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Analysis on Riemannian foliations of bounded geometry

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Tribute to Christopher Deninger for his 60th birthday

Abstract. A leafwise Hodge decomposition was proved by Sanguiao for Riemannian foliations of bounded geometry. Its proof is explained again in terms of our study of bounded geometry for Riemannian foliations. It is used to associate smoothing operators to foliated flows, and describe their Schwartz kernels. All of this is extended to a leafwise version of the Novikov differential complex.

Contents

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1. Introduction

Christopher Deninger has proposed a program to study arithmetic zeta functions by finding an interpretation of the so-called explicit formulas as a (dynamical) Lefschetz trace formula for foliated flows on suitable foliated spaces [16, 17, 18, 19, 20]. Hypothetically, the action of the flow on some reduced leafwise cohomology should have some Lefschetz distribution. Then the trace formula would describe it using local data from the fixed points and closed orbits. The precise expression of these contributions was previously suggested by Guillemin [27]. Further developments of these ideas were made in [22, 35, 36, 33].

Deninger's program needs the existence of foliated spaces of arithmetic nature, where the application of the trace formula has arithmetic consequences. Perhaps some generalization of foliated spaces should be considered. Anyway, to begin with, we consider a simple foliated flow $\phi = \{\phi^t\}$ on a smooth closed foliated manifold $(M, \mathcal{F})$. We assume that $\mathcal{F}$ is of codimension one and the orbits of ϕ are transverse to the leaves without fixed points.

The first two authors proved such a trace formula when $\{\phi^t\}$ has no fixed points [3]. A generalization for transverse actions of Lie groups was also given [4]. It uses the space $C^\infty(M; \Lambda\mathcal{F})$ of leafwise forms (smooth sections of $\Lambda\mathcal{F} = \bigwedge T\mathcal{F}^*$ over M), which is a differential complex with the leafwise derivative $d_\mathcal{F}$. Its reduced cohomology is denoted by $\bar{H}^*(\mathcal{F})$ (the leafwise reduced cohomology). Since ϕ is foliated, there are induced actions $\phi^* = \{\phi^{t*}\}$ on $C^\infty(M; \Lambda\mathcal{F})$ and $\bar{H}^*(\mathcal{F})$. In this case, $\mathcal{F}$ is Riemannian, and therefore it has a leafwise Hodge decomposition [2],

$$C^\infty(M; \Lambda\mathcal{F}) = \ker \Delta_\mathcal{F} \oplus \overline{\operatorname{im} d_\mathcal{F}} \oplus \overline{\operatorname{im} \delta_\mathcal{F}}, \tag{1}$$

where $\delta_\mathcal{F}$ and $\Delta_\mathcal{F}$ are the leafwise coderivative and leafwise Laplacian. Moreover, the leafwise heat operator $e^{-u\Delta_\mathcal{F}}$ defines a continuous map

$$C^\infty(M; \Lambda\mathcal{F}) \times [0, \infty] \to C^\infty(M; \Lambda\mathcal{F}), \quad (\alpha, t) \mapsto e^{-u\Delta_\mathcal{F}}\alpha, \tag{2}$$

where $\Pi_\mathcal{F} = e^{-\infty\Delta_\mathcal{F}}$ is the projection to $\ker \Delta_\mathcal{F}$ given by (1). This projection induces a leafwise Hodge isomorphism

$$\bar{H}^*(\mathcal{F}) \cong \ker \Delta_\mathcal{F}. \tag{3}$$

These properties are rather surprising because the differential complex $d_\mathcal{F}$ is only leafwise elliptic. Of course, the condition on the foliation to be Riemannian is crucial to make up for the lack of transverse ellipticity. The decomposition (1) may not be valid for non-Riemannian foliations [21].

On the other hand, the action ϕ^* on $\bar{H}^*(\mathcal{F})$ satisfies the following properties [3, 4]. For all $f \in C^\infty_c(\mathbb{R})$ and $0 < u \leq \infty$, the operator

$$P_{u,f} = \int_\mathbb{R} \phi^{t*} e^{-u\Delta_\mathcal{F}} f(t)\, dt \tag{4}$$

is smoothing, and therefore it is of trace class since M is closed. Moreover, its super-trace, $\operatorname{Tr}^s P_{u,f}$, depends continuously on f and is independent of u, and the limit of $\operatorname{Tr}^{\mathrm{s}} P_{u,f}$ as $u \downarrow 0$ gives the expected contribution of the closed orbits. But, by (3) and (4), the mapping $f \mapsto \operatorname{Tr}^{\mathrm{s}} P_{\infty,f}$ can be considered as a distributional version of the super-trace of ϕ^* on $\bar{H}^*(\mathcal{F})$; i.e., the Lefschetz distribution $L_{\mathrm{dis}}(\phi)$, solving the problem in this case.

We would like to extend the trace formula to the case where ϕ has fixed points, which are very relevant in Deninger's program. But their existence prevents the foliation from being Riemannian, except in trivial cases. However, the foliations with simple foliated flows have a precise description [6]. For example, the $\mathcal{F}$-saturation of the fixed point set of ϕ is a finite union M^0 of compact leaves, and the restriction $\mathcal{F}^1$ of $\mathcal{F}$ to $M^1 = M \setminus M^0$ is a Riemannian foliation. Moreover, $\mathcal{F}^1$ has bounded geometry in the sense of [44, 5] for certain bundle-like metric g^1 on M^1. Then, instead of $C^\infty(M; \Lambda\mathcal{F})$, we consider in [7] the space $I(M, M^0; \Lambda\mathcal{F})$ of distributional leafwise forms conormal to M^0 (the best possible singularities). This is a complex with the continuous extension of $d_{\mathcal{F}}$, and we have a short exact sequence of complexes,

$$0 \to K(M, M^0; \Lambda\mathcal{F}) \hookrightarrow I(M, M^0; \Lambda\mathcal{F}) \to J(M, M^0; \Lambda\mathcal{F}) \to 0,$$

where $K(M, M^0; \Lambda\mathcal{F})$ is the subcomplex supported in M^0, and $J(M, M^0; \Lambda\mathcal{F})$ is defined by restriction to M^1. A key result of [7] is that we also have a short exact sequence in (reduced) cohomology,

$$0 \to H^*K(\mathcal{F}) \to \bar{H}^*I(\mathcal{F}) \to \bar{H}^*J(\mathcal{F}) \to 0,$$

with corresponding actions $\phi^* = \{\phi^{t*}\}$ induced by ϕ. Thus we can now define $L_{\mathrm{dis}}(\phi) = L_{\mathrm{dis},K}(\phi) + L_{\mathrm{dis},J}(\phi)$, using distributional versions of the super-traces of ϕ^* on $H^*K(\mathcal{F})$ and $\bar{H}^*J(\mathcal{F})$.

On the one hand, $H^*K(\mathcal{F})$ can be described using Novikov cohomologies on M^0. Under some conditions and taking coefficients in the normal density bundle, we can define $L_{\mathrm{dis},K}(\phi)$ in this way, with the expected contribution from the fixed points.

On the other hand, $\bar{H}^*J(\mathcal{F})$ can be described using the reduced cohomology $\bar{H}^*H^\infty(\mathcal{F}^1)$ of the cochain complex defined by $d_{\mathcal{F}^1}$ on the Sobolev space $H^\infty(M^1; \Lambda\mathcal{F}^1)$ (defined with g^1); actually, leafwise Novikov versions of this complex are also needed. At this point, to define $L_{\mathrm{dis},J}(\phi)$, we need a generalization of (1)–(4) for Riemannian foliations of bounded geometry using this type of cochain complex. This generalization is the purpose of this paper.

Precisely, let $\mathcal{F}$ be a Riemannian foliation of bounded geometry on an open manifold M with a bundle-like metric. Then Sanguiao [44] proved versions of (1)–(3) using $H^\infty(M; \Lambda\mathcal{F})$ and $\bar{H}^*H^\infty(\mathcal{F})$ instead of $C^\infty(M; \Lambda\mathcal{F})$ and $\bar{H}^*(\mathcal{F})$. We explain again the proof in terms of our study of Riemannian foliations of bounded geometry [5].

Moreover, let ϕ be a simple foliated flow on M transverse to the leaves. If the infinitesimal generator of ϕ is C^∞ uniformly bounded, then we also get that (4) defines a smoothing operator $P_{u,f}$, and whose Schwartz kernel is described for $0 < u < \infty$. But now the operators $P_{u,f}$ are not of trace class because M is not compact. So additional tools will be used in [7] to define and study $L_{\mathrm{dis},J}(\phi)$ (see also [34]).

Finally, we show how to extend these results to leafwise versions of the Novikov complex, as needed in [7].

2. Preliminaries on section spaces and differential operators

Let us recall some analytic concepts and fix their notation.

2.1. **Distributional sections.** Let M be a (smooth, i.e., C^∞) manifold of dimension n, and let E be a (smooth complex) vector bundle over M. The space of smooth sections, $C^\infty(M;E)$, is equipped with the (weak) C^∞ topology (see, e.g., [32]). This notation will be also used for the space of smooth sections of other types of fiber bundles. If we consider only compactly supported sections, we get the space $C^\infty_{\mathrm{c}}(M;E)$, with the compactly supported C^∞ topology.

Let $\Omega^a E$ ($a \in \mathbb{R}$) denote the line bundle of a-densities of E, and let $\Omega E = \Omega^1 E$. Let TM and T^*M be the (complex) tangent and cotangent vector bundles, respectively, $\Lambda M = \bigwedge T^*M$, $\Omega^a M = \Omega^a TM$ and $\Omega M = \Omega^1 M$. Moreover, let $\mathfrak{X}(M) = C^\infty(M;TM)$ and $\mathfrak{X}_{\mathrm{c}}(M) = C^\infty_{\mathrm{c}}(M;TM)$. The restriction of vector bundles to any submanifold $L \subset M$ may be denoted with a subindex, like E_L, $T_L M$, $T^*_L M$ and $\Omega^a_L M$. Redundant notation will be removed; for instance, $C^\infty(L;E)$ and $C^\infty(M;\Omega^a)$ will be used instead of $C^\infty(L;E_L)$ and $C^\infty(M;\Omega^a M)$. We may also use the notation $C^\infty(E) = C^\infty(M;E)$ and $C^\infty_{\mathrm{c}}(E) = C^\infty_{\mathrm{c}}(M;E)$ if there is no danger of confusion. As usual, the trivial line bundle is omitted from this notation: the spaces of smooth (complex) functions and its compactly supported version are denoted by $C^\infty(M)$ and $C^\infty_{\mathrm{c}}(M)$.

A similar notation is used for other section spaces. For instance, consider also the spaces of distributional (or generalized) sections of E, and its compactly supported version,

$$C^{-\infty}(M;E) = C^\infty_{\mathrm{c}}(M;E^* \otimes \Omega)',$$
$$C^{-\infty}_{\mathrm{c}}(M;E) = C^\infty(M;E^* \otimes \Omega)',$$

where we take the topological[1] dual spaces with the weak-$*$ topology. A continuous injection $C^\infty(M;E) \subset C^{-\infty}(M;E)$ is defined by $\langle u, v\rangle = \int_M uv$ for $u \in C^\infty(M;E)$ and $v \in C^\infty_{\mathrm{c}}(M;E^* \otimes \Omega)$, using the canonical pairing of E and E^*. There is a similar continuous injection $C^\infty_{\mathrm{c}}(M;E) \subset C^{-\infty}_{\mathrm{c}}(M;E)$. If E is endowed with a Hermitian structure, we can also consider the Banach space $L^\infty(M;E)$ of its essentially bounded sections, whose norm is denoted by $\|\cdot\|_{L^\infty}$. If M is compact, then the equivalence class of $\|\cdot\|_{L^\infty}$ is independent of

[1] This term is added to algebraic concepts on topological vector spaces to mean that they are compatible with the topologies. For instance, the *topological dual* V' consists of continuous linear maps $V \to \mathbb{C}$, an isomorphism is called *topological* if it is also a homeomorphism, and a direct sum is called *topological* if it has the product topology.

the Hermitian structure. Also, for any[2] $m \in \mathbb{N}_0$, $C^m(M;E)$ denotes the space of C^m sections.

When explicitly indicated, we will also consider real objects with the same notation: real vector bundles, Euclidean structures, real tangent vectors and vector fields, real densities, real functions and distributions, etc.

2.2. **Operators on section spaces.** Let E and F be vector bundles over M, and let $A\colon C^\infty_{\mathrm{c}}(M;E) \to C^\infty(M;F)$ be a continuous linear operator. The transpose of A,

$$A^{\mathrm{t}}\colon C^{-\infty}_{\mathrm{c}}(M;F^*\otimes\Omega) \to C^{-\infty}(M;E^*\otimes\Omega),$$

is given by $\langle A^{\mathrm{t}}v,u\rangle = \langle v,Au\rangle$ for $u \in C^\infty_{\mathrm{c}}(M;E)$ and $v \in C^{-\infty}_{\mathrm{c}}(M;F^*\otimes\Omega)$. For instance, the transpose of the continuous dense injection $C^\infty_{\mathrm{c}}(M;E^*\otimes\Omega) \subset C^\infty(M;E^*\otimes\Omega)$ is the continuous dense injection $C^{-\infty}_{\mathrm{c}}(M;E) \subset C^{-\infty}(M;E)$. If there is a restriction $A^{\mathrm{t}}\colon C^\infty_{\mathrm{c}}(M;F^*\otimes\Omega) \to C^\infty(M;E^*\otimes\Omega)$, then $A^{\mathrm{tt}}: C^{-\infty}_{\mathrm{c}}(M;E) \to C^{-\infty}(M;F)$ is a continuous extension of A, also denoted by A. The Schwartz kernel, $K_A \in C^{-\infty}(M^2;F\boxtimes(E^*\otimes\Omega))$, is determined by the condition $\langle K_A, v\otimes u\rangle = \langle v,Au\rangle$ for $u \in C^\infty_{\mathrm{c}}(M;E)$ and $v \in C^\infty_{\mathrm{c}}(M;F^*\otimes\Omega)$. The mapping $A \mapsto K_A$ defines a bijection (the Schwartz kernel theorem)[3]

$$L(C^\infty_{\mathrm{c}}(M;E), C^{-\infty}(M;F)) \to C^{-\infty}(M^2;F\boxtimes(E^*\otimes\Omega)).$$

Note that

$$K_{A^{\mathrm{t}}} = R^*K_A \in C^{-\infty}(M^2;(E^*\otimes\Omega)\boxtimes F),$$

where $R\colon M^2 \to M^2$ is given by $R(x,y) = (y,x)$.

There are obvious versions of the construction of A^{t} and A^{tt} when both the domain and target of A have compact support, or no support restriction.

2.3. **Differential operators.** Let $\mathrm{Diff}(M) \subset \mathrm{End}(C^\infty(M))$ be the $C^\infty(M)$-submodule and subalgebra of differential operators, filtered by the order. Every $\mathrm{Diff}^m(M)$ ($m \in \mathbb{N}_0$) is $C^\infty(M)$-spanned by all compositions of up to m tangent vector fields, where $\mathfrak{X}(M)$ is considered as the Lie algebra of derivations of $C^\infty(M)$. In particular, $\mathrm{Diff}^0(M) \equiv C^\infty(M)$. Any $A \in \mathrm{Diff}^m(M)$ has the following local description. Given a chart (U,x) of M with $x = (x^1,\dots,x^n)$, let $\partial_j = \frac{\partial}{\partial x^j}$ and $D_j = \frac{1}{i}\partial_j$. For any multi-index $I = (i_1,\dots,i_n) \in \mathbb{N}_0^n$, let $\partial_I = \partial_1^{i_1}\cdots\partial_n^{i_n}$, $D^I = D^I_x = D_1^{i_1}\cdots D_n^{i_n}$ and $|I| = i_1+\dots+i_n$. Then $A = \sum_{|I|\le m} a_I D^I$ on $C^\infty_{\mathrm{c}}(U)$ for some local coefficients $a_I \in C^\infty(U)$.

On the other hand, let $P(T^*M) \subset C^\infty(T^*M)$ be the graded $C^\infty(M)$-module and subalgebra of functions on T^*M whose restriction to the fibers are polynomials, with the grading defined by the degree of the polynomials. In particular, $P^{[0]}(T^*M) \equiv C^\infty(M)$ and $P^{[1]}(T^*M) \equiv \mathfrak{X}(M)$. The principal symbol of any $X \in \mathfrak{X}(M) \subset \mathrm{Diff}^1(M)$ is $\sigma_1(X) = iX \in P^{[1]}(T^*M)$. The

[2] We use the notation $\mathbb{N} = \mathbb{Z}^+$ and $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

[3] For locally convex (topological vector) spaces X and Y, the notation $L(X,Y)$ is used for the space of continuous linear operators $X \to Y$ with the topology of bounded convergence. $\mathrm{End}(X) := L(X,X)$ is an associative algebra with the operation of composition.

map σ_1 can be extended to a homomorphism of $C^\infty(M)$-modules and algebras, $\sigma\colon \operatorname{Diff}(M) \to P(T^*M)$, obtaining for every m the principal symbol surjection

$$\sigma_m : \operatorname{Diff}^m(M) \to P^{[m]}(T^*M), \tag{5}$$

with kernel $\operatorname{Diff}^{m-1}(M)$.

For vector bundles E and F over M, the above concepts can be extended by taking the $C^\infty(M)$-tensor product with $C^\infty(M; F\otimes E^*)$, obtaining the filtered $C^\infty(M)$-submodule

$$\operatorname{Diff}(M; E, F) \subset L(C^\infty(M; E), C^\infty(M; F)),$$

the graded $C^\infty(M)$-submodule

$$P^{[m]}(T^*M; F\otimes E^*) \subset C^\infty(T^*M; \pi^*(F\otimes E^*)),$$

where $\pi\colon T^*M \to M$ is the vector bundle projection, and the principal symbol surjection

$$\sigma_m\colon \operatorname{Diff}^m(M; E, F) \to P^{[m]}(T^*M; F\otimes E^*), \tag{6}$$

with kernel $\operatorname{Diff}^{m-1}(M; E, F)$. Using local trivializations of E and F, any $A \in \operatorname{Diff}^m(M; E, F)$ has local expressions

$$A = \sum_{|I|\le m} a_I D^I \colon C^\infty_{\mathrm{c}}(U, \mathbb{C}^l) \to C^\infty_{\mathrm{c}}(U, \mathbb{C}^{l'}),$$

as above, where $l = \operatorname{rank} E$ and $l' = \operatorname{rank} F$, with local coefficients $a_I \in C^\infty(U; \mathbb{C}^{l'} \otimes \mathbb{C}^{l*})$. If $E = F$, then we use the notation $\operatorname{Diff}(M; E)$, which is also a filtered algebra with the operation of composition. Recall that $A \in \operatorname{Diff}^m(M; E, F)$ is called elliptic if $\sigma_m(A)(p,\xi) \in F_p\otimes E^*_p \equiv \operatorname{Hom}(E_p, F_p)$ is an isomorphism for all $p \in M$ and $0 \ne \xi \in T^*_pM$.

Using integration by parts, it follows that the class of differential operators is closed by transposition. So any $A \in \operatorname{Diff}(M; E, F)$ defines continuous linear maps (Section 2.2),

$$A : C^{-\infty}(M; E) \to C^{-\infty}(M; F),$$
$$A : C^{-\infty}_{\mathrm{c}}(M; E) \to C^{-\infty}_{\mathrm{c}}(M; F).$$

2.4. **Sobolev spaces.** The Hilbert space $L^2(M; \Omega^{1/2})$ is the completion $C^\infty_{\mathrm{c}}(M; \Omega^{1/2})$ with the scalar product $\langle u, v\rangle = \int_M u\bar{v}$. There is a continuous inclusion $L^2(M; \Omega^{1/2}) \subset C^{-\infty}(M; \Omega^{1/2})$.

Suppose first that M is compact. Then $L^2(M; \Omega^{1/2})$ is also a $C^\infty(M)$-module. Thus the Sobolev space of order $m \in \mathbb{N}_0$,

$$H^m(M; \Omega^{\frac12}) = \{u \in L^2(M; \Omega^{\frac12}) \mid \operatorname{Diff}^m(M; \Omega^{\frac12})\, u \subset L^2(M; \Omega^{\frac12})\}, \tag{7}$$

is also a $C^\infty(M)$-module. In particular, $H^0(M; \Omega^{1/2}) = L^2(M; \Omega^{1/2})$. By the elliptic estimate, an elliptic operator $P \in \operatorname{Diff}^1(M; \Omega^{1/2})$ can be used to equip $H^m(M; \Omega^{1/2})$ with the Hilbert space structure defined by

$$\langle u, v\rangle_m = \langle (1 + P^*P)^m u, v\rangle. \tag{8}$$

The equivalence class of the corresponding norm $\|\cdot\|_m$ is independent of the choice of P. Thus $H^m(M;\Omega^{1/2})$ is a Hilbertian space with no canonical choice of a scalar product in general. Now, the Sobolev space of order $-m$ is the Hilbertian space

$$H^{-m}(M;\Omega^{\frac{1}{2}}) = H^m(M;\Omega^{\frac{1}{2}})' \equiv \operatorname{Diff}^m(M;\Omega^{\frac{1}{2}})\,L^2(M;\Omega^{\frac{1}{2}}). \tag{9}$$

For any vector bundle E, the $C^\infty(M)$-module $H^{\pm m}(M;E)$ can be defined as the $C^\infty(M)$-tensor product of $H^{\pm m}(M;\Omega^{1/2})$ with $C^\infty(M;E\otimes\Omega^{-1/2})$; in particular, this defines $L^2(M;E)$. A Hermitian structure $(\cdot,\cdot)$ on E and a non-vanishing smooth density ω on M can be used to define an obvious scalar product $\langle\cdot,\cdot\rangle$ on $L^2(M;E)$. Using, moreover, an elliptic operator $P\in\operatorname{Diff}^1(M;E)$, we get a scalar product $\langle\cdot,\cdot\rangle_m$ on $H^m(M;E)$ like in (8), with norm $\|\cdot\|_m$. Indeed, this scalar product makes sense on $C^\infty(M;E)$ for any order $m\in\mathbb{R}$, where $(1+P^*P)^m$ is defined by the functional calculus given by the spectral theorem. Then, taking the corresponding completion of $C^\infty(M;E)$, we get the Sobolev space $H^m(M;E)$ of order $m\in\mathbb{R}$. In particular, $H^{-m}(M;E)\equiv H^m(M;E^*\otimes\Omega)'$.

When M is not compact, any choice of P, $(\cdot,\cdot)$ and ω can be used to equip $C^\infty_{\mathrm{c}}(M;E)$ with a scalar product $\langle\cdot,\cdot\rangle_m$ as above, and the corresponding Hilbert space completion can be denoted by $H^m(M;E)$; in particular, this defines $L^2(M;E)=H^0(M;E)$. But now the equivalence class of $\|\cdot\|_m$ (and therefore $H^m(M;E)$) depends on the choices. However, their compactly supported and their local versions, $H^m_{\mathrm{c}}(M;E)$ and $H^m_{\mathrm{loc}}(M;E)$, are independent of the choices involved. In particular, we have $L^2_{\mathrm{c}}(M;E)$ and $L^2_{\mathrm{loc}}(M;E)$. The formal adjoint of any differential operator is locally defined like in the compact case.

In any case, the notation $\|\cdot\|_{m,m'}$ (or $\|\cdot\|_m$ if $m=m'$) is used for the induced norm of operators $H^m(M;E)\to H^{m'}(M;E)$ ($m,m'\in\mathbb{R}$). For example, when M is compact, any $A\in\operatorname{Diff}^m(M;E)$ defines a bounded operator $A\colon H^{m+s}(M;E)\to H^m(M;E)$ for all $s\in\mathbb{R}$. Taking $s=0$, we can consider A as a densely defined linear operator in $L^2(M;E)$ with domain $H^m(M;E)$. Its adjoint A^* in $L^2(M;E)$ is defined by the formal adjoint $A^*\in\operatorname{Diff}^m(M;E)$, which is locally determined using integration by parts. Recall that A is called formally selfadjoint or symmetric if it is equal to its formal adjoint.

2.5. **Differential complexes.** A *topological* (*cochain*) *complex* (C,d) is a cochain complex, where C is a graded topological vector space and d is continuous. Then the cohomology $H(C,d)=\ker d/\operatorname{im} d$ has an induced topology, whose maximal Hausdorff quotient, $\bar H(C,d) := H(C,d)/\overline{0} \equiv \ker d/\overline{\operatorname{im} d}$, is called the *reduced cohomology*. The elements in $H(C,d)$ and $\bar H(C,d)$ defined by some $u\in\ker d$ will be denoted by $[u]$ and $\overline{[u]}$, respectively. The continuous cochain maps between topological complexes induce continuous homomorphisms between the corresponding (reduced) cohomologies. *Topological graded differential algebras* can be similarly defined by assuming that their product is continuous.

Recall that a *differential complex* of *order* m is a topological complex of the form $(C^\infty(M;E),d)$, where $E=\bigoplus_r E^r$ and $d=\bigoplus_r d_r$, for a finite sequence of differential operators of the same order m,

$$C^\infty(M;E^0)\xrightarrow{d_0}C^\infty(M;E^1)\xrightarrow{d_1}\cdots\xrightarrow{d_{N-1}}C^\infty(M;E^N).$$

The compactly supported version $(C^\infty_{\mathrm{c}}(M;E),d)$ may be also considered. Negative or decreasing degrees may be also considered without essential change. Such a differential complex is called *elliptic* if the symbol sequence,

$$0\to E^0_p\xrightarrow{\sigma_m(d_0)(p,\xi)}E^1_p\xrightarrow{\sigma_m(d_1)(p,\xi)}\cdots\xrightarrow{\sigma_m(d_{N-1})(p,\xi)}E^N_p\to 0,$$

is exact for all $p\in M$ and $0\neq\xi\in T^*_pM$.

Suppose that every E^r is equipped with a Hermitian structure, and M with a distinguished non-vanishing smooth density. Then the formal adjoint $\delta=d^*$ also defines a differential complex, giving rise to symmetric operators $D=d+\delta$ and $\Delta=D^2=d\delta+\delta d$ (a generalized Laplacian) in the Hilbert space $L^2(M;E)$. The differential complex d is elliptic if and only if the differential complex δ is elliptic, and if and only if the differential operator D (or Δ) is elliptic. If d is elliptic and M is closed, then D and Δ have discrete spectra, giving rise to a topological and orthogonal decomposition (a generalized Hodge decomposition)

$$C^\infty(M;E)=\ker\Delta\oplus\operatorname{im}\delta\oplus\operatorname{im}d, \tag{10}$$

which induces a topological isomorphism (a Hodge isomorphism)

$$H(C^\infty(M;E),d)\cong\ker\Delta. \tag{11}$$

Thus $H(C^\infty(M;E),d)$ is of finite dimension and Hausdorff.

2.6. **Novikov differential complex.** The most typical example of elliptic differential complex is given by the de Rham derivative d on $C^\infty(M;\Lambda)$, defining the de Rham cohomology $H^*(M)=H^*(M;\mathbb{C})$. Suppose that M is endowed with a Riemannian metric g, which defines a Hermitian structure on TM. Then we have the de Rham coderivative $\delta=d^*$, and the symmetric operators $D=d+\delta$ and $\Delta=D^2=d\delta+\delta d$ (the Laplacian).

With more generality, take any closed $\theta\in C^\infty(M;\Lambda^1)$. For the sake of simplicity, assume that θ is real. Let $V\in\mathfrak{X}(M)$ be determined by $g(V,\cdot\,)=\theta$, let $\mathcal{L}_V$ denote the Lie derivative with respect to V, and let $\theta\lrcorner=-(\theta\wedge)^*=-\iota_V$. Then we have the *Novikov operators* defined by θ, depending on $z\in\mathbb{C}$,

$$\begin{aligned}
&d_z=d+z\,\theta\wedge,\quad \delta_z=d_z^*=\delta-\bar z\,\theta\lrcorner,\\
&D_z=d_z+\delta_z=D+\Re z\,R_\theta+i\,\Im z\,L_\theta,\\
&\Delta_z=D_z^2=d_z\delta_z+\delta_zd_z=\Delta+\Re z\,(\mathcal{L}_V+\mathcal{L}_V^*)-i\,\Im z\,(\mathcal{L}_V-\mathcal{L}_V^*)+|z|^2|\theta|^2,
\end{aligned}$$

where, for $\alpha\in C^\infty(M;\Lambda^rM)$,

$$L_\theta\alpha=\theta\cdot\alpha=\theta\wedge\alpha+\theta\lrcorner\,\alpha,\quad R_\theta\alpha=(-1)^r\alpha\cdot\theta=\theta\wedge\alpha-\theta\lrcorner\,\alpha.$$

The subindex "M" may be added to this notation if needed. Here, the dot denotes Clifford multiplication defined via the linear identity $\Lambda M\equiv\mathrm{Cl}(T^*M)$.

We may write $\theta\cdot = L_\theta$. The differential operator $\mathcal{L}_V + \mathcal{L}_V^*$ is of order zero, and $\mathcal{L}_V - \mathcal{L}_V^*$ is of order one. The differential complex $(C^\infty(M;\Lambda), d_z)$ is elliptic; indeed, it has the same principal symbol as the de Rham differential complex. Actually, Δ_z is a generalized Laplacian [9, Definition 2.2], and therefore D_z is a generalized Dirac operator, and d_z is a generalized Dirac complex. The terms *Novikov differential complex* and *Novikov cohomology* are used for $(C^\infty(M;\Lambda), d_z)$ and its cohomology, $H_z^*(M) = H_z^*(M;\mathbb{C})$.

If θ is exact, say $\theta = dF$ for some $\mathbb{R}$-valued $F \in C^\infty(M)$, then the Novikov operators are called *Witten operators*; in particular, $d_z = e^{-zF}\, d\, e^{zF}$ and $\delta_z = e^{\bar{z}F}\,\delta\, e^{-\bar{z}F}$. Thus the multiplication operator e^{zF} on $C^\infty(M;\Lambda)$ induces an isomorphism $H_z^*(M) \cong H^*(M)$ in this case.

In the general case, the above kind of argument shows that the isomorphism class of $H_z^*(M)$ depends only on $[\theta] \in H^1(M)$. We can also take a regular covering $\pi\colon \widetilde{M} \to M$ so that the lift $\tilde{\theta} = \pi^*\theta$ is exact, say $\tilde{\theta} = dF$ for some $\mathbb{R}$-valued $F \in C^\infty(\widetilde{M})$. Thus we get the Witten derivative $d_{\widetilde{M},z} = e^{-zF}\, d_{\widetilde{M}}\, e^{zF}$ on $C^\infty(\widetilde{M};\Lambda)$, which corresponds to the Novikov derivative $d_{M,z}$ on $C^\infty(M;\Lambda)$ via π^*.

For any smooth map $\phi\colon M \to M$, take a lift $\tilde{\phi}\colon \widetilde{M} \to \widetilde{M}$; i.e., $\pi\tilde{\phi} = \phi\pi$. Then $\tilde{\phi}_z^* = e^{-zF}\,\tilde{\phi}^*\, e^{zF} = e^{z(\tilde{\phi}^*F - F)}\,\tilde{\phi}^*$ is an endomorphism of Witten differential complex $(C^\infty(\widetilde{M};\Lambda), d_{\widetilde{M},z})$, which can be called a *Witten perturbation* of $\tilde{\phi}^*$. For all $\gamma \in \Gamma$, we have $T_\gamma^*(\tilde{\phi}^*F - F) = \tilde{\phi}^*F - F$, obtaining $T_\gamma^*\tilde{\phi}_z^* = \tilde{\phi}_z^* T_\gamma^*$. Therefore $\tilde{\phi}_z^*$ induces an endomorphism ϕ_z^* of Witten differential complex $d_{M,z}$ on $C^\infty(M;\Lambda)$, which can be considered as a *Novikov perturbation* of ϕ^*. This ϕ_z^* depends on the choice of the lift $\tilde{\phi}$ of ϕ. However, any flow $\phi = \{\phi^t\}$ has a unique lift to a flow $\tilde{\phi} = \{\tilde{\phi}^t\}$ on $\widetilde{M}$, giving rise to a canonical choice of ϕ_z^{t*}, called the *Novikov perturbation* of ϕ^{t*}.

If M is oriented, then

$$\theta\lrcorner = (-1)^{nr+n+1} \star \theta\wedge\star, \quad \delta = (-1)^{nr+n+1} \star d\,\star, \tag{12}$$

on $C^\infty(M;\Lambda^r)$, using the Hodge operator $\star$ on ΛM. So

$$\delta_z = (-1)^{nr+n+1} \star d_{-\bar{z}} \star . \tag{13}$$

3. Preliminaries on bounded geometry

The concepts recalled here become relevant when M is not compact. Equip M with a Riemannian metric g, and let ∇ denote its Levi-Civita connection, R its curvature, and $\mathrm{inj}_M\colon M \to \mathbb{R}^+$ its injectivity radius function. Suppose that M is connected, obtaining an induced distance function d. Actually, in the non-connected case, we can take $d(p,q) = \infty$ if p and q belong to different connected components. Observe that M is complete if $\inf \mathrm{inj}_M > 0$. For $r > 0$, $p \in M$ and $S \subset M$, let $B(p,r)$ and $\overline{B}(p,r)$ denote the open and closed r-balls centered at p, and let $\mathrm{Pen}(S,r)$ and $\overline{\mathrm{Pen}}(S,r)$ denote the open and closed r-penumbras of S (defined by the conditions $d(\cdot,S) < r$ and $d(\cdot,S) \le r$, respectively). We may add the subindex "M" to this notation if needed, or a subindex "a" if we are referring to a family of Riemannian manifolds M_a.

3.1. **Manifolds and vector bundles of bounded geometry.** It is said that M is of *bounded geometry* if $\inf \operatorname{inj}_M > 0$ and $\sup|\nabla^m R| < \infty$ for every $m \in \mathbb{N}_0$. This concept has the following chart description.

Theorem 3.2 (Eichhorn [23]; see also [42, 45, 46])**.** *M is of bounded geometry if and only if, for some open ball $B \subset \mathbb{R}^n$ centered at 0, there are normal coordinates $y_p\colon V_p \to B$ at every $p \in M$ such that the corresponding Christoffel symbols Γ^i_{jk}, as a family of functions on B parametrized by i, j, k and p, lie in a bounded set of the Fréchet space $C^\infty(B)$. This equivalence holds as well replacing the Christoffel symbols with the metric coefficients g_{ij}.*

Remark 3.3. Any non-connected Riemannian manifold of bounded geometry can be considered as a family of Riemannian manifolds (the connected components), which are of *equi-bounded geometry* in the sense that they satisfy the condition of bounded geometry with the same bounds. Conversely, any disjoint union of Riemannian manifolds of equi-bounded geometry is of bounded geometry.

Assume that M is of bounded geometry and consider the charts $y_p\colon V_p \to B$ given by Theorem 3.2. The radius of B will be denoted by r_0.

Proposition 3.4 (Schick [45, Theorem A.22], [46, Proposition 3.3])**.** *For every multi-index I, the function $|\partial_I(y_q y_p^{-1})|$ is bounded on $y_p(V_p \cap V_q)$, uniformly on $p, q \in M$.*

Proposition 3.5 (Shubin [47, Appendices A1.2 and A1.3]; see also [46, Proposition 3.2])**.** *For any $0 < 2r \leq r_0$, there are a subset $\{p_k\} \subset M$ and some $N \in \mathbb{N}$ such that the balls $B(p_k, r)$ cover M, and every intersection of $N+1$ sets $B(p_k, 2r)$ is empty. Moreover, there is a partition of unity $\{f_k\}$ subordinated to the open covering $\{B(p_k, 2r)\}$, which is bounded in the Fréchet space[4] $C^\infty_{\mathrm{ub}}(M)$.*

A vector bundle E of rank l over M is said to be of *bounded geometry* when it is equipped with a family of local trivializations over the charts (V_p, y_p), for small enough r_0, with corresponding defining cocycle $a_{pq}\colon V_p \cap V_q \to \mathrm{GL}(\mathbb{C}, l) \subset \mathbb{C}^{l^2}$, such that, for every multi-index I, the function $|\partial_I(a_{pq} y_p^{-1})|$ is bounded on $y_p(V_p \cap V_q)$, uniformly on $p, q \in M$. When referring to local trivializations of a vector bundle of bounded geometry, we always mean that they satisfy the above condition. If the corresponding defining cocycle is valued in $\mathrm{U}(l)$, then E is said to be of *bounded geometry* as a Hermitian vector bundle.

Example 3.6.
1. If E is associated to the principal $\mathrm{O}(n)$-bundle P of orthonormal frames of M and any unitary representation of $\mathrm{O}(n)$, then it is of bounded geometry in a canonical way. In particular, this applies to TM and ΛM.
2. The properties of 1 can be extended to the case where E is associated to any reduction Q of P with structural group $H \subset \mathrm{O}(n)$, and any unitary representation of H.

[4] The definition of $C^\infty_{\mathrm{ub}}(M)$ is given in Section 3.7.

(3) The condition of bounded geometry is preserved by operations of vector bundles induced by operations of vector spaces, like dual vector bundles, direct sums, tensor products, exterior products, etc.

3.7. **Uniform spaces.** For every $m \in \mathbb{N}_0$, a function $u \in C^m(M)$ is said to be *C^m-uniformly bounded* if there is some $C_m \ge 0$ with $|\nabla^{m'} u| \le C_m$ on M for all $m' \le m$. These functions form the *uniform C^m space*[5] $C^m_{\mathrm{ub}}(M)$, which is a Banach space with the norm $\|\cdot\|_{C^m_{\mathrm{ub}}}$ defined by the best constant C_m. As usual, the super-index "m" may be removed from this notation if $m = 0$, and we have $C_{\mathrm{ub}}(M) = C(M) \cap L^\infty(M)$. Equivalently, we may take the norm $\|\cdot\|'_{C^m_{\mathrm{ub}}}$ defined by the best constant $C'_m \ge 0$ such that $|\partial_I(uy_p^{-1})| \le C'_m$ on B for all $p \in M$ and $|I| \le m$; in fact, by Proposition 3.4, it is enough to consider any subset of points p so that $\{V_p\}$ covers M. The *uniform C^∞ space* is $C^\infty_{\mathrm{ub}}(M) = \bigcap_m C^m_{\mathrm{ub}}(M)$, with the inverse limit topology, called *uniform C^∞ topology*. It consists of the functions $u \in C^\infty(M)$ such that all functions uy_p^{-1} lie in a bounded set of $C^\infty(B)$, which are said to be *C^∞-uniformly bounded*. Of course, if M is compact, then the C^m_{ub} topology is just the C^m topology, and the notation $\|\cdot\|_{C^m}$ and $\|\cdot\|'_{C^m}$ is preferred. On the other hand, the definition of uniform spaces with covariant derivatives can be also considered for non-complete Riemannian manifolds.

For a Hermitian vector bundle E of bounded geometry over M, the *uniform C^m space* $C^m_{\mathrm{ub}}(M;E)$, of *C^m-uniformly bounded* sections, can be defined by introducing $\|\cdot\|'_{C^m_{\mathrm{ub}}}$ like the case of functions, using local trivializations of E to consider every uy_p^{-1} in $C^m(B,\mathbb{C}^l)$ for all $u \in C^m(M;E)$. Then, as above, we get the *uniform C^∞ space* $C^\infty_{\mathrm{ub}}(M;E)$ of *C^∞-uniformly bounded* sections, which are the sections $u \in C^\infty(M;E)$ such that all functions uy_p^{-1} define a bounded set of $C^\infty(B;\mathbb{C}^l)$, equipped with the *uniform C^∞ topology*. In particular, $\mathfrak{X}_{\mathrm{ub}}(M) := C^\infty_{\mathrm{ub}}(M;TM)$ is a $C^\infty_{\mathrm{ub}}(M)$-submodule and Lie subalgebra of $\mathfrak{X}(M)$. Observe that

$$C^m_{\mathrm{ub}}(M) = \{u \in C^m(M) \mid \mathfrak{X}_{\mathrm{ub}}(M) \overset{(m)}{\cdots} \mathfrak{X}_{\mathrm{ub}}(M)\, u \subset L^\infty(M)\}. \tag{14}$$

Let $\mathfrak{X}_{\mathrm{com}}(M) \subset \mathfrak{X}(M)$ be the subset of complete vector fields.

Proposition 3.8. *We have $\mathfrak{X}_{\mathrm{ub}}(M) \subset \mathfrak{X}_{\mathrm{com}}(M)$.*

Proof. Let $X \in \mathfrak{X}_{\mathrm{ub}}(M)$. The maximal domain of the local flow ϕ of X is an open neighborhood Ω of $M \times \{0\}$ in $M \times \mathbb{R}$. By the Picard–Lindelöf theorem (see, e.g., [29, Theorem II.1.1]) and the C^∞-uniform boundedness of X, there is some $\alpha > 0$ such that $\{p\} \times (-\alpha,\alpha) \subset \Omega$ for all $p \in M$. So $\Omega = M \times \mathbb{R}$, since ϕ is a local flow. □

3.9. **Differential operators of bounded geometry.** Like in Section 2.3, by using $\mathfrak{X}_{\mathrm{ub}}(M)$ and $C^\infty_{\mathrm{ub}}(M)$ instead of $\mathfrak{X}(M)$ and $C^\infty(M)$, we get the filtered subalgebra and $C^\infty_{\mathrm{ub}}(M)$-submodule $\mathrm{Diff}_{\mathrm{ub}}(M) \subset \mathrm{Diff}(M)$ of differential

[5] Here, the subindex "ub" is used instead of the common subindex "b" to avoid similarities with the notation of b-calculus used in [7].

operators of *bounded geometry*. Let $P_{\mathrm{ub}}(T^*M) \subset P(T^*M)$ be the graded subalgebra generated by $P_{\mathrm{ub}}^{[0]}(T^*M) \equiv C_{\mathrm{ub}}^\infty(M)$ and $P_{\mathrm{ub}}^{[1]}(T^*M) \equiv \mathfrak{X}_{\mathrm{ub}}(M)$, which is also a graded $C_{\mathrm{ub}}^\infty(M)$-submodule. Then (5) restricts to a surjection $\sigma_m\colon \mathrm{Diff}_{\mathrm{ub}}^m(M) \to P_{\mathrm{ub}}^{[m]}(T^*M)$ whose kernel is $\mathrm{Diff}_{\mathrm{ub}}^{m-1}(M)$. These concepts can be extended to vector bundles of bounded geometry E and F over M by taking the $C_{\mathrm{ub}}^\infty(M)$-tensor product with $C_{\mathrm{ub}}^\infty(M; F \otimes E^*)$, obtaining the filtered $C_{\mathrm{ub}}^\infty(M)$-modules $\mathrm{Diff}_{\mathrm{ub}}(M; E, F)$ (or $\mathrm{Diff}_{\mathrm{ub}}(M; E)$ if $E = F$) and $P_{\mathrm{ub}}(T^*M; F \otimes E^*)$, and the surjective restriction

$$\sigma_m\colon \mathrm{Diff}_{\mathrm{ub}}^m(M; E, F) \to P_{\mathrm{ub}}^{[m]}(T^*M; F \otimes E^*)$$

of (6), whose kernel is $\mathrm{Diff}_{\mathrm{ub}}^{m-1}(M; E, F)$. Bounded geometry of differential operators is preserved by compositions and by taking transposes, and by taking formal adjoints in the case of Hermitian vector bundles of bounded geometry; in particular, $\mathrm{Diff}_{\mathrm{ub}}(M; E)$ is a filtered subalgebra of $\mathrm{Diff}(M; E)$. Using local trivializations of E and F over the charts (V_p, y_p), we get a local description of any element of $\mathrm{Diff}_{\mathrm{ub}}^m(M; E, F)$ by requiring the local coefficients to define a bounded subset of the Fréchet space $C^\infty(B, \mathbb{C}^{l'} \otimes \mathbb{C}^{l*})$, where l and l' are the ranks of E and F. Using the norms $\|\cdot\|'_{C_{\mathrm{ub}}^m}$, it easily follows that every $A \in \mathrm{Diff}_{\mathrm{ub}}^m(M; E, F)$ defines bounded operators $A\colon C_{\mathrm{ub}}^{m+s}(M; E) \to C_{\mathrm{ub}}^s(M; F)$ ($s \in \mathbb{N}_0$), which induce a continuous operator $A\colon C_{\mathrm{ub}}^\infty(M; E) \to C_{\mathrm{ub}}^\infty(M; F)$.

Example 3.10. (i) In Example 3.6 1, the Levi-Civita connection ∇ induces a connection of bounded geometry on E, also denoted by ∇. In particular, ∇ itself is of bounded geometry on TM, and induces a connection ∇ of bounded geometry on ΛM. This holds as well for the connection on E induced by any other Riemannian connection of bounded geometry on TM.

(ii) In Example 3.6 2, if a Riemannian connection of bounded geometry on TM is given by a connection on Q, then the induced connection on E is of bounded geometry.

(iii) In Example 3.6 3, bounded geometry of connections is preserved by taking the induced connections in the indicated operations with vector bundles of bounded geometry.

(iv) The standard expression of the de Rham derivative d on local coordinates shows that it is of bounded geometry, and therefore δ is also of bounded geometry.

Let E and F be Hermitian vector bundles of bounded geometry. Then any unitary connection ∇ of bounded geometry on E can be used to define an equivalent norm $\|\cdot\|_{C_{\mathrm{ub}}^m}$ on every Banach space $C_{\mathrm{ub}}^m(M; E)$, like in the case of $C_{\mathrm{ub}}^m(M)$.

It is said that $A \in \mathrm{Diff}^m(M; E, F)$ is *uniformly elliptic* if there is some $C \geq 1$ such that, for all $p \in M$ and $\xi \in T_p^*M$,

$$C^{-1}|\xi|^m \leq |\sigma_m(A)(p, \xi)| \leq C|\xi|^m.$$

This condition is independent of the choice of the Hermitian metrics of bounded geometry on E and F. Any $A \in \operatorname{Diff}^m_{\mathrm{ub}}(M;E,F)$ satisfies the second inequality.

3.11. **Sobolev spaces of manifolds of bounded geometry.** For any Hermitian vector bundle E of bounded geometry over M, any choice of a uniformly elliptic $P \in \operatorname{Diff}^1_{\mathrm{ub}}(M;E)$, besides the Riemannian density and the Hermitian structure, can be used to define the Sobolev space $H^m(M;E)$ ($m \in \mathbb{R}$) (Section 2.4). Any choice of P defines the same Hilbertian space $H^m(M;E)$, which is a $C^\infty_{\mathrm{ub}}(M)$-module. Every $A \in \operatorname{Diff}^m_{\mathrm{ub}}(M;E,F)$ defines bounded operators $A\colon H^{m+s}(M;E) \to H^s(M;F)$ ($s \in \mathbb{R}$), which induce continuous maps $A\colon H^{\pm\infty}(M;E) \to H^{\pm\infty}(M;F)$.

Proposition 3.12 (Roe [42, Proposition 2.8]). *If $m' > m + n/2$, then we have $H^{m'}(M;E) \subset C^m_{\mathrm{ub}}(M;E)$, continuously. Thus $H^\infty(M;E) \subset C^\infty_{\mathrm{ub}}(M;E)$, continuously.*

3.13. **Schwartz kernels on manifolds of bounded geometry.** Let E and F be Hermitian vector bundles of bounded geometry over M.

Proposition 3.14 (Roe [42, Proposition 2.9]). *The Schwartz kernel mapping, $A \mapsto K_A$, defines a continuous linear map*

$$L(H^{-\infty}(M;E), H^\infty(M;F)) \to C^\infty_{\mathrm{ub}}(M; F \boxtimes (E^* \otimes \Omega)).$$

Remark 3.15. Let $A \in L(H^{-\infty}(M;E), H^\infty(M;F))$ and $r > 0$. Obviously,

$$\operatorname{supp} K_A \subset \{(p,q) \in M^2 \mid d(p,q) \le r\}$$

if and only if, for all $u \in H^{-\infty}(M;E)$,

$$\operatorname{supp} Au \subset \overline{\operatorname{Pen}}(\operatorname{supp} u, r).$$

Recall that a function $\psi \in C(\mathbb{R})$ is called *rapidly decreasing* if, for all $k \in \mathbb{N}_0$, there is some $C_k \ge 0$ so that $|\psi(x)| \le C_k(1+|x|)^{-k}$. They form a Fréchet space denoted by $\mathcal{R} = \mathcal{R}(\mathbb{R})$, with the best constants C_k as semi-norms. If $P \in \operatorname{Diff}^m_{\mathrm{ub}}(M;E)$ is uniformly elliptic and formally selfadjoint, then it is selfadjoint as an unbounded operator in the Hilbert space $L^2(M;E)$, and the functional calculus given by the spectral theorem defines a continuous linear map

$$\mathcal{R} \to L(H^{-\infty}(M;E), H^\infty(M;E)), \quad \psi \mapsto \psi(P).$$

Thus the linear map

$$\mathcal{R} \to C^\infty_{\mathrm{ub}}(M; E \boxtimes (E^* \otimes \Omega)), \quad \psi \mapsto K_{\psi(P)}, \tag{15}$$

is continuous by Proposition 3.14, see [42, Proposition 2.10].

For any closed real $\theta \in C^\infty_{\mathrm{ub}}(M;\Lambda^1)$ and $z \in \mathbb{C}$, we have the corresponding Novikov operators, $D_z \in \operatorname{Diff}^1_{\mathrm{ub}}(M;\Lambda)$ and $\Delta_z \in \operatorname{Diff}^2_{\mathrm{ub}}(M;\Lambda)$ (Section 2.6), which are uniformly elliptic and formally selfadjoint; indeed, D_z is a generalized Dirac operator. Thus, for the symmetric hyperbolic equation

$$\partial_t \alpha_t = i D_z \alpha_t, \quad \alpha_0 = \alpha,$$

on any open subset of M and with t in any interval containing 0, any solution satisfies the finite propagation speed property, see the proof of [14, Proposition 1.1] (see also [13, Theorem 1.4] and the proof of [43, Proposition 7.20]),

$$\operatorname{supp}\alpha_t \subset \operatorname{Pen}(\operatorname{supp}\alpha, |t|). \tag{16}$$

In particular, for the *Novikov wave operator* e^{itD_z} and any $\alpha \in C^\infty(M;\Lambda)$, $\alpha_t = e^{itD_z}\alpha$ satisfies (16). On the other hand, using the expression of the inverse Fourier transform, we get

$$\psi(D_z) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{i\xi D_z}\hat{\psi}(\xi)\,d\xi. \tag{17}$$

According to Remark 3.15, it follows from (16) and (17) that, for all $r > 0$,

$$\operatorname{supp}\hat{\psi} \subset [-r,r] \Rightarrow \operatorname{supp}K_{\psi(D_z)} \subset \{(p,q)\in M^2 \mid d(p,q)\le r\}. \tag{18}$$

For $\psi \in \mathcal{R}$, the operator $\psi(D_z)$ is smoothing and we may use the notation $k_z = k_{\psi,z} = K_{\psi(D_z)}$. We may also use the notation $k_{u,z} = k_{\psi_u,z}$ for any family of functions $\psi_u \in \mathcal{R}$ depending on a parameter u. For instance, for $\psi_u(x) = e^{-ux^2}$ ($u > 0$), we get the *Novikov heat kernel* $k_{u,z} = K_{e^{-u\Delta_z}}$.

3.16. **Maps of bounded geometry.** For $a \in \{1,2\}$, let M_a be a Riemannian manifold of bounded geometry, of dimension n_a. Consider a normal chart $y_{a,p}\colon V_{a,p} \to B_a$ at every $p \in M_a$ satisfying the statement of Theorem 3.2. Let r_a denote the radius of B_a. For $0 < r \le r_a$, let $B_{a,r} \subset \mathbb{R}^{n_a}$ denote the ball centered at the origin with radius r. We have $B_a(p,r) = y_{a,p}^{-1}(B_{a,r})$.

A smooth map $\phi\colon M_1 \to M_2$ is said to be of *bounded geometry* if, for some $0 < r < r_1$ and all $p \in M_1$, we have $\phi(B_1(p,r)) \subset V_{2,\phi(p)}$, and the compositions $y_{2,\phi(p)}\phi y_{1,p}^{-1}$ define a bounded family of the Fréchet space $C^\infty(B_{1,r},\mathbb{R}^{n_2})$. This condition is preserved by composition of maps. The family of smooth maps $M_1 \to M_2$ of bounded geometry is denoted by $C^\infty_{\mathrm{ub}}(M_1,M_2)$.

For $m \in \mathbb{N}_0$ and $\phi \in C^\infty_{\mathrm{ub}}(M_1,M_2)$, using $\|\cdot\|'_{C^m_{\mathrm{ub}}}$, we easily get that ϕ^* induces a bounded homomorphism

$$\phi^*\colon C^m_{\mathrm{ub}}(M_2;\Lambda) \to C^m_{\mathrm{ub}}(M_1;\Lambda), \tag{19}$$

obtaining a continuous homomorphism

$$\phi^*\colon C^\infty_{\mathrm{ub}}(M_2;\Lambda) \to C^\infty_{\mathrm{ub}}(M_1;\Lambda). \tag{20}$$

On the other hand, recall that ϕ is called *uniformly metrically proper* if, for any $s \ge 0$, there is some $t_s \ge 0$, so that, for all $p,q \in M_1$,

$$d_2(\phi(p),\phi(q)) \le s \Rightarrow d_1(p,q) \le t_s.$$

For $0 < 2r \le r_1, r_2$, take $\{p_{1,k}\} \subset M_1$, $\{p_{2,l}\} \subset M_2$ and $N \in \mathbb{N}$ satisfying the statement of Proposition 3.5. Then $\phi \in C^\infty(M_1,M_2)$ is uniformly metrically proper if and only if there is some $N' \in \mathbb{N}$ such that every set $\phi^{-1}(B_2(p_{2,l},r))$ meets at most N' sets $B_1(p_{1,k},r)$. Using $\|\cdot\|'_m$ on $H^m(M_a;\Lambda)$ (Section 3.7), it follows that, if ϕ is of bounded geometry and uniformly metrically proper, then ϕ^* induces a bounded homomorphism

$$\phi^*\colon H^m(M_2;\Lambda)\to H^m(M_1;\Lambda) \tag{21}$$

for all m, obtaining induced continuous homomorphisms

$$\phi^*\colon H^{\pm\infty}(M_2;\Lambda)\to H^{\pm\infty}(M_1;\Lambda). \tag{22}$$

If $\phi\in\operatorname{Diffeo}(M_1,M_2)$, and both ϕ and ϕ^{-1} are of bounded geometry, then ϕ is uniformly metrically proper.

3.17. **Smooth families of bounded geometry.** Consider the notation of Sections 3.1, 3.7 and 3.9. Let T be a manifold, and let $\operatorname{pr}_1\colon M\times T\to M$ denote the first factor projection. A section $u\in C^\infty(M\times T;\operatorname{pr}_1^*E)$ is called a *smooth family of smooth sections* of E (*parametrized* by T), and we may use the notation $u=\{u_t\mid t\in T\}\subset C^\infty(M;E)$, where $u_t=u(\,\cdot\,,t)$. Its T*-support* is $\overline{\{t\in T\mid u_t\neq 0\}}$. If the T-support is compact, then u is said to be T*-compactly supported*. It is said that u is T*-locally* C^∞*-uniformly bounded* if any $t\in T$ is in some chart (O,z) of T such that the maps $u(y_p\times z)^{-1}$ define a bounded subset of the Fréchet space $C^\infty(B\times z(O),\mathbb{C}^l)$, using local trivializations of E over the normal charts (V_p,y_p).

In particular, we can consider smooth families of $\mathbb{C}$-valued functions, tangent vector fields and sections of $C^\infty(M;F\otimes E^*)$, which can be used to define a *smooth family of differential operators*, $A=\{A_t\mid t\in T\}\subset\operatorname{Diff}(M;E,F)$, like in Section 2.3. The T*-support* of A and the property of being T*-compactly supported* is defined like in the case of sections. Adapting Section 3.9, if the smooth families of functions, tangent vector fields and sections used to describe A are T-locally C^∞-uniformly bounded, then it is said that A is of T*-local bounded geometry*.

On the other hand, with the notation of Section 3.16, a smooth map $\phi\colon M_1\times T\to M_2$ is called a *smooth family of smooth maps* $M_1\to M_2$ (*parametrized* by T). It may be denoted by $\phi=\{\phi^t\mid t\in T\}$, where $\phi^t=\phi(\,\cdot\,,t)\colon M_1\to M_2$. It is said that ϕ is of T*-local bounded geometry* if every $t\in T$ is in some chart (O,z) of T such that, for some $0<r<r_1$, we have $\phi(B_1(p,r)\times O)\subset V_{2,\phi(p)}$ for all $p\in M_1$, and the compositions $y_{2,\phi(p)}\phi(y_{1,p}\times z)^{-1}$, for $p\in M_1$, define a bounded subset of the Fréchet space $C^\infty(B_{1,r}\times z(O),\mathbb{R}^{n_2})$. The composition of smooth families of maps parametrized by T has the obvious sense and preserves the T-local bounded geometry condition. In particular, for a flow $\phi=\{\phi^t\mid t\in\mathbb{R}\}=\{\phi^t\}$ on M, it makes sense to consider the $\mathbb{R}$-local bounded geometry condition. The following result complements Proposition 3.8.

Proposition 3.18. *Let $X\in\mathfrak{X}_{\mathrm{com}}(M)$ with flow ϕ. Then $X\in\mathfrak{X}_{\mathrm{ub}}(M)$ if and only if ϕ is of $\mathbb{R}$-local bounded geometry.*

Proof. The "if" part is obvious because $X_p=\phi_*(0_p,\frac{d}{dt}(0))$ for all $p\in M$, where 0_p denotes the zero element of T_pM.

Let us prove the "only if" part. First, given $0<r<r_0$ and $t_0\in\mathbb{R}$, since $|X|$ is uniformly bounded on M, there is some $\epsilon>0$ such that $\phi^t(B(p,r))\subset$

$B(\phi^{t_0}(p), r_0) = V_p$ for all $p \in M$ if $t \in O = (t_0 - \epsilon, t_0 + \epsilon)$; in particular, the compositions $y_{\phi^{t_0}(p)}\phi(y_p \times \mathrm{id}_O)^{-1}$, for all $p \in M$, define a bounded subset of $C(B_r \times O, \mathbb{R}^n)$. Then the argument of the proof of [29, Theorem V.3.1] shows that the maps $y_{\phi^{t_0}(p)}\phi(y_p \times \mathrm{id}_O)^{-1}$, for all $p \in M$, define a bounded subset of $C^1(B_r \times O, \mathbb{R}^n)$, where $B_r \subset \mathbb{R}^n$ is the ball centered at the origin of radius r. Continuing by induction on m, it also follows, like in the proofs of [29, Corollary V.3.2 and Theorem V.4.1], that the maps $y_{\phi^{t_0}(p)}\phi(y_p \times \mathrm{id}_O)^{-1}$, for all $p \in M$, define a bounded subset of $C^m(B_r \times O, \mathbb{R}^n)$ for all m. □

4. Preliminaries on foliations

Standard references on foliations are [30, 31, 10, 26, 11, 12, 49].

4.1. **Foliations.** Recall that a (*smooth*) *foliation* $\mathcal{F}$ on manifold M, with *codimension* $\operatorname{codim}\mathcal{F} = n'$ and *dimension* $\dim\mathcal{F} = n''$, can be described by a *foliated atlas* of M, which consists of charts (U_k, x_k) of the smooth structure of M, called *foliated charts* or *foliated coordinates*, with

$$x_k = (x'_k, x''_k) \colon U_k \to x_k(U_k) = \Sigma_k \times B''_k \subset \mathbb{R}^{n'} \times \mathbb{R}^{n''} \equiv \mathbb{R}^n, \tag{23}$$

such that Σ_k is open in $\mathbb{R}^{n'}$ and B''_k is an open ball in $\mathbb{R}^{n''}$, and the corresponding changes of coordinates are locally of the form

$$x_l x_k^{-1}(u, v) = (h_{lk}(u), g_{lk}(u, v)). \tag{24}$$

The notation

$$x_k = (x_k^1, \dots, x_k^n) = (x_k'^1, \dots, x_k'^{n'}, x_k''^{n'+1}, \dots, x_k''^n)$$

will be also used. In the case of codimension one, the notation $(x, y) = (x, y^1, \dots, y^{n-1})$ will be often used instead of (x', x''). It may be also said that $(M, \mathcal{F})$ is a *foliated manifold*. The open sets U_k and the projections $x'_k \colon U_k \to \Sigma_k$ are said to be *distinguished*, and the fibers of x'_k are called *plaques*. The smooth submanifolds $x_k''^{-1}(v) \subset U_k$ $(v \in B''_k)$ are called *local transversals* defined by (U_k, x_k), which can be identified with Σ_k via x'_k. All possible plaques form a base of a finer topology on M, becoming a smooth manifold of dimension n'' with the obviously induced charts whose connected components are called *leaves*. The leaf through any point p may be denoted by L_p. Foliations on manifolds with boundary are similarly defined, where the boundary is tangent or transverse to the leaves. The $\mathcal{F}$*-saturation* of a subset $S \subset M$, denoted by $\mathcal{F}(S)$, is the union of leaves that meet S.

If a smooth map $\phi \colon M' \to M$ is transverse to (the leaves of) $\mathcal{F}$, then the connected components of the inverse images $\phi^{-1}(L)$ of the leaves L of $\mathcal{F}$ are the leaves of a smooth foliation $\phi^*\mathcal{F}$ on M' of codimension n', called *pullback* of $\mathcal{F}$ by ϕ. In particular, for the inclusion map of any open subset, $\iota \colon U \hookrightarrow M$, the pullback $\iota^*\mathcal{F}$ is the *restriction* $\mathcal{F}|_U$, which can be defined by the charts of $\mathcal{F}$ with domain in U.

Given foliations $\mathcal{F}_a$ on manifolds M_a $(a = 1, 2)$, the products of leaves of $\mathcal{F}_1$ and $\mathcal{F}_2$ are the leaves of the *product foliation* $\mathcal{F}_1 \times \mathcal{F}_2$, whose charts can be defined using products of charts of $\mathcal{F}_1$ and $\mathcal{F}_2$. Any connected manifold M' can be considered as a foliation with one leaf. We can also consider the foliation by points on M', denoted by M'_{pt}. Thus we get the foliations $\mathcal{F} \times M'$ and $\mathcal{F} \times M'_{\mathrm{pt}}$ on $M \times M'$.

4.2. **Holonomy.** After considering a possible refinement, we can assume that the foliated atlas $\{U_k, x_k\}$ is *regular* in the following sense: it is locally finite; for every k, there is a foliated chart $(\widetilde{U}_k, \tilde{x}_k)$ such that $\overline{U_k} \subset \widetilde{U}_k$ and $\tilde{x}_k$ extends x_k; and, if $U_{kl} := U_k \cap U_l \neq \varnothing$, then there is another foliated chart (U, x) such that $\overline{U_k} \cup \overline{U_l} \subset U$. In this case, (24) holds on the whole of U_{kl}, obtaining diffeomorphisms $h_{kl}\colon x'_l(U_{kl}) \to x'_k(U_{kl})$ determined by the condition $h_{kl} x'_l = x'_k$ on U_{kl}, called *elementary holonomy transformations*. The collection $\{U_k, x'_k, h_{kl}\}$ is called a *defining cocycle*. The elementary holonomy transformations h_{kl} generate the so-called *holonomy pseudogroup* $\mathcal{H}$ on $\Sigma := \bigsqcup_k \Sigma_k$, which is unique up to certain *equivalence* of pseudogroups [28]. The $\mathcal{H}$-orbit of every $\bar{p} \in \Sigma$ is denoted by $\mathcal{H}(\bar{p})$. The maps x'_k define a homeomorphism between the leaf space, $M/\mathcal{F}$, and the orbit space, $\Sigma/\mathcal{H}$.

The paths in the leaves are called *leafwise paths* when considered in M. Let $c\colon I := [0,1] \to M$ be a leafwise path with $p := c(0) \in U_k$ and $q := c(1) \in U_l$, and let $\bar{p} = x'_k(p) \in \Sigma_k$ and $\bar{q} = x'_k(q) \in \Sigma_l$. There is a partition of I, $0 = t_0 < t_1 < \cdots < t_m = 1$, and a sequence of indices, $k = k_1, k_2, \dots, k_m = l$, such that $c([t_{i-1}, t_i]) \subset U_{k_i}$ for $i = 1, \dots, m$. The composition $h_c = h_{k_m k_{m-1}} \cdots h_{k_2 k_1}$ is a diffeomorphism with $\bar{p} \in \operatorname{dom} h_c \subset \Sigma_k$ and $\bar{q} = h_c(\bar{p}) \in \operatorname{im} h_c \subset \Sigma_l$. The tangent map $h_{c*}\colon T_{\bar{p}}\Sigma_k \to T_{\bar{q}}\Sigma_l$ is called *infinitesimal holonomy* of c. The germ $\mathbf{h}_c$ of h_c at $\bar{p}$, called *germinal holonomy* of c, depends only on $\mathcal{F}$ and the end-point homotopy class of c in $L := L_p$.

4.3. **Infinitesimal transformations and transverse vector fields.** The vectors tangent to the leaves form the *tangent bundle* $T\mathcal{F} \subset TM$, obtaining also the *normal bundle* $N\mathcal{F} = TM/T\mathcal{F}$, the *cotangent bundle* $T^*\mathcal{F} = (T\mathcal{F})^*$ and the *conormal bundle* $N^*\mathcal{F} = (N\mathcal{F})^*$, the *tangent/normal density bundles*, $\Omega^a\mathcal{F} = \Omega^a T\mathcal{F}$ ($a \in \mathbb{R}$) and $\Omega^a N\mathcal{F}$ (removing "a" from the notation when it is 1), and the *tangent/normal exterior bundles*, $\Lambda\mathcal{F} = \bigwedge T^*\mathcal{F}$ and $\Lambda N\mathcal{F} = \bigwedge N^*\mathcal{F}$. The complex versions of these vector bundles are taken, unless it is explicitly indicated that the real versions are considered. The terms *tangent/normal* vector fields, densities and differential forms are used for their smooth sections. Sometimes, the terms "*leafwise*" or "*vertical*" are used instead of "tangent". By composition with the canonical projection $TM \to N\mathcal{F}$, any X in TM or $\mathfrak{X}(M)$ defines an element of $N\mathcal{F}$ or $C^\infty(M; N\mathcal{F})$ denoted by $\overline{X}$. For any smooth local transversal Σ of $\mathcal{F}$ through a point $p \in M$, there is a canonical isomorphism $T_p\Sigma \cong N_p\mathcal{F}$.

A smooth vector bundle E over M, endowed with a flat $T\mathcal{F}$-partial connection, is said to be *$\mathcal{F}$-flat*. For instance, $N\mathcal{F}$ is $\mathcal{F}$-flat with the $T\mathcal{F}$-partial connection $\nabla^{\mathcal{F}}$ given by $\nabla^{\mathcal{F}}_V \overline{X} = \overline{[V, X]}$ for $V \in \mathfrak{X}(\mathcal{F}) := C^\infty(M; T\mathcal{F})$ and $X \in \mathfrak{X}(M)$. For every leafwise path c from p to q, its infinitesimal holonomy can be considered as a homomorphism $h_{c*}\colon N_p\mathcal{F} \to N_q\mathcal{F}$, which equals the $\nabla^{\mathcal{F}}$-parallel transport along c.

$\mathfrak{X}(\mathcal{F})$ is a Lie subalgebra and $C^\infty(M)$-submodule of $\mathfrak{X}(M)$, whose normalizer is denoted by $\mathfrak{X}(M,\mathcal{F})$, obtaining the quotient Lie algebra $\overline{\mathfrak{X}}(M,\mathcal{F}) = \mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F})$. The elements of $\mathfrak{X}(M,\mathcal{F})$ (respectively, $\overline{\mathfrak{X}}(M,\mathcal{F})$) are called *infinitesimal transformations* (respectively, *transverse vector fields*) of $(M,\mathcal{F})$. The projection of every $X \in \mathfrak{X}(M,\mathcal{F})$ to $\overline{\mathfrak{X}}(M,\mathcal{F})$ is also denoted by $\overline{X}$; in fact, $\overline{\mathfrak{X}}(M,\mathcal{F})$ can be identified with the linear subspace of $C^\infty(M;N\mathcal{F})$, consisting of the $\nabla^{\mathcal{F}}$-parallel normal vector fields (those that are invariant by infinitesimal holonomy). Any $X \in \mathfrak{X}(M)$ is in $\mathfrak{X}(M,\mathcal{F})$ if and only if every restriction $X|_{U_k}$ can be projected by x'_k, defining an $\mathcal{H}$-invariant vector field on Σ, also denoted by $\overline{X}$. This induces a canonical isomorphism of $\overline{\mathfrak{X}}(M,\mathcal{F})$ to the Lie algebra $\mathfrak{X}(\Sigma,\mathcal{H})$ of $\mathcal{H}$-invariant tangent vector fields on Σ.

When M is not closed, we can consider the subsets of complete vector fields, $\mathfrak{X}_{\mathrm{com}}(\mathcal{F}) \subset \mathfrak{X}(\mathcal{F})$ and $\mathfrak{X}_{\mathrm{com}}(M,\mathcal{F}) \subset \mathfrak{X}(M,\mathcal{F})$. Let $\overline{\mathfrak{X}}_{\mathrm{com}}(M,\mathcal{F}) \subset \overline{\mathfrak{X}}(M,\mathcal{F})$ be the projection of $\mathfrak{X}_{\mathrm{com}}(M,\mathcal{F})$.

4.4. **Holonomy groupoid.** On the space of leafwise paths in M, with the compact-open topology, two leafwise paths are declared to be equivalent if they have the same end points and the same germinal holonomy. This is an equivalence relation, and the corresponding quotient space, $\mathfrak{G} = \operatorname{Hol}(M,\mathcal{F})$, becomes a smooth manifold of dimension $n+n''$ in the following way. An open neighborhood $\mathfrak{U}$ of a class $[c]$ in $\mathfrak{G}$, with $c(0) \in U_k$ and $c(1) \in U_l$, is defined by the leafwise paths d such that $d(0) \in U_k$, $d(1) \in U_l$, $x'_k d(0) \in \operatorname{dom} h_c$, and h_d and h_c have the same germ at $x'_k d(0)$. Local coordinates on $\mathfrak{U}$ are given by $[d] \mapsto (x_k d(0), x''_l d(1))$. Moreover, $\mathfrak{G}$ is a Lie groupoid, called the *holonomy groupoid*, where the space of units $\mathfrak{G}^{(0)} \equiv M$ is defined by the constant paths, the source and range projections $\mathbf{s},\mathbf{r}\colon \mathfrak{G} \to M$ are given by the first and last points of the paths, and the operation is induced by the opposite of the usual path product[6]. Note that $\mathfrak{G}$ is Hausdorff if and only if $\mathcal{H}$ is *quasi-analytic* in the sense that, for any $h \in \mathcal{H}$ and every open $O \subset \Sigma$ with $\overline{O} \subset \operatorname{dom} h$, if $h|_O = \mathrm{id}_O$, then h is the identity on some neighborhood of $\overline{O}$. Observe also that $\mathbf{s},\mathbf{r}\colon \mathfrak{G} \to M$ are smooth submersions, and $(\mathbf{r},\mathbf{s})\colon \mathfrak{G} \to M^2$ is a smooth immersion. Let $\mathcal{R}_{\mathcal{F}} = \{(p,q) \in M^2 \mid L_p = L_q\} \subset M^2$, which is not a regular submanifold in general, and let $\Delta \subset M^2$ be the diagonal. We have $(\mathbf{r},\mathbf{s})(\mathfrak{G}) = \mathcal{R}_{\mathcal{F}}$ and $(\mathbf{r},\mathbf{s})(\mathfrak{G}^{(0)}) = \Delta$. For any leaf L and $p \in L$, we have $\operatorname{Hol}(L,p) = \mathbf{s}^{-1}(p) \cap \mathbf{r}^{-1}(p)$, the map $\mathbf{r}\colon \mathbf{s}^{-1}(p) \to L$ is the covering projection $\widetilde{L}^{\mathrm{hol}} \to L$, and $\mathbf{s}\colon \mathbf{r}^{-1}(p) \to L$ corresponds to $\mathbf{r}\colon \mathbf{s}^{-1}(p) \to L$ by the inversion of $\mathfrak{G}$. Thus $(\mathbf{r},\mathbf{s})\colon \mathfrak{G} \to M^2$ is injective if and only if all leaves have trivial holonomy groups, but, even in this case, it may not be a topological embedding. The fibers of $\mathbf{s}$ and $\mathbf{r}$ define smooth foliations of codimension n on $\mathfrak{G}$. We also have the smooth foliation $\mathbf{s}^*\mathcal{F} = \mathbf{r}^*\mathcal{F}$ of codimension n' with

[6]A product of leafwise paths, $c_2 \cdot c_1$, is defined if $c_1(1) = c_2(0)$, and it is equal to c_1 followed by c_2, reparametrized in the usual way.

leaves $\mathbf{s}^{-1}(L) = \mathbf{r}^{-1}(L) = (\mathbf{r},\mathbf{s})^{-1}(L^2)$ for leaves L of $\mathcal{F}$, and every restriction $(\mathbf{r},\mathbf{s})\colon (\mathbf{r},\mathbf{s})^{-1}(L^2) \to L^2$ is a smooth covering projection.

Let $\mathcal{F}_k = \mathcal{F}|_{U_k}$, $\mathfrak{G}_k = \mathrm{Hol}(U_k,\mathcal{F}_k)$ and $\mathcal{R}_k = \mathcal{R}_{\mathcal{F}_k}$. Then $\bigcup_k \mathfrak{G}_k$ (respectively, $\bigcup_k \mathcal{R}_k$) is an open neighborhood of $\mathfrak{G}^{(0)}$ in $\mathfrak{G}$ (respectively, of Δ in $\mathcal{R}_{\mathcal{F}}$). Furthermore, by the regularity of $\{U_k, x_k\}$, the map $(\mathbf{r},\mathbf{s})\colon \bigcup_k \mathfrak{G}_k \to M^2$ is a smooth embedding with image $\bigcup_k \mathcal{R}_k$; we will write $\bigcup_k \mathfrak{G}_k \equiv \bigcup_k \mathcal{R}_k$.

4.5. **The convolution algebra on $\mathfrak{G}$ and its global action.** Consider the notation of Section 4.4. For the sake of simplicity, assume that $\mathfrak{G}$ is Hausdorff. The extension of the following concepts to the case where $\mathfrak{G}$ is not Hausdorff can be made like in [15].

Given a vector bundle E over M, consider the vector bundle $S = \mathbf{r}^*E \otimes \mathbf{s}^*(E^* \otimes \Omega\mathcal{F})$ over $\mathfrak{G}$. Let $C^\infty_{\mathrm{cs}}(\mathfrak{G};S) \subset C^\infty(\mathfrak{G};S)$ denote the subspace of sections $k \in C^\infty(\mathfrak{G};S)$ such that $\operatorname{supp} k \cap \mathbf{s}^{-1}(K)$ is compact for all compact $K \subset M$; in particular, $C^\infty_{\mathrm{cs}}(\mathfrak{G};S) = C^\infty_{\mathrm{c}}(\mathfrak{G};S)$ if M is compact. Similarly, define $C^\infty_{\mathrm{cr}}(\mathfrak{G};S)$ by using $\mathbf{r}$ instead of $\mathbf{s}$. Both $C^\infty_{\mathrm{cs}}(\mathfrak{G};S)$ and $C^\infty_{\mathrm{cr}}(\mathfrak{G};S)$ are associative algebras with the *convolution* product defined by

$$\begin{aligned}(k_1 * k_2)(\gamma) &= \int_{\delta\epsilon=\gamma} k_1(\delta)\, k_2(\epsilon)\\ &= \int_{\mathbf{s}(\epsilon)=\mathbf{s}(\gamma)} k_1(\gamma\epsilon^{-1})\, k_2(\epsilon) = \int_{\mathbf{r}(\delta)=\mathbf{r}(\gamma)} k_1(\delta)\, k_2(\delta^{-1}\gamma),\end{aligned}$$

and $C^\infty_{\mathrm{csr}}(\mathfrak{G};S) := C^\infty_{\mathrm{cs}}(\mathfrak{G};S) \cap C^\infty_{\mathrm{cr}}(\mathfrak{G};S)$ and $C^\infty_{\mathrm{c}}(\mathfrak{G};S)$ are subalgebras.

There is a *global action* of $C^\infty_{\mathrm{cs}}(\mathfrak{G};S)$ on $C^\infty(M;E)$ defined by

$$(k \cdot u)(p) = \int_{\mathbf{r}(\gamma)=p} k(\gamma)\, u(\mathbf{s}(\gamma)).$$

In this way, $C^\infty_{\mathrm{cs}}(\mathfrak{G};S)$ can be understood as an algebra of operators on $C^\infty(M;E)$. Moreover, $C^\infty_{\mathrm{csr}}(\mathfrak{G};S)$ preserves $C^\infty_{\mathrm{c}}(M;E)$, obtaining an algebra of operators on $C^\infty_{\mathrm{c}}(M;E)$. It can be said that these operators are defined by a leafwise version of the Schwartz kernel (cp. Section 2.2).

Let S' be defined like S with $E^* \otimes \Omega\mathcal{F}$ instead of E. Then the mapping $k \mapsto k^{\mathrm{t}}$, where $k^{\mathrm{t}}(\gamma) = k(\gamma^{-1})$, defines the anti-homomorphisms $C^\infty_{\mathrm{cs/cr}}(\mathfrak{G};S) \to C^\infty_{\mathrm{cr/cs}}(\mathfrak{G};S')$ and $C^\infty_{\mathrm{csr}}(\mathfrak{G};S) \to C^\infty_{\mathrm{csr}}(\mathfrak{G};S')$, obtaining a leafwise version of the transposition of operators (cp. Section 2.2). Similarly, using $E = \Omega^{1/2}\mathcal{F}$, or if E has a Hermitian structure and we fix a non-vanishing leafwise density, we get a leafwise version of the adjointness of operators.

4.6. **Leafwise distance.** Assume that M is a Riemannian manifold, and consider the induced Riemannian metric on the leaves. The *leafwise distance* is the map $d_{\mathcal{F}}\colon M^2 \to [0,\infty]$ given by the distance function of the leaves on $\mathcal{R}_{\mathcal{F}}$, and with $d_{\mathcal{F}}(M^2 \setminus \mathcal{R}_{\mathcal{F}}) = \infty$. Note that $d_{\mathcal{F}} \ge d_M$. Given $p \in M$, $S \subset M$ and $r > 0$, the *open* and *closed leafwise balls*, $B_{\mathcal{F}}(p,r)$ and $\overline{B}_{\mathcal{F}}(p,r)$, and the *open* and *closed leafwise penumbras*, $\mathrm{Pen}_{\mathcal{F}}(S,r)$ and $\overline{\mathrm{Pen}}_{\mathcal{F}}(S,r)$, are defined with $d_{\mathcal{F}}$ like in the case of Riemannian metrics (see Section 3).

Equip $\mathfrak{G}$ with the Riemannian structure so that the smooth immersion $(\mathbf{r},\mathbf{s})\colon \mathfrak{G}\to M^2$ is isometric. Let $d_{\mathbf{r}}\colon \mathfrak{G}\to[0,\infty]$ denote the leafwise distance for the foliation on $\mathfrak{G}$ defined by the fibers of $\mathbf{r}$, and consider the corresponding open and closed leafwise penumbras, $\operatorname{Pen}_{\mathbf{r}}(\mathfrak{G}^{(0)},r)$ and $\overline{\operatorname{Pen}}_{\mathbf{r}}(\mathfrak{G}^{(0)},r)$. Note that we get the same penumbras by using $\mathbf{s}$ instead of $\mathbf{r}$; indeed, they are defined by the conditions $d_{\mathcal{F}}^{\mathrm{hol}}<r$ and $d_{\mathcal{F}}^{\mathrm{hol}}\le r$, respectively, where $d_{\mathcal{F}}^{\mathrm{hol}}\colon \mathfrak{G}\to[0,\infty)$ is defined by $d_{\mathcal{F}}^{\mathrm{hol}}(\gamma)=\inf_c \operatorname{length}(c)$, with c running in the piecewise smooth representatives of γ.

We have $d_{\mathcal{F}}^{\mathrm{hol}}\equiv d_{\mathcal{F}}$ on $\bigcup_k \mathfrak{G}_k\equiv\bigcup_k \mathcal{R}_k$. Using the convexity radius (see, e.g., [39, Section 6.3.2]), it follows that, after refining $\{U_k,x_k\}$ if necessary, we can assume $d_{\mathcal{F}}$ is continuous on $\bigcup_k \mathcal{R}_k$.

From now on, suppose that the leaves are complete Riemannian manifolds. Then the exponential maps of the leaves define a smooth map, $\exp_{\mathcal{F}}\colon T\mathcal{F}\to M$, on the real tangent bundle of $\mathcal{F}$.

Lemma 4.7. *For all compact $Q\subset M$ and $r>0$, $\operatorname{Pen}_{\mathcal{F}}(Q,r)$ is relatively compact in M, and $\operatorname{Pen}_{\mathbf{r}}(\mathfrak{G}^{(0)},r)\cap\mathbf{s}^{-1}(Q)$ and $\operatorname{Pen}_{\mathbf{s}}(\mathfrak{G}^{(0)},r)\cap\mathbf{r}^{-1}(Q)$ are relatively compact in $\mathfrak{G}$.*

Proof. The set $E=\{v\in T\mathcal{F}\mid \|v\|\le r\}$ is a subbundle of $T\mathcal{F}$ with compact typical fiber, $\overline{B}_{\mathbb{R}^{n''}}(0,r)$. So its restriction E_Q is compact, obtaining that $\exp_{\mathcal{F}}(E_Q)=\overline{\operatorname{Pen}}_{\mathcal{F}}(Q,r)$ is compact.

For every $v\in T\mathcal{F}$, let $c_v\colon I\to M$ denote the leafwise path defined by $c_v(t)=\exp_{\mathcal{F}}(tv)$. A smooth map $\sigma\colon T\mathcal{F}\to\mathfrak{G}$ is defined by $\sigma(v)=[c_v]$. We get that $\sigma(E_Q)=\overline{\operatorname{Pen}}_{\mathcal{F}}(\mathfrak{G}^{(0)},r)\cap\mathbf{s}^{-1}(Q)$ is compact, as well as $\overline{\operatorname{Pen}}_{\mathbf{s}}(\mathfrak{G}^{(0)},r)\cap\mathbf{r}^{-1}(Q)=(\overline{\operatorname{Pen}}_{\mathbf{r}}(\mathfrak{G}^{(0)},r)\cap\mathbf{s}^{-1}(Q))^{-1}$. □

With the notation of Section 4.5, let $C^\infty_{\mathrm{p}}(\mathfrak{G};S)\subset C^\infty(\mathfrak{G};S)$ denote the subspace of sections supported in leafwise penumbras of $\mathfrak{G}^{(0)}$. By Lemma 4.7, this is a subalgebra of $C^\infty_{\mathrm{csr}}(\mathfrak{G};S)$, and the leafwise transposition restricts to an anti-homomorphism $C^\infty_{\mathrm{p}}(\mathfrak{G};S)\to C^\infty_{\mathrm{p}}(\mathfrak{G};S')$.

4.8. **Foliated maps and foliated flows.** A *foliated map* $\phi\colon(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ is a map $\phi\colon M_1\to M_2$ that maps leaves of $\mathcal{F}_1$ to leaves of $\mathcal{F}_2$. In this case, assuming that ϕ is smooth, its tangent map defines morphisms $\phi_*\colon T\mathcal{F}_1\to T\mathcal{F}_2$ and $\phi_*\colon N\mathcal{F}_1\to N\mathcal{F}_2$, where the second one is compatible with the corresponding flat partial connections. We also get an induced Lie groupoid homomorphism $\operatorname{Hol}(\phi)\colon \operatorname{Hol}(M_1,\mathcal{F}_1)\to\operatorname{Hol}(M_2,\mathcal{F}_2)$, defined by $\operatorname{Hol}(\phi)([c])=[\phi c]$. The set of smooth foliated maps $(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ is denoted by $C^\infty(M_1,\mathcal{F}_1;M_2,\mathcal{F}_2)$. A smooth family $\phi=\{\phi^t\mid t\in T\}$ of foliated maps $(M_1,\mathcal{F}_1)\to(M_2,\mathcal{F}_2)$ can be considered as the smooth foliated map $\phi\colon(M_1\times T,\mathcal{F}_1\times T_{\mathrm{pt}})\to(M_2,\mathcal{F}_2)$.

For example, given another manifold M', if a smooth map $\psi\colon M'\to M$ is transverse to $\mathcal{F}$, then it is a foliated map $(M',\psi^*\mathcal{F})\to(M,\mathcal{F})$.

Let $\operatorname{Diffeo}(M,\mathcal{F})$ be the group of foliated diffeomorphisms (or transformations) of $(M,\mathcal{F})$. A smooth flow $\phi=\{\phi^t\}$ on M is called *foliated* if $\phi^t\in\operatorname{Diffeo}(M,\mathcal{F})$ for all $t\in\mathbb{R}$. More generally, a local flow $\phi\colon\Omega\to M$, defined on some open neighborhood Ω of $M \times \{0\}$ in $M \times \mathbb{R}$, is called *foliated* if it is a foliated map $(\Omega, (\mathcal{F} \times \mathbb{R}_{\mathrm{pt}})|_\Omega) \to (M, \mathcal{F})$. Then $\mathfrak{X}(M, \mathcal{F})$ consists of the smooth vector fields whose local flow is foliated, and $\mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$ consists of the complete smooth vector fields whose flow is foliated.

Let $X \in \mathfrak{X}_{\mathrm{com}}(M, \mathcal{F})$, with foliated flow $\phi = \{\phi^t\}$. With the notation of Sections 4.2 and 4.3, let $\bar\phi$ be the local flow on Σ generated by $\overline{X} \in \mathfrak{X}(\Sigma, \mathcal{H})$. Since $X|_{U_k}$ corresponds to $\overline{X}|_{\Sigma_k}$ via $x'_k \colon U_k \to \Sigma_k$, the local flow defined by ϕ on every U_k also corresponds to the restriction of $\bar\phi$ to Σ_k. Hence $\bar\phi$ is $\mathcal{H}$-equivariant in an obvious sense.

4.9. **Differential operators on foliated manifolds.** Like in Section 2.3, using $\mathfrak{X}(\mathcal{F})$ instead of $\mathfrak{X}(M)$, we get the filtered $C^\infty(M)$-submodule and subalgebra of *leafwise differential operators*, $\mathrm{Diff}(\mathcal{F}) \subset \mathrm{Diff}(M)$, and a *leafwise principal symbol* surjection for every order m,

$$^{\mathcal{F}}\!\sigma_m \colon \mathrm{Diff}^m(\mathcal{F}) \to P^{[m]}(T^*\mathcal{F}) \to 0,$$

whose kernel is $\mathrm{Diff}^{m-1}(\mathcal{F})$. Moreover, these concepts can be extended to vector bundles E and F over M like in Section 2.3, obtaining the filtered $C^\infty(M)$-submodule $\mathrm{Diff}(\mathcal{F}; E, F)$ (or $\mathrm{Diff}(\mathcal{F}; E)$ if $E = F$) of $\mathrm{Diff}(M; E, F)$, and the *leafwise principal symbol* surjection

$$^{\mathcal{F}}\!\sigma_m \colon \mathrm{Diff}^m(\mathcal{F}; E, F) \to P^{[m]}(T^*\mathcal{F}; F \otimes E^*),$$

whose kernel is $\mathrm{Diff}^{m-1}(\mathcal{F}; E, F)$. The diagram

$$\begin{array}{ccc} \mathrm{Diff}^m(\mathcal{F}; E, F) & \xrightarrow{^{\mathcal{F}}\!\sigma_m} & P^{[m]}(T^*\mathcal{F}; F \otimes E^*) \\ \downarrow & & \downarrow \\ \mathrm{Diff}^m(M; E, F) & \xrightarrow{\sigma_m} & P^{[m]}(T^*M; F \otimes E^*) \end{array} \tag{25}$$

is commutative, where the left-hand side vertical arrow denotes the inclusion homomorphism, and the right-hand side vertical arrow is defined by the restriction morphism $T^*M \to T^*\mathcal{F}$. The condition of being a leafwise differential operator is preserved by compositions and by taking transposes, and by taking formal adjoints in the case of Hermitian vector bundles; in particular, $\mathrm{Diff}(\mathcal{F}; E)$ is a filtered subalgebra of $\mathrm{Diff}(M; E)$. It is said that $A \in \mathrm{Diff}^m(\mathcal{F}; E, F)$ is *leafwisely elliptic* if the leafwise symbol $^{\mathcal{F}}\!\sigma_m(A)(p, \xi)$ is an isomorphism for all $p \in M$ and $0 \neq \xi \in T^*_p\mathcal{F}$.

A smooth family of leafwise differential operators, $A = \{A_t \mid t \in T\}$ with $A_t \in \mathrm{Diff}^m(\mathcal{F}; E, F)$, can be canonically considered as a leafwise differential operator $A \in \mathrm{Diff}^m(\mathcal{F} \times T_{\mathrm{pt}}; \mathrm{pr}_1^* E, \mathrm{pr}_1^* F)$, where $\mathrm{pr}_1 \colon M \times T \to M$ is the first factor projection.

On the other hand, using the canonical injection $N^*\mathcal{F} \subset T^*M$, it is said that $A \in \mathrm{Diff}^m(M; E, F)$ is *transversely elliptic* if the symbol $\sigma_m(A)(p, \xi)$ is an isomorphism for all $p \in M$ and $0 \neq \xi \in N^*_p\mathcal{F}$.

4.10. **Riemannian foliations.** The $\mathcal{H}$-invariant structures on Σ are called (*invariant*) *transverse structures*. For instance, we will use the concepts of a *transverse orientation*, a *transverse Riemannian metric*, and a *transverse parallelism*. The existence of these transverse structures defines the classes of *transversely orientable*, (*transversely*) *Riemannian*, and *transversely parallelizable* (*TP*) foliations. If a transverse parallelism of $\mathcal{F}$ is a base of a Lie subalgebra $\mathfrak{g} \subset \mathfrak{X}(\Sigma, \mathcal{H})$, it gives rise to the concepts of *transverse Lie structure* and ($\mathfrak{g}$-)*Lie foliation*. If G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, then $\mathcal{F}$ is a $\mathfrak{g}$-Lie foliation just when $\mathcal{H}$ is equivalent to some pseudogroup on G generated by some left translations.

By using the canonical isomorphism $\overline{\mathfrak{X}}(M, \mathcal{F}) \cong \mathfrak{X}(\Sigma, \mathcal{H})$, the condition on $\mathcal{F}$ to be TP means that there is a global frame of $N\mathcal{F}$ consisting of transverse vector fields $\overline{X_1}, \dots, \overline{X_{n'}}$, also called a *transverse parallelism*; and the condition on $\mathcal{F}$ to be a $\mathfrak{g}$-Lie foliation means that, moreover, $\overline{X_1}, \dots, \overline{X_{n'}}$ form a base of a Lie subalgebra of $\mathfrak{g} \subset \overline{\mathfrak{X}}(M, \mathcal{F})$. With this point of view, if, moreover, $\overline{X_1}, \dots, \overline{X_{n'}} \in \overline{\mathfrak{X}}_{\mathrm{com}}(M, \mathcal{F})$, then the TP or Lie foliation $\mathcal{F}$ is called *complete*.

Similarly, a transverse Riemannian metric can be described as a Euclidean structure on $N\mathcal{F}$ that is invariant by infinitesimal holonomy. In turn, this is induced by a Riemannian metric on M such that every $x'_k \colon U_k \to \Sigma_k$ is a Riemannian submersion, called *bundle-like metric*. Thus $\mathcal{F}$ is Riemannian if and only if there is a bundle-like metric on M.

It is said that $\mathcal{F}$ is *transitive at* a point $p \in M$ when the evaluation map $\mathrm{ev}_p \colon \mathfrak{X}(M, \mathcal{F}) \to T_pM$ is surjective, or, equivalently, the evaluation map $\overline{\mathrm{ev}}_p \colon \overline{\mathfrak{X}}(M, \mathcal{F}) \subset C^\infty(M; N\mathcal{F}) \to N_p\mathcal{F}$ is surjective. The transitive point set is open and saturated. The foliation $\mathcal{F}$ is called *transitive* if it is transitive at every point. It is said that $\mathcal{F}$ is *transversely complete* (*TC*) if $\mathrm{ev}_p(\mathfrak{X}_{\mathrm{com}}(M, \mathcal{F}))$ generates T_pM for all $p \in M$. Since the evaluation map $\mathfrak{X}_{\mathrm{com}}(\mathcal{F}) \to T_p\mathcal{F}$ is surjective [37, Section 4.5], $\mathcal{F}$ is TC if and only if $\overline{\mathrm{ev}}_p(\overline{\mathfrak{X}}_{\mathrm{com}}(M, \mathcal{F}))$ generates $N_p\mathcal{F}$ for all $p \in M$.

All TP foliations are transitive, and all transitive foliations are Riemannian. On the other hand, Molino's theory [37] describes Riemannian foliations in terms of TP foliations. A Riemannian foliation is called *complete* if, using Molino's theory, the corresponding TP foliation is TC. Furthermore, Molino's theory describes TC foliations in terms of complete Lie foliations with dense leaves. In turn, complete Lie foliations have the following description due to Fedida [24, 25] (see also [37, Theorem 4.1 and Lemma 4.5]). Assume that M is connected and $\mathcal{F}$ is a complete $\mathfrak{g}$-Lie foliation. Let G be the simply connected Lie group whose Lie algebra (of left-invariant vector fields) is (isomorphic to) $\mathfrak{g}$. Then there is a regular covering space, $\pi \colon \widetilde{M} \to M$, a fiber bundle $D \colon \widetilde{M} \to G$ (the *developing* map) and a monomorphism $h \colon \Gamma := \mathrm{Aut}(\pi) \equiv \pi_1 L / \pi_1 \widetilde{L} \to G$ (the *holonomy* homomorphism) such that the leaves of $\widetilde{\mathcal{F}} := \pi^*\mathcal{F}$ are the fibers of D, and D is h-equivariant with respect to the left action of G on itself by left translations. As a consequence, π restricts to diffeomorphisms between the leaves of $\widetilde{\mathcal{F}}$ and $\mathcal{F}$. The subgroup $\mathrm{Hol}\,\mathcal{F} = \mathrm{im}\,h \subset G$, isomorphic to Γ, is called the *global holonomy group*. The $\widetilde{\mathcal{F}}$-leaf through every $\tilde{p} \in \widetilde{M}$ will be denoted by $\widetilde{L}_{\tilde{p}}$. Since D induces an identity $\widetilde{M}/\widetilde{\mathcal{F}} \equiv G$, the π-lift and D-projection of vector fields define the identities[7]

$$\overline{\mathfrak{X}}(M,\mathcal{F}) \equiv \overline{\mathfrak{X}}(\widetilde{M},\widetilde{\mathcal{F}},\Gamma) \equiv \mathfrak{X}(G,\operatorname{Hol}\mathcal{F}). \tag{26}$$

These identities give a precise realization of $\mathfrak{g} \subset \overline{\mathfrak{X}}(M,\mathcal{F})$ as the Lie algebra of left invariant vector fields on G. The holonomy pseudogroup of $\mathcal{F}$ is equivalent to the pseudogroup on G generated by the action of $\operatorname{Hol}\mathcal{F}$ by left translations. Thus the leaves are dense if and only if $\operatorname{Hol}\mathcal{F}$ is dense in G, which means $\mathfrak{g} = \overline{\mathfrak{X}}(M,\mathcal{F})$.

4.11. **Differential forms on foliated manifolds.**

4.11.1. *The leafwise complex.* Let $d_{\mathcal{F}} \in \operatorname{Diff}^1(\mathcal{F};\Lambda\mathcal{F})$ be given by $(d_{\mathcal{F}}\xi)|_L = d_L(\xi|_L)$ for every leaf L and $\xi \in C^\infty(M;\Lambda\mathcal{F})$. Then $(C^\infty(M;\Lambda\mathcal{F}),d_{\mathcal{F}})$ is a differential complex, called the *leafwise* (*de Rham*) *complex*. This gives rise to the (*reduced*) *leafwise cohomology*[8] (with complex coefficients), $H^*(\mathcal{F}) = H^*(\mathcal{F};\mathbb{C})$ and $\bar{H}^*(\mathcal{F}) = \bar{H}^*(\mathcal{F};\mathbb{C})$. Compactly supported versions may be also considered when M is not compact.

Similarly, we can take coefficients in any complex $\mathcal{F}$-flat vector bundle E over M, obtaining the differential complex $(C^\infty(M;\Lambda\mathcal{F}\otimes E),d_{\mathcal{F}})$, with $d_{\mathcal{F}} \in \operatorname{Diff}^1(\mathcal{F};\Lambda\mathcal{F}\otimes E)$, and the corresponding (*reduced*) *leafwise cohomology* with coefficients in E, $H^*(\mathcal{F};E)$ and $\bar{H}^*(\mathcal{F};E)$. For example, we can consider the vector bundle E defined by the $\mathrm{GL}(n')$-principal bundle of (real) normal frames and any unitary representation of $\mathrm{GL}(n')$, with the $\mathcal{F}$-flat structure induced by the $\mathcal{F}$-flat structure of $N\mathcal{F}$. A particular case is $\Lambda N\mathcal{F}$, which gives rise to the differential complex $(C^\infty(M;\Lambda\mathcal{F}\otimes\Lambda N\mathcal{F}),d_{\mathcal{F}})$ and its compactly supported version. Note that

$$\Lambda\mathcal{F} \equiv \Lambda\mathcal{F}\otimes\Lambda^0 N\mathcal{F} \subset \Lambda\mathcal{F}\otimes\Lambda N\mathcal{F},$$

inducing an injection of topological complexes and their (reduced) cohomologies, and the same holds for the compactly supported versions. In fact, these are topological graded differential algebras with the exterior product, and the above injections are compatible with the product structures.

For any $\phi \in C^\infty(M_1,\mathcal{F}_1;M_2,\mathcal{F}_2)$, the morphisms $\phi_*\colon T\mathcal{F}_1 \to T\mathcal{F}_2$ and $\phi_*\colon N\mathcal{F}_1 \to N\mathcal{F}_2$ induce a morphism

$$\phi^*\colon \phi^*(\Lambda\mathcal{F}_2\otimes\Lambda N\mathcal{F}_2) \to \Lambda\mathcal{F}_1\otimes\Lambda N\mathcal{F}_1$$

over id_{M_1}, which in turn induces a continuous homomorphism of graded differential algebras,

$$\phi^*\colon C^\infty(M_2;\Lambda\mathcal{F}_2\otimes\Lambda N\mathcal{F}_2) \to C^\infty(M_1;\Lambda\mathcal{F}_1\otimes\Lambda N\mathcal{F}_1), \tag{27}$$

[7]Given an action, the group is added to the notation of a space of vector fields to indicate the subspace of invariant elements.

[8]The term *tangential cohomology* is also used.

and continuous homomorphisms between the corresponding (reduced) leafwise cohomologies. By restriction, we get the homomorphism

$$\phi^*\colon C^\infty(M_2;\Lambda\mathcal{F}_2)\to C^\infty(M_1;\Lambda\mathcal{F}_1),$$

with analogous properties.

4.11.2. *Bigrading.* Consider any splitting

$$TM = T\mathcal{F}\oplus\mathbf{H}\cong T\mathcal{F}\oplus N\mathcal{F}, \tag{28}$$

given by a transverse distribution $\mathbf{H}\subset TM$, and let $\Lambda\mathbf{H}=\bigwedge\mathbf{H}^*$. It induces a decomposition

$$\Lambda M\equiv\Lambda\mathcal{F}\otimes\Lambda\mathbf{H}\cong\Lambda\mathcal{F}\otimes\Lambda N\mathcal{F}, \tag{29}$$

giving rise to the bigrading of ΛM defined by[9]

$$\Lambda^{u,v}M\equiv\Lambda^v\mathcal{F}\otimes\Lambda^u\mathbf{H}\cong\Lambda^v\mathcal{F}\otimes\Lambda^u N\mathcal{F},$$

and the corresponding bigrading of $C^\infty(M;\Lambda)$ with terms

$$C^\infty(M;\Lambda^{u,v})\equiv C^\infty(M;\Lambda^v\mathcal{F}\otimes\Lambda^u N\mathcal{F}).$$

This bigrading depends on $\mathbf{H}$, but the spaces $\Lambda^{\geq u,\cdot}M$ and $C^\infty(M;\Lambda^{\geq u,\cdot})$ are independent of $\mathbf{H}$ (see, e.g., [1]). In particular, every $\Lambda^{\geq u,\cdot}M/\Lambda^{\geq u+1,\cdot}M$ is independent of $\mathbf{H}$; indeed, there are canonical identities

$$\Lambda^{\geq u,\cdot}M/\Lambda^{\geq u+1,\cdot}M\equiv\Lambda^{u,\cdot}M\equiv\Lambda\mathcal{F}\otimes\Lambda^u N\mathcal{F}, \tag{30}$$

where only the middle bundle depends on $\mathbf{H}$. The de Rham derivative on $C^\infty(M;\Lambda)$ decomposes into bi-homogeneous components,

$$d=d_{0,1}+d_{1,0}+d_{2,-1}, \tag{31}$$

where the double subindex denotes the corresponding bi-degree. We have

$$d_{0,1}\in\operatorname{Diff}^1(\mathcal{F};\Lambda M),\quad d_{1,0}\in\operatorname{Diff}^1(M;\Lambda),\quad d_{2,-1}\in\operatorname{Diff}^0(M;\Lambda).$$

Moreover,[10]

$$d_{0,1}\equiv d_{\mathcal{F}}, \tag{32}$$

via (29), and $d_{2,-1}=0$ if and only if $\mathbf{H}$ is completely integrable. Note that

$$d_{0,1}=d\colon C^\infty(M;\Lambda^{n',\cdot})\to C^\infty(M;\Lambda^{n',\cdot}). \tag{33}$$

By comparing bi-degrees in $d^2=0$, we get (see, e.g., [1])

$$d_{0,1}^2=d_{0,1}d_{1,0}+d_{1,0}d_{0,1}=0. \tag{34}$$

For any $\phi\in C^\infty(M_1,\mathcal{F}_1;M_2,\mathcal{F}_2)$, we have restrictions

$$\phi^*\colon C^\infty(M_2;\Lambda^{\geq u,\cdot})\to C^\infty(M_1;\Lambda^{\geq u,\cdot})$$

[9]We have reversed the order given in [2] for the factors of the tensor product in the definition of $\Lambda^{u,v}M$ because the signs in some expressions become simpler. But we keep the same order for the "transverse degree" u and the "tangential degree" v in $\Lambda^{u,v}M$ because this is the usual order in the Leray spectral sequence of fiber bundles, generalized to foliations.

[10]The sign of [2, Lemma 3.4] is omitted here by our change in the definition of $\Lambda^{u,v}M$.

of $\phi^*\colon C^\infty(M_2;\Lambda)\to C^\infty(M_1;\Lambda)$, which induce (27) using (30). Like in (31) and (32),

$$\phi^*=\phi^*_{0,0}+\phi^*_{1,-1}+\cdots\colon C^\infty(M_2;\Lambda)\to C^\infty(M_1;\Lambda) \tag{35}$$

and

$$\phi^*_{0,0}\equiv\phi^*, \tag{36}$$

via (29), where the right-hand side is (27).

For any $X\in\mathfrak{X}(M)$, let ι_X denote the corresponding inner product, and let $\mathbf{V}\colon TM\to T\mathcal{F}$ and $\mathbf{H}\colon TM\to\mathbf{H}$ denote the projections defined by (28). By comparing bi-degrees in Cartan's formula, $\mathcal{L}_X=d\iota_X+\iota_X d$, we get a decomposition into bi-homogeneous components,

$$\mathcal{L}_X=\mathcal{L}_{X,-1,1}+\mathcal{L}_{X,0,0}+\mathcal{L}_{X,1,-1}+\mathcal{L}_{X,2,-2};$$

for instance,

$$\mathcal{L}_{X,0,0}=d_{0,1}\iota_{\mathbf{V}X}+\iota_{\mathbf{V}X}d_{0,1}+d_{1,0}\iota_{\mathbf{H}X}+\iota_{\mathbf{H}X}d_{1,0}. \tag{37}$$

It is easy to check that $\mathcal{L}_{X,-1,1}$, $\mathcal{L}_{X,1,-1}$ and $\mathcal{L}_{X,2,-2}$ are of order zero, and

$$\mathcal{L}_{X,0,0}(\alpha\wedge\beta)=\mathcal{L}_{X,0,0}\alpha\wedge\beta+\alpha\wedge\mathcal{L}_{X,0,0}\beta. \tag{38}$$

Moreover, $\mathcal{L}_{X,0,0}=X$ and $\mathcal{L}_{X,-1,1}=\mathcal{L}_{X,1,-1}=\mathcal{L}_{X,2,-2}=0$ on $C^\infty(M)$.

Assume that $X\in\mathfrak{X}(M,\mathcal{F})$ from now on. Then $\mathcal{L}_{X,-1,1}=0$ by (35), and therefore

$$\mathcal{L}_{X,0,0}d_{0,1}=d_{0,1}\mathcal{L}_{X,0,0}, \tag{39}$$

by comparing bi-degrees in the formula $\mathcal{L}_Xd=d\mathcal{L}_X$. Let Θ_X be the operator on $C^\infty(M;\Lambda\mathcal{F}\otimes\Lambda N\mathcal{F})$ that corresponds to $\mathcal{L}_{X,0,0}$ via (29). By (38) and (39),

$$\Theta_X(\xi\wedge\zeta)=\Theta_X\xi\wedge\zeta+\xi\wedge\Theta_X\zeta,\quad \Theta_Xd_{\mathcal{F}}=d_{\mathcal{F}}\Theta_X.$$

Let (U,x) be a foliated chart of $\mathcal{F}$, with $x=(x',x'')$, like in (23). To emphasize the difference between the coordinates x' and x'', we use the following notation on U or $x(U)$. Let $x'^i=x^i$ and $\partial'_i=\partial_i$ for $i\le n'$, and $x''^i=x^i$ and $\partial''_i=\partial_i$ for $i>n'$. Thus, when using x'^i or ∂'_i, it will be understood that i runs in $\{1,\dots,n'\}$, and, when using x''^i or ∂''_i, it will be understood that i runs in $\{n'+1,\dots,n\}$. For multi-indices of the form $I=(i_1,\dots,i_n)\in\mathbb{N}_0^n$, write $\partial_I=\partial'_I\partial''_I$, where $\partial'_I=\partial_1^{i_1}\cdots\partial_{n'}^{i_{n'}}$ and $\partial''_I=\partial_{n'+1}^{i_{n'+1}}\cdots\partial_n^{i_n}$. For multi-indices of the form $J=\{j_1,\dots,j_r\}$ with $1\le j_1<\cdots<j_r\le n$, let $dx^J=dx^{j_1}\wedge\cdots\wedge dx^{j_r}$ be denoted by dx'^J or dx''^J if J only contains indices in $\{1,\dots,n'\}$ or $\{n'+1,\dots,n\}$, respectively. Using functions $f_I,f_{IJ}\in C^\infty(U)$, $d_{\mathcal{F}}$ can be locally described by

$$d_{\mathcal{F}}(f_I\,dx''^I)=\partial''_jf_I\,dx''^j\wedge dx''^I, \tag{40}$$

and (32) means that

$$d_{0,1}(f_{IJ}\,dx''^I\wedge dx'^J)=d_{\mathcal{F}}(f_{IJ}\,dx''^I)\wedge dx'^J. \tag{41}$$

4.11.3. *Compatibility of orientations.* A transverse orientation of $\mathcal{F}$ can be described as a (necessarily $\nabla^{\mathcal{F}}$-invariant) orientation of $N\mathcal{F}$. It is determined by a non-vanishing real form $\omega \in C^\infty(M; \Lambda^{n'} N\mathcal{F})$; i.e., some real $\omega \in C^\infty(M; \Lambda^{n'})$ with

$$T\mathcal{F} = \{Y \in TM \mid \iota_Y \omega = 0\}.$$

On the other hand, an orientation of $T\mathcal{F}$ is called an *orientation* of $\mathcal{F}$, which can be described by a non-vanishing form $\chi \in C^\infty(M; \Lambda^{n''}\mathcal{F}) \equiv C^\infty(M; \Lambda^{0,n''})$. When $\mathcal{F}$ is equipped with a transverse orientation (respectively, an orientation), it is said to be *transversely oriented* (respectively, *oriented*). Given transverse and tangential orientations of $\mathcal{F}$ described by forms ω and χ as above, we get an induced orientation of M defined by the non-vanishing form $\chi \wedge \omega \in C^\infty(M; \Lambda^{n',n''}) = C^\infty(M; \Lambda^n)$.

Suppose that, moreover, M is a Riemannian manifold, and take $\mathbf{H} = T\mathcal{F}^\perp$. Then, using (29), the induced Hodge star operators, $\star$ on ΛM, $\star_{\mathcal{F}}$ on $\Lambda\mathcal{F}$ and $\star_\perp$ on $\Lambda\mathbf{H}$ satisfy[11] [8, Lemma 4.8], [2, Lemma 3.2],

$$\star \equiv (-1)^{u(n''-v)} \star_{\mathcal{F}} \otimes \star_\perp \colon \Lambda^{u,v} M \to \Lambda^{n'-u,n''-v} M. \tag{42}$$

If we take $\omega = \star_\perp 1 \in C^\infty(M; \Lambda^{n'}\mathbf{H}) \equiv C^\infty(M; \Lambda^{n',0})$ and $\chi = \star_{\mathcal{F}} 1 \in C^\infty(M; \Lambda^{n''}\mathcal{F}) \equiv C^\infty(M; \Lambda^{0,n''})$, then $\chi \wedge \omega = \star 1 \in C^\infty(M; \Lambda^n)$.

4.11.4. *Bihomogeneous components of the coderivative.* Let g be a Riemannian metric on M. On the one hand, g induces a Hermitian structure on $\Lambda\mathcal{F} \otimes \Lambda N\mathcal{F}$, and we can consider $\delta_{\mathcal{F}} = d_{\mathcal{F}}^*$ on $C^\infty(M; \Lambda\mathcal{F} \otimes \Lambda N\mathcal{F})$. On the other hand, by taking formal adjoints in (31) with $\mathbf{H} = T\mathcal{F}^\perp$, we get the decomposition into bi-homogeneous components,

$$\delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1}, \tag{43}$$

where $\delta_{-i,-j} = d_{i,j}^*$. From (34), it follows that

$$\delta_{0,-1}^2 = \delta_{0,-1}\delta_{-1,0} + \delta_{-1,0}\delta_{0,-1} = 0. \tag{44}$$

Lemma 4.12. *The metric g is bundle-like if and only if $\delta_{0,-1} \equiv \delta_{\mathcal{F}}$ via* (29).

Proof. By working locally, we can assume that $\mathcal{F}$ is transversely oriented and oriented, and consider the induced orientation of M according to Section 4.11.3. By (41) and (42), and since $\star_\perp$ determines $g|_{\mathbf{H}}$, we get that g is bundle like if and only if $d_{0,1}$ commutes with $1 \otimes \star_\perp$, which is equivalent to $\delta_{0,-1} \equiv \delta_{\mathcal{F}}$ by (12). □

With the notation of (41), the equality $\delta_{0,-1} \equiv \delta_{\mathcal{F}}$ means that

$$\delta_{0,-1}(f_{IJ}\, dx''^I \wedge dx'^J) = \delta_{\mathcal{F}}(f_{IJ}\, dx''^I) \wedge dx'^J. \tag{45}$$

[11] The sign of this expression is different in [2, Lemma 3.2] by the different choice of induced orientation of M, given by $\omega \wedge \chi$ in that paper.

5. Riemannian foliations of bounded geometry

With the notation of Section 4, suppose that $\mathcal{F}$ is Riemannian. Let g be a bundle-like metric on M, ∇ its Levi-Civita connection and R its curvature.

The vector subbundle $T\mathcal{F}^\perp \subset TM$ is called *horizontal*, giving rise to the concepts of *horizontal* vectors, vector fields and frames. Now, we take $\mathbf{H} = T\mathcal{F}^\perp$ in (28), and therefore $\mathbf{V}\colon TM \to T\mathcal{F}$ and $\mathbf{H}\colon TM \to \mathbf{H}$ are the orthogonal projections. The O'Neill tensors [38] of the local Riemannian submersions defining $\mathcal{F}$ can be combined to produce $(1,2)$-tensors T and A on M, defined by

$$\mathsf{T}_E F = \mathbf{H}\nabla_{\mathbf{V}E}(\mathbf{V}F) + \mathbf{V}\nabla_{\mathbf{V}E}(\mathbf{H}F),$$
$$\mathsf{A}_E F = \mathbf{H}\nabla_{\mathbf{H}E}(\mathbf{V}F) + \mathbf{V}\nabla_{\mathbf{H}E}(\mathbf{H}F),$$

for all $E, F \in \mathfrak{X}(M)$. According to [38, Theorem 4], if M is connected, given g and any $p \in M$, the foliation $\mathcal{F}$ is determined by T, A and $T_p\mathcal{F}$.

A Riemannian connection $\mathring{\nabla}$ on M, called *adapted*, is defined by [8]

$$\mathring{\nabla}_E F = \mathbf{V}\nabla_E(\mathbf{V}F) + \mathbf{H}\nabla_E(\mathbf{H}F),$$

for all $E, F \in \mathfrak{X}(M)$. For $V, W \in \mathfrak{X}(\mathcal{F})$ and $X \in C^\infty(M; \mathbf{H})$, we have

$$\nabla_V - \mathring{\nabla}_V = \mathsf{T}_V, \quad \nabla_X - \mathring{\nabla}_X = \mathsf{A}_X, \tag{46}$$

and (see [5, Eqs. (3.8)–(3.10)])

$$\nabla^{\mathcal{F}}_V W = \mathring{\nabla}_V W, \tag{47}$$
$$\nabla^{\mathcal{F}}_V \overline{X} = \overline{\mathring{\nabla}_V X - \mathsf{A}_X V},$$
$$\mathbf{V}([X, V]) = \mathring{\nabla}_X V - \mathsf{T}_V X. \tag{48}$$

By (47), the $\mathring{\nabla}$-geodesics that are tangent to the leaves at some point remain tangent to the leaves at every point, and they are the geodesics of the leaves. So the leaves are $\mathring{\nabla}$-totally geodesic, but not necessarily ∇-totally geodesic. By the second equality of (46) and [38, Lemma 2], $\mathring{\nabla}$ and ∇ have the same geodesics orthogonal to the leaves.

Given any $p \in M$, let $x'\colon U \to \Sigma$ be a distinguished submersion so that $p \in U$. Consider the Riemannian metric on Σ such that x' is a Riemannian submersion, and let $\check{\nabla}$ and $\check{\exp}$ denote the corresponding Levi-Civita connection and exponential map of Σ. From [38, Lemma 1 (3)], it follows that $\mathring{\nabla}_X Y \in \mathfrak{X}(U, \mathcal{F}|_U)$ for all horizontal $X, Y \in \mathfrak{X}(U, \mathcal{F}|_U)$ and, moreover,

$$\overline{\mathring{\nabla}_X Y} = \check{\nabla}_{\overline{X}} \overline{Y}. \tag{49}$$

Let $\mathring{\exp}$ denote the exponential map of the geodesic spray of $\mathring{\nabla}$ (see, e.g., [40, pp. 96–99]). Observe that the exponential map of the leaves is a restriction of $\mathring{\exp}$. The maps $\mathring{\exp}$ and $\check{\exp}$ restrict to diffeomorphisms of some open neighborhoods, V of 0 in T_pM and $\check{V}$ of 0 in $T_{x'(p)}\Sigma$, to some open neighborhoods,

O of p in M and $\check O$ of $x'(p)$ in Σ. Moreover, we can suppose that $O \subset U$, $x'_*(V) \subset \check V$ and $x'(O) \subset \check O$. By (49),

$$x' \mathring{\exp} = \check{\exp} x'_* \tag{50}$$

on $V \cap T_p\mathcal{F}^\perp$. Let κ_p (or simply κ) be the smooth map of some neighborhood W of 0 in T_pM to M defined by

$$\kappa_p(X) = \mathring{\exp}_q(\mathring{P}_{\mathbf{H}X}\mathbf{V}X),$$

where $q = \mathring{\exp}_p(\mathbf{H}X)$, and $\mathring{P}_{\mathbf{H}X} \colon T_pM \to T_qM$ denotes the $\mathring{\nabla}$-parallel transport along the $\mathring{\nabla}$-geodesic $t \mapsto \mathring{\exp}_p(t\mathbf{H}X)$, $0 \le t \le 1$, which is also a ∇-geodesic because it is orthogonal to the leaves. By choosing W small enough, we have $W \subset V$ and $\kappa(W) \subset O$; thus $x'_*(W) \subset \check V$ and $x'\kappa(W) \subset \check O$. For $X, Y \in W$, we have $X - Y \in T_p\mathcal{F}$ if and only if $\kappa(X)$ and $\kappa(Y)$ belong to the same plaque of U [5, Proposition 6.1]. We also have $x'\kappa(X) = \check{\exp} x'_*(X)$ for all $X \in W \cap T_p\mathcal{F}^\perp$ by (50). Furthermore, κ defines a diffeomorphism of some neighborhood of 0 in T_pM to some neighborhood of p in M. By choosing horizontal and vertical orthonormal frames at p, we get identities $T_p\mathcal{F}^\perp \equiv \mathbb{R}^{n'}$ and $T_p\mathcal{F} \equiv \mathbb{R}^{n''}$. Then, for some open balls centered at the origin, B' in $\mathbb{R}^{n'}$ and B'' in $\mathbb{R}^{n''}$, we can assume that κ is a diffeomorphism of $B' \times B''$ to some open neighborhood of p. From now on, we use the notation $U = \kappa(B' \times B'')$ and $\kappa^{-1} = x = (x', x'')$ on U, like in (23). This foliated chart (U, x) is called *normal*, as well as the foliated coordinates x. As usual, g_{ij} denotes the corresponding coefficients of the bundle-like metric, and let $(g^{ij}) = (g_{ij})^{-1}$. On U, we have[12]

$$\mathbf{V} = g_{ik}g^{kj}\,\partial''_j \otimes dx'^i + \partial''_i \otimes dx''^i, \tag{51}$$

$$\mathbf{H} = \partial'_i \otimes dx'^i - g_{ik}g^{kj}\,\partial''_j \otimes dx'^i, \tag{52}$$

where k runs in $\{n'+1, \dots, n\}$, see [5, Eq. (7.2)].

It will be said that $\mathcal{F}$ has *positive injectivity bi-radius*[13] if there are normal foliated coordinates $x_p \colon U_p \to B' \times B''$ at every $p \in M$ such that the balls B' and B'' are independent of p.

Definition 5.1 (Alvarez–Kordyukov–Leichtnam [5, Definition 8.1])**.** It is said that $\mathcal{F}$ is of *bounded geometry* if it has positive injectivity bi-radius, and the functions $|\nabla^m R|$, $|\nabla^m \mathsf{T}|$ and $|\nabla^m \mathsf{A}|$ are uniformly bounded on M for every $m \in \mathbb{N}_0$.

Another definition of bounded geometry for Riemannian foliations was given by Sanguiao [44, Definition 1.7]. Definition 5.1 also has the following chart characterization, which is at least as strong as Sanguiao's definition [5, Remark 8.5].

[12] We use the convention that repeated indices are summed.

[13] In [5, Section 8], the concept of transverse injectivity radii was introduced for a defining cocycle, and it was wrongly stated that its positivity is independent of the defining cocycle. Then some step in the proof of [5, Theorem 8.4] does not work. This problem is clearly solved with the new concept of positive injectivity bi-radius.

Theorem 5.2 (Álvarez–Kordyukov–Leichtnam [5, Theorem 8.4])**.** *With the above notation, $\mathcal{F}$ is of bounded geometry if and only if there is a normal foliated chart $x_p\colon U_p \to B' \times B''$ at every $p \in M$ such that the balls B' and B'' are independent of p, and the corresponding coefficients g_{ij} and g^{ij}, as family of smooth functions on $B' \times B''$ parametrized by i, j and p, lie in a bounded subset of the Fréchet space $C^\infty(B' \times B'')$.*

In this section, assume from now on that $\mathcal{F}$ is of bounded geometry. Then M and the disjoint union of the leaves are of bounded geometry [5, Remark 8.2 and Proposition 8.6]. Consider the foliated charts $x_p\colon U_p \to B' \times B''$ given by Theorem 5.2. The radii of the balls B' and B'' will be denoted by r_0' and r_0''. By the usual expression of the Christoffel symbols Γ_{ij}^k of ∇ in terms of the metric coefficients g_{ij} and g^{ij}, and by (51) and (52), it follows that the Christoffel symbols $\mathring{\Gamma}_{ij}^k$ of $\mathring{\nabla}$, as family of smooth functions on $B' \times B''$ parametrized by i, j, k and p, also lie in a bounded subset of the Fréchet space $C^\infty(B' \times B'')$.

Proposition 5.3 (Álvarez–Kordyukov–Leichtnam [5, Proposition 8.6])**.** *For some $r > 0$, we have $B(p,r) \subset U_p$ for all $p \in M$.*

Proposition 5.4 (Álvarez–Kordyukov–Leichtnam [5, Proposition 8.7])**.** *For every multi-index I, the function $|\partial_I(x_q x_p^{-1})|$ is bounded on $x_p(U_p \cap U_q)$, uniformly on $p, q \in M$.*

For $0 < r' \le r_0'$ and $0 < r'' \le r_0''$, let $B'_{r'}$ and $B''_{r''}$ denote the balls in $\mathbb{R}^{n'}$ and $\mathbb{R}^{n''}$ centered at the origin with radii r' and r'', respectively, and set $U_{p,r',r''} = x_p^{-1}(B'_{r'} \times B''_{r''})$.

Proposition 5.5 (Álvarez–Kordyukov–Leichtnam [5, Proposition 8.8])**.** *Let $r', r'' > 0$ with $2r' \le r_0'$ and $2r'' \le r_0''$. Then there is a collection of points p_k in M, and there is some $N \in \mathbb{N}$ such that the sets $U_{p_k,r',r''}$ cover M, and every intersection of $N+1$ sets $U_{p_k,2r',2r''}$ is empty. Moreover, there is a partition of unity $\{f_k\}$ subordinated to the open covering $\{U_{p_k,2r',2r''}\}$, which is bounded in the Fréchet space $C^\infty_{\mathrm{ub}}(M)$.*

Let $y_p\colon V_p \to B$ be normal coordinates satisfying the statement of Theorem 3.2. The radius of B is denoted by r_0. According to Proposition 5.3, we can assume that $V_p \subset U_p$ for all p.

Proposition 5.6. *The functions $x_p y_p^{-1}$, for $p \in M$, define a bounded subset of the Fréchet space $C^\infty(B, \mathbb{R}^{n'} \times \mathbb{R}^{n''})$.*

Proof. By Theorem 3.2, the statement is equivalent to requiring that, for all $m \in \mathbb{N}_0$, the functions $|\nabla^m x_p|$ are bounded on V_p, uniformly on $p \in M$. By (46), this in turn is equivalent to requiring that the functions $|\mathring{\nabla}^m x_p|$ are bounded on V_p, uniformly on $p \in M$. But this follows from Theorem 5.2, since the functions $x_p x_p^{-1} = \mathrm{id}_{B' \times B''}$ obviously define a bounded subset of the Fréchet space $C^\infty(B' \times B'', \mathbb{R}^{n'} \times \mathbb{R}^{n''})$. □

Take $0 < r' < r'_0$ and $0 < r'' < r''_0$ such that $r' + r'' < r_0$. Then $U_{p,r',r''} \subset V_p$ for all $p \in M$ by the triangle inequality. The proof of the following result is similar to the proof of Proposition 5.6.

Proposition 5.7. *The functions $y_p x_p^{-1}$, for $p \in M$, define a bounded subset of the Fréchet space $C^\infty(B'_{r'} \times B''_{r''}, \mathbb{R}^n)$.*

Let E be the Hermitian vector bundle of bounded geometry associated to the principal $\mathrm{O}(n)$-bundle of orthonormal frames on M and a unitary representation of $\mathrm{O}(n)$ (Example 3.6 1). Since ∇ on TM is of bounded geometry, it follows from (46) that $\mathring{\nabla}$ is also of bounded geometry. Thus we get induced connections ∇ and $\mathring{\nabla}$ of bounded geometry on E (Example 3.10 i). By (46), we also get that $\mathring{\nabla}$ can be used instead of ∇ to define equivalent versions of $\|\cdot\|_{C^m_{\mathrm{ub}}}$ and $\langle\cdot,\cdot\rangle_m$ in the spaces $C^m_{\mathrm{ub}}(M;E)$ and $H^m(M;E)$ (Sections 3.7 and 3.11). By Propositions 5.6 and 5.7, if B' and B'' are small enough, then we can use the coordinates (U_p, x_p) instead of (V_p, y_p) to define equivalent versions of $\|\cdot\|'_{C^m_{\mathrm{ub}}}$ and $\langle\cdot,\cdot\rangle'_m$. Similarly, given another bundle F like E, we can use the coordinates (U_p, x_p) instead of (V_p, y_p) to describe $\mathrm{Diff}^m_{\mathrm{ub}}(M;E,F)$ (Section 3.9) by requiring that the local coefficients form a bounded subset of the Fréchet space $C^\infty(B' \times B''; \mathbb{C}^{l'} \otimes \mathbb{C}^{l*})$, where l and l' are the ranks of E and F.

The conditions of being leafwise differential operators and having bounded geometry are preserved by compositions, and by taking transposes and formal adjoints. Moreover,

$$\mathrm{Diff}_{\mathrm{ub}}(\mathcal{F};E,F) = \mathrm{Diff}(\mathcal{F};E,F) \cap \mathrm{Diff}_{\mathrm{ub}}(M;E,F)$$

is a filtered $C^\infty_{\mathrm{ub}}(M)$-submodule of $\mathrm{Diff}(\mathcal{F};E,F)$. The notation $\mathrm{Diff}_{\mathrm{ub}}(\mathcal{F};E)$ is used if $E = F$; this is a graded subalgebra of $\mathrm{Diff}(\mathcal{F};E)$. The concepts of *uniform leafwise ellipticity* for operators in $\mathrm{Diff}(\mathcal{F};E,F)$, and *uniform transverse ellipticity* for operators in $\mathrm{Diff}(M;E,F)$, can be defined like uniform ellipticity (Section 3.9). If $P \in \mathrm{Diff}^1_{\mathrm{ub}}(\mathcal{F};E)$ is uniformly leafwise elliptic and $Q \in \mathrm{Diff}^1_{\mathrm{ub}}(M;E)$ is uniformly transversely elliptic, then $H^m(M;E)$ can be described with the scalar product $\langle u,v\rangle_m = \langle (1 + P^*P + Q^*Q)^m u, v\rangle$ ($m \in \mathbb{R}$).

The normal foliated coordinates (U_p, x_p) can be used in a standard way to endow $T\mathcal{F}$ with the structure of a vector bundle of bounded geometry, and let $\mathfrak{X}_{\mathrm{ub}}(\mathcal{F}) = C^\infty_{\mathrm{ub}}(M;T\mathcal{F})$, which equals $\mathfrak{X}_{\mathrm{ub}}(M) \cap \mathfrak{X}(\mathcal{F})$. On the other hand, let $\mathfrak{X}_{\mathrm{ub}}(M,\mathcal{F}) = \mathfrak{X}_{\mathrm{ub}}(M) \cap \mathfrak{X}(M,\mathcal{F})$.

6. Operators of bounded geometry on differential forms

The principal $\mathrm{O}(n)$-bundle P of orthonormal frames of M has a reduction Q with structural group $\mathrm{O}(n') \times \mathrm{O}(n'') \subset \mathrm{O}(n)$, which consists of the frames of the form (e', e''), where e' and e'' are orthonormal frames in $\mathbf{H}$ and $T\mathcal{F}$, respectively. Then $\mathbf{H}$ and $T\mathcal{F}$ are associated to Q and the unitary representations of $\mathrm{O}(n') \times \mathrm{O}(n'')$ on $\mathbb{C}^{n'}$ and $\mathbb{C}^{n''}$ induced by the canonical unitary representations of $\mathrm{O}(n')$ and $\mathrm{O}(n'')$. Thus $\mathbf{H}$ and $T\mathcal{F}$ are of bounded geometry

(Example 3.6 2). Moreover, $\mathring{\nabla}$ can be restricted to connections on $\mathbf{H}$ and $T\mathcal{F}$, which are of bounded geometry because they are induced by the restriction to Q of the connection on P defined by $\mathring{\nabla}$ (Example 3.10 ii). Thus every $\Lambda^{u,v}M$ is of bounded geometry (Example 3.6 3), and the connection $\mathring{\nabla}$ on $\Lambda^{u,v}M$ is of bounded geometry (Example 3.10 iii).

Consider the induced connections ∇ and $\mathring{\nabla}$ of bounded geometry on ΛM. By using $\mathring{\nabla}$ instead of ∇ in the definitions of $\|\cdot\|_{C^m_{\mathrm{ub}}}$ and $\langle\cdot,\cdot\rangle_m$, it follows that the spaces $C^m_{\mathrm{ub}}(M;\Lambda)$ and $H^m(M;\Lambda)$ inherit the bigrading of ΛM, and therefore $C^\infty_{\mathrm{ub}}(M;\Lambda)$ and $H^{\pm\infty}(M;\Lambda)$ have an induced bigrading: their terms of bi-degree (u,v) are the uniform and Sobolev spaces for $\Lambda^{u,v}M$. In particular, all of this applies to $\Lambda\mathcal{F}\equiv\Lambda^{0,\cdot}M$.

Lemma 6.1. *The canonical projection of ΛM to every $\Lambda^{u,v}M$ is of bounded geometry for all u and v.*

Proof. This follows from (51), (52) and Theorem 5.2. ☐

Using the decompositions (31) and (43), let

$$D_0=d_{0,1}+\delta_{0,-1},\quad D_\perp=d_{1,0}+\delta_{-1,0},$$
$$\Delta_0=D_0^2=d_{0,1}\delta_{0,-1}+\delta_{0,-1}d_{0,1}.$$

Note that $D_0\in\mathrm{Diff}^1(\mathcal{F};\Lambda M)$, $D_\perp\in\mathrm{Diff}^1(M;\Lambda)$ and $\Delta_0\in\mathrm{Diff}^2(\mathcal{F};\Lambda M)$.

Corollary 6.2. *The differential operators $d_{i,j}$, $\delta_{-i,-j}$, D_0, $D_\perp$ and Δ_0 are of bounded geometry.*

Proof. This follows from Lemma 6.1, since d is of bounded geometry, and this property is preserved by taking formal adjoints and compositions. ☐

It is elementary that

$$\begin{aligned}{}^{\mathcal{F}}\sigma(d_{0,1})(p,\xi)&=i\xi\wedge,\quad \sigma(d_{-1,0})(p,\zeta)=i\zeta\wedge,\\ {}^{\mathcal{F}}\sigma(\delta_{0,-1})(p,\xi)&=i\xi\lrcorner,\quad \sigma(\delta_{0,-1})(p,\zeta)=i\zeta\lrcorner,\end{aligned}\tag{53}$$

for all $p\in M$, $\xi\in T^*_p\mathcal{F}$ and $\zeta\in N^*_p\mathcal{F}$. So

$$\begin{gathered}{}^{\mathcal{F}}\sigma(D_0)(p,\xi)=i(\xi\wedge+\xi\lrcorner),\quad {}^{\mathcal{F}}\sigma(\Delta_0)(p,\xi)=|\xi|^2,\\ \sigma(D_\perp)(p,\zeta)=i(\zeta\wedge+\zeta\lrcorner).\end{gathered}\tag{54}$$

Thus we get the following.

Proposition 6.3. *D_0 and Δ_0 are uniformly leafwise elliptic, and $D_\perp$ is uniformly transversely elliptic.*

Let us extend the arguments of [2, Section 3] to open manifolds using bounded geometry. The expression (48) defines a differential operator

$$\Theta\colon\mathfrak{X}(\mathcal{F})\to C^\infty(M;\mathbf{H}^*\otimes T\mathcal{F}),\quad \Theta_XV=\mathbf{V}([X,V]),$$

for $X \in C^\infty(M;\mathbf{H})$ and $V \in \mathfrak{X}(\mathcal{F})$. It induces a differential operator

$$\Theta\colon C^\infty(M;\Lambda\mathcal{F}) \to C^\infty(M;\mathbf{H}^*\otimes\Lambda\mathcal{F}),$$
$$(\Theta_X\alpha)(V_1,\dots,V_r) = X\alpha(V_1,\dots,V_r) - \sum_{j=1}^{r}\alpha(V_1,\dots,\Theta_X V_j,\dots,V_r),$$

for $X \in C^\infty(M;\mathbf{H})$, $\alpha \in C^\infty(M;\Lambda^r\mathcal{F})$ and $V_j \in \mathfrak{X}(\mathcal{F})$. If $X \in \mathfrak{X}(M,\mathcal{F}) \cap C^\infty(M;\mathbf{H})$, then this expression agrees with the operator Θ_X of Section 4.11. According to [2, Lemma 3.3], a zero order differential operator

$$\Xi\colon C^\infty(M;\Lambda\mathcal{F}) \to C^\infty(M;\mathbf{H}^*\otimes\Lambda\mathcal{F})$$

is locally defined by

$$\Xi_X = (-1)^{(n''-v)v}[\Theta_X, \star_{\mathcal{F}}]\star_{\mathcal{F}}$$

on $C^\infty(M;\Lambda^v\mathcal{F})$ for any $X \in C^\infty(M;\mathbf{H})$, where $\star_{\mathcal{F}}$ is the local leafwise star operator determined by g and any local orientation of $\mathcal{F}$. This Ξ can be considered as a vector bundle morphism $\Lambda\mathcal{F} \to \mathbf{H}^*\otimes\Lambda\mathcal{F}$. By tensoring Ξ with the identity morphism on $\Lambda\mathbf{H}$, we get a vector bundle morphism $\Lambda M \to \mathbf{H}^*\otimes\Lambda M$ according to (29), which is also denoted by Ξ. On any normal foliated chart (U,x), let K be the endomorphism of ΛU given by

$$K = dx'^i \wedge \Xi_{\mathbf{H}\partial'_i}.$$

This local definition gives rise to a global endomorphism K of ΛM.

Proposition 6.4. *The endomorphism K is of bounded geometry.*

Proof. Take a normal foliated chart (U,x). By (52),

$$\Theta_{\mathbf{H}\partial'_i}\partial''_b = \mathbf{V}([\mathbf{H}\partial'_i,\partial''_b]) = \mathbf{V}([\partial'_i - g_{ik}g^{kj}\partial''_j,\partial''_b]) = \partial''_b(g_{ik}g^{kj})\partial''_j,$$

where k runs in $\{n'+1,\dots,n\}$. Hence

$$\begin{aligned}(\Theta_{\mathbf{H}\partial'_i}dx''^a)(\partial''_b) &= \mathbf{H}\partial'_i(dx''^a(\partial''_b)) - dx''^a(\Theta_{\mathbf{H}\partial'_i}\partial''_b)\\ &= -\partial''_b(g_{ik}g^{kj})\,dx''^a(\partial''_j) = -\partial''_b(g_{ik}g^{k\alpha}),\end{aligned}$$

giving

$$\Theta_{\mathbf{H}\partial'_i}dx''^a = -\partial''_b(g_{ik}g^{ka})\,dx''^b.$$

It follows that $\Theta_{\mathbf{H}\partial'_i}dx''^I = f_{iIK}\,dx''^K$, where the functions f_{iIK} are universal polynomial expressions of the functions g_{ab} and g^{ab}, and their partial derivatives. On the other hand, for any choice of an orientation of $\mathcal{F}$ on U, we have $\star_{\mathcal{F}}dx''^I = h_{IK}\,dx''^K$, where the functions h_{IK} are universal expressions of the functions g_{ab} and g^{ab}. So

$$\begin{aligned}\Xi_{\mathbf{H}\partial'_i} dx''^I &= (-1)^{(n''-v)v}[\Theta_{\mathbf{H}\partial'_i}, \star_{\mathcal{F}}] \star_{\mathcal{F}} dx''^I \\ &= \Theta_{\mathbf{H}\partial'_i} dx''^I - (-1)^{(n''-v)v} \star_{\mathcal{F}} \Theta_{\mathbf{H}\partial'_i} \star_{\mathcal{F}} dx''^I \\ &= f_{iIK}\, dx''^K - (-1)^{(n''-v)v} \star_{\mathcal{F}} \Theta_{\mathbf{H}\partial'_i}(h_{IL}\, dx''^A) \\ &= f_{iIK}\, dx''^K - (-1)^{(n''-v)v} \star_{\mathcal{F}} \big((\partial'_i - g_{ik} g^{kj} \partial''_j)(h_{IA})\, dx''^A \\ &\qquad + h_{IA} f_{iAB}\, dx''^B\big) \\ &= \big(f_{iIK} - (-1)^{(n''-v)v}\big((\partial'_i h_{IA} - g_{ik} g^{kj} \partial''_j h_{IA}) h_{AK} \\ &\qquad + h_{IA} f_{iAB} h_{BK}\big)\big)\, dx''^K. \qquad \square\end{aligned}$$

Like in the case of compact manifolds [2, Proposition 3.1], using the local expression $\delta_{\mathcal{F}} = (-1)^v \star_{\mathcal{F}} d_{\mathcal{F}} \star_{\mathcal{F}}$ on $C^\infty(U; \Lambda^v \mathcal{F})$, and (34), (41), (44) and (45), we get

$$D_\perp D_0 + D_0 D_\perp = K D_0 + D_0 K. \tag{55}$$

7. Leafwise Hodge decomposition

With the notation of Section 5, since M is complete, by (54) and the commutativity of (25), given any $\alpha \in C^\infty_{\mathrm{c}}(M; \Lambda)$, the hyperbolic equation

$$\partial_t \alpha_t = i D_0 \alpha_t, \quad \alpha_0 = \alpha, \tag{56}$$

has a unique solution $\alpha_t \in C^\infty_{\mathrm{c}}(M; \Lambda)$ depending smoothly on $t \in \mathbb{R}$, see [14, Theorem 1.3]. The solutions of (56) defined on any open subset of M and for t in any interval containing zero satisfy (see [41, Proposition 1.2])

$$\operatorname{supp} \alpha_t \subset \operatorname{Pen}_{\mathcal{F}}(\operatorname{supp} \alpha, |t|). \tag{57}$$

This can be proved like (16), or it also follows from (16) by taking restrictions to the leaves.

The operators D_0 and Δ_0, with domain $C^\infty_{\mathrm{c}}(M; \Lambda)$, are essentially selfadjoint in $L^2(M; \Lambda)$ [14, Theorem 2.2], and their selfadjoint extensions are also denoted by D_0 and Δ_0. Using the functional calculus of D_0 given by the spectral theorem, we get a unitary operator e^{itD_0} and a bounded selfadjoint operator $e^{-t\Delta_0}$ on $L^2(M; \Lambda)$ with $\|e^{-t\Delta_0}\| \le 1$. The solution of (56) is given by $\alpha_t = e^{itD_0}\alpha$. Let Π_0 (or $e^{-\infty\Delta_0}$) denote the orthogonal projection of $L^2(M; \Lambda)$ to the kernel of Δ_0 in $L^2(M; \Lambda)$.

Proposition 7.1. *For every $m \in \mathbb{N}_0$, there is some $C_m \ge 0$ such that, for all $\alpha \in C^\infty_{\mathrm{c}}(M; \Lambda)$ and $t \in \mathbb{R}$,*

$$\|e^{itD_0}\alpha\|_m \le e^{C_m|t|} \|\alpha\|_m.$$

Proof. We adapt arguments from [48, Section IV.2]. The case where M is compact is stated in [41, Proposition 1.4] with more generality.

By Proposition 6.3, we can assume that, for all $\alpha \in C^\infty_{\mathrm{c}}(M; \Lambda)$,

$$\|\alpha\|_m = \|\alpha\| + \|D_0^m \alpha\| + \|D_\perp^m \alpha\|.$$

Writing $\alpha_t = e^{itD_0}\alpha$, we have

$$\begin{aligned}\frac{d}{dt}\|D_0^m\alpha_t\|^2 &= \langle iD_0^{m+1}\alpha_t, D_0^m\alpha_t\rangle + \langle D_0^m\alpha_t, iD_0^{m+1}\alpha_t\rangle = 0,\\ \frac{d}{dt}\|D_\perp^m\alpha_t\|^2 &= \langle iD_\perp^m D_0\alpha_t, D_\perp^m\alpha_t\rangle + \langle D_\perp^m\alpha_t, iD_\perp^m D_0\alpha_t\rangle\\ &= i\langle[\Delta_\perp^m, D_0]\alpha_t, \alpha_t\rangle.\end{aligned}$$

But, by (55),

$$\begin{aligned}[\Delta_\perp^m, D_0] &= \sum_{j=0}^{m-1}\Delta_\perp^{m-1-j}[D_\perp, D_\perp D_0 + D_0D_\perp]\Delta_\perp^j\\ &= \sum_{j=0}^{m-1}\Delta_\perp^{m-1-j}[D_\perp, KD_0 + D_0K]\Delta_\perp^j.\end{aligned}$$

This expression can be written as a sum of $4m$ terms, $\sum_l P_lQ_l$, where, up to sign, P_l and Q_l are operators of the one of the following forms: $D_\perp^a KD_0D_\perp^b$, $D_\perp^a D_0KD_\perp^b$ or $D_\perp^m$, for $a, b \in \mathbb{N}_0$ with $a + b + 1 = m$. In particular, $P_l, Q_l \in \operatorname{Diff}_{\mathrm{ub}}^m(M;\Lambda)$ by Corollary 6.2 and Lemma 6.4. Hence there is some $C_m > 0$, independent of α, such that

$$\frac{d}{dt}\|D_\perp^m\alpha_t\|^2 \le \sum_l|\langle Q_l\alpha_t, P_l^*\alpha\rangle| \le \sum_l\|Q_l\alpha_t\|\cdot\|P_l^*\alpha\| \le C_m\|\alpha_t\|_m^2.$$

Therefore

$$\frac{d}{dt}\|\alpha_t\|_m^2 \le C_m\|\alpha_t\|_m^2,$$

and the result follows by using Gronwall's inequality. □

Recall also that the Schwartz space $\mathcal{S} = \mathcal{S}(\mathbb{R})$ is the Fréchet space of functions $\psi \in C^\infty(\mathbb{R})$ such that $\psi^{(m)} \in \mathcal{R}$ for all $m \in \mathbb{N}_0$, with the semi-norms defined by applying the semi-norms of $\mathcal{R}$ to derivatives of arbitrary order. Let $\mathcal{A}$ denote the Fréchet algebra and $\mathbb{C}[z]$-module of functions $\psi\colon \mathbb{R} \to \mathbb{C}$ that can be extended to entire functions on $\mathbb{C}$ such that, for every compact $K \subset \mathbb{R}$, the set $\{x \mapsto \psi(x+iy) \mid y \in K\}$ is bounded in $\mathcal{S}$, see [41, Section 4]. It contains all functions with compactly supported smooth Fourier transform, as well as the Gaussian $x \mapsto e^{-x^2}$. Furthermore, if $\psi \in \mathcal{A}$ and $u > 0$, then $\psi_u \in \mathcal{A}$, where $\psi_u(x) = \psi(ux)$. By the Paley–Wiener theorem, for every $\psi \in \mathcal{A}$ and $c > 0$, there is some $A_c > 0$ such that, for all $\xi \in \mathbb{R}$,

$$|\hat{\psi}(\xi)| \le A_c e^{-c|\xi|}. \tag{58}$$

The semi-norms on $\mathcal{A}$, $\|\cdot\|_{\mathcal{A},C,r}$ ($C > 0$ and $r \in \mathbb{N}_0$), can be defined by

$$\|\psi\|_{\mathcal{A},C,r} = \max_{j+k\le r}\int_\infty^\infty|\xi^j\partial_\xi^k\hat{\psi}(\xi)|\,e^{C|\xi|}\,d\xi.$$

Proposition 7.2. *The functional calculus map, $\psi \mapsto \psi(D_0)$, restricts to a continuous homomorphism $\mathcal{A} \to \operatorname{End}(H^\infty(M;\Lambda))$ of $\mathbb{C}[z]$-modules and algebras.*

Proof. This follows like in the case where M is compact [41, Proposition 4.1]. Precisely, for every $\psi \in \mathcal{A}$, it follows from the inverse Fourier transform that

$$\psi(D_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\psi}(\xi) e^{i\xi D_0} \, d\xi. \tag{59}$$

So, by Proposition 7.1 and (58), $\psi(D_0)$ defines an endomorphism of every $H^m(M;\Lambda)$ with

$$\|\psi(D_0)\|_m \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{\psi}(\xi)| \, e^{C_m|\xi|} \, d\xi \leq \frac{A_k}{2\pi} \int_{-\infty}^{\infty} e^{(C_m-k)|\xi|} \, d\xi, \tag{60}$$

which is finite for $k > C_m$. □

According to Proposition 7.2, the operator $e^{-t\Delta_0}$ $(t > 0)$ restricts to a continuous endomorphism of $H^\infty(M;\Lambda)$. As pointed out in [44], by using (55), Corollary 6.2, and Propositions 6.3, 6.4 and 7.2, the arguments of the proof of [2, Theorem A] can be adapted to show the following result, where Δ_0 is considered on $H^\infty(M;\Lambda)$.

Theorem 7.3 (Sanguiao [44]). *There is a topological direct sum decomposition,*

$$H^\infty(M;\Lambda) = \ker \Delta_0 \oplus \overline{\operatorname{im} d_{0,1}} \oplus \overline{\operatorname{im} \delta_{0,-1}}. \tag{61}$$

Moreover, $(t,\alpha) \mapsto e^{-t\Delta_0}\alpha$ *defines a continuous map*

$$[0,\infty] \times H^\infty(M;\Lambda) \to H^\infty(M;\Lambda).$$

By Corollary 6.2, $(H^\infty(M;\Lambda), d_{0,1})$ is a topological complex. The terms of the direct sum decomposition (61) are orthogonal in $L^2(M;\Lambda)$. Thus Π_0 has a restriction $H^\infty(M;\Lambda) \to \ker \Delta_0$, which induces the isomorphism stated in the following corollary. Its inverse is induced by the inclusion map $\ker \Delta_0 \hookrightarrow H^\infty(M;\Lambda)$.

Corollary 7.4 (Sanguiao [44]). *As topological vector spaces,*

$$\bar{H}^*(H^\infty(M;\Lambda), d_{0,1}) \cong \ker \Delta_0.$$

By (29) and (32), we can consider $(H^\infty(M;\Lambda\mathcal{F}), d_{\mathcal{F}})$ as a topological subcomplex of $(H^\infty(M;\Lambda), d_{0,1})$, and the notation $H^*H^\infty(\mathcal{F})$ and $\bar{H}^*H^\infty(\mathcal{F})$ is used for its (reduced) cohomology. By Lemma 4.12, $\delta_{\mathcal{F}}$ on $H^\infty(M;\Lambda\mathcal{F})$ is also given by $\delta_{0,-1}$. Thus we get the operators $D_{\mathcal{F}} = d_{\mathcal{F}} + \delta_{\mathcal{F}}$ and $\Delta_{\mathcal{F}} = D_{\mathcal{F}}^2 = \delta_{\mathcal{F}} d_{\mathcal{F}} + d_{\mathcal{F}} \delta_{\mathcal{F}}$ on $H^\infty(M;\Lambda\mathcal{F})$, which are essentially selfadjoint in $L^2(M;\Lambda\mathcal{F})$. Then Propositions 7.1 and 7.2, Theorem 7.3 and Corollary 7.4 have obvious versions for $d_{\mathcal{F}}$, $\delta_{\mathcal{F}}$, $D_{\mathcal{F}}$ and $\Delta_{\mathcal{F}}$; in particular, we get the following.

Theorem 7.5 (Sanguiao [44]). *Let $\mathcal{F}$ be a Riemannian foliation of bounded geometry on a Riemannian manifold M with a bundle-like metric. Then there is a topological direct sum decomposition,*

$$H^\infty(M;\Lambda\mathcal{F}) = \ker \Delta_{\mathcal{F}} \oplus \overline{\operatorname{im} d_{\mathcal{F}}} \oplus \overline{\operatorname{im} \delta_{\mathcal{F}}}. \tag{62}$$

Moreover, $(t,\alpha) \mapsto e^{-t\Delta_{\mathcal{F}}}\alpha$ *defines a continuous map*

$$[0,\infty] \times H^\infty(M;\Lambda\mathcal{F}) \to H^\infty(M;\Lambda\mathcal{F}).$$

Corollary 7.6 (Sanguiao [44]). *We have $\bar{H}^*H^\infty(\mathcal{F}) \cong \ker\Delta_{\mathcal{F}}$, as topological vector spaces.*

Theorem 7.3 and Corollary 7.4 can be considered as versions of Theorem 7.5 and Corollary 7.6 with coefficients in $\Lambda N\mathcal{F}$. We could also take leafwise differential forms with coefficients in other Hermitian vector bundles associated to $N\mathcal{F}$, like the bundles of transverse densities or transverse symmetric tensors[14].

Compare Theorem 7.3 and Corollary 7.6 with (10) and (11) (the case of an elliptic complex on a closed manifold).

8. Foliated maps of bounded geometry

For $a = 1, 2$, let $\mathcal{F}_a$ be a Riemannian foliation of bounded geometry on a manifold M_a with a bundle-like metric. Set $n'_a = \operatorname{codim}\mathcal{F}_a$ and $n''_a = \dim\mathcal{F}_a$. Consider a normal foliated chart $x_{a,p}\colon U_{a,p} \to B'_a \times B''_a$ at every $p \in M_a$ satisfying the conditions of Theorem 5.2. Let r'_a and r''_a denote the radii of B'_a and B''_a, respectively. For $0 < r' < r'_a$ and $0 < r'' < r''_a$, let $U_{a,p,r',r''} = x_{a,p}^{-1}(B'_{a,r'} \times B''_{a,r''})$, where $B'_{a,r'} \subset \mathbb{R}^{n'_a}$ and $B''_{a,r''} \subset \mathbb{R}^{n''_a}$ denote the balls centered at the origin with respective radii r' and r''. Like in the cases of $C^m_{\mathrm{ub}}(M;E)$, $H^m(M;E)$ and $\operatorname{Diff}^m_{\mathrm{ub}}(M;E,F)$ (Section 5), in the definition of bounded geometry for maps $M_1 \to M_2$ (Section 3.16), we can replace the charts $(V_{1,p}, y_{1,p})$ and $(V_{2,\phi(p)}, y_{2,\phi(p)})$ with the charts $(U_{1,p}, x_{1,p})$ and $(U_{2,\phi(p)}, x_{2,\phi(p)})$, and we can replace the sets $B_1(p,r)$ with the sets $U_{1,p,r',r''}$. Let

$$C^\infty_{\mathrm{ub}}(M_1,\mathcal{F}_1;M_2,\mathcal{F}_2) = C^\infty(M_1,\mathcal{F}_1;M_2,\mathcal{F}_2) \cap C^\infty_{\mathrm{ub}}(M_1,M_2).$$

For $m \in \mathbb{N}_0$ and $\phi \in C^\infty_{\mathrm{ub}}(M_1,\mathcal{F}_1;M_2,\mathcal{F}_2)$, using the version of $\|\cdot\|'_{C^m_{\mathrm{ub}}}$ defined with the charts (U_p, x_p), it follows that ϕ^* induces a bounded homomorphism

$$\phi^*\colon C^m_{\mathrm{ub}}(M_2;\Lambda\mathcal{F}_2) \to C^m_{\mathrm{ub}}(M_1;\Lambda\mathcal{F}_1),$$

obtaining a continuous homomorphism

$$\phi^*\colon C^\infty_{\mathrm{ub}}(M_2;\Lambda\mathcal{F}_2) \to C^\infty_{\mathrm{ub}}(M_1;\Lambda\mathcal{F}_1).$$

These homomorphisms are induced by (19) and (20) via (29). Similarly, if ϕ is also uniformly metrically proper, then ϕ^* induces a bounded homomorphism

$$\phi^*\colon H^m(M_2;\Lambda\mathcal{F}_2) \to H^m(M_1;\Lambda\mathcal{F}_1)$$

for all m, and therefore it induces a continuous homomorphism

$$\phi^*\colon H^{\pm\infty}(M_2;\Lambda\mathcal{F}_2) \to H^{\pm\infty}(M_1;\Lambda\mathcal{F}_1).$$

By (36), these homomorphisms are induced by (21) and (22) via (29).

Now, let $\phi = \{\phi^t\}$ be a foliated flow on $(M,\mathcal{F})$ of $\mathbb{R}$-local bounded geometry, and let $Z \in \mathfrak{X}_{\mathrm{ub}}(M,\mathcal{F})$ be its infinitesimal generator (Proposition 3.18).

[14]However, this is not true for any Hermitian vector bundle with a flat Riemannian $\mathcal{F}$-partial connection, contrary to what was wrongly asserted in [2, Corollary C] when M is compact. A counterexample was provided to the first two authors by S. Goette.

Every ϕ^t is uniformly metrically proper because $\phi^{\pm t}$ is of bounded geometry (Section 3.16). Thus ϕ^{t*} induces continuous homomorphisms,

$$C^\infty_{\mathrm{ub}}(M;\Lambda\mathcal{F}) \to C^\infty_{\mathrm{ub}}(M;\Lambda\mathcal{F}), \quad H^{\pm\infty}(M;\Lambda\mathcal{F}) \to H^{\pm\infty}(M;\Lambda\mathcal{F}).$$

Proposition 8.1. *We have* $\Theta_Z \in \mathrm{Diff}^1_{\mathrm{ub}}(M;\Lambda\mathcal{F})$.

Proof. Since $d \in \mathrm{Diff}^1_{\mathrm{ub}}(M;\Lambda\mathcal{F})$ and $Z \in \mathfrak{X}_{\mathrm{ub}}(M,\mathcal{F})$, we obtain that $\mathcal{L}_Z = d\iota_Z + \iota_Z d$ is of bounded geometry. So $\Theta_Z \equiv \mathcal{L}_{Z,0,0}$ is of bounded geometry by Lemma 6.1 and using the identity (29). □

By (37) and (53),

$$\sigma(\Theta_Z)(p,\zeta) = i\zeta(Z)$$

for all $p \in M$ and $\zeta \in N^*_pM$, obtaining the following.

Proposition 8.2. Θ_Z *is uniformly transversely elliptic if* $\inf_M|\overline{Z}| > 0$.

9. A class of smoothing operators

Suppose that $\mathcal{F}$ is of codimension one[15]. Assume also that M is equipped with a bundle-like metric g so that $\mathcal{F}$ is of bounded geometry. Let $\phi = \{\phi^t\}$ be a foliated flow of $\mathbb{R}$-local bounded geometry, whose infinitesimal generator is $Z \in \mathfrak{X}_{\mathrm{ub}}(M,\mathcal{F})$ (Proposition 3.18). Suppose that $\inf_M|\overline{Z}| > 0$; in particular, the orbits of ϕ are transverse to the leaves. Moreover, let $A = \{A_t \mid t \in \mathbb{R}\} \subset \mathrm{Diff}^m(\mathcal{F};\Lambda\mathcal{F})$ be a smooth $\mathbb{R}$-compactly supported family of $\mathbb{R}$-local bounded geometry. For every $\psi \in \mathcal{A}$, the operator

$$P = \int_{\mathbb{R}} \phi^{t*} A_t \, dt \, \psi(D_{\mathcal{F}}) \tag{63}$$

on $H^{-\infty}(M;\Lambda\mathcal{F})$ is defined by the version of Proposition 7.2 for $D_{\mathcal{F}}$. The subindex "ψ" may be added to the notation of P if needed, or the subindex "u" in the case of functions $\psi_u \in \mathcal{A}$ depending on a parameter u.

Proposition 9.1. P_ψ *is a smoothing operator, and the linear map*

$$\mathcal{A} \to L(H^{-\infty}(M;\Lambda\mathcal{F}), H^\infty(M;\Lambda\mathcal{F})), \quad \psi \mapsto P_\psi,$$

is continuous.

Proof. According to the proof of Proposition 7.2, $\psi(D_{\mathcal{F}})$ defines a bounded operator on every $H^m(M;\Lambda\mathcal{F})$. Since, moreover, ϕ and A are of $\mathbb{R}$-local bounded geometry, and A is $\mathbb{R}$-compactly supported, it follows that P also defines a bounded operator on every $H^m(M;\Lambda\mathcal{F})$.

Since $\Theta_Z \in \mathrm{Diff}^1_{\mathrm{ub}}(M;\Lambda\mathcal{F})$ is uniformly transversely elliptic (see Propositions 8.1 and 8.2) and $D_{\mathcal{F}} \in \mathrm{Diff}^1_{\mathrm{ub}}(\mathcal{F};\Lambda\mathcal{F})$ is uniformly leafwise elliptic (Corollary 6.2 and Proposition 6.3), to get that P is smoothing, it suffices to prove that $\Theta_Z^N P$ and $D_{\mathcal{F}}^N P$ are of the form (63) for all $N \in \mathbb{N}_0$. In turn, this follows by showing that $\Theta_Z P$ and QP are of the form (63) for any $Q \in \mathrm{Diff}_{\mathrm{ub}}(\mathcal{F};\Lambda\mathcal{F})$.

[15]The higher dimensional case could be treated like in [4], but we only consider codimension one here for the sake of simplicity.

We have

$$QP = \int_{\mathbb{R}} \phi^{t*} B_t \, dt \, \psi(D_{\mathcal{F}}),$$

where $B_t = \phi^{-t*} Q \phi^{t*} A_t$. Since ϕ^t is a foliated map, this defines a smooth family $B = \{B_t \mid t \in \mathbb{R}\} \subset \mathrm{Diff}(\mathcal{F}; \Lambda\mathcal{F})$. Moreover, B is $\mathbb{R}$-compactly supported and of $\mathbb{R}$-local bounded geometry because A is $\mathbb{R}$-compactly supported, and ϕ, Q and A are of $\mathbb{R}$-local bounded geometry. Thus QP is of the form (63).

Let $C = \{C_t \mid t \in \mathbb{R}\} \subset \mathrm{Diff}(\mathcal{F}; \Lambda\mathcal{F})$ be the smooth family given by $C_t = \frac{d}{ds} A_{t-s}|_{s=0}$. Note that C is of $\mathbb{R}$-local bounded geometry because the family A is of $\mathbb{R}$-local bounded geometry and $\mathbb{R}$-compactly supported. Like in the proof of [4, Proposition 6.1], we get

$$\begin{aligned}\Theta_Z P &= \frac{d}{ds} \int_{\mathbb{R}} \phi^{t+s*} A_t \, dt \Big|_{s=0} \psi(D_{\mathcal{F}}) \\ &= \frac{d}{ds} \int_{\mathbb{R}} \phi^{r*} A_{r-s} \, dr \Big|_{s=0} \psi(D_{\mathcal{F}}) = \int_{\mathbb{R}} \phi^{t*} C_t \, dt \, \psi(D_{\mathcal{F}}),\end{aligned}$$

which is of the form (63).

By (60), for any $N \in \mathbb{N}_0$ and $\psi \in \mathcal{A}$, the operator $(1 + \Delta_{\mathcal{F}})^N \psi(D_{\mathcal{F}})$ extends to a bounded operator on every $H^m(M; \Lambda\mathcal{F})$ with

$$\|(1 + \Delta_{\mathcal{F}})^N \psi(D_{\mathcal{F}})\|_m \le \frac{1}{2\pi} \int_{-\infty}^{\infty} |(1 - \partial_\xi^2)^N \hat{\psi}(\xi)| \, e^{C_m |\xi|} \, d\xi. \tag{64}$$

Hence, by the above argument, it can be easily seen that, for integers $m \le m'$, there are some $C, C' > 0$ and $N \in \mathbb{N}_0$ such that

$$\|P\|_{m,m'} \le C' \int |(\mathrm{id} - \partial_\xi^2)^N \hat{\psi}(\xi)| \, e^{C|\xi|} \, d\xi \le C' \|\psi\|_{\mathcal{A},C,2N}. \tag{65}$$

Here, C depends on m and m', and C' depends on m, m' and A. Then the mapping $\psi \mapsto P_\psi$ of the statement is continuous. □

Corollary 9.2. *The linear map*

$$\mathcal{A} \to C^\infty_{\mathrm{ub}}(M^2; \Lambda\mathcal{F} \boxtimes (\Lambda\mathcal{F}^* \otimes \Omega M)), \quad \psi \mapsto K_{P_\psi},$$

is continuous.

Proof. This follows from Propositions 9.1 and 3.14. □

Now, consider the particular case where $\psi_u(x) = e^{-ux^2}$, and the corresponding operators P_u $(u > 0)$, defined on $H^\infty(M; \Lambda\mathcal{F})$. Let also

$$P_\infty = \int_{\mathbb{R}} \phi^{t*} A_t \, dt \, \Pi_{\mathcal{F}}$$

on $H^\infty(M; \Lambda\mathcal{F})$, where $\Pi_{\mathcal{F}}$ is the orthogonal projection to $\ker \Delta_{\mathcal{F}}$.

Corollary 9.3. *P_∞ is a smoothing operator.*

Proof. This follows from Theorem 7.5 and Proposition 9.1, since we have $P_\infty = P_u \Pi_{\mathcal{F}}$. □

Proposition 9.4. *$P_u \to P_\infty$ in $L(H^{-\infty}(M;\Lambda\mathcal{F}), H^\infty(M;\Lambda\mathcal{F}))$ as $u \uparrow \infty$.*

Proof. By Theorem 7.3, $e^{-u\Delta_\mathcal{F}} - \Pi_\mathcal{F} \to 0$ in $\operatorname{End}(H^\infty(M;\Lambda\mathcal{F}))$ as $u \uparrow \infty$. Therefore this convergence also holds in $\operatorname{End}(H^{-\infty}(M;\Lambda\mathcal{F}))$, by taking dual spaces and dual operators. Hence

$$P_u - P_\infty = P_1(e^{-(u-1)\Delta_\mathcal{F}} - \Pi_\mathcal{F}) \to 0$$

in $L(H^{-\infty}(M;\Lambda\mathcal{F}), H^\infty(M;\Lambda\mathcal{F}))$ as $u \uparrow \infty$. □

Corollary 9.5. *$K_{P_u} \to K_{P_\infty}$ in $C^\infty_{\mathrm{ub}}(M^2;\Lambda\mathcal{F} \boxtimes (\Lambda\mathcal{F}^* \otimes \Omega M))$ as $u \uparrow \infty$.*

Proof. This follows from Propositions 3.14 and 9.4. □

From now on, consider only the case where $A = f \in C^\infty_\mathrm{c}(\mathbb{R})$, obtaining the smoothing operator

$$P = \int_\mathbb{R} \phi^{t*} f(t)\, dt\, \psi(D_\mathcal{F}), \tag{66}$$

as well as its versions, P_u if ψ_u is used, and P_∞ if $\Pi_\mathcal{F}$ is used. The proof of [4, Proposition 6.1] clearly extends to the open manifold case, showing the following improvement of (65).

Proposition 9.6. *For any compact $I \subset \mathbb{R}$ containing $\operatorname{supp} f$, and for all $m, m' \in \mathbb{N}_0$, there are some $C, C' > 0$ and $N \in \mathbb{N}_0$, depending on m, m' and I, such that*

$$\|P\|_{m,m'} \le C' \, \|f\|_{C^N,I} \|\psi\|_{\mathcal{A},C,N}.$$

Here, $\|\cdot\|_{C^N,I}$ is the semi-norm on $C^N(\mathbb{R})$ defined by

$$\|f\|_{C^N,I} = \max\{|f^{(m)}(x)| \mid x \in I,\ m = 0,\dots,N\}.$$

10. Description of some Schwartz kernels

Here, we will keep the setting of Section 9. The transverse vector field $\overline{Z}$ defines the structure of a transversely complete $\mathbb{R}$-Lie foliation on $\mathcal{F}$ (Section 4.10). The corresponding Fedida's description of $\mathcal{F}$ is given by a regular covering map $\pi\colon \widetilde{M} \to M$, a holonomy homomorphism $h\colon \Gamma := \operatorname{Aut}(\pi) \to \mathbb{R}$, and the developing map $D\colon \widetilde{M} \to \mathbb{R}$ (Section 4.10). The lift of the bundle-like metric g to $\widetilde{M}$ is a bundle-like metric $\tilde{g}$ of $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$, and let $\tilde{\phi}\colon \widetilde{M} \times \mathbb{R} \to \widetilde{M}$ and $\widetilde{Z} \in \mathfrak{X}_\mathrm{ub}(\widetilde{M}, \widetilde{\mathcal{F}})$ be the lifts of ϕ and Z. Then $\tilde{g}$ and $\widetilde{Z}$ are Γ-invariant, and $\tilde{\phi}$ is Γ-equivariant. Moreover, $\widetilde{Z}$ is D-projectable, and we can assume that $D_*\widetilde{Z} = \partial_x \in \mathfrak{X}(\mathbb{R})$, where x denotes the standard global coordinate of $\mathbb{R}$. Thus ϕ induces via D the flow $\bar{\phi}$ on $\mathbb{R}$ defined by $\bar{\phi}^t(x) = t + x$. Considering the equivalence between the holonomy pseudogroup and the pseudogroup generated by the action of $\operatorname{Hol}\mathcal{F}$ on $\mathbb{R}$ by translations, this $\bar{\phi}$ corresponds to the equivariant local flow $\bar{\phi}$ induced by ϕ on the holonomy pseudogroup. Since $\bar{\phi}^t$ preserves every $\operatorname{Hol}\mathcal{F}$-orbit in $\mathbb{R}$ if and only if $t \in \operatorname{Hol}\mathcal{F}$, it follows that ϕ^t preserves every leaf of $\mathcal{F}$ if and only if $t \in \operatorname{Hol}\mathcal{F}$.

For any $\psi \in \mathcal{A}$ and $f \in C^\infty_\mathrm{c}(\mathbb{R})$, we have the smoothing operator P given by (66), and a similar smoothing operator $\widetilde{P}$ defined with $\tilde{\phi}$ and $\widetilde{\mathcal{F}}$ instead of ϕ and $\mathcal{F}$. We are going to describe their smoothing kernels under some assumptions.

Let $\mathfrak{G} = \mathrm{Hol}(M, \mathcal{F})$ and $\widetilde{\mathfrak{G}} = \mathrm{Hol}(\widetilde{M}, \widetilde{\mathcal{F}})$, whose source and range maps are denoted by $\mathbf{s}, \mathbf{r} \colon \mathfrak{G} \to M$ and $\tilde{\mathbf{s}}, \tilde{\mathbf{r}} \colon \widetilde{\mathfrak{G}} \to \widetilde{M}$ (Section 4.4). Since the leaves of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ have trivial holonomy groups, the smooth immersions $(\mathbf{r}, \mathbf{s}) \colon \mathfrak{G} \to M^2$ and $(\tilde{\mathbf{r}}, \tilde{\mathbf{s}}) \colon \widetilde{\mathfrak{G}} \to \widetilde{M}^2$ are injective, with images $\mathcal{R}_{\mathcal{F}}$ and $\mathcal{R}_{\widetilde{\mathcal{F}}}$. Via these injections, the restriction $\pi \times \pi \colon \mathcal{R}_{\widetilde{\mathcal{F}}} \to \mathcal{R}_{\mathcal{F}}$ corresponds to the Lie groupoid homomorphism $\pi_{\mathfrak{G}} := \mathrm{Hol}(\pi) \colon \widetilde{\mathfrak{G}} \to \mathfrak{G}$ (Section 4.8), which is a covering map with $\mathrm{Aut}(\pi_{\mathfrak{G}}) \equiv \Gamma$. In fact, since $\widetilde{\mathcal{F}}$ is defined by the fiber bundle D, we get that $\mathcal{R}_{\widetilde{\mathcal{F}}}$ is a regular submanifold of $\widetilde{M}^2$, and $(\tilde{\mathbf{r}}, \tilde{\mathbf{s}}) \colon \widetilde{\mathfrak{G}} \to \mathcal{R}_{\widetilde{\mathcal{F}}}$ is a diffeomorphism. We may write $\mathfrak{G} \equiv \mathcal{R}_{\mathcal{F}}$ and $\widetilde{\mathfrak{G}} \equiv \mathcal{R}_{\widetilde{\mathcal{F}}}$.

Consider the C^∞ vector bundles, $S = \mathbf{r}^* \Lambda\mathcal{F} \otimes \mathbf{s}^*(\Lambda\mathcal{F} \otimes \Omega\mathcal{F})$ over $\mathfrak{G}$ and $\widetilde{S} = \tilde{\mathbf{r}}^* \Lambda\widetilde{\mathcal{F}} \otimes \tilde{\mathbf{s}}^*(\Lambda\widetilde{\mathcal{F}} \otimes \Omega\widetilde{\mathcal{F}})$ over $\widetilde{\mathfrak{G}}$. Note that $\widetilde{S} \equiv \pi_{\mathfrak{G}}^* S$, and any $k \in C^\infty(\mathfrak{G}; S)$ lifts via $\pi_{\mathfrak{G}}$ to a section $\tilde{k} \in C^\infty(\widetilde{\mathfrak{G}}; \widetilde{S})$. Since π restricts to diffeomorphisms of the leaves of $\widetilde{\mathcal{F}}$ to the leaves of $\mathcal{F}$, it follows that $\tilde{k} \in C^\infty_{\mathrm{p}}(\widetilde{\mathfrak{G}}; \widetilde{S})$ if and only if $k \in C^\infty_{\mathrm{p}}(\mathfrak{G}; S)$.

For any $\psi \in \mathcal{R}$, the collection of Schwartz kernels $k_L := K_{\psi(D_L)}$, for all leaves L of $\mathcal{F}$, defines a section $k = k_\psi$ of S called *leafwise Schwartz kernel*, which *a priori* may not be continuous. This also applies to the operators $\psi(D_{\widetilde{L}})$ on the leaves $\widetilde{L}$ of $\widetilde{\mathcal{F}}$, obtaining the leafwise Schwartz kernel $\tilde{k} = \tilde{k}_\psi$, which is a possibly discontinuous section of $\widetilde{S}$.

Proposition 10.1. *If $\hat{\psi} \in C^\infty_{\mathrm{c}}(\mathbb{R})$, then $k_\psi \in C^\infty_{\mathrm{p}}(\mathfrak{G}; S)$, and the global action of k_ψ on $C^\infty_{\mathrm{c}}(M; \Lambda\mathcal{F})$ (Section 4.5) agrees with the restriction of the operator $\psi(D_{\mathcal{F}})$ on $H^\infty(M; \Lambda\mathcal{F})$, defined by Proposition 7.2 and* (29).

Proof. This follows with the arguments of [41, Theorem 2.1], using $C^\infty_{\mathrm{p}}(\mathfrak{G}; S)$ instead of $C^\infty_{\mathrm{c}}(\mathfrak{G}; S)$ when M is not compact. ☐

Remark 10.2. In Proposition 10.1, more precisely, if $\operatorname{supp} \hat{\psi} \subset [-R, R]$ for some $R > 0$, then $\operatorname{supp} k_\psi \subset \overline{\mathrm{Pen}}_{\mathcal{F}}(\mathfrak{G}^{(0)}, R)$ by (18). Hence $\operatorname{supp} \psi(D_{\mathcal{F}})\alpha \subset \overline{\mathrm{Pen}}_{\mathcal{F}}(\operatorname{supp} \alpha, R)$ for all $\alpha \in H^{-\infty}(M; \Lambda\mathcal{F})$ by Remark 3.15.

Suppose for a while that $\hat{\psi} \in C^\infty_{\mathrm{c}}(\mathbb{R})$. Then Proposition 10.1 also applies to $\widetilde{\mathcal{F}}$, obtaining that $\tilde{k} \in C^\infty_{\mathrm{p}}(\widetilde{\mathfrak{G}}; \widetilde{S})$, and the global action of $\tilde{k}$ on $C^\infty_{\mathrm{c}}(\widetilde{M}; \Lambda\widetilde{\mathcal{F}})$ is the restriction of the operator $\psi(D_{\widetilde{\mathcal{F}}})$ on $H^\infty(\widetilde{M}; \Lambda\widetilde{\mathcal{F}})$ defined by Proposition 7.2 and (29). Indeed, since π restricts to a diffeomorphism between the leaves of $\widetilde{\mathcal{F}}$ and the leaves of $\mathcal{F}$, we get that $\tilde{k}$ is the lift of k, and therefore the diagram

$$
\begin{array}{ccc}
C^\infty_{\mathrm{c}}(\widetilde{M}; \Lambda\widetilde{\mathcal{F}}) & \xrightarrow{\psi(D_{\widetilde{\mathcal{F}}})} & C^\infty_{\mathrm{c}}(\widetilde{M}; \Lambda\widetilde{\mathcal{F}}) \\
{\scriptstyle \pi_*}\downarrow & & \downarrow{\scriptstyle \pi_*} \\
C^\infty_{\mathrm{c}}(M; \Lambda\mathcal{F}) & \xrightarrow{\psi(D_{\mathcal{F}})} & C^\infty_{\mathrm{c}}(M; \Lambda\mathcal{F})
\end{array}
\tag{67}
$$

is commutative, where

$$\pi_*\tilde\alpha \equiv \sum_{\gamma\in\Gamma} T_\gamma^*\tilde\alpha \tag{68}$$

for all $\tilde\alpha \in C^\infty_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F})$, using the notation T_γ for the action of every $\gamma\in\Gamma$ on $\widetilde M$. Locally, the series of (68) only has a finite number of nonzero terms. The same expression also defines $\pi_*\colon C^{-\infty}_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F}) \to C^{-\infty}_{\mathrm{c}}(M;\Lambda\mathcal F)$.

Take some $R>0$ such that $\operatorname{supp}\hat\psi \subset [-R,R]$. By Remark 10.2 and since ϕ is of $\mathbb R$-local bounded geometry, there is some $R'>0$ such that $\operatorname{supp} P\alpha \subset \overline{\operatorname{Pen}}(\operatorname{supp}\alpha, R')$ for all $\alpha\in H^{-\infty}(M;\Lambda\mathcal F)$. Thus P defines a continuous homomorphism $C^{-\infty}_{\mathrm{c}}(M;\Lambda\mathcal F)\to C^\infty_{\mathrm{c}}(M;\Lambda\mathcal F)$. Similarly, $\widetilde P$ defines a continuous homomorphism $C^{-\infty}_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F}) \to C^\infty_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F})$. Moreover, the commutativity of (67) yields the commutativity of the diagram

$$\begin{array}{ccc} C^{-\infty}_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F}) & \xrightarrow{\ \widetilde P\ } & C^\infty_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F}) \\ {\scriptstyle\pi_*}\downarrow & & \downarrow{\scriptstyle\pi_*} \\ C^{-\infty}_{\mathrm{c}}(M;\Lambda\mathcal F) & \xrightarrow{\ P\ } & C^\infty_{\mathrm{c}}(M;\Lambda\mathcal F). \end{array} \tag{69}$$

Let $\widetilde\Lambda = D^*dx \equiv dx$, which is a transverse invariant volume form of $\widetilde{\mathcal F}$ defining the same transverse orientation as $\overline{\widetilde Z}$. Since $\widetilde\Lambda$ is Γ-invariant by the h-equivariance of D, it defines a transverse volume form Λ of $\mathcal F$, which defines the same transverse orientation as $\overline Z$. These $\widetilde\Lambda$ and Λ define transverse invariant densities $|\widetilde\Lambda|$ and $|\Lambda|$ of $\widetilde{\mathcal F}$ and $\mathcal F$.

Proposition 10.3. *Let $\psi\in\mathcal A$ and $\tilde p,\tilde q\in\widetilde M$ over $p,q\in M$. Then, writing $t_{\tilde p,\tilde q} = D(\tilde q)-D(\tilde p)$ and using the identity $\widetilde S_{(\tilde p,\tilde q)}\equiv S_{(p,q)}$, we have*[16]

$$K_P(p,q) \equiv \sum_{\gamma\in\Gamma} T_\gamma^*\,\tilde\phi^{t_{\tilde p,\tilde q}-h(\gamma)*}\tilde k(T_\gamma\tilde\phi^{t_{\tilde p,\tilde q}-h(\gamma)}(\tilde p),\tilde q)\,f(t_{\tilde p,\tilde q})\,|\Lambda|(q),$$

defining a convergent series in $C^\infty_{\mathrm{ub}}(M^2;S)$.

Proof. We can assume that $\hat\psi\in C^\infty_{\mathrm{c}}(\mathbb R)$ by Propositions 9.6 and 3.14, and because $C^\infty_{\mathrm{c}}(\mathbb R)$ is dense in $\mathcal A$. Then, by Proposition 10.1, for all $\tilde\alpha\in C^\infty_{\mathrm{c}}(\widetilde M;\Lambda\widetilde{\mathcal F})$,

$$\begin{aligned}(\widetilde P\tilde\alpha)(\tilde p) &= \int_{\mathbb R} \big(\tilde\phi^{t*}\,\psi(D_{\widetilde{\mathcal F}})\tilde\alpha\big)(\tilde p)\,f(t)\,dt \\ &= \int_{\mathbb R} \tilde\phi^{t*}\big(\psi(D_{\widetilde{\mathcal F}})\tilde\alpha\big)\big(\tilde\phi^t(\tilde p)\big)\,f(t)\,dt \\ &= \int_{t\in\mathbb R}\int_{\tilde q\in D^{-1}(t)} \tilde\phi^{t*}\tilde k(\tilde\phi^t(\tilde p),\tilde q)\tilde\alpha(\tilde q)\,f(t)\,dt \\ &= \int_{\tilde q\in\widetilde M} \tilde\phi^{t_{\tilde p,\tilde q}*}\tilde k(\tilde\phi^{t_{\tilde p,\tilde q}}(\tilde p),\tilde q)\tilde\alpha(\tilde q)\,f(t_{\tilde p,\tilde q})\,\big|\widetilde\Lambda\big|(\tilde q),\end{aligned}$$

[16]The leafwise part of the density of $K_P(\,\cdot\,,q)$ is given by the density of $\tilde k(\,\cdot\,,\tilde q)$.

because $D\tilde{\phi}^t(\tilde{p}) = D(\tilde{q})$ if and only if $t = t_{\tilde{p},\tilde{q}}$. Therefore

$$K_{\widetilde{P}}(\tilde{p},\tilde{q}) = \tilde{\phi}^{t_{\tilde{p},\tilde{q}}*}\tilde{k}(\tilde{\phi}^{t_{\tilde{p},\tilde{q}}}(\tilde{p}),\tilde{q})\, f(t_{\tilde{p},\tilde{q}})\, \big|\widetilde{\Lambda}\big|(\tilde{q}).$$

On the other hand, by (68) and the commutativity of (69),

$$K_P(p,q) \equiv \sum_{\gamma\in\Gamma} T_\gamma^* K_{\widetilde{P}}(\gamma\cdot\tilde{p},\tilde{q}).$$

Locally, this series only has a finite number of nonzero terms. This is the series of the statement because $D(\tilde{q}) - D(\gamma\cdot\tilde{p}) = D(\tilde{q}) - D(\tilde{p}) - h(\gamma)$ by the h-equivariance of D. □

11. Extension to the leafwise Novikov differential complex

Consider the notation of Section 2.6 and Section 4.11, and assume that $\theta \in C^\infty_{\text{ub}}(M;\Lambda^{0,1}) \equiv C^\infty_{\text{ub}}(M;\Lambda^1\mathcal{F})$. Then, like in (31) and (43), we get the decompositions into bi-homogeneous components,

$$d_z = d_{z,0,1} + d_{1,0} + d_{2,-1}, \quad \delta_z = \delta_{z,0,-1} + \delta_{-1,0} + \delta_{-2,1},$$

where $d_{z,0,1} = d_{0,1} + z\,\theta\wedge$ and $\delta_{z,0,-1} = \delta_{0,-1} - \bar{z}\,\theta\lrcorner$, which are of bounded geometry by Corollary 6.2 and because $\theta \in C^\infty_{\text{ub}}(M;\Lambda^{0,1})$. Since θ is closed, we get $d_{i,j}\theta = 0$ for all i,j. So, by (34) and (44),

$$d_{z,0,1}^2 = d_{z,0,1}d_{1,0} + d_{1,0}d_{z,0,1} = 0, \tag{70}$$

$$\delta_{z,0,-1}^2 = \delta_{z,0,-1}\delta_{-1,0} + \delta_{-1,0}\delta_{z,0,-1} = 0. \tag{71}$$

Let

$$D_{0,z} = d_{z,0,1} + \delta_{z,0,-1}, \quad \Delta_{0,z} = D_{0,z}^2 = d_{z,0,1}\delta_{z,0,-1} + d_{z,0,1}\delta_{z,0,-1}.$$

On the other hand, we can also consider the leafwise version of the Novikov differential complex, $d_{\mathcal{F},z} = d_{\mathcal{F}} + z\,\theta\wedge$ on $C^\infty(M;\Lambda\mathcal{F})$, or on $C^\infty(M;\Lambda\mathcal{F}\otimes\Lambda N\mathcal{F})$, as well as its formal adjoint $\delta_{\mathcal{F},z} = \delta_{\mathcal{F}} - \bar{z}\,\theta\lrcorner$. They satisfy the obvious versions of (32) and Lemma 4.12, yielding obvious versions of (41) and (45). Furthermore, for any choice of an orientation of $\mathcal{F}$ on a distinguished open set U, we have

$$\delta_{\mathcal{F},z} = (-1)^{n''v+n''+1} \star_{\mathcal{F}} d_{\mathcal{F},-\bar{z}} \star_{\mathcal{F}}$$

on $C^\infty(U;\Lambda^v\mathcal{F})$ by (13). So, using also (70) and (71), we get the following version of (55):

$$D_\perp D_{0,z} + D_{0,z}D_\perp = KD_{0,z} + D_{0,z}K.$$

This yields straight-forward generalizations of all results and proofs of Section 7 for the *leafwise Novikov operators*, $d_{z,0,1}$, $D_{0,z}$, $\Delta_{0,z}$, $d_{\mathcal{F},z}$, $D_{\mathcal{F},z}$ and $\Delta_{\mathcal{F},z}$. Let $\Pi_{0,z}$ and $\Pi_{\mathcal{F},z}$ denote the corresponding versions of Π_0 and $\Pi_{\mathcal{F}}$. The term *leafwise Witten operators* should be used if θ is leafwise exact.

Let $\phi\colon (M,\mathcal{F}) \to (M,\mathcal{F})$ be a smooth foliated map, let $\widetilde{M}$ be a regular covering of M so that the lift $\tilde{\theta}$ of θ is exact, and let $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$ to $\widetilde{M}$. Like in the case of the Novikov differential complex (Section 2.6), using (27), any lift $\tilde{\phi}\colon (\widetilde{M},\widetilde{\mathcal{F}}) \to (\widetilde{M},\widetilde{\mathcal{F}})$ of ϕ determines an endomorphism ϕ_z^{t*} of the leafwise Novikov differential complex $d_{\mathcal{F},z}$ on $C^\infty(M;\Lambda\mathcal{F})$, or on $C^\infty(M;\Lambda\mathcal{F}\otimes\Lambda N\mathcal{F})$, which can be called a *leafwise Novikov perturbation* of ϕ^*. With this definition, there is an obvious generalization of (36), using the bi-homogeneous component $\phi^*_{z,0,0}$ of ϕ^*_z on $C^\infty(M;\Lambda)$. For every foliated flow $\phi=\{\phi^t\}$ on $(M,\mathcal{F})$, using its unique lift to a foliated flow $\tilde\phi=\{\tilde\phi^t\}$ on $(\widetilde{M},\widetilde{\mathcal{F}})$, we get a unique determination of ϕ^{t*}_z on $C^\infty(M;\Lambda\mathcal{F})$, or on $C^\infty(M;\Lambda\mathcal{F}\otimes\Lambda N\mathcal{F})$, called *the* Novikov perturbation of ϕ^{t*}. Then, in Sections 9 and 10, the definitions of P, P_u, P_∞, k, $\tilde k$, k_u and $\tilde k_u$ can be extended by using ϕ^{t*}_z and $D_{\mathcal{F},z}$ instead of ϕ^{t*} and $D_{\mathcal{F}}$. The subindex "z" may be added to their notation if needed. Moreover, the results, proofs and observations of Sections 9 and 10 have straight-forward generalizations to this setting, using the indicated extensions of the tools.

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