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A combinatorial model for tame frieze patterns

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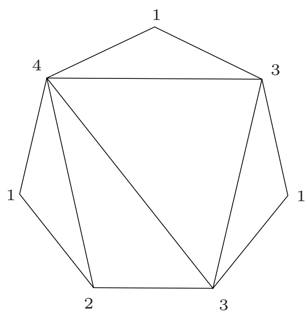
Abstract. Let R be an arbitrary subset of a commutative ring. We introduce a combinatorial model for the set of tame frieze patterns with entries in R based on a notion of irreducibility of frieze patterns. When R is a ring, then a frieze pattern is reducible if and only if it contains an entry (not on the border) which is 1 or -1. To my knowledge, this model generalizes simultaneously all previously presented models for tame frieze patterns bounded by 0s and 1s.

1. Introduction

Conway and Coxeter [1] introduced a combinatorial model for the so-called 'frieze patterns', which are bounded arrays of numbers in which any adjacent 2×2 matrix has determinant 1. Their patterns, consisting entirely of positive numbers within the frieze, are in one-to-one correspondence to triangulations of a convex polygon by non-intersecting diagonals, for example (the numbers of triangles at each vertex are the entries in the third diagonal):

0	1	1	3	2	1	1	0						
	0	1	4	3	2	3	1	0					
		0	1	1	1	2	1	1	0				
			0	1	2	5	3	4	1	0			
				0	1	3	2	3	1	1	0		
					0	1	1	2	1	2	1	0	
						0	1	3	2	5	3	1	0



This gives a connection between specializations of the variables of cluster algebras of type A to positive integers on one side (see, for example, [3]), and Catalan combinatorics on the other side.

Since then, many generalizations of these concepts were considered (see [7] for a survey). In the present note, for each set R of numbers, we present a combinatorial model which is associated to the set of tame frieze patterns with entries in this set R. Hence, we generalize the above connection to arbitrary specializations of the variables in the cluster algebras of type A.

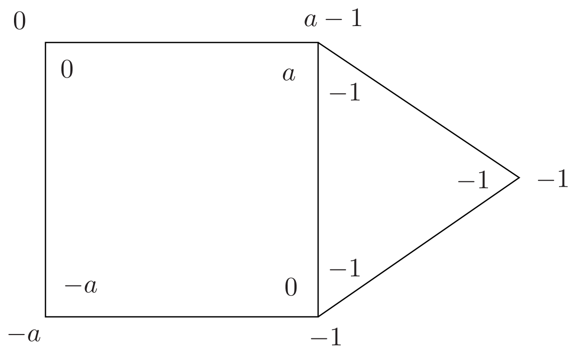


FIGURE 1. $(a,0,-a,0) \oplus (-1,-1,-1) = (a-1,0,-a,-1,-1)$.

To this end, we introduce a notion of irreducibility of frieze patterns, Definition 2.9. Every frieze pattern has a (not necessarily unique) decomposition into irreducible frieze patterns. In the combinatorial model, irreducible patterns become polygons that may be glued together to produce arbitrary frieze patterns (see, for example, Figure 1). This gluing of friezes was presented first in [4, Lem. 3.2] using a different terminology.

The problem of understanding this type of combinatorics for a given set R thus reduces to the problem of classifying the irreducible patterns. It turns out that a frieze pattern is reducible over a ring R if and only if it contains an entry (not on the border) which is 1 or -1 (see Lemma 2.13).

2. QUIDDITY CYCLES

Definition 2.1. For c in a commutative ring, let

$$\eta(c) := \begin{pmatrix} c & -1 \\ 1 & 0 \end{pmatrix}.$$

Definition 2.2. Let R be a subset of a commutative ring and $\lambda \in \{\pm 1\}$. A λ-*quiddity cycle*[1] over R is a sequence $(c_1, \dots, c_m) \in R^m$ satisfying

$$\prod_{k=1}^{m} \eta(c_k) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda \,\mathrm{id}. \tag{1}$$

A (-1)-quiddity cycle is called a *quiddity cycle* for short.

[1]Notice that the case $R = \mathbb{N}_{>0}$ was also recently considered in [8].

Remark 2.3. We agree that $m > 0$ in Definition 2.2. In fact, $m > 1$ by Definition 2.1.

Example 2.4. Consider the commutative ring $\mathbb{C}$ and $R = \mathbb{C}$.

(1) $(0, 0)$ is the only λ-quiddity cycle of length 2.
(2) $(1, 1, 1)$ and $(-1, -1, -1)$ are the only λ-quiddity cycles of length 3.
(3) $(t, 2/t, t, 2/t)$, t a unit, and $(a, 0, -a, 0)$, a arbitrary, are the only λ-quiddity cycles of length 4.

Definition 2.5. Let D_n be the dihedral group with $2n$ elements acting on $\{1, \ldots, n\}$. If $\underline{c} = (c_1, \ldots, c_n)$ is a λ-quiddity cycle, then we write

$$\underline{c}^\sigma := (c_1, \ldots, c_n)^\sigma := (c_{\sigma(1)}, \ldots, c_{\sigma(n)})$$

for $\sigma \in D_n$.

Proposition 2.6. *Let $\underline{c} = (c_1, \ldots, c_m)$ be a λ-quiddity cycle. Then for any $\sigma \in D_n$, the cycle $\underline{c}^\sigma$ is a λ-quiddity cycle as well.*

Proof. Since the matrix $\lambda\,\mathrm{id}$ commutes with every matrix, rotating this cycle is again a λ-quiddity cycle. Reversing a λ-quiddity cycle is also a λ-quiddity cycle, see, for example, [2, Prop. 5.3 (3)]. ☐

When thinking about a λ-quiddity cycle $\underline{c}$, in general, we do not care which element in $D_n \cdot \underline{c}$ we consider. In the following lemma however, we have to be careful. We introduce a *sum* of λ-quiddity cycles which is not invariant under the action of the dihedral group. Note that this "gluing" of frieze patterns was already described in [4, Lem. 3.2] for real entries and that other variants were proposed, for instance, in [5, Section 3] for 2-friezes.

Lemma 2.7. *Let $(a_1, \ldots, a_k)$ be a λ'-quiddity cycle and $(b_1, \ldots, b_\ell)$ be a λ''-quiddity cycle. Then*

$$(a_1 + b_\ell, a_2, \ldots, a_{k-1}, a_k + b_1, b_2, \ldots, b_{\ell-1})$$

is a $(-\lambda'\lambda'')$-quiddity cycle of length $k + \ell - 2$ which we call the sum*:*

$$(a_1, \ldots, a_k) \oplus (b_1, \ldots, b_\ell) := (a_1 + b_\ell, a_2, \ldots, a_{k-1}, a_k + b_1, b_2, \ldots, b_{\ell-1}).$$

Proof. We use the identities $\eta(a+b) = -\eta(a)\eta(0)\eta(b)$ and $\eta(0)^2 = -\mathrm{id}$ (which are easy to check, see also [3, Lem. 4.1]):

$$\begin{aligned}
&\eta(a_1 + b_\ell)\eta(a_2)\cdots\eta(a_{k-1})\eta(a_k + b_1)\eta(b_2)\cdots\eta(b_{\ell-1})\\
&\qquad = \eta(b_\ell)\eta(0)\eta(a_1)\eta(a_2)\cdots\eta(a_{k-1})\eta(a_k)\eta(0)\eta(b_1)\eta(b_2)\cdots\eta(b_{\ell-1})\\
&\qquad = \lambda'\eta(b_\ell)\eta(0)\eta(0)\eta(b_1)\eta(b_2)\cdots\eta(b_{\ell-1})\\
&\qquad = -\lambda'\eta(b_\ell)\eta(b_1)\eta(b_2)\cdots\eta(b_{\ell-1}) = -\lambda'\lambda''\,\mathrm{id}.
\end{aligned}$$

☐

Example 2.8. (1) If $(a_1, \ldots, a_m)$ is a quiddity cycle, then

$$(a_1, \ldots, a_m) \oplus (0, 0) = (a_1, \ldots, a_m).$$

(2) For $a \in \mathbb{C}$, $(a, 0, -a, 0)$ and $(-1, -1, -1)$ are 1-quiddity cycles, their sum is $(a-1, 0, -a, -1, -1)$ and it is a quiddity cycle (see also Figure 1).

The following are the central notions of *reducibility* and *irreducibility* of quiddity cycles mentioned in the introduction.

Definition 2.9. Let R be a subset of a commutative ring. A λ-quiddity cycle $(c_1,\dots,c_m)\in R^m$, $m>2$, is called *reducible over* R if there exist a λ'-quiddity cycle $(a_1,\dots,a_k)\in R^k$, a λ''-quiddity cycle $(b_1,\dots,b_\ell)\in R^\ell$, and $\sigma\in D_m$ such that $\lambda=-\lambda'\lambda''$, $k,\ell>2$ and

$$\begin{aligned}(c_1,\dots,c_m)^\sigma &= (a_1+b_\ell,a_2,\dots,a_{k-1},a_k+b_1,b_2,\dots,b_{\ell-1})\\ &= (a_1,\dots,a_k)\oplus(b_1,\dots,b_\ell).\end{aligned}$$

A λ-quiddity cycle of length $m>2$ is called *irreducible over* R if it is not reducible.

Remark 2.10. There is no need to consider the cycle of length $m<3$ (which is $(0,0)$) in Definition 2.9.

Definition 2.11. Consider a λ-quiddity cycle $\underline{c}=(c_1,\dots,c_m)$ and define c_k for all $k\in\mathbb{Z}$ by repeating $\underline{c}$ periodically. For $i,j\in\mathbb{Z}$, let

$$x_{i,j}:=\left(\prod_{k=i}^{j-2}\eta(c_k)\right)_{1,1}\qquad\text{if } i\le j-2,$$

$x_{i,i+1}:=1$, and $x_{i,i}:=0$. Notice that $x_{i,i+2}=c_i$. Then we call the array $\mathcal{F}=(x_{i,j})_{i\le j\le i+m}$ the *frieze pattern* of $\underline{c}$. The *entries* of the frieze pattern of $\underline{c}$ are the numbers $x_{i,j}$ with $i+2\le j\le i+m-2$. We say that the frieze pattern of $\underline{c}$ is *reducible* (resp. *irreducible*) if $\underline{c}$ is *reducible* (resp. *irreducible*).

Remark 2.12. (a) If $\underline{c}$ is a quiddity cycle, then we obtain what we usually call the frieze pattern. In fact, in this way we exactly obtain all *tame* frieze patterns, i.e., those for which every adjacent 3×3 determinant is zero (see for example [3, Prop. 2.4]). In the original definition, Coxeter assumes that all entries in the frieze are positive in which case the frieze is automatically tame. Starting with a 1-quiddity cycle, one obtains a frieze pattern with 1s on one border and -1s on the other border, i.e., $x_{i,i+m-1}=-1$ for all i.

(b) The entries $x_{i,j}$ of a frieze pattern are specialized cluster variables of a cluster algebra of Dynkin type A (see, for example, [3, Section 5]).

(c) Notice that if $\underline{c}$ is a λ-quiddity cycle over R, then its frieze pattern may have entries which are not in R. It is an interesting question to determine the set of entries of frieze patterns of λ-quiddity cycles for a fixed set R. For example, if R is a ring then all entries in the frieze patterns are in R.

The following lemma explains the appearance of 1s and -1s in friezes. Some similar statement is contained implicitly for the case $R=\mathbb{N}_{>0}$ in [6, Cor. 1.11] for Coxeter friezes.

Lemma 2.13. *Let R be a commutative ring. A λ-quiddity cycle is reducible over R if and only if the corresponding tame frieze pattern contains an entry 1 or -1.*

Proof. Reducibility requires that the length m of the cycle is at least 4; since there are no entries in a frieze pattern with λ-quiddity cycle of length less than 4, we may assume $m \geq 4$.

Assume first the existence of an entry $\varepsilon = \pm 1$, i.e., without loss of generality (rotating the cycle if necessary), there are $i, j \in \{1, \ldots, m\}$, with $i < j-1$, $j - i < m - 1$, and $M_{1,1} = \varepsilon$ for $M = \prod_{k=i}^{j-2} \eta(c_k)$. Since $\det(M) = 1$, with $a := \varepsilon M_{2,1}$, $b := -\varepsilon M_{1,2}$, we have

$$M = \begin{pmatrix} \varepsilon & -\varepsilon b \\ \varepsilon a & -\varepsilon ab + \varepsilon \end{pmatrix} = -\varepsilon \begin{pmatrix} -1 & b \\ -a & ab-1 \end{pmatrix} = -\varepsilon \eta(a)^{-1} \eta(b)^{-1}.$$

We obtain

$$\eta(a) \Big(\prod_{k=i}^{j-2} \eta(c_k) \Big) \eta(b) = -\varepsilon \, \mathrm{id},$$

so $(a, c_i, \ldots, c_{j-2}, b)$ is a $(-\varepsilon)$-quiddity cycle. Because the cycle is a λ-quiddity cycle we also get $(\prod_{k=j-1}^{m} \eta(c_k))(\prod_{k=1}^{i-1} \eta(c_k)) = -\lambda\varepsilon\eta(b)\eta(a)$, which implies

$$\begin{aligned} &\eta(c_{j-1} - b) \Big(\prod_{k=j}^{m} \eta(c_k) \Big) \Big(\prod_{k=1}^{i-2} \eta(c_k) \Big) \eta(c_{i-1} - a) \\ &\quad = (-1)^2 \eta(-b)\eta(0) \Big(\prod_{k=j-1}^{m} \eta(c_k) \Big) \Big(\prod_{k=1}^{i-1} \eta(c_k) \Big) \eta(0)\eta(-a) \\ &\quad = -\lambda\varepsilon\eta(-b)\eta(0)\eta(b)\eta(a)\eta(0)\eta(-a) = -\lambda\varepsilon\eta(0)^2 = \lambda\varepsilon \, \mathrm{id}, \end{aligned}$$

and thus $(c_{j-1} - b, c_j, \ldots, c_m, c_1, \ldots, c_{i-2}, c_{i-1} - a)$ is a $(\lambda\varepsilon)$-quiddity cycle of length $m - j + i + 1 \geq 3$, since $j - i < m - 1$; thus, the cycle is reducible:

$$\begin{aligned} (2)\quad &(c_{i-1}, \ldots, c_m, c_1, \ldots, c_{i-2}) \\ &\quad = (a, c_i, \ldots, c_{j-2}, b) \oplus (c_{j-1} - b, c_j, \ldots, c_m, c_1, \ldots, c_{i-2}, c_{i-1} - a). \end{aligned}$$

For the converse, assume that we have a decomposition into a sum as in Equation 2 for some i, j with $i < j - 2$. Then the same argument as above shows $(\prod_{k=i}^{j-2} \eta(c_k))_{1,1} \in \{\pm 1\}$, which gives an entry ± 1 in the pattern. $\square$

3. Examples of subsets

Some classifications of irreducible λ-quiddity cycles are already known. For example, every quiddity cycle over $\mathbb{N}_{>0}$ contains a 1. Thus, any quiddity cycle over $\mathbb{N}_{>0}$ of length greater than 3 has a summand $(1, 1, 1)$ (cf. [1]), although the other summand only has positive entries if the original frieze pattern has no entry zero. In general, the following theorem holds.

Theorem 3.1. *The only irreducible λ-quiddity cycles over $\mathbb{Z}_{\geq 0}$ are* $(0, 0, 0, 0)$ *and* $(1, 1, 1)$.

Proof. Let $\underline{c} = (c_1, \ldots, c_m) \in \mathbb{Z}_{\geq 0}^m$, $m > 2$, be a λ-quiddity cycle.

If $c_i > 0$ for all i, then, by [3, Cor. 3.3], there exists a $j \in \{1, \dots, m\}$ with $c_j = 1$; without loss of generality, $j = 2$. But then $\underline{c} = (1,1,1) \oplus \underline{c}'$, where $\underline{c}' = (c_3 - 1, c_4, \dots, c_m, c_1 - 1) \in \mathbb{Z}_{\geq 0}^{m-1}$.

Otherwise, there are zeros in $\underline{c}$. If $\underline{c}$ contains two adjacent zeros, say $c_2 = c_3 = 0$, then $\underline{c} = (0,0,0,0) \oplus \underline{c}'$, where $\underline{c}' = (c_4, \dots, c_m, c_1) \in \mathbb{Z}_{\geq 0}^{m-2}$.

The last case is when there are zeros, but none of them has an adjacent zero. Notice first that since $\eta(a)\eta(0)\eta(b) = -\eta(a+b)$ for all a, b (cf. [3, Lem. 4.1]), if $(c_1, 0, c_3, \dots, c_m)$ is a λ-quiddity cycle, then $(c_1 + c_3, \dots, c_m)$ is a $(-\lambda)$-quiddity cycle. Applying this transformation to all zeros simultaneously yields a λ-quiddity cycle $\underline{c}''$ in which only the entries coming from $\underline{c}$ which were not adjacent to a zero may be ≤ 1. But, by [3, Cor. 3.3], there exists an entry ≤ 1 in $\underline{c}''$, so we find a 1 in $\underline{c}$ which has nonzero adjacent entries, hence $\underline{c}^\sigma = (1,1,1) \oplus \underline{c}'$ for some $\underline{c}' \in \mathbb{Z}_{\geq 0}^{m-1}$ and $\sigma \in D_m$ as in the first case. □

If we allow entries in the set of all integers, the situation is slightly more complicated.

Theorem 3.2 ([3, Thm. 6.2]). *The set of irreducible λ-quiddity cycles over $\mathbb{Z}$ is*

$$\big\{(1,1,1), (-1,-1,-1), (a,0,-a,0), (0,a,0,-a) \mid a \in \mathbb{Z} \setminus \{\pm 1\}\big\}.$$

Proposition 3.3. *Let $k \in \mathbb{N}_{>0}$ and $\mathrm{i} = \sqrt{-1}$. Then*

$$\underline{c} = \big(2\mathrm{i}, -\mathrm{i}+1, \underbrace{2, \dots, 2}_{2k\text{-times}}, \mathrm{i}+1, -2\mathrm{i}, \mathrm{i}-1, \underbrace{-2, \dots, -2}_{2k\text{-times}}, -\mathrm{i}-1\big)$$

is an irreducible quiddity cycle over $\mathbb{Z}[\mathrm{i}]$.

Proof. Notice first that

$$\eta(2)^\ell = \begin{pmatrix} \ell+1 & -\ell \\ \ell & 1-\ell \end{pmatrix}, \quad \eta(-2)^\ell = (-1)^\ell \begin{pmatrix} \ell+1 & \ell \\ -\ell & 1-\ell \end{pmatrix}$$

for $\ell \in \mathbb{N}_{>0}$. It is then easy to check that $\underline{c}$ is a quiddity cycle. Further, using the same identities, we can compute each type of entry in the frieze pattern. We compute $x_{1,2k+5}$ as an example:

$$\prod_{i=1}^{2k+3} \eta(c_i) = \eta(2\mathrm{i})\eta(-\mathrm{i}+1)\eta(2)^{2k}\eta(\mathrm{i}+1) = \begin{pmatrix} 2\mathrm{i}k + \mathrm{i} - 1 & -2k - 2\mathrm{i} - 1 \\ 2k+1 & 2\mathrm{i}k + \mathrm{i} - 1 \end{pmatrix},$$

and thus $x_{1,2k+5} = 2\mathrm{i}k + \mathrm{i} - 1$. It turns out that none of them is ± 1 and hence it is irreducible by Lemma 2.13. □

This immediately yields the following corollary.

Corollary 3.4. *There are infinitely many irreducible λ-quiddity cycles over the Gaussian numbers $\mathbb{Z}[\mathrm{i}]$.*

4. Combinatorial model

Let $(a_1,\dots,a_k)$ be a λ'-quiddity cycle and $(b_1,\dots,b_\ell)$ be a λ''-quiddity cycle. If we represent these two cycles as polygons, then gluing them together yields a larger polygon representing their sum, see Figure 2.

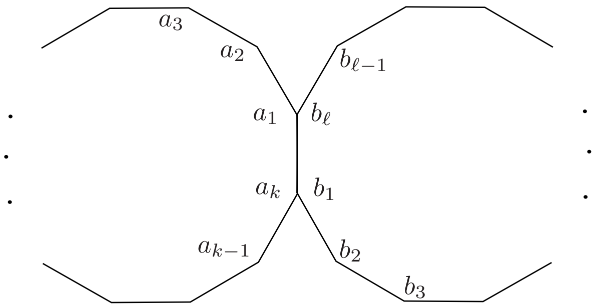


Figure 2. $(a_1,\dots,a_k)\oplus(b_1,\dots,b_\ell)$.

We see the sum

$$(a_1,\dots,a_k)\oplus(b_1,\dots,b_\ell)=\big(a_1+b_\ell,a_2,\dots,a_{k-1},a_k+b_1,b_2,\dots,b_{\ell-1}\big)$$

in the new polygon when adding the entries at the vertices which are glued together.

Hence, the decomposition of a λ-quiddity cycle into a sum of irreducible ones translates in a natural way into a polygon decomposed into building blocks which correspond to some irreducible summands.

Since the only irreducible λ-quiddity cycles for $R=\mathbb{N}_{\geq 0}$ are $(0,0,0,0)$ and $(1,1,1)$, in this special case we recover the Catalan combinatorics originally proposed by Conway and Coxeter. It is easy to prove that if the frieze pattern of a λ-quiddity cycle $\underline{c}$ for $R=\mathbb{N}_{>0}$ has only positive entries, then $\underline{c}$ is a sum of quiddity cycles $(1,1,1)$. The $(0,0,0,0)$-polygons are the parts that glue classical Conway–Coxeter friezes together; they produce zeros within the corresponding frieze pattern.

We close this note with a somewhat vague task.

Open Problem 4.1. Classify irreducible quiddity cycles for some of the most interesting sets $R\subseteq\mathbb{C}$.

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References

[1] J. H. Conway and H. S. M. Coxeter, Triangulated polygons and frieze patterns, Math. Gaz. **57** (1973), no. 401, 175–183. MR0461270

[2] M. Cuntz and I. Heckenberger, Reflection groupoids of rank two and cluster algebras of type A, J. Combin. Theory Ser. A **118** (2011), no. 4, 1350–1363. MR2755086

[3] M. Cuntz and T. Holm, Frieze patterns over integers and other subsets of the complex numbers, arXiv:1711.03724v1 [math.CO] (2017).

[4] T. Holm and P. Jørgensen, A p-angulated generalisation of conway and coxeters theorem on frieze patterns, arXiv:1709.09861v3 [math.CO] (2018); to appear in Int. Math. Res. Not. IMRN.

[5] S. Morier-Genoud, Arithmetics of 2-friezes, J. Algebraic Combin. **36** (2012), no. 4, 515–539. MR2984155

[6] S. Morier-Genoud, Frises et algèbres non-associatives, Mémoire (2014), 79 pp. `https://webusers.imj-prg.fr/~sophie.morier-genoud/HDR_main.pdf`.

[7] S. Morier-Genoud, Coxeter's frieze patterns at the crossroads of algebra, geometry and combinatorics, Bull. Lond. Math. Soc. **47** (2015), no. 6, 895–938. MR3431573

[8] V. Ovsienko, Partitions of unity in $\mathrm{SL}(2,\mathbb{Z})$, negative continued fractions, and dissections of polygons, arXiv:1710.02996v5 [math.CO] (2018).

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