{O}_K \twoheadrightarrow \mathcal{O}_K/\mathfrak{P}$, and therefore $\ker \gamma = \mathfrak{P} \cap A = \mathfrak{p}$. Hence all $\mathfrak{p}_v$ dividing (π) will indeed lie above $\mathfrak{p}$.

We therefore obtain a decomposition of Ψ in the following way: Write Ψ_v for the set of morphism $\phi : E \to \mathbb{C}_\infty$ relating $\mathfrak{P}$ and $\mathfrak{p}_v$ and d_v for the dimension of the corresponding subspace of coker $\tau_{M'}$, i. e.

$$\begin{aligned} H_v &:= \{\psi \in H_E \mid \psi^{-1}(\mathfrak{P}) = \mathfrak{p}_v\}, \\ d_v &:= \sum_{\psi \in \Psi_v} d_\psi. \end{aligned}$$

Obviously,

$$\sum_{v|\mathfrak{p}} d_v = \sum_{\psi\in\Psi} d_\psi = d.$$

Write $\mathfrak{q} := \mathrm{N}_{K/Q}(\mathfrak{P}) = \mathfrak{p}^{[\mathbb{F}_\mathfrak{P}:\mathbb{F}_\mathfrak{p}]} = \mathfrak{p}^{f(\mathfrak{P}/\mathfrak{p})}$. We now claim that

$$\mathrm{N}_{E/Q}\left(\mathfrak{p}_v^{n_v}\right) = \mathfrak{p}^{d_v\cdot[\mathbb{F}_\mathfrak{P}:\mathbb{F}_\mathfrak{p}]} = \mathfrak{q}^{d_v}. \tag{$*$}$$

Assuming this to be true, we obtain

$$\begin{aligned} \mathrm{N}_{E/Q}\left(\mathfrak{p}_v^{n_v}\right) &= (\mathfrak{q})^{d_v} \\ &= \left(\mathrm{N}_{K/Q}\,\mathfrak{P}\right)^{d_v} \\ &= \prod_{\psi\in\Psi_v} \left(\mathrm{N}_{K/Q}\,\mathfrak{P}\right)^{d_\psi} \\ &= \prod_{\psi\in\Psi_v} \left(\mathrm{N}_{E/Q}(\psi^{-1}(\mathrm{N}_{K/\psi E}(\mathfrak{P}))\right)^{d_\psi} \\ &= \mathrm{N}_{E/Q}\left(\prod_{\psi\in\Psi_v} \left(\psi^{-1}(\mathrm{N}_{K/\psi E}\,\mathfrak{P})\right)^{d_\psi}\right). \end{aligned}$$

Since prime ideal factorization is unique and we know that $\prod_{\psi\in\Psi_v}\left(\psi^{-1}(\mathrm{N}_{K/\psi E}\,\mathfrak{P})\right)^{d_\psi}$ is a power of $\mathfrak{p}_v$ by construction, we obtain

$$\mathfrak{p}_v^{n_v} = \prod_{\psi\in\Psi_v} \left(\psi^{-1}(\mathrm{N}_{K/\psi E}\,\mathfrak{P})\right)^{d_\psi}$$

and therefore the statement of the theorem, once we form the product over all Ψ_v.

It remains to show the claim in $(*)$. We pick an integer h such that

$$(\mathfrak{p}_v^{n_v})^h = (\beta_v) \subset \mathcal{O}_E.$$

Such an h exists, since $E \cong \prod E_i$ is a finite product of fields with finite class numbers. We now look at the induced map $\beta_v : \operatorname{coker}\tau_{M'} \to \operatorname{coker}\tau_{M'}$. We claim that for some suitable positive integer n, we get a decomposition

$$\ker(\beta_v^n : \operatorname{coker}\tau_{M'} \to \operatorname{coker}\tau_{M'}) = \bigoplus_{\psi\in\Psi_v} (\operatorname{coker}\tau_{M'})_\psi\,.$$

To see this, let us consider the action of β_v on coker $\tau_{M'}$. If we write $z_\psi := y_\psi - \psi(y_\psi)$, the action of β_v is given by multiplication with β_v , which we write as

$$\psi(\beta_v) + (\beta_v - \psi(\beta_v))$$

on the factor corresponding to ψ. Therefore, β_v^n for some $n > 0$ acts as multiplication with

$$\big(\psi(\beta_v) + (\beta_v - \psi(\beta_v))\big)^n.$$

Since we are working in characteristic $p > 0$, if we pick for n a p-power $n = p^* > d$, the $(\beta_v - \psi(\beta_v))$-term vanishes into the ideal $(y_\psi - \psi(y_\psi))^{d_\psi}$, and the action of β_v^n collapses into multiplication with $\psi(\beta_v)^n$. Since

$$\operatorname{coker} \tau_{M'} \cong \operatorname{coker} \tau_{\mathcal{M}} / (\mathfrak{P} \cdot \operatorname{coker} \tau_{\mathcal{M}}),$$

this action will be zero if and only if $\psi \in \Psi_v$. Hence the kernel of β_v^n is indeed given by

$$\bigoplus_{\psi \in \Psi_v} (\operatorname{coker} \tau_{M'})_\psi,$$

as was to be shown.
Therefore

$$\begin{aligned} \dim_k \operatorname{coker} \beta_v^n &= \dim_k \bigoplus_{\psi \in \Psi_v} (\operatorname{coker} \tau_{M'})_\psi \\ &= \operatorname{rk}_{\mathcal{O}_K} \bigoplus_{\psi \in \Psi_v} (\operatorname{coker} \tau_{\mathcal{M}})_\psi \\ &= \dim_K \bigoplus_{\psi \in \Psi_v} (\operatorname{coker} \tau_M)_\psi \\ &= d_v. \end{aligned}$$

Since $\pi^{hn} = (\tau_{M'})^{h \cdot [\mathbb{F}_\mathfrak{P} : \mathbb{F}_\mathfrak{p}] \cdot n}$ holds, we obtain

$$\dim_k \operatorname{coker} \beta_v \leq h \cdot d_v \cdot [\mathbb{F}_\mathfrak{P} : \mathbb{F}_\mathfrak{p}].$$

Using[3]

$$\deg(\beta_v) = \mathfrak{p}^{\frac{\dim_{\mathbb{F}_\mathfrak{P}} \operatorname{coker} \beta_v}{[\mathbb{F}_\mathfrak{P} : \mathbb{F}_q]}},$$

we end up with[4]

$$\mathrm{N}_{E/Q}(\mathfrak{p}_v^{h \cdot n_v}) = \mathrm{N}_{E/Q}(\beta_v) = \deg \beta_v \subseteq \mathfrak{p}^{h \cdot d_v \cdot [\mathbb{F}_\mathfrak{P} : \mathfrak{p}]},$$

and we can remove the h from the exponent. But now we are done, because by forming the product over all v and using $\sum d_v = d$ we see at once that the inclusion must be an equality. ⌟

[3]see [BH09, Cor. 7.5] or [Pel09, Kor. 6.1.4]
[4]The second equality follows from [BH09, Thm. 7.3]

2.2.2. Revised version at the characteristic place

The approach in the last section has two drawbacks. First, it only works for unramified prime ideals $\mathfrak{p}$, i. e. the A-motive $\underline{M}'$ has to have an unramified characteristic ideal ε, which severely limits scope of the statement's usefulness with regards to our purposes. Second, it uses results from [BH11, BH09] obtained in the context of pure motives only. We will now explain how to avoid both of these problems by giving another proof totally independent from the arguments in the last section. Our approach will follow the general outline of Tate's work in [Tat68] (for a fleshed out version, see [Con04]) for abelian varieties. His ingenious idea was to reformulate the formula of Taniyama and Shimura for abelian varieties in terms of their p-divisible groups. For us, p-divisible groups are replaced by local shtukas.

We begin by restating the Taniyama-Shimura formula in more accessible, concrete terms.

Proposition 2.19. *With notations and assumptions as before, the Taniyama-Shimura-formula is equivalent to each of the following two statements:*

1. *For all primes $\mathfrak{p}_v \subset \mathcal{O}_E$ dividing $\mathfrak{p} = \mathfrak{P} \cap A$, we have*

$$\operatorname{ord}_{\mathfrak{p}_v}(\pi) = \sum_{\psi \in \Psi_v} d_\psi \cdot f(\mathfrak{P}/\psi(\mathfrak{p}_{\mathfrak{v}}))$$

and zero otherwise.

2. *For all primes $\mathfrak{p}_v \subset \mathcal{O}_E$ dividing $\mathfrak{p}$, we have*

$$\frac{\operatorname{ord}_{\mathfrak{p}_v}(\pi)}{\operatorname{ord}_{\mathfrak{p}_v}(\mathfrak{q})} = \frac{d_v}{\# H_v}$$

and zero otherwise, where $H_v := \{\phi : E \to K \mid \phi^{-1}(\mathfrak{P}) = \mathfrak{p}_v\}$, $\mathfrak{q} := \mathrm{N}_{K/Q}(\mathfrak{P}) = \mathfrak{p}^{f(\mathfrak{P}/\mathfrak{p})}$ and Ψ_v as before.

Proof. The first statement is immediately seen to be equivalent to (TSF), since

$$\operatorname{ord}_{\mathfrak{p}_v}(\psi^{-1}\big(\mathrm{N}_{K/\psi E)}\,\mathfrak{P}\big)^{d_\psi}) = d_\psi \cdot \operatorname{ord}_{\psi(\mathfrak{p}_v)}\big(\mathrm{N}_{K/\psi E}\,\mathfrak{P}\big) = d_\psi \cdot f(\mathfrak{P}/\psi(\mathfrak{p}_v))$$

if and only if $\psi^{-1}(\mathfrak{P}) = \mathfrak{p}_v$, and zero otherwise. As for the second statement, note that

$$\operatorname{ord}_{\mathfrak{p}_v}(\mathfrak{q}) = \mathrm{f}(\mathfrak{P} \mid \mathfrak{p}) \cdot \operatorname{ord}_{\mathfrak{p}_v}(\mathfrak{p}) = \mathrm{f}(\mathfrak{P}/\mathfrak{p}) \cdot \mathrm{e}(\mathfrak{p}_v \mid \mathfrak{p})$$

and

$$\# H_v = f(\mathfrak{p}_v/\mathfrak{p}) \cdot \mathrm{e}(\mathfrak{p}_v \mid \mathfrak{p}).$$

Using the first equality we just proved, we obtain

$$\frac{\operatorname{ord}_{\mathfrak{p}_v}(\pi)}{\operatorname{ord}_{\mathfrak{p}_v}(\mathfrak{q})} = \sum_{\psi \in \Psi_v} \frac{d_\psi \cdot f(\mathfrak{P}/\psi(\mathfrak{p}_v))}{f(\mathfrak{P}/\mathfrak{p}) \cdot \mathrm{e}(\mathfrak{p}_v \mid \mathfrak{p})} = \sum_{\psi \in \Psi_v} \frac{d_\psi}{f(\mathfrak{p}_v/\mathfrak{p}) \cdot \mathrm{e}(\mathfrak{p}_v \mid \mathfrak{p})} = \frac{d_v}{\# H_v},$$

and vice versa. Hence the second statement is equivalent to the first statement, which was already seen to being a reformulation of the Taniyama-Shimura formula. ⌟

The reader will notice that the proof of equivalence does not require the prime $\mathfrak{p}$ to be unramified. We are therefore left to prove the following revised theorem

Theorem 2.20. *Let $\underline{M}$ be an effective A-motive over an A-field $(K, A \subset K)$ with (CM) given by a semisimple, separable Q-algebra E, such that K is finite over Q. Assume that K contains $\psi(E)$ for all $\psi \in \mathrm{Hom}(E, \mathbb{C}_\infty) = H_E$. Let $\mathfrak{P}$ be a prime ideal in the integral closure $\mathcal{O}_K$ of A in K, such that $\underline{M}$ has good reduction $\underline{M}_\mathfrak{P}$ at $\mathfrak{P}$, and write $\mathfrak{p} := \mathfrak{P} \cap A$. Assume that $\mathcal{O}_E = E \cap \mathrm{End}(\underline{M})$. Let π be the Frobenius endomorphism of the reduction $\underline{M}_\mathfrak{P}$. For all places $u \in Spec(\mathcal{O}_E)$ above $\mathfrak{p}$, we have*

$$\frac{\mathrm{ord}_u(\pi)}{\mathrm{ord}_u(\mathfrak{q})} = \frac{d_{u,\mathfrak{p}}}{h_u}, \tag{rTSF}$$

where $(d_{u,\mathfrak{p}})$ is defined via $d_{u,\mathfrak{p}} = \sum_{\psi \in H_u} d_\psi$ for the CM-type $\Psi = (d_\psi)_{\psi \in H_E}$ of $\underline{M}$, $H_u = \{\psi \in H_E; \psi^{-1}(\mathfrak{P}) = u\}$, $h_u := \# H_u = [E_u : Q_\mathfrak{p}]$ and $\mathfrak{q} := \mathrm{N}_{K/Q}(\mathfrak{P}) = \mathfrak{p}^{f(\mathfrak{P}/\mathfrak{p})}$.

Remark 2.21. *The presentation of formula* (rTSF) *is chosen to show the analogy with the Shimura-Taniyama formula for abelian varieties. If one simply wants to calculate the behaviour of the Frobenius endomorphism π at places above $\mathfrak{p} = \mathfrak{P} \cap A$, the formula can be written more explicitly as*

$$\mathrm{ord}_u(\pi) = d_{u,\mathfrak{p}} \cdot \frac{[\mathbb{F}_\mathfrak{P} : \mathbb{F}_\mathfrak{p}]}{[\mathbb{F}_u : \mathbb{F}_\mathfrak{p}]}.$$

We will not prove this for the moment, as it is an easy calculation, and will in fact be part of the proof of formula (rTSF).

Remark 2.22. *Since we are for the moment only concerned with the structure of the Frobenius at the characteristic ideal $\mathfrak{p}$ of the reduction $\underline{M}_\mathfrak{P}$, we will remove the subscript $\mathfrak{p}$ from the notation, and simply write d_u for $d_{u,\mathfrak{p}}$.*

Any A-motive with CM has potentially good reduction everywhere, and up to isogeny we can always assume that $\mathcal{O}_E \subset \mathrm{End}_K(\underline{M})$. We may therefore without loss of generality assume that K is large enough to not only contain all images $\psi(E)$, but also to guarantee good reduction at a prime ideal $\mathfrak{P}$ above $\varepsilon = \mathfrak{p}$. We write $\underline{\mathcal{M}}$ to denote the model of $\underline{M}$ over $\mathcal{O}_K$. As always, we may consider $\mathcal{M}$ as a free $\mathcal{O}_{E,K} := \mathcal{O}_E \otimes_{\mathbb{F}_q} \mathcal{O}_K$-module of rank one and $\tau_\mathcal{M}$ as an isomorphism $\sigma^*\mathcal{M} \xrightarrow{\sim} \mathcal{M}$ on $Spec(\mathcal{O}_{E,K})$ outside the zero locus of the canonical ideal $\mathcal{J}$.

Since $E = E_1 \times \cdots \times E_s$ is a finite product of fields, the set of homomorphisms $H_E = \mathrm{Hom}(E, K)$ decomposes into a disjoint union of $H_{E_i} = \mathrm{Hom}(E_i, K)$. We also get a disjoint decomposition

$$H_E = \bigcup_{u|\mathfrak{p}} H_{E_u}$$

with $H_{E_w} := \mathrm{Hom}_{Q_\mathfrak{p}}(E_w, K_\mathfrak{P})$, where $K_\mathfrak{P}$ is defined to be the closure of K in $\widehat{Q_\mathfrak{p}^{\mathrm{alg}}}$. We will also use the notation $\mathbb{F}_\mathfrak{P}$ to denote the residue field of $K_\mathfrak{P}$. The ring of units $\mathcal{O}_{K_\mathfrak{P}}$ of $K_\mathfrak{P}$ is isomorphic to $\mathbb{F}_\mathfrak{P}[[\xi]]$ for some uniformizer ξ of $\mathcal{O}_\mathfrak{P}$.

Let us fix some place $u \mid \mathfrak{p}$ of E, and write $\widetilde{A}_u := \widehat{\mathcal{O}_{E,u}}$; thus $\widetilde{A}_u \cong \mathbb{F}_u[\![y_u]\!]$ for some uniformizer y_u of E at u and $A_\mathfrak{p} \subset \widetilde{A}_u$. Furthermore, write

$$\widetilde{A}_{u,\mathcal{O}_{K_\mathfrak{P}}} := \big(\tilde{A}_u \otimes_{\mathbb{F}_q} \mathcal{O}_{K_\mathfrak{P}}\big)^{\wedge_u} \cong \big(\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_\mathfrak{P}[\![\xi]\!]\big)[\![y_u]\!].$$

Consider now the ring $\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_\mathfrak{P}[\![\xi]\!]$. This decomposes into a finite product

$$\prod_{\tilde{i} \in \mathbb{Z}/\tilde{f}\,\mathbb{Z}} \big(\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_\mathfrak{P}[\![\xi]\!]\big)/\tilde{\mathfrak{a}}_{\tilde{i}}\,,$$

where $\tilde{f} := [\mathbb{F}_u : \mathbb{F}_q]$ and

$$\tilde{\mathfrak{a}}_{\tilde{i}} := (\lambda \otimes 1 - 1 \otimes \lambda^{q^{\tilde{i}}} \;:\; \lambda \in \mathbb{F}_u).$$

For $i \in \mathbb{Z}/[\mathbb{F}_\mathfrak{p} : \mathbb{F}_q]\,\mathbb{Z}$, let also

$$\mathfrak{a}_i := (a \otimes 1 - 1 \otimes a^{q^i} \;:\; a \in \mathbb{F}_\mathfrak{p}).$$

Note that the number of ideals $\tilde{\mathfrak{a}}_*$ and the number of ideals $\mathfrak{a}_*$ differs by a factor of $[\mathbb{F}_u : \mathbb{F}_\mathfrak{p}]$, that is, for every $\mathfrak{a}_i$ there are exactly $[\mathbb{F}_u : \mathbb{F}_\mathfrak{p}]$ ideals $\tilde{\mathfrak{a}}_{\tilde{i}}$ lying above it.

The local shtuka $\underline{\hat{M}}_\mathfrak{p}(\underline{\mathcal{M}})$ can now be obtained as

$$\underline{\hat{M}}_\mathfrak{p}(\underline{\mathcal{M}}) = \underline{\mathcal{M}} \otimes_{A_{\mathcal{O}_{K_\mathfrak{P}}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}}/\mathfrak{a}_0\,.$$

The underlying module may be written in the form

$$\mathcal{M} \otimes_{A_{\mathcal{O}_{K_\mathfrak{P}}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}} \cong \mathcal{M} \otimes_{\mathcal{O}_E \otimes_{\mathbb{F}_q} \mathcal{O}_{K_\mathfrak{P}}} \big(\mathcal{O}_E \otimes_{\mathbb{F}_q} \mathcal{O}_{K_\mathfrak{P}}\big) \otimes_{A \otimes_{\mathbb{F}_q} \mathcal{O}_{K_\mathfrak{P}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}}\,.$$

Hence, we have to compute

$$\begin{aligned}\big(\mathcal{O}_E \otimes_{\mathbb{F}_q} \mathcal{O}_{K_\mathfrak{P}}\big) \otimes_{A \otimes_{\mathbb{F}_q} \mathcal{O}_{K_\mathfrak{P}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}} &\cong \mathcal{O}_E \otimes_A A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}} \\ &\cong (\mathcal{O}_E \otimes_A A_\mathfrak{p}) \otimes_{A_\mathfrak{p}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}} \\ &\cong \prod_{u \mid \mathfrak{p}} \widehat{\mathcal{O}}_{E,u} \otimes_{A_\mathfrak{p}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}}\end{aligned}$$

We obtain

$$\mathcal{M} \otimes_{A_{\mathcal{O}_{K_\mathfrak{P}}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}}/\mathfrak{a}_0 = \prod_{u \mid \mathfrak{p}} \big(\mathcal{M} \otimes_{\mathcal{O}_E \otimes \mathcal{O}_{K_\mathfrak{P}}} \tilde{A}_{u,\mathcal{O}_{K_\mathfrak{P}}}\big)/\mathfrak{a}_0\,.$$

We summarize our findings as follows:

Lemma 2.23. *The module $\mathcal{M} \otimes_{A_{\mathcal{O}_{K_\mathfrak{P}}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}}/\mathfrak{a}_0$ decomposes into a finite product*

$$\mathcal{M} \otimes_{A_{\mathcal{O}_{K_\mathfrak{P}}}} A_{\mathfrak{p},\mathcal{O}_{K_\mathfrak{P}}}/\mathfrak{a}_0 = \prod_{u \mid \mathfrak{p}} \hat{\mathrm{M}}_u(\mathcal{M})$$

where

$$\hat{\mathrm{M}}_u(\mathcal{M}) := \big(\mathcal{M} \otimes_{\mathcal{O}_E \otimes \mathcal{O}_{K_\mathfrak{P}}} \tilde{A}_{u,\mathcal{O}_{K_\mathfrak{P}}}\big)/\mathfrak{a}_0$$

We now claim in fact this decomposition produces is compatible with the local-shtuka-structure in the following way:

Lemma 2.24. *Each factor* $\hat{\mathrm{M}}_u(\mathcal{M})$ *is in a natural way the underlying module of a local* $\hat{\sigma}$*-shtuka* $\underline{\hat{M}}_u(\underline{\mathcal{M}})$ *at* $\mathfrak{p}$ *of dimension* d_u *and height (or rank)* h_u.

Proof. With $\tau_{\mathcal{M}}$ being the underlying morphism of $\underline{\mathcal{M}}$, the underlying morphism of its associated local $\hat{\sigma}$-shtuka at $\mathfrak{p}$ is given by $\hat{\tau} := (\tau_{\mathcal{M}})^{[F_{\mathfrak{p}}:\mathbb{F}_q]}$. (Here $\hat{\sigma}$ denotes $\sigma^{[\mathbb{F}_{\mathfrak{p}}:\mathbb{F}_q]}$)

As $\mathcal{M}$ is a locally free $\mathcal{O}_{E,K_{\mathfrak{P}}}$-module of rank one, $\hat{\mathrm{M}}_u(\mathcal{M})$ is a locally free $\tilde{A}_{u,\mathcal{O}_{K_{\mathfrak{P}}}}$-module of rank one. Since $\tilde{A}_u$ is isomorphic to $A_{\mathfrak{p}}^{\oplus\#H_u}$ as $A_{\mathfrak{p}}$-module and E was assumed to be separable, $\hat{\mathrm{M}}_u(\mathcal{M})$ is a locally free $A_{\mathfrak{p},\mathcal{O}_{K_{\mathfrak{P}}}}$-module of rank $[\tilde{A}_u : A_{\mathfrak{p}}] = \#H_u$. By restricting $\hat{\tau}$ to each factor, each $\hat{\mathrm{M}}_u(\mathcal{M})$ therefore defines a local $\hat{\sigma}$-shtuka

$$\underline{\hat{M}}_u(\underline{\mathcal{M}}) = (\hat{\mathrm{M}}_u(\mathcal{M}), \hat{\tau}_u)$$

at $\mathfrak{p}$, which is of rank $\#H_u$ at $\mathfrak{p}$. (Note that $\sigma_{|E} = \mathrm{id}_E$ and hence σ deserves each factor.) It remains to determine the dimension of $\underline{\hat{M}}_u(\underline{\mathcal{M}})$. In other words, we need to calculate the dimension of the cokernel $\operatorname{coker}\hat{\tau}_u$. In the definition of the CM-type $(d_\psi)_{\psi\in H_E}$, we picked a uniformizer y_u for every $u \mid \mathfrak{p}$ and for every $\psi \in H_{E_u}$, we considered the difference $z_{u,\psi} := y_u - \psi(y_u)$. To each $z_{u,\psi}$ we obtain a point

$$x_{u,\psi} := \mathrm{V}(z_{u,\psi}) = \mathrm{V}(y_u - \psi(y_u))$$

lying above the generic fibre $\mathit{Spec}\,(K_{\mathfrak{P}})$ of $\mathit{Spec}\,(\mathcal{O}_{E,K_{\mathfrak{P}}})$. We claim that[5]

$$\operatorname{coker}\hat{\tau}_u \otimes_{\mathcal{O}_{K_{\mathfrak{P}}}} K_{\mathfrak{P}} \cong \widetilde{A}_{u,K_{\mathfrak{P}}} / \prod_{\psi\in H_{E_u}} (y_u - \psi(y_u)^{q^{j(\psi)}})^{d_\psi}$$

for a suitable set of integers $\{j(\psi) : \psi \in H_{E_u}\}$. Let us prove this statement. Consider the map

$$\left(\prod_{\psi\in H_u} (y_u - \psi(y_u))^{-d_\psi}\right) \cdot \tau_{\mathcal{M}}\,.$$

Locally on all $\mathit{Spec}\,(\tilde{A}_{u,\mathcal{O}_{K_{\mathfrak{P}}}} \otimes_{\mathcal{O}_{K_{\mathfrak{P}}}} K_{\mathfrak{P}}) \subset \mathit{Spec}\,(\mathcal{O}_E \otimes_{\mathbb{F}_q} K_{\mathfrak{P}})$, this defines an isomorphism $\mathcal{M} \xrightarrow{\sim} \mathcal{M}$.

Let us take a closer look at the points above $\mathit{Spec}\,(\mathbb{F}_{\mathfrak{P}})$. We have a decomposition

$$\begin{aligned}\mathit{Spec}\,(\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_{\mathfrak{P}}) &\cong \coprod_{i\in\mathbb{Z}/[\mathbb{F}_{\mathfrak{p}}:\mathbb{F}_q]} \mathit{Spec}\,(\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_{\mathfrak{P}})/\mathfrak{a}_i \\ &\cong \coprod_{\tilde{i}\in\mathbb{Z}/[\mathbb{F}_u:\mathbb{F}_q]} \mathit{Spec}\,(\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_{\mathfrak{P}})/\tilde{\mathfrak{a}}_{\tilde{i}}\,.\end{aligned}$$

Note that every $\mathbb{F}_u \otimes_{\mathbb{F}_q} \mathbb{F}_{\mathfrak{P}}/\tilde{\mathfrak{a}}_{\tilde{i}}$ is isomorphic to $\mathbb{F}_{\mathfrak{P}}$ and therefore defines a single point represented by $\tilde{\mathfrak{a}}_i$, which is a member of a well-defined set of points represented by the

[5] The calculation is done only over $K_{\mathfrak{P}}$. Note that this suffices, since $\operatorname{coker}\hat{\tau}_u$ is a free $\mathcal{O}_{K_{\mathfrak{P}}}$-module by [HK18, Lemma 2.3].

ideal $\mathfrak{a}_i$. There is an Frobenius-induced operation on the points $\tilde{\mathfrak{a}}_{\tilde{i}}$ and therefore on the sets $\{\mathfrak{a}_i\}$ given by

$$\mathfrak{a}_i \ni \tilde{\mathfrak{a}}_i \mapsto \sigma^*(\tilde{\mathfrak{a}}_i) \in \mathfrak{a}_{i+1} .$$

On the points $x_{u,\psi}$, this corresponds to

$$\mathfrak{a}_0 \ni x_\psi = \mathrm{V}(y_u - \psi(y_u)) \mapsto \mathrm{V}(y_u - \psi(y_u)^q) \in \mathfrak{a}_1$$

In this way, if $\psi_0 \in H_u$ gives a point $x_{u,\psi} \in \tilde{\mathfrak{a}}_0$, we get a conjugated set of points

$$\sigma^{k*}(x_{u,\psi}) \in \tilde{\mathfrak{a}}_k \qquad (0 \leq k < [\mathbb{F}_\mathfrak{p} : \mathbb{F}_q]) .$$

If ψ is any another element of H_u, we get a similar conjugated set of points, starting in some set $\tilde{\mathfrak{a}}_{\tilde{i}(\psi)} \in \mathfrak{a}_{i(\psi)}$. We therefore get maps $\tilde{i}(\cdot)$ and $i(\cdot)$, relating an embedding ψ to it's ideals $\tilde{\mathfrak{a}}_{\tilde{i}(\psi)}$ and $\mathfrak{a}_{i(\psi)}$. The map $\tilde{i}(\cdot)$ can be calculated in the following manner: Let ψ be some element of H_u, i. e., some $Q_\mathfrak{p}$-morphism $E_u \to K_\mathfrak{P}$. The natural diagram

$$\begin{array}{ccccc} \mathbb{F}_u & \hookrightarrow & \widetilde{A}_u & \hookleftarrow & E_u \\ \downarrow{\scriptstyle \psi_{|\mathbb{F}_u}} & & & & \downarrow{\scriptstyle \psi} \\ \mathbb{F}_u & \hookrightarrow & \mathcal{O}_{K_\mathfrak{P}} & \hookrightarrow & K_\mathfrak{P} \end{array}$$

yields an induced endomorphism $\psi_{|\mathbb{F}_u}$, which is given by some exponent

$$\mathbb{F}_u \ni \lambda \mapsto \lambda^{q^s} .$$

The exponent s is precisely $\tilde{i}(\psi)$, and $i(\psi)$ can be obtained as the residue class of $\tilde{i}(\psi)$ in $\mathbb{Z}/[\mathbb{F}_\mathfrak{p} : \mathbb{F}_q] \cdot \mathbb{Z}$. Writing

$$j(\psi) = [\mathbb{F}_u : \mathbb{F}_q] - \tilde{i}(\psi)$$

for $0 < \tilde{i}(\psi) < [\mathbb{F}_u : \mathbb{F}_u]$ and

$$j(\psi) = 0$$

for $\tilde{i}(\psi) = 0$, we conclude that

$$\prod_{\psi \in H_u} (y_u - \psi(y_u)^{j(\psi)})^{-d_\psi} \cdot \hat{\tau}_u$$

defines an isomorphism

$$\widehat{\sigma}^* \underline{\hat{M}}_u(\underline{\mathcal{M}}) \otimes_{\mathcal{O}_{K_\mathfrak{P}}} K_\mathfrak{P} \xrightarrow{\sim} \underline{\hat{M}}_u(\underline{\mathcal{M}}) \otimes_{\mathcal{O}_{K_\mathfrak{P}}} K_\mathfrak{P} .$$

In particular, we obtain the desired statement

$$\operatorname{coker} \hat{\tau}_u \cong \widetilde{A}_{u,\mathcal{O}_{K_\mathfrak{P}}} / \prod_{\psi \in H_{E_u}} (y_u - \psi(y_u)^{q^{j(\psi)}})^{d_\psi},$$

and counting dimensions at every $\psi \in H_u$, we end up with

$$\dim \underline{\hat{M}}_u(\underline{\mathcal{M}}) = \operatorname{rk} \operatorname{coker} \hat{\tau}_u = \sum_{\psi \in H_u} d_\psi = d_u ,$$

as was to be shown.

⌟

Lemma 2.25. *Under the assumptions of the revised TSF-theorem, the following equality holds:*

$$\frac{\operatorname{ord}_{\mathfrak{p}_u}(\pi)}{\operatorname{ord}_{\mathfrak{p}_u}(\mathfrak{q})} = \frac{\dim(\underline{\hat{M}}_u(\underline{\mathcal{M}})}{\operatorname{rk}(\underline{\hat{M}}_u(\underline{\mathcal{M}})}.$$

Proof. Let us begin with calculating the numerator of the left-hand-side quotient. For $\pi \in \mathcal{O}_E$ being defined as the Frobenius $\tau_{\mathcal{M}}^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_q]}$ of $\mathcal{M} \otimes_{\mathcal{O}_{K_{\mathfrak{P}}}} \mathbb{F}_{\mathfrak{P}}$, we know by construction that $\pi = \widehat{\tau}^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]}$. Furthermore, there exists a unit $b_u \in \mathcal{O}_{E_u}^{\times}$ such that

$$\pi = y_u^{\operatorname{ord}_u(\pi)} \cdot b_u.$$

We consider the map given by

$$\prod_{l \in \mathbb{Z}/[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]} \hat{\sigma}^l \Bigg(\prod_{\psi \in H_u} (y_u - \psi(y_u))^{-d_\psi} \cdot \hat{\tau}_u \Bigg) \cdot \pi\,.$$

This defines an isomorphism

$$\hat{\sigma}^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]*} \underline{\hat{M}}_u(\underline{\mathcal{M}}) \xrightarrow{\sim} \underline{\hat{M}}_u(\underline{\mathcal{M}}).$$

By changing the base to $\mathbb{F}_{\mathfrak{P}}$, we see that

$$\prod_{\psi \in H_u} y_u^{-d_\psi \cdot [\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]} \cdot \pi$$

defines an isomorphism

$$\hat{\sigma}^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]*} \big(\underline{\hat{M}}_u(\underline{\mathcal{M}})_{\mathbb{F}_{\mathfrak{P}}}\big) \xrightarrow{\sim} \underline{\hat{M}}_u(\underline{\mathcal{M}})_{\mathbb{F}_{\mathfrak{P}}}$$

at every unit disk around the points $\tilde{\mathfrak{a}}_{\tilde{i}}$. Focusing at the disk containing the point $\tilde{\mathfrak{a}}_0$ and using the observation about the number of ideals $\mathfrak{a}_*$ and $\tilde{\mathfrak{a}}_*$, we conclude that

$$\underbrace{\Bigg(\prod_{\psi \in H_u} y_u^{-d_\psi \cdot [\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]} \Bigg)}_{= y_u^{-d_u \cdot [\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]}} \cdot \pi = y_u^{[\mathbb{F}_u:\mathbb{F}_{\mathfrak{p}}] \cdot \operatorname{ord}_u(\pi) - d_u \cdot [\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]} \cdot b_u$$

must be a unit, which is equivalent to the vanishing of the exponent of y_u, i. e.

$$\operatorname{ord}_u(\pi) = d_u \cdot \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]}{[\mathbb{F}_u : \mathbb{F}_{\mathfrak{p}}]}\,. \tag{2.1}$$

Note that this is precisely the formula asserted in Remark 2.21.

As for the denominator, since $\mathfrak{q}$ was defined as $\mathrm{N}_{\mathcal{O}_K/A}(\mathfrak{P})$, we have the equality

$$\operatorname{ord}_u \mathfrak{q} = \operatorname{ord}_u \mathfrak{p}^{f(\mathfrak{P}\,/\,\mathfrak{p})} = \mathrm{e}(u \mid \mathfrak{p}) \cdot f(\mathfrak{P}\,/\,\mathfrak{p}) = \mathrm{e}(u \mid \mathfrak{p}) \cdot [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}].$$

Therefore,

$$\frac{\operatorname{ord}_u(\pi)}{\operatorname{ord}_u(\mathfrak{q})} = \frac{d_u \cdot [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]}{\mathrm{e}(u \mid \mathfrak{p}) \cdot [\mathbb{F}_u : \mathbb{F}_{\mathfrak{p}}] \cdot [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]} = \frac{d_u}{h_u}\,,$$

using

$$\# H_u = h_u = \mathrm{e}(u \mid \mathfrak{p}) \cdot [\mathbb{F}_u : \mathbb{F}_{\mathfrak{p}}].$$

Together with the results from the last Lemma, we obtain the desired statement. ⌟

Remark 2.26. *Informally, the valuation of π at some place $u \mid \mathfrak{p}$ is given by the weight of the local shtuka at this place, up to a factor from the field extension.*

The revised Taniyama-Shimura-formula (rTSF) of Thm. 2.17 follows now immediately by combining the preceeding three results.

Remark 2.27. *Note that we did not really need the A-motive structure of $\underline{M}$ for any of the preceeding arguments. Therefore, one could just as well start with a global shtuka $\underline{\mathcal{N}}$ and make the same argument for the induced local shtuka at the characteristic places of $\underline{\mathcal{N}}$.*

Remark 2.28. *Note that the argument for A-motives works entirely with local shtukas, which take the place that p-divisible groups take in the theory of abelian varieties.*

2.2.3. The formula at the infinite place

The TSF allows us to understand the valuation of the Frobenius endomorphism of a reduction at all primes above the characteristic place ε by relating them to the cokernel of the underlying morphism τ_M. However, to understand the A-motive $\underline{M}$, we also need to understand it's behaviour at the hidden characteristic ∞.

Theorem 2.29. *Let $\underline{M}$ be an effective A-motive over an A-field $(K, A \subset K)$ with (CM) given by a semisimple, separable Q-algebra E, such that K is finite over Q. Assume that K contains $\psi(E)$ for all $\psi \in \mathrm{Hom}(E, \mathbb{C}_\infty) = H_E$. Let $\mathfrak{P}$ be a prime ideal in the integral closure $\mathcal{O}_K$ of A in K, such that $\underline{M}$ has good reduction $\underline{M}_{\mathfrak{P}}$ at $\mathfrak{P}$. Assume that $\mathcal{O}_E = E \cap \mathrm{End}(\underline{M})$. Let π be the Frobenius endomorphism of the reduction $\underline{M}_{\mathfrak{P}}$. Let $\mathbb{D}_\infty = (d_{u,\infty})$ be the local CM-divisor of $\underline{M}_{\mathfrak{P}}$ at ∞. For all places $u \in Spec\,(\mathcal{O}_E)$ above ∞, we have*

$$\mathrm{ord}_u(\pi) = d_{u,\infty} \cdot \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]}{[\mathbb{F}_u : \mathbb{F}_q]}\,. \tag{rTSFi}$$

Proof. The asserted formula follows from [Fra17, Lemma 2.2.3.4]. Due to its importance for us, we give a short review of the argument.

On the curve $\tilde{C}$, let u be a point above $\infty \in C$. On the curve $\tilde{C}_{\mathbb{F}_{fP}}$, there are precisely $[\mathbb{F}_u : \mathbb{F}_q]$ many points P lying aboce each such u, which are cyclically permutated by σ^*. Let $\mathcal{N}$ be an extension of M to $\tilde{C}_{\mathbb{F}_{\mathfrak{P}}}$, and denote by $\mathcal{N}_P$ the restriction of $\mathcal{N}$ to $\tilde{C}_{\mathbb{F}_{\mathfrak{P}}} \setminus \{P\}$ Hence every P is a fixpoint under the action of $\sigma^{[\mathbb{F}_u:\mathbb{F}_q]*}$, and therefore the order

$$\mathrm{ord}_P\left(\tau^{[\mathbb{F}_u:\mathbb{F}_q]*}\mathcal{N}_P \to \mathcal{N}_P\right)$$

does not depend on the choice of $\mathcal{N}$ and P. Therefore we can define

$$d_{u,\infty} := \mathrm{ord}_P\left(\tau^{[\mathbb{F}_u:\mathbb{F}_q]*}\mathcal{N}_P \to \mathcal{N}_P\right),$$

and we conclude

$$\mathrm{ord}_u(\pi) = \mathrm{ord}_u(\tau^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_q]})) = [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_u] \cdot d_{u,\infty} = d_{u,\infty} \cdot \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]}{[\mathbb{F}_u : \mathbb{F}_q]}\,.$$

⌟

2.2.4. Generalization to Global Shtukas

Finally, we would like to develop a similar understanding for the structure of the CM-types of global shtukas at their characteristic places.

Theorem 2.30. *Let* $\underline{\mathcal{N}} = (\mathcal{N}, \underline{c}, \tau_{\mathcal{N}})$ *be a global shtuka in the sense of Definition 1.19 over a finite field* $k \cong \mathbb{F}_s \cong \mathbb{F}_{q^e}$ *with Frobenius* π *and (CM) via some commutative* Q*-algebra* E*. Write* $F = Q(\pi)$*. Let* $\mathbb{D}_{\underline{\mathcal{N}}} = (d_{u,i})$ *denote the total CM-divisor of* $\underline{\mathcal{N}}$ *at all places of* E *above* $\underline{c}$*. Let* w_i *be some place of* Q *in* $\underline{c}$*, and let* u *be some place of* E *above* w_i*. Then*

$$\frac{\operatorname{ord}_u(\pi)}{d_{u,i}} = \frac{[k : \mathbb{F}_q]}{[\mathbb{F}_u : \mathbb{F}_q]} .$$

Proof. Works the same as the proof for Theorem 2.29. ⌟

3. The Honda-Tate Correspondence

In this chapter we will explain the setup of the map associating a field element to an abelian Anderson-Motive. At first, we will give a short overview of the analogous parts on the number field side.

3.1. The Correspondence on the Number Field Side

A (complex) abelian variety $A \cong V/\Lambda$ of dimension $n = \dim_{\mathbb{C}} V$ has rank $2n$, if one defines the rank of an abelian variety as the $\mathbb{R}$-rank of its defining lattive Λ. The *weight* $\frac{\dim}{\text{rk}}$ of an abelian variety is therefore always $1/2$.

Definition 3.1.

1. *Let $a > 0$ be a positive integer and p a prime number. Write $q = p^a$. A **p-Weil-Number of order** a (or simply q-Weil number) is an algebraic integer α such that*
$$\sigma(\alpha) \cdot \overline{\sigma(\alpha)} = q$$
for every embedding $\sigma : \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$.

2. *Let α, α' be two q-Weil-numbers. Then α and α' are called **conjugate to each other**, written $\alpha \sim \alpha'$, if there exists a $\mathbb{Q}$-isomorphism*
$$\mathbb{Q}[\alpha] \to \mathbb{Q}[\alpha'], \quad \alpha \mapsto \alpha'.$$
Equivalently,
$$\alpha \sim \alpha' \quad \Longleftrightarrow \quad \min_{\alpha,\mathbb{Q}} = \min_{\alpha',\mathbb{Q}}.$$

3.1.1. Weil Numbers

Theorem 3.2. *There is one-to-one correspondence between the set of conjugacy classes of p-Weil numbers of order a to the set of isogeny classes of simple abelian varieties over $\mathbb{F}_{p^a}$, induced by the Frobenius.*

3.1.2. Injectivity

In order to proof the injectivity of the map, one needs to show that isogeny classes of abelian varieties may be characterized via their Frobenius polynomials, that is, two abelian varieties A and A' are isogenous if and only if their minimal polynomials $\min_A$ and $\min_{A'}$ are equal. This was proven 1966 by John Tate [Tat66].

3.1.3. Surjectivity

The surjectivity of the map was shown by Honda [Hon68]; other sources are [eisentraeger, oort, etc]. The principial strategy is as follows:

1. Choose a Weil-Number α.
2. Determine the (CM)-type associated to α.
3. Find an complex abelian variety A with (CM), that has the "correct" endomorphism ring E.
4. Show that A is already defined over an algebraic number field.
5. Reduce A to a abelian variety defined over a finite field.
6. Find a simple abelian subvariety A_p in A with Frobenius α^N.
7. Use descent theory to find an abelian variety with Frobenius α.

3.2. The Correspondence on the Function Field Side

From now on, fix an algebraic closure Q^{alg} of $Q = \mathrm{Frac}(A)$.

3.2.1. Drinfeld Modules

Let $\mathfrak{p} \subset A$ be a maximal ideal and let $a, r > 0$ be positive integers. An element $\alpha \in Q^{\mathrm{alg}}$ is called an **$\mathfrak{p}$-Drinfeld-Weil-number of degree a and rank r**, if is satisfies the following five conditions:

1. The element α is integral over A, i. e. its minimal polynomial $\min_{\alpha,Q}$ has coefficients in A.
2. There is only one finite place of $Q(\alpha)$ lying above $(\alpha) \subset \mathcal{O}_{Q(\alpha)}$, and this place lies above the characteristic point $\mathfrak{p}$. Here $\mathcal{O}_{Q(\alpha)}$ denotes the integral closure of A in $Q(\alpha)$.
3. There is only one place of $Q(\alpha)$ lying above ∞.
4. The element α has absolute value $|\alpha|_\infty = (\# \mathbb{F}_\mathfrak{p})^{a/r}$ at ∞.
5. The degree $[Q(\alpha) : Q]$ divides r.

As before, two Weil-numbers are called **conjugated**, if they have the same minimal polynomial over Q. Then the following analogue of Honda-Tate-Theory was proven by Jiu-Kang Yu:

Theorem 3.3 ([Yu95, Gos96]). *There is a one-to-one correspondence between isogeny-classes of Drinfeld-modules of rank r over $\mathbb{F}_{p^a}$ and conjugacy classes of Weil-numbers of degree a and rank r, induced from $\phi \mapsto \min_{\phi,Q}$.*

Remark 3.4. *Note that due to the exponent $1/r$ in the definition of Weil-numbers it is always possible to immediately tell the rank of a Drinfeld-module from its Weil-number (and technically also its dimension as an abelian Anderson-motive). This is not the case for abelian varieties in general, since the quotient of dimension and rank always equates to $1/2$.*

3.2.2. A-Motives

While there are many striking analogies between abelian varieties and A-motives, there is also a number of important differences resulting in certain adjustments one has to make.

We have the following

Weil Numbers

We will now give our definition of the analogue of Weil-numbers for A-motives $\underline{M}$ over a finite field k, which we will call *Weil-numbers*. In principle, these will be elements of an appropriate field of absolute value $(\#k)^{\mathrm{wt}\,\underline{M}}$. Note that there is no immediately obvious way to tell rank r and dimension d from α, as was the case for Drinfeld-modules (where always $d = 1$ holds). Extracting rank and dimension from a Weil-number will be the subject of the next subsection.

Pure Weil Numbers

We will first look at the pure case, that is, the Weil numbers corresponding to pure simple A-motives over finite fields.

Definition 3.5. *Let $\mu \in \mathbb{Q}_{\geq 0}$ be a rational number. Let A be an admissible ring via the curve $C'/\mathbb{F}_q$, and let Q be its field of fractions. Let $\mathfrak{p} \subset A$ be a maximal ideal, and let $a > 0$ be a positive integer. Let Q^{alg} denote an algebraic closure of Q. A **pure $\mathfrak{p}$-Weil-number α of order a and weight** $\mathrm{wt}(\alpha) := \mu$ is an element of Q^{alg} satisfying the following conditions:*

1. *The minimal polynomial $\min_{\alpha,Q}$ of α over Q has coefficients in A.*
2. *The element α does not lie over any place w outside $\mathfrak{p}$ and ∞, i. e.*

$$\mathrm{ord}_w(\alpha) = 0 \quad (\forall\, w \nmid \mathfrak{p} \cdot \infty).$$

3. *The element α has absolute value*

$$|\alpha| = \left(\mathbb{F}_{\mathfrak{p}}\right)^{a \cdot \mu}$$

at all extensions of the place $\infty \in C = Spec\,(A) \cup \{\infty\}$ to $Q(\alpha)$.

*We call two such Weil-numbers α, α' **conjugated**, and write $\alpha \sim \alpha'$, if they have the same minimal polynomial $\min_{\alpha,Q} = \min_{\alpha',Q}$ over Q. We denote the set of pure $\mathfrak{p}$-Weil numbers of order a and weight μ by*

$$W_\mu(A, \mathfrak{p}, a)$$

and the set of conjugacy classes of Weil numbers of weight μ by

$$W^{\sim}_{\mu}(A, \mathfrak{p}, a)\,.$$

Weil Numbers for general A-motives

Let us now consider the more general case of Weil numbers for arbitrary, i.e. simple, but not necessarily pure, A-motives.

Definition 3.6. *Let A be an admissible ring via the curve $C'/\mathbb{F}_q$, and let Q be its field of fractions. Fix an $\mathbb{F}_q$-rational point ∞ on C'. Let $\mathfrak{p} \subset A$ be a maximal ideal, and let $a > 0$ be a positive integer. Let Q^{alg} denote an algebraic closure of Q. An $\mathfrak{p}$-**Weil-number** α **of order** a is an element of Q^{alg} satisfying the following conditions:*

1. *The minimal polynomial $\min_{\alpha,Q}$ of α over Q has coefficients in A.*
2. *The element α does not lie over any place w outside $\mathfrak{p}$ and ∞, i.e.*

$$\operatorname{ord}_w(\alpha) = 0 \quad (\forall\, w \nmid \mathfrak{p} \cdot \infty).$$

*We call two such Weil-numbers α, α' **conjugated**, and write $\alpha \sim \alpha'$, if they have the same minimal polynomial $\min_{\alpha,Q} = \min_{\alpha',Q}$ over Q. We denote the set of $\mathfrak{p}$-Weil numbers of order a by*

$$\mathrm{W}(A, \mathfrak{p}, a)$$

and the set of conjugacy classes of Weil numbers by

$$\mathrm{W}^{\sim}(A, \mathfrak{p}, a).$$

Remark 3.7.
1. *Please note that the definition above does not, in fact, depend on the choice of a. However, the bijection between Weil numbers and A-motives will depend on the ground field k, given by $k \cong \mathbb{F}_{q^a}$, and the same Weil number α would therefore correspond to multiple A-motives, depending on the choice of k. To avoid misunderstandings, we have therefore decided to include the a in the definition of Weil-number.*
2. *Also, the reader may have already noticed that our definition of Weil-numbers for A-motives, pure or otherwise, does not include any provision on the number of places above ε or ∞, opposed to Def. 3.2.1 of Drinfeld-Weil numbers. It was proven by E.-U. Gekeler in [Gek91, Th. 2.9] for Drinfeld-modules, there is only one place of F above ε and ∞, respectively. For an A-motive of higher dimension, no such requirement is needed.*

3.2.3. Global Shtukas

Definition 3.8. *Let $n \geq 2$ be a natural number, k be some finite field, and $\underline{c} := (c_1, \ldots, c_n)$ be a tuple of closed points on C. We then call any element $\alpha \in Q^{alg}$ a **Weil number of characteristic** $\underline{c}$ **and order** a, if the following single condition holds:*

- *The element α does not lie over any place u outside $\underline{c}$, i. e.*

$$\operatorname{ord}_u(\alpha) = 0 \quad (\forall\, u \notin \underline{c}).$$

We call two such Weil-numbers α, α' ***conjugated****, and write $\alpha \sim \alpha'$, if they have the same minimal polynomial $\min_{\alpha,Q} = \min_{\alpha',Q}$ over Q. We denote the set of Weil numbers of characteristic $\underline{c}$ by*

$$\mathrm{W}_{\underline{c}}(A, a)$$

and the set of conjugacy classes of Weil numbers of characteristic $\underline{c}$ by

$$\mathrm{W}_{\underline{c}}^{\sim}(A, a).$$

3.2.4. Questions

Given the correspondence for abelian varieties and Drinfeld-modules, it is natural to ask the following questions:

1. Let $k/\mathbb{F}_{\mathfrak{p}}$ be a field extension of degree a. Given an $\mathfrak{p}$-Weil-number α of order a, does there exist an effective A-motive $\underline{M}_\alpha$ over k of characteristic $\mathfrak{p}$ such that it's Frobenius endomorphism π is conjugated to α?

2. Will any two A-motives $\underline{M}$ and $\underline{M}'$ with Weil-number α be isogenous?

3. If α is a pure Weil-number of weight μ, can we find a pure A-motive $\underline{M}_\alpha$ of weight μ over k of characteristic $\mathfrak{p}$ with Frobenius $\pi \sim \alpha$?

4. If $\underline{M}_\alpha$ exists, can we determine its dimension and rank directly from α?

5. Let k be a finite field and let $\underline{c} = (c_1, \dots, c_n)$ be a tuple of $\mathbb{F}_q$-homomorphisms $c_i^* : \mathbb{F}_{v_i} \to k$. Given a Weil number α of characteristic $\underline{c}$, does there exist a global shtuka over k whose k-Frobenius is conjugated to α?

Most of these questions will be answered in later chapters. However, we can answer the fourth one, assuming that such an A-motive can indeed be found.

3.2.5. Dimension and Rank

As in the case for Drinfeld-modules, i. e. $d = 1$, we would like to control the rank r and dimension d of $\underline{M}$ as well. Since r and d do not need to be relatively prime, however, one has to be careful if one wishes to extract (d, r) from the weight of $\underline{M}$ alone. Therefore, additional insights are required.

Let $\underline{M}$ be a simple A-motive of dimension d and rank r over a finite A-field (k, γ). Write $E := \operatorname{QEnd}(\underline{M})$ and $F := Q(\pi)$, where π is the Frobenius of $\underline{M}$. Let h denote the degree of F/Q, i. e. the degree of the Frobenius polynomial of $\underline{M}$.

We know from Theorem 1.77 that E is a skew field with center F. We know from [BH09], that the degree $[E : F]$ is precisely $\frac{r^2}{h^2}$. If we can find a way to calculate $[E : F]$ from Q and F alone, we can calculate r.

Furthermore, we know from Theorem 1.77 the local Hasse invariants $\mathrm{inv}_v(E)$ for all places v of F. More precisely, we have for all places v of F the explicit formula

$$\mathrm{inv}_v(E) = -\frac{[\mathbb{F}_v : \mathbb{F}_q]}{a} \cdot \mathrm{ord}_v(\pi).$$

Note that the expression on the right hand side does depend only on $F = Q(\pi)$ and k. Therefore, given an $\mathfrak{p}$-Weil-number α of order a, we can calculate all the local invariants of E from knowledge of $F = Q(\alpha)$ and a. Write $h(\alpha) := [Q(\alpha) : Q]$. Global class field theory yields the short exact sequence

$$0 \to \mathbf{Br}(F) \to \bigoplus_v \mathbf{Br}(F_v) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

where the first non-trivial map is given by scalar extension and the second by summation. In particular, given all local invariants of the central simple algebra E, we can calculate the order of $[E] \in \mathbf{Br}(F) \hookrightarrow \oplus_v \mathbf{Br}(F_v)$ as the least common multiple

$$l(\alpha) := \mathrm{lcm}_v(\mathrm{ord}_{\mathbf{Br}\, F_v}(\mathrm{inv}_v[E])) = \mathrm{lcm}_v \left(\mathrm{ord}_{\mathbb{Q}/\mathbb{Z}} \left(-\frac{[\mathbb{F}_v : \mathbb{F}_\mathfrak{p}]}{a} \cdot \mathrm{ord}_v(\pi) \right) \right)$$

of the local orders in $\mathbf{Br}(Fv)$, that is $\mathbb{Q}/\mathbb{Z}$. However, the order of $[E]$ in $\mathbf{Br}(F)$ is precisely $\sqrt{[E : F]}$ via the theorem below; therefore, with

$$\mathrm{rk}(\alpha) := h(\alpha) \cdot l(\alpha)$$

and for α pure of weight $\mathrm{wt}(\alpha) = \mu$

$$\dim(\alpha) := \mathrm{wt}(\alpha) \cdot \mathrm{rk}(\alpha),$$

any simple pure A-motive with Frobenius $\pi = \alpha$ must have rank $\mathrm{rk}(\alpha)$ and dimension $\dim(\alpha)$. For for non-pure α, the rank-calculation works just the same as above, but the dimension needs to be calculated differently; since the dimension of any A-motive (M, τ_M) is just the dimension of the cokernel of τ_M, which according to the discussion in Sections 2.1.6 and 2.2.2 can be calculated in the following way:

$$\dim(\alpha) := \sum_{u|\varepsilon} \frac{h_u}{a} \cdot \frac{\mathrm{ord}_u(\alpha)}{\mathrm{ord}_u(\mathfrak{p})},$$

with notation as in Thm. 2.20. We denote the set of Weil numbers of rank r and dimension d w. r. t. an A-field k of degree $[k : \mathbb{F}_\mathfrak{p}] = a$ by

$$\mathrm{W}(A, \mathfrak{p}, a, d, r) \subset \mathrm{W}(A, \mathfrak{p}, a)$$

and the set of conjugacy classes of Weil numbers of rank r and dimension d by

$$\mathrm{W}^{\sim}(A, \mathfrak{p}, a, d, r).$$

The equivalent sets for pure Weil-numbers of dimension d and rank r and weight $\mu = d/r$ are denoted by

$$W_\mu(A, \mathfrak{p}, a, d, r) \quad \text{and} \quad W_\mu^{\sim}(A, \mathfrak{p}, a, d, r)$$

Remark 3.9. *Note that the equation $r(\alpha) := h(\alpha) \cdot l(\alpha)$ implies ($h(\alpha)$ and $l(\alpha)$ being integers by definition) in particular, that the degree $h(\alpha)$ of $\min_{\alpha,Q}$ of a Weil-number divides its rank $r(\alpha)$, echoing the last condition for Weil-numbers of Drinfeld-modules.*

3.2.6. Statement of Honda-Tate Correspondence

Theorem 3.10. *There is a bijection between the set of isogeny classes of simple A-motives over a fixed finite A-field* (k,γ) *with fixed characteristic points* $(\mathfrak{p} = \ker\gamma, \infty)$ *with* $[k : \mathbb{F}_\mathfrak{p}] = a$ *and the set of conjugacy classes of Weil-numbers* α *in the sense of Definition 3.6 induced by* $\underline{M} \mapsto \min_{\pi,Q}$*:*

$$A\text{-Mot}^{\sim}(\mathfrak{p}, a) \stackrel{1:1}{\longleftrightarrow} \mathrm{W}^{\sim}(A, \mathfrak{p}, a)$$

Moreover, this bijection induces bijections

$$A\text{-Mot}^{\sim}(\mathfrak{p}, a, d, r) \stackrel{1:1}{\longleftrightarrow} \mathrm{W}^{\sim}(A, \mathfrak{p}, a, d, r)$$

for all $d \geq 0, r > 0$.

Theorem 3.11. *The bijections of the preceeding theorem are compatible with purity definitions on both sides, that is, for all* $d \geq 0, r > 0$ *there is a bijection*

$$A\text{-Mot}^{\sim}_{\mathrm{p,s}}(\mathfrak{p}, a, d, r) \stackrel{1:1}{\longleftrightarrow} W^{\sim}_{d/r}(\mathfrak{p}, a),$$

induced via $\underline{M} \mapsto \pi$*. In particular, for* $d \geq 0, r > 0$ *we also get bijections*

$$A\text{-Mot}^{\sim}_{\mathrm{p,s}}(\mathfrak{p}, a, d, r) \stackrel{1:1}{\longleftrightarrow} W^{\sim}_{d/r}(A, \mathfrak{p}, a, d, r)\,.$$

Theorem 3.12. *Let us denote the set of isogeny classes of simple global shtukas over* $k \cong \mathbb{F}_{q^a}$ *of characteristic* $\underline{c}$ *by*

$$\mathrm{N}^{\sim}_{\mathrm{s}}(\underline{c}, a)\,.$$

Let $k \cong \mathbb{F}_{q^a}$ *be a finite field of characteristic* p*. There is a bijection between the set of isogeny classes of simple global shtukas over* $S = Spec\,(k)$ *with fixed paws paws* $\underline{c}$*, and the set of conjugacy classes of Weil-numbers* α *in the sense of Definition 3.8 induced by* $\underline{\mathcal{N}} \mapsto \min_{\pi,Q}$*:*

$$\mathrm{N}^{\sim}_{\mathrm{s}}(\underline{c}, a) \stackrel{1:1}{\longleftrightarrow} \mathrm{W}^{\sim}_{\underline{c}}(a)$$

4. Proof of Injectivity

This chapter is rather short, as basically all the work has already been done. In particular, the injectivity of the Honda-Tate correspondence was proven for pure A-motives by Hartl and Bornhofen in [BH09]. Specifically, we have the following statement:

Theorem 4.1. *Let $\underline{M}$ and $\underline{M}'$ be two simple A-motives over a finite A-field k with Frobenius endomorphisms π and π', respectively. Let $\min_{\pi,Q}$ and $\min_{\pi',Q}$ denote their respective Frobenius polynomials as elements of $A[X]$. Let furthermore ν be a place of A away from the A-characteristic ε, and denote by χ_ν and χ'_ν the characteristic polynomials of $\pi_\nu = \underline{\hat{V}}_\nu(\underline{\pi}) \in \mathrm{End}_{Q_v}(\underline{\hat{V}}_\nu(\underline{M}))$ and $\pi'_\nu = \underline{\hat{V}}_\nu(\underline{\pi}') \in \mathrm{End}_{Q_v}(\underline{\hat{V}}_\nu(\underline{M}'))$, respectively. Then $\underline{M}$ and $\underline{M}'$ are isogenous if and only they have the same characteristic polynomial, or equivalently if and only if they are of the same rank r and have the same Frobenius polynomial, that is if $\min_{\pi,Q} = \min_{\pi',Q}$, i. e.*

$$\underline{M} \sim \underline{M}' \quad \Longleftrightarrow \quad \chi_\nu = \chi'_\nu \quad \Longleftrightarrow \quad r = r' \text{ and } \min_{\pi,Q} = \min_{\pi',Q}\,.$$

The proof, while not explicitly stated there, is easily derived from [BH09]. For the sake of completeness, we will give an explicit write-up.

Proof. The statement follows immediately from [BH09, Thm. 8.1, part 1], as long as we can convince ourselves that π_ν and $\pi_{\nu'}$ are semisimple and μ_π is irreducible, i. e. $Q(\pi)$ is a field. Since our A-motives are assumed to be simple, we know from [BH09, Thm. 6.11] that the quasi-endomorphism algebra $E := \mathrm{QEnd}(\underline{M})$ forms a division algebra over Q. In particular, E is semisimple and therefore π and also π_ν are semisimple according to [BH09, Prop. 6.8]. Finally, [BH09, Cor. 6.10] informs us that the algebra $Q(\pi)$ is the center of $\mathrm{QEnd}(\underline{M})$ and therefore a field, as was to be shown. Note that none of the cited arguments in [BH09] need purity assumptions, as all calculations are done away from ∞. ⌟

Corollary 4.2. *Let $k \cong \mathbb{F}_s \cong \mathbb{F}_{q^e}$ be a finite field of characteristic p and let $\underline{\mathcal{N}}$ and $\underline{\mathcal{N}}'$ be two simple global shtukas over $S = Spec\,(k)$ with the same set of paws $\underline{c}$. Let π and π' denote their Frobenius elements, respectively. Then the same statement holds as for A-motives, i. e. $\underline{\mathcal{N}}$ and $\underline{\mathcal{N}}'$ are isogenous if and only if they have the same rank r and their Frobenius polynimals $\min_{\pi,Q}$ and $\min_{\pi',Q}$ coincide.*

Proof. The proof of the necessary parts of [BH09, Thm. 8.1] is based on local arguments away from ∞, and therefore carries over to global shtukas without problems. In particular, the Tate conjecture remains true for global shtukas, see Thm. (1.73). ⌟

Hence we obtain as immediate consequence the following

Corollary 4.3. *The map associating to a pure abelian A-motive $\underline{M}$, respectively to a global shtuka $\underline{\mathcal{N}}$ over a finite field its Frobenius element is well-defined and injective on isogeny classes.*

5. Proof of Surjectivity

We now aim to prove the central result of this thesis, that is, all Weil numbers are representatives stem from A-motives (or global shtukas, respectively). Our strategy[1] for A-motives is based on the arguments Tate used in [Tat68] and follows the same essential pattern - we first construct the correct endomorphism algebra from a $\mathfrak{p}^a$-Weil-number α and find in it a CM-algebra, which we use to define an A-motive $\underline{M}_{\mathbb{C}_\infty}$ of generic characteristic above $\mathbb{C}_\infty$. Using results about CM-motives and reduction theory from the first chapter, we restrict $\underline{M}_{\mathbb{C}_\infty}$ to a finite extension field K of Q and reduce it to an A-motive $\underline{M}_{k'}$ over some finite field $k'/\mathbb{F}_\mathfrak{p}$. Using the Taniyama-Shimura formula proven in Chapter 2, we can choose $\underline{M}_{k'}$ such that its Frobenius endomorphism will be some power α^m of α. Finally, we use descent to obtain an A-motive over the correct field k of degree $[k : \mathbb{F}_\mathfrak{p}] = a$ such that one of its simple quotient motives has Frobenius conjugate to α. Finally, we explain how to adjust the arguments used in order to make them work for the more general case of global shtukas as well.

5.1. The Endomorphism Algebra E_α

From now on, let A be some admissible ring via some curve C, and let $\mathfrak{p} \subset A$ be some maximal ideal. Fix a positive integer a, and let α be an $\mathfrak{p}$-Weil-number of degree a, that is

$$\alpha \in \mathrm{W}^\sim(A, \mathfrak{p}, a)\,.$$

5.1.1. Construction of E_α

We write $F_\alpha := Q(\alpha)$. Consider the set of rational numbers

$$i_v = -\frac{[\mathbb{F}_v : \mathbb{F}_q]}{a} \cdot \mathrm{ord}_v(\alpha)$$

for all places v of F_α. Note that all the mathematical objects on the right hand side of this equation are solely dependend on the information given by F_α and the degree of α. Therefore, the tupel $(i_v)_v$ is well defined. Let us now choose a semisimple Q-algebra E'_α with center F_α such that

$$\mathrm{inv}_v(E'_\alpha) = i_v$$

for all places v of F_α. By construction of the Brauer group (see Appendix, part D.1), we can always pick E'_α to be a skew field. In fact, E'_α is even simple, since its center is by construction a field.

[1] I also made use of seminar notes on Honda-Tate theory written by Brian Conrad [Con04], Kirsten Eisentraeger [Eis04] and Frans Oort [Oor08], which have the benefit of being somewhat more detailed than Tate's paper, and also of being written in English.

5.1.2. Existence of a Commutative Subalgebra of rank $\mathrm{rk}(\alpha)$

Since we need our A-motive to possess (CM), we need to show that its algebra of quasi-endomorphisms contains a commutative subalgebra of rank $\mathrm{rk}(\alpha)$. We will find this algebra in E'_α. Note that $Q(\alpha) \subset E'_\alpha$ will not in general suffice, since we can only expect the relation $[Q(\alpha) : Q] \mid \mathrm{rk}(\alpha)$ to hold. However, it is well known (see Prop. D.1 in the appendix) that the algebra E'_α contains a separable splitting field of degree $\mathrm{rk}(\alpha)/h$ over its center $Q(\alpha)$ and therefore a commutative Q-subalgebra of degree $\mathrm{rk}(\alpha)$ over Q. Let us denote this splitting field by E_α (was K_α).

5.1.3. The Case of inseparable E

Our construction so far has not made any assumptions on α that would guarantee separability of E_α; in particular, α could be inseparable over Q. We did, however, develop our theory in earlier chapters for separable E only, since the Taniyama-Shimura-formula in particular makes use of the existence of sufficiently many embeddings $E_\alpha \hookrightarrow Q^{\mathrm{alg}}$. As we will now explain, this is not a fundamental problem, since we can always reduce[2] to the case of separable E. Define F' to be the field $Q(\alpha^{p^n})$, where p^n is the inseparability degree $[Q(\alpha) : Q]_{\mathrm{insep}}$ of $Q(\alpha)$ over Q. In particular, α^{p^n} will be separable over Q, so F'/Q is a separable extension. Now define F'' to be field $F'(\beta^{p^n})$, where β is a root of $X^{p^n} - \alpha^{p^n} \in F'[X]$. The field F' embeds into F'' via $x \mapsto x^{p^n}$. The picture of field inclusions now looks like this:

$$\begin{array}{ccc}
F' = Q(\alpha^{p^n}) & \xrightarrow{x \mapsto x^{p^n}} & F'' = F'(\beta^{p^n}) \\
\uparrow & & \uparrow \\
A \subset Q & \xrightarrow{x \mapsto x^{p^n}} & Q'' = Q \supset A'' = A
\end{array}$$

Note that both horizontal extensions are purely inseparable of degree p^n, while both vertical extensions are separable (see [HS17, Lemma A.2]). In particular, we can use β^{p^n} instead of α as Weil-number and construct an A''-motive for the field extension F''/Q'', which we will consider as A-motive under the natural inclusion $A \hookrightarrow A''$. In other words, we may assume without loss of generality, that F/Q is already separable.

5.2. Construction of A-motive over a finite extension of Q

5.2.1. Construction over $\mathbb{C}_\infty$

Our next step is to find an A-motive $\underline{M}$ defined over $\mathbb{C}_\infty$ with (CM) via E_α. We understand $\mathbb{C}_\infty$ as an A-field via the canonical embedding

$$A \hookrightarrow Q \hookrightarrow \mathbb{C}_\infty\,.$$

There exists a projective and regular curve $\widetilde{C}$ defined over $\mathbb{F}_q$ with function field $\mathbb{F}_q(\widetilde{C}) = E_\alpha$. Let us denote the base-change to $\mathbb{C}_\infty$ by $\widetilde{C}_{\mathbb{C}_\infty} := \widetilde{C} \times_{\mathbb{F}_q} \mathbb{C}_\infty$. For some algebraic closure $\mathbb{F}_q^{\mathrm{alg}} \subset \mathbb{C}_\infty$ we obtain the following diagrams:

[2] This trick was pointed out to me by U. Hartl.

$$\begin{array}{ccccc} \widetilde{C}_{\mathbb{C}_\infty} & \longrightarrow & \widetilde{C}_{\mathbb{F}_q^{\mathrm{alg}}} & \longrightarrow & \widetilde{C} \\ \downarrow & & \downarrow & & \downarrow \\ Spec\,(\mathbb{C}_\infty) & \longrightarrow & Spec\,(\mathbb{F}_q^{\mathrm{alg}}) & \longrightarrow & Spec\,(\mathbb{F}_q) \end{array}$$

and

$$\begin{array}{ccc} \widetilde{C}_{\mathbb{C}_\infty} & \longrightarrow & C_{\mathbb{C}_\infty} \\ \downarrow & & \downarrow \\ \widetilde{C} & \longrightarrow & C \end{array}$$

Choose divisors $D_{\mathfrak{p}}$ and D_∞ on $\widetilde{C}_{\mathbb{C}_\infty}$ (lying over the two different points $\mathrm{V}(\mathcal{J})$ and ∞ of C) of the same degree $d \geq 0$ and form the divisor $D' := D_\infty - D_{\mathfrak{p}}$ of degree zero. Now, if we consider the corresponding invertible sheaf and use $\deg D' = 0$, we get

$$\mathcal{O}(D') \in \mathrm{Pic}^0_{\widetilde{C}/\mathbb{F}_q}(\mathbb{C}_\infty) \subset \mathrm{Pic}_{\widetilde{C}/\mathbb{F}_q}(\mathbb{C}_\infty).$$

We know that $f := \mathrm{Frob}_q - id$ is surjective as endomorphism of the algebraic group $\mathrm{Pic}^0_{\widetilde{C}/\mathbb{F}_q}(\mathbb{C}_\infty)$ and can therefore choose an invertible sheaf $\mathcal{M}_{\mathbb{C}_\infty}$ corresponding to a divisor D such that

$$\mathrm{Pic}^0_{\widetilde{C}/\mathbb{F}_q}(\mathbb{C}_\infty) \ni \mathcal{M}_{\mathbb{C}_\infty} \mapsto \mathcal{O}(D') \in \mathrm{Pic}^0_{\widetilde{C}/\mathbb{F}_q}(\mathbb{C}_\infty)$$

under f. On the level of divisors, this means that

$$\sigma^*(D) - D = D' = D_\infty - D_{\mathfrak{p}}.$$

Hence, we get an isomorphism

$$\sigma^*\mathcal{M}_{\mathbb{C}_\infty} = \mathcal{O}(\sigma^* D) \to \mathcal{O}(D + D') = \mathcal{M}(D_\infty - D_{\mathfrak{p}}),$$

which we will label $\tau_{\mathcal{M}_{\mathbb{C}_\infty}}$.

By construction, we have $Spec\,(\mathcal{O}_{E_\alpha,\mathbb{C}_\infty}) \subset \widetilde{C}_{\mathbb{C}_\infty}$. Then

$$\underline{M} := (\Gamma(Spec\,(\mathcal{O}_{E_\alpha,\mathcal{M}}), \mathcal{M}), \tau_{\mathcal{M}_{\mathbb{C}_\infty}}(Spec\,(\mathcal{O}_{E_\alpha,\mathbb{C}_\infty}))$$

forms a τ-module of rank 1 over $\mathcal{O}_{E_\alpha,\mathbb{C}_\infty}$.

Lemma 5.1.

$$\underline{M} := (\Gamma(Spec\,(\mathcal{O}_{E_\alpha,\mathcal{M}}), \mathcal{M}), \tau_{\mathcal{M}_{\mathbb{C}_\infty}}(Spec\,(\mathcal{O}_{E_\alpha,\mathbb{C}_\infty}))$$

forms a τ-module of rank 1 *over* $\mathcal{O}_{E_\alpha,\mathbb{C}_\infty}$.

Furthermore, we claim that $\underline{M}$, when viewed as $A_{\mathbb{C}_\infty}$-module, carries the structure of an A-motive of over $\mathbb{C}_\infty$, and that

$$E_\alpha \subset \mathrm{QEnd}_{\mathbb{C}_\infty}(\underline{M})$$

defines a semisimple CM-algebra for $\underline{M}$ of rank $r = \mathrm{rk}(\alpha)$ over Q. More precisely, we have the following diagram of inclusions:

$$\begin{array}{ccccccc}
A & \hookrightarrow & Q & \xhookrightarrow{\operatorname{rk} h} & Q(\alpha) & \xleftrightarrow{=} & Center(E'_\alpha) \\
\big\downarrow{\scriptstyle \operatorname{rk} r} & & \big\downarrow{\scriptstyle \operatorname{rk} r} & & \big\downarrow{\scriptstyle \operatorname{rk} r/h} & & \big\downarrow{\scriptstyle \operatorname{rk} r^2/h^2} \\
\mathcal{O}_{E_\alpha} & \hookrightarrow & E_\alpha & \xleftrightarrow{=} & E_\alpha & \hookrightarrow & E'_\alpha \\
\big\downarrow & & & & \big\downarrow & & \\
\operatorname{End}_{\mathbb{C}_\infty}(\underline{M}) & & & & \operatorname{QEnd}_{\mathbb{C}_\infty}(\underline{M}) & &
\end{array}$$

Here, the first and second rows are true by assumption and construction. The first row of vertical arrows is immediately derived from the theory of central simple algebras, see above. The injection

$$\mathcal{O}_{E_\alpha} \hookrightarrow \operatorname{End}_{\mathbb{C}_\infty}(\underline{M})$$

follows from the fact that M is a $\mathcal{O}_{E_\alpha} \otimes \mathbb{C}_\infty$-module with τ-compatible action. The second inclusion

$$E_\alpha \hookrightarrow \operatorname{QEnd}_{\mathbb{C}_\infty}(\underline{M})$$

follows from the first and the fact that

$$\mathcal{O}_{E_\alpha} \otimes Q = E_\alpha.$$

We summarize our findings in the following

Proposition 5.2. *The τ-module $\underline{M}$ defines an A-motive $\underline{M}_{\mathbb{C}_\infty}$ over $\mathbb{C}_\infty$ of rank* $\operatorname{rk}(\alpha)$*. If $D_\mathfrak{p}$ is effective of degree $d \geq 0$, then $\underline{M}$ is an A-motive of dimension d. Furthermore, $\underline{M}$ has Complex Multiplication via E_α.*

Good reduction over a finite Q-extension

Since $\underline{M}_{\mathbb{C}_\infty}$ is defined over the algebraically closed field $\mathbb{C}_\infty$ and has (CM) via $\mathcal{O}_{E_\alpha}$ by construction, we can apply section (2.1.4). In our situation, the A-motive $\underline{M}_{\mathbb{C}_\infty}$ has generic characteristic, i. e. the kernel ε of $A \to \mathbb{C}_\infty$ vanishes, and therefore $\mathbb{F}_\varepsilon = Q$. Hence there exists a finite extension field K of Q and an A-motive $\underline{M}_K$ defined over K such that

$$\underline{M}_K \otimes \mathbb{C}_\infty \cong \underline{M}_{\mathbb{C}_\infty}.$$

Furthemore, by Theorem 2.13 the motive $\underline{M}_K$ has potentially good reduction everywhere, that is, for any given prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ there exists a finite field extension K'/K such that $\underline{M}_{K'}$ has good reduction at a prime above $\mathfrak{P}$, i. e., we can reduce to an A-motive $\underline{M}_{\kappa(\mathfrak{P})}$ with $\kappa(\mathfrak{P}) = \mathcal{O}_{K'}/\mathfrak{P}$. So we can without loss of generality assume that $\underline{M}_K$ has already good reduction.

5.2.2. Choice of divisors $D_\mathfrak{p}$ and D_∞

The construction works for any divisors $D_\mathfrak{p}$ and D_∞ of the same degree d lying above two distinct closed points $\mathfrak{p}$ and ∞ of the curve C. We continue on to make specific choices for D_∞ and $D_\mathfrak{p}$ in our setting.

Consider the ideal

$$\mathcal{J}_{\mathbb{C}_\infty} = (a \otimes 1 - 1 \otimes \gamma(a) : a \in A) \subset A_{\mathbb{C}_\infty}$$

and its corresponding point

$$P_{\mathcal{J}} := \mathrm{V}(\mathcal{J}) \in C_{\mathbb{C}_\infty}.$$

In particular, $P_{\mathcal{J}}$ is lying above the characteristic ideal of $\underline{M}_{\mathbb{C}_\infty}$, which is the generic point η on C. Let

$$\tilde{P}_1, \ldots, \tilde{P}_r \in \widetilde{C}_{\mathbb{C}_\infty}$$

be the points lying above P. Similarly, let

$$\infty_1, \ldots, \infty_s \in C_{\mathbb{C}_\infty}$$

denote the points of $C_{\mathbb{C}_\infty}$ lying above $\infty \in C$, and let $\{\infty_{i,j}\}$ for all $1 \leq i \leq s$ denote the finite set of points in $\widetilde{C}_{\mathbb{C}_\infty}$ lying above each $\infty_i \in C_{\mathbb{C}_\infty}$. Therefore,

$$\begin{array}{ccccc} \overbrace{\tilde{P}_i}^{\in \widetilde{C}_{\mathbb{C}_\infty}} & \to & \overbrace{P}^{=V(J) \in C_{\mathbb{C}_\infty}} & \to \eta, & \\ \underbrace{\tilde{\infty}_{i,j}}_{\in \widetilde{C}_{\mathbb{C}_\infty}} & \to & \underbrace{\infty_i}_{C_{\mathbb{C}_\infty}} & \to & \underbrace{\infty}_{\in C}. \end{array}$$

The Divisor above $\mathfrak{p}$

We now have to pick a CM-type $(E, (d_\psi)_{\psi \in H_E})$ at $\mathfrak{p}$. According to the revised Taniyama-Shimura formula (rTSF) of Theorem 2.20,

$$\frac{\mathrm{ord}_u(\pi)}{\mathrm{ord}_u(\mathfrak{q})} = \frac{d_u}{h_u},$$

where π is the Frobenius of the reduction, $h_u := \# H_u = [E_u : Q_{\mathfrak{p}}]$ and $\mathfrak{q} := \mathfrak{p}^a$. Write $\mathfrak{q}_\alpha := \mathfrak{p}^a$. We define the tuple of integers $(d_{u,\mathfrak{p}})_u$ indexed by the places u of E_α lying above $\mathfrak{p}$ to be

$$\begin{aligned} d_{u,\mathfrak{p}} &= \frac{h_u \cdot \mathrm{ord}_u(\alpha)}{\mathrm{ord}_u(\mathfrak{q}_\alpha)} \\ &= \frac{f(u \mid \mathfrak{p}) \cdot \mathrm{e}(u \mid \mathfrak{p}) \cdot \mathrm{ord}_u(\alpha)}{a \cdot \mathrm{e}(u \mid \mathfrak{p})} \\ &= \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_{\mathfrak{p}}]}{a}. \end{aligned}$$

Then all $d_{u,\mathfrak{p}}$ are integers: Since the CM-algebra E_α is in particular a splitting field for E'_α, all invariants of E'_α at the places of E_α vanish. Pick a place u of E_α above a place v of F_α above the place $w = \mathfrak{p}$ of Q, that is, $u|v|w = \mathfrak{p}$. Then

$$\mathbb{Q}/\mathbb{Z} \ni 0 = \mathrm{inv}_u(E_\alpha \otimes_F E'_\alpha) = -\frac{[\mathbb{F}_v : \mathbb{F}_{\mathfrak{p}}]}{a} \cdot \mathrm{ord}_v(\alpha) \cdot [E_{\alpha,u} : F_{\alpha,v}]$$

using the general formula

$$\mathrm{inv}_u(E_\alpha \otimes_{F_\alpha} E'_\alpha) = [E_{\alpha,u} : F_{\alpha,v}] \cdot \mathrm{inv}_v(E'_\alpha)$$

from the appendix, section D.2 and

$$\mathrm{inv}_v(E'_\alpha) = -\frac{[\mathbb{F}_v : \mathbb{F}_q]}{[\mathbb{F}_s : \mathbb{F}_q]} \cdot \mathrm{ord}_v(\pi)$$

from Thm. 1.77. Now

$$\begin{aligned} d_{u,\mathfrak{p}} &= \frac{h_u \cdot \mathrm{ord}_u(\alpha)}{\mathrm{ord}_u(\mathfrak{q}_\alpha)} \\ &= [E_{\alpha,u} : Q_\mathfrak{p}] \cdot \frac{\mathrm{ord}_u(\alpha)}{a \cdot \mathrm{e}(u \mid \mathfrak{p})} \\ &= [E_{\alpha,u} : F_{\alpha,v}] \cdot [F_{\alpha,v} : Q_\mathfrak{p}] \cdot \frac{\mathrm{ord}_v(\alpha) \cdot \mathrm{e}(u \mid v)}{a \cdot \mathrm{e}(u \mid v) \cdot \mathrm{e}(v \mid p)} \\ &= [E_{\alpha,u} : F_{\alpha,v}] \cdot \frac{[F_{\alpha,v} : Q_\mathfrak{p}]}{\mathrm{e}(v \mid \mathfrak{p})} \cdot \frac{\mathrm{ord}_v(\alpha)}{a} \\ &= \frac{[\mathbb{F}_v : \mathbb{F}_\mathfrak{p}]}{a} \cdot \mathrm{ord}_v(\alpha) \cdot [E_{\alpha,u} : F_v]\,. \end{aligned}$$

Ergo

$$d_{u,\mathfrak{p}} = -\,\mathrm{inv}_u(E_\alpha \otimes_F E'_\alpha) = 0 \in \mathbb{Q}\,/\,\mathbb{Z}\,,$$

and hence all $d_{u,\mathfrak{p}}$ are indeed integers. Note that all the factors on the right hand side of this equation are known to us from the knowledge of α.

Define H_u to be the elements ψ of $E_{E'_\alpha}$ satisfying $\psi^{-1}(\mathrm{V}(\mathcal{J})) = u$. Now pick for all $u \mid \mathfrak{p}$ a set of integers d_ψ indexed by the elements of H_u such that

$$d_{u,\mathfrak{p}} = \sum_{\psi \in H_u} d_\psi\,.$$

Since the points $\tilde{P} \mid P = \mathrm{V}(\mathcal{J})$ and the elements ψ of H_E correspond to each other bijectively, we can form the divisor $D_\mathfrak{p}$ as

$$D_\mathfrak{p} := \sum_{\tilde{P} \mid \mathrm{V}(\mathcal{J})} d_\psi \cdot (\tilde{P})\,.$$

Remark 5.3. *Note that the choice of coefficients d_ψ is in general not unique. However, all the choices result in isogenous A-motives, since their endomorphism algebras are by construction isomorphic.*

The Divisor above ∞

We define a CM-type at ∞ in a similar fashion, using Theorem 2.29. Define

$$d_{u,\infty} := \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_q]}{a \cdot [\mathbb{F}_\mathfrak{p} : \mathbb{F}_q]} = \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_\mathfrak{p}]}{a}$$

at all places u of E laying above ∞. The same argument as above shows that all the $d_{u,\infty}$ are integers, so we can form the divisor

$$D_\infty := \sum_{v|\infty} -d_{u,\infty} \cdot (v)\,.$$

Then the divisor

$$D := D_\mathfrak{p} - D_\infty$$

has degree zero, since

$$\begin{aligned}
\deg D &= \sum_{\tilde{P}|P} d_\psi + \sum_{u|\infty} d_{u,\infty} \\
&= \sum_{u|\mathfrak{p}} d_{u,\mathfrak{p}} + \sum_{u|\infty} + d_{u,\infty} \\
&= \sum_{u|\mathfrak{p}} \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_\mathfrak{p}]}{a} + \sum_{u|\infty} \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_\mathfrak{p}]}{a} \\
&= \frac{1}{a} \sum_{u|\mathfrak{p}\cdot\infty} \mathrm{ord}_u(\alpha) \cdot [\mathbb{F}_u : \mathbb{F}_\mathfrak{p}] \\
&= \frac{1}{a} \sum_{u \text{ place of } E_\alpha} \mathrm{ord}_u(\alpha) \cdot [\mathbb{F}_u : \mathbb{F}_\mathfrak{p}] \\
&= \frac{1}{a} \cdot \underbrace{\deg(\mathrm{div}(\alpha))}_{=0} \\
&= 0
\end{aligned}$$

5.3. Surjectivity up to a power

Proposition 5.4. *There exists a prime ideal $\mathfrak{P} \subset \mathcal{O}_K$, such that the reduction of $\underline{M}_K$ at $\mathfrak{P}$, namely the A-motive*

$$\underline{M}_\mathfrak{P} := \underline{M}_K \otimes \mathcal{O}_K/\mathfrak{P}$$

over $k'' := \mathcal{O}_K/\mathfrak{P}$ satisfies the following condition: There are positive integers $n, m \in \mathbb{Z}_{>0}$, such that the Frobenius endomorphism $\pi_\mathfrak{P}$ of $\underline{M}_\mathfrak{P}$ is a zero of $X^n - \alpha^m \in Q(\alpha[X]$.

Proof. We had chosen K such that $\underline{M}_K$ has good reduction everywhere, which allows us to reduce $\underline{M}$ at any prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$ above $\mathfrak{p}$ and apply the Taniyama-Shimura formula. The revised Taniyama-Shimura formula of Theorem 2.20 does not need $\mathfrak{P} \cap A$ to be unramified, hence any prime ideal $\mathfrak{P}$ with good reduction lying above the ideal $\mathfrak{p}$, i. e. dividing $\mathfrak{p} \cdot \mathcal{O}_K$ may be used. We need to show that our chosen Weil number α, when taken to the m-th power, is equal to a power of the Frobenius endomorphism $\pi_\mathfrak{P}$ of the reduction $\underline{M}_\mathfrak{P}$. It suffices to show that for some powers α' of α and $\pi'_\mathfrak{P}$ of $\pi_\mathfrak{P}$ the fraction $\pi'_\mathfrak{P}/\alpha'$ is a root of unity, since then some power $(\pi'_\mathfrak{P}/\alpha')^N$ is equal to 1, or in other words, $\left(\pi'_\mathfrak{P}\right)^N = \left(\alpha'\right)^N$. Since we are working over a function field with finite field of coefficients, every unit is a root of unity, according to [Has80, Chapter 22, p.

356]. It is therefore sufficient find suitable powers $\pi'_{\mathfrak{P}}$ and α' and prove that $\pi'_{\mathfrak{P}}/\alpha'$ is a unit, i. e. that

$$\operatorname{ord}_u(\alpha') = \operatorname{ord}_u(\pi'_{\mathfrak{P}})$$

for all places u of E_α. By construction of $\underline{M}$ and the definition of Weil-numbers, π and α do not have prime divisors outside $\mathfrak{p}$ or ∞, hence we already know

$$\operatorname{ord}_u(\alpha') = 0 = \operatorname{ord}_u(\pi'_{\mathfrak{P}}) \qquad (\forall u \nmid \mathfrak{p} \cdot \infty)$$

for any two powers α' and $\pi'_{\mathfrak{P}}$. For pure Weil numbers, at all places u above ∞, the value of $\pi_{\mathfrak{P}}$ is by construction

$$|\pi_{\mathfrak{P}}|_u = \big(\#\,\mathbb{F}_{\mathfrak{P}}\big)^{\dim(\alpha)/\operatorname{rk}(\alpha)},$$

while the value of α is by assumption

$$|\alpha|_u = \big(\mathbb{F}_{\mathfrak{p}}\big)^{a\cdot\mu(\alpha)} = \big(\mathbb{F}_{\mathfrak{p}}\big)^{a\cdot\frac{\dim(\alpha)}{\operatorname{rk}(\alpha)}}.$$

both π and α have the same value $q^{d/r}$. Hence

$$|\alpha^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]}|_u = \big(|\alpha|_u\big)^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_{\mathfrak{p}}]} = |\pi_{\mathfrak{P}}|_u$$

everywhere.

Let us pick some suitable ideal $\mathfrak{P}$.

According to (rTSF), the Frobenius $\pi_{\mathfrak{P}}$ of the reduction-motive $\underline{M}_{\mathfrak{P}}$ then satisfies

$$\operatorname{ord}_u(\pi_{\mathfrak{P}}) = d_{u,\mathfrak{p}} \cdot \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]}{[\mathbb{F}_u : \mathbb{F}_{\mathfrak{p}}]}$$

at all prime places u of E_α above the characteristic ideal $\mathfrak{p}$, and

$$\operatorname{ord}_u(\pi_{\mathfrak{P}}) = d_{u,\infty} \cdot \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]}{[\mathbb{F}_u : \mathbb{F}_q]}$$

at all places $u \mid \infty$.

When we constructed $\underline{M}_{\mathbb{C}_\infty}$ from the fixed Weil number α, we picked the $d_{v,\mathfrak{p}}$ and $d_{w,\infty}$ precisely to satisfy this formula evaluated for α as Weil-$\mathfrak{p}$-number of order a. Hence using the Taniyama-Shimura formulas for $\pi_{\mathfrak{P}}$ at some ideal $\mathfrak{P}$, we get the equalities

$$\frac{\operatorname{ord}_u(\pi_{\mathfrak{P}})}{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]} = \frac{d_{u,\infty}}{[\mathbb{F}_u : \mathbb{F}_q]} = \frac{\operatorname{ord}_u(\alpha)}{a \cdot [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_q]}$$

for all $u \mid \infty$ and

$$\frac{\operatorname{ord}_u(\pi_{\mathfrak{P}})}{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]} = \frac{d_{u,\mathfrak{p}}}{[\mathbb{F}_u : \mathbb{F}_q]} = \frac{\operatorname{ord}_u(\alpha)}{a \cdot [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_q]}$$

for all $u \mid \mathfrak{p}$ and thus

$$\frac{\operatorname{ord}_u(\pi_{\mathfrak{P}})}{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]} = \frac{\operatorname{ord}_u(\alpha)}{a \cdot [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_q]}$$

or

$$\operatorname{ord}_u(\pi_{\mathfrak{P}}) \cdot a = \operatorname{ord}_u(\alpha) \cdot [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]$$

for all $u \mid \mathfrak{p} \cdot \infty$. Writing $\pi' := \pi_{\mathfrak{P}}^{a \cdot [\mathbb{F}_{\mathfrak{p}}:\mathbb{F}_q]}$ and $\alpha' := \alpha^{[\mathbb{F}_{\mathfrak{P}}:\mathbb{F}_q]}$ we obtain

$$\operatorname{ord}_u(\pi') = \operatorname{ord}_u(\alpha')$$

at all places $u \mid \mathfrak{p} \cdot \infty$, and 0 otherwise. But then the fraction $\frac{\pi'}{\alpha'}$ must be a root of unity, so we can find some other natural number N such that

$$(\pi')^N = (\alpha')^N .$$

Thus with $n := a \cdot [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_q] \cdot N$ and $m := [\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q] \cdot N$, the Frobenius $\pi_{\mathfrak{P}}$ is indeed a zero of

$$X^n - \alpha^m ,$$

as was to be shown. ⌟

Let k be an extension of $\mathbb{F}_{\mathfrak{p}}$ of order a.

Corollary 5.5. *There exists an extension k'/k of degree m such that the Weil-number α^m is conjugated to the Frobenius $\pi_{k'}$ of some semisimple A-motive $\underline{M}_{k'}$ defined over k'. In other words,*

$$\pi_{k'} \sim \alpha^{[k':k]} .$$

Proof. Using the notation from the proof of Theorem 5.4 and writing k' for the field $\mathbb{F}_{\mathfrak{P}}$, we know that $\pi_{\mathfrak{P}}$ is a zero of $X^n - \alpha^m \in Q(\alpha)[X]$ with

$$\frac{m}{n} = \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q]}{a \cdot [\mathbb{F}_{\mathfrak{p}} : \mathbb{F}_q]} = \frac{[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_{\mathfrak{p}}]}{a} = [k' : k] .$$

Choosing k'/k sufficiently large enough such that $\underline{M}_{\mathfrak{P}} \otimes k'$ is semisimple and $[k' : k]$ is a multiple of $[\mathbb{F}_{\mathfrak{P}} : \mathbb{F}_q] \cdot N$, we can immediately conclude

$$\pi_{k'} = \alpha^{[k':k]}$$

as desired. ⌟

5.4. Descent: from α^m to α

So far, we have succeeded in proving that some power α^m of our chosen Weil number α may be written as Frobenius endomorphism of some A-motive $\underline{M}'$. It remains to argue that α itself is a simple Frobenius.

Remark 5.6. *Note that the last Proposition is completely obvious when the A-motive $\underline{M}$ is pure, since in this case all valuations at $v \mid \infty$ are equal to $(\#k)^{-\operatorname{wt}(\underline{M}_{k'})}$, and we have a direct relation between Weil-numbers of weight μ and degree e and pure simple A-motives of weight μ over $\mathbb{F}_{q^e}$. This precisely matches the argument for abelian varieties. Since non-pure A-motives do not allow such a direct relation between the field k and the Weil-number, more care has to be taken.*

We can now proceed to argue that α itself is a Weil number, using the Weil restriction functor from k' to k, as explained in the Appendix, part (C.2): Form the module defined via

$$\widetilde{M} := \bigoplus_{i=0}^{[k':k]-1} \sigma_k^{i*} M',$$

where σ_k denotes the k-Frobenius in $\mathrm{Aut}_k(k'/k)$. There exists a permutation isomorphism

$$\sigma_k^* \underline{\widetilde{M}} \xrightarrow{\sim} \underline{\widetilde{M}},$$

i. e. given by the matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & 1 & 0 \end{pmatrix}$$

This module has rank $[k' : k] \cdot \mathrm{rk}(\underline{M}')$, and together with

$$\tau_{\widetilde{M}} : \bigoplus \sigma_k^{i*} \tau_M : \sigma^* \widetilde{M} \to \widetilde{M}$$

it forms an A-motive $\underline{\widetilde{M}} = (\widetilde{M}, \tau_{\widetilde{M}})$.

Proposition 5.7. *The A-motive $\underline{\widetilde{M}}$ over k'/k with Frobenius $\tilde{\pi}_{k'}$ descends to an A-motive $\underline{\widetilde{M}}_k$ over k with Frobenius $\tilde{\pi}_k$ such that*

$$(\tilde{\pi}_k)^{[k':k]} = \tilde{\pi}_{k'}.$$

.

With

$$\tilde{\pi}_k := \tau_{\widetilde{M}}^{[k:\mathbb{F}_q]},$$

we obtain

$$(\tilde{\pi}_k)^{[k':k]} = \tilde{\pi}_{k'}.$$

As per construction, we have obtained an A-motive $\underline{\widetilde{M}}$ over k such that its Frobenius endomorphism $\tilde{\pi}_k$ satisfies $\tilde{\pi}_K^{[k':k]} = \pi_{k'} = \alpha^{[k':k]}$. Pick some place $v \neq \mathfrak{p}$ of A and consider the characteristic polynomials $\chi_{k',v}$ and $\chi_{k,v}$ of $\tilde{\pi}_{k',v}$ and $\tilde{\pi}_{k',v}$, respectively. Then

$$\chi_{k,v}(T^{[k':k]}) = \chi_{k',v}(T).$$

Now, the gist of the argument is as follows: Since $\alpha^{[k':k]}$ is a zero of $\chi_{k',v}$, the Weil number α is a zero of $\chi_{k,v}$, and hence of some irreducible polynomial g dividing $\chi_{v,k}$, corresponding to some simple factor motive $\underline{M}$ of $\underline{\widetilde{M}}$. Therefore, $\underline{M}$ must have Frobenius π conjugated to α. There is, however, no direct statement in the literature that immediately gives the necessary relationship between irredible factors of χ_v and quotient A-motives, so we have chosen to include a more detailed argument for our specific situation.

Proposition 5.8. *There exists a simple quotient A-motive $\underline{M}$ of $\underline{\tilde{M}}$ such that the Frobenius π of $\underline{M}$ is conjugated to α.*

Proof. Since α is a root of $\chi_{k,v}$, it has to be a root of $\min_{\tilde{\pi}_{k,v},Q}$ and therefore of $\min_{\tilde{\pi}_k,Q}$. Ergo $\min_{\alpha,Q}$ divides $\min_{\tilde{\pi}_k,Q}$. Let us write

$$\min_{\tilde{\pi}_k,Q} = (\min_{\alpha,Q})^l \cdot g\,,$$

with $(g, \min_{\alpha,Q}) = 1$, and choose polynomials $a, b \in Q[X]$ such that $a\cdot(\min_{\alpha,Q})^l + b\cdot g = 1$. Now put

$$\beta := a(\tilde{\pi}_k) \cdot (\min_{\alpha,Q})^l(\tilde{\pi}_k)$$

and

$$\delta := b(\tilde{\pi}_k) \cdot g(\tilde{\pi}_k)\,,$$

both of which are elements of $\mathrm{QEnd}_k(\underline{\tilde{M}})$. Choose some $c \in A$ such that $\delta' = c \cdot \delta \in \mathrm{End}_k(\underline{\tilde{M}})$, and write $\underline{N}$ for the image of δ'. Then we have a canonical projection-inclusion sequence

$$\underline{\tilde{M}}_k \overset{p}{\twoheadrightarrow} \underline{N} \overset{i}{\hookrightarrow} \underline{\tilde{M}}_k$$

such that $i \circ p = \delta'$. By construction,

$$\beta + \delta = \mathrm{id}, \quad \beta \circ \delta = 0, \quad \beta^2 = \beta, \quad \delta^2 = \delta\,.$$

Hence

$$i \circ p \circ i \circ p = \delta^2 = \delta = i \circ p, \,,$$

and, by injectivity of i,

$$p = p \circ i \circ p = p \circ \delta'\,.$$

Furthermore,

$$(p \circ \tilde{\pi}_k \circ i)^2 = p \circ \tilde{\pi}_k \circ \underbrace{(i \circ p)}_{=\delta'} \circ \tilde{\pi}_k \circ i = p \circ \delta' \circ \tilde{\pi}_k^2 \circ i$$

and then

$$\min^l_{\alpha,Q}\left((\tilde{\pi}_k)_{|\mathrm{Im}(\delta')}\right) = (\min_{\alpha,Q})^l(p \circ \tilde{\pi}_k \circ i) = p \circ \underbrace{\delta' \circ (\min_{\alpha,Q})^l(\tilde{\pi}_k)}_{=0} \circ i = 0\,.$$

In particular, if $\underline{M}$ is a simple quotient A-motive of $\mathrm{Im}(\delta')$ with Frobenius π, then $(\min_{\alpha,Q})^l = 0$, thus π is a root of $\min_{\alpha,Q}$, and hence

$$\pi \sim \alpha$$

as desired.

⌟

5.4.1. Purity

We did also claim that pure Weil-numbers correspond exactly to pure A-motives, that is, the Honda-Tate correspondence induces a bijectipn

$$A\text{-Mot}^{\sim}_{\mathrm{p,s}}(A,\mathfrak{p},a,d,r) \xrightarrow{\sim} W^{\sim}_{d/r}(A,\mathfrak{p},a)$$

Let us first note that is suffices to show purity for the A-motive $\underline{M}_{\mathbb{C}_\infty}$ defined over $\mathbb{C}_\infty$ we constructed first, since none of the reduction steps we took afterwards changed the behaviour at the point ∞. Now, purity for $\underline{M}_{\mathbb{C}_\infty}$ means the following: Let z be a uniformizing parameter at $\infty \in C$; then $Q_\infty = \mathbb{F}_\infty[\![z]\!][1/z] = \hat{\mathcal{O}}_{C,\infty}$, and $\mathbb{F}_\infty / \mathbb{F}_q$ is a finite field extension of degree f_∞. Write $Q_{\infty,\mathbb{C}_\infty} = (\mathbb{C}_\infty \otimes_{\mathbb{F}_q} \mathbb{F}_\infty)(\!(z)\!)$. Then

$$Q_{\infty,\mathbb{C}_\infty} = \prod_{i\in\mathbb{Z}/f_\infty} (\mathbb{C}_\infty \otimes_{\mathbb{F}_q} \mathbb{F}_\infty)(\!(z)\!)/\mathfrak{a}_i$$

for $\mathfrak{a}_i = (a\otimes 1 - 1\otimes \gamma(a)^{q^i} : a \in \mathbb{F}_\infty)$, corresponding to the points ∞_i above ∞ (compare the section 1.5.1 on local shtukas). And then the local isoshtuka at ∞ splits into standard-isoshtukas as follows (also compare the proof of Thm. 2.20):

$$\begin{aligned}\mathcal{M} \otimes_{\mathcal{O}_{\mathbb{C}_\infty}} Q_{\infty,\mathbb{C}_\infty}/\mathfrak{a}_0 &\cong (\mathbb{C}_\infty(\!(z)\!)^{\oplus r}, \tau_{\mathcal{M}}^{f_\infty}) \\ &\cong \left(\bigoplus_i (\mathbb{C}_\infty(\!(z)\!)^{\oplus r_i}, \tau = \begin{pmatrix} 0 & & \dots & z^{d_i} \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}\right)\right)\end{aligned}$$

So $\underline{M}_{\mathbb{C}_\infty}$ is pure if and only if the fractions $\frac{d_i}{r_i}$ are identical for all i, that is, if the pole of $\tau_{\mathcal{M}}^{f_\infty}$ at a place $\infty_{0,j}$ on $\tilde{C}_{\mathbb{C}_\infty}$ above ∞_0 does not depend on j. But this follows from our construction of D_∞ and the definition of pure Weil-numbers, hence $\underline{M}_{\mathbb{C}_\infty}$ is indeed pure if α was chosen to be pure.

5.5. Extension to Global Shtukas

Let $\alpha \in Q$ be a Weil-number of characteristic $\underline{c}$ global shtukas in the sense of Definition 3.8 for some finite field $k \cong \mathbb{F}_s \cong \mathbb{F}_{q^e}$ and a set of paws $\underline{c} = (c_1,\dots,c_n)$ on the curve C. As before, write F for the field $Q(\alpha)$. Let E'_α be the commutative simple Q-algebra with center F defined by α via its Hasse invariants

$$\mathrm{inv}_u(E'_\alpha) = -\operatorname{ord}_v(\alpha)\cdot\frac{[\mathbb{F}_u : \mathbb{F}_q]}{[k:\mathbb{F}_q]}$$

using the structure theorem (1.77) for Global Shtukas, and pick a splitting field E_α inside E'_α. Write $\tilde{C}$ for the curve above C corresponding to E_α.

Construcion of Global Shtuka over k^{alg}

Pick a place w of Q in $\underline{c}$ and let u be a place of E_α above w. Write v for the place of F below u and above w.

Define
$$d_u := \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_q]}{[k : \mathbb{F}_q]}\,.$$

Then d_u is a rational integer, using the same argument as before. Hence we can find integers $d_{\tilde{u}}$ for $\tilde{u}$ the points on $\tilde{C}_{k^{\mathrm{alg}}}$ lying above $u \in \tilde{C}$ such that
$$\sum_{\tilde{u}|u} d_{\tilde{u}} = d_u\,.$$

Choose any such set of integers $d_{\tilde{u}}$ to define the divisor
$$D_{\underline{c}} = \sum_{\tilde{u}|\underline{c}} d_{\tilde{u}} \cdot [\tilde{u}]\,.$$

Let $D_{\underline{c}}$ be a divisor of the curve $\tilde{C}_{k^{\mathrm{alg}}}$ given via a tuple $(d_{\tilde{u}})_{\tilde{u}}$ of integers, indexed over the points $\tilde{u}$ on the curve $\tilde{C}_{k^{\mathrm{alg}}}$, via
$$D_{\underline{c}} = \sum_{\tilde{u}|\underline{c}} d_{\tilde{u}} \cdot [\tilde{u}]\,,$$

such that for all places u of E_α we have
$$\sum_{\tilde{u}|u} d_{\tilde{u}} = d_u = \mathrm{ord}_u(\alpha) \cdot \frac{[\mathbb{F}_u : \mathbb{F}_q]}{[k : \mathbb{F}_q]}\,.$$

Assume that $D_{\underline{c}}$ has degree 0. Then similar to before, we define
$$\mathcal{L}' := \mathcal{O}_{\tilde{C}_{k^{\mathrm{alg}}}}(D_{\underline{c}}) \in \mathrm{Pic}^0_{\tilde{C}/\mathbb{F}_q}(k^{\mathrm{alg}})\,.$$

Using the isomorphism $\mathrm{id} - \mathrm{Frob}_q$ on $\mathrm{Pic}^0_{\tilde{C}/\mathbb{F}_q}(k^{\mathrm{alg}})$, we find an element $\mathcal{L}$ of $\mathrm{Pic}^0_{\tilde{C}/\mathbb{F}_q}(k^{\mathrm{alg}})$ with
$$\mathcal{L} \cong \sigma^*\mathcal{L} \otimes \mathcal{L}'\,.$$

Since we are working over the algebraic closure of the finite field $k \cong \mathbb{F}_s$ of characteristic p, the line bundles $\mathcal{L}'$ and $\mathcal{L}$ are defined already over a finite extension field of $\mathbb{F}_q$. Call this field of definition k'. Write $\mathcal{N}_{k'}$ for the global shtuka given by
$$(\mathcal{L}, \tau_{\mathcal{N}} := \sigma^*\mathcal{L} \to \mathcal{L} \otimes \mathcal{O}(-D_{\underline{c}}), \underline{c})\,.$$

Frobenius and Descent

Lemma 5.9. *Let $\underline{\mathcal{N}}_{k'}$ be the global shtuka over k' constructed above, and let $\pi_{k'}$ be its Frobenius endomorphism. Then*
$$\mathrm{ord}_u(\pi_{k'}) = \mathrm{ord}_u(\alpha^{[k':k']})$$

at all places $u \in \tilde{C}_{k'}$.

Proof. From Theorem 2.30, we know

$$\operatorname{ord}_u(\pi_{k'}) = d_u \cdot \frac{[k' : \mathbb{F}_q]}{[\mathbb{F}_u : \mathbb{F}_q]}$$

above any of the characteristic places, and zero otherwise. We chose the integers d_u to satisfy

$$\operatorname{ord}_u(\alpha) = d_u \cdot \frac{[k : \mathbb{F}_q]}{[\mathbb{F}_u : \mathbb{F}_q]}$$

above any of the characteristic places, and zero otherwise. Together, $\operatorname{ord}_u(\pi/\alpha^{[k':k']}) = 0$ at all places u on $\tilde{C}_{k'}$. ⌟

We have now found a global shtuka $\underline{\mathcal{N}}$ over a finite extension k' of k with

$$\operatorname{ord}_u(\pi_{k'}) = \operatorname{ord}_u(\alpha^{[k':k']})$$

i. e. we are in the situation of Proposition,5.4. Using the same argument as in Proposition 5.5, we conclude that the Frobenius π of $\underline{\mathcal{N}}_{k'}$ is (conjugated to) a power α^m of α, and that m is precisely the degree $[k' : k]$. Ergo, using Weil restriction again, we can descent $\underline{\mathcal{N}}_{k'}$ to a global shtuka $\underline{\mathcal{N}}_k$ over k, and find a simple factor shtuka $\underline{\mathcal{N}}$ of $\underline{\mathcal{N}}_k$ with Frobenius conjugated to α.

Appendices

A. Anderson-A-modules

A.1. Definitions

We will now proceed to give the first higher-dimensional generalization of Drinfeld modules, which was invented by Anderson [And86] under the name of *t-modules*.

Definition A.1.

1. *An* ***abelian Anderson-A-Module*** $\underline{G} = (G, \varphi)$ *over an A-field* (k, γ) *is a geometrically regular, affine group scheme G together with a ring homomorphism* $\varphi : A \to \mathrm{End}_k(G)$, $a \mapsto \varphi_a$, *such that the following conditions are met:*
   - *(i)* $G \times_L Spec\,(L^{alg}) \cong \mathbb{G}_{a,k^{alg}}$.
   - *(ii)* $(\mathrm{T}_0(\varphi_a) - \gamma(a))^d = 0$ *on* $\mathrm{T}_0\,G = k^d$.
   - *(iii)* $M := M(G, \varphi) := \mathrm{Hom}(G, \mathbb{G}_{a,k})$ *is a locally free* A_k*-module of rank r via* $(a, m) \mapsto m \circ \varphi_a$ *and* $(b, m) \mapsto b \circ m$ *for all* $a \in A$, $b \in k$ *and* $m \in M$*. It is furthermore a* $k\{\tau\}$*-module, with* τ *operating as q-Frobenius on* M.
2. *A morphism of Anderson-A-modules is a morphism of group schemes compatible (in the natural way) with the A-action on both sides.*

Remark A.2. *Every Drinfeld-module* ϕ *naturally defines an Anderson-A-module* $M(\phi)$*. In fact, every Anderson-A-module of dimension* 1 *arises in this manner, and we have an equivalence of categories*

$$\{ \text{ Drinfeld-A-modules of rank } r \ \} \xrightarrow{\sim} \{ \text{ Anderson-A-modules of dim. } 1 \text{ and rank } r \}$$

If we avoid the technicalities of scheme-theory in the definition of Anderson-modules, the similarities between Drinfeld-modules and Anderson-modules are readily apparent. More precisely, an Anderson-A-module of dimension d may be defined as a ring morphism

$$\phi : A \to k\{\tau\}^{d\times d}, \quad a \mapsto \phi_a$$

such that

$$(B_0 - \gamma(a))^d = 0\,, \qquad \text{where } \phi_a = \sum_{i=0}^{n} B_i \cdot \tau^i$$

and $k\{\tau\}^{1\times d}$ *is locally free as* A_k*-module under*

$$A \times M \ni (a, m) \mapsto m \cdot \phi_a \in M\,,$$
$$k \times M \ni (b, m) \mapsto b \cdot m \in M\,.$$

A.2. Equivalence of Anderson-A-modules and A-motives

Via descent theory (see Appendix C.1), we give a sketch argument for the equivalence of our two notions of analogues of abelian variety. From [Gos96] we know that both notions are equivalent over perfect fields. If $\underline{G} = (G, \varphi)$ is an abelian Anderson-A-module over k, let

$$\underline{M}(\underline{G}) := (\mathrm{Hom}_{k,\mathbb{F}_q}(G, \mathbb{G}_{\mathrm{a},k}), \tau_M),$$

where τ_M is given by

$$\sigma^* M \ni m \otimes 1 \mapsto \tau \circ m \in M.$$

.

Theorem A.3. *The mapping $\underline{G} \mapsto \underline{M}(\underline{G})$ defines a fully faithful, contravariant functor from the category of abelian Anderson-A-modules over k to the category of abelian A-motives over k, with essential image consisting of the subcategory of abelian effective Anderson-A-motives. In addition, the functor respects rank and dimension.*

Sketch of Proof. (See [And86, Thm.1] and [Har16, Thm.3.5].) Via descent theory the category of Anderson-A-modules (resp. -motives) is equivalent to the category of abelian Anderson-A-modules (resp. -motives) over k^{alg} with descent datum. Since k^{alg} is perfect, we know from [Gos96, Thm. 5.4.11] that these categories are equivalent. All that remains is to proof that the resulting square diagram of functors commutes, which is merely a technical exercise. ⌟

A.3. CM-types for A-modules

There is a different way to define CM-types of A-motives at the point $\varepsilon = \ker \gamma$, which was originally used by Schindler [Sch09] to classify certain A-motives and by Pelzer [Pel09] to prove a version of the Taniyama-Shimura formula. We will briefly recap those definitions and explain how they give the same result.

Definition A.4. *Let E be a Q-algebra. Let K/Q be a field extension sufficiently large, such that $\varphi(E) \subset K$ for all $\varphi \in \hom(E, Q^{alg})$.*

1. *A pair $(E, \mathfrak{q})$ is called a* ***CM-type****, if the following conditions are met:*
   - *(i) The Q-algebra E is semisimple and commutative.*
   - *(ii) There exists an $z \in Q$ such that $I = (z - \gamma(z)) \subset A_K$ and*

     $$\mathfrak{q} \subset E \otimes K[\![z - \gamma(z)]\!]$$

     is a submodule of rank $\dim_Q E$.
2. *Two CM-types $(E, \mathfrak{q})$ and $(E', \mathfrak{q}')$ are said to be isomorphic, if there exists a Q-algebra isomorphism $E \to E'$ mapping $\mathfrak{q}$ to $\mathfrak{q}'$.*

Remark A.5. *The connection to the definition of (CM)-types presented in Chapter 2 is as follows: Let $(E, (d_\psi)_{\psi \in H_E})$ be a CM-type as defined in section 2.1.6. We've already used that the tensor product $E \otimes K[\![z - \gamma(z)]\!]$ splits as*

$$E \otimes K[\![z - \gamma(z)]\!] \cong \prod_{\psi \in H_E} K[\![x_\psi - \psi(x_\psi)]\!] \,,$$

where the x_ψ are local uniformizers of E with $\psi(x_\psi) \neq 0$. Then form the ideal

$$\mathfrak{q} := \bigoplus_{\psi \in H_E} (x_\psi - \psi(x_\psi))^{-d_\psi} \cdot K[\![x_\psi - \psi(x_\psi)]\!]$$

Then obviously

$$\mathfrak{q} \subset E \otimes K[\![z - \gamma(z)]\!] \,,$$

and the pair $(E, \mathfrak{q})$ forms a CM-type as defined above, and vice versa.

Definition A.6. *Let $\underline{G}$ be a pure Anderson-A-Module with (CM) via a semisimple Q-algebra E. We define the* ***generalized eigenspace*** $(\mathrm{T}_0(\underline{G}))_\varphi$ *of $\varphi \in \mathrm{Hom}(E, K)$ as*

$$(\mathrm{T}_0(\underline{G}))_\varphi := \{x \in \mathrm{T}_0 \, \underline{G} \,; \forall g \in \mathcal{O}_E \, \exists n \in \mathbb{N}_0 \,:\, (\mathrm{T}_0 \, g - \varphi(g))^n(x) = 0\}.$$

Furthermore, we define the ***CM-set*** Φ *of $\underline{G}$ as*

$$\Phi := \{(\varphi, d_\varphi) \,; \, \varphi \in \mathrm{Hom}(E, K), (\mathrm{T}_0 \, \underline{G})_\varphi \neq 0, d_\varphi = \dim(\mathrm{T}_0 \, \underline{G})_\varphi\}.$$

Lemma A.7. *If E is semisimple, the generalized eigenspaces of $\underline{G}$ span the tangent space $\mathrm{T}_0 \, \underline{G}$ of $\underline{G}$ at zero, i. e.*

$$\mathrm{T}_0 \, \underline{G} = \bigoplus_{\varphi \in \Phi} (\mathrm{T}_0 \, \underline{G})_\varphi.$$

Note that the dual $\mathrm{Hom}_K(\mathrm{T}_0 \, \underline{G}, K)$ of $\mathrm{T}_0 \, \underline{G}$ is canonically isomorphic to the cokernel of τ_M, see [Har16, Thm. 3.5].

Remark A.8.
1. *The two notions of CM-type are equivalent whenever both are defined. More precisely, for an abelian Anderson A-motive $\underline{M}$ with associated uniformizable Anderson A-module $\underline{G}$, the CM-types obtained from the decomposition of $\mathfrak{q}_{\underline{M}}$ and from the decomposition of $\mathrm{T}_0 \, \underline{G}$ are equivalent, and both give rise to equivalent decompositions of the cokernel of τ_M. In particular, the sets of numbers $\Psi = (d_\psi)_{\psi \in H_E}$ and $\Phi = \{(\varphi, d_\varphi) \mid \varphi \in \mathrm{Hom}(E, \mathbb{C}_\infty)\}$ are equivalent.*
2. *In the original definition of CM-types and the proof of the (ramified) Taniyama-Shimura formula in [Sch09, Pel09], there is an additional assumption of* uniformizability *of Anderson-A-modules. Uniformizability of an A-module $\underline{G}$ (and it's associated A-motive) over $\mathbb{C}_\infty$ means that, similar to complex abelian varieties, $\underline{G}$ has a lattice representation $G(\mathbb{C}_\infty) \cong \mathrm{T}_0(G)/\Lambda$, induced by the exponential function $\exp_{\underline{G}} : \mathrm{T}_0(G) \to G(\mathbb{C}_\infty)$. The reader may wonder what happened to this assumption of uniformizability in the theory we presented. The answer is that we found the CM-type of an A-motive $\underline{M}$ in section 2.1.6 in its*

deRham-cohomology $\mathbf{H}^1_{dR}(\underline{M}, k[\![z-\zeta]\!])$*, while the calculations [Sch09, Pel09] were (without explicitly stating that) done in its so-called* Betti cohomology realization $\mathbf{H}^1_{Betti}(M, A)$ *(see [HS17, after Def. 1.1]). If both defined, i. e. if* $\underline{M}$ *is uniformizable, there is a comparison isomorphism (*loc. cit.*)*

$$h_{Betti,dR} : \mathbf{H}^1_{Betti}(M, A) \otimes_A \mathbb{C}_\infty \xrightarrow{\sim} \mathbf{H}^1_{dR}(\underline{M}, k) \otimes_k \mathbb{C}_\infty \,,$$

which explains why the two notions of CM-type give the same result. Avoiding Betti cohomology allows us to avoid uniformizability considerations.

B. Complex Multiplication for Abelian Varieties

We give a brief overview of the theory Complex Multiplication for abelian varieties.[1]

B.1. CM-algebras and CM-types

Let K be some number field, i. e. a finite extension of $\mathbb{Q}$. Then K is called **totally real**, if the image of every embedding $K \hookrightarrow \mathbb{C}$ is contained in the real numbers $\mathbb{R}$. If no such image is contained in $\mathbb{R}$, then K is called **totally imaginary**. A **CM-field** K is a totally imaginary quadratic extension of a totally real number field F. We have the following classifcation of CM-fields:

Proposition B.1. *Let K be a number field. The following conditions on K are equivalent:*
(a) K is CM.
(b) There exists an automorphism $\iota_K \neq \mathrm{id}$ in $\mathrm{Aut}(K)$ such that for all embeddings $\rho : K \hookrightarrow \mathbb{C}$ we have

$$\rho \circ \iota_K = \iota_{\mathbb{C}} \circ \rho \,,$$

where $\iota_{\mathbb{C}}$ is the complex conjugation automorphism on $\mathbb{C}$.
(c) There exists some element $x \in K$ such that $K = F[x]$ for some totally real F and $x^2 \in F$ and $\rho(x^2) < 0$ for all embeddings $\rho : F \hookrightarrow \mathbb{C}$.

A **CM-algebra** E is a semisimple algebra that is a finite product of CM-fields.

Let $E \cong K_1 \times \cdots \times K_n$ be such a CM-algebra, and let H_E be the set of Q-algebra homomorphisms $E \to \mathbb{C}$. Then H_E consists of conjugated pairs $(\psi, \iota_E \circ \psi)$, where ι_E is the unique conjugation automorphism on E.

Definition B.2. *A **CM-type** of a CM-algebra E is any subset $\Psi \subset H_E$ such that Ψ includes precisely one member of each conjugated pair $(\psi, \iota \circ \psi)$. The pair (E, Ψ) is then called a **CM-pair**. An isomorphism f of CM-pairs $(E', \Psi') \to (E, \Psi)$ is an isomorphism $f : E' \to E$ such that $\psi \circ f \in \Psi'$ for all $\psi \in \Psi$.*

B.2. Abelian Varieties with Complex Multiplication

Definition B.3. *A complete, connected smooth group variety $\mathcal{A}$ over some field k is called an **abelian variety**. A **morphism between abelian varieties** is a morphism*

[1] The best modern source known to the author are the (as yet) unpublished notes [Mil06] on CM and [Mil07] the fundamental theorem of CM by J. S. Milne .

of algebraic groups. An ***isogeny between abelian varieties*** *is a morphism of algebraic groups that is surjective with finite kernel.*

Remark B.4. *In particular, elliptic curves are abelian varieties of dimension* 1.

Let $\mathcal{A}$ be an abelian variety over some field k. Let E be a semisimple $\mathbb{Q}$-algebra of degree $2 \cdot \dim_k \mathcal{A}$. If there exists an embedding

$$i : E \hookrightarrow \mathrm{QEnd}_k(\mathcal{A}) = \mathrm{End}_k(\mathcal{A}) \otimes \mathbb{Q}\,,$$

we say that $\mathcal{A}$ has **Complex Multiplication by** E **over** k, and the pair $(\mathcal{A}, i)$ is called an **abelian variety with CM via** E **over** k.

Theorem B.5. *There exists a bijection from the set of isogeny classes of abelian varieties over* $\mathbb{C}$ *with CM to the set of isomorphism classes of CM-pairs.*

The Taniyama-Shimura formula for Abelian Varieties

Theorem B.6. *Let* $\mathcal{A}$ *be an abelian variety with CM via* E *over a number field* k*, that is assumed sufficiently large to contain all images* $\varphi(E)$ *for* $\varphi \in \mathrm{Hom}(E, \mathbb{Q}^{alg})$*. Let* $\mathfrak{P}$ *be a prime ideal of* $\mathcal{O}_k$ *at which* $\mathcal{A}$ *has good reduction. Assume furthermore that* $\mathrm{End}(\mathcal{A}) \cap E = \mathcal{O}_E$*. Then there exists an element* $\pi \in \mathcal{O}_E$ *inducing the Frobenius endomorphism on the reduction* $A_{\mathfrak{P}}$ *of* A *at* $\mathfrak{P}$*, and we have the following equality of* $\mathcal{O}_E$*-ideals:*

$$(\pi) = \prod_{\psi \in \Psi} \psi^{-1}(\mathrm{N}_{k/\psi E}\,\mathfrak{P}).$$

Equivalently, for all primes v *of* E *dividing* p*, the formula*

$$\frac{\mathrm{ord}_v(\pi)}{\mathrm{ord}_v(q)} = \frac{\#\Psi_v}{\#H_v}$$

holds, where $H_v := \{\rho : E \to k \mid \rho^{-1}(\mathfrak{P}) = \mathfrak{p}_v\}$ *and* $\Psi_v := \Psi \cap H_v$ *and* $q := \#\mathcal{O}_k/\mathfrak{P}$.

C. Descent Theory

We will give a short description of the well-known descent theory of modules. Our primary source is [BLR90].

C.1. General Theory

Let k^{alg} denote an algebraic closure of k. Let $S := Spec\,(k)$ and $S' := Spec\,(k^{\mathrm{alg}})$. Let p denote the induced morphism $S' \to S$, which is obviously quasi-compact (S and S' being affine) as well as surjective and flat (k being a field), hence faithfully flat. Now write $S'' := Spec\,(k^{\mathrm{alg}} \otimes_k k^{\mathrm{alg}})$ and denote by p_i the projection morphism onto the i-th factor of S''.

Let now $\mathcal{M}'$ be a quasi-coherent S'-module.

Definition C.1. *A* ***covering datum*** *for a quasi-coherent S'-module $\mathcal{M}'$ is a S''-isomorphism*

$$\beta : p_1^*\mathcal{M}' \to p_2^*\mathcal{M}'.$$

The pairs $(\mathcal{M}', \beta)$ form a category with morphisms $(\mathcal{M}', \beta_{\mathcal{M}'}) \to (\mathcal{N}', \beta_{\mathcal{N}'})$ consisting of those S'-morphisms $\mathcal{M}' \xrightarrow{f} \mathcal{N}'$ compatible with the covering data on both sides, i. e. the following diagram is commutative:

$$\begin{array}{ccc} p_1^*\mathcal{M}' & \xrightarrow{\beta_{\mathcal{M}'}} & p_2^*\mathcal{M}' \\ {\scriptstyle p_1^* f}\big\downarrow & & \big\downarrow{\scriptstyle p_2^* f} \\ p_1^*\mathcal{N}' & \xrightarrow{\beta_{\mathcal{N}'}} & p_2^*\mathcal{N}' \end{array}$$

Now write $S''' := Spec\,(k^{\mathrm{alg}} \otimes_k k^{\mathrm{alg}} \otimes_k k^{\mathrm{alg}})$ and denote by $p_{i,j}$ the projection morphism $S''' \to S''$ onto the i-th and j-th component, respectively:

$$S''' \;\begin{array}{c} \xrightarrow{p_{1,2}} \\ \xrightarrow{p_{1,3}} \\ \xrightarrow{p_{2,3}} \end{array}\; S'' \;\begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array}\; S' \xrightarrow{p} S$$

A covering datum is not in general compatible with these additional projections; by compatible, we mean that the following diagram is commutative:

$$\begin{array}{ccccccc} p_{1,2}^* p_1^*\mathcal{M}' & \xrightarrow{p_{1,2}^*\beta} & p_{1,2}^* p_2^*\mathcal{M}' & \xleftrightarrow{=} & p_{2,3}^* p_1^*\mathcal{M}' & \xrightarrow{p_{2,3}^*\beta} & p_{2,3}^* p_2^*\mathcal{M}' \\ & {\scriptstyle =}\searrow\!\!\nwarrow & & & & \nearrow\!\!\swarrow{\scriptstyle =} & \\ & & p_{1,3}^* p_1^*\mathcal{M}' & \xrightarrow{p_{1,3}^*\beta} & p_{1,3}^* p_2^*\mathcal{M}' & & \end{array}$$

Definition C.2. *A* ***descent datum*** *on $\mathcal{M}'$ is a covering datum β on $\mathcal{M}'$, which is compatible with the projections $p_{i,j}$ as defined above. A descent datum β on $\mathcal{M}'$ is called* ***effective****, if it comes from a quasi-coherent S-module $\mathcal{M}$, i. e. $(\mathcal{M}', \beta) \cong (p^*\mathcal{M}, \iota)$ where ι is the canonical descent datum on $p^*\mathcal{M}$ induced by $p \circ p_1 = p \circ p_2$.*

Remark C.3. *We can equip both Anderson-modules and Anderson-motives over k^{alg} with descent data in a natural way.*

The following theoren now states, that faithfully flat descent of quasi-coherent modules is effective, that is, every descent datum β for $\mathcal{M}'$ on $\mathcal{S}'$ is effective:

Theorem C.4. *Write $S = Spec\,(A_k)$ and $S' = Spec\,(A_{k'})$ for some field extension k'/k. Let $p : S' \to S$ be faithfully flat and quasi-compact. Then the induced functor $\mathcal{F} \mapsto p^*\mathcal{F}$ from the category of quasi-coherent S-moduls to the category of quasi-coherent S'-moduls with descent datum is an equivalence of categories.*

Theorem C.5. *Let $p : S' \to S$ be faithfully flat and quasi-compact. Then the pull-back functor $p^* : X \mapsto p^*X$ is fully faithful. Moreover, if S and S' are affine, a descent datum β on a S'-scheme X' is effective if and only if X' may be covered by quasi-affine subschemes stable under β.*

C.2. Weil Restriction

Let $h : S' \to S$ denote some morphism of schemes, and define the following contravariant functor associated to some S'-scheme Y as

$$\mathrm{Res}_{S'/S}(Y) : T \mapsto \mathrm{Hom}_{S'}(T \times_S S', Y)\,,$$

where T is some connected S-scheme. If this functor is representable, it is called the **Weil Restriction** of Y (w. r. t. h). In other words, the Weil restriction is adjoint to the base change functor, i. e. there is a functorial isomorphism

$$\mathrm{Hom}_S(T, \mathrm{Res}_{S'/S}(Y)) \xrightarrow{\sim} \mathrm{Hom}_{S'}(T \times_S S', Y).$$

The Weil restriction exists for instance under the following conditions:

Theorem C.6. *Let h as above be finite and locally free, and let Y be some S'-scheme. Assume that for each $s \in S$ and any finite subset $P \subset Y \otimes_S \kappa(s)$, there exists an open affine subscheme $U \subset Y$ containing P. Then the Weil restriction $\mathrm{Res}_{S'/S}(Y)$ exists.*

Let us now briefly consider the special case of a separable field extension L/K, i. e. $Spec\,(K) = S$ and $Spec\,(L) = S'$, with h the canonical map induced by the inclusion morphism $K \hookrightarrow L$. Assume that Y is some S'-scheme such that $\mathrm{Res}\,Y$ exists (for instance Y affine of finite type over L). In this scenario the Weil restriction can be made very explicit (going back to Weil) in the following way: Let M/K be some finite Galois extension with Galois group G containing a Galois closure of L/K. Form the scheme

$$\tilde{Y} := \prod_{\iota : L \hookrightarrow M} Y \otimes_L M_\iota\,,$$

where M_ι denotes M as L-algebra via the embedding $\iota : L \hookrightarrow M$. Then G acts on the set of embeddings ι, and we obtain a permutation of factors for each automorphism $\sigma \in G$, that is an isomorphism

$$f_\sigma : \sigma^* \tilde{Y} \to \tilde{Y}.$$

Proposition C.7. *The collection $(f_\sigma)_{\sigma \in G}$ forms an effective descent datum on Y, with the resulting scheme isomorphic to the Weil restriction* $\mathrm{Res}(Y)$*. In particular, if Y is geometrically connected or smooth over L, then so is* $\mathrm{Res}(Y)$ *over K.*

In particular, for a finite Galois extension L/K and an L-scheme Y for which the Weil restriction exists, we have

$$\mathrm{Res}_{L/K}(Y) \otimes_K L \cong \prod_{\sigma \in \mathrm{Aut}_K(L/K)} Y \otimes_L L_\sigma \,.$$

Theorem C.8. *Let k be a finite field and k'/k some finite extension with Galois group G. Let A be some projective variety defined over k'. Assume that for every pair of Galois automorphisms $(\alpha, \beta) \in G \times G$ we have an isomorphism*

$$f_{\alpha,\beta} : (A)^\beta \to (A)^\alpha$$

such that

$$\begin{aligned} f_{\alpha,\beta} \circ f_{\beta,\gamma} &= f_{\alpha,\gamma} \,, \\ f_{\alpha\gamma,\beta\gamma} &= (f_{\alpha,\beta})^\gamma \,, \\ f_{\alpha,\alpha} &= \mathrm{id} \;. \end{aligned}$$

Then there exists a projective variety A_k defined over k and an k'-isomorphism $f : A \to A_k$ such that

$$f_{\alpha,\beta} = (f^\beta)^{-1} \circ f^\alpha$$

for all pairs (α, β) as above. Furthermore, f is unique up to k-isomorphism.

Proof. This is the content of [Wei56, Section I], in particular Thm. 3. ⌟

Let B be some algebraic variety over k', which does not necessarily satisfy the conditions of the last theorem. In order to apply it, one forms the variety

$$A := B \times \sigma^* B \times \cdots \times \sigma^{(b-1)*} B \,,$$

where $\sigma \in \mathrm{Gal}(k'/k)$ denotes the k-Frobenius automorphism and $\sigma^*(B)$ denotes the variety obtained by the natural k-action of σ on B (defined, for instance, as applying σ to the defining polynomial equations) and b is the degree of the field extension $[k' : k]$. In particular there is a natural cyclical permutaion action of σ on the factors of A, giving us varieties $\sigma^* A, \ldots, \sigma^{(b-1)*} A$, and we see that A satisfies the conditions of Thm. C.8, and ergo A descends to k, i. e. there exists a variety A_k defined over k that becomes isomorphic to A after base-change to k'.

D. The Brauer Group

There are various accounts of Brauer group theory in the literatur; we refer to the book "Central Simple Algebas and Galois Cohomology" from Gille and Szamuely, see [GS06].

D.1. Construction of $\mathrm{Br}(k)$

Let B be an algebra over a field k. Then B is called **central**, if its centre is k, and it is called **simple**, if it is semisimple and has no two-sided non-trivial ideals. Let K/k be a field extension. If $B \otimes K$ is isomorphic to a matrix algebra $\mathrm{M}_n(K)$ for some $n \in \mathbb{N}$, we call K a **splitting field** for B.

Proposition D.1 (Noether, Köthe)**.** *Any central simple k-algebra has a splitting field that is separable over k.*

Proof. See [GS06, Prop. 2.2.5]. ⌟

The **Brauer group $\mathbf{Br}(k)$** of some field k can be defined as the group of Morita equivalence classes of central simple algebras over k as follows: Using the Artin-Wedderburn classification theorem (or the Noether-Köthe splitting field theorem from above), we can write any central simple algebra over k as matrix algebra $\mathrm{M}_n(D)$ for some $n \in \mathbb{N}$ and some division algebra D, and we call two central simple algebras over k equivalent, if they can be represented as matrix algebras over the same division ring D. The tensor product functor on k-algebras is compatible with these equivalence classes, since central simple algebras over k are precisely those k-algebras that become isomorphic to a matrix ring after base change to an algebraic closure k^{alg}. Since the equivalence class of the opposite algebra k^{op} acts as an inverse to the class of k, the set of equivalence classes of central simple algebras together with the group law induced by tensor product forms a group, the Brauer group $\mathbf{Br}\,k$. The **degree** of a central simple algebra B is defined as the square root of its dimension, its **period** is defined as the order of the class of B in $\mathbf{Br}\,k$, and the **index** of B is defined as the degree of the division algebra D that is Brauer equivalent to B.

Theorem D.2. *Let E be a central division algebra over a global field F. The period of E, is equal to the index of E, i. e. the order of $[E]$ in $\mathbf{Br}(F)$ is $\sqrt{[E:F]}$.*

Proof. cf. [GS06, Prop. 6.3.10, Rem. 6.5.5]. ⌟

D.2. Class Field Theory

Let k be a global field, and let B be a central simple algebra over k. Let v be some place of k, and denote by k_v the completion of k at v. The tensor product $B \otimes_k k_v$ is then a central simple algebra over k_v. We obtain a homomorphism $\mathbf{Br}(k) \to \mathbf{Br}(k_v)$, and since any central simple algebra splits at all but finitely many places, we also get a homomorphism $\mathbf{Br}(k) \to \bigoplus_v \mathbf{Br}(k_v)$, where the right hand side is the summation over all valuations v of k. Furthermore, from local class field theory we have the canonical isomorphism $\mathrm{inv}_v(k_v) : \mathbf{Br}(k_v) \to \mathbb{Q}/\mathbb{Z}$, the so-called **local Hasse-invariant at** v. These morphisms fit together into the following short exact sequence[1]):

$$0 \to \mathbf{Br}(k) \to \bigoplus_v \mathbf{Br}(k_v) \to \mathbb{Q}/\mathbb{Z} \to 0\,.$$

Here the third map is summation $\sum \mathrm{inv}_v(\cdot)$ over all the local Hasse-invariants. We also make note of a useful property of local invariants which we need in Chapter 5. Namely, for any field k complete under a discrete valuation with perfect residue field and a finite extension k'/k, the restriction map $\mathbf{Br}(k) \to \mathbf{Br}(k')$ is compatible with the ismorphisms $\mathbf{Br}(k) \to \mathbb{Q}/\mathbb{Z}$ and $\mathbf{Br}(k') \to \mathbb{Q}/\mathbb{Z}$ in the following way (see [GS06, Prop. 6.3.9, Rem. 6.3.11]): There is a commutative diagram

$$\begin{array}{ccc} \mathbf{Br}(k) & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{Res}(\cdot)} & & \downarrow{\scriptstyle [k':k']} \\ \mathbf{Br}(k') & \xrightarrow{\cong} & \mathbb{Q}/\mathbb{Z} \end{array}$$

In particular, the statement is true for local fields. Now let E be some central simple algebra over a global field K, and let L/K be a finite field extension. Then the local Hasse invariant of $L \otimes_K E$ at a place v of K and a place u of E above v can be calculated as

$$\mathrm{inv}_u(L \otimes_K E) = [L_u : K_v] \cdot \mathrm{inv}_v(E)\,.$$

[1] originally proven by Hasse; for a modern discussion, see [GS06, Cor. 6.5.4]

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