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Sums of commutators in pure C*-algebras

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Abstract. In a pure C*-algebra (i.e., one having suitable regularity properties in its Cuntz semigroup), any element on which all bounded traces vanish is a sum of 7 commutators.

1. Introduction

This paper is concerned with the problem of representing trace zero elements in a C*-algebra as sums of commutators. This problem has a long history, going back to the result by Shoda that a matrix of zero trace is expressible as a single commutator (i.e., has the form $xy - yx$). In a general C*-algebra, one can deduce from the Hahn–Banach theorem that the elements that vanish on every bounded trace belong to the norm closure of the linear span of the commutators. One can even arrange, by a result of Cuntz and Pedersen [7], for a series of commutators converging in norm to any given trace zero element. A problem that has occupied numerous authors [9, 15, 29, 22, 17, 25] is that of turning this infinite sum of commutators into a finite one. Examples in [21] and more recently [25] show that this is not always possible; not even for simple nuclear C*-algebras with a unique tracial state. Marcoux [15], continuing work of Fack [9] and Thomsen [29], was the first to show that C*-algebraic regularity properties, such as Blackadar's strict comparison of projections, could be used to obtain a positive answer. This idea has proven fruitful, and the present paper extends further the work in this direction. We prove our results in the setting of pure C*-algebras; i.e., C*-algebras whose Cuntz semigroups have certain algebraic regularity properties. The class of pure C*-algebras includes all $\mathcal{Z}$-stable C*-algebras (i.e., those tensorially absorbing the Jiang–Su algebra) and tensorially prime examples such as the reduced C*-algebra of the free group in infinitely many generators.

Let us fix some notation: Let A be a C*-algebra. By a commutator in A we understand an element of the form $xy - yx$; we denote it by $[x, y]$. We denote by $[A, A]$ the linear span of the commutators of A.

Following Winter [31], we say that A is pure if its Cuntz semigroup has the properties of almost divisibility and almost unperforation. The latter property is equivalent to the strict comparison of positive elements by lower semi-continuous 2-quasitraces (see Section 3). We prove the following theorem:

Theorem 1.1. *Let A be a pure C^*-algebra whose lower semi-continuous $[0,\infty]$-valued 2-quasitraces are traces. Then for every $h \in \overline{[A,A]}$ we have $h = \sum_{i=1}^{7}[x_i, y_i]$, with $x_i, y_i \in A$ such that $\|x_i\| \cdot \|y_i\| \leqslant C\|h\|$ for all i, and where C is a universal constant.*

A significant departure in this theorem from past results is that the existence of a unit in the C*-algebra is not assumed. This brings new technical difficulties that can nevertheless be overcome. The assumption of simplicity for the C*-algebra, typically present in previous results on this question, has also been dropped.

Part of the motivation for this paper has been to investigate pure C*-algebras for their own sake. Indeed, toward the proof of Theorem 1.1, we establish a number of results on pure C*-algebras of intrinsic interest. Pure C*-algebras arise naturally in the classification program for simple nuclear C*-algebras and in various C*-algebra constructions. For example, the tensor product of any C*-algebra with the Jiang–Su algebra $\mathcal{Z}$ is pure. However, while it is reasonable to expect that many naturally occurring simple C*-algebras are pure, there is no evidence that $\mathcal{Z}$-stability is a prevalent property beyond the realm of nuclear C*-algebras. Theorem 1.1 applies to infinite reduced free products which can be both nonexact and tensorially prime (see Example 4.11).

On the way to proving Theorem 1.1, we investigate traces of products and ultraproducts of C*-algebras; a topic also of independent interest. Traces of ultraproducts show up in the recent work on the Toms–Winter conjectures: [13, 30, 6]. Given a C*-algebra A let us denote by $\mathrm{T}_1(A)$ the traces on A of norm at most one (endowed with the weak-* topology). We prove the following theorem:

Theorem 1.2. *Let $A_1, A_2, \dots$ be C^*-algebras with strict comparison of positive elements by traces. The following are true:*

(i) *The convex hull of the sets $\mathrm{T}_1(A_1), \mathrm{T}_1(A_2), \dots$ is weak-* dense in the set $\mathrm{T}_1(\prod_{n=1}^{\infty} A_n)$.*

(ii) *For any free ultrafilter U in $\mathbb{N}$ we have $\mathrm{T}_1(\prod_U A_i) = \prod_U \mathrm{T}_1(A_i)$.*

Both (i) *and* (ii) *also hold if we instead assume that the C^*-algebras $A_1, A_2, \dots$ all have strict comparison of full positive elements by bounded traces and that their primitive spectra are compact.*

A special case of the theorem above is [20, Thm. 8], where the C*-algebras are unital, $\mathcal{Z}$-stable, and exact. Here, $\mathcal{Z}$-stability and exactness have been replaced by strict comparison by traces (which we show implies that "2-quasitraces are traces"; see Theorem 3.6).

Here is a brief overview of the paper: In Section 2 we introduce the notion of "commutator bounds" for a C*-algebra and discuss its basic properties. We then go over a number of techniques, particularly a method originally due to Fack, for proving that a C*-algebra has finite commutator bounds. We take special care to adapt these techniques to the nonunital case. In Section 3 we investigate the property of strict comparison by traces and some variations on it. We show that strict comparison by lower semi-continuous traces implies that (lower semi-continuous) 2-quasitraces are traces. In Section 4 we prove Theorems 1.1 and 1.2. In Section 5 we use nilpotents of order 2, rather than commutators, to represent trace zero elements of a pure C*-algebra. In Section 6 we look at multiplicative commutators of unitaries and the kernel of the de la Harpe–Skandalis determinant in a pure C*-algebra.

2. Commutator bounds

Let us start by fixing some notation. Let A be a C*-algebra. Let A_{sa} and A_+ denote the sets of selfadjoint and positive elements of A respectively. Let $A^\sim$ denote the minimal unitization of A and $M(A)$ the multiplier C*-algebra of A.

By a commutator in A we understand an element of the form $[x,y] := xy - yx$, with $x, y \in A$. We denote the linear span of the commutators by $[A,A]$. We regard $A/\overline{[A,A]}$ as a Banach space under the quotient norm and let $\mathrm{Tr}\colon A \to A/\overline{[A,A]}$ denote the quotient map (called the universal trace on A). We regard Tr as also defined on $M_n(A)$ for all $n \in \mathbb{N}$ by

$$\mathrm{Tr}((a_{i,j})_{i,j=1}^n) = \mathrm{Tr}\Big(\sum_{i=1}^n a_{ii}\Big).$$

We denote by $\mathrm{T}_1(A)$ the traces on A of norm at most 1; i.e., the positive linear functionals on A that vanish on $[A,A]$ and have norm at most 1. It follows from Hahn–Banach's theorem and the Jordan decomposition of bounded traces that

$$\|\mathrm{Tr}(a)\| = \sup\{|\tau(a)| \mid \tau \in \mathrm{T}_1(A)\}$$

(see [7, Thm. 2.9] and the proof of [29, Lem. 3.1]). In particular,

$$\overline{[A,A]} = \ker \mathrm{Tr} = \bigcap\{\ker \tau \mid \tau \in \mathrm{T}_1(A)\}.$$

We will often write $a \sim_{\mathrm{Tr}} b$ meaning that $\mathrm{Tr}(a-b) = 0$; i.e., $a - b \in \overline{[A,A]}$.

In [15], Marcoux calls commutator index of a C*-algebra the least $m \in \mathbb{N}$ such that every element $h \sim_{\mathrm{Tr}} 0$ is expressible as a sum of m commutators. We introduce here a variation on this concept where only approximation by sums of commutators is required. Furthermore, we keep track of the norms of the elements appearing in the commutators.

Definition 2.1. Let us say that a C*-algebra A has commutator bounds (m, C) if for all $h \in \overline{[A,A]}$ and $\varepsilon > 0$, there exist $x_1, y_1, \ldots, x_m, y_m \in A$

such that

$$\Big\| h - \sum_{i=1}^{m} [x_i, y_i] \Big\| < \varepsilon \tag{1}$$

and

$$\sum_{i=1}^{m} \|x_i\| \cdot \|y_i\| \leqslant C\|h\|. \tag{2}$$

If (1) and (2) hold with $\varepsilon = 0$ for some $x_i, y_i \in A$, then we say that A has commutator bounds (m, C) with no approximations.

Remark 2.2. We can alternatively define commutator bounds without assuming $h \in \overline{[A, A]}$ as follows: for each $h \in A$ and $\varepsilon > 0$ there exist $x_i, y_i \in A$, with $i = 1, \ldots, m$ such that

$$\Big\| h - \sum_{i=1}^{m} [x_i, y_i] \Big\| \leqslant \|\mathrm{Tr}(h)\| + \varepsilon$$

and (2) hold.

Many classes of C*-algebras can be shown to have finite commutator bounds: unital C*-algebras with no bounded traces have finite commutator bounds with no approximations [22]; C*-algebras of nuclear dimension $m \in \mathbb{N}$ have commutator bounds $(m+1, m+1)$ (see [25, Rem. 3.2]); by Theorem 1.1 from the introduction (proven below), pure C*-algebras whose 2-quasitraces are traces have commutator bounds $(7, C)$ with no approximations. On the other hand, even among simple unital nuclear C*-algebras there are some that have no finite commutator bounds [25, Thm. 1.4].

Before going over a number of results on the computation of commutator bounds, let us discuss an application of this concept to traces of products and ultraproducts. Let A_i, $i = 1, 2, \ldots$ be C*-algebras. Recall that the product C*-algebra $\prod_{i=1}^{\infty} A_i$ is the C*-algebra of bounded sequences $(a_i)_{i=1}^{\infty}$, with $a_i \in A_i$ for all i. For a given free ultrafilter U of the positive integers, the ultraproduct $\prod_U A_i$ is the quotient of $\prod_{i=1}^{\infty} A_i$ by the ideal of sequences $(a_i)_{i=1}^{\infty}$ such that $\lim_U a_i = 0$. For each $n \in \mathbb{N}$, let us view $\tau \in \mathrm{T}_1(A_n)$ as an element of $\mathrm{T}_1(\prod_{i=1}^{\infty} A_i)$ by $\tau((a_i)_i) = \tau(a_n)$. For each sequence of traces $(\tau_i)_i$, with $\tau_i \in \mathrm{T}_1(A_i)$ for all i, there is a trace on $\prod_U A_i$ given by

$$\mathrm{T}_1\Big(\prod_U A_i\Big) \ni \overline{(a_i)_i} \mapsto \lim_U \tau_i(a_i).$$

Let us denote by $\prod_U \mathrm{T}_1(A_i)$ the weak-* closure (in $\mathrm{T}_1(\prod_U A_i)$) of the set of traces that arise in this way.

Proposition 2.3. *Let A_i, with $i = 1, 2, \ldots,$ be C*-algebras, all with commutator bounds (m, C) for some $m \in \mathbb{N}$ and $C > 0$. Then the convex span of the sets $\mathrm{T}_1(A_i)$, with $i = 1, 2, \ldots,$ is weak-* dense in $\mathrm{T}_1(\prod_{i=1}^{\infty} A_i)$. Moreover, we have $\prod_U \mathrm{T}_1(A_i) = \mathrm{T}_1(\prod_U A_i)$ for any free ultrafilter U.*

Proof. The proof is the same as that of [20, Thm. 8] by Ozawa, except that [20, Thm. 6] is replaced with this paper's Remark 2.2. (Notice that Ozawa denotes by $\prod_U \mathrm{T}_1(A_i)$ the set of tracial states obtained as limits along the ultrafilter U rather than its weak-* closure.) □

Let us now look into permanence properties for the commutator bounds:

Proposition 2.4. *Let A be a C^*-algebra and I a closed two-sided ideal of A.*

(i) *If A has commutator bounds (m, C), then so do I and A/I.*
(ii) *If I and A/I have commutator bounds (m, C) and (n, D), respectively, then A has commutator bounds $(m+n, C+D)$.*
(iii) *Let A be the inductive limit of C^*-algebras $(A_\lambda)_{\lambda\in\Lambda}$, each with commutator bounds (m, C). Then A has commutator bounds (m, C) too.*

Proof. (i) Let $h \in \overline{[I, I]}$. Since A has commutator bounds (m, C), we can find $x_1, y_1, \ldots, x_m, y_m \in A$ that satisfy (1) and (2). Let $(e_\lambda)_\lambda$ be an approximately central approximate unit of I. Then for λ large enough $x_i' = x_i e_\lambda$ and $y_i' = y_i e_\lambda$ satisfy (1) and (2) and belong to I.

Let us suppose now that $h \in A/I$ and $h \sim_{\mathrm{Tr}} 0$. It suffices to assume that $\|h\| = 1$. Let $\varepsilon > 0$ be given. By [18, Lem. 2.1 (i)], there exists a lift $h' \in A$ of h such that $h' \sim_{\mathrm{Tr}} 0$ and $\|h'\| \leqslant 1 + \frac{\varepsilon}{2}$. Since A has commutator bounds (m, C), there exist $x_1, y_1, \ldots, x_m, y_m \in A$ such that

$$\Big\|\Big(1+\frac{\varepsilon}{2}\Big)^{-1} h' - \sum_{i=1}^{m} [x_i, y_i]\Big\| < \frac{\varepsilon}{2} \tag{3}$$

and

$$\sum_{i=1}^{m} \|x_i\| \cdot \|y_i\| \leqslant C.$$

Observe now that $\|h' - (1+\frac{\varepsilon}{2})^{-1}h'\| = \|h'\| \cdot \|1 - (1+\frac{\varepsilon}{2})^{-1}\| \leqslant \frac{\varepsilon}{2}$. Hence, from inequality (3) we get

$$\Big\|h' - \sum_{i=1}^{m} [x_i, y_i]\Big\| < \varepsilon.$$

Thus, the images in the quotient A/I of $x_1, y_1, \ldots, x_m, y_m$, and h' satisfy (1) and (2), as desired.

(ii) Let $h \in A$ with $h \sim_{\mathrm{Tr}} 0$. Let $h' \in A/I$ denote the image of h in A/I. Let $\varepsilon > 0$ be given. Since the quotient C^*-algebra A/I has commutator bounds (n, D), there exist $x_1', y_1', \ldots, x_n', y_n' \in A/I$ such that

$$\Big\|h' - \sum_{i=1}^{n} [x_i', y_i']\Big\| < \frac{\varepsilon}{3} \quad \text{and} \quad \sum_{i=1}^{n} \|x_i'\| \cdot \|y_i'\| \leqslant D\|h'\|.$$

For $i = 1, \ldots, n$, let us find lifts x_i'' and y_i'' in A of x_i' and y_i' , respectively, such that $\|x_i''\| = \|x_i'\|$ and $\|y_i''\| = \|y_i'\|$. Let $(e_\lambda)_\lambda$ be an approximately central approximate unit of the ideal I. We can choose an index λ such that

$$\|h - e_\lambda h e_\lambda - (1-e_\lambda)h(1-e_\lambda)\| < \frac{\varepsilon}{3},$$

and

$$\Big\|(1-e_\lambda)h(1-e_\lambda)-\sum_{i=1}^{n}[x_i,y_i]\Big\|<\frac{\varepsilon}{3},$$

where $x_i=(1-e_\lambda)^{1/2}x_i''(1-e_\lambda)^{1/2}$ and $y_i=(1-e_\lambda)^{1/2}y_i''(1-e_\lambda)^{1/2}$ for $i=1,\dots,n$. Note that

$$\sum_{i=1}^{n}\|x_i\|\cdot\|y_i\|\leqslant\sum_{i=1}^{n}\|x_i''\|\cdot\|y_i''\|\leqslant D\|h'\|\leqslant D\|h\|.$$

Since $e_\lambda he_\lambda\in I$ and the ideal I has commutator index (m,C), we can find elements $x_{n+1},y_{n+1},\dots,x_{n+m},y_{n+m}\in I$ such that

$$\Big\|e_\lambda he_\lambda-\sum_{i=n+1}^{n+m}[x_i,y_i]\Big\|<\frac{\varepsilon}{3}$$

and

$$\sum_{i=n+1}^{n+m}\|x_i\|\cdot\|y_i\|\leqslant C\|e_\lambda he_\lambda\|\leqslant C\|h\|.$$

Hence,

$$\sum_{i=1}^{m+n}\|x_i\|\cdot\|y_i\|\leqslant(C+D)\|h\|$$

and

$$\begin{aligned}\Big\|h-\sum_{i=1}^{m+n}[x_i,y_i]\Big\|&\leqslant\|h-e_\lambda he_\lambda-(1-e_\lambda)h(1-e_\lambda)\|\\&\quad+\Big\|(1-e_\lambda)h(1-e_\lambda)-\sum_{i=1}^{n}[x_i,y_i]\Big\|+\Big\|e_\lambda he_\lambda-\sum_{i=n+1}^{n+m}[x_i,y_i]\Big\|<\varepsilon.\end{aligned}$$

(iii) Since we have already shown that the commutator bounds pass to quotients, we may assume that $A_\lambda\subseteq A$ for all λ and that $\overline{\bigcup_{\lambda\in\Lambda}A_\lambda}=A$.

Let $h\in A$ be such that $h\sim_{\mathrm{Tr}}0$. Let $\varepsilon>0$ be given. Let us prove the existence of $x_1,y_1,\dots,x_m,y_m\in A$ satisfying (1) and (2). It is clear that we may reduce ourselves to the case $\|h\|=1$. We claim there exist $\lambda\in\Lambda$ and a contraction $h'\in A_\lambda$ such that $\|h-h'\|<\varepsilon/2$ and $h'\sim_{\mathrm{Tr}}0$ in A_λ. To prove this, we first approximate h sufficiently by a finite sum of commutators:

$$\Big\|h-\sum_{j=1}^{k}[v_j,w_j]\Big\|<\frac{\varepsilon}{4}.$$

Next, we choose $\lambda\in\Lambda$ and $v_1',w_1',\dots,v_k',w_k'\in A_\lambda$ such that $\|v_j-v_j'\|$ and $\|w_j-w_j'\|$ are sufficiently small for all j, so that

$$\Big\|h-\sum_{j=1}^{k}[v_j',w_j']\Big\|<\frac{\varepsilon}{4}.$$

Finally, we set

$$h' = \Big(1 + \frac{\varepsilon}{4}\Big)^{-1} \sum_{j=1}^{k} [v'_j, w'_j].$$

Notice then that

$$\|h - h'\| \leqslant \Big\|h - \Big(1 + \frac{\varepsilon}{4}\Big)h'\Big\| + \frac{\varepsilon}{4}\|h'\| < \frac{\varepsilon}{2},$$

and that $h' \sim_{\mathrm{Tr}} 0$ in A_λ, as desired.

Since A_λ has commutator bounds (m, C), there exist $x_1, y_1, \ldots, x_m, y_m \in A_\lambda$ such that

$$\Big\|h' - \sum_{j=1}^{m} [x_i, y_i]\Big\| < \frac{\varepsilon}{2} \quad \text{and} \quad \sum_{i=1}^{m} \|x_i\| \cdot \|y_i\| \leqslant C\|h'\| \leqslant C.$$

These are the desired elements. □

It is possible to reduce the number of commutators by passing from a C*-algebra A with commutator bounds (m, C) to a matrix algebra $M_n(A)$. This, however, is achieved at the expense of increasing the constant C. Marcoux obtains such a reduction for unital C*-algebras in [16, Lem. 4.1] and [15, Lem. 2.2]. Here, we cover the nonunital case and give explicit bounds for the norms of the commutators.

Lemma 2.5. *Let $d_1, \ldots, d_n$ in A be such that $\sum_{i=1}^n d_i = 0$. Then there exist $X, Y \in M_n(A)$, with $\|X\| \cdot \|Y\| \leqslant 4n \max_i \|d_i\|$, such that the main diagonal of $[X, Y]$ equals $(d_1, \ldots, d_n)$.*

Proof. For $k = 1, \ldots, n-1$, let $s_k = \sum_{i=1}^k d_i$. Let us write $s_k = q_k r_k$ for some $q_k, r_k \in A$ such that $\|q_k\| = \|r_k\| = \|s_k\|^{1/2}$ (e.g., $q_k = v|s_k|^{1/2}$ and $r_k = |s_k|^{1/2}$, where $s_k = v|s_k|$ is the polar decomposition of s_k in A^{**}). Let

$$X = \begin{pmatrix} 0 & q_1 & & & \\ & r_1 & q_2 & & \\ & & r_2 & \ddots & \\ & & & \ddots & q_{n-1} \\ & & & & r_{n-1} \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & & & & \\ r_1 & q_1 & & & \\ & r_2 & q_2 & & \\ & & \ddots & \ddots & \\ & & & r_{n-1} & q_{n-1} \end{pmatrix}.$$

A straight-forward computation shows that X and Y are as required. For the convenience of the reader, here is the 3×3 case:

$$\begin{aligned} XY - YX &= \begin{pmatrix} 0 & q_1 & 0 \\ 0 & r_1 & q_2 \\ 0 & 0 & r_2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ r_1 & q_1 & 0 \\ 0 & r_2 & q_2 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ r_1 & q_1 & 0 \\ 0 & r_2 & q_2 \end{pmatrix} \begin{pmatrix} 0 & q_1 & 0 \\ 0 & r_1 & q_2 \\ 0 & 0 & r_2 \end{pmatrix} \\ &= \begin{pmatrix} q_1 r_1 & * & * \\ * & r_1 q_1 + q_2 r_2 & * \\ * & * & r_2 q_2 \end{pmatrix} - \begin{pmatrix} 0 & * & * \\ * & r_1 q_1 + q_1 r_1 & * \\ * & * & r_2 q_2 + q_2 r_2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} q_1r_1 & * & * \\ * & q_2r_2 - q_1r_1 & * \\ * & * & -q_2r_2 \end{pmatrix}$$
$$= \begin{pmatrix} d_1 & * & * \\ * & d_2 & * \\ * & * & d_3 \end{pmatrix}. \qquad \square$$

Lemma 2.6. *Let $h \in M_n(A)$, with $h = (h_{j,k})_{j,k}$. Suppose that $h_{j,j} = [x_j, y_j]$ for some $x_j, y_j \in A$ for $j = 1, \ldots, n$. Then $h = [X, Y]$ for some $X, Y \in M_n(A)$ such that*

$$\|X\| \cdot \|Y\| \leqslant 36n^2(n-1)\|h\| + 3n\sum_{k=1}^{n}\|x_k\| \cdot \|y_k\|.$$

Proof. Let $h \in M_n(A)$ and $x_j, y_j \in A$, with $j = 1, \ldots, n$, be as in the statement of the lemma. Let us also assume that x_j is a contraction for all j (replacing y_j with $\|x_j\|y_j$ if necessary). Let $\lambda_j = 3(j-1)$ and $d_j = x_j + \lambda_j 1 \in A^\sim$ for $j = 1, \ldots, n$. Notice that the spectrum of d_j is contained in $\{z \in \mathbb{C} \mid |z - \lambda_j| < 1\}$ for all j. In particular, the spectra of the d_j's are pairwise disjoint. Notice also that $[d_j, b] = [x_j, b]$ for all $b \in A^\sim$ and $j = 1, \ldots, n$. Let us fix k, j with $1 \leqslant k, j \leqslant n$ and $k \neq j$. By [12, Cor. 3.2], the Rosenblum operator $T_{k,j}\colon A^\sim \to A^\sim$, defined by $T_{k,j}(b) = d_k b - b d_j$, is invertible. Let $b_{k,j} \in A^\sim$ be such that $T_{k,j}(b_{k,j}) = d_k b_{k,j} - b_{k,j} d_j = h_{k,j}$. Since $h_{k,j} \in A$, and by our choice of λ_k and λ_j, we must have that $b_{k,j} \in A$.

Let us define $b_{k,k} := y_k$ for $k = 1, \ldots, n$. Let $X, Y \in M_n(A)$ be given by

$$X = \operatorname{diag}(d_1, \ldots, d_n), \quad Y = (b_{k,j})_{k,j=1}^n.$$

It is a straight-forward computation to show that $[X, Y] = a$ (see [15, Lem. 2.2]).

Let us find bounds for the norms of X and Y. The bound $\|X\| = \max_j \|d_j\| \leqslant 3n$ is straight-forward. In order to bound $\|Y\|$, we first estimate $\|b_{k,j}\|$. Fix k, j such that $1 \leqslant k, j \leqslant n$ and $k \neq j$. By [12, Cor. 3.20],

$$b_{k,j} = \frac{1}{2\pi i}\int_{\Gamma_k} (d_k - \alpha 1)^{-1} h_{k,j} (d_j - \alpha 1)^{-1} d\alpha, \tag{4}$$

where Γ_k is the positively oriented simple closed contour given by $\Gamma_k(t) = \lambda_k + \frac{3}{2}e^{it}$ for $t \in [0, 2\pi]$. We have

$$(d_k - \alpha 1)^{-1} = \frac{1}{\lambda_k - \alpha}\Big(1 - \frac{d_k - \lambda_k 1}{\alpha - \lambda_k}\Big)^{-1}.$$

But $\|d_k - \lambda_k 1\| = \|x_k\| \leqslant 1$ and $|\alpha - \lambda_k| = 3/2$ for all $\alpha \in \Gamma_k$. So,

$$\Big\|\frac{d_k - \lambda_k 1}{\lambda_k - \alpha}\Big\| \leqslant \frac{1}{3/2} = \frac{2}{3},$$

from which we deduce that

$$\|(d_k - \alpha 1)^{-1}\| \leqslant \frac{2}{3}\sum_{l=0}^{\infty}\Big(\frac{2}{3}\Big)^l = 2$$

for all $\alpha \in \Gamma_k$. Next, since $k \neq j$, we have $|\lambda_j - \alpha| \geqslant 3/2$ for all $\alpha \in \Gamma_k$. Also, $\|d_j - \lambda_j 1\| = \|x_j\| \leqslant 1$. Hence,

$$\|(d_j - \alpha 1)^{-1}\| = \frac{1}{\|\lambda_j - \alpha\|}\left\|\left(1 - \frac{d_j - \lambda_j 1}{\alpha - \lambda_j}\right)^{-1}\right\| \leqslant \frac{2}{3}\sum_{l=0}^{\infty}\left(\frac{2}{3}\right)^l = 2$$

for all $\alpha \in \Gamma_k$. Thus, $\|(d_k - \alpha 1)^{-1}\| \leqslant 2$ and $\|(d_j - \alpha 1)^{-1}\| \leqslant 2$ for all $\alpha \in \Gamma_k$. From this and (4), we get

$$\|b_{k,j}\| \leqslant \left(\frac{1}{2\pi}\right)4\|h_{k,j}\| \cdot \mathrm{length}(\Gamma_k) = \left(\frac{1}{2\pi}\right)4\|h_{k,j}\|(6\pi) = 12\|h_{k,j}\| \leqslant 12\|h\|$$

for all $k \neq j$. Recall that, by our conventions, $\|b_{k,k}\| = \|y_k\| = \|x_k\| \cdot \|y_k\|$. Then

$$\|Y\| \leqslant n(n-1)12\|h\| + \sum_{k=1}^{n}\|x_k\| \cdot \|y_k\|.$$

This, together with $\|X\| \leqslant 3n$, proves the lemma. □

Theorem 2.7. *Let A be a* C**-algebra and $n \in \mathbb{N}$.*

(i) *If $M_n(A)$ has commutator bounds (m, C) (with no approximations), then A has commutator bounds (mn^2, Cn) (with no approximations).*
(ii) *If A has commutator bounds (m, C) (with no approximations), then $M_n(A)$ has commutator bounds $(2, C')$ (with no approximations) for all $n \geqslant m$, where $C' \leqslant 36n^3 + (2C - 36)n^2 + n$.*

Proof. (i) Let $h \in A$ with $h \sim_{\mathrm{Tr}} 0$. Then $a \otimes 1_n \in M_n(A)$ and $a \otimes 1_n \sim_{\mathrm{Tr}} 0$ in $M_n(A)$. Let $\varepsilon > 0$ be given. Since $M_n(A)$ has commutator bounds (m, C), there exist $X_j, Y_j \in M_n(A)$ $(1 \leqslant j \leqslant m)$ such that

$$\left\|h \otimes 1_n - \sum_{j=1}^{m}[X_j, Y_j]\right\| < \varepsilon \tag{5}$$

and

$$\sum_{j=1}^{m}\|X_j\| \cdot \|Y_j\| \leqslant C\|a \otimes 1_n\| = C\|h\|. \tag{6}$$

Averaging along the main diagonal in (5), we get

$$\left\|h - \frac{1}{n}\sum_{j=1}^{m}\sum_{k,l=1}^{n}[x_{j,k,l}, y_{j,l,k}]\right\| < \varepsilon,$$

where $X_j = (x_{j,k,l})_{k,l=1}^n$ and $Y_j = (y_{j,k,l})_{k,l=1}^n$. On the other hand, using (6), we get

$$\frac{1}{n}\sum_{j=1}^{m}\sum_{k,l=1}^{n}\|x_{j,k,l}\| \cdot \|y_{j,l,k}\| \leqslant \frac{1}{n}\sum_{j=1}^{m}\sum_{k,l=1}^{n}\|X_j\| \cdot \|Y_j\|$$
$$= n\sum_{j=1}^{m}\|X_j\| \cdot \|Y_j\| \leqslant nC\|h\|,$$

as required. The same arguments as above, but with $\varepsilon = 0$, prove the result for commutator bounds with no approximations.

(ii) Let us deal with the case of commutator bounds with no approximations. Let $h \in M_n(A)$ be such that $h \sim_{\mathrm{Tr}} 0$. Then $\sum_{i=1}^n h_{i,i} \sim_{\mathrm{Tr}} 0$ in A. But A has commutator bounds (m, C) with no approximations. Hence, $\sum_{i=1}^n h_{i,i} = \sum_{i=1}^m [x_i, y_i]$ for some $x_i, y_i \in A$. By Lemma 2.5, there exist $X_1, Y_1 \in M_n(A)$ such that the entries along the main diagonal of $[X_1, Y_1]$ equal

$$h_{1,1} - [x_1, y_1], \ldots, h_{m,m} - [x_m, y_m], h_{m+1,m+1}, \ldots, h_{n,n}.$$

Now, by Lemma 2.6, $h - [X_1, Y_1] = [X_2, Y_2]$ for some $X_2, Y_2 \in M_n(A)$. The bound on C' follows from the norm bounds in Lemmas 2.5 and 2.6.

In the case that the algebra A has commutator bounds (m, C) with approximations, the initial element $h \sim_{\mathrm{Tr}} 0$ can be slightly perturbed along the main diagonal so that, for the perturbed element, the sum of the diagonal entries is exactly a sum of m commutators. The arguments above then show that the perturbed element is a sum of two commutators. □

The proof of Theorem 2.9 below relies on a technique first used by Fack in [9]. Despite its technical statement, Theorem 2.9 constitutes our main tool in proving that a C*-algebra has finite commutator bounds with no approximations. Before stating the theorem, we introduce some definitions and prove a lemma.

Let us define the direct sum of positive elements in $A \otimes \mathcal{K}$. Fix isometries $v_1, v_2 \in B(\ell_2)$ generating the Cuntz algebra $\mathcal{O}_2$. Let us regard them as elements of the multiplier algebra $M(A \otimes \mathcal{K})$ via the natural embeddings $1 \otimes B(\ell_2) \subseteq M(A) \otimes B(\ell_2) \subseteq M(A \otimes \mathcal{K})$. Then, given $a, b \in (A \otimes \mathcal{K})_+$, let us define $a \oplus b \in (A \otimes \mathcal{K})_+$ by $a \oplus b = v_1 a v_1^* + v_2 b v_2^*$.

Next, let us introduce a preorder relation on the positive element of a C*-algebra. Let $a, b \in A_+$. Let us write $a \preceq b$ if $a = x^*x$ and $xx^* \in \mathrm{her}(b)$ for some $x \in A$. In [19], this relation is called Blackadar's relation. It can be alternatively described as saying that the right ideal $\overline{aA}$ embeds into $\overline{bA}$ as a Hilbert module. (Thus, it is clearly transitive.) We will make repeated use of the following fact (see [19, Prop. 4.6]): Say $a = x^*x$ and $xx^* \in \mathrm{her}(b)$. Let $x = v|x|$ be the polar decomposition of x in A^{**}. Then $vy \in \overline{bA}$ for any $y \in \overline{aA}$.

Lemma 2.8. *Let $a, b \in A_+$ be such that $a \preceq b^{\oplus n}$ (in $A \otimes \mathcal{K}$). Then for all $h \in \mathrm{her}(a)$ there exist $z_1, w_1, \ldots, z_n, w_n \in A$ and $h' \in \mathrm{her}(b)$ such that $h = \sum_{i=1}^n [z_j, w_j] + h'$, $\|z_j\| \cdot \|w_j\| \leqslant \|h\|$ for all j, and $\|h'\| \leqslant n\|h\|$.*

Proof. Let us regard A as a subalgebra of $M_n(A)$ embedded in the top left corner. The assumption $a \preceq b^{\oplus n}$ can be rephrased as $a \preceq b \otimes 1_n$ in $M_n(A)$. That is, there exists $x \in M_n(A)$ such that $a = x^*x$ and $xx^* \in \mathrm{her}(b \otimes 1_n)$. Let $x = v|x|$, with $v \in M_n(A)^{**}$, be the polar decomposition of x. Recall that $M_n(A)^{**}$ is canonically isomorphic to $M_n(A^{**})$ so we may regard v as an element of the latter. Let $(v_1, \ldots, v_n)$ denote the first row of v (the rest of the rows are 0). Finally, let us write $h = h_1 h_2$, with $h_1, h_2 \in \mathrm{her}(a)$ such that

$\|h_1\|\cdot\|h_2\| = \|h\|$ (e.g., as in the proof of Lemma 2.5). Then

$$h = h_1\Big(\sum_{j=1}^{n} v_j^* v_j\Big)h_2 = \sum_{j=1}^{n}[h_1v_j^*, v_jh_2] + \sum_{j=1}^{n} v_jh_2h_1v_j^*.$$

Hence, the elements $z_j = h_1v_j^*$ and $w_j = v_jh_2$ for $j = 1,\dots,n$, as well as $h' = \sum_{j=1}^{n} v_jh_2h_1v_j^*$ are as required. □

Theorem 2.9. *Let A be a* C*-*algebra and $e_0 \in A_+$ a strictly positive element. Suppose that the following are true:*

(i) *There exist an integer $L \geqslant 1$ and pairwise orthogonal positive elements $e_1, e_2, \ldots \in A_+$ such that $e_j \precsim e_{j+1}^{\oplus L}$ for $j = 0, 1, \ldots$.*

(ii) *There exist constants $C > 0$, $M \in \mathbb{N}$, and $0 < \lambda < 1$, such that for all $j \in \{0, 1, \ldots\}$ and $h \in \mathrm{her}(e_j)$ such that $h \sim_{\mathrm{Tr}} 0$ (in $\mathrm{her}(e_j)$), there exist $x_1, y_1, \ldots, x_M, y_M \in \mathrm{her}(e_j)$ such that*

$$\Big\|h - \sum_{i=1}^{M}[x_i, y_i]\Big\| \leqslant \lambda\|h\|,$$

and $\|x_i\|\cdot\|y_i\| \leqslant C\|h\|$ for all i.

Then A has finite commutator bounds $(\overline{M}, \overline{C})$ with no approximations, where $\overline{M}$ and $\overline{C}$ depend only on L, M, λ, C.

Proof. Let us choose $L_1 \in \mathbb{N}$ such that $\lambda^{L_1} < 1/(2L)$. From hypothesis (ii) we deduce the following:

(ii′) For all $j \in \{0, 1, \ldots\}$ and $h \in \mathrm{her}(e_j)$ such that $h \sim_{\mathrm{Tr}} 0$ (in $\mathrm{her}(e_j)$), there exist $x_1, y_1, \ldots, x_{L_1M}, y_{L_1M} \in \mathrm{her}(e_j)$ such that

$$\Big\|h - \sum_{i=1}^{L_1M}[x_i, y_i]\Big\| < \lambda^{L_1}\|h\| < \frac{1}{2L}\|h\|,$$

and $\|x_i\|, \|y_i\| \leqslant C^{1/2}\|h\|^{1/2}$ for all i.

Let $h \in A$ be such that $h \sim_{\mathrm{Tr}} 0$. By hypothesis (i) and Lemma 2.8, we have

$$h = \sum_{j=1}^{L}[z_j, w_j] + h_1,$$

where $h_1 \in \mathrm{her}(e_1)$ and $\|h_1\| \leqslant L\|h\|$. By (ii′) above, applied in the hereditary algebra $\mathrm{her}(e_1)$, there exist $x_1^{(1)}, y_1^{(1)}, \ldots, x_{L_1M}^{(1)}, y_{L_1M}^{(1)}, h_1' \in \mathrm{her}(e_1)$ such that

$$h_1 = \sum_{i=1}^{L_1M}[x_i^{(1)}, y_i^{(1)}] + h_1',$$

and $\|h_1'\| \leqslant \frac{1}{2L}\|h_1\| \leqslant \frac{\|h\|}{2}$. Again by hypothesis (i) and Lemma 2.8, we have

$$h_1' = \sum_{j=1}^{L}[z_j^{(1)}, w_j^{(1)}] + h_2,$$

where $h_2 \in \mathrm{her}(e_2)$, $z_j^{(1)}, w_j^{(1)} \in \mathrm{her}(e_1 + e_2)$ for all j, and $\|h_2\| \leqslant L\|h_1'\|$. Applying (ii′) in $\mathrm{her}(e_2)$, we get

$$h_2 = \sum_{i=1}^{L_1 M} [x_i^{(2)}, y_i^{(2)}] + h_2',$$

with $\|h_2'\| \leqslant \frac{\|h_1'\|}{2} \leqslant \frac{\|h\|}{4}$. Continuing in this way, we construct, for each $n \in \mathbb{N}$, elements

(i) $h_n, h_n' \in \mathrm{her}(e_n)$ such that

$$\|h_n'\| \leqslant \frac{1}{2^n}\|h\|, \quad \|h_n\| \leqslant \frac{L}{2^{n-1}}\|h\|,$$

(ii) $x_1^{(n)}, y_1^{(n)}, \ldots, x_{L_1M}^{(n)}, y_{L_1M}^{(n)} \in \mathrm{her}(e_n)$, such that

$$\|x_i^{(n)}\|, \|y_i^{(n)}\| \leqslant C^{\frac{1}{2}}\|h_n\|^{\frac{1}{2}} \quad \text{for all } i,$$

and

$$h_n = \sum_{i=1}^{L_1 M} [x_i^{(n)}, y_i^{(n)}] + h_n',$$

(iii) $z_1^{(n)}, w_1^{(n)}, \ldots, z_L^{(n)}, w_L^{(n)} \in \mathrm{her}(e_n + e_{n+1})$, such that

$$\|z_j^{(n)}\|, \|w_j^{(n)}\| \leqslant \|h_n'\|^{\frac{1}{2}} \quad \text{for all } j,$$

and

$$h_n' = \sum_{j=1}^{L} [z_j^{(n)}, w_j^{(n)}] + h_{n+1}.$$

It follows that

$$h_1 = \sum_{k=1}^{n} \sum_{i=1}^{L_1 M} [x_i^{(k)}, y_i^{(k)}] + \sum_{k=1}^{n} \sum_{j=1}^{L} [z_j^{(k)}, w_j^{(k)}] + h_{n+1}.$$

We can gather terms belonging to orthogonal hereditary subalgebras and define

$$X_i = \sum_{n=1}^{\infty} x_i^{(n)}, \quad Y_i = \sum_{n=1}^{\infty} y_i^{(n)},$$

for $i = 1, \ldots, L_1 M$, and

$$Z_{j,0} = \sum_{n \text{ odd}}^{\infty} z_j^{(n)}, \quad Z_{j,1} = \sum_{n \text{ even}}^{\infty} z_j^{(n)},$$

$$W_{j,0} = \sum_{n \text{ odd}}^{\infty} w_j^{(n)}, \quad W_{j,1} = \sum_{n \text{ even}}^{\infty} w_j^{(n)},$$

for $j = 1, \ldots, L$. Note that the terms in the series defining the elements $X_i, Y_i, Z_{j,k}, W_{j,k}$ are pairwise orthogonal. Also, the norm estimates on the elements $x_i^{(n)}, y_i^{(n)}, z_j^{(n)}, w_j^{(n)}$ guarantee that these series converge. Furthermore, it is clear that norm estimates on $X_i, Y_i, Z_{j,k}, W_{j,k}$ can be obtained from those on $x_i^{(n)}, y_i^{(n)}, z_j^{(n)}, w_j^{(n)}$. We have

$$h = \sum_{j=1}^{L}[z_j, w_j] + \sum_{i=1}^{L_1 M}[X_i, Y_i] + \sum_{j=1}^{L}[Z_{j,0}, W_{j,0}] + \sum_{j=1}^{L}[Z_{j,1}, W_{j,1}].$$

This shows that A has commutator bounds $(3L + L_1 M, C')$ with no approximations, for some C'. □

Remark 2.10. From the proof of Theorem 2.9, we can see that the commutator bounds $\overline{M}$ and C' in the statement of Theorem 2.9 can be chosen as $\overline{M} = 3L + L_1 M$ and $C' = 3L + LL_1 MC$.

3. Strict comparison of positive elements

Here we review and explore the strict comparison of positive elements by traces and 2-quasitraces. Some of these results will be used in the proof of Theorem 1.1 in the next section.

Let us start by recalling the definition of the Cuntz semigroup. Let A be a C*-algebra. Let $a, b \in (A \otimes \mathcal{K})_+$. Let us write $a \preceq_{\mathrm{Cu}} b$ if $d_n^* b d_n \to a$ for some $d_n \in A \otimes \mathcal{K}$. In this case we say that a is Cuntz smaller than b. Let us write $a \sim_{\mathrm{Cu}} b$ if $a \preceq_{\mathrm{Cu}} b$ and $b \preceq_{\mathrm{Cu}} a$, in which case we say that a and b are Cuntz equivalent. Let $[a]$ denote the Cuntz class of $a \in (A \otimes \mathcal{K})_+$.

The Cuntz semigroup of A, denoted by $\mathrm{Cu}(A)$, is defined as the quotient set $(A \otimes \mathcal{K})/{\sim_{\mathrm{Cu}}}$, endowed with the following order and addition: $[a] \leqslant [b]$ if $a \preceq_{\mathrm{Cu}} b$ and $[a] + [b] = [a \oplus b]$, with the direct sum $a \oplus b \in (A \otimes \mathcal{K})_+$ as defined in the previous section. The reader is referred to [1, 2] for the basic theory of the Cuntz semigroup (some of which will be used below).

Let us denote by $\mathrm{T}(A)$ the cone of lower semi-continuous $[0,\infty]$-valued traces on A; i.e., the lower semi-continuous maps $\tau\colon A_+ \to [0,\infty]$ that are additive, homogeneous, map 0 to 0, and satisfy that $\tau(x^*x) = \tau(xx^*)$ for all $x \in A$. Let us denote by $\mathrm{QT}(A)$ the lower semi-continuous $[0,\infty]$-valued 2-quasitraces on A_+. Traces and 2-quasitraces extend uniquely to traces and quasitraces on $(A \otimes \mathcal{K})_+$, and we shall regard them as defined on this domain (see [5, Rem. 2.27 (viii)]). Recall from the previous section that we denote by $\mathrm{T}_1(A)$ the convex set of traces on A of norm at most 1.

A topology on $\mathrm{QT}(A)$ can be defined as follows: Let $(\tau_\lambda)_\lambda$ be a net in $\mathrm{QT}(A)$ and $\tau \in \mathrm{QT}(A)$. Let us say that $\tau_\lambda \to \tau$ if for any $a \in (A \otimes \mathcal{K})_+$ and $\varepsilon > 0$ we have

$$\limsup_\lambda \tau_\lambda((a-\varepsilon)_+) \leqslant \tau(a) \leqslant \liminf_\lambda \tau_\lambda(a).$$

In this way $\mathrm{QT}(A)$ is a compact Hausdorff space and $\mathrm{T}(A)$ and $\mathrm{T}_1(A)$ are closed subsets of $\mathrm{QT}(A)$ (see [8, Sec. 3.2 and 4.1]).

The dimension function associated to $\tau \in \mathrm{QT}(A)$ is defined as

$$d_\tau(a) = \lim_n \tau(a^{1/n}) = \|\tau|_{\mathrm{her}(a)}\|$$

for all $a \in (A \otimes \mathcal{K})_+$. The value $d_\tau(a)$ depends only on the Cuntz class of the positive element a. Thus, by a slight abuse of notation, we also write $d_\tau([a])$.

The ordered semigroup $\mathrm{Cu}(A)$ is called *almost unperforated* if

$$(k+1)x \leqslant ky \quad \Rightarrow \quad x \leqslant y$$

for all $k \in \mathbb{N}$ and $x, y \in \mathrm{Cu}(A)$. The ordered semigroup $\mathrm{Cu}(A)$ is called *almost divisible* if for all $n \in \mathbb{N}$, $x \in \mathrm{Cu}(A)$ and $x' \ll x$ (i.e., x' compactly contained in x), there exists $y \in \mathrm{Cu}(A)$ such that

$$ny \leqslant x \quad \text{and} \quad x' \leqslant (n+1)y.$$

The C*-algebra A is called *pure* if $\mathrm{Cu}(A)$ is both almost unperforated and almost divisible. By [26], C*-algebras that absorb tensorially the Jiang–Su algebra are pure. There are, however, tensorially prime pure C*-algebras.

It is shown in [8, Prop. 6.2] (and in [26, Cor. 4.6] for simple C*-algebras) that almost unperforation in $\mathrm{Cu}(A)$ is equivalent to the property of strict comparison of positive elements by 2-quasitraces. We consider here the following generalization of the latter property:

Definition 3.1. Let A be a C*-algebra and $K \subseteq \mathrm{QT}(A)$ a compact subset. Let us say that A has strict comparison of positive elements by 2-quasitraces in K if $d_\tau(a) \leqslant (1-\gamma)d_\tau(b)$ for all $\tau \in K$ and some $\gamma > 0$ implies that $[a] \leqslant [b]$ for all $a, b \in (A \otimes \mathcal{K})_+$.

For $K = \mathrm{QT}(A)$, this notion agrees with the strict comparison of positive elements mentioned above. Another case of interest is $K = \mathrm{T}(A)$. In this case we say that A has strict comparison of positive elements by traces. (In the context of simple unital C*-algebras, strict comparison by traces is often taken to mean that the inequality $d_\tau(a) < d_\tau(b)$ for all bounded traces τ implies that $[a] \leqslant [b]$. Although the definition of strict comparison that we have given above is formally weaker than this property, they can be seen to be equivalent in the simple unital case.) If A is unital or more generally $\mathrm{Prim}(A)$ is compact, it is also interesting to consider the property of strict comparison by traces restricted to full positive elements only (i.e., those generating A as a two-sided ideal). Let us define this more formally:

Definition 3.2. Let A be a C*-algebra such that $\mathrm{Prim}(A)$ is compact. Let us say that A has strict comparison of full positive elements by traces if $d_\tau(a) \leqslant (1-\gamma)d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$ and some $\gamma > 0$ implies that $[a] \leqslant [b]$ for all full positive elements $a, b \in (A \otimes \mathcal{K})_+$.

Lemma 3.3. *Let $K \subseteq \mathrm{QT}(A)$ be compact and $a, b \in (A \otimes \mathcal{K})_+$. Suppose that $d_\tau(a) \leqslant (1-\gamma)d_\tau(b)$ for all $\tau \in K$ and some $\gamma > 0$. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$d_\tau((a-\varepsilon)_+) \leqslant \Big(1 - \frac{\gamma}{2}\Big) d_\tau((b-\delta)_+) \quad \text{for all } \tau \in K.$$

Proof. As shown in the proof of [8, Lem. 5.11], we have

$$\overline{\{\tau \in K \mid d_\tau((a-\varepsilon)_+) > 1\}} \subseteq \Big\{\tau \in K \mid \Big(1 - \frac{\gamma}{2}\Big) d_\tau(b) > 1\Big\}.$$

The left side is compact while the right side is covered by the open sets

$$\Big\{\tau \in K \mid \Big(1-\frac{\gamma}{2}\Big)d_\tau\Big(\Big(b-\frac{1}{n}\Big)_+\Big) > 1\Big\}, \quad n = 1, 2, \ldots .$$

Thus, one of these open sets covers $\{\tau \in K \mid d_\tau((a-\varepsilon)_+) > 1\}$. This proves the lemma. $\square$

Lemma 3.4. *Let A be a C^*-algebra with strict comparison of positive elements by K, where $K \subseteq \mathrm{QT}(A)$ is compact. Let $a, b \in (A \otimes \mathcal{K})_+$.*

(i) *If $d_\tau(a) \leqslant d_\tau(b)$ for all $\tau \in K$, then $d_\tau(a) \leqslant d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$.*
(ii) *If $\tau(a) \leqslant \tau(b)$ for all $\tau \in K$, then $\tau(a) \leqslant \tau(b)$ for all $\tau \in \mathrm{QT}(A)$.*

Proof. (i) Let $k \in \mathbb{N}$. Then $d_\tau(a \otimes 1_k) \leqslant (1-\frac{1}{k+1})d_\tau(b \otimes 1_{k+1})$ for all $\tau \in K$. Since A has strict comparison by 2-quasitraces in K, we have $a \otimes 1_k \precsim b \otimes 1_{k+1}$, which in turn implies that $d_\tau(a) \leqslant (1+\frac{1}{k+1})d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$. Letting $k \to \infty$, we get that $d_\tau(a) \leqslant d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$, as desired.

(ii) Let $\varepsilon > 0$. By a compactness argument as in the proof of Lemma 3.3, we find that there exists $\delta > 0$ such that $\tau((a-\varepsilon)_+) \leqslant \tau((b-\delta)_+)$ for all $\tau \in K$ (see [8, Prop. 5.1 and 5.3]). Let $f_\varepsilon, g, g_\delta \colon \mathrm{QT}(A) \to [0, \infty]$ be as follows: $f_\varepsilon(\tau) = \tau((a-\varepsilon)_+)$, $g(\tau) = \tau(b)$, and $g_\delta(\tau) = \tau((b-\delta)_+)$ for all τ. The functions f_ε and g (and also g_δ) belong to the realification of $\mathrm{Cu}(A)$, as defined in [24]. That is, $f_\varepsilon = f_n \uparrow$ and $g = g_n \uparrow$, where $f_n(\tau) = \frac{1}{r_n}d_\tau([a_n])$ and $g_n(\tau) = \frac{1}{s_n}d_\tau([b_n])$ for some $[a_n], [b_n] \in \mathrm{Cu}(A)$ and $r_n, s_n \in \mathbb{N}$ for $n = 1, \ldots$. (This follows from the fact that

$$\tau(c) = \int_0^{\|c\|} d_\tau([(c-t)_+])dt$$

for all $c \in (A \otimes \mathcal{K})_+$ and $\tau \in \mathrm{QT}(A)$.) By [8, Prop. 5.1 and 5.3], the function g_δ is way below g, so that $g_\delta \leqslant g_n$ for some n. Hence, $f_m(\tau) \leqslant g_n(\tau)$ for all $\tau \in K$ and all $m \in \mathbb{N}$; i.e., $\frac{1}{r_m}d_\tau(a_m) \leqslant \frac{1}{s_n}d_\tau(b_n)$ for all $\tau \in K$ and m. By (i), this same inequality holds for all $\tau \in \mathrm{QT}(A)$; whence, $f_m \leqslant g$ for all m. Passing to the supremum over m, we get $\tau(a-\varepsilon)_+ \leqslant \tau(b)$ for all $\tau \in \mathrm{QT}(A)$ and $\varepsilon > 0$. Now passing to the supremum over all $\varepsilon > 0$, we get $\tau(a) \leqslant \tau(b)$ for all $\tau \in \mathrm{QT}(A)$, as desired. $\square$

There is a version of the previous lemma for strict comparison of full positive elements by traces:

Lemma 3.5. *Let A be a C^*-algebra with* $\mathrm{Prim}(A)$ *compact and with strict comparison of full positive elements by traces. Let $a, b \in (A \otimes \mathcal{K})_+$ be full positive elements.*

(i) *If $d_\tau(a) \leqslant d_\tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $d_\tau(a) \leqslant d_\tau(b)$ for all $\tau \in \mathrm{QT}(A)$.*
(ii) *If $\tau(a) \leqslant \tau(b)$ for all $\tau \in \mathrm{T}(A)$, then $\tau(a) \leqslant \tau(b)$ for all $\tau \in \mathrm{QT}(A)$.*

Note: Since b is full, the inequality $d_\tau(a) \leqslant d_\tau(b)$ need only be verified on densely finite traces, for otherwise $d_\tau(b) = \infty$.

Proof. The same proof as the one of Lemma 3.4, with the obvious modifications, works here. When taking functional calculus cut-downs $(a-\varepsilon)_+$ and $(b-\delta)_+$, we must take care to choose them so that they are still full (which is possible by the compactness of $\mathrm{Prim}(A)$). ☐

Theorem 3.6. *Let A be a C^*-algebra.*

(i) *If A has strict comparison of positive elements by traces, then every lower semi-continuous 2-quasitrace on A is a trace.*
(ii) *If $\mathrm{Prim}(A)$ is compact and A has strict comparison of full positive elements by traces, then every densely finite lower semi-continuous 2-quasitrace on A is a trace.*

Proof. (i) Let $a, b \in A_+$. Let $c, d \in M_2(A)$ be defined as

$$c = \begin{pmatrix} a+b & \\ & 0 \end{pmatrix}, \quad d = \begin{pmatrix} a & \\ & b \end{pmatrix}.$$

Then $\tau(c) = \tau(d)$ for all $\tau \in \mathrm{T}(A)$. By Lemma 3.4 (ii), we get that $\tau(c) = \tau(d)$ for all $\tau \in \mathrm{QT}(A)$. But $\tau(c) = \tau(a+b)$ and $\tau(d) = \tau(a) + \tau(b)$. So all $\tau \in \mathrm{QT}(A)$ are additive, as desired.

(ii) The same proof as in (i), but relying now on Lemma 3.5 (ii), shows in this case that the lower semi-continuous 2-quasitraces on A are additive on pairs of full positive elements. Let us now prove that the densely finite ones are additive on any pair of positive elements. Let τ be one such 2-quasitrace and let $a, b \in A_+$. Say $w \in A_+$ is full (whose existence is guaranteed by the compactness of $\mathrm{Prim}(A)$) and let $e_1, e_2, \ldots$ be an approximate unit of $C^*(a, b, w)$ such that $e_{n+1}e_n = e_n$ for all n. Notice that e_n is full for large enough n by the compactness of $\mathrm{Prim}(A)$. So

$$\begin{aligned} \tau(e_n a e_n + e_n b e_n) + 2\tau(e_{n+1}) &= \tau(e_n a e_n + e_{n+1} + e_n b e_n + e_{n+1}) \\ &= \tau(e_n a e_n + e_{n+1}) + \tau(e_n b e_n + e_{n+1}) \\ &= \tau(e_n a e_n) + \tau(e_n b e_n) + 2\tau(e_{n+1}). \end{aligned}$$

In the first and third equalities we have used the additivity of τ on commutative C^*-algebras and in the middle equality the additivity of τ on pairs of full elements. Since $\tau(e_{n+1}) < \infty$, we get

$$\tau(e_n a e_n + e_n b e_n) = \tau(e_n a e_n) + \tau(e_n b e_n)$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ and using the lower semicontinuity of τ, we obtain that $\tau(a+b) = \tau(a) + \tau(b)$, as desired. ☐

Remark 3.7. In view of part (i) of the previous theorem, the property of strict comparison of positive elements by traces is equivalent to "strict comparison by 2-quasitraces" and "2-quasitraces are traces" (all traces and 2-quasitraces are assumed to be lower semi-continuous). Observe also that if A has strict comparison of positive elements by 2-quasitraces and its densely finite 2-quasitraces are traces, then A has strict comparison of full positive elements by traces. Indeed, if $b \in (A \otimes \mathcal{K})_+$ is a full element, the inequality $d_\tau(a) \leqslant (1-\varepsilon)d_\tau(b)$ need only be verified on all densely finite traces (otherwise $\tau(b) = \infty$).

4. Proof of Theorem 1.1

In this section we prove Theorem 1.1 from the introduction. The proof is preceded by a number of preparatory results.

Theorem 4.1. *Let $A_1, A_2, \ldots$ be* C*-*algebras with strict comparison by traces. Let $h \in \prod_{n=1}^{\infty} A_n$ be such that $h_n \sim_{\mathrm{Tr}} 0$ for all $n \in \mathbb{N}$. Then $h \sim_{\mathrm{Tr}} 0$.*

Proof. Let $h \in \prod_{n=1}^{\infty} A_n$ be such that $h_n \sim_{\mathrm{Tr}} 0$ for all n. First, let us show how to reduce ourselves to the case that the C*-algebras A_n are σ-unital for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $h_n \in \overline{[A_n, A_n]}$, there exist finite sums of commutators $\sum_{i=1}^{N_k} [x_i^{(k)}, y_i^{(k)}]$, with $x_i^{(k)}, y_i^{(k)} \in A_n$ for all i, k, converging to h_n as $k \to \infty$. Consider the C*-subalgebra

$$B_n = \mathrm{her}\big(\{x_i^{(k)}, y_i^{(k)} \mid i = 1, \ldots, N_k,\, k \in \mathbb{N}\}\big).$$

Then B_n has strict comparison by traces and is σ-unital. Furthermore, $h_n \in \overline{[B_n, B_n]}$. To prove the theorem, it suffices to show that $h \sim_{\mathrm{Tr}} 0$ in $\prod_{n=1}^{\infty} B_n$. Thus, from this point on, we assume that the C*-algebras $A_1, A_2, \ldots$ are all σ-unital.

Let us set $\prod_{n=1}^{\infty} A_n = A$. As before, let $h \in A$ be such that $h_n \sim_{\mathrm{Tr}} 0$ for all n. We may assume that $\|h\| \leqslant 1$. Let us suppose, for the sake of contradiction, that $\mu(h) \neq 0$ for some trace μ on A of norm 1. Notice then that $(h_n)_+ \sim_{\mathrm{Tr}} (h_n)_-$ for all $n \in \mathbb{N}$, but $\mu(h_+) \neq \mu(h_-)$. We wish, however, to find positive elements $a_n, b_n \in A_n$ agreeing on all traces in $\mathrm{T}(A_n)$ (i.e., lower semicontinuous and $[0,\infty]$-valued) while at the same time $\mu((a_n)_n) \neq \mu((b_n)_n)$. Let us show how to achieve this: Assume, without loss of generality, that $\mu(h) > 0$. Set $\mu(h) = \delta$. Fix $n \in \mathbb{N}$. Let $(e_n^{(i)})_i$ be an approximate unit of A_n such that

$$e_n^{(i+1)} e_n^{(i)} = e_n^{(i)} \quad \text{for all } i.$$

We can find $h_n^{(i)} \in \mathrm{her}(e_n^{(i)})$ for i large enough such that $\|h_n^{(i)} - h_n\| < \frac{\delta}{2}$ and $h_n^{(i)} \sim_{\mathrm{Tr}} 0$ in $\mathrm{her}(e_n^{(i)})$. (This is achieved as follows: first, sufficiently approximate h_n by a finite sum of commutators; next, multiply the elements in these commutators by $e_n^{(i)}$ and let $i \to \infty$.) Let

$$a_n := (h_n^{(i)})_+ + e_n^{(i+1)},$$
$$b_n := (h_n^{(i)})_- + e_n^{(i+1)}.$$

Let $\tau \in \mathrm{T}(A_n)$. Suppose first that $\tau(e_n^{(i+1)}) < \infty$. Then τ is bounded on $\mathrm{her}(e_n^{(i)})$, and so $\tau(h_n^{(i)}) = 0$, because $h_n^{(i)} \sim_{\mathrm{Tr}} 0$ in $\mathrm{her}(e_n^{(i)})$. Hence,

$$\begin{aligned}\tau(a_n) &= \tau((h_n^{(i)})_+) + \tau(e_n^{(i+1)}) \\ &= \tau((h_n^{(i)})_-) + \tau(e_n^{(i+1)}) \\ &= \tau(b_n).\end{aligned}$$

On the other hand, if $\tau(e_n^{(i+1)}) = \infty$, then again we find that $\tau(a_n) = \infty = \tau(b_n)$. Furthermore,

$$\begin{aligned}\tau((a_n - t)_+) &= \tau((h_n^{(i)})_+ + (e_n^{(i+1)} - t)_+)\\ &= \tau((h_n^{(i)})_- + (e_n^{(i+1)} - t)_+)\\ &= \tau((b_n - t)_+)\end{aligned}$$

for all $0 \leqslant t < 1$ and all $\tau \in \mathrm{T}(A_n)$. (This equality is again verified both if $\tau((e_n^{(i+1)} - t)_+) < \infty$ and if $\tau((e_n^{(i+1)} - t)_+) = \infty$.)

Let $a = (a_n)_n$ and $b = (b_n)_n$. Notice that $\mu(a) - \mu(b) = \mu((h_n^{(i)})_n) > \delta/2$. Let $K = \overline{\bigcup_{n=1}^{\infty} \mathrm{T}(A_n)}$. Let us show that $\tau(a) = \tau(b)$ for all $\tau \in K$. Consider the set

$$Q = \{\tau \in \mathrm{T}(A) \mid \tau((a - t)_+) = \tau((b - t)_+) \text{ for all } 0 \leqslant t < 1\}.$$

Clearly, $\tau(a) = \tau(b)$ for all $\tau \in Q$. So it suffices to show that $K \subseteq Q$. By our construction of a and b, we have $\bigcup_{n=1}^{\infty} \mathrm{T}(A_n) \subseteq Q$. We will be done once we have shown that Q is closed in $\mathrm{T}(A)$. Suppose that $\tau_\lambda \to \tau$ in $\mathrm{T}(A)$, with $\tau_\lambda \in Q$ for all λ. Let $0 \leqslant t < 1$ and choose $t < t' < 1$. Then

$$\tau((a - t')_+) \leqslant \liminf \tau_\lambda((a - t')_+) \leqslant \limsup \tau_\lambda(b - t')_+ \leqslant \tau(b - t)_+.$$

Passing to the supremum over all $t' > t$ on the left, we get that $\tau((a - t)_+) \leqslant \tau((b-t)_+)$. By symmetry, we also have $\tau((b-t)_+) \leqslant \tau((a-t)_+)$. Thus, $\tau \in Q$ as desired.

Let us now show that A has strict comparison of positive elements by K (as defined in Definition 3.1). Let $c, d \in (A \otimes \mathcal{K})_+$ be such that $d_\tau(c) \leqslant (1-\gamma) d_\tau(d)$ for all $\tau \in K$ and some $\gamma > 0$. In order to show that $[c] \leqslant [d]$, it suffices to show that $[(c - \varepsilon)_+] \leqslant [d]$ for all $\varepsilon > 0$. But, for each $\varepsilon > 0$ and $\delta > 0$, we have $(c - \varepsilon)_+ \sim_{\mathrm{Cu}} c' \in M_N(A)$ and $(d - \delta)_+ \sim_{\mathrm{Cu}} d' \in M_N(A)$ for some $N > 0$. Thus, applying Lemma 3.3, we may reduce the proof to the case that $c, d \in M_N(A)$ for some $N \in \mathbb{N}$. Let us assume this. Let us fix $\varepsilon > 0$. Again by Lemma 3.3, there exists $\delta > 0$ such that

$$d_\tau((c - \varepsilon)_+) \leqslant \Big(1 - \frac{\gamma}{2}\Big) d_\tau((d - \delta)_+) \quad \text{for all } \tau \in K.$$

Since $M_N(A) \cong \prod_n M_N(A_n)$, we can write $c = (c_n)_n$ and $d = (d_n)_n$, with $c_n, d_n \in M_N(A_n)$ for all n. Projecting onto A_n, we get

$$d_\tau((c_n - \varepsilon)_+) \leqslant \Big(1 - \frac{\gamma}{2}\Big) d_\tau((d_n - \delta)_+) \quad \text{for all } \tau \in \mathrm{T}(A_n).$$

Since the C*-algebra A_n has strict comparison of positive elements by traces, we get that $[(c_n - \varepsilon)_+] \leqslant [(d_n - \delta)_+]$. Thus, $(c_n - 2\varepsilon)_+ = x_n^* x_n$ and $x_n x_n^* \in \mathrm{her}((d_n - \delta)_+)$ for some $x_n \in A_n$. Then $(c - 2\varepsilon)_+ = x^* x$ and $xx^* \leqslant Md$ for some $M > 0$, where $x = (x_n)_n$. Hence, $[(c - 2\varepsilon)_+] \leqslant [d]$ for all $\varepsilon > 0$. Letting $\varepsilon \to 0$, we get $[c] \leqslant [d]$, as desired.

We now know that $\tau(a) = \tau(b)$ for all $\tau \in K$ and that A has strict comparison by K. By Lemma 3.4 (ii), we conclude that $\tau(a) = \tau(b)$ for all $\tau \in \mathrm{T}(A)$. But this contradicts that $\mu(a) \neq \mu(b)$, which completes the proof. ☐

Essentially the same proof, with some modifications, yields the following theorem:

Theorem 4.2. *Let $A_1, A_2, \ldots$ be* C^**-algebras with* $\mathrm{Prim}(A_n)$ *compact and with strict comparison of full positive elements by traces. Let $h \in \prod_{n=1}^{\infty} A_n$ be such that $h_n \sim_{\mathrm{Tr}} 0$ for all $n \in \mathbb{N}$. Then $h \sim_{\mathrm{Tr}} 0$.*

Proof. Let us sketch the necessary modifications to the proof of Theorem 4.1 that yield a proof of the present theorem: As before, we can reduce to the case that the C*-algebras $A_1, A_2, \ldots$ are σ-unital. To this end, we use that $\mathrm{Prim}(A_n)$ is compact if and only if there exist $c_n \in (A_n)_+$ and $\varepsilon > 0$ such that $(c_n - \varepsilon)_+$ is full. Now defining

$$B_n = \mathrm{her}\big(\{c_n, x_i^{(k)}, y_i^{(k)} \mid i = 1, \ldots, N_k,\, k \in \mathbb{N}\}\big),$$

we guarantee that B_n has compact primitive spectrum for all n and is σ-unital. Next, the elements a_n and b_n in the first part of the proof are constructed as before, except that we take care that they be full elements. This is possible since the elements of the approximate unit $(e_n^{(i)})_i$ are full for large enough i. The definitions of the sets K and Q remain unchanged, and again we find that $K \subseteq Q$ and that Q is closed. In the next segment of the proof of Theorem 4.1 it is shown that A has strict comparison of positive elements by K. The same arguments can be used to show that, in the present case, the strict comparison by K holds for *full* positive elements; i.e., assuming that c and d are full. We finish the proof as before, now relying on Lemma 3.5, rather than Lemma 3.4. □

We deduce from the previous theorems the following corollaries:

Corollary 4.3. *There exists $N \in \mathbb{N}$ such that if A is a* C^**-algebra with strict comparison of positive elements by traces and $h \in A$ is such that $h \sim_{\mathrm{Tr}} 0$, then*

$$\Big\| h - \sum_{i=1}^{N} [x_i, y_i] \Big\| \leqslant \frac{1}{2} \|h\|$$

for some $x_i, y_i \in A$ such that $\|x_i\| \cdot \|y_i\| \leqslant \|h\|$ for all i.

Proof. Let us suppose for the sake of contradiction that no such N exists. Then there exist C*-algebras $A_1, A_2, A_3, \ldots$ with strict comparison by traces and contractions $h_n \in A_n$ such that $h_n \sim_{\mathrm{Tr}} 0$ and the distance from h_n to elements of the form $\sum_{i=1}^{n} [u_i, v_i]$, with $\|u_i\|, \|v_i\| \leqslant 1$ for all i, is at least $1/2$ for all $n \in \mathbb{N}$. Let $h = (h_n) \in \prod_{n=1}^{\infty} A_n$. By Theorem 4.1, $h \sim_{\mathrm{Tr}} 0$. Hence, $\|h - \sum_{j=1}^{N} [x_j, y_j]\| < 1/2$ for some N and some $x_1, y_1, \ldots, x_N, y_N \in \prod_{n=1}^{\infty} A_n$. Increasing N if necessary, we may assume that $\|x_j\|, \|y_j\| \leqslant 1$ for all $j = 1, \ldots, N$. We get a contradiction projecting onto the N-th coordinate. □

The same proof, now relying on Theorem 4.2, yields the following corollary:

Corollary 4.4. *There exists $N' \in \mathbb{N}$ such that if A is a* C*-*algebra with* $\mathrm{Prim}(A)$ *compact and with strict comparison of full positive elements by traces, and $h \in A$ is such that $h \sim_{\mathrm{Tr}} 0$, then*

$$\Big\| h - \sum_{i=1}^{N'} [x_i, y_i] \Big\| \leqslant \frac{1}{2}\|h\|$$

for some $x_i, y_i \in A$ such that $\|x_i\| \cdot \|y_i\| \leqslant \|h\|$ for all i.

From these corollaries we deduce the theorem on traces of products and ultraproducts stated in the introduction:

Proof of Theorem 1.2. Imitate the proof of [20, Thm. 8], now relying on Corollary 4.3 or Corollary 4.4 instead of on [20, Thm. 6]. □

In order to obtain finite commutator bounds for a pure C*-algebra whose 2-quasitraces are traces, we intend to apply Theorem 2.9. We have already shown that condition (ii) of that theorem is met by this class of C*-algebras (in Corollary 4.3). In the next lemmas we establish the existence of a sequence of pairwise orthogonal positive elements as in Theorem 2.9 (i).

Recall that, given positive elements a and b, by $b \preceq a$ we mean that $b = x^*x$ and $xx^* \in \mathrm{her}(a)$ for some $x \in A$.

Lemma 4.5. *Let A be a pure* C*-*algebra and $a, b \in (A \otimes \mathcal{K})_+$. If $d_\tau(b) \leqslant \gamma d_\tau(a)$ for all $\tau \in \mathrm{QT}(A)$ and some $\gamma < 1/2$, then $b \preceq a$.*

Proof. The proof follows closely that of [4, Thm. 4.4.1], but we take care to remove the assumption on finite quotients needed there. First, using functional calculus, let us find $b_i, b_i', b_i'' \in C^*(b)_+$, with $i = 1, 2, \ldots$, such that

(i) $b_i' b_i = b_i$ and $b_i'' b_i' = b_i'$ for all i,
(ii) $b_i'' \perp b_j''$ for all i, j such that $i \neq j$ and $i - j$ is even,
(iii) $b = \sum_{i=1}^\infty b_i$,
(iv) $\|b_i'\| = 1$.

Since $b \preceq \bigoplus_{i=1}^\infty \frac{1}{i} b_i$, it suffices to show that $\bigoplus_{i=1}^\infty \frac{1}{i} b_i \preceq a$. Let us prove this.

We have that

$$\sum_{i=1}^\infty d_\tau(b_i'') \leqslant \sum_{i=1}^\infty d_\tau(b_{2i}'') + \sum_{i=1}^\infty d_\tau(b_{2i-1}'') \leqslant 2d_\tau(b) \leqslant 2\gamma d_\tau(a)$$

for all $\tau \in \mathrm{QT}(A)$. Since A is pure, for each $N \in \mathbb{N}$ there exists $d \in (A \otimes \mathcal{K})_+$ such that $N[d] \leqslant [a] \leqslant (N+1)[d]$. By choosing N large enough we can arrange that $d_\tau(b_1'' \oplus d) \leqslant \gamma_1 d_\tau(a)$ for some $\gamma_1 \in (2\gamma, 1)$ and all $\tau \in \mathrm{QT}(A)$. It follows by the strict comparison property of A that $[b_1'' \oplus d] \leqslant [a]$. Notice also that, by our choice of d, we have that $[b_1''] \leqslant k[d]$ for some $k \in \mathbb{N}$. Since $[b_1'] \ll [b_1'']$ (where $\ll$ is the relation of compact containment in $\mathrm{Cu}(A)$), there exists $\varepsilon > 0$ such that $[b_1'] \leqslant k[(d-\varepsilon)_+]$. Let $d' = (d-\varepsilon)_+$. We then have that $b_1' \oplus d' \preceq (a-\delta)_+$ for some $\delta > 0$. Let $v \in (A \otimes \mathcal{K})^{**}$ be a partial isometry implementing this subequivalence. Let $c_1', e \in \mathrm{her}((a-\delta)_+)$ be given by $c_1' = v b_1' v^*$ and $e = v d' v^*$. Let $g_\delta(a) \in C^*(a)$ be strictly positive and such that $g_\delta(a)(a-\delta)_+ = (a-\delta)_+$. Notice that $g_\delta(a)$ acts as a multiplicative unit for every element in $\mathrm{her}((a-\delta)_+)$. In particular, $g_\delta(a)c_1' = c_1'$. Since c_1' is a contraction (because b_1' is one), we have that $g_\delta(a) - c_1' \geqslant 0$. Let us set $a_1 = g_\delta(a) - c_1'$. Then

$$\sum_{i=2}^{\infty} d_\tau(b_i'') + d_\tau(c_1') \leqslant 2\gamma d_\tau(a) \leqslant 2\gamma d_\tau(a_1) + d_\tau(c_1')$$

for all $\tau \in \mathrm{QT}(A)$. If $d_\tau(c_1') < \infty$, we get

$$\sum_{i=2}^{\infty} d_\tau(b_i'') \leqslant 2\gamma d_\tau(a_1), \tag{7}$$

Suppose that $d_\tau(c_1') = \infty$. Then $d_\tau(b_1') = d_\tau(c_1') = \infty$. Since $[b_1'] \leqslant k[d']$, we also have $d_\tau(e) = d_\tau(d') = \infty$. But $a_1 e = (g_\delta(a) - c_1')e = e$. Hence, $d_\tau(a_1) = \infty$. So again we have (7). Let $c_1 = vb_1v^*$ and notice that $a_1 \perp c_1$. We can repeat the same arguments, now finding positive elements $c_2, a_2 \in \mathrm{her}(a_1)$ such that $b_2 \sim c_2$, $a_2 \perp c_2$ and $\sum_{i=3}^{\infty} d_\tau(b_i'') \leqslant 2\gamma d_\tau(a_2)$. Continuing this process ad infinitum, we obtain $c_1, c_2, \ldots \in \mathrm{her}(a)$ such that $b_i \sim c_i$ for all i and $c_i \perp c_j$ for all $i \neq j$. Hence,

$$\bigoplus_{i=1}^{\infty} \frac{1}{i} b_i \sim \sum_{i=1}^{\infty} \frac{1}{i} c_i \in \mathrm{her}(a),$$

which proves the lemma. □

The previous lemma implies that in a pure C*-algebra the ordered semigroup $W(A)$ is hereditary in $\mathrm{Cu}(A)$. This will not be needed later on but has independent interest. Recall that $\mathrm{W}(A)$ is defined as

$$\mathrm{W}(A) = \Big\{[a] \in \mathrm{Cu}(A) \mid a \in \bigcup_{n=1}^{\infty} M_n(A)\Big\}.$$

Corollary 4.6. *Let A be a pure* C**-algebra. The ordered semigroup* $\mathrm{W}(A)$ *is a hereditary (in the order-theoretic sense) subsemigroup of* $\mathrm{Cu}(A)$.

Proof. Let $a \in A \otimes \mathcal{K}$ and $b \in M_n(A)$ be positive elements such that $[a] \leqslant [b]$. By Lemma 4.5, $a \preceq b^{\oplus 3} \in M_{3n}(B)$. Hence $a \sim a' \in \mathrm{her}(b^{\oplus 3}) \subseteq M_{3n}(B)$, which in turn implies that $[a] = [a'] \in \mathrm{W}(A)$. □

In the following lemma we make use of the abundance of *soft* elements in the Cuntz semigroup of a pure C*-algebra (see [1]). It will not be necessary here to recall their definition and multiple properties. We will merely need the following fact:

Let A be a pure C**-algebra and $[a] \in \mathrm{Cu}(A)$. Then there exists $[a]_s \in \mathrm{Cu}(A)$ such that $[a]_s \leqslant [a]$, $d_\tau([a]_s) = d_\tau([a])$ for all $\tau \in \mathrm{QT}(A)$, and $[a]_s$ is exactly divisible by all $n \in \mathbb{N}$; i.e., for each $n \in \mathbb{N}$ there exists $[b] \in \mathrm{Cu}(A)$ such that $n[b] = [a]_s$.*

A proof of this fact can be extracted from [1] as follows: By [1, Thm. 7.3.11], the Cuntz semigroup of a pure C*-algebra has Z-multiplication (in the sense of [1, Def. 7.1.3]). Here Z denotes the Cuntz semigroup of the Jiang–Su C*-algebra $\mathcal{Z}$. Then $[a]_s = 1' \cdot [a]$, with $1' \in Z$ denoting the "soft" 1, has the desired properties. See [1, Prop. 7.3.16].

Lemma 4.7. *Let A be a σ-unital pure* C*-*algebra and $e_0 \in A_+$ a strictly positive element. Then there exist pairwise orthogonal positive elements $e_1, e_2, \ldots \in A_+$ such that $e_i \preceq e_{i+1}^{\oplus 5}$ for all $i \geqslant 1$ and $e_0 \preceq e_1^{\oplus 11}$.*

Proof. Let $[e_0]_s \leqslant [e_0]$ be the soft element associated to the Cuntz semigroup element $[e_0]$. Let us find $[f_1], [f_2], \ldots \in \mathrm{Cu}(A)$ such that $5[f_1] = [e_0]_s$ and $2[f_{i+1}] = [f_i]$ for all $i \geqslant 1$. Let $f \in (A \otimes \mathcal{K})_+$ be given by $f = f_1 \oplus \frac{1}{2} f_2 \oplus \frac{1}{3} f_3 \oplus \cdots$. Then

$$d_\tau(f_i) = \frac{2}{5} d_\tau(f_{i+1}^{\oplus 5}),$$
$$d_\tau(e_0) = \frac{5}{11} d_\tau(f_1^{\oplus 11}),$$
$$d_\tau(f) = \frac{2}{5} d_\tau(e_0),$$

for all $\tau \in \mathrm{QT}(A)$. Hence, by Lemma 4.5, $f_i \preceq f_{i+1}^{\oplus 5}$, $e_0 \preceq f_1^{\oplus 11}$, and $f \preceq e_0$. Let $v \in (A \otimes \mathcal{K})^{**}$ be the partial isometry implementing the comparison $f \preceq e_0$. Then, the positive elements $e_i = v f_i v^*$, with $i = 1, 2, \ldots$, have the desired properties. □

Let us now prove Theorem 1.1 from the introduction.

Proof of Theorem 1.1. We first prove that every C*-algebra A as in the theorem has finite commutator bounds with no approximations. We then reduce the number of commutators to 7.

Let $h \in \overline{[A, A]}$. Then $\sum_{i=1}^{k_n} [x_i^{(n)}, y_i^{(n)}] \to h$ for some $x_i^{(n)}, y_i^{(n)} \in A$. Passing to the hereditary C*-subalgebra $\mathrm{her}(\{x_i^{(n)}, y_i^{(n)} \mid i = 1, \ldots, k_n,\ n = 1, \ldots\})$ if necessary, we may assume that A is σ-unital (since hereditary subalgebras of pure C*-algebras are again pure). Let $e_0 \in A_+$ be a strictly positive element. By Lemma 4.7, A contains a sequence of pairwise orthogonal positive elements $(e_i)_{i=1}^\infty$ such that $e_i \preceq e_{i+1}^{\oplus 7}$ for all i and $e_0 \preceq e_1^{\oplus 11}$. Furthermore, Corollary 4.3 is applicable to every hereditary subalgebra of A. Thus, Theorem 2.9 is applicable to A. That is, A has finite commutator bounds (n, C) with no approximations, for some $n \in \mathbb{N}$ and $C > 0$.

Let us now reduce the number of commutators to 7. Since A is pure, we can find $b \in (A \otimes \mathcal{K})_+$ such that $(2n+1)[b] \leqslant [e_0] \leqslant (2n+2)[b]$ (with n as in the end of the previous paragraph). Then

$$d_\tau(b^{\oplus n}) \leqslant \frac{n}{2n+1} d_\tau(e_0) \quad \text{and} \quad d_\tau(e_0) \leqslant \frac{2n+2}{5n} d_\tau(b^{\oplus 5n}).$$

Hence, by Lemma 4.5, there exists $f \in A_+$ such that $b^{\oplus n} \sim f$ and $e_0 \preceq f^{\oplus 5}$. Now let $h \in A$ be such that $h \sim_{\mathrm{Tr}} 0$. From Lemma 2.8 we obtain that $h = \sum_{i=1}^5 [x_i, y_i] + h'$ for some $h' \in \mathrm{her}(f)$ such that $h' \sim_{\mathrm{Tr}} 0$ in $\mathrm{her}(f)$. But $\mathrm{her}(f) \cong M_n(\mathrm{her}(b))$. Thus, by Theorem 2.7 (ii), $h' = [x_6, y_6] + [x_7, y_7]$.

Furthermore, the sum $\sum_{i=1}^{7}\|x_i\|\cdot\|y_i\|$ is bounded by $C'\|h\|$ for some $C'>0$, as can be seen from the statements of Lemma 2.8 and Theorem 2.7. □

In Theorem 1.1, it is possible to reduce further the number of commutators under a variety of additional assumptions. We show in Theorem 4.10 below that if the C*-algebra is assumed to be unital, then three commutators suffice. We need a couple of lemmas.

Lemma 4.8. *Let $d\in M(A)$ be a multiplier positive contraction satisfying that $d\oplus d\oplus d\precsim 1-d$ in $M(A)$. Then for each $h\in A$ there exist $x,y\in A$ and $h'\in\mathrm{her}(1-d)$ such that $h=[x,y]+h'$ and $\|x\|\cdot\|y\|\leqslant 2\|h\|$.*

Proof. We have $h=dh+(1-d)hd+(1-d)h(1-d)$. Let

$$\begin{aligned} f&=dh+(1-d)hd,\\ e&=d+h^*d^2h+(1-d)hd^2h^*(1-d). \end{aligned}$$

Since $dh\in\mathrm{her}(d+h^*d^2h)$ and $(1-d)hd\in\mathrm{her}(d+(1-d)hd^2h^*(1-d))$, we have that $f\in\mathrm{her}(e)$. Also, $e\precsim d\oplus d\oplus d\precsim 1-d$. We can apply Lemma 2.8 (with $n=1$) to $f\in\mathrm{her}(e)$. We get $f=[x,y]+f'$, with $f'\in\mathrm{her}(1-d)$ and $\|x\|\cdot\|y\|\leqslant\|f\|\leqslant 2\|h\|$. Hence, $h=[x,y]+(1-d)h(1-d)+f'$, which proves the lemma. □

Let $\mathcal{Z}_{n-1,n}$ denote the dimension drop C*-algebra:

$$\mathcal{Z}_{n-1,n}=\{f\in M_{(n-1)n}(C[0,1])\mid f(0)\in M_{n-1}\otimes 1_n,\ f(1)\in M_n\otimes 1_{n-1}\}.$$

Proposition 4.9. *Let A be a pure unital C*-algebra. Then for all $n\in\mathbb{N}$ there exists a unital homomorphism $\phi\colon\mathcal{Z}_{n-1,n}\to A$.*

Proof. Let $n\in\mathbb{N}$. Since $\mathrm{Cu}(A)$ is almost divisible, we can find $[a]\in\mathrm{Cu}(A)$ such that $n[a]\leqslant[1]$ and $[1]\leqslant(n+1)[(a-\varepsilon)_+]$ for some $\varepsilon>0$. In turn, this implies that there exist pairwise orthogonal positive elements $b_1,b_2,\dots,b_n\in A_+$ such that $(a-\frac{\varepsilon}{2})_+\sim b_1\sim\cdots\sim b_n$. Using functional calculus, let us modify b_i so that there exists $b_i'\in\mathrm{her}(b_i)_+$, with $b_ib_i'=b_i'$ and $b_i'\sim(a-\varepsilon)_+$ for all i. Set $\sum_{i=1}^{n}b_i=b$. Then $(1-b)b_i'=(1-b_i)b_i'=0$ for all i. That is, $1-b\perp b_i'$ for all i. We have

$$d_\tau(1-b)+nd_\tau(b_1')=d_\tau(1-b)+d_\tau\Big(\sum_{i=1}^{n}b_i'\Big)\leqslant d_\tau(1)\leqslant(n+1)d_\tau(b_1')$$

for all $\tau\in\mathrm{QT}(A)$. Hence,

$$d_\tau(1-b)\leqslant d_\tau(b_1')=\frac{1}{3}d_\tau(b_1'\oplus b_2'\oplus b_3').$$

By Lemma 4.5, this implies that $1-b\precsim b_1'\oplus b_2'\oplus b_3'$. Let us assume now that $n=3k$ for some $k\in\mathbb{N}$. By [27, Prop. 5.1], there exists a unital homomorphism from the dimension drop C*-algebra $\mathcal{Z}_{k,k+1}$ into A. □

Theorem 4.10. *Let A be a pure* C*-*algebra with compact* $\mathrm{Prim}(A)$ *and with strict comparison of full positive elements by traces. Then A has commutator bounds $(7, C_1)$, with no approximations, for some universal constant C_1. If A is unital, then it has commutator bounds $(3, C_2)$, with no approximations, for some universal constant C_2.*

Proof. Let us first show that A has finite commutator bounds. To this end, we proceed as in the proof of Theorem 1.1, but with a few small modifications. (The main difference with Theorem 1.1 being that now we only require strict comparison by traces on full positive elements.) Lemma 4.7 is applicable to A, yielding a sequence of pairwise orthogonal positive elements $(e_i)_{i=1}^{\infty}$ such that $e_0 \preceq e_1^{\oplus 11}$ and $e_i \preceq e_{i+1}^{\oplus 7}$ for all i. Here $e_0 \in A_+$ is strictly positive. The hereditary subalgebras $\mathrm{her}(e_i)$ have compact primitive spectrum for all i (since they are full in B). Hence, Corollary 4.4 is applicable in each of them. Now Theorem 2.9 implies that A has finite commutator bounds (n, C) with no approximations.

The arguments for reducing the number of commutators to 7 in the proof of Theorem 1.1 apply here as well.

Let us now show that if A is unital, then the number of commutators can be reduced to 3. By Proposition 4.9, the dimension drop C*-algebra $\mathcal{Z}_{n,n+1}$ maps unitally into A. By [27, Prop. 5.1], there exist mutually orthogonal $f_1, \dots, f_n \in A_+$ such that $f_1 \sim f_i$ for all i and $1 - \sum_{i=1}^{n} f_i \preceq (f_1 - \varepsilon)_+$ for some $\varepsilon > 0$. Let us assume without loss of generality that $n \geqslant 3$. By Lemma 4.8, for each $h \in \overline{[A, A]}$ we have $h = [x_1, y_1] + h'$, with $h' \in \mathrm{her}(f_1, \dots, f_n)$. But $\mathrm{her}(f_1, \dots, f_n) \cong M_n(\mathrm{her}(f_1))$, and $\mathrm{her}(f_1)$ has commutator bounds (n, C) with no approximations. It follows by Theorem 2.7 that $h' = [x_2, y_2] + [x_3, y_3]$. □

Example 4.11. Let (A_i, τ_i), with $i = 1, 2, \dots$, be unital C*-algebras with faithful tracial states. Assume that for infinitely many indices $i_1, i_2, \dots$ there exist unitaries $u_{i_n} \in A_{i_n}$ such that $\tau_{i_n}(u_{i_n}) = 0$ for all n. Let $A = A_1 * A_2 * \cdots$ and $\tau = \tau_1 * \tau_2 * \cdots$ be the reduced free product C*-algebra and tracial state. It is known that A is simple and τ is its unique tracial state (by [3]). Furthermore, by [23, Prop. 6.3.2], A has strict comparison of positive elements by the trace τ. It follows that A is pure and, by Theorem 3.6, that the only bounded 2-quasitraces on A are the scalar multiples of τ. By Theorem 4.10, if $h \in A$ is such that $\tau(h) = 0$, then h is a sum of three commutators.

5. Sums of nilpotents of order 2

Let A be a C*-algebra. Let $N_2 = \{x \in A \mid x^2 = 0\}$; i.e., N_2 denotes the set of nilpotent elements of order 2 in A (a.k.a, square zero elements).

Lemma 5.1. *Let $z \in N_2$. Then $z = [u, v]$ and $z + z^* = [w^*, w]$ for some $u, v, w \in A$.*

Proof. We may assume that $\|z\| = 1$. The universal C*-algebra generated by a square zero contraction is $M_2(C_0(0,1])$. Thus, there exists a homomorphism $\phi\colon M_2(C_0(0,1]) \to A$ such that $\tilde{z} := \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right) \overset{\phi}{\mapsto} z$. So it suffices to express $\tilde{z}$ and $\tilde{z} + \tilde{z}^*$ as commutators. Indeed,

$$\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} = \left[\begin{pmatrix} t^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & t^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}\right],$$

$$\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} = \frac{1}{4}\left[\begin{pmatrix} t^{\frac{1}{2}} & -t^{\frac{1}{2}} \\ t^{\frac{1}{2}} & -t^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} t^{\frac{1}{2}} & t^{\frac{1}{2}} \\ -t^{\frac{1}{2}} & -t^{\frac{1}{2}} \end{pmatrix}\right]. \qquad \square$$

The preceding lemma implies that the linear span of N_2 is contained in $[A, A]$. In Theorem 5.3 below we show, conversely, that if A is a pure unital C*-algebra, then every commutator is expressible as a sum of at most 14×256 order 2 nilpotents.

Lemma 5.2. *Let A be a* C*-*algebra and $a, b \in A$.*

(i) *If $a, b \in N_2$, then $[a, b]$ is a sum of three order 2 nilpotents.*
(ii) *If $A = M_2(B)$ or $A = M_3(B)$, then $[a, b]$ is at most a sum of 14 order 2 nilpotents.*

In both cases, the norm of the nilpotents is bounded by $C\|a\| \cdot \|b\|$, where C is a universal constant.

Proof. (i) Let $a, b \in N_2$. Normalizing a and b if necessary, we may assume that they are contractions. Then, as pointed out in [14, Lem. 3.2],

$$[a, b] = (ab + aba - b - ba) + (-aba) + (b),$$

and each term on the right-hand side is an order 2 nilpotent of norm at most 4.

(ii) This is proven by Marcoux in [14, Thm. 5.6 (ii)] for $n = 2$, and in [14, Thm. 3.5 (ii)] for $n = 3$ (see also the remarks after [15, Thm. 5.1]). Although in the statements of these theorems Marcoux assumes that B is unital, a quick inspection of the proofs reveals that this is not used. $\square$

Theorem 5.3. *Let A be a pure unital* C*-*algebra and $a, b \in A$.*

(i) *Then $[a, b]$ is a sum of at most 14×256 nilpotents of order 2. The norm of the order 2 nilpotents is bounded by $C\|a\| \cdot \|b\|$, where C is a universal constant.*
(ii) *If $[a, b]$ is selfadjoint, then it is a sum of 14×256 commutators of the form $[x^*, x]$ with $x \in N_2$, with $\|x\|^2 \leqslant C'\|a\| \cdot \|b\|$ and C' is a universal constant.*

Proof. (i) Let us choose s_1 and s_2 such that $0 < s_1 < s_2 < 1$. Let $f_1, f_2, f_3, f_4 \in C([0,1])_+$ be functions such that f_1 is supported on $[0, s_1)$, f_2 is supported on $(0, s_2)$, f_3 is supported on $(s_1, 1)$, f_4 is supported on $(s_2, 1]$, and $f_1 + f_2 + f_3 + f_4 = 1$. Let us regard $C([0,1])$ embedded in $\mathcal{Z}_{2,3}$ via the map $f \mapsto f \cdot 1_6$. Further, by Proposition 4.9, $\mathcal{Z}_{2,3}$ maps unitally into A. In this way, we can view f_1, f_2, f_3, f_4 as elements of A. Then

$$[a,b] = \sum_{1 \leqslant i,j,k,l \leqslant 4} [f_i a f_j, f_k b f_l].$$

We will show that each of the 256 terms on the right-hand side is expressible as a sum of at most 14 nilpotents of order 2.

Let us examine the commutator $[f_i a f_j, f_k b f_l]$. Since $\mathrm{her}_A(f_1 + f_2 + f_3) \cong M_2(B)$ and $\mathrm{her}_A(f_2 + f_3 + f_4) \cong M_3(B')$ for some $B, B' \subseteq A$, if not all four functions appear in $[f_i a f_j, f_k b f_l]$, then by Lemma 5.2 this commutator is expressible as a sum of at most 14 order 2 nilpotents.

Let us examine the commutators $[f_i a f_j, f_k b f_l]$ where all four functions appear; i.e., such that $\{i,j,k,l\} = \{1,2,3,4\}$. Let us assume that $i = 1$. If $l = 3$ or $l = 4$, then $[f_i a f_j, f_k b f_l]$ is itself an order 2 nilpotent and we are done. Suppose that $l = 2$. Since we are assuming that all four indices must appear, either $k = 4$ and $j = 3$, or $k = 3$ and $j = 4$. Suppose that $k = 4$ and $j = 3$. Since $f_1 a f_3$ and $f_4 b f_2$ are both order 2 nilpotents, $[f_1 a f_3, f_4 b f_2]$ is a sum of three order 2 nilpotents by Lemma 5.2 (i). Suppose now that $k = 3$ and $j = 4$. The commutator is then $[f_1 a f_4, f_3 b f_2]$, which can be dealt with as follows:

$$\begin{aligned}[f_1 a f_4, f_3 b f_2] = {} & [f_1 a (f_4 f_3)^{\frac{1}{2}}, (f_4 f_3)^{\frac{1}{2}} b f_2] \\ & + [(f_4 f_3)^{\frac{1}{2}} b (f_2 f_1)^{\frac{1}{2}}, (f_2 f_1)^{\frac{1}{2}} a (f_4 f_3)^{\frac{1}{2}}] \\ & + [(f_2 f_1)^{\frac{1}{2}} a f_4, f_3 b (f_2 f_1)^{\frac{1}{2}}].\end{aligned}$$

Each of the commutators on the right side is a commutator of order 2 nilpotents and is thus expressible as a sum of three order 2 nilpotents. So, $[f_1 a f_4, f_3 b f_2]$ is expressible as a sum of 9 nilpotents of order 2.

Let us assume now that $i \neq 1$. As argued above, we may reduce ourselves to the case that one of the indices j, k, l is 1. On the grounds of the symmetry of our set-up, any of these cases can be dealt with just as we did above for $i = 1$. (Notice that only the orthogonality relations between the functions were used in our analysis; the asymmetry of the dimension drop C*-algebra played no role.) We are thus done.

(ii) Suppose that $[a,b]$ is selfadjoint. By (i), it can be written as a sum with 14×256 terms, each of the form $z + z^*$, with $z \in N_2$. In turn, each of these terms is expressible as a commutator of the form $[x^*, x]$, with $x \in N_2$, by Lemma 5.1. □

Theorem 5.4. *Let A be a pure unital* C*-*algebra whose bounded* 2-*quasitraces are traces. Then the sets $\overline{[A,A]}$, $[A,A]$, and the linear span of N_2, are equal. Each $h \in A$ such that $h \sim_{\mathrm{Tr}} 0$ is expressible as a sum of* $3 \times 14 \times 256$ *square zero elements. If h is selfadjoint, then it is also expressible as sum of* $3 \times 14 \times 256$ *commutators of the form $[x^*, x]$, with $x \in N_2$.*

Proof. The assumptions on A imply that it has strict comparison of full positive elements by traces (see Remark 3.7). By Theorem 4.10, each $h \in \overline{[A,A]}$ is expressible as a sum of three commutators. Each of these commutators, in turn, is expressible as a sum of 14×256 square zero elements. If h is selfadjoint, then it is also expressible as a sum of 14×256 terms of the form $z + z^*$, with $z \in N_2$, and each of these is a commutator $[x^*, x]$ with $x \in N_2$. □

6. The kernel of the determinant map

Let us briefly recall the definition of the de la Harpe–Skandalis determinant, as defined in [10]. Let A be a C*-algebra. Let $\mathrm{GL}_\infty(A)$ denote the infinite general linear group of A and $\mathrm{GL}_\infty^0(A)$ the connected component of the identity. (If A is nonunital, the general linear groups $\mathrm{GL}_n(A)$ are defined as the subgroup of $\mathrm{GL}_n(A^\sim)$ of elements of the form $1_n + x$, with $x \in M_n(A)$, and $\mathrm{GL}_\infty(A)$ is the direct limit.) Let $x \in \mathrm{GL}_\infty^0(A)$. Let $\eta\colon [0,1] \to \mathrm{GL}_\infty^0(A)$ be a path such that $\eta(0) = 1$ and $\eta(1) = x$. Let

$$\widetilde{\Delta}_\eta(x) = \frac{1}{2\pi i}\mathrm{Tr}\Big(\int_0^1 \eta'(t)\eta(t)^{-1}\Big) \in A/\overline{[A,A]}.$$

If ζ is another path connecting 1 to x, then $\widetilde{\Delta}_\eta(x) - \widetilde{\Delta}_\zeta(x)$ belongs to the (additive) subgroup $\{\mathrm{Tr}(p) - \mathrm{Tr}(q) \mid p, q \text{ projections in } M_\infty(A)\}$, which we denote by $\underline{\mathrm{Tr}}(\mathrm{K}_0(A))$. The de la Harpe–Skandalis determinant $\Delta_{\mathrm{Tr}}(x)$ is defined as the image of $\widetilde{\Delta}_\eta(x)$ in the quotient of $A/\overline{[A,A]}$ by $\underline{\mathrm{Tr}}(\mathrm{K}_0(A))$. If $x = \prod_{k=1}^n e^{ih_k}$, then $\Delta_{\mathrm{Tr}}(x)$ can be computed to be the image of $\sum_{k=1}^n h_k$ in this quotient.

Let $\mathrm{U}(A)$ denote the unitary group of A and $\mathrm{U}^0(A)$ the connected component of the identity. (If A is nonunital, $\mathrm{U}(A)$ is defined as the unitaries in $\mathrm{U}(A^\sim)$ of the form $1 + x$, with $x \in A$.) Given unitaries $u, v \in \mathrm{U}^0(A)$, let us denote by (u, v) the multiplicative commutator $uvu^{-1}v^{-1}$. Let $\mathrm{DU}^0(A)$ denote the derived or commutator subgroup of $\mathrm{U}^0(A)$. It is clear that $\mathrm{DU}^0(A)$ is contained in the kernel of Δ_{Tr} (since Δ_{Tr} is a group homomorphism with abelian codomain). In this section we prove the following theorem:

Theorem 6.1. *Let $A = M_3(B)$, where B is a unital pure* C*-*algebra whose bounded 2-quasitraces are traces. Then* $\ker \Delta_{\mathrm{Tr}} \cap \mathrm{U}^0(A) = \mathrm{DU}^0(A)$.

The proof is preceded by a number of lemmas.

Lemma 6.2. *Let A be a pure unital* C*-*algebra and $p \in M_m(A)$ a projection. Then there exists $h \in A_{\mathrm{sa}}$ such that $h \sim_{\mathrm{Tr}} p$ and $e^{ih} = (u, v)$ for some unitaries $u, v \in \mathrm{U}^0(A)$.*

Proof. Let $B = pM_m(A)p$. Choose $n \in \mathbb{N}$. Since B is pure and unital, there exists a unital homomorphism $\phi\colon \mathcal{Z}_{n-1,n} \to B$, by Proposition 4.9. Let $e \in \mathcal{Z}_{n-1,n}$ be a positive element such that $\mathrm{rank}(e(t)) = n$ for all $t \in (0, 1]$ and $\mathrm{rank}(e(0)) = n - 1$ (so that $(n - 1)[e] \leqslant [1] \leqslant n[e]$ in the Cuntz semigroup of $\mathcal{Z}_{n-1,n}$). In the proof of [18, Lem. 5.4] a selfadjoint element $h \in \mathrm{her}(e)$ is constructed such that $h \sim_{\mathrm{Tr}} 1$ (in $\mathcal{Z}_{n-1,n}$) and $e^{ih} = (u, v)$ for some unitaries $u, v \in \mathrm{U}^0(\mathrm{her}(e))$. Moving these elements with the homomorphism ϕ, we get

$e' \in B_+$, $h' \in \mathrm{her}(e')_{\mathrm{sa}}$, and $u', v' \in U^0(\mathrm{her}(e'))$, such that

$$(n-1)[e'] \leqslant [p] \leqslant n[e'], \quad h' \sim_{\mathrm{Tr}} p, \quad \text{and} \quad e^{ih'} = (u', v').$$

Now choose $n > 2m$. Then $(2m+1)[e'] \leqslant [p] \leqslant m[1]$, where 1 is the unit of A. By Lemma 4.5, this implies that $e' \preceq 1$; i.e., there exists $x \in M_m(A)$ such that $x^*x = e'$ and $xx^* \in A$. Let $x = w|x|$ be the polar decomposition of x (in $M_n(A)^{**}$). Then the selfadjoint $h'' = wh'w^*$, and the unitaries $u'' = wuw^*$ and $v'' = wvw^*$ have the desired properties. □

Lemma 6.3. *Let A be a pure unital* C*-*algebra. Let $u \in \mathrm{U}^0(A)$ be such that $\Delta_{\mathrm{Tr}}(u) = 0$. Then $u = \prod_{j=1}^{M}(u_j, v_j) \cdot e^{ih}$ for some $u_1, v_1, \dots, u_M, v_M \in \mathrm{U}^0(A)$ and some $h \in A_{\mathrm{sa}}$ such that $h \sim_{\mathrm{Tr}} 0$.*

Proof. Since $u \in \mathrm{U}^0(A)$ we have $u = \prod_{j=1}^{n} e^{ih_j}$, where $h_1, \dots, h_n \in A_{\mathrm{sa}}$, and since $\Delta_{\mathrm{Tr}}(u) = 0$ we also have $\sum_{j=1}^{n} h_j \sim_{\mathrm{Tr}} p - q$ for some projections $p, q \in M_m(A)$. Applying the previous lemma, we can write

$$u = (u', v')(u'', v'') \prod_{j=1}^{n+2} e^{ih_j},$$

where now $\sum_{j=1}^{n+2} h_j \sim_{\mathrm{Tr}} 0$. It thus suffices to prove the lemma for the unitary $\prod_{j=1}^{n+2} e^{ih_j}$. Let $N \in \mathbb{N}$. Then

$$\prod_{j=1}^{n+2} e^{ih_j} = \prod_{j=1}^{n+2} (e^{ih_j/N})^N = \prod_{j=1}^{M} (u_j, v_j) \cdot \Big(\prod_{j=1}^{n+2} e^{ih_j/N}\Big)^N.$$

Here the commutators (u_j, v_j) result simply from rearranging the factors of the first product. In particular, $u_j, v_j \in \mathrm{U}^0(A)$ for all j. We can choose N large enough so that $\prod_{j=1}^{n+2} e^{ih_j/N} = e^{ih}$ for some $h \in A_{\mathrm{sa}}$. By [10, Lem. 3 (b)], the trace of the logarithm of a product of $n+2$ unitaries belonging to a sufficiently small neighborhood of the identity is equal to the sum of the trace of the logarithm of each of the unitaries. Thus, for N large enough, we have that $h \sim_{\mathrm{Tr}} \sum_{j=1}^{n+2} h_j/N \sim_{\mathrm{Tr}} 0$. Therefore,

$$u = \prod_{j=1}^{M} (u_j, v_j) \cdot e^{iNh},$$

with $h \in A_{\mathrm{sa}}$ such that $h \sim_{\mathrm{Tr}} 0$. This proves the lemma □

Lemma 6.4. *Let $m \in \mathbb{N}$, $R > 0$, and $\varepsilon > 0$. Then there exists $M \in \mathbb{N}$ with the following property: If A is a* C*-*algebra and $a_1, \dots, a_m \in A_{\mathrm{sa}}$ are such that $\|a_i\| \leqslant R$ for all i, then there exist $x_1, y_1, \dots, x_M, y_M \in A_{\mathrm{sa}}$ and $c \in A_{\mathrm{sa}}$ such that*

$$e^{i(a_1 + \cdots + a_m)} = \prod_{k=1}^{M} (e^{ix_k}, e^{iy_k}) \cdot e^{ia_1} \cdots e^{ia_m} \cdot e^{ic},$$

$$\|x_k\|, \|y_k\| \leqslant \varepsilon \cdot \sum_{j=1}^{m} \|a_j\| \quad \textit{for } k = 1, \dots, M,$$

$$\|c\| \leqslant \varepsilon \cdot \sum_{j=1}^{m} \|a_j\|, \quad \textit{and} \quad c \sim_{\mathrm{Tr}} 0.$$

Proof. By [28, Thm. 2], for $\lambda \in \mathbb{R}$ such that $|\lambda| < \frac{1}{mR} \cdot (\ln 2 - \frac{1}{2})$ we have

$$e^{i\lambda a_1 + \dots + i\lambda a_m} = e^{i\lambda a_1} e^{i\lambda a_2} \cdots e^{i\lambda a_m} \cdot e^{ic(\lambda)},$$

where $\|c(\lambda)\| \leqslant L\lambda^2 \max_j \|a_j\|^2$ and the constant $L > 0$ is dependent on m and R only. Furthermore, by [10, Lem. 3 (b)], for $|\lambda|$ small enough (depending only on m and R), the trace of the logarithm of the right side is equal to $\sum_{j=1}^{m} i\lambda \mathrm{Tr}(a_j) + \mathrm{Tr}(c(\lambda))$; whence $c(\lambda) \sim_{\mathrm{Tr}} 0$. Now let us choose $\lambda = \frac{1}{N}$, with $N \in \mathbb{N}$. Then for N large enough (depending only on m and R) we have that

$$e^{ia_1/N + \dots + ia_m/N} = e^{ia_1/N} e^{ia_2/N} \cdots e^{ia_m/N} \cdot e^{c_N}, \tag{8}$$

where $c_N \in A_{\mathrm{sa}}$ satisfies that $c_N \sim_{\mathrm{Tr}} 0$ and

$$\|c_N\| \leqslant \frac{L}{N^2} \cdot \max_i \|a_i\|^2 \leqslant \frac{LR}{N^2} \cdot \max_i \|a_i\|.$$

Raising to the N on both sides of (8) we get

$$\begin{aligned} e^{ia_1 + \dots + ia_m} &= (e^{a_1/N} e^{a_2/N} \cdots e^{a_m/N} e^{c_N})^N \\ &= \prod_{k=1}^{M} (e^{ix_k}, e^{iy_k}) \cdot e^{a_1} e^{a_2} \cdots e^{a_m} e^{Nc_N}. \end{aligned}$$

The commutators (e^{ix_k}, e^{iy_k}) in the last expression result from rearranging the terms in $(e^{a_1/N} e^{a_2/N} \cdots e^{a_m/N} \cdot e^{c})^N$. Notice that M depends only on N and m. Choosing $N > \frac{1}{\varepsilon}$ we arrange for $\|x_k\|, \|y_k\| \leqslant \varepsilon \max_j \|a_j\|$. Choosing $N > \frac{LR}{\varepsilon}$ we also get that

$$\|Nc_N\| \leqslant \frac{LR}{N} \cdot \max_i \|a_i\| \leqslant \varepsilon \max_j \|a_j\|. \qquad \square$$

Proposition 6.5. *There exists $N \in \mathbb{N}$ such that the following holds: If $A = M_2(B)$, where B is a pure* C**-algebra with compact* Prim(B) *and whose bounded* 2*-quasitraces are traces, and $h \in A_{\mathrm{sa}}$ is such that $h \sim_{\mathrm{Tr}} 0$ and $\|h\| \leqslant 1$, then*

$$e^{ih} = \prod_{j=1}^{N} (u_j, v_j) \cdot e^{ic}$$

for some $c \in A_{\mathrm{sa}}$ and some $u_1, v_1, \dots, u_N, v_N \in \mathrm{U}^0(A)$ such that

$$c \sim_{\mathrm{Tr}} 0, \quad \|c\| \leqslant \frac{1}{2}\|h\|,$$

and

$$\|u_j - 1\|, \|v_j - 1\| \leqslant \|e^{ih} - 1\|^{\frac{1}{2}} \quad \textit{for all } j = 1, \dots, N.$$

Proof. Let $h \in A_{\mathrm{sa}}$ be such that $h \sim_{\mathrm{Tr}} 0$ and $\|h\| \leqslant 1$. By Theorem 4.10, h is a sum of seven commutators, and by Lemma 5.2 (ii), each of these commutators is a sum of at most 14 nilpotents of order 2. Furthermore, since h is selfadjoint, we can assume that these nilpotent elements have the form $[z^*, z]$, with $z^2 = 0$. We thus have $h = \sum_{k=1}^{m}[z_k^*, z_k]$ for some $z_1, \ldots, z_m \in A$ such that $z_k^2 = 0$. Here $m = 7 \times 14$ and $\|z_k\|^2 \leqslant C'\|h\|$ for all k, where C' is a universal constant. However, dividing the elements z_k by a large natural number ($> C'^{1/2}$) and enlarging m, we can assume instead that $\|z_k\|^2 \leqslant \|h\|$ for all $k = 1, \ldots, m$. Let us assume this. Notice that now $\|[z_k, z_k^*]\| \leqslant \|h\| \leqslant 1$. By Lemma 6.4 applied with $\varepsilon = \frac{1}{2}$, m, and $R = \frac{1}{2}$, we have

$$e^{ih} = e^{i[z_1,z_1^*]+\cdots+i[z_m,z_m^*]} = \prod_{k=1}^{M}(e^{ix_k}, e^{iy_k}) \cdot \prod_{k=1}^{m} e^{i[z_k^*,z_k]} \cdot e^{ic},$$

where $x_1, y_1, \ldots, x_M, y_M \in A$ and $c \in A_{\mathrm{sa}}$ are such that $c \sim_{\mathrm{Tr}} 0$, $\|c\| \leqslant \frac{1}{2}\|h\|$, and $\|x_k\|, \|y_k\| \leqslant \frac{1}{2}\|h\|$ for all k. Notice that $\|e^{ix_k}-1\|, \|e^{iy_k}-1\| \leqslant \|e^{ih}-1\|^{1/2}$ for all $k = 1, \ldots, M$. It remains to show that the terms $e^{i[z_k,z_k^*]}$ are also expressible as commutators. By [18, Lem. 2.4 (ii)], for all $z \in A$ such that $z^2 = 0$ and $\|z\|^2 \leqslant \frac{\pi}{2}$ we have $e^{i[z^*,z]} = (u, v)$ for some unitaries $u, v \in \mathrm{U}^0(A)$ such that $\|u-1\|, \|v-1\| \leqslant \|e^{[z,z^*]-1}\|^{1/2}$. Applying this to each z_k, we get $e^{i[z_k,z_k^*]} = (u_k, v_k)$, where $u_k, v_k \in \mathrm{U}^0(A)$ are such that

$$\|u_k - 1\|, \|v_k - 1\| \leqslant \|e^{i[z_k,z_k^*]} - 1\|^{\frac{1}{2}} \leqslant \|e^{ih} - 1\|^{\frac{1}{2}}$$

for all $k = 1, \ldots, m$. □

Lemma 6.6. *Let A be a pure* C**-algebra with strictly positive element $d \in A_+$. Let $\varepsilon > 0$. Then there exist pairwise orthogonal positive elements such that $a_i \sim b_i$ and $b_i \preceq a_{i+1} + b_{i+1}$ for all i, and $[(d-\varepsilon)_+] \leqslant 11[a_1]$.*

Proof. Let $[d]_s \in \mathrm{Cu}(A)$ denote the "soft" element associated to $[d]$. Using that this element is infinitely divisible, we can find $[c_1], [c_2], [c_3], \ldots \in \mathrm{Cu}(A)$ such that $[c_i] = 2[c_{i+1}]$ for all i and $10[c_1] = [d_s]$. Let us pick the representatives c_i such that $\|c_i\| \to 0$. Since $[d] \leqslant 11[c_1]$, there exists $\delta_1 > 0$ such that $[(d-\varepsilon)_+] \leqslant 11[(c_1 - \delta_+1)]$. Let us continue choosing $\delta_2, \delta_3, \ldots$ such that

$$(c_i - \delta_i)_+ \preceq (c_{i+1} - \delta_{i+1})_+ \oplus (c_{i+1} - \delta_{i+1})_+$$

for all i. Consider the element

$$c = \bigoplus_{i=1}^{\infty}(c_i \oplus c_i) \in (A \otimes \mathcal{K})_+.$$

Notice that $d_\tau(c) \leqslant \frac{2}{5}d_\tau(d)$ for all $\tau \in \mathrm{QT}(A)$. By Lemma 4.5, $c \preceq d$. Let $v \in \mathrm{her}(d)^{**}$ denote the partial isometry implementing this subequivalence. Let us define

$$a_i = v((c_i - \delta_i)_+ \oplus 0)v^*,$$
$$b_i = v(0 \oplus (c_i - \delta_i)_+)v^*,$$

for all i. The elements $a_1, b_1, a_2, b_2, \dots$ are pairwise orthogonal. They satisfy $a_i \sim b_i$ and $b_i \preceq a_{i+1} + b_{i+1}$ for all i and $[(d-\varepsilon)_+] \leqslant 11[(c_1 - \delta_1)+] = 11[a_1]$, as desired. □

Lemma 6.7. *Let A be a pure* C*-*algebra with* Prim(A) *compact and whose bounded 2-quasitraces are traces. Let $a_1, b_1, a_2, b_2, \dots \in A_+$ be pairwise orthogonal positive elements as in the previous lemma. Let $h \in \mathrm{her}(a_1)$ be a selfadjoint element such that $h \sim_{\mathrm{Tr}} 0$. Then $e^{ih} \in \mathrm{DU}^0(A)$.*

Proof. The proof uses the multiplicative version of "Fack's technique", as applied in [18, Lem. 6.5].

Since $e^{ih} = (e^{ih/N})^N$ for all $N \in \mathbb{N}$ we can assume that $\|h\| < \delta$ for any prescribed δ. Let us choose δ such that [11, Prop. 5.18] is applicable to any unitary within a distance of at most δ of 1.

By Proposition 6.5 applied in her$(a_1 + b_1)$, there exist unitaries $u_i^{(1)}, v_i^{(1)}$ in $\mathrm{U}^0(\mathrm{her}(a_1 + b_1))$ for $i = 1, \dots, N$, such that

$$e^{ih} = \prod_{i=1}^{N} (u_i^{(1)}, v_i^{(1)}) e^{ih_1'},$$

where $h_1' \in \mathrm{her}(a_1 + b_1)_{\mathrm{sa}}$, $h_1' \sim_{\mathrm{Tr}} 0$, and $\|h_1\|' < \|h\|/2^1$.

Next, by [11, Prop. 5.18] (see also [18, Lem. 6.4]), there exist unitaries $w_1^{(1)}, x_1^{(1)}, w_2^{(1)}, x_2^{(1)}$ in $\mathrm{U}^0(\mathrm{her}(a_1 + b_1))$ such that

$$e^{ih_1'} = (w_1^{(1)}, x_1^{(1)})(w_2^{(1)}, x_2^{(1)}) e^{ih_1''},$$

and $h_1'' \in \mathrm{her}(b_2)_{\mathrm{sa}}$. Finally, by [11, Lem. 5.17] applied in her$(b_1 + a_2 + b_2)$, we have $e^{ih_1''} = (y^{(1)}, z^{(1)}) e^{ih_2}$, with $y^{(1)}, z^{(1)} \in \mathrm{U}^0(\mathrm{her}(b_1 + a_2 + b_2))$ and $h_2 \in \mathrm{her}(a_2 + b_2)_{\mathrm{sa}}$. Next, we apply again Proposition 6.5 in her$(a_2 + b_2)$:

$$e^{ih_2} = \prod_{i=1}^{N} (u_i^{(2)}, v_i^{(2)}) e^{ih_2'},$$

with $h_2' \sim_{\mathrm{Tr}} 0$ and $\|h_2'\| < \frac{1}{2^2}$, followed by applications of [11, Prop. 5.18] and [11, Lem. 5.17]:

$$\begin{aligned} e^{ih_2'} &= (w_1^{(2)}, x_1^{(2)})(w_2^{(2)}, x_2^{(2)}) e^{ih_2''}, \\ &= (w_1^{(2)}, x_1^{(2)})(w_2^{(2)}, x_2^{(2)})(y^{(2)}, z^{(2)}) e^{ih_3}, \end{aligned}$$

where $h_2'' \in \mathrm{her}(b_2)_{\mathrm{sa}}$ and $h_2'' \sim_{\mathrm{Tr}} 0$, and $h_3 \in \mathrm{her}(a_3 + b_3)_{\mathrm{sa}}$ and $h_3 \sim_{\mathrm{Tr}} 0$. Continuing this strategy, we construct, for each $n \in \mathbb{N}$,

(i) unitaries $u_1^{(n)}, v_1^{(n)}, \dots, u_N^{(n)}, v_N^{(n)}$ and $w_1^{(n)}, x_1^{(n)}, w_2^{(n)}, x_2^{(n)}$ in $U^0(\mathrm{her}(b_n))$,
(ii) unitaries $y^{(n)}, z^{(n)}$ in $\mathrm{U}^0(\mathrm{her}(b_n + a_{n+1} + b_{n+1})$, and
(iii) a selfadjoint $h_n \in \mathrm{her}(e_n)_{\mathrm{sa}}$,

such that $h_n \sim_{\mathrm{Tr}} 0$ and

$$e^{ih} = \prod_{k=1}^{n-1} \Big(\prod_{i=1}^{N} (u_i^{(k)}, v_i^{(k)}) \Big) (w_1^{(k)}, x_1^{(k)})(w_1^{(k)}, x_1^{(k)})(y^{(k)}, z^{(k)}) \cdot e^{ih_n}.$$

Notice that $h_n \to 0$. Thus, the above formula yields an expression of e^{ih} as an infinite product of commutators. By the pairwise orthogonality of the elements a_n and b_n, we can gather the terms of this infinite product into subsequences, each of them equal to a finite product of commutators. This is done in the same manner as in the proof of [11, Prop. 6.1]. First, we group together the commutators $(y^{(k)}, z^{(k)})$ in the product above:

$$e^{ih} = \prod_{k=1}^{n-1} \Big(\prod_{i=1}^{N} (\tilde{u}_i^{(k)}, \tilde{v}_i^{(k)}) \Big) (\tilde{w}_1^{(k)}, \tilde{x}_1^{(k)})(\tilde{w}_1^{(k)}, \tilde{x}_1^{(k)}) \prod_{k=1}^{n-1} (y^{(k)}, z^{(k)}) \cdot e^{ih_n},$$

where

$$\begin{aligned} \tilde{u}_i^{(k)} &= (y^{(k-1)}, z^{(k-1)}) u_i^{(k)} (y^{(k-1)}, z^{(k-1)})^{-1}, \\ \tilde{v}_i^{(k)} &= (y^{(k-1)}, z^{(k-1)}) v_i^{(k)} (y^{(k-1)}, z^{(k-1)})^{-1}, \\ \tilde{w}_j^{(k)} &= (y^{(k-1)}, z^{(k-1)}) w_j^{(k)} (y^{(k-1)}, z^{(k-1)})^{-1}, \\ \tilde{x}_j^{(k)} &= (y^{(k-1)}, z^{(k-1)}) x_j^{(k)} (y^{(k-1)}, z^{(k-1)})^{-1}, \end{aligned}$$

for all $i = 1, \dots, N$, $j = 1, 2$, and $k = 2, \dots, n$. Since $(y^{(k)}, z^{(k)}) - 1$ belongs to $\mathrm{her}(b_k) + \mathrm{her}(a_{k+1} + b_{k+1})$, the modified unitaries $\tilde{u}_i^{(k)}, \tilde{v}_i^{(k)}, \tilde{w}_j^{(k)}, x_j^{(k)}$ continue to belong to $\mathrm{U}^0(\mathrm{her}(a_k + b_k))$. Therefore,

(i) $\prod_{k=1}^{\infty} \big(\prod_{i=1}^{N} (\tilde{u}_i^{(k)}, \tilde{v}_i^{(k)}) \big) \cdot (\tilde{w}_1^{(k)}, \tilde{x}_1^{(k)}) \cdot (\tilde{w}_2^{(k)}, \tilde{x}_2^{(k)})$ is a product of $N + 2$ commutators,

(ii) $\prod_{k=1}^{\infty} (y^{(2k-1)}), z^{(2k-1)})$ is a single commutator,

(iii) $\prod_{k=1}^{\infty} (y^{(2k)}), z^{(2k)})$ is a single commutator.

We thus arrive at an expression of e^{ih} as a product of $N + 4$ commutators. □

Proof of Theorem 6.1. It is clear that every unitary in $\mathrm{DU}^0(A)$ is in $\ker \Delta_{\mathrm{Tr}}$. To prove the converse, it suffices, by Lemma 6.3, to show that e^{ih}, with $h \sim_{\mathrm{Tr}} 0$ is a finite product of commutators. Writing $e^{ih} = (e^{ih/N})^N$, we can assume that $\|h\| < \delta$ for any prescribed δ. We will specify how small should δ be soon. By [11, Prop. 5.18], e^{ih} is a product of commutators times $e^{ih'}$, with $h' \in \mathrm{her}(e_{1,1})$ and $h' \sim_{\mathrm{Tr}} 0$. Let us choose $d \in \mathrm{her}(e_{2,2} + e_{3,3})$ and $\varepsilon > 0$ such that

$$N[d] \leqslant [e_{2,2} + e_{3,3}] \leqslant (N + 1)[(d - 2\varepsilon)_+],$$

for N large enough (how large to be specified soon). Let us find pairwise orthogonal elements $d_1, \dots, d_N \in \mathrm{her}(e_{2,2} + e_{3,3})$ such that $d_i \sim (d - \varepsilon)_+$ for all i. Since $[e_{1,1}] \leqslant N[(d - 2\varepsilon)_+]$, we have $e_{1,1} \preceq (\sum_i d_i - \varepsilon)_+$. By repeated application of [11], we can express $e^{ih'}$ as a finite product of commutators times $e^{ih''}$, with $h'' \in \mathrm{her}((d_1 - \varepsilon)_+)$ selfadjoint such that $h'' \sim_{\mathrm{Tr}} 0$ (in $\mathrm{her}((d_1 - \varepsilon)_+)$). Let us choose a sequence $a_1, b_1, a_2, b_2 \ldots \in \mathrm{her}(d_2 + \cdots + d_{13})$ as in Lemma 6.6. Then $12[d_1] = [d_2 + \cdots + d_{13}] \leqslant 11[a_1]$. Thus, $(d_1 - \varepsilon)_+ \preceq a_1$. We can therefore express $e^{ih''}$ as a commutator times $e^{ih'''}$, with $h''' \in \mathrm{her}(a_1)$ and $h''' \sim_{\mathrm{Tr}} 0$. By Lemma 6.7, $e^{ih'''}$ is a finite product of commutators. □

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