Induction and restriction of (φ, Γ)-modules

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Abstract. Let L be a nonarchimedean local field of characteristic 0. We present a variant of the theory of (φ, Γ)-modules associated with Lubin–Tate groups, developed by Kisin and Ren [11], in which we replace the Lubin–Tate tower by the maximal abelian extension $\Gamma = \mathrm{Gal}(L^{ab}/L)$. This variation allows us to compute the functors of induction and restriction for (φ, Γ)-modules, when the ground field L changes. We also give a self-contained account of the Cherbonnier–Colmez theorem on overconvergence in our setting.

Let L be a finite extension of the field $\mathbb{Q}_p$. Let $\mathcal{O}$ denote its ring of integers, κ its residue field, q the cardinality of κ, $G = \mathrm{Gal}(\overline{L}/L)$ the absolute Galois group, and $\mathrm{Rep}_{\mathcal{O}}(L)$ the category of finitely generated $\mathcal{O}$-modules equipped with a continuous G-action. In order to study $\mathrm{Rep}_{\mathcal{O}}(L)$, Fontaine introduced in [6] the category $\mathrm{Mod}_{\varphi, \Gamma}(\mathbf{A}_L)$ of (φ, Γ)-modules over the "period ring" $\mathbf{A}_L$. He then constructed a functorial equivalence between the rather complicated category $\mathrm{Rep}_{\mathcal{O}}(L)$, and the seemingly simpler full subcategory $\mathrm{Mod}_{\varphi, \Gamma}^{\acute{e}t}(\mathbf{A}_L)$ of $\mathrm{Mod}_{\varphi, \Gamma}(\mathbf{A}_L)$ consisting of *étale* (φ, Γ)-modules.

The ring $\mathbf{A}_L$ and the category $\mathrm{Mod}_{\varphi, \Gamma}(\mathbf{A}_L)$ depend on the choice of a certain Γ-extension L_∞/L, which in Fontaine's work, and in most of the applications that followed, was taken to be the cyclotomic $\mathbb{Z}_p$-extension of L. Kisin and Ren [11] introduced a variant, in which L_∞/L is the extension generated by the torsion points of a Lubin–Tate group $\mathcal{G}$ defined over $\mathcal{O}$. Using this new variant they were able to generalize results of Berger, Colmez and Wach on crystalline representations, which were previously available only for L unramified over $\mathbb{Q}_p$ and L_∞ its cyclotomic extension.

A full account of the Kisin–Ren theory was provided in the recent monograph [14] of Schneider. Among other things, this book substitutes Scholze's notion of "tilting" for the original "field of norms" approach. However, the constructions in both these references depend strongly on the Lubin–Tate group law, and in particular on the choice of a uniformizer of L. This makes it difficult to express the functors of restriction and induction in the language of (φ, Γ)-modules, when we let the base field L change. Similar difficulties, arising from the incompatibility of Lubin–Tate theory with base-field extensions, were

encountered in Iwasawa's development of local class field theory [8], and in the generalization of local class field theory to meta-abelian extensions studied in [12].

The goal of this note is to present yet another variant of the Kisin–Ren theory, in which the extension L_∞/L is replaced by the maximal abelian extension L^{ab}/L. Thus, $\Gamma = \mathrm{Gal}(L^{ab}/L)$. As such, no choice of a uniformizer or a Lubin–Tate group is involved, and the ambiguity mentioned above is resolved. For our purpose it is also necessary to replace the fields of norms of Fontaine and Wintenberger by their *completed perfections*, which we shall denote by the letter F. More precisely, if the field of norms of a Lubin–Tate tower is (non-canonically) isomorphic to $\kappa((\omega))$, then our F will be isomorphic to the completion of the perfection of $\overline{\kappa}((\omega))$. Intrinsically, F is defined to be the tilt $(\widehat{L^{ab}})^\flat$ of the completion $\widehat{L^{ab}}$ of L^{ab}. Such "complete perfections" of norm-fields were already shown to be useful in the work of Cherbonnier and Colmez [4] and Kedlaya and Liu [9, 10], who nevertheless avoided the extension of scalars from κ to $\overline{\kappa}$, and did not consider these objects in the Lubin–Tate setting. The coefficient ring for our (φ,Γ)-modules is modified accordingly. As F is perfect, one can simply take as coefficients the ring $\widetilde{\mathbf{A}}_L = W(F)_L$ of Witt vectors of F, tensored over $W(\kappa)$ with $\mathcal{O}$, instead of the smaller $\mathbf{A}_L$, whose construction would require further work.

The structure of $\widetilde{\mathbf{A}}_L$ and the new category $\mathrm{Mod}_{\varphi,\Gamma}(L)$ of (φ,Γ)-modules over it are elucidated in Section 1. The main theorem on the equivalence between $\mathrm{Rep}_{\mathcal{O}}(L)$ and the full subcategory $\mathrm{Mod}_{\varphi,\Gamma}^{\text{ét}}(L)$ is given in Section 2. These two sections repeat well-known results. As we follow [14] closely, proofs are omitted. Once we have established the new variant, the computations of the functors of induction and restriction are straight-forward, and are given in Section 3. The main result concerning these functors is Theorem 3.7.

We remark that for "cyclotomic (φ,Γ)-modules", i.e., when Γ is the cyclotomic extension, a similar result was obtained in Liu's thesis in the framework of (φ,Γ)-modules over the Robba ring, cp. [13, Prop. 2.1]. Recall that the Robba ring does not admit an integral structure, and the appropriate (φ,Γ)-modules over the Robba ring classify L-vector space representations of G rather than $\mathcal{O}$-module representations. More importantly, in the Kisin–Ren setting, the fields of norms attached to two Lubin–Tate towers over L_1 and L_2, where $L_1 \subset L_2$ is a finite extension, are not comparable, even if the associated primes are. See [5]. Replacing the fields of norms by $F = (\widehat{L^{ab}})^\flat$ is essential for the inclusion $\widetilde{\mathbf{A}}_{L_1} \subset \widetilde{\mathbf{A}}_{L_2}$, without which one can not proceed.

Besides compatibility with induction and restriction, working with $\widetilde{\mathbf{A}}_L$ as coefficients instead of $\mathbf{A}_L$ (and the full abelian extension instead of the cyclotomic or Lubin–Tate tower) has another advantage over the Kisin–Ren modules studied in [11] and [14]. As observed in [7] and [3], the Cherbonnier–Colmez theorem, asserting that étale (φ,Γ)-modules are overconvergent [4], no longer holds for $\mathbf{A}_L$-modules in the Lubin–Tate setting, as soon as $L \neq \mathbb{Q}_p$. This has to do with the fact that now $\Gamma \simeq \mathcal{O}_L^\times$ rather than $\mathbb{Z}_p^\times$, and is related to

the question of L-analyticity. See the discussion in [14, Section 4.3]. In our setting, the Cherbonnier–Colmez theorem *does hold*.

In Section 4 we introduce the ring of overconvergent periods $\widetilde{\mathbf{A}}_L^\dagger$ and the category $\mathrm{Mod}_{\varphi,\Gamma}^\dagger(L)$ of overconvergent (φ,Γ)-modules. Just as $\mathbf{A}_L^\dagger$ can be realized as a ring of power series converging in some annulus $R < |X| < 1$, $\widetilde{\mathbf{A}}_L^\dagger$ can be realized as a ring of power series converging in some *(pre)perfectoid* annulus. Base change from $\widetilde{\mathbf{A}}_L^\dagger$ to $\widetilde{\mathbf{A}}_L$ induces an equivalence of categories between the full subcategory $\mathrm{Mod}_{\varphi,\Gamma}^{\dagger \acute{e}t}(L)$ of overconvergent étale (φ,Γ)-modules, and $\mathrm{Mod}_{\varphi,\Gamma}^{\acute{e}t}(L)$. In Section 4 we give an elementary, self-contained proof of this result. Once again we are motivated by Kedlaya's paper [9], but the reader familiar with his proof will notice that we prove the basic lemmas directly over L, rather than after a base change L'/L trivializing the (φ,Γ)-module modulo p, thus avoiding the need for Galois descent. As explained by Kedlaya, if the field of norms is replaced by its completed perfection and the ring $\mathbf{A}_L$ by $\widetilde{\mathbf{A}}_L$, then overconvergence can be deduced solely from an analysis of the matrix representing φ (at least if there is no p-torsion, but torsion modules are easy to deal with). The structure of Γ intervenes only at the second stage, when one tries to descend to $\mathbf{A}_L$ and its overconvergent subring. It is this second stage that works for the cyclotomic Γ but fails in the Lubin–Tate setting if $L \neq \mathbb{Q}_p$.

In this regard we want to mention the work of Berger and Colmez on locally analytic vectors, and the differential operators associated to Γ in the Lubin–Tate case [2, Section 4.1]. This plays a role in [1], where Berger proves that the "Lubin–Tate" (φ,Γ)-modules attached to *L-analytic Galois representations* stay overconvergent even when descended from $\widetilde{\mathbf{A}}_L$ to $\mathbf{A}_L$. As the referee pointed out, it might be possible to define such differential operators in our set-up too.

As should be clear from the introduction, our improvement over what is already in the literature is modest, and this paper replicates ideas and results scattered throughout many of the sources that we have cited. Nevertheless, the better compatibility with induction and restriction, as well as with overconvergence, makes one wonder whether $\widetilde{\mathbf{A}}_L$ should not substitute for $\mathbf{A}_L$ as the basic period ring. Development of p-adic Hodge theory so far relied on descent from $\widetilde{\mathbf{A}}_L$ to $\mathbf{A}_L$. For example, the very definition of the operator ψ relies on φ *not* being bijective, and the study of locally analytic vectors and p-adic differential equations is also conducted over the *non-perfectoid* Robba ring. However, the predominance of perfectoid rings in Scholze's work and their many applications, together with the two observations made above, support such thoughts.

1. (φ,Γ)-modules over L

1.1. **Tilting.** The following fundamental constructions are due to Fontaine and Scholze. Let L be a finite extension of $\mathbb{Q}_p$ and write $\mathbb{C}_p$ for the completion

of a fixed algebraic closure $\overline{L}$ of L. If

$$L \subset K \subset \mathbb{C}_p$$

is any complete intermediate field, we write $K^\flat$ for the collection of sequences

$$x = (\ldots, x_2, x_1, x_0),$$

where $x_i \in K$ and $x_{i+1}^p = x_i$. We define $x \cdot y$ by component-wise multiplication and $x + y = z$, where

$$z_i = \lim_{j \to \infty} (x_{i+j} + y_{i+j})^{p^j}$$

(the limit exists). We let $|x|_\flat = |x_0|$. Then $K^\flat$ becomes a field of characteristic p and $|\cdot|_\flat$ is a complete nonarchimedean absolute value on $K^\flat$. For example, $L^\flat \simeq \mathbb{F}_q$, the residue field of $\mathcal{O} = \mathcal{O}_L$. The field $K^\flat \subset \mathbb{C}_p^\flat$ is called the *tilt* of K.

Recall that K is called a *perfectoid* [15] if it is in addition non-discrete and every element of the ring $\mathcal{O}_K/p\mathcal{O}_K$ is a p-th power. In this case, $K^\flat$ is perfect and can be identified with the field of fractions of the perfection of $\mathcal{O}_K/p\mathcal{O}_K$. The field $\mathbb{C}_p$ itself is such a perfectoid.

The group $\mathrm{Aut}_{\mathrm{cont}}(K/L)$ of continuous automorphisms of K over L acts by functoriality on $K^\flat$.

As an example, consider a Lubin–Tate formal group $\mathcal{G}$ over $\mathcal{O}$, associated with the prime π of $\mathcal{O}$. We fix a formal parameter X on $\mathcal{G}$ and denote by $[a]$ the endomorphism of $\mathcal{G}$ whose expression in X starts with $aX + (\text{higher terms})$. We let $\phi = [\pi]$, so that

$$\phi(X) = \pi X + \cdots \equiv X^q \bmod \pi.$$

For $n \geq 1$ let $\omega_n \in \mathcal{G}[\pi^n]$ be such that $\omega_1 \neq 0$ and $[\pi](\omega_{n+1}) = \omega_n$. Let $L_n = L(\omega_n)$ and $L_\infty = \bigcup L_n$. Its completion $K = \widehat{L_\infty}$ is a perfectoid subfield of $\mathbb{C}_p$. Since $\omega_{n+1}^q \equiv \omega_n \bmod \pi$, the choice of ω_n defines an element $\omega \in K^\flat$ and $\kappa((\omega)) \subset K^\flat$, where $\kappa = \mathcal{O}/\pi\mathcal{O} \simeq \mathbb{F}_q$. To be precise, letting $q = p^f$, ω is the unique element $x \in K^\flat$ in which $x_{nf} \equiv \omega_{n+m}^{q^m} \bmod p$ for large enough m. Moreover (see [14, Prop. 1.4.17]), $K^\flat$ is the completion of the perfection of $\kappa((\omega))$. Thus, elements of $K^\flat$ are formal power series $\sum a_m \omega^m$, where $m \in \mathbb{Z}[p^{-1}]$, $a_m \in \kappa$, and for any real number M, there are only finitely many $m \in \mathbb{Z}[p^{-1}]$ with $m < M$ and $a_m \neq 0$. The field $\kappa((\omega))$ is called the *field of norms* of the extension L_∞/L, and is independent of the choice of ω. Note that

$$|\omega|_\flat = |\pi|^{q/(q-1)}.$$

In this work, however, we consider $L^{ab} = L^{nr}L_\infty$, the maximal abelian extension of L in $\overline{L}$, and let

$$K = \widehat{L^{ab}}, \quad F = K^\flat.$$

Proposition 1.2. (i) *The field K is a perfectoid.*

(ii) *The field F can be identified with the field of formal power series $\sum a_m \omega^m$, where $m \in \mathbb{Z}[p^{-1}]$, $a_m \in \overline{\kappa}$, and for any real number M, there are only finitely many $m \in \mathbb{Z}[p^{-1}]$ with $m < M$ and $a_m \neq 0$. Alternatively, it is the completed perfection of $\overline{\kappa}((\omega))$.*

Proof. (i) K is complete and non-discrete. The ring $\mathcal{O}_K/p\mathcal{O}_K$ is the union of $\mathcal{O}_{L_d^{nr} L_\infty}/p\mathcal{O}_{L_d^{nr} L_\infty}$ for $d = 1, 2, \ldots$, where L_d^{nr} is the unramified extension of L of degree d. The extension $L_d^{nr} L_\infty/L$ is arithmetically profinite (its ramification groups, in the upper numbering, are open), hence its completion is a perfectoid field, and every element of $\mathcal{O}_{L_d^{nr} L_\infty}/p\mathcal{O}_{L_d^{nr} L_\infty}$ is a p-th power. It follows that every element of $\mathcal{O}_K/p\mathcal{O}_K$ is a p-th power as well.

(ii) The proof in [14, Prop. 1.4.17] carries over to our case with minor modifications. $\square$

The field F is a non-discrete complete valuation field. Its ring of integers $\mathcal{O}_F$ is the perfection of $\mathcal{O}_K/p\mathcal{O}_K$. For $M > 0$, we denote by $\mathfrak{m}_F^{(M)}$ the ideal of $\mathcal{O}_F$ consisting of all x with

$$|x|_\flat \leq |\pi|^{Mq/(q-1)}.$$

Thus, $\mathfrak{m}_F^{(M)}$ can be identified with the ideal of all the formal power series $\sum a_m \omega^m$ as above, with $a_m = 0$ for $m < M$.

The group $\Gamma = \mathrm{Gal}(L^{ab}/L) = \mathrm{Aut}_{\mathrm{cont}}(K/L)$ is isomorphic to the profinite completion $\widehat{L^\times}$ of $L^\times$. We denote by $[a, L^{ab}/L]$ the local Artin symbol of $a \in \widehat{L^\times}$ and by $\chi_L \colon \Gamma \simeq \widehat{L^\times}$ the character defined by

$$\chi_L(\gamma) = a \Leftrightarrow \gamma = [a^{-1}, L^{ab}/L].$$

Thus, if γ is the geometric Frobenius of L^{ab}/L_∞, then $\chi_L(\gamma) = \pi$. Lubin–Tate theory tells us, on the other hand, see [8], that if $\gamma \in \mathrm{Gal}(L^{ab}/L^{nr})$, then $\chi_L(\gamma) \in \mathcal{O}_L^\times$ and

$$\gamma(\omega_n) = [\chi_L(\gamma)](\omega_n).$$

Let

$$H = \mathrm{Gal}(\overline{L}/L^{ab}) = \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p/K), \quad G = \mathrm{Gal}(\overline{L}/L) = \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p/L),$$

so that $G/H = \Gamma$. Let F^{sep} denote the separable closure of F in $\mathbb{C}_p^\flat$. The following Proposition is well-known, see [15, Thm. 3.7].

Proposition 1.3. (i) *The field F^{sep} is algebraically closed and dense in $\mathbb{C}_p^\flat$.*

(ii) *The action of G on $\mathbb{C}_p^\flat$ induces an isomorphism*

$$H \simeq \mathrm{Gal}(F^{\mathrm{sep}}/F) = \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p^\flat/F).$$

(iii) *If K' is a finite extension of K in $\mathbb{C}_p$, then K' is a perfectoid, and $[K'^\flat : F] = [K' : K]$. Every finite extension of F in $\mathbb{C}_p^\flat$ is $K'^\flat$ for a unique K' as above, and if K' is Galois, $\mathrm{Gal}(K'/K) \simeq \mathrm{Gal}(K'^\flat/F)$.*

(iv) *(Ax–Sen–Tate theorem) $\mathbb{C}_p^G = L$ and $\mathbb{C}_p^H = K$. Likewise, $(\mathbb{C}_p^\flat)^H = F$.*

If $\alpha \in F$, we shall sometimes write $|\alpha|$ for $|\alpha|_\flat$ to ease the notation. We also write $\varphi(\alpha) = \alpha^q$ for the Frobenius automorphism of order q. If $\alpha \neq 0$,

$$\lim_{n \to \infty} |\varphi^{-n}(\alpha)| = 1.$$

If $A = (\alpha_{ij}) \in M_d(F)$, we let $|A| = \max\{|\alpha_{ij}|\}$. Then

$$|A + B| \leq \max\{|A|, |B|\}, \quad |AB| \leq |A||B|.$$

The following technical lemma on matrices will be needed in the section on overconvergence. The reader interested only in the formulas for induction and restriction of (φ, Γ)-modules, can skip it.

Lemma 1.4. *Let* $A \in \mathrm{GL}_d(F)$ *be given. Then there exists a constant* c, *depending only on* A, *so that for every* $B \in M_d(F)$, *there exist* $U, V \in M_d(F)$ *with* $|V| \leq c$ *such that*

$$A^{-1}\varphi(U)A - U = B - V.$$

Proof. Let

$$U = \sum_{i=1}^{N} \varphi^{-1}(A)\varphi^{-2}(A) \cdots \varphi^{-i}(A) \cdot \varphi^{-i}(B) \cdot \varphi^{-i}(A)^{-1} \cdots \varphi^{-2}(A)^{-1}\varphi^{-1}(A)^{-1}.$$

Then

$$A^{-1}\varphi(U)A - U = B - V,$$

where

$$V = \varphi^{-1}(A)\varphi^{-2}(A) \cdots \varphi^{-N}(A) \cdot \varphi^{-N}(B) \cdot \varphi^{-N}(A)^{-1} \cdots \varphi^{-2}(A)^{-1}\varphi^{-1}(A)^{-1}.$$

Now

$$|\varphi^{-1}(A)\varphi^{-2}(A) \cdots \varphi^{-N}(A)| \leq |A|^{q^{-1}+q^{-2}+\cdots+q^{-N}}$$

is bounded independently of N, and similarly for $|\varphi^{-N}(A)^{-1} \cdots \varphi^{-1}(A)^{-1}|$. On the other hand, by selecting N large enough, we can let $|\varphi^{-N}(B)|$ be as close as we want to 1. This concludes the proof, in fact with any $c > (|A||A^{-1}|)^{1/(q-1)}$. $\qquad\square$

1.5. **The coefficient ring.** Consider the usual ring of Witt vectors $W(F)$. It contains the subring $W(\kappa)$, which is the ring of integers of $L^0 = L \cap \mathbb{Q}_p^{nr}$. We let

$$\widetilde{\mathbf{A}}_L = W(F)_L = \mathcal{O} \otimes_{W(\kappa)} W(F).$$

The action of Γ on F defines an action of Γ on $W(F)$, and as it fixes $W(\kappa)$ point-wise, it extends to $\widetilde{\mathbf{A}}_L$ $\mathcal{O}$-linearly. Similarly, letting $\varphi(x) = x^q$ be the Frobenius automorphism of order q of F, we denote by φ the induced $\mathcal{O}$-linear automorphism of $\widetilde{\mathbf{A}}_L$. The actions of Γ and φ on $\widetilde{\mathbf{A}}_L$ commute with each other. The structure of $\widetilde{\mathbf{A}}_L$ is given in the next proposition. Let v_p be the p-adic valuation on $\mathbb{C}_p^\times$, normalized by $v_p(p) = 1$, fix a Lubin–Tate group $\mathcal{G}$ over $\mathcal{O}$ associated with the prime π, and let L_∞/L be its Lubin–Tate tower, $L_n = L(\omega_n)$ as before.

Proposition 1.6.   (i) *Let W_L be the completion of the ring of integers in the maximal unramified extension of L, i.e., $W_L = \mathcal{O} \otimes_{W(\kappa)} W(\overline{\kappa})$. The $\mathcal{O}$-algebra $\widetilde{\mathbf{A}}_L$ is isomorphic to the ring of all power series $\sum a_m X^m$, where $m \in \mathbb{Z}[p^{-1}]$, $a_m \in W_L$, and for any real number M, there are only finitely many m with $m < M$ and $v_p(a_m) < M$. Under reduction modulo π (a prime of $\mathcal{O}$) X^m goes to ω^m.*

(ii) *Under the isomorphism in (i), if $\gamma \in \mathrm{Gal}(L^{ab}/L^{nr})$, then*

$$\gamma(a_m) = a_m, \quad \gamma(X) = \lim_{n \to \infty} ([\chi_L(\gamma)](X^{1/q^n}))^{q^n},$$

$\mathrm{Gal}(L^{ab}/L_\infty) \simeq \mathrm{Gal}(L^{nr}/L)$ *acts on a_m via the natural action on W_L and trivially on X, while φ acts naturally on W_L and*

$$\varphi(X) = X^q.$$

Proof. (i) Temporarily, let $\mathscr{A}_L$ denote the power series ring in (i). It is readily checked that $\mathscr{A}_L$ is a strict p-ring (in the category of $\mathcal{O}$-algebras) with $\mathscr{A}_L/\pi\mathscr{A}_L \simeq F$, the isomorphism sending $a \in W_L$ to its reduction modulo π and X^m to ω^m. This suffices to establish the existence of a unique isomorphism

$$\iota \colon \mathscr{A}_L \simeq \widetilde{\mathbf{A}}_L$$

compatible with the given isomorphism after reduction modulo π.

(ii) If $x \in F$, we let $\tau(x)$ be its Teichmüller representative in $\widetilde{\mathbf{A}}_L$. For $m \in \mathbb{Z}[p^{-1}]$, we have $X^m = \tau(\omega^m)$. The formulas in (ii) are then clear. $\square$

We remark that in [14] the Teichmüller representative $\tau(\omega)$ is modified to obtain $\omega_\phi \in \mathbf{A}_L$ satisfying $\gamma(\omega_\phi) = [\chi_L(\gamma)](\omega_\phi)$ and $\varphi(\omega_\phi) = [\pi](\omega_\phi)$. The Cohen ring $\mathbf{A}_L$ is then the p-adic completion of $W_L((\omega_\phi))$. This ω_ϕ however can not be raised to a power $m \in \mathbb{Z}[p^{-1}]$.

The discrete valuation ring $\widetilde{\mathbf{A}}_L$ has two topologies. The *strong* topology is the one given by the valuation. The *weak* topology is induced on $W(F)$ via the natural bijection with $F^{\mathbb{N}_0}$ from the product topology on the latter. It is then extended naturally to $W(F)_L$, which, as an additive group, is isomorphic to $W(F)^{[L:L^0]}$. A basis of open neighborhoods at 0 in the weak topology is given by

$$U_{n,m} = \pi^n W(F)_L + W(\mathfrak{m}_F^{(m)})_L$$

for $n, m \geq 0$. The weak topology is a complete Hausdorff topology, but unlike the situation in [14], the subring $W(\mathcal{O}_F)_L$ is not compact. The automorphisms φ, φ^{-1} and the action $\Gamma \times \widetilde{\mathbf{A}}_L \to \widetilde{\mathbf{A}}_L$ are continuous in the weak topology of $\widetilde{\mathbf{A}}_L$, but the orbit map $\Gamma \to \widetilde{\mathbf{A}}_L$, $\gamma \mapsto \gamma x$, is not continuous, for a general $x \in \widetilde{\mathbf{A}}_L$, in the strong topology.

1.7. **(φ, Γ)-modules over L.** As in [14, Exercise 2.2.3], any finitely generated $\widetilde{\mathbf{A}}_L$-module carries a canonical topology called the *weak topology*, which may be defined as the quotient topology of any surjective homomorphism $\widetilde{\mathbf{A}}_L^n \twoheadrightarrow M$. We make the following definition.

Definition 1.8.   (i) A (φ, Γ)-module over L is a finitely generated $\widetilde{\mathbf{A}}_L$-module M equipped with a φ-semilinear endomorphism

$$\varphi_M \colon M \to M$$

and a Γ-semilinear action

$$\Gamma \times M \to M,$$

which is bi-continuous (when Γ is given its Krull topology and M its canonical weak topology), and which commutes with φ_M.

  (ii) A homomorphism between (φ, Γ)-modules over L is a homomorphism of $\widetilde{\mathbf{A}}_L$-modules $\alpha : M \to N$ which commutes with the Γ-action and satisfies $\alpha \circ \varphi_M = \varphi_N \circ \alpha$.

 (iii) A (φ, Γ)-module M over L is called *étale* if φ_M is bijective.

Remark (about topologies).   (i) A homomorphism α as above, as well as the semilinear φ_M, are automatically continuous. The arguments from [14, Rem. 2.2.5] remain valid.

  (ii) In [14, Thm. 2.2.8] it is shown that if M is an étale φ-module, then every semilinear Γ-action which commutes with φ is automatically bi-continuous. The proof of this useful fact relies on the local compactness of the field of norms. As our F is not locally compact, we do not know if we can give up the continuity assumption in the definition, even if M is étale.

We denote by

$$\mathrm{Mod}_{\varphi, \Gamma}^{ét}(L) \subset \mathrm{Mod}_{\varphi, \Gamma}(L)$$

the category of (φ, Γ)-modules over L, and its full subcategory of étale (φ, Γ)-modules. These are abelian categories and the forgetful functors from them to the category of $\widetilde{\mathbf{A}}_L$-modules are exact.

2. Equivalence of categories

2.1. **The functors.** Let

$$\widetilde{\mathbf{A}} = \mathcal{O} \otimes_{W(\kappa)} W(F^{\mathrm{sep}}),$$

and let φ continue to denote the q-power Frobenius of $W(F^{\mathrm{sep}})$, extended linearly to $\widetilde{\mathbf{A}}$. The Galois group G acts on F^{sep}, hence on $\widetilde{\mathbf{A}}$. The weak topology of $\widetilde{\mathbf{A}}$ is defined as it was defined for $\widetilde{\mathbf{A}}_L$. As before, φ and the G-action on $\widetilde{\mathbf{A}}$ are continuous for the weak topology, and

$$(1) \qquad\qquad \widetilde{\mathbf{A}}^H = \widetilde{\mathbf{A}}_L, \quad \widetilde{\mathbf{A}}^\varphi = \mathcal{O}.$$

Given $V \in \mathrm{Rep}_{\mathcal{O}}(L)$, let

$$\mathcal{D}(V) = (\widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V)^H,$$

where the fixed points of H are taken with respect to the diagonal action. By (1), $\mathcal{D}(V)$ is an $\widetilde{\mathbf{A}}_L$-module. The diagonal action of G yields a residual semilinear Γ-action on $\mathcal{D}(V)$, and since $\varphi \otimes 1$ and H commute in their action on $\widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V$, $\varphi \otimes 1$ induces a semilinear endomorphism $\varphi_{\mathcal{D}(V)}$ on $\mathcal{D}(V)$.

Lemma 2.2. *With the above definitions, $\mathcal{D}(V) \in \mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L)$.*

Proof. The proof that $\mathcal{D}(V)$ is an *étale* φ-module is straight-forward, since φ is bijective on $\widetilde{\mathbf{A}}$. The key step is the proof that $\mathcal{D}(V)$ is a finitely generated module over the discrete valuation ring $\widetilde{\mathbf{A}}_L$, and that the homomorphism

$$\mathrm{ad}_V \colon \widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} \mathcal{D}(V) \to \widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V, \quad a \otimes m \mapsto am$$

is bijective. This is done first under the assumption that V is killed by π, with the help of Hilbert's theorem 90, then by dévissage for torsion V's, and finally, taking inverse limits, for general V. For the details, see [14]. These two facts also imply (see [14, Lem. 3.1.10]) that the G-action on $\widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V$ is continuous, hence the Γ-action on $\mathcal{D}(V)$ is continuous. $\qquad\square$

We next define a functor in the opposite direction. Let $M \in \mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L)$, and define

$$\mathcal{V}(M) = (\widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M)^{\varphi \otimes \varphi_M}.$$

By (1), this is an $\mathcal{O}$-module. Since the diagonal Galois action of G on $\widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M$ commutes with $\varphi \otimes \varphi_M$, $\mathcal{V}(M)$ carries a G-action.

Lemma 2.3. *With the above definition, $\mathcal{V}(M) \in \mathrm{Rep}_{\mathcal{O}}(L)$.*

Proof. Once again, the key is the proof that $\mathcal{V}(M)$ is finitely generated over $\mathcal{O}$, and that the homomorphism

$$\mathrm{ad}_M \colon \widetilde{\mathbf{A}} \otimes_{\mathcal{O}} \mathcal{V}(M) \to \widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M, \quad a \otimes v \mapsto av,$$

is bijective. This is done first under the assumption that M is killed by π (i.e., is an F-vector space), using [14, Prop. 3.2.4], then when M is killed by some π^n by dévissage, and finally, taking inverse limits, for general M. Compare with [14, Prop. 3.3.9]. $\qquad\square$

2.4. The equivalence of categories.

The main theorem is the following.

Theorem 2.5. *The functors $\mathcal{D}$ and $\mathcal{V}$ are equivalences of categories between $\mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L)$ and $\mathrm{Rep}_{\mathcal{O}}(L)$, and are quasi-inverse to each other.*

Proof. One first proves that $\mathcal{D}(\mathcal{V}(M)) = M$ and $\mathcal{V}(\mathcal{D}(V)) = V$, under the assumption that M and V are killed by π. Here the key is that ad_V and ad_M are both bijective. Next, one checks that the two functors are exact and commute with inverse limits, and then one concludes as in [14, Thm. 3.3.10]. $\qquad\square$

2.6. Elementary divisors.

If $(R, \mathfrak{m})$ is a discrete valuation ring and X is a finitely generated R-module we write

$$[X : R] = (r; e_1, \ldots, e_n)$$

if $r \geq 0$, $e_1 \geq e_2 \geq \cdots \geq e_n \geq 1$ and $X \simeq R^r \oplus (R/\mathfrak{m}^{e_1}) \oplus \cdots \oplus (R/\mathfrak{m}^{e_n})$. The rank r and the elementary divisors e_i are uniquely determined and characterize X up to isomorphism. The following is well-known and easy.

Lemma 2.7. *If $[X : R] = [Y : R]$ and $\alpha \colon X \to Y$ is a surjective homomorphism, then α is an isomorphism.*

Proposition 2.8. *Let $M \in \mathrm{Mod}_{\varphi, \Gamma}^{\acute{e}t}(L)$ and $V \in \mathrm{Rep}_{\mathcal{O}}(L)$ correspond to each other under $\mathcal{V}$ and $\mathcal{D}$. Then*

$$[M : \widetilde{\mathbf{A}}_L] = [V : \mathcal{O}].$$

Proof. We have an obvious string of equalities

$$[V : \mathcal{O}] = [\widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V : \widetilde{\mathbf{A}}] = [\widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M : \widetilde{\mathbf{A}}] = [M : \widetilde{\mathbf{A}}_L].$$

The middle equality stems of course from the fact that ad_M and its inverse ad_V are isomorphisms. $\qquad\square$

2.9. **A remark on the use of $\widetilde{\mathbf{A}}$.** It is possible to define the functors $\mathcal{D}$ and $\mathcal{V}$ using, instead of $\widetilde{\mathbf{A}} = \mathcal{O} \otimes_{W(\kappa)} W(F^{\mathrm{sep}})$, the larger ring

$$\widehat{\mathbf{A}} = \mathcal{O} \otimes_{W(\kappa)} W(\mathbb{C}_p^\flat).$$

The reason is that for $V \in \mathrm{Rep}_{\mathcal{O}}(L)$ and $M = \mathcal{D}(V)$,

$$(\widehat{\mathbf{A}} \otimes_{\mathcal{O}} V)^H = (\widehat{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}} (\widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V))^H = (\widehat{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}} (\widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M))^H$$

$$= (\widehat{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M)^H = \widehat{\mathbf{A}}^H \otimes_{\widetilde{\mathbf{A}}_L} M = M,$$

since $(\mathbb{C}_p^\flat)^H = F$. A similar argument works for the functor $\mathcal{V}$.

3. Restriction and Induction

3.1. **Definitions of the two functors.** Let $L_1 \subset L_2$ be two finite extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. Let $\mathcal{O}_1 \subset \mathcal{O}_2$ be their rings of integers, $\kappa_1 \subset \kappa_2$ their residue fields, and let G_i, H_i and Γ_i be the groups defined before, with L_i as L. Write

$$d = [\kappa_2 : \kappa_1]$$

for the inertial degree of L_2/L_1. Letting φ_i denote the Frobenius automorphism of L_i^{nr}/L_i, we have that

$$\varphi_2|_{L_1^{nr}} = \varphi_1^d.$$

Let

$$\mathrm{Res}_{L_1}^{L_2} \colon \mathrm{Rep}_{\mathcal{O}_1}(L_1) \to \mathrm{Rep}_{\mathcal{O}_2}(L_2)$$

be the functor $\mathrm{Res}_{L_1}^{L_2}(V) = \mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathrm{Res}_{G_1}^{G_2}(V)$. Thus, we restrict the group action to a smaller subgroup and extend scalars. Similarly, let

$$\mathrm{Ind}_{L_1}^{L_2} \colon \mathrm{Rep}_{\mathcal{O}_2}(L_2) \to \mathrm{Rep}_{\mathcal{O}_1}(L_1)$$

be the functor $\mathrm{Ind}_{L_1}^{L_2}(W) = \mathrm{Res}_{\mathcal{O}_1}^{\mathcal{O}_2}(\mathrm{Ind}_{G_1}^{G_2}(W))$. In this case, we take the induced module, which is an $\mathcal{O}_2$-module with a G_1-action, but view it solely as an $\mathcal{O}_1$-module.

The two functors are adjoints of each other: there is a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}_{\mathcal{O}_1}(L_1)}(V, \mathrm{Ind}_{L_1}^{L_2}(W)) \simeq \mathrm{Hom}_{\mathrm{Rep}_{\mathcal{O}_2}(L_2)}(\mathrm{Res}_{L_1}^{L_2}(V), W).$$

Our goal is to describe the corresponding functors between the categories $\mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L_i)$. We shall construct functors

$$\mathcal{R}^{L_2}_{L_1}\colon \mathrm{Mod}_{\varphi,\Gamma}(L_1) \to \mathrm{Mod}_{\varphi,\Gamma}(L_2), \quad \mathcal{I}^{L_2}_{L_1}\colon \mathrm{Mod}_{\varphi,\Gamma}(L_2) \to \mathrm{Mod}_{\varphi,\Gamma}(L_1),$$

show that they respect the full subcategories of étale (φ,Γ)-modules, and that the following diagram is commutative:

$$(2) \qquad \begin{array}{ccccc}
\mathrm{Rep}_{\mathcal{O}_1}(L_1) & \xrightarrow{\mathrm{Res}^{L_2}_{L_1}} & \mathrm{Rep}_{\mathcal{O}_2}(L_2) & \xrightarrow{\mathrm{Ind}^{L_2}_{L_1}} & \mathrm{Rep}_{\mathcal{O}_1}(L_1) \\
\mathcal{V}_1 \big\uparrow \big\downarrow \mathcal{D}_1 & & \mathcal{V}_2 \big\uparrow \big\downarrow \mathcal{D}_2 & & \mathcal{V}_1 \big\uparrow \big\downarrow \mathcal{D}_1 \\
\mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L_1) & \xrightarrow{\mathcal{R}^{L_2}_{L_1}} & \mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L_2) & \xrightarrow{\mathcal{I}^{L_2}_{L_1}} & \mathrm{Mod}^{\acute{e}t}_{\varphi,\Gamma}(L_1).
\end{array}$$

Chasing a diagram of functors as above means that we have to check commutativity both on objects and on morphisms. We shall do it on objects, leaving out verifications, e.g., that $\mathcal{D} \circ \mathrm{Ind}^{L_2}_{L_1} \circ \mathcal{V}$ agrees with $\mathcal{I}^{L_2}_{L_1}$ on morphisms, to the reader.

Note first that since $L_1 \subset L_2$, also $L_1^{ab} \subset L_2^{ab}$, hence $K_1 \subset K_2 \subset \mathbb{C}_p$ and $F_1 \subset F_2 \subset \mathbb{C}_p^{\flat}$. It follows that

$$\widetilde{\mathbf{A}}_{L_1} \subset \widetilde{\mathbf{A}}_{L_2}, \quad \widetilde{\mathbf{A}}_1 \subset \widetilde{\mathbf{A}}_2.$$

Second, there is a natural group homomorphism

$$r\colon \Gamma_2 \to \Gamma_1$$

given by $r(\gamma_2) = \gamma_2|_{L_1^{ab}}$. Its image is $\mathrm{Gal}(L_1^{ab}/L_2 \cap L_1^{ab})$, which is of finite index in Γ_1. Its kernel

$$\Gamma_{12} = \ker(r)$$

is the subgroup $\mathrm{Gal}(L_2^{ab}/L_1^{ab}L_2)$. If we let $H_{12} = \mathrm{Gal}(\overline{L}_1/L_1^{ab}L_2)$, then this kernel is identified with H_{12}/H_2. Via the local Artin maps, r is identified with the norm map

$$N_{L_2/L_1}\colon \widehat{L_2^{\times}} \to \widehat{L_1^{\times}}.$$

The inclusion $\widetilde{\mathbf{A}}_{L_1} \subset \widetilde{\mathbf{A}}_{L_2}$ is compatible with the homomorphism r.

To define $\mathcal{R}^{L_2}_{L_1}$, let $M_1 \in \mathrm{Mod}_{\varphi,\Gamma}(L_1)$. Then simply put

$$M_2 = \mathcal{R}^{L_2}_{L_1}(M_1) = \widetilde{\mathbf{A}}_{L_2} \otimes_{\widetilde{\mathbf{A}}_{L_1}} M_1$$

with the Γ_2-action

$$\gamma_2(\lambda \otimes m) = \gamma_2(\lambda) \otimes r(\gamma_2)(m)$$

and φ_{M_2} given by

$$\varphi_{M_2}(\lambda \otimes m) = \varphi_2(\lambda) \otimes \varphi^d_{M_1}(m).$$

One easily checks that $M_2 \in \mathrm{Mod}_{\varphi,\Gamma}(L_2)$.

The definition of $\mathcal{I}^{L_2}_{L_1}$ is a little more subtle, as this is a case of "semilinear induction" not too common in the literature. Let Φ be a variable and let Φ^d act on $M_2 \in \mathrm{Mod}_{\varphi,\Gamma}(L_2)$ as φ_{M_2}. We set

$$M_1 = \mathcal{I}^{L_2}_{L_1}(M_2) = \big\{ f\colon \Gamma_1 \to \mathcal{O}_1[\Phi] \otimes_{\mathcal{O}_1[\Phi^d]} M_2 \mid f(r(\gamma_2)\gamma) = (1 \otimes \gamma_2)f(\gamma) \big\}$$

(for all $\gamma \in \Gamma_1$, $\gamma_2 \in \Gamma_2$). Note that for any $f \in M_1$, its image lies in $\mathcal{O}_1[\Phi] \otimes_{\mathcal{O}_1[\Phi^d]} M_2^{\Gamma_{12}}$. The structure of M_1 as an $\widetilde{\mathbf{A}}_{L_1}$-module is given as follows. For $\lambda \in \widetilde{\mathbf{A}}_{L_1}$ and $f(\gamma) = \sum_{i=0}^{d-1} \Phi^i \otimes m_i(\gamma)$ ($m_i(\gamma) \in M_2$), we put

$$(\lambda f)(\gamma) = \sum_{i=0}^{d-1} \Phi^i \otimes (\varphi_1^{-i} \circ \gamma)(\lambda) \cdot m_i(\gamma).$$

The Γ_1-action is given as usual by right translation

$$(\gamma_1 f)(\gamma) = f(\gamma \gamma_1).$$

Finally, φ_{M_1} is given by

$$(\varphi_{M_1} f)(\gamma) = (\Phi \otimes 1) f(\gamma).$$

Proposition 3.2. *We have $M_1 \in \mathrm{Mod}_{\varphi,\Gamma}(L_1)$.*

Everything is easy to check, except that M_1 is finitely generated over $\widetilde{\mathbf{A}}_{L_1}$. Since $r(\Gamma_2)$ is of finite index in Γ_1, it is enough to show that $M_2^{\Gamma_{12}}$ is a finitely generated $\widetilde{\mathbf{A}}_{L_1}$-module. We prove this fact in a sequence of lemmas.

By Proposition 1.3, the field $K_1 L_2$ is a perfectoid,

$$F_{12} = (K_1 L_2)^{\flat}$$

is a finite extension of F_1, the inclusion $F_1^{\mathrm{sep}} \cap F_2 \subset F_2$ is dense,

$$\Gamma_{12} \simeq \mathrm{Gal}(F_1^{\mathrm{sep}} \cap F_2 / F_{12}) = \mathrm{Aut}_{\mathrm{cont}}(F_2 / F_{12}),$$

and $F_2^{\Gamma_{12}} = F_{12}$. For simplicity, write $F = F_{12}$, $E = F_2$ and $\Gamma = \Gamma_{12}$.

Lemma 3.3. *Let N be a finite-dimensional vector space over E equipped with a semilinear action of Γ. Put $N_0 = N^{\Gamma}$. Then*

$$E \otimes_F N_0 \simeq N.$$

In particular, $\dim_F N_0 < \infty$.

Proof. Showing that

$$E \otimes_F N_0 \to N, \quad a \otimes m \mapsto am,$$

is injective is standard: By way of contradiction, assume $\sum_{i=1}^t a_i m_i = 0$, where $m_i \in N_0$, $a_i \in E$ and t is minimal. We may assume that $a_1 = 1$. Applying $\gamma \in \Gamma$ to the relation and subtracting we get a shorter relation, contradicting the minimality of t, unless all $a_i \in E^{\Gamma} = F$. But this means that $t = 1$ so $m_1 = 0$.

The surjectivity is equivalent to the statement that $H_{\mathrm{cont}}^1(\Gamma, \mathrm{GL}_n(E)) = 0$ (where $n = \dim_E N$). The proof of this is similar to the proof of [17, Prop. 4]. The role of *loc. cit.* Prop. 1 is played by the almost-étaleness of the extension $\mathcal{O}_{F'}/\mathcal{O}_F$ ($[F' : F] < \infty$), cp. [15, Prop 5.23]. This is the only place where the assumption that F is a perfectoid is used. $\qquad\square$

Lemma 3.4. *Let N be as above. Then $H_{\mathrm{cont}}^1(\Gamma, N) = 0$.*

Proof. The previous Lemma reduces this to the statement that $H^1_{\text{cont}}(\Gamma, E) = 0$, whose proof is again similar to that of the analog statement in classical Sen–Tate theory. $\qquad\square$

Lemma 3.5. *We have* $\pi_1 M_2^{\Gamma_{12}} = (\pi_1 M_2)^{\Gamma_{12}}$.

Proof. Consider the long exact sequence in cohomology associated to the short exact sequence

$$0 \to M_2[\pi_1] \to M_2 \xrightarrow{\pi_1} \pi_1 M_2 \to 0.$$

It is enough to show that $H^1_{\text{cont}}(\Gamma_{12}, M_2[\pi_1]) = 0$. Noting that $M_2[\pi_1] = M_2[\pi_2^e]$, where e is the index of ramification of L_2/L_1, this last fact follows by dévissage from the previous lemma. $\qquad\square$

We can now conclude the proof of Proposition 3.2. No nonzero element of $M_2^{\Gamma_{12}}$ is divisible by π_1^n for all n, because the same is true in M_2, which is finitely generated over the DVR $\widetilde{A}_{L_2}$. By a well-known version of Nakayama's lemma, it is enough to prove that $M_2^{\Gamma_{12}}/\pi_1 M_2^{\Gamma_{12}}$ is finite-dimensional over $\widetilde{A}_{L_1}/\pi_1\widetilde{A}_{L_1} = F_1$. By the last lemma,

$$M_2^{\Gamma_{12}}/\pi_1 M_2^{\Gamma_{12}} = M_2^{\Gamma_{12}}/(\pi_1 M_2)^{\Gamma_{12}} \hookrightarrow (M_2/\pi_1 M_2)^{\Gamma_{12}},$$

so it is enough to prove that $(M_2/\pi_1 M_2)^{\Gamma_{12}}$ is finite-dimensional over F_1. By dévissage, it is enough to prove that $(M_2/\pi_2 M_2)^{\Gamma_{12}}$ is finite-dimensional over F_{12}. This was established in Lemma 3.3.

With Proposition 3.2 being settled, we have checked that the two functors $\mathcal{R}^{L_2}_{L_1}$ and $\mathcal{I}^{L_2}_{L_1}$ are well-defined on objects. Their definition on morphisms is self-evident and is left to the reader.

3.6. **The main theorem.** It is the following.

Theorem 3.7. *The two functors* $\mathcal{R}^{L_2}_{L_1}$ *and* $\mathcal{I}^{L_2}_{L_1}$ *respect the full subcategories of étale* (φ, Γ)-*modules, and diagram* (2) *commutes.*

Proof. We shall show that for $M_1 \in \text{Mod}^{\text{ét}}_{\varphi, \Gamma}(L_1)$,

$$(3) \qquad \mathcal{R}^{L_2}_{L_1}(M_1) = \mathcal{D}_2 \circ \text{Res}^{L_2}_{L_1} \circ \mathcal{V}_1(M_1),$$

and that for $M_2 \in \text{Mod}^{\text{ét}}_{\varphi, \Gamma}(L_2)$,

$$(4) \qquad \mathcal{I}^{L_2}_{L_1}(M_2) = \mathcal{D}_1 \circ \text{Ind}^{L_2}_{L_1} \circ \mathcal{V}_2(M_2).$$

This will imply both that the functors respect étale (φ, Γ)-modules, and that the diagram commutes on objects. As mentioned above, we leave to the reader to check that it commutes on morphisms as well.

We start with $\mathcal{R}^{L_2}_{L_1}$ and note that (3) is equivalent to the statement that for $V_1 = \mathcal{V}_1(M_1) \in \text{Rep}_{\mathcal{O}_1}(L_1)$,

$$\left(\widetilde{A}_2 \otimes_{\mathcal{O}_2} (\mathcal{O}_2 \otimes_{\mathcal{O}_1} V_1)\right)^{H_2} = \widetilde{A}_{L_2} \otimes_{\widetilde{A}_{L_1}} (\widetilde{A}_1 \otimes_{\mathcal{O}_1} V_1)^{H_1},$$

as submodules of $\widetilde{A}_2 \otimes_{\mathcal{O}_1} V_1$. Thus, we have to show that the natural map

$$\alpha \colon \widetilde{A}_{L_2} \otimes_{\widetilde{A}_{L_1}} (\widetilde{A}_1 \otimes_{\mathcal{O}_1} V_1)^{H_1} \to (\widetilde{A}_2 \otimes_{\mathcal{O}_1} V_1)^{H_2}$$

is bijective. Let $[V_1 : \mathcal{O}_1] = (r; e_1, \ldots, e_n)$, and let X be either the source or the target of α. Then Proposition 2.8 implies that

$$[X : \widetilde{\mathbf{A}}_{L_2}] = (r; ee_1, \ldots, ee_n),$$

where e is the ramification index of L_2/L_1. By Lemma 2.7, it is enough to prove that α is surjective, and for that, it is enough to prove, by Nakayama's lemma, that it is surjective modulo π_2. We may therefore assume that V_1 is a κ_1-vector space representation of G_1 and show that

$$\alpha \colon F_2 \otimes_{F_1} (F_1^{\mathrm{sep}} \otimes_{\kappa_1} V_1)^{H_1} \to (F_2^{\mathrm{sep}} \otimes_{\kappa_1} V_1)^{H_2}$$

is an isomorphism of F_2-vector spaces. Pick a basis $w_1, \ldots, w_t$ of $F_1^{\mathrm{sep}} \otimes_{\kappa_1} V_1$ over F_1^{sep} which is fixed by H_1. The vectors $1 \otimes w_i$ then form a basis of the left-hand side over F_2 and are mapped by α to a basis of the right-hand side over F_2. This concludes the proof of (3).

We next show (4) by a direct computation. An alternative approach, which works equally well, and which we do not pursue, is to prove the adjointness of $\mathcal{I}_{L_1}^{L_2}$ and $\mathcal{R}_{L_1}^{L_2}$. Below, it will be convenient to make use of the remark from Section 2.9, replacing $\widetilde{\mathbf{A}}_i$ with $\widehat{\mathbf{A}}_i$. Start with $M_2 \in \mathrm{Mod}_{\varphi,\Gamma}^{\text{ét}}(L_2)$, $V_2 = \mathcal{V}_2(M_2)$, as usual. Then $\mathcal{I}_{L_1}^{L_2}(M_2) = \mathcal{D}_1 \circ \mathrm{Ind}_{L_1}^{L_2} \circ \mathcal{V}_2(M_2)$ is

$$\left\{ \widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} \mathrm{Fun}_{G_2}\big(G_1, (\widehat{\mathbf{A}}_2 \otimes_{\widetilde{\mathbf{A}}_{L_2}} M_2)^{\varphi^d \otimes \varphi_2} \big) \right\}^{H_1},$$

where $\mathrm{Ind}_{L_1}^{L_2} V_2 = \mathrm{Fun}_{G_2}(G_1, V_2)$ is the space of functions $f \colon G_1 \to V_2$ satisfying $f(\gamma_2 \gamma) = \gamma_2 f(\gamma)$ for all $\gamma_2 \in G_2$. We regard this space merely as an $\mathcal{O}_1$-module and let G_1 act on it by right translation: $(\gamma_1 f)(\gamma) = f(\gamma \gamma_1)$. Since $[G_1 : G_2] < \infty$, the above module is the same as

$$\mathrm{Fun}_{G_2}(G_1, \widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} \widehat{\mathbf{A}}_2 \otimes_{\widetilde{\mathbf{A}}_{L_2}} M_2)^{1 \otimes \varphi^d \otimes \varphi_2, H_1}.$$

Here the action of G_2 on the triple tensor product is via the last two factors only. More precisely, $\gamma_2 \in G_2$ acts on it via $1 \otimes \gamma_2 \otimes \overline{\gamma}_2$, where $\overline{\gamma}_2$ is its image in $\Gamma_2 = G_2/H_2$. The action of $\gamma_1 \in H_1$ is via right translation on G_1 and via $\gamma_1 \otimes 1 \otimes 1$ on the triple tensor product, namely, $(\gamma_1 f)(\gamma) = (\gamma_1 \otimes 1 \otimes 1)(f(\gamma \gamma_1))$.

For f in the last space of functions define $f^\sharp$ by

$$f^\sharp(\gamma) = (\gamma \otimes 1 \otimes 1)(f(\gamma))$$

($\gamma \in G_1$). Note that $f \mapsto f^\sharp$ is *not* $\widetilde{\mathbf{A}}_{L_1}$-linear. However, the H_1-invariance of f is translated to the invariance of $f^\sharp$ under right translation by H_1, so we can regard $f^\sharp$ as a function on $G_1/H_1 = \Gamma_1$. In addition, the condition $f(\gamma_2 \gamma) = (1 \otimes \gamma_2 \otimes \overline{\gamma}_2)(f(\gamma))$ ($\gamma_2 \in G_2$) gets translated to the condition

$$f^\sharp(\gamma_2 \gamma) = (\gamma_2 \gamma \otimes 1 \otimes 1)(f(\gamma_2 \gamma)) = (\gamma_2 \gamma \otimes \gamma_2 \otimes \overline{\gamma}_2)(f(\gamma)) = (\gamma_2 \otimes \gamma_2 \otimes \overline{\gamma}_2)(f^\sharp(\gamma)).$$

Now if $\gamma_2 \in H_2$, it also lies in H_1, which is normalized by G_1, so $f^\sharp(\gamma_2 \gamma) = f^\sharp(\gamma \gamma^{-1} \gamma_2 \gamma) = f^\sharp(\gamma)$ by the right-invariance of $f^\sharp$ under H_1. We conclude that

$$f^\sharp(\gamma) \in (\widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} \widehat{\mathbf{A}}_2 \otimes_{\widetilde{\mathbf{A}}_{L_2}} M_2)^{1 \otimes \varphi^d \otimes \varphi_2, H_2},$$

where the action of H_2 this time is diagonal, on all three factors. The group on the right is nothing but $(\widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} V_2)^{H_2}$.

We have reached the following description. With $V_2 = \mathcal{V}_2(M_2)$, $\mathcal{I}_{L_1^2}^{L_2}(M_2)$ may be identified with the space of functions $f^\sharp \colon \Gamma_1 \to (\widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} V_2)^{H_2}$ satisfying $f^\sharp(r(\gamma_2)\gamma) = \gamma_2 f^\sharp(\gamma)$ for every $\gamma_2 \in \Gamma_2$ and $\gamma \in \Gamma_1$. It remains to identify the group $(\widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} V_2)^{H_2}$ with $\mathcal{O}_1[\Phi] \otimes_{\mathcal{O}_1[\Phi^d]} M_2$ and to calculate the resulting actions of $\widetilde{\mathbf{A}}_{L_1}$, φ_1 and Γ_1.

Write $\widehat{\mathbf{A}} = W(\mathbb{C}_p^\flat)$. This ring does not depend on L and $\widehat{\mathbf{A}}_i = \mathcal{O}_i \otimes_{W(\kappa_i)} \widehat{\mathbf{A}}$ $(i = 1, 2)$. We get

$$(\widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} V_2)^{H_2} = (\widehat{\mathbf{A}} \otimes_{W(\kappa_1)} V_2)^{H_2}$$
$$= (\widehat{\mathbf{A}} \otimes_{W(\kappa_2)} [W(\kappa_2) \otimes_{W(\kappa_1)} W(\kappa_2)] \otimes_{W(\kappa_2)} V_2)^{H_2}.$$

Now use the isomorphism

$$W(\kappa_2) \otimes_{W(\kappa_1)} W(\kappa_2) \simeq W(\kappa_2)^d, \quad a \otimes b \mapsto (a\varphi_1^i(b))_{i=0}^{d-1},$$

to identify the module $(\widehat{\mathbf{A}}_1 \otimes_{\mathcal{O}_1} V_2)^{H_2}$ with the direct sum of d copies of $M_2 = (\widehat{\mathbf{A}}_2 \otimes_{\mathcal{O}_2} V_2)^{H_2} = (\widehat{\mathbf{A}} \otimes_{W(\kappa_2)} V_2)^{H_2}$, as desired.

We leave as an exercise tracing the various identifications and verifying that the resulting actions of $\widetilde{\mathbf{A}}_{L_1}$, Γ_1 and φ_1 are as indicated in the definition of $\mathcal{I}_{L_1^2}^{L_2}(M_2)$. The non-obvious action of $\widetilde{\mathbf{A}}_{L_1}$ results from the replacement of f with $f^\sharp$. $\qquad\square$

4. The ring $\widetilde{\mathbf{A}}_L^\dagger$ and overconvergence

4.1. The rings of overconvergent periods.
We recall the definition of the subring $\widetilde{\mathbf{A}}_L^\dagger$ (resp. $\widetilde{\mathbf{A}}^\dagger$) of $\widetilde{\mathbf{A}}_L$ (resp. $\widetilde{\mathbf{A}}$) consisting of overconvergent periods. This will still be a discrete valuation ring with uniformizer π and residue field F (resp. F^{sep}), dense in $\widetilde{\mathbf{A}}_L$ (resp. $\widetilde{\mathbf{A}}^\dagger$) in the weak topology. The automorphism φ and the action of Γ (resp. G) will be induced by the corresponding actions on $\widetilde{\mathbf{A}}_L$ (resp. $\widetilde{\mathbf{A}}$).

To define this ring introduce, for $0 \le r \le \infty$ and $x \in \widetilde{\mathbf{A}}_L$, a "norm" $|x|_r \ge 0$, which nevertheless may be infinite. Write $x = \sum_{n=0}^{\infty} \pi^n \tau(\xi_n)$ with $\xi_n \in F$ and let, if $r < \infty$,

$$|x|_r = \sup_n \{q^{-n} |\xi_n|_\flat^r\}.$$

If $r = 0$, $|\xi|_\flat^r = 1$ if $\xi \ne 0$ and is 0 if $\xi = 0$. Thus, $|x|_0$ is always finite and is nothing but the norm associated with the discrete valuation. When $0 \le s < r$, we have

$$|x|_s \le |x|_r^{s/r},$$

so if $|x|_r < \infty$, also $|x|_s < \infty$. Finally, if $r = \infty$, we put

$$|x|_\infty = \sup_n \{|\xi_n|_\flat\},$$

so that $|x|_r \le |x|_\infty^r$. Note that if $x \in W(\mathcal{O}_F)_L$, then $|x|_r < \infty$ for any $0 \le r \le \infty$.

It is not hard to see ([9, Lem. 1.7.2]) that each $|\cdot|_r$ is a multiplicative norm on the subring $\widetilde{\mathbf{A}}_L^{(0,r)}$ of $x \in \widetilde{\mathbf{A}}_L$ for which $|x|_r < \infty$. This subring grows when r decreases, and we let $\widetilde{\mathbf{A}}_L^\dagger$ be the ring of all $x \in \widetilde{\mathbf{A}}_L$ for which there exists an $r > 0$ with $|x|_r < \infty$, i.e.,

$$\widetilde{\mathbf{A}}_L^\dagger = \bigcup_{r>0} \widetilde{\mathbf{A}}_L^{(0,r)}.$$

The ring $\widetilde{\mathbf{A}}^\dagger$ is defined in precisely the same manner, using the extension of the norm $|\xi|_\flat$ to $\xi \in F^{\mathrm{sep}}$. It is clear that

$$\pi^n \widetilde{\mathbf{A}}_L^\dagger = \widetilde{\mathbf{A}}_L^\dagger \cap \pi^n \widetilde{\mathbf{A}}_L.$$

We claim that $\widetilde{\mathbf{A}}_L^\dagger$ is a discrete valuation ring with π as a prime. To see this, it is enough to check that if $x \in \widetilde{\mathbf{A}}_L^\dagger$ is invertible in $\widetilde{\mathbf{A}}_L$, then its inverse $x^{-1} \in \widetilde{\mathbf{A}}_L^\dagger$. This follows at once from the multiplicativity of the norm $|\cdot|_r$. The residue field of $\widetilde{\mathbf{A}}_L^\dagger$ is contained in the residue field of $\widetilde{\mathbf{A}}_L$, i.e., in F, and contains $W(\mathcal{O}_F)_L / \pi W(\mathcal{O}_F)_L = \mathcal{O}_F$, hence must be equal to F. Similar arguments apply to $\widetilde{\mathbf{A}}^\dagger$.

The automorphism φ intertwines $|\cdot|_r$ and $|\cdot|_{qr}$, while Γ or G preserve $|\cdot|_r$. This implies that $\widetilde{\mathbf{A}}_L^\dagger$ and $\widetilde{\mathbf{A}}^\dagger$ are invariant under φ and the Γ or G action, which are continuous in the weak topology.

We conclude by giving an alternative description of $\widetilde{\mathbf{A}}_L^\dagger$ which explains the name *overconvergent* periods. For $0 \le r < \infty$, let

$$R_r = q^{-rq/(q-1)}.$$

As $r \to 0$ from above, $R_r \to 1$ from below.

Proposition 4.2. *Let $x \in \widetilde{\mathbf{A}}_L$ correspond to the power series*

$$f = \sum_{m \in \mathbb{Z}[p^{-1}]} a_m X^m$$

($a_m \in W_L$) under the isomorphism of Proposition 1.6. Then for $0 \le r < \infty$,

$$|x|_r = \|f\|_r := \sup_m \{|a_m| R_r^m\}.$$

Proof. Let

$$x = \sum_{n=0}^\infty \pi^n \tau(\xi_n) \quad \text{and} \quad f = \sum_{n=0}^\infty \sum_{m \in \mathbb{Z}[p^{-1}]} \pi^n \tau(\alpha_{n,m}) X^m,$$

where $\alpha_{n,m} \in \overline{\kappa}$ are the "digits" of a_m. Clearly, $\|f\|_r$ is the supremum of $q^{-n} R_r^m$ over the pairs (n, m) such that $\alpha_{n,m} \ne 0$. Reduction modulo π shows that $\xi_0 = \sum_m \alpha_{0,m} \omega^m$, hence

$$|\tau(\xi_0)|_r = |\xi_0|_\flat^r = \sup\{R_r^m\},$$

where the sup is over the set of m such that $\alpha_{0,m} \ne 0$. But this is also

$$|\tau(\xi_0)|_r = \|f_0\|_r,$$

where $f_0 = \sum_{m \in \mathbb{Z}[p^{-1}]} \tau(\alpha_{0,m}) X^m$. Consider

$$x' = x - \tau(\xi_0), \quad f'' = f - f_0.$$

Then

$$|x|_r = \sup\{|\tau(\xi_0)|_r, |x'|_r\}, \quad \|f\|_r = \sup\{\|f_0\|_r, \|f''\|_r\}.$$

The power series corresponding to x' is not f'' but rather $f' = f - \tau(\xi_0)$. However,

$$\|f_0 - \tau(\xi_0)\|_r \leq \|f_0\|_r,$$

so $\|f\|_r = \sup\{\|f_0\|_r, \|f'\|_r\}$ as well. We may now divide both x' and f' by π and continue recursively to get the desired equality $|x|_r = \|f\|_r$. $\qquad \square$

Let $V(R, 1) = \{X \mid R < |X| < 1\}$, regarded as a rigid analytic annulus over (the fraction field of) W_L. Taking inverse limit with respect to $X \mapsto X^p$ gives

$$\widetilde{V}(R, 1) = \varprojlim \left(V(R, 1) \xleftarrow{p} V(R^{1/p}, 1) \xleftarrow{p} \cdots \right),$$

which we regard as a *preperfectoid space* in the sense of [16]. A point of $\widetilde{V}(R, 1)$ in some analytic field containing L amounts to giving compatible values to X^m, for every $m \in \mathbb{Z}[p^{-1}]$.

Corollary 4.3.   (i) *The ring $\widetilde{\mathbf{A}}_L^{\dagger}$ consists of those $f \in \widetilde{\mathbf{A}}_L$ which converge on $\widetilde{V}(R, 1)$ for some $R < 1$.*

(ii) *For $0 < r < \infty$ the ring $\widetilde{\mathbf{A}}_L^{(0,r)}$ consists of those $f \in \widetilde{\mathbf{A}}_L$ which converge and are bounded on $\widetilde{V}(R_r, 1)$.*

(iii) *The ring $\widetilde{\mathbf{A}}_L^{(0,\infty)}$ consists of those $f \in \widetilde{\mathbf{A}}_L$ which converge on $\widetilde{V}(0, 1)$ and have a pole at 0.*

Proof. Write $f = f^+ + f^-$, where

$$f^+ = \sum_{0 \leq m \in \mathbb{Z}[p^{-1}]} a_m X^m, \quad f^- = \sum_{0 > m \in \mathbb{Z}[p^{-1}]} a_m X^m.$$

As $a_m \in W_L$, f^+ is convergent on the open unit disk and $\|f^+\|_r \leq 1$, while if $r < \infty$, f^- is convergent on $\widetilde{V}(R_r, 1)$ and bounded there if and only if $\|f^-\|_r < \infty$. For (iii) note that $f \in \widetilde{\mathbf{A}}_L^{(0,\infty)}$ if and only if for some n, $X^n f \in W(\mathcal{O}_F)_L$. $\qquad \square$

The ring of all bounded rigid analytic functions on some $\widetilde{V}(R, 1)$ is the *bounded (preperfectoid) Robba ring* $\widetilde{\mathbf{B}}_L^{\dagger} = \widetilde{\mathbf{A}}_L^{\dagger}[p^{-1}]$. It is actually a field, the fraction field of $\widetilde{\mathbf{A}}_L^{\dagger}$, but we shall have no use for it.

4.4. **Overconvergent (φ, Γ)-modules.** The definition of an overconvergent (φ, Γ)-module is the same as the one of a formal (φ, Γ)-module, substituting the ring $\widetilde{\mathbf{A}}_L^{\dagger}$ for $\widetilde{\mathbf{A}}_L$. It is sometimes customary to impose in the definition a further continuity condition on the action of Γ, on which we comment now. In addition to the weak topology inherited from $\widetilde{\mathbf{A}}_L$, the ring $\widetilde{\mathbf{A}}_L^{\dagger}$ has a "limit of Fréchet" (LF) topology, resulting from the Fréchet topologies of uniform convergence on affinoid sub-annuli on each $\widetilde{\mathbf{A}}_L^{(0,r)}$. Just as for the weak topology, every

finitely generated $\widetilde{\mathbf{A}}_L^\dagger$-module is endowed then with a canonical LF topology, and one requires the action of Γ to be continuous in it as well. For étale (φ, Γ)-modules over $\widetilde{\mathbf{A}}_L^\dagger$, continuity of the action of Γ in the weak topology most probably implies its continuity in the LF topology (see [9, Lem. 2.4.3], under the assumption that the module is trivial modulo p, an assumption that ought to be irrelevant). We therefore do not impose continuity in the LF topology as part of our definition.

We denote by

$$\mathrm{Mod}_{\varphi, \Gamma}^{\dagger \acute{e}t}(L) \subset \mathrm{Mod}_{\varphi, \Gamma}^\dagger(L)$$

the category of overconvergent (φ, Γ)-modules, and its full subcategory of overconvergent étale (φ, Γ)-modules. The Cherbonnier–Colmez theorem, in our setting, is the following.

Theorem 4.5.   (i) *Base change from* $\widetilde{\mathbf{A}}_L^\dagger$ *to* $\widetilde{\mathbf{A}}_L$ *induces an equivalence of categories between* $\mathrm{Mod}_{\varphi, \Gamma}^{\dagger \acute{e}t}(L)$ *and* $\mathrm{Mod}_{\varphi, \Gamma}^{\acute{e}t}(L)$.

    (ii) *Let* $V \in \mathrm{Rep}_{\mathcal{O}}(L)$ *and put* $\mathcal{D}^\dagger(V) := (\widetilde{\mathbf{A}}_L^\dagger \otimes_{\mathcal{O}} V)^H$. *Then* $\mathcal{D}^\dagger(V) \in \mathrm{Mod}_{\varphi, \Gamma}^{\dagger \acute{e}t}(L)$ *and* $\mathcal{D}(V)$ *is the base change from* $\widetilde{\mathbf{A}}_L^\dagger$ *to* $\widetilde{\mathbf{A}}_L$ *of* $\mathcal{D}^\dagger(V)$.

See [9, Thm. 2.4.5]. The structure of Γ is irrelevant in the proof of that theorem (unlike the proof of *loc. cit.* Thm. 2.6.2) so although it is phrased in the cyclotomic setting, it works for $\Gamma = \mathrm{Gal}(L^{ab}/L)$ as well. Also, [9] treats only torsion-free representations and torsion-free modules, but our claim follows from this easily, since $\widetilde{\mathbf{A}}_L^\dagger / \pi^n \widetilde{\mathbf{A}}_L^\dagger = \widetilde{\mathbf{A}}_L / \pi^n \widetilde{\mathbf{A}}_L$, so torsion modules are the same for the two rings.

As explained in the introduction, the Cherbonnier–Colmez theorem *fails* with $\mathbf{A}_L$ and $\mathbf{A}_L^\dagger$ replacing $\widetilde{\mathbf{A}}_L$ and $\widetilde{\mathbf{A}}_L^\dagger$ whenever $L \neq \mathbb{Q}_p$.

For completeness we give a self-contained proof of the theorem, based on Lemma 1.4. Kedlaya uses a similar estimate, but only for (φ, Γ)-modules which are trivial modulo p, something that can be achieved (in view of Theorem 2.5) after restriction to a finite Galois extension L' of L. He then ends up using Galois descent to go back from L' to L. We believe that our proof is a little more transparent.

The notion of a φ-module over the ring $\widetilde{\mathbf{A}}_L = \mathcal{O} \otimes_{W(\kappa)} W(F)$ is defined as before, without any reference to the action of Γ, and in fact makes sense (over $W(F)$) for any perfectoid field F in characteristic p, whether realized as $K^\flat$ for some characteristic 0 perfectoid field K or not. A φ-module M is étale if φ is bijective. Similarly, one defines the notion of an overconvergent (general or étale) φ-module over $\widetilde{\mathbf{A}}_L^\dagger$.

Lemma 4.6. *Let M be an étale φ-module over $\widetilde{\mathbf{A}}_L$. Then there exists an overconvergent étale φ-module $M^\dagger$ over $\widetilde{\mathbf{A}}_L^\dagger$, contained in M, such that the canonical map*

$$\widetilde{\mathbf{A}}_L \otimes_{\widetilde{\mathbf{A}}_L^\dagger} M^\dagger \to M$$

is an isomorphism.

Remark. We actually prove a stronger statement, that there exists an étale φ-module $M^{(0,\infty)}$ over $\widetilde{\mathbf{A}}_L^{(0,\infty)}$ for which

$$\widetilde{\mathbf{A}}_L \otimes_{\widetilde{\mathbf{A}}_L^{(0,\infty)}} M^{(0,\infty)} \to M$$

is an isomorphism. Note that $\widetilde{\mathbf{A}}_L^{(0,\infty)}$ is stable under φ, but $\widetilde{\mathbf{A}}_L^{(0,r)}$, for $0 < r < \infty$, is not.

Proof. Since $\widetilde{\mathbf{A}}_L^{\dagger}/\pi^n\widetilde{\mathbf{A}}_L^{\dagger} = \widetilde{\mathbf{A}}_L/\pi^n\widetilde{\mathbf{A}}_L$, if M is torsion, we can take $M^{\dagger} = M$. Suppose we prove the lemma when M is torsion-free. We can then consider the exact sequence

$$0 \to M_{\mathrm{tor}} \to M \xrightarrow{\mathrm{pr}} N \to 0,$$

where N is torsion-free and let $M^{\dagger} = \mathrm{pr}^{-1}(N^{\dagger})$. We may therefore assume that M is freely generated by $e_1, \ldots, e_d$ over $\widetilde{\mathbf{A}}_L$ and that the matrix of φ in this basis is given by $A = (a_{ij}) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L)$, i.e.,

$$\varphi(e_j) = \sum_{i=1}^d a_{ij} e_i.$$

If $U = (u_{ij}) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L)$ and $e'_j = \sum_{i=1}^d u_{ij} e_i$, then the matrix of φ in the basis $\{e'_j\}$ is $U^{-1}A\varphi(U)$. Our goal is to find U such that $C = U^{-1}A\varphi(U) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L^{\dagger})$, because then we can take $M^{\dagger} = \sum_{j=1}^d \widetilde{\mathbf{A}}_L^{\dagger} e'_j$. We shall in fact find a U such that $C \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L^{(0,\infty)})$ (see the remark preceding the proof).

Let us write $A = \sum_{n=0}^{\infty} \pi^n \tau(A_n)$, where $A_n \in M_d(F)$ and $\tau(A_n)$ is the matrix obtained by taking the Teichmüller representatives of the entries of A_n one-by-one. Note that $A_0 \in \mathrm{GL}_d(F)$. Similarly, write $U = \sum_{n=0}^{\infty} \pi^n \tau(U_n)$ and

$$C = U^{-1}A\varphi(U) = \sum_{n=0}^{\infty} \pi^n \tau(C_n).$$

It is enough to construct U so that $|C_n|_{\flat}$ is bounded, as the entries of C will then lie in $\widetilde{\mathbf{A}}_L^{(0,\infty)}$. Recall that the norm of a matrix with entries from F is defined to be the maximum of the norms of its entries.

Let $U_0 = I$ and suppose $U_0, \ldots, U_{n-1}$ $(n \geq 1)$ have been defined. Let $U' = \sum_{i=0}^{n-1} \pi^i \tau(U_i)$, and let $C_0 = A_0, C_1, \ldots, C_{n-1} \in M_d(F)$ be the matrices such that

$$U'^{-1}A\varphi(U') \equiv \sum_{i=0}^{n-1} \pi^i \tau(C_i) \bmod \pi^n.$$

Write $U'^{-1}A\varphi(U') - \sum_{i=0}^{n-1} \pi^i \tau(C_i) = \pi^n B$ and look for $U_n \in M_d(F)$ so that

$$(U' + \pi^n \tau(U_n))^{-1} A\varphi(U' + \pi^n \tau(U_n)) \equiv \sum_{i=0}^n \pi^i \tau(C_i) \bmod \pi^{n+1},$$

with $|C_n|_{\flat}$ small. If we denote by $\overline{B} \in M_d(F)$ the reduction of B modulo π, the above equation is equivalent to

$$U_n - A_0\varphi(U_n)A_0^{-1} = \overline{B}A_0^{-1} - C_nA_0^{-1}.$$

Lemma 1.4 guarantees that U_n can be chosen so that $|C_n A_0^{-1}|_\flat$, hence also $|C_n|_\flat$, is bounded uniformly in n. The bound depends only on A_0. This concludes the induction step, and with it the proof of the lemma. $\qquad\square$

The next lemma is a manifestation of the "contracting" property of Frobenius.

Lemma 4.7. *Let $A \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L^\dagger)$, $B \in \mathrm{GL}_e(\widetilde{\mathbf{A}}_L^\dagger)$ and $U \in M_{d \times e}(\widetilde{\mathbf{A}}_L)$ satisfy*

$$A \cdot \varphi(U) = U \cdot B.$$

Then $U \in M_{d \times e}(\widetilde{\mathbf{A}}_L^\dagger)$.

Remark. If A and B have entries in $\widetilde{\mathbf{A}}_L^{(0,r)}$ for some $0 < r \le \infty$, the proof will show that so does U.

Proof. Let $0 < r \le \infty$ be such that A and B have entries in $\widetilde{\mathbf{A}}_L^{(0,r)}$. We let $|B|_r$ be the maximum of $|b_{ij}|_r$, where b_{ij} are the entries of B, and similarly for A. Suppose that $|B|_r \le 1$. Then for $i \ge 1$ $|\varphi^{-i}(B)|_r \le 1$ and for any integer N,

$$|\varphi^{-N}(B) \cdots \varphi^{-2}(B)\varphi^{-1}(B)|_r \le |\varphi^{-N}(B)|_r \cdots |\varphi^{-2}(B)|_r |\varphi^{-1}(B)|_r \le 1.$$

In general, B can be written as $\tau(\beta)B_0$, with $\beta \in F$ and $|B_0|_r \le 1$, so (if $r < \infty$)

$$|\varphi^{-N}(B) \cdots \varphi^{-2}(B)\varphi^{-1}(B)|_r \le |\beta|_\flat^{r(q^{-1} + \cdots + q^{-N})}$$

is bounded independently of N, and similarly (with another bound) if $r = \infty$.

From the equation $U = (\varphi^{-1}(A))^{-1} \cdot \varphi^{-1}(U) \cdot \varphi^{-1}(B)$, we get by iteration

$$U = (\varphi^{-1}(A))^{-1} \cdots (\varphi^{-N}(A))^{-1} \cdot \varphi^{-N}(U) \cdot \varphi^{-N}(B) \cdots \varphi^{-1}(B),$$

which we write as $U = A_N^{-1}\varphi^{-N}(U)B_N$ with $|A_N^{-1}|_r$ and $|B_N|_r$ bounded independently of N. Let $U^{(n)}$ be the truncation of U modulo π^n, i.e., if $U = \sum_{i=0}^\infty \pi^i \tau(U_i)$, then $U^{(n)} = \sum_{i=0}^{n-1} \pi^i \tau(U_i)$. We then have

$$U^{(n)} \equiv A_N^{-1} \cdot \varphi^{-N}(U^{(n)}) \cdot B_N \bmod \pi^n.$$

Fixing n and choosing N large, we can make $|A_N^{-1} \cdot \varphi^{-N}(U^{(n)}) \cdot B_N|_r \le c$, where c is a constant depending only of A and B. But if $V = \sum_{i=0}^\infty \pi^i \tau(V_i)$ is a matrix, then

$$|V|_r = \sup_i \{q^{-i}|V_i|_\flat^r\}$$

if $r < \infty$ and $|V|_\infty = \sup_i\{|V_i|_\flat\}$, so clearly $|V^{(n)}|_r \le |V|_r$. We conclude that $|U^{(n)}|_r \le c$. As this is true for every n, $|U|_r \le c < \infty$, as was to be shown. $\quad\square$

Corollary 4.8. *The submodule $M^\dagger$ whose existence is guaranteed by Lemma 4.6 is unique.*

Proof. As usual, we may assume that M is torsion-free. Let $M^\dagger$ be freely generated over $\widetilde{\mathbf{A}}_L^\dagger$ by $e_1, \ldots, e_d$ and suppose $e_j' = \sum_{i=1}^d u_{ij}e_i$ span another φ-stable $\widetilde{\mathbf{A}}_L^\dagger$-submodule, where $U = (u_{ij}) \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L)$. Our goal is to show that $U \in \mathrm{GL}_d(\widetilde{\mathbf{A}}_L^\dagger)$. Letting A and B be the matrices from $\mathrm{GL}_d(\widetilde{\mathbf{A}}_L^\dagger)$ expressing φ

in the bases $\{e_i\}$ and $\{e'_i\}$, respectively, we get the relation $A \cdot \varphi(U) = U \cdot B$, hence the corollary follows from the lemma. $\qquad\square$

Corollary 4.9. *Let M and $M^\dagger$ be as in Lemma 4.6. Then*

$$M^\varphi = (M^\dagger)^\varphi.$$

Proof. We first reduce to the case where M is torsion-free. Let $N = M/M_{\mathrm{tor}}$ and $N^\dagger = M^\dagger/M^\dagger_{\mathrm{tor}}$. Recall that $M^\dagger_{\mathrm{tor}} = M_{\mathrm{tor}}$. Consider the commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M^{\dagger\varphi}_{\mathrm{tor}} & \longrightarrow & M^{\dagger\varphi} & \longrightarrow & N^{\dagger\varphi} & \longrightarrow & M^\dagger_{\mathrm{tor}}/(\varphi-1) \\
& & \| & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & M^{\varphi}_{\mathrm{tor}} & \longrightarrow & M^{\varphi} & \longrightarrow & N^{\varphi} & \longrightarrow & M_{\mathrm{tor}}/(\varphi-1),
\end{array}
$$

whose rows are exact. If $N^\varphi = (N^\dagger)^\varphi$, then the same would hold with M.

Assume therefore that M is torsion-free, let $e_1, \ldots, e_d$ be a basis of $M^\dagger$ and $A \in \mathrm{GL}_d(\widetilde{\mathbf{A}}^\dagger_L)$ the matrix of φ in this basis. If $m = \sum_{i=1}^d u_i e_i \in M^\varphi$, the coordinate vector u satisfies

$$A \cdot \varphi(u) = u,$$

so by the lemma, $u_i \in \widetilde{\mathbf{A}}^\dagger_L$ and $m \in (M^\dagger)^\varphi$. $\qquad\square$

We can summarize the discussion so far in the following proposition.

Proposition 4.10. (i) *Let $\mathrm{Mod}_\varphi(L)$ be the category of φ-modules over $\widetilde{\mathbf{A}}_L$ and $\mathrm{Mod}^\dagger_\varphi(L)$ the category of overconvergent φ-modules over $\widetilde{\mathbf{A}}^\dagger_L$. Then the functors $M^\dagger \mapsto M := \widetilde{\mathbf{A}}_L \otimes_{\widetilde{\mathbf{A}}^\dagger_L} M^\dagger$ and $M \mapsto M^\dagger$ (given by Lemma 4.6) induce an equivalence of categories between the full subcategories $\mathrm{Mod}^{\text{ét}}_\varphi(L)$ and $\mathrm{Mod}^{\dagger\text{ét}}_\varphi(L)$ of étale submodules.*

(ii) *Let $V \in \mathrm{Rep}_\mathcal{O}(F)$ be a continuous representation of $H = \mathrm{Gal}(F^{\mathrm{sep}}/F)$ on a finitely generated $\mathcal{O}$-module V. If $M = \mathcal{D}(V) := (\widetilde{\mathbf{A}} \otimes_\mathcal{O} V)^H$ is the corresponding étale φ-module, then $M^\dagger = \mathcal{D}^\dagger(V) := (\widetilde{\mathbf{A}}^\dagger \otimes_\mathcal{O} V)^H$.*

Proof. (i) Lemma 4.6 and Corollary 4.8 imply that the two functors induce a bijection between the objects of the two categories. The categories $\mathrm{Mod}^{\text{ét}}_\varphi(L)$ and $\mathrm{Mod}^{\dagger\text{ét}}_\varphi(L)$ have tensor products and internal Hom's (for the latter we need φ to be bijective!) and the two functors commute with them. Since

$$\mathrm{Hom}_\varphi(M, N) = \underline{\mathrm{Hom}}(M, N)^\varphi$$

(the first Hom is the group of morphisms in the category, the second underlined Hom is the internal Hom), and since the same holds in the overconvergent category, the equality $\mathrm{Hom}_\varphi(M, N) = \mathrm{Hom}_\varphi(M^\dagger, N^\dagger)$ follows from the obvious fact that $\underline{\mathrm{Hom}}(M, N)^\dagger = \underline{\mathrm{Hom}}(M^\dagger, N^\dagger)$ and from Corollary 4.9.

(ii) Lemma 4.6 and Corollary 4.8 remain valid, with the same proof, for finitely generated étale φ-modules X over $\widetilde{\mathbf{A}}$ and their overconvergent variants over $\widetilde{\mathbf{A}}^\dagger$. Consider

$$X = \widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}_L} M = \widetilde{\mathbf{A}} \otimes_{\mathcal{O}} V.$$

Both $\widetilde{\mathbf{A}}^\dagger \otimes_{\widetilde{\mathbf{A}}_L^\dagger} M^\dagger$ and $\widetilde{\mathbf{A}}^\dagger \otimes_{\mathcal{O}} V$ are finitely generated $\widetilde{\mathbf{A}}^\dagger$-submodules $X^\dagger$ of X satisfying

$$\widetilde{\mathbf{A}} \otimes_{\widetilde{\mathbf{A}}^\dagger} X^\dagger = X.$$

By the uniqueness of $X^\dagger$, they are equal: $\widetilde{\mathbf{A}}^\dagger \otimes_{\widetilde{\mathbf{A}}_L^\dagger} M^\dagger = \widetilde{\mathbf{A}}^\dagger \otimes_{\mathcal{O}} V$. Taking H-invariants we get the desired formula

$$M^\dagger = \mathcal{D}^\dagger(V). \qquad \square$$

We can now conclude the proof of Theorem 4.5.

Proof of Theorem 4.5. Assume now that $M^\dagger \in \mathrm{Mod}_{\varphi,\Gamma}^{\dagger\,\text{ét}}(L)$. Then

$$M = \widetilde{\mathbf{A}}_L \otimes_{\widetilde{\mathbf{A}}_L^\dagger} M^\dagger$$

carries a semi-linear continuous action of Γ commuting with φ, hence $M \in \mathrm{Mod}_{\varphi,\Gamma}^{\text{ét}}(L)$. Conversely, if $M \in \mathrm{Mod}_{\varphi,\Gamma}^{\text{ét}}(L)$ and $M^\dagger$ is the φ-submodule constructed in Lemma 4.6, then for any $\gamma \in \Gamma$, the φ-submodule $\gamma(M^\dagger)$ also satisfies the conditions of the lemma, so by uniqueness (Corollary 4.8) $M^\dagger$ is stable under Γ. The action of Γ on $M^\dagger$ is clearly continuous in the weak topology inherited from M, so $M^\dagger \in \mathrm{Mod}_{\varphi,\Gamma}^{\dagger\,\text{ét}}(L)$. Finally, the morphisms between two (φ, Γ)-modules are the morphisms in the category of φ-modules, which furthermore commute with Γ. The same applies to overconvergent (φ, Γ)-modules. If M and N are étale, we have already shown that $\mathrm{Hom}_\varphi(M, N) = \mathrm{Hom}_\varphi(M^\dagger, N^\dagger)$, hence we also have $\mathrm{Hom}_{\varphi,\Gamma}(M, N) = \mathrm{Hom}_{\varphi,\Gamma}(M^\dagger, N^\dagger)$.

Part (ii) of the theorem follows from part (ii) of Proposition 4.10, because the functors $\mathcal{D}$ and $\mathcal{D}^\dagger$ do not involve the Γ-action. $\qquad \square$

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