Münster J. of Math. **7** (2014), 363–379
urn:nbn:de:hbz:6-22359592466
Münster Journal of Mathematics

The pro-p-Iwahori–Hecke algebra of a reductive p-adic group, II

Marie-France Vignéras

(Communicated by Christopher Deninger)

This paper is dedicated to Peter Schneider on the occasion of its 60-th birthday, in appreciation for his insights, precision and generosity in mathematics.

Abstract. For any commutative ring R and any reductive p-adic group G, we describe the center of the pro-p-Iwahori–Hecke R-algebra of G. We show that the pro-p-Iwahori–Hecke algebra is a finitely generated module over its center and is a finitely generated R-algebra. When the ring R is noetherian, the center is a finitely generated R-algebra and the pro-p-Iwahori–Hecke R-algebra is noetherian. This generalizes results known only for split groups.

1. Results

Let R be a commutative ring, let F be a local nonarchimedean field, of finite residue field k with q elements and of characteristic p, let G, T, Z, N be the groups of F-points of a connected reductive F-group $\mathbf{G}$ of maximal F-split torus $\mathbf{T}$ of $\mathbf{G}$-centralizer $\mathbf{Z}$ and $\mathbf{G}$-normalizer $\mathbf{N}$. The group Z admits a unique parahoric subgroup Z_0, and Z_0 admits a unique pro-p-Sylow $Z_0(1)$. The unicity of these groups imply that they are normalized by N. The group $Z_k = Z_0/Z_0(1)$ is the group of k-points of a k-torus.

Let $W_0 = N/Z$, $W = N/Z_0$ and $W(1) = N/Z_0(1)$. Then W_0 is the relative finite Weyl group of G, the quotient map $W \to W_0$ of kernel $\Lambda = Z/Z_0$ splits, the group $W(1)$ is an extension of W by Z_k.

We will denote by $X(1)$ the inverse image in $W(1)$ of a subset X of W, and we write $\tilde{w}$ for an arbitrary element of $W(1)$ of image $w \in W$.

The group Λ is commutative but the group $\Lambda(1) = Z/Z_0(1)$ may be not commutative. A conjugacy class C of W is contained in Λ or disjoint from Λ. The same is true for $W(1)$ and $\Lambda(1)$.

The Iwahori–Hecke R-algebra $\mathcal{H}_R$ of G is a deformation of the group R-algebra $R[W]$ and the pro-p-Iwahori–Hecke R-algebra $\mathcal{H}_R(1)$ of G is a deformation of the group R-algebra $R[W(1)]$, cp. [7].

The center of the R-algebra of the group W or of $W(1)$ can be easily determined using the following following property:

Lemma 1.1. *A conjugacy class C of W is finite if and only if C is contained in Λ. The same is true for $W(1)$ and $\Lambda(1)$.*

As a corollary, the center of the R-algebra $R[W]$ is equal to $R[\Lambda]^{W_0}$ and is R-free of basis

$$z_C = \sum_{\lambda \in C} \lambda,$$

for all finite conjugacy classes C of W. The group algebra $R[W]$ is a finitely generated module over its center. If the ring R is noetherian, the center is a finitely generated algebra and $R[W]$ is a noetherian algebra. The same is true for $R[W(1)]$. We will prove that these properties remain true for the Iwahori and pro-p-Iwahori R-algebras.

For each (spherical) orientation o, we denote by $(E_o(w))_{w\in W}$, resp. $(E_o(w))_{w\in W(1)}$, the (alcove walk or Bernstein) basis of $\mathcal{H}_R$, resp. $\mathcal{H}_R(1)$ (cp. [7, Cor. 5.26, Cor. 5.28]).

Theorem 1.2. *The center $\mathcal{Z}_R$ of the Iwahori–Hecke R-algebra $\mathcal{H}_R$ is R-free of basis*

$$E(C) = \sum_{\lambda \in C} E_o(\lambda)$$

for all finite conjugacy classes C of W. The basis elements $E(C)$ do not depend on the choice of the orientation o.

The Iwahori–Hecke R-algebra is finitely generated and is a finitely generated module over its center. If the ring R is noetherian, the center is a finitely generated algebra, and the Iwahori–Hecke R-algebra is noetherian.

The center $\mathcal{Z}_R(1)$ of the pro-p Iwahori–Hecke R-algebra $\mathcal{H}_R(1)$ satisfies the same properties.

This theorem was proved by Bernstein for affine Hecke complex algebras [3], or when the group G is F-split [5], [6]. In his unpublished and nice diplomarbeit [4], Schmidt gave this theorem for certain pro-p-Iwahori–Hecke algebras, although his proof contains some gaps but the pattern of his proof is correct and we follow it. The Iwahori–Hecke or pro-p-Iwahori–Hecke algebras attached to reductive p-adic groups are more general than the affine Hecke algebras of Lusztig and than the pro-p-Iwahori–Hecke algebras of Schmidt.

We will prove the theorem for the R-algebras $\mathcal{H}_R(q_s, c_s)$ defined in the part I of our work [7] generalizing the pro-p-Iwahori–Hecke algebra. This allows to reduce some proofs to the simpler case $q_s = 1$.

We recall the definition of $\mathcal{H}_R(q_s, c_s)$, cp. [7]. There exists an affine Weyl Coxeter system $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$ and a finitely generated commutative subgroup Ω normalizing S^{aff}, such that

$$W = W^{\mathrm{aff}} \rtimes \Omega.$$

We write $s \sim s'$ for two elements $s, s' \in S^{\mathrm{aff}}$ which are W-conjugate. Let $(q_s)_{s \in S^{\mathrm{aff}}/\sim}$ in R, and $(c_s)_{s\in S^{\mathrm{aff}}(1)}$ in $R[Z_k]$ satisfying

$$c_{\tilde{s}t} = c_s t,\ c_{s'} = w c_s w^{-1},\ \ (t \in Z_k,\ s' = wsw^{-1},\ s, s' \in S^{\mathrm{aff}}(1),\ w \in W(1)).$$

We set $q_{\tilde{s}} = q_s$. The R-algebra $\mathcal{H}_R(q_s, c_s)$ is the free R-module of basis $(T_w)_{w\in W(1)}$ endowed with the product satisfying

- The braid relations:

(1) $$T_w T_{w'} = T_{ww'}\ \ (w, w' \in W(1), \ell(w) + \ell(w') = \ell(ww')),$$

  where $\ell(w) = \ell(w^{\mathrm{aff}})$ if the image of w in W is $w^{\mathrm{aff}} u,\ w^{\mathrm{aff}} \in W^{\mathrm{aff}}, u \in \Omega$ and ℓ the length of $(W^{\mathrm{aff}}, S^{\mathrm{aff}})$. Therefore the linear map sending t to T_t identifies $R[Z_k]$ to a subalgebra of $\mathcal{H}_R(q_s, c_s)$, and $s^2 \in Z_k$ and $c_s \in R[Z_k]$ identify to elements of $\mathcal{H}_R(q_s, c_s)$.

- The quadratic relations:

(2) $$T_s^2 = q_s s^2 + c_s T_s\ (s \in S^{\mathrm{aff}}(1)).$$

In our basic example, the pro-p-Iwahori–Hecke algebra, $q_s = [IsI : I] = [I(1)sI(1) : I(1)]$ where I is an Iwahori group of pro-p-Sylow $I(1)$; when s belongs to a certain coset $sZ_{k,s} = Z_{k,s}s$ in $W(1)$ of a subgroup $Z_{k,s}$ of Z_k defined in [7],

$$c_s = (q_s - 1)|Z_{k,s}|^{-1} \sum_{t\in Z_{k,s}} t.$$

The R-algebra $\mathcal{H}_R(q_s)$ generalizing the Iwahori–Hecke algebra is simpler. It admits the same definition with $c_s = q_s - 1$ and W instead of $W(1)$. We have

$$\mathcal{H}_R(q_s) = R \otimes_{R[Z_k]} \mathcal{H}_R(q_s, q_s - 1)$$

for the homomorphism sending $t \in Z_k$ to 1.

The affine Weyl group W^{aff} is generated by the orthogonal reflections with respect to a set of affine hyperplanes in an Euclidean real vector space V. The finite Weyl group W_0 identifies with the subgroup of W^{aff} generated by the the reflections with respect to the hyperplanes containing 0. The group W acts by conjugation on Λ and

$$W = \Lambda \rtimes W_0.$$

The simply transitive action of the group W_0 on the Weyl chambers inflates to an action of W trivial on Λ and to an action of $W(1)$ trivial on $\Lambda(1)$. The orientations are in bijection with the Weyl chambers, and inherit this action. For an orientation o and for w in W_0 or W or $W(1)$, we denote by $o \bullet w$ the image of o by w^{-1}. We denote by S_o the set of reflections with respect to the walls of the Weyl chamber defining o. The R-basis $(E_o(w))_{w\in W(1)}$ of $\mathcal{H}_R(q_s, c_s)$ associated to o satisfies:

- the product formula [7, Thm. 5.25]:

(3) $$E_o(w)E_{o\bullet w}(w') = q_{w,w'} E_o(ww')\ \ (w, w' \in W(1)),$$

  where $q_{w,w'} = (q_w q_{w'} q_{ww'}^{-1})^{1/2}$ and $q_w = q_{s_1} \dots q_{s_r}$ if $w = \tilde{s}_1 \dots \tilde{s}_r \tilde{u}$ with $s_i \in S^{\mathrm{aff}}, u \in \Omega$ is a reduced decomposition (r minimal).

- the Bernstein relations [7, Thm. 5.45]:

$$E_o(s)E_o(\lambda) - E_o(s\lambda s^{-1})E_o(s) = \sum_{\mu\in\Lambda(1)} a_{o,s,\lambda}(\mu)E_o(\mu) \tag{4}$$

for $\lambda \in \Lambda(1)$, $s \in S_o(1)$, with explicit coefficients $a_{o,s,\lambda}(\mu) \in R$.

We have the subalgebra $\mathcal{A}_o(1)$ of basis $(E_o(\lambda))_{\lambda\in\Lambda(1)}$ with the natural action of $W(1)$ coming from the conjugation on $\Lambda(1)$, (cp. [7, Cor. 5.26, Cor. 5.28]). Theorem 1.2 is a corollary of:

Theorem 1.3. *The center $\mathcal{Z}_R(q_s, c_s)$ of $\mathcal{H}_R(q_s, c_s)$ is the ring $\mathcal{A}_o(1)^{W(1)}$ of $W(1)$-invariant elements of $\mathcal{A}_o(1)$. It is a free R-module of basis*

$$E(C) = \sum_{\lambda\in C} E_o(\lambda)$$

for all finite conjugacy classes C of $W(1)$. The basis elements $E(C)$ do not depend on the choice of the orientation o.

The R-algebra $\mathcal{H}_R(q_s, c_s)$ is finitely generated and is a finitely generated module over its center $\mathcal{Z}_R(q_s, c_s)$. If the ring R is noetherian, the center $\mathcal{Z}_R(q_s, c_s)$ is a finitely generated R-algebra, and the R-algebra $\mathcal{H}_R(q_s, c_s)$ is noetherian.

Remark 1.4. The action of W on Λ factorizes through W_0. The center of $\mathcal{H}_R(q_s)$ is $\mathcal{A}_o^{W_0}$ where $\mathcal{A}_o$ is the commutative subring of basis $E_o(\lambda)$ for $\lambda \in \Lambda$.

This theorem generalizes our results on the center for G split [5]. In a sequel of this paper, when C is an algebraically closed field of characteristic p, this theorem will be used to generalize to any reductive connected group G, some results of Ollivier on pro-p-Iwahori–Hecke C-algebra when G is split: the embedding of the weighted spherical algebras in the pro-p Iwahori–Hecke R-algebra $\mathcal{H}_C(1)$, the inverse of the Satake isomorphism considered in [2] and the classification of the supersingular $\mathcal{H}_C(1)$-modules.

2. Proofs

We denote by T the maximal split torus of G, and by $X^*(T)$ and $X_*(T)$ the lattices of characters and of cocharacters of T. Let ω be the valuation of F such that $\omega(F^\times) = \mathbb{Z}$. The kernel of the canonical map $t \mapsto (\chi \mapsto (\omega\circ\chi)(t))$: $T \to \mathrm{Hom}(X^*(T), \mathbb{Z}) \simeq X_*(T)$ is the maximal compact subgroup T_0 of T. The choice of an uniformizer p_F defines a splitting of the exact sequence

$$1 \to T_0 \to T \to X_*(T) \to 1$$

sending $\mu \in X_*(T)$ to $\mu(p_F)$. We have $Z_0 \cap T = T_0$ and $Z_0(1) \cap T = T_0(1)$ is the unique pro-p-Sylow of T. The quotient $T_k = T_0/T_0(1)$ is the group of k-points of a split k-torus. The splitting induces isomorphisms

$$T \simeq T_0 \times X_*(T),\ T/T_o(1) \simeq T_k \times X_*(T).$$

The commutative finitely generated group $T/T_0(1)$ is central of finite index in $\Lambda(1)$. We recall the homomorphism

(5) $\nu : Z \to V = \mathbb{R} \otimes_{\mathbb{Z}} Q(\Phi^\vee) \mid \alpha \circ \nu(t) = -\omega \circ \alpha(t)$ for $\alpha \in \Phi,\ t \in T$,

where $\Phi \subset X^*(T)$ is the set of roots of T in G. The homomorphism ν factorizes through $\Lambda = Z/Z_0$ and $\Lambda(1) = Z/Z_0(1)$.

Lemma 2.1. *A conjugacy class of Λ is finite.*

A conjugacy class of W is finite and contained in Λ, or infinite and disjoint from Λ.

The same is true for $W(1)$ and $\Lambda(1)$.

Proof. A conjugacy class of W is contained in Λ or disjoint from Λ because Λ is normal in W. The same is true for $W(1)$ and $\Lambda(1)$.

The group Λ is commutative and the conjugacy classes of W contained in Λ are the orbits of the finite Weyl group W_0 acting on Λ. They are of course finite.

The group $\Lambda(1)$ is not commutative but the conjugacy classes of $\Lambda(1)$ are finite, as the center of $\Lambda(1)$ contains $T/T_0(1)$ hence has a finite index. For $\lambda \in \Lambda(1)$ we denote by $c(\lambda)$ the conjugacy class of $\lambda \in \Lambda(1)$. We have $W(1) = \Lambda(1)W_0(1)$. Obviously $tc(\lambda)t^{-1} = c(\lambda)$ for $t \in Z_k$, and $W_0 = W_0(1)/Z_k$ acts on the conjugacy classes of $\Lambda(1)$, with orbits the conjugacy classes of $W(1)$ contained in $\Lambda(1)$. They are of course finite.

The image in W of a conjugacy class of $W(1)$ not contained in $\Lambda(1)$ is a conjugacy class of W not contained in Λ.

We show that a conjugacy class of W not contained in Λ is infinite. Let $\lambda \in \Lambda$, $w \in W_0$, $w \neq 1$. The conjugacy class of λw in W is infinite because, for $x \in T/T_0$, $x\lambda w x^{-1} = x\lambda w x^{-1} w^{-1} w$ and $\nu(x\lambda w x^{-1} w^{-1}) = \nu(x) - w(\nu(x)) + \nu(\lambda)$, and the set of $\nu(x) - w(\nu(x))$ for $x \in T/T_0$ contains the set of $y - w(y)$ for y in the subgroup $Q(\Phi^\vee)$ of $X_*(T)$ generated by the set $\Phi^\vee$ of coroots of T in G. This latter set is infinite. ☐

It is now easy to see that the center of the group R-algebra $R[W(1)]$ is $R[\Lambda(1)]^{W(1)}$:

Lemma 2.2. *The center of $R[W(1)]$ is the free R-module of basis*

$$z_C = \sum_{\lambda \in C} \lambda,$$

for all finite conjugacy classes C of $W(1)$.

Proof. Let $z = \sum_{u \in W(1)} z(u)u \in R[W(1)]$ where the function $z : W(1) \to R$ has finite support. The following properties are equivalent:

1. $zw = wz$ for all $w \in W(1)$,
2. $z(w^{-1}u) = z(uw^{-1})$ for all $u, w \in W(1)$,
3. $z(.)$ is constant on C for all conjugacy classes C of $W(1)$.

☐

We want now to determine the center of the R-algebra $\mathcal{H}_R(q_s, c_s)$, described in the first part of Theorem 1.3. The proof is long and we formulate the different steps as propositions, lemmas, and corollaries. We fix an orientation o and we determine first the center of $\mathcal{A}_o(1)$.

Proposition 2.3. *The center of $\mathcal{A}_o(1)$ is a R-free module of basis*

$$E_o(c) = \sum_{\lambda \in c} E_o(\lambda),$$

for all conjugacy classes c of $\Lambda(1)$.

Proof. Let $z = \sum_{x \in \Lambda(1)} z(x) E_o(x)$ be an element in $\mathcal{A}_o(1)$ where $z : \Lambda(1) \to R$ is a function of finite support, and let $\lambda \in \Lambda(1)$. Let ([7, Def. 4.14]):

$$q_{w,w'} = (q_w q_{w'} q_{ww'}^{-1})^{1/2} \quad (w, w' \in W(1)). \tag{6}$$

By [7, Cor. 5.28],

$$E_o(x) E_o(\lambda) = q_{x,\lambda} E_o(x\lambda)$$

and $q_{\lambda x \lambda^{-1},\lambda} = q_{x,\lambda} = q_{\lambda,x}$ because q_λ depends only on the image of λ in Λ, and Λ is commutative. Then z commutes with $E_o(\lambda)$ if and only if

$$\sum_{x \in \Lambda(1)} z(x) q_{x,\lambda} E_o(x\lambda) = \sum_{x \in \Lambda(1)} z(x) q_{\lambda,x} E_o(\lambda x).$$

Replacing x by $\lambda x \lambda^{-1}$, the left hand side is equal to

$$\sum_{x \in \Lambda(1)} z(\lambda x \lambda^{-1}) q_{\lambda,x} E_o(\lambda x).$$

Hence z belongs to the center of $\mathcal{A}_o(1)$ if and only if

$$z(\lambda x \lambda^{-1}) q_{x,\lambda} = z(x) q_{x,\lambda} \text{ for all } x, \lambda \in \Lambda(1).$$

If $z(.)$ is constant on the conjugacy classes of $\Lambda(1)$, then z central in $\mathcal{A}_o(1)$. For the converse, there is a problem when $q_{x,\lambda}$ is not invertible in R, but it can be raised. We have $q_{x,\lambda} = 1$ if and only if $\ell(\lambda x) = \ell(\lambda) + \ell(x)$. If z central in $\mathcal{A}_o$, the next lemma implies that $z(.)$ is constant on the conjugacy classes of $\Lambda(1)$. Admitting the next lemma, the proposition is proved. □

Lemma 2.4. *Let x, x' be two conjugate elements of $\Lambda(1)$. There exists $\lambda \in \Lambda(1)$ such that $\ell(\lambda x) = \ell(\lambda) + \ell(x)$ and $x' = \lambda x \lambda^{-1}$.*

Proof. We choose an arbitrary element $\lambda \in \Lambda(1)$ such that $x' = \lambda x \lambda^{-1}$. We choose $\lambda' \in T/T_o(1)$ such that $\nu(\lambda\lambda')$ and $\nu(x)$ belong to the same closed Weyl chamber. We have $\ell(\lambda\lambda' x) = \ell(\lambda\lambda') + \ell(x)$ (cp. [7, Ex. 5.12]) and $x' = (\lambda\lambda') x (\lambda\lambda')^{-1}$ because λ' belongs to the center of $\Lambda(1)$. □

The orientation o is associated to a Weyl chamber $\mathfrak{D}_o$ of V. We denote by Δ_o the set of reduced roots $\alpha \in \Phi$ positive on $\mathfrak{D}_o$ such that $\operatorname{Ker}\alpha$ is a wall of $\mathfrak{D}_o$, by $S_o \subset W_0$ the set of reflections s_α for $\alpha \in \Delta_o$. The opposite orientation associated to the opposite Weyl chamber $-\mathfrak{D}_o$ gives the same set S_o. The set $S = S^{\text{aff}} \cap W_0$ is associated to the dominant and antidominant Weyl chambers

$\mathfrak{D}^+$ and $-\mathfrak{D}^+$. For $s \in S_o(1)$, there exists a unique root $\alpha \in \Delta_o \cup -\Delta_o$ taking positive values on $\mathfrak{D}^+$ such that the image of s in S_o is s_α. We say that the root α of Φ is associated to (o, s).

For $\gamma \in \Phi$ let $e_\gamma \in \mathbb{N}_{>}0$ be the positive integer such that the image of Φ by the map $\gamma \mapsto e_\gamma \gamma$ is a reduced root system Σ of affine Weyl group W^{aff}. If α is the root of Φ is associated to (o, s), the root $\beta = e_\alpha \alpha$ of Σ is called associated to (o, s).

Let z_o be a central element of $\mathcal{A}_o(1)$. By Proposition 2.3:

$$z_o = \sum_c z_o(c) E_o(c) \quad (z_o(c) \in R). \tag{7}$$

Let $s \in S_o(1)$. In the R-basis $(E_o(w))_{w \in W(1)}$ of $\mathcal{H}_R(q_s, c_s)$, we write

$$-z_o E_o(s) + E_o(s) z_o = z_o' + z_o''$$

where $z_o'' \in \mathcal{A}_o(1)$ and the component of z_o' in $\mathcal{A}_o(1)$ is 0. The elements z_o and $E_o(s)$ commute if and only if $z_o' = z_o'' = 0$. We denote by scs^{-1} the set of $s\lambda s^{-1}$ for $\lambda \in c$.

Lemma 2.5. *We have*

$$z_o'' = E_o(s) \sum_c z_o(c)(E_o(c) - E_{o \bullet s}(c)),$$
$$z_o' = E_o(s) \sum_c (z_o(c) - z_o(scs^{-1})) E_{o \bullet s}(c) = \sum_c z_o'(sc) E_o(sc),$$

with $z_o'(sc) = q_{s,c}(z_o(c) - z_o(scs^{-1}))$.

Proof. The equality $q_{scs^{-1},s} = q_{s,c}$ deduced from (6), implies by the product formula (3)

$$E_o(scs^{-1}) E_o(s) = E_o(s) E_{o \bullet s}(c).$$

Writing $z_o E_o(s) = E_o(s) \sum_c z_o(c) E_{o \bullet s}(s^{-1} c s) = E_o(s) \sum_c z_o(scs^{-1}) E_{o \bullet s}(c)$ we obtain

$$-z_o E_o(s) + E_o(s) z_o = E_o(s) \sum_c z_o(c) E_o(c) - z_o(scs^{-1}) E_{o \bullet s}(c),$$

from which we deduce $-z_o E_o(s) + E_o(s) z_o = z_o' + z_o''$ with z_o', z_o'' as in the lemma. The element z_o'' belong to $\mathcal{A}_o(1)$ by the Bernstein relations (4) and the component of

$$z_o' = \sum_c z_o'(sc) E_o(sc), \; z_o'(sc) = q_{s,c}(z_o(c) - z_o(scs^{-1}))$$

in $\mathcal{A}_o(1)$ is 0 because $s\Lambda(1) \cap \Lambda(1) = \varnothing$. □

Proposition 2.6. *Let $s \in S_o(1)$ and let c be a conjugacy class in $\Lambda(1)$. Then*
$E_o(s)$ and $E_o(c) + E_o(scs^{-1})$ commute.
$E_o(s)$ and $E_o(c)$ commute if and only if $c = scs^{-1}$.

Proof. a) Let β be the root of Σ associated to (o,s). We note that ν is constant on a conjugacy class of $\Lambda(1)$, scs^{-1} is a conjugacy class of $\Lambda(1)$, $\nu(s_\beta \lambda s_\beta^{-1}) = s_\beta(\nu(\lambda))$ for $\lambda \in \Lambda$ and $s_\beta(x) = x - \beta(x)\beta^\vee$ for $x \in V$ where $\beta^\vee$ is the coroot of β. Therefore we have

$$\nu(scs^{-1}) = \nu(c) - \beta \circ \nu(c)\beta^\vee.$$

If $c = scs^{-1}$ then $\beta \circ \nu(c) = 0$.

If $\beta \circ \nu(c) = 0$ then $q_{s,c} = q_{c,s} = 1$ and $E_{o\bullet s}(c) = E_o(c)$. This implies

$$E_o(s)E_o(c) - E_o(c)E_o(s) = E_o(s)E_{o\bullet s}(c) - E_o(c)E_o(s) = E_o(sc) - E_o(cs).$$

We deduce that $E_o(s)$ and $E_o(c)$ commute when $c = scs^{-1}$, and that $E_o(s)$ and $E_o(c)$ do not commute when $c \neq scs^{-1}$ and $\beta \circ \nu(c) = 0$.

b) We suppose $\beta \circ \nu(c) \neq 0$. We prove that either $E_o(c)$ or $E_o(scs^{-1})$ does not commutes with $E_o(s)$.

When $\ell(sc) = 1 + \ell(c)$, take $z_o = E_o(c)$ in Lemma 2.5. The coefficient of z'_o on $E_o(sc)$ is $q_{s,c} = (q_s q_c q_{sc}^{-1})^{1/2} = 1$. We have $z'_o \neq 0$ hence $E_o(c)$ does not commute with $E_o(s)$.

When $\ell(sc) = -1 + \ell(c)$ we have $\ell(cs) = 1 + \ell(c)$. Take $z_o = E_o(scs^{-1})$. We have $E_o(scs^{-1}) = E_o(s^{-1}cs)$ because $s^2 \in Z_k$. We have $\beta \circ \nu(c) = -\beta \circ \nu(scs^{-1}) \neq 0$. The coefficient of z'_o on $E_o(cs)$ is $q_{s,s^{-1}cs} = (q_s q_{s^{-1}cs} q_{cs}^{-1})^{1/2} = 1$. We have $z'_o \neq 0$ hence $E_o(scs^{-1})$ does not commute with $E_o(s)$.

c) We show that $E_o(c) + E_o(scs^{-1})$ commutes with $E_o(s)$. In Lemma 2.5 we take $z_o = E_o(c) + E_o(scs^{-1})$. We have obviously $z'_o = 0$.

When the $q_s = 1$ we show that $z''_o = 0$. By symetry, we can suppose $\beta \circ \nu(c) < 0$ and $\beta \circ \nu(scs^{-1}) > 0$. The Bernstein relations (4) give an explicit element $B_{o,n} \in \mathcal{A}_o(1)$ and different signs $\epsilon_{o\bullet s}(1,s) \neq \epsilon_o(1,s)$ such that

$E_o(s)(E_{o\bullet s}(c) - E_o(c)) = \epsilon_{o\bullet s}(1,s)B_{o,n}E_o(c)$,

$E_o(s)(E_{o\bullet s}(scs^{-1}) - E_o(scs^{-1})) = \epsilon_o(1,s)E_o(c)B_{o,n}$.

As $E_o(c)$ is central in $\mathcal{A}_o$,

$$-z''_o = E_o(s)(E_{o\bullet s}(c) - E_o(c) + E_{o\bullet s}(scs^{-1}) - E_o(scs^{-1})) = 0.$$

We proved that $E_o(s)$ commutes with $E_o(c) + E_o(scs^{-1})$ when the q_s are 1.

We choose indeterminates $\mathfrak{q}_s$ for $s \in S^{\text{aff}}/\sim$ of square $\mathfrak{q}_s^2 = \mathbf{q}_s$. By Lemma 5.43 in [7], we deduce that $\mathfrak{q}_s^{-1}E_o(s)$ commutes with $\mathfrak{q}_c^{-1}E_o(c) + \mathfrak{q}_{scs^{-1}}^{-1}E_o(scs^{-1})$ in the algebra $\mathcal{H}_{R[(\mathfrak{q}_s,\mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)]$. We have $\mathfrak{q}_c = \mathfrak{q}_{scs^{-1}}$ (cp. [7, Prop. 5.13]). Hence $E_o(s)$ commutes with $E_o(c) + E_o(scs^{-1})$ in the generic subalgebra $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)]$. We specialize $\mathbf{q}_s \mapsto q_s$ hence $E_o(s)$ commutes with $E_o(c) + E_o(scs^{-1})$ in $\mathcal{H}_R(q_s, c_s)$.

d) We deduce from b) and c) that both $E_o(c)$ and $E_o(scs^{-1})$ do not commutes with $E_o(s)$ when $\beta \circ \nu(c) \neq 0$.

□

Proposition 2.7. *For any conjugacy class C of $W(1)$ contained in $\Lambda(1)$, the element*

$$E(C) = E_o(C) = \sum_{\lambda \in C} E_o(\lambda),$$

does not depend on the choice of the orientation o.

The R-algebra $\mathcal{A}_o(1)^{W(1)}$ does not depend on the choice of o, is R-free of basis $(E(C))$ for the conjugacy classes C of $W(1)$ contained in $\Lambda(1)$, and is contained in the center $\mathcal{Z}_R(q_s, c_s)$.

Proof. Let $s \in S_o(1)$. By Proposition 2.6 and its proof, $z_o = E_o(C)$ commutes with $E_o(s)$ and $z_o'' = E_o(s)(E_{o\bullet s}(C) - E_o(C)) = 0$. When $q_s = 1$ for all s, $E_o(s)$ is invertible and $z_o'' = 0$ implies $E_{o\bullet s}(C) = E_o(C)$. As $\mathfrak{q}_{\mathbf{C}} = \mathfrak{q}_{\mathbf{w}}$ is constant on C we have $E_{o\bullet s}(C) = E_o(C)$ in the generic algebra $\mathcal{H}_{R[(\mathfrak{q}_s, \mathfrak{q}_s^{-1})]}(\mathbf{q}_s, c_s)]$, hence in the generic subalgebra $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)]$. This is valid for any $s \in S_o(1)$, and the set $S_o(1)$ generates the group $W_0(1)$. The spherical orientations are $o \bullet w$ for $w \in W_0(1)$. Hence $E_{o\bullet w}(C) = E_o(C)$ for any $w \in W_0(1)$. We denote this element by $E(C)$.

We show that $E(C)$ belongs to the center of $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)$. Indeed, for the orientation o associated to the antidominant Weyl chamber $-\mathfrak{D}^+$, we have: $S_o = S^{\text{aff}} \cap W_0$, $E_o(s) = T_s$ for $s \in S_o(1)$, and $E(C)$ commutes with $\mathcal{A}_o(1)$ and T_s for $s \in S_o(1)$. By the braid relations (1), $E(C)$ commutes with T_w for $w \in W_0(1)$. We have $W(1) = \Lambda(1)W_0(1)$ with $Z_k = \Lambda(1) \cap W_0(1)$. The product formula (3) implies that $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)]$ is generated by $\mathcal{A}_o(1)$ and T_w for $w \in W_0(1)$.

The proposition is proved in $\mathcal{H}_{R[(\mathfrak{q}_s)]}(\mathbf{q}_s, c_s)$. By specialization $\mathbf{q}_s \mapsto q_s$ the proposition is true in $\mathcal{H}_R(q_s, c_s)$. ☐

The following proposition implies that the intersection of $\mathcal{A}_o(1)$ with $\mathcal{Z}_R(q_s, c_s)$ is contained in $\mathcal{A}_o(1)^{W(1)}$. We recall the reduced root system $\Sigma = \{e_\alpha \alpha \mid \alpha \in \Phi\}$ attached to W^{aff} [7].

Proposition 2.8. *Let $s \in S_o(1)$, let $\beta \in \Sigma$ be the root associated to (o, s) and let z_o be a central element of $\mathcal{A}_o(1)$ as in (7). Then $E_o(s)$ commutes with z_o if and only if $z_o(c) = z_o(scs^{-1})$ for all c such that $\beta \circ \nu(c) \neq 0$.*

Proof. We suppose that $z_0 \neq 0$, and thanks to Proposition 2.6, that $z_o(c) \neq 0$ implies $z_o(scs^{-1}) = 0$. We will prove that z_o' or z_o'' does not vanish. This implies that z_o does not commute with $E_o(s)$ by Lemma 2.5.

If there exists c with $z_o(c) \neq 0$ and $\beta \circ \nu(c) = 0$, then $\ell(sc) = \ell(c) + 1$, $q_{s,c} = 1$ and $z_o'(sc) = z_o(c)$. Hence $z_o' \neq 0$.

We suppose, as we may, that $z_o(c) \neq 0$ implies $\beta \circ \nu(c) \neq 0$ and $z_o(scs^{-1}) = 0$. We prove this time that $z_o'' \neq 0$.

We analyze $z_o'' = -\sum_c z_o(c) \sum_{\lambda \in c} E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda))$ using the Bernstein relations (4) (cp. [7, Thm. 5.45 and Rem. 5.46]) for $\lambda \in c$:

$$E_o(s)(E_{o\bullet s}(\lambda) - E_o(\lambda)) = \epsilon_o(1, s)\epsilon_\beta(c) \sum_{k=0}^{n(c)-1} q(k, \lambda)c(k, \lambda)E_o(\mu(k, \lambda))$$

where $\beta \circ \nu(c) = \epsilon_\beta(c)n(c)$ with $\epsilon_\beta(c) \in \{\pm 1\}$ and $n(c) > 0$, and $\beta \circ \nu(\mu(k, \lambda)) = 2k - n(c)$.

Let $m = \max\{n(c) \mid z_o(c) \neq 0\}$. To prove the proposition it suffices to show that the component $z''_{o,m}$ of z''_o in $\oplus_{|\beta\circ\nu(\lambda)|=m} RE_o(\lambda)$ is not 0.

Only the terms with $k = 0$ in the expansion of $E_o(s)(E_{o\bullet s}(\lambda) - E_{o\bullet s}(\lambda))$ for $\lambda \in c$ with $n(c) = m$ and $z_o(c) \neq 0$ contribute to $z''_{o,m}$. Up to multiplication by a sign, $z''_{o,m}$ is equal to

$$\sum_{c,n(c)=m} z_o(c)\epsilon_\beta(c) \sum_{\lambda\in c} q(0,\lambda)c(0,\lambda)E_o(\mu(0,\lambda)).$$

By [7, Rem. 5.46], $q(0,\lambda)c(0,\lambda)E_o(\mu(0,\lambda))$ is equal to

$$c_s E_o(\lambda) \text{ if } \epsilon_\beta(c) = -1, \quad E_o(s\lambda s^{-1})c_s \text{ if } \epsilon_\beta(c) = 1.$$

We deduce

$$\pm z''_{o,m} = \sum_{c,\beta\circ\nu(c)=-m} z_o(c)\epsilon_\beta(c)c_s E_o(c) + \sum_{c,\beta\circ\nu(c)=m} z_o(c)\epsilon_\beta(c)E_o(scs^{-1})c_s.$$

As $\beta\circ\nu(c) = -\beta\circ\nu(scs^{-1})$, and $z_o(c) \neq 0$ implies $z_o(scs^{-1}) = 0$, we obtain $z''_{o,m} \neq 0$. □

Corollary 2.9. *The intersection of $\mathcal{A}_o(1)$ with $\mathcal{Z}_R(q_s, c_s)$ is $\mathcal{A}_o(1)^{W(1)}$.*

Proof. Proposition 2.8 and 2.7. □

The last step of the proof of the equality $\mathcal{Z}_R(q_s, c_s) = \mathcal{A}_o(1)^{W(1)}$ is given by:

Proposition 2.10. *When o is the orientation associated to the anti-dominant Weyl chamber $-\mathfrak{D}^+$, an element of $\mathcal{H}_R(q_s, c_s)$ which commutes with each element of $\mathcal{A}_o(1)$ is contained in $\mathcal{A}_o(1)$.*

Proof. We pick a nonzero element in $\mathcal{H}_R(q_s, c_s) - \mathcal{A}_o(1)$,

$$z = \sum_{x\in W(1)} z(x)E_o(x),$$

which commute with $E_o(\lambda)$ for all $\lambda \in \Lambda(1)$. We pick an element $w \in W(1)$ of maximal length in the nonempty set of $x \in W(1) - \Lambda(1)$ with $z(x) \neq 0$. We will show that there exists $\lambda \in \Lambda(1)$ such that

$$\ell(w) < \ell(\lambda w\lambda^{-1}),\ \ell(\lambda w) = \ell(\lambda) + \ell(w). \tag{8}$$

We have $q_{\lambda,w} = 1$ (cp. [7, Lemma 4.13]) and $E_o(\lambda w) = E_o(\lambda)E_o(w)$. We have $E_o(\lambda)z = zE_o(\lambda)$ and the coefficient of $E_o(\lambda)z$ on $E_o(\lambda w)$ is $z(w) \neq 0$.

We will show that the coefficient of $E_o(\lambda w)$ in $zE_o(\lambda)$ is $z(\lambda w\lambda^{-1})q_{\lambda w\lambda^{-1},\lambda}$. We have $z(\lambda w\lambda^{-1}) = 0$ because $\lambda w\lambda^{-1}$ does not belong to $\Lambda(1)$ and $\ell(w) < \ell(\lambda w\lambda^{-1})$. Our hypothesis was absurd and the proposition is proved if we admit the existence of λ and the value of the coefficient of $E_o(\lambda w)$ in $zE_o(\lambda)$, given by the next two lemmas. □

Lemma 2.11. *Let $w \in W$ not in Λ. There exists $\lambda \in \Lambda$ such that*

$$\ell(w^{-1}\lambda w\lambda^{-1}) > 2\ell(w),\ \ell(\lambda w) = \ell(\lambda) + \ell(w) > \ell(w). \tag{9}$$

The lemma is stronger than the claim in the proof of Proposition 2.10 because (9) implies $2\ell(w) < \ell(w^{-1}\lambda w\lambda^{-1}) \leq \ell(w) + \ell(\lambda w\lambda^{-1})$ hence $\ell(w) < \ell(\lambda w\lambda^{-1})$.

Proof. Replacing λ by $w^{-1}\lambda w$ which has the same length, (9) is replaced by

$$\ell(\lambda w\lambda^{-1}w^{-1}) > 2\ell(w),\ \ell(w\lambda) = \ell(\lambda) + \ell(w) > \ell(w). \tag{10}$$

We show first that there exists $\lambda \in \Lambda$ satisfying the weaker property

$$\ell(\lambda w\lambda^{-1}w^{-1}) > 0,\ \ell(w\lambda) = \ell(\lambda) + \ell(w) > \ell(w). \tag{11}$$

We write $w = w_0\lambda_0 \in W_0 \ltimes \Lambda$ with $w_0 \neq 1$. The length formula [7, Cor. 5.10] shows that $\ell(w\lambda) = \ell(w) + \ell(\lambda)$ is equivalent to

1) $(\gamma \circ \nu)(\lambda)$, $(\gamma \circ \nu)(\lambda_0)$ have the same sign when $\gamma \in \Sigma^+$, $w_0(\gamma) > 0$,
2) $(\gamma \circ \nu)(\lambda)$, $1 + (\gamma \circ \nu)(\lambda_0)$ have the same sign when $\gamma \in \Sigma^+$, $w_0(\gamma) < 0$.

We have:
$\ell(\lambda w\lambda^{-1}w^{-1}) = \ell(\lambda w_0\lambda^{-1}w_0^{-1})$ because Λ is commutative,
$\ell(\lambda w_0\lambda^{-1}w_0^{-1}) > 0 \Leftrightarrow (\gamma \circ \nu)(\lambda w_0\lambda^{-1}w_0^{-1}) \neq 0$ for some $\gamma \in \Sigma$, by the length formula [7, Cor. 5.10],
$\Leftrightarrow \nu(\lambda w_0\lambda^{-1}w_0^{-1}) \neq 0 \Leftrightarrow \nu(\lambda) \neq w_0(\nu(\lambda))$.

The existence of $\lambda \in \Lambda$ satisfying (11) is equivalent to the existence of $\lambda \in \Lambda$ satisfying

$$0 \neq \nu(\lambda) \neq w_0(\nu(\lambda)),\ \ell(\lambda w) = \ell(\lambda) + \ell(w). \tag{12}$$

It is obvious that there are infinitely many $\lambda \in \Lambda$ satisfying these conditions.

In fact, the existence of $\lambda \in \Lambda$ satisfying (11) is not a weaker property than the existence of $\lambda \in \Lambda$ satisfying (10). Indeed, let $\lambda \in \Lambda$ satisfying (11), we show that, for a large odd integer n, λ^n satisfies (10).

We have $(\lambda w\lambda^{-1}w^{-1})^n = \lambda^n w\lambda^{-n}w^{-1}$ because Λ is commutative. If n is large and $\ell(\lambda w\lambda w^{-1}) > 0$ then $\ell(\lambda^n w\lambda^{-n}w^{-1}) = n\ell(\lambda w\lambda w^{-1}) > 2\ell(w)$. In particular, $\ell(w\lambda) = \ell(\lambda) + \ell(w)$ implies $\ell(w\lambda^n) = \ell(\lambda^n) + \ell(w)$ for odd $n > 0$. Obviously $\ell(\lambda) + \ell(w) > \ell(w)$ implies $\ell(\lambda^n) + \ell(w) > \ell(w)$. ☐

Lemma 2.12. *The coefficient of $E_o(\lambda w)$ in $zE_o(\lambda)$ is $z(\lambda w\lambda^{-1})q_{\lambda w\lambda^{-1},\lambda}$, for z, λ, w in the proof of Proposition 2.10.*

Proof. We write $E_o(x)E_o(\lambda)$ for $x \in W(1)$, as

$$E_o(x)E_{o\bullet x}(\lambda)+E_o(x)(E_o(\lambda)-E_{o\bullet x}(\lambda)) = q_{x,\lambda}E_o(x\lambda)+E_o(x)(E_o(\lambda)-E_{o\bullet x}(\lambda)),$$

and $zE_o(\lambda) = z_1 + z_2$ where

$$z_1 = \sum_{x\in W(1)} z(x)q_{x,\lambda}E_o(x\lambda),\ z_2 = \sum_{x\in W(1)} z(x)E_o(x)(E_o(\lambda) - E_{o\bullet x}(\lambda)).$$

The coefficient of z_1 on $E_o(\lambda w)$ is $z(\lambda w\lambda^{-1})q_{\lambda w\lambda^{-1},\lambda}$ as $x\lambda = \lambda w \Leftrightarrow x = \lambda w\lambda^{-1}$. The coefficient of z_2 on $E_o(\lambda w)$ is 0 because

a) if $x \in \Lambda(1)$ we have $o \bullet x = o$.

b) if $x \in W(1) - \Lambda(1)$, we have (cp. [7, Thm. 4.5, Cor. 5.26])

$$\begin{aligned} E_o(x) &\in T_x + \sum_{y<x} RT_y, \\ E_o(\lambda) - E_{o\bullet x}(\lambda) &\in \sum_{u<\lambda} RT_u, \\ T_y T_u &\in \sum_{v \leq yu} RT_v, \\ E_o(x)(E_o(\lambda) - E_{o\bullet x}(\lambda)) &\in \sum_{v \leq yu, u<\lambda, y<x} RT_v, \end{aligned}$$

and from $v \leq yu, u < \lambda, y < x$, we have $\ell(v) < \ell(\lambda) + \ell(x) \leq \ell(w) + \ell(\lambda) = \ell(\lambda w)$. □

This ends the proof of the first part of Theorem 1.3 describing the center $\mathfrak{Z}_R(q_s, c_s)$. The second part generalizes the following finiteness properties:

Proposition 2.13. *The group algebra $R[W]$ is a finitely generated module over its center. If the ring R is noetherian, the center is a finitely generated algebra, and $R[W]$ is a noetherian algebra.*

The group algebra $R[W(1)]$ satisfies the same properties.

Proof. i) $T/T_0(1)$ is a free, finitely generated commutative group which is normalized by $W(1)$. The action of $W(1)$ by conjugation on $T/T_0(1)$ factorizes by W_0. By a general theorem ([1, AC V.1.9 Thm. 2 p. 29]) for any finitely generated commutative R-algebra with an action of a group with finite orbits, $R[T/T_0(1)]$ is a finitely generated $R[T/T_0(1)]^{W_0}$-module; moreover, if R is noetherian, $R[T/T_0(1)]^{W_0}$ is a finitely generated R-algebra.

ii) The center $R[\Lambda(1)]^{W(1)}$ of $R[W(1)]$ contains $R[T/T_0(1)]^{W_0}$, $R[W(1)]$ is a finitely generated $R[\Lambda(1)]$-module and $R[\Lambda(1)]$ is a finitely generated $R[T/T_0(1)]$-module, because the index of $T/T_0(1)$ in $\Lambda(1)$ and the index of $\Lambda(1)$ in $W(1)$ are finite.

iii) $R[W(1)]$ is finitely generated over $R[\Lambda(1)]^{W(1)}$ and over $R[T/T_0(1)]^{W_0}$. If the ring R is noetherian, then $R[W(1)]$ is noetherian, $R[\Lambda(1)]^{W(1)}$ is a finitely generated $R[T/T_0(1)]^{W_0}$-module hence a finitely generated R-algebra. □

The proof of these finiteness properties for the R-algebra $\mathcal{H}_R(q_s, c_s)$ (the second part of Theorem 1.3) follows the same pattern. We pick a spherical orientation o. The R-module $\mathcal{A}_o(T/T_0(1))$ generated by $E_o(\lambda)$ for $\lambda \in T/T_0(1)$ is a commutative algebra with an action of $W(1)$ factorizing through W_0. We claim:

a) The R-algebra $\mathcal{A}_o(T/T_0(1))$ is finitely generated.

Therefore, $\mathcal{A}_o(T/T_0(1))$ is a finitely generated $\mathcal{A}_o(T/T_0(1))^{W_0}$-module; moreover, if R is noetherian, $\mathcal{A}_o(T/T_0(1))^{W_0}$ is a finitely generated R-algebra (cp. [1, AC V.1.9 Thm. 2 p. 29]). The center $\mathcal{A}_o^{W(1)} = \mathcal{Z}_R(q_s, c_s)$ contains $\mathcal{A}_o(T/T_0(1))^{W_0}$. Theorem 1.3 follows from:

b) The left $\mathcal{A}_o(1)$-module $\mathcal{H}_R(q_s, c_s)$ is finitely generated,
c) The left $\mathcal{A}_o(T/T_0(1))$-module $\mathcal{A}_o(1)$ is finitely generated.

It remains to prove the claims a), b) and c). When the homomorphism $\nu : \Lambda(1) \to V$ defined by (5) sends $\lambda, \lambda' \in \Lambda(1)$ to the same closed Weyl chamber, we have $\ell(\lambda) + \ell(\lambda') = \ell(\lambda\lambda')$ and the product formula (3) is simply

$$E_o(\lambda)E_o(\lambda') = E_o(\lambda\lambda').$$

We denote by $\Lambda(1)_{\mathfrak{D}}$ the inverse image by ν of the closure of a Weyl chamber $\mathfrak{D}$ of V. The maximal subgroup of the monoid $\Lambda(1)_{\mathfrak{D}}$ is the kernel $\Omega(1) \cap \Lambda(1)$ of ν.

We denote $L = T/T_0(1)$ and $L_{\mathfrak{D}} = L \cap \Lambda(1)_{\mathfrak{D}}$. The R-module of basis $E_o(\lambda)$ for $\lambda \in \Lambda(1)_{\mathfrak{D}}$ is a subalgebra $\mathcal{A}_{o,\mathfrak{D}}$ and the R-algebra $\mathcal{A}_o(L_{\mathfrak{D}})$ of basis $E_o(\lambda)$ for $\lambda \in L_{\mathfrak{D}}$ satisfy

$$\mathcal{A}_{o,\mathfrak{D}} \simeq R[\Lambda(1)_{\mathfrak{D}}], \ \mathcal{A}_o(L_{\mathfrak{D}}) \simeq R[L_{\mathfrak{D}}], \tag{13}$$

$$\mathcal{A}_o(1) = \cup_{\mathfrak{D}} \mathcal{A}_{o,\mathfrak{D}}, \ \mathcal{A}_o(L) = \cup_{\mathfrak{D}} \mathcal{A}_o(L_{\mathfrak{D}}), \tag{14}$$

where $\mathfrak{D}$ runs over all the Weyl chambers of V.

Lemma 2.14. *$L_{\mathfrak{D}}$ is a finitely generated monoid of finite index in $\Lambda(1)_{\mathfrak{D}}$.*

Proof. We have exact sequences

$$1 \to L \cap \Omega(1) \to L_{\mathfrak{D}} \to \nu(L) \cap \mathfrak{D} \to 1,$$
$$1 \to \Lambda(1) \cap \Omega(1) \to \Lambda(1)_{\mathfrak{D}} \to \nu(\Lambda(1)) \cap \mathfrak{D} \to 1.$$

The monoid $\nu(L) \cap \mathfrak{D}$ is finitely generated of finite index in the monoid $\nu(\Lambda(1)) \cap \mathfrak{D}$. The commutative group $L \cap \Omega(1)$ is finitely generated because any subgroup of L is free and finitely generated. The index of $L \cap \Omega(1)$ in $\Lambda(1) \cap \Omega(1)$ is finite because $\Lambda(1)/L$ is finite. □

We deduce the claims a) and c):

Lemma 2.15. *The R-algebra $\mathcal{A}_o(T/T_0(1))$ is finitely generated and $\mathcal{A}_o(1)$ is a finitely generated left and right $\mathcal{A}_o(T/T_0(1))$-module. In particular, the R-algebra $\mathcal{A}_o(1)$ is finitely generated.*

Proof. By the Lemma 2.14, $\mathcal{A}_o(L_{\mathfrak{D}})$ is a finitely generated R-algebra and $\mathcal{A}_{o,\mathfrak{D}}$ is a finitely generated $\mathcal{A}_o(L_{\mathfrak{D}})$-module. □

Lemma 2.16. *Λ contains a finite set X such that, for any $(\lambda, w) \in \Lambda \times W_0$, there exists $\mu \in X$ such that*

$$\ell(\lambda w) = \ell(\lambda\mu^{-1}) + \ell(\mu w).$$

Proof. It suffices to prove the lemma for an arbitrary fixed element of the finite group W_0. Let $w \in W_0$ and let $\lambda, \mu \in \Lambda$. By the length formula [7, Cor. 5.10] $\ell(\lambda w)$ is equal to the sum of $|\alpha \circ \nu(\lambda)|$ over the positive roots α such that $w^{-1}(\alpha)$ is positive, plus the sum of $|\beta \circ \nu(\lambda) - 1|$ over the positive roots β such that $w^{-1}(\beta)$ is negative. We deduce that $\ell(\lambda w) = \ell(\lambda\mu^{-1}) + \ell(\mu w)$ if and only if

$\alpha \circ \nu(\mu) > 0$ implies $\alpha \circ \nu(\lambda) \geq \alpha \circ \nu(\mu)$,
$\alpha \circ \nu(\mu) < 0$ implies $\alpha \circ \nu(\lambda) \leq \alpha \circ \nu(\mu)$,
$\beta \circ \nu(\mu) > 1$ implies $\beta \circ \nu(\lambda) \geq \beta \circ \nu(\mu)$,
$\beta \circ \nu(\mu) < 1$ implies $\beta \circ \nu(\lambda) \leq \beta \circ \nu(\mu)$.
for all α, β as above. It is clear that, for any $\lambda \in \Lambda$, there exists a positive integer N such that $\ell(\lambda w) = \ell(\lambda\mu^{-1}) + \ell(\mu w)$ for some μ with $|\gamma \circ \nu(\mu)| \leq N$, for all positive roots γ. There are finitely many choices of $x(\mu) = (\gamma \circ \nu(\mu))_{\gamma>0}$ with $|\gamma \circ \nu(\mu)| \leq N$ for all positive roots γ, and $\mu \in \Lambda$. We deduce that there are finitely many elements $\mu_1, \ldots, \mu_r$ such that, for any $\lambda \in \Lambda$, there exists μ_i such that $\ell(\lambda w) = \ell(\lambda\mu_i^{-1}) + \ell(\mu_i w)$. □

We choose a finite set $X \subset \Lambda$ as in Lemma 2.16, and we denote by $X(1)$ and $XW_0(1)$ the inverse image in $W(1)$ of the finite sets X and $\{\mu w \mid \mu \in X,\ w \in W_0\}$.

Lemma 2.17. *The left $\mathcal{A}_o(1)$-module $\mathcal{H}_R(q_s, c_s)$ is generated by the elements $E_o(w)$ for w in the finite set $XW_0(1)$, and the R-algebra $\mathcal{H}_R(q_s, c_s)$ is finitely generated.*

Proof. $W(1) = \Lambda(1)W_0(1)$ and for $w = \lambda w_0 \in W(1)$ with $\lambda \in \Lambda(1), w_0 \in W_0(1)$, there exists $\mu \in X(1)$ such that $\ell(\lambda w_0) = \ell(\lambda\mu^{-1}) + \ell(\mu w_0)$. Hence $q_{\lambda\mu^{-1},\mu w_0} = 1$ and

$$E_o(w) = E_o(\lambda w_0) = E_o(\lambda\mu^{-1})E_o(\mu w_0).$$

We deduce the first part of the lemma. As the R-algebra $\mathcal{A}_o(1)$ is finitely generated by Lemma 2.15, the same is true for the R-algebra $\mathcal{H}_R(q_s, c_s)$. □

We deduce the claim b). The proof of Theorem 1.3 is complete.

3. Remarks

3.1. When the order of the finite commutative group Z_k is invertible in R, and when R contains a root of unity of order the least common multiple of the orders of the elements of Z_k (we say that R splits Z_k), the idempotents of $R[Z_k]$

$$e_\chi = |Z_k|^{-1} \sum_{t \in Z_k} \chi^{-1}(t)t \tag{15}$$

for all R-characters χ of Z_k, are orthogonal of sum 1. When X is a W_0-orbit of characters of Z_k, the idempotent $e_X = \sum_{\chi \in X} e_\chi$ is central in $\mathcal{H}_R(q_s, c_s)$.

Lemma 3.2. *When R splits Z_k, the R-algebra $\mathcal{H}_R(q_s, c_s)$ is the direct sum of the subalgebras $e_X\mathcal{H}_R(q_s, c_s)$ when X runs over the W_0-orbits of characters of Z_k. As a R-module,*

$$e_X\mathcal{H}_R(q_s, c_s) = \oplus_{\chi \in X}\ e_\chi\mathcal{H}_R(q_s, c_s), \quad e_\chi\mathcal{H}_R(q_s, c_s) \simeq \chi \otimes_{R[Z_k]} \mathcal{H}_R(q_s, c_s).$$

3.3. When $q_s = 1$ for $s \in S^{\mathrm{aff}}$,

$$E_o(\lambda)E_o(w) = E_o(\lambda w) \text{ for } \lambda \in \Lambda(1) \text{ and } w \in W(1).$$

The linear map $\lambda \mapsto E_o(\lambda)$ from $R[T/T_o(1)] \subset R[\Lambda(1)]$ to $\mathcal{A}_o(T/T_o(1)) \subset \mathcal{A}_o(1)$ are algebra isomorphisms and

$$\mathcal{H}_R(1, c_s) = \sum_{w \in W_0(1)} \mathcal{A}_o(1)E_o(w).$$

3.4. Set $\mathcal{J}_o(1) = \sum_{s \in S^{\mathrm{aff}}/\sim} q_s \mathcal{A}_o(1)$. In the generic algebra $\mathcal{H}_{R[(\mathbf{q}_s)]}(\mathbf{q}_s, c_s)$, for any orientation o, we have the product formula (3)

$$E_o(\lambda)E_o(\lambda') = \mathbf{q}_{\lambda,\lambda'} E_o(\lambda\lambda'),$$

and $\mathbf{q}_{\lambda,\lambda'} = 1$ if and only if $\lambda, \lambda' \in \Lambda(1)_{\mathfrak{D}}$ for some Weyl chamber $\mathfrak{D}$. By specialization of the indeterminates $\mathbf{q}_s$ to q_s, we obtain in $\mathcal{H}_R(q_s, c_s)$,

$$E_o(\lambda)E_o(\lambda') = E_o(\lambda\lambda') \text{ if } \lambda, \lambda' \in \Lambda(1)_{\mathfrak{D}} \text{ for some } \mathfrak{D},$$
$$E_o(\lambda)E_o(\lambda') \in \mathcal{J}_o(1) \text{ otherwise}.$$

We denote by $W_0(\mu)$ the W_0-orbit of μ in $X_*(T)$ and by $X_{*,\mathfrak{D}}(T)$ the monoid of $\mu \in X_*(T)$ such that $\nu(\mu(p_F))$ belongs to the closure of $\mathfrak{D}$. We denote by $C(\mu)$ the conjugacy class in $W(1)$ of the image λ of $\mu(p_F)$ in $\Lambda(1)$, and we set $E_o(\mu) = E_o(\lambda)$. We have

$$E(C(\mu)) = \sum_{\mu' \in W_0(\mu)} E_o(\mu').$$

Proposition 3.5. *Let $\mu_1, \mu_2 \in X_{*,\mathfrak{D}}(T)$. In $\mathcal{H}_R(q_s, c_s)$ we have*

$$E(C(\mu_1))E(C(\mu_2)) \in E(C(\mu_1 + \mu_2)) + \mathcal{J}_o(1).$$

Proof. We fix the Weyl chamber $\mathfrak{D}$. For another Weyl chamber $\mathfrak{D}'$ we denote by $w_{\mathfrak{D}'}$ the unique element of W_0 such that $w_{\mathfrak{D}'}(\mathfrak{D}) = \mathfrak{D}'$. For $\mu \in X_{*,\mathfrak{D}}(T)$, $\mu_{\mathfrak{D}'} = w_{\mathfrak{D}'}(\mu)$ belongs to $X_{*,\mathfrak{D}'}(T)$. We have $\mu_{1,\mathfrak{D}'} + \mu_{2,\mathfrak{D}'} = (\mu_1 + \mu_2)_{\mathfrak{D}'}$ and

$$E(C(\mu)) = \sum_{\mu_{\mathfrak{D}'}} E_o(\mu_{\mathfrak{D}'})$$

(the sum is not over the Weyl chambers $\mathfrak{D}'$, but over the distinct elements $\mu_{\mathfrak{D}'}$).

$E(C(\mu_1))E(C(\mu_2))$ is equal modulo $\mathcal{J}_o(1)$ to the sum of $E_o(\mu_1' + \mu_2')$ over the pairs $(\mu_1', \mu_2') \in W_0(\mu_1) \times W_0(\mu_2)$ which belong to $X_{*,\mathfrak{D}'}(T)$ for the same Weyl chamber $\mathfrak{D}'$.

□

3.6. We recall the involutive R automorphism of $\mathcal{H}_R(q_s, c_s)$ defined by

$$\iota(T_w) = (-1)^{\ell(w)} T_w^* \text{ for } w \in W(1),$$

where $T_w^* = (T_{s_1} - c_{s_1}) \dots (T_{s_r} - c_{s_r}) T_u$ if $w = s_1 \dots s_r u$ is a reduced decomposition of w, $s_i \in S^{\mathrm{aff}}(1)$, $u \in \Omega(1)$, $\ell(w) = r$ (cp. [7, Prop. 4.23]). Let o be a spherical orientation attached to a Weyl chamber $\mathfrak{D}$, and $\overline{o}$ the spherical orientation attached to the opposite Weyl chamber $-\mathfrak{D}$. We recall (cp. [7, Lemma 5.31])

$$\iota(E_o(w)) = (-1)^{\ell(w)} E_{\overline{o}}(w) \text{ for } w \in W(1).$$

Proposition 3.7. *If C is a finite conjugacy class of $W(1)$, we have*

$$\iota(E(C)) = (-1)^{\ell(C)} E(C).$$

Proof. The length is constant on C hence

$$\iota(E(C)) = \sum_{\mu \in C} \iota(E_o(\mu)) = (-1)^{\ell(C)} \sum_{\mu \in C} E_{\overline{o}}(\mu) = (-1)^{\ell(C)} E(C),$$

as $E(C) = E_o(C)$ does not depend on the choice of the orientation o. □

Remark 3.8. When $\mu \in X_*(T)$, $\ell(C(\mu))$ is an even number (cp. [6]), hence $E(C(\mu))$ is fixed by ι.

ADDED IN PROOF ON AUGUST 14, 2014

1) The Bernstein relations (4), therefore also Lemma 2.5, Proposition 2.6, and Proposition 2.8, are valid only for $s \in (S \cap S_o)(1)$ where $S = W_0 \cap S^{\mathrm{aff}}$ and not for $s \in S_o(1)$ [7, Thm. 5.45], hence the arguments for $E_o(C)$ being independent of the orientation o in Proposition 2.7 are not valid.

Proof that $E_o(C)$ is independent of o. We reduce to $q_s = 1$ for $s \in S^{\mathrm{aff}}$. Then, this results from the formula

$$E_o(w)^{-1} E_o(\lambda) E_o(w) = E_{o \bullet w}(w^{-1} \lambda w) \quad (w \in W_0(1), \lambda \in \Lambda(1)), \tag{16}$$

for the anti-dominant orientation o, because $E_o(C)$ is central and $W_0(1)$ acts transitively on the (spherical) orientations.

2) The proof of Lemma 2.16 must be replaced by:

Proof of Lemma 2.16. Let $L = \{\vec{\ell}(w) : \gamma \mapsto \ell_\gamma(w) : \Sigma^+ \to \mathbb{Z} \mid w \in W\}$ where for $(\lambda, w_0) \in \Lambda \times W_0$, $\ell_\gamma(\lambda w_0)$ is equal to ([7, Cor. 5.9])

$$\alpha \circ \nu(\lambda) \text{ if } \gamma \in w_0(\Sigma^+), \quad \gamma \circ \nu(\lambda) - 1 \text{ if } \gamma \in w_0(\Sigma^-).$$

For $w, w' \in W$, we write $\vec{\ell}(w) \leq \vec{\ell}(w')$ if $|\ell_\gamma(w)| = |\ell_\gamma(w')| + |\ell_\gamma(w) - \ell_\gamma(w')|$ for all $\gamma \in \Sigma^+$. We say that $\vec{\ell}(w)$ is minimal if $\Lambda w = \Lambda w'$ and $\vec{\ell}(w') \leq \vec{\ell}(w)$ implies $\vec{\ell}(w') = \vec{\ell}(w)$. As in [4, Lem. 4.2] one shows that the set L_{min} of minimal elements of L is finite. The finite subset $X = \cup_{w_0 \in W_0} X(w_0)$ of Λ where $L_{min} = \cup_{w_0 \in W_0} X(w_0) w_0$ satisfies Lemma 2.16 because

$$\Lambda w = \Lambda w' \text{ and } \vec{\ell}(w') \leq \vec{\ell}(w) \Rightarrow \ell(w) = \ell(w') + \ell(w w'^{-1}). \tag{17}$$

In the left hand side of (17), $ww'^{-1} \in \Lambda$ implies $\ell_\gamma(ww'^{-1}) = \ell_\gamma(w) - \ell_\gamma(w')$ for all $\gamma \in \Sigma^+$; with $\vec{\ell}(w') \leq \vec{\ell}(w)$ we have $|\ell_\gamma(w)| = |\ell_\gamma(w')| + |\ell_\gamma(w) - \ell_\gamma(w')| = |\ell_\gamma(w')| + |\ell_\gamma(ww'^{-1})|$. Apply the length formula [7, Prop. 5.7]

$$\ell(w) = \sum_{\gamma \in \Sigma^+} |\ell_\gamma(w)|$$

to obtain (17).

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Received January 10, 2014; accepted January 13, 2014; Errata added April 27, 2014

Marie-France Vignéras
Université de Paris 7, UMR 7586, Institut de mathématiques de Jussieu
4, place Jussieu, Case 247, 75005 Paris, France
E-mail: vigneras@math.jussieu.fr
URL: http://www.math.jussieu.fr/~vigneras/