Mathematik

Chern Characters for Topological Groups

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Introduction

To win war, you gotta become war.

(John J. Rambo)

One of the first questions in algebraic topology might be which data determine a homology theory $\mathcal{H}_*$ completely. One might conjecture that there exists a natural equivalence of homology theories (restricted to CW-complexes)

$$\mathrm{ch}_*\colon H_*(?;\mathcal{H}_*(\{\bullet\})) \to \mathcal{H}_*(?),$$

where the left hand side is cellular homology. This turns out to be too ambitious, but if $\mathcal{H}_*$ has rational coefficients, it is a theorem of Dold [19]. A similar statement can be made in the cohomological case.

For proper actions of discrete groups, Lück [29, 30] generalized this theorem to the equivariant case. We give a brief survey of his work in Section 5.4. As input, we have proper equivariant homology theories (1.5.8), and cellular homology is replaced by Bredon homology (1.6.2). In this case, the coefficients are modules over the subgroup category. This approach splits up into two parts. Lück constructed a natural map

$$\widetilde{\mathrm{ch}}_*^G\colon H_*(X^?/C_G?) \otimes_{\mathsf{Sub}_{\mathcal{FIN}}(G)} \mathcal{H}_*^G(G/?) \to \mathcal{H}_*(X) \tag{$*$}$$

and then identified the left hand side with Bredon homology by showing the flatness of $\mathcal{H}_*^G(G/?)$ (provided a Mackey structure (1.5.10) exists). Similar considerations can be made in the cohomological case.

The starting point of this thesis was the question if one can generalize this approach to proper smooth actions of totally disconnected groups.

We can construct a similar map to $(*)$ using the orbit category

$$\widetilde{\mathrm{ch}}_*^G\colon H_*(X^?) \otimes_{\mathsf{Or}_{\mathcal{CO}}(G)} \mathcal{H}_*^G(G/?) \to \mathcal{H}_*(X) \tag{$**$}$$

instead of the subgroup category even for topological groups. If G is unimodular and the semigroups $\mathrm{mor}_{\mathsf{Or}_{\mathcal{CO}}(G)}(G/H, G/H)$ are finite for any compact open subgroup $H \subseteq G$ and a Mackey structure exists, then the coefficient modules are flat (over the orbit category) and we get the desired theorem. In special cases, e.g., the Borel construction, we can weaken this assumption to locally finite. Note that passing to the orbit category is a severe restriction and, consequently, this approach does not apply to discrete groups in general. Indeed, even for p-adic Lie groups, there exist examples (3.3.4, 4.3.2), where the coefficient module is flat viewed as a module over the subgroup category but fails to be flat over the orbit category. It is even worse since in Example 4.3.2 a Chern character does not exist at all. Therefore, we cannot use the subgroup category but we must be satisfied with the orbit category. However, this approach does work for semisimple p-adic groups. Hence the main theorem (4.1.7) of this thesis is

Theorem. *Let R be a semisimple commutative ring with $\mathbb{Q} \subseteq R$. Let $\mathcal{H}_*^?$ be an equivariant proper smooth homology theory with values in R-modules which has a Mackey structure on coefficients. Let G be a semisimple p-adic group. Then there is an isomorphism of equivariant proper smooth homology theories*

$$\mathrm{ch}_*^G \colon \mathcal{BH}_*^G(X, A) \to \mathcal{H}_*^G(X, A)$$

which is natural in (X, A) and compatible with the boundary maps.

An analogous statement can be made for any topological group if we consider equivariant smooth coproper homology theories. As a corollary (4.1.8) we obtain

Theorem. *Let G be a semisimple p-adic group. Then we get an isomorphism*

$$\bigoplus_{k\in\mathbb{Z}} CH_{2k+n}^G(\beta G) \cong K_n(C_r^* G) \otimes \mathbb{C},$$

where CH_*^G denotes cosheaf homology and βG the affine Bruhat-Tits building.

In the cohomological case, the first part carries over directly. In the second part, we have to prove injectivity instead of flatness, which turns out to be a more restrictive condition. Basically, this means that a $\lim^1$-term comes into play, which has to vanish. Unfortunately, this seems to happen very rarely. Even in very basic examples (3.3.8) the derived limit does not vanish.

Meanwhile, Christian Voigt has constructed for l-groups a bivariant Chern character for K-theory which we will discuss in Section 5.6.

The structure of this thesis is as follows. Chapter 1 introduces the basic terms. Along the way, we discuss under which conditions the semigroup $\mathrm{mor}_{\mathrm{Or}_{\mathcal{CO}}(G)}(G/H, G/H)$ is a group, which turns out to be a necessary condition to apply the machinery in Chapter 3. Furthermore, we establish some finiteness results for $\mathrm{mor}_{\mathrm{Or}_{\mathcal{CO}}(G)}(G/H, G/K)$ which are needed in Chapter 3, too. In Chapter 2, we introduce the Borel construction and equivariant K-theory. We prove that they yield equivariant (co)homology theories and admit a Mackey structure on coefficients. In Chapter 3, we prove flatness and injectivity results for modules over the orbit category and over the subgroup category, respectively. This is denoted by the "second part" in the above discussion. In Chapter 4, we construct the map (∗∗) and obtain the main results. In Chapter 5, we compare our Chern character with several ones which were known before. It turns out that all these constructions coincide if they exist.

Conventions

Finally, we want to state some global conventions. Throughout this thesis we will work in the category of compactly generated Hausdorff spaces (see [50] and [58, I.4]). In particular, this implies that every topological group is Hausdorff and whenever we consider a homogenous space G/H, the subgroup $H \subseteq G$ is closed. Moreover, all rings are assumed to be associative and to have a unit.

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1 The Basic Setup

In this chapter, we introduce the basic definitions and develop the basic tools which will be needed in the sequel.

First, we introduce totally disconnected groups and discuss their basic properties. Although they are not needed for our Chern character, they will appear in the last chapter. Moreover, we will often deal with group actions which have compact open isotropy groups. In many cases, totally disconnected groups provide such actions. In the second section, we introduce the orbit category. This category appears naturally in the study of G-CW-complexes for a group G. We develop some basic properties. Furthermore, it will be important whether the endomorphism sets of the orbit category are (locally) finite groups. This is discussed, too. In the third section, we introduce modules over a category. In that section, the category might be arbitrary but the most important example is given by the orbit category. In particular, modules over the orbit category are the basic ingredient for the construction of a cellular homology theory for G-CW-complexes, which is done in the sixth section. Before we come to the sixth section, we introduce G-CW-complexes in Section 1.4. In the following section, we introduce G-(co)homology theories and equivariant (co)homology theories. The latter are the basic input for our Chern character. Finally, we construct cellular (co)homology for G-CW-complexes, which is called Bredon (co)homology. As in the non-equivariant case, Bredon (co)homology can be constructed for an arbitrary coefficient module M. However, in this case the coefficient module is a module over the orbit category. If the coefficient module provides some extra structure, we show that Bredon (co)homology yields an equivariant (co)homology theory. In the last section, we introduce linear algebraic p-adic groups, which are an important class of totally disconnected groups. In particular, we show that the orbit category of a semisimple p-adic group has finite morphism sets. Semisimple p-adic groups are important examples of linear algebraic p-adic groups.

1.1 Totally Disconnected Groups

Definition 1.1.1. Let X be a topological space. Then X is *totally disconnected* if each connected component consists only of a single point. An *l-space* is a locally compact totally disconnected space. An *l-group* is a topological group whose underlying topological space is an l-space.

Example 1.1.2. The following groups are l-groups:

(i) Discrete groups.

(ii) $\mathbb{Q}_p$, $\mathbb{Z}_p$, $GL_n(\mathbb{Q}_p)$ and $SL_n(\mathbb{Q}_p)$, where p is a prime and $n \in \mathbb{N}$.

(iii) Profinite groups, i.e., groups arising as limits $\lim_{i\in I} G_i$ with G_i finite. These groups occur naturally as Galois groups $\mathrm{Gal}(L/K)$ for a Galois extension L/K (see [27, p.51]).

Proposition 1.1.3. *Let G be locally compact. Then the following are equivalent:*

(i) G is totally disconnected (and hence an l-group);

(ii) G admits a basis of topology which consists of compact open subgroups.

Proof. The implication (i) $\Rightarrow$ (ii) is done by Hewitt and Ross [21, Thm. 7.7].

Now we prove (ii) $\Rightarrow$ (i). Let $x, y \in G$ be two distinct points and $U \subseteq G$ be a subset such that $x, y \in U$. By (ii) and the fact that G is Hausdorff, there exists an open closed set $x \in \tilde{U}_x$ in G such that $y \in \tilde{U}_x^c$. Now we have a decomposition

$$U = (U \cap \tilde{U}_x) \coprod (U \cap \tilde{U}_x^c)$$

into two disjoint open sets. Consequently, U cannot be connected. □

Corollary 1.1.4. *Let G be an l-group and $H \subseteq G$ be a compact subgroup. Then there exists a compact open subgroup $H \subseteq K \subseteq G$. In particular, every maximal compact subgroup is open.*

Proof. Let L be a compact open subgroup, which exists by the previous proposition. Then there exist finitely many $h_1, \ldots, h_n$ such that

$$H \subseteq \bigcup_{i=1}^{n} h_i L.$$

Now we define $M := \bigcap_{i=1}^{n} h_i L h_i^{-1}$. We get $hMh^{-1} = M$ for any $h \in H$, and M is a compact open subgroup. Then $K = MH$ is open. It is compact because $M \times H$ is compact. □

The subgroups of Proposition 1.1.3 need not be normal. In particular there exist l-groups which do not have any compact open normal subgroup.

Example. Let $F = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z}$. We get the representation

$$F = \langle a, b, h | a^2 = b^2 = h^2 = 1,\ ab = ba,\ ha = bh \rangle .$$

Now let G be the group of all functions $g \colon \mathbb{N} \to F$ such that $g(n) \in \{1, a\}$ for all but finitely many n. We equip G with the topology induced by the (compact) open sets

$$U(n, g) = \{ f \in G \mid f(m) = g(m) \text{ for } m \leq n,\ f(m) \in \{1, a\} \text{ for } m > n \} .$$

Then G is an l-group. Let $H \subseteq G$ be an open normal subgroup. There exists an integer n such that H contains $U(n,1)$. Since H is normal and $hah^{-1} = b$, it follows that for each $m > n$ there exists a $g_m \in H$ such that

$$g_m(k) = \begin{cases} 1, & \text{if } k \neq m, \\ b, & \text{if } k = m. \end{cases}$$

Hence H cannot be compact because $g_m \notin U(m-1, f)$ for any $f \in F$. Finally, G contains no compact open normal subgroup.

Remark 1.1.5. Suppose we have an l-group G and an element $g \in G$ which does not normalize any compact open subgroup. Then Willis [59] showed that G contains a closed subgroup which contains g and does not have any open compact normal subgroup.

However, we have the following characterization [46, Lem. 1.3/5].

Proposition 1.1.6. *Let G be locally compact. Then the following are equivalent:*

(i) G is a limit of discrete groups, i.e., $G = \lim_{i \in I} G_i$ with G_i discrete;

(ii) G admits a basis of topology which consists of compact open normal subgroups.

These groups are called prodiscrete.

Proof. (i) $\Rightarrow$ (ii): Let $G = \lim_{i \in I} G_i$, $\mathrm{pr}_i \colon G \to G_i$ the canonical projection and $e_i \in G_i$ the unit element. Then $\{ \mathrm{pr}_i^{-1}(e_i) \mid i \in I \}$ forms a basis of compact open normal subgroups.

(ii) $\Rightarrow$ (i): We set

$$N(G) = \{ N \subseteq G \mid N \text{ compact open normal} \}$$

and obtain a continuous map

$$\mathrm{pr}_* \colon G \to \lim_{N \in N(G)} G/N.$$

We have to show that pr_* is open and bijective. Since G is Hausdorff, pr_* is injective. To see that the map is surjective, pick an element $(g_N N)_N \in \lim_{N \in N(G)} G/N$. Since $\bigcap_{N \in N(G)} g_N N$ can be interpreted as an inverse limit of non-empty compact Hausdorff spaces, it is non-empty [44, Prop. 1.4]. Hence any element in the intersection can serve as a preimage. Finally, any compact open normal subgroup H is mapped to the intersection of $\lim_{N \in N(G)} G/N$ and the open set $\mathrm{pr}_{G/H}^{-1}(1) \subseteq \prod_{N \in N(G)} G/N$. Hence H is mapped to an open set, as required. $\square$

Remark 1.1.7. Any nilpotent compactly generated l-group is prodiscrete by a result of Willis [60].

Lemma 1.1.8. *Let G be a compact l-group. Then G is profinite.*

Proof. Let $K \subseteq G$ be compact open. We set

$$H = \bigcap_{[g] \in K \backslash G / K} gKg^{-1}.$$

Since G is compact, the above index set is finite. Thus $H \subseteq K$ is a compact open normal subgroup. The compact open normal subgroups form a basis of the topology of G, and we obtain, analogously to the previous proposition,

$$G = \lim_{N \in N(G)} G/N, \quad \text{where} \quad N(G) = \{ H \subseteq G \mid \ H \text{ compact open normal} \},$$

as required. □

Prodiscrete groups need not be locally compact; there are easy counterexamples, e.g., $\prod_{\mathbb{Z}} \mathbb{Z}$. We have the following characterization of (second countable) locally compact prodiscrete groups.

Lemma 1.1.9. *Let $G = \lim_{i \in I} G_i$ be a second countable prodiscrete group. Then G is locally compact if and only if the kernels of the structure maps are finite almost everywhere.*

Proof. Since G is second countable, we can consider $I = \mathbb{N}$ for simplicity (cf. Remark 3.2.8). We denote the structure maps by d.

First, we assume that $\ker d$ is always finite. Then $d^{-n}(1)$ is finite, too. Therefore the set $(\dots, d^{-2}(1), d^{-1}(1), 1)$ is compact. However, the set is open by construction and we get a compact open neighborhood of the unit element. Hence G is locally compact.

Now let $\ker d$ be infinite for infinitely many d. Since the sets

$$(\dots, d^{-2}(1), d^{-1}(1), 1, \dots, 1)$$

are a basis of topology and closed (open subgroups are closed), it suffices to show that these sets are not compact. Let $\ker d$ be infinite and $F_n \subseteq \ker d$ be an exhaustive properly increasing filtration of $\ker d$. Then $(\dots, d^{-2}(F_n), d^{-1}(F_n), F_n, 1, \dots, 1)$ is an open cover of $(\dots, d^{-3}, d^{-2}(1), d^{-1}(1), 1, \dots, 1)$, which admits no finite subcover. □

The great benefit of l-groups is the existence of Hecke algebras, which we will introduce now.

Definition 1.1.10. Let G be an l-group and choose a Haar measure μ on G. Then the *Hecke algebra* $\mathcal{H}(G, \mu)$ is defined by

$$\mathcal{H}(G, \mu) = \{ f : G \to \mathbb{C} \mid f \text{ locally constant and has compact support} \},$$

where the algebra structure is given by the *convolution product*:

$$(\varphi * \psi)(\gamma) = \int_G \varphi(\gamma g) \psi(g^{-1}) d\mu(g), \quad \gamma \in G, \ \varphi, \psi \in \mathcal{H}(G).$$

The choice of a different Haar measure ν leads to a Hecke algebra $\mathcal{H}(G, \nu)$ which is isomorphic to $\mathcal{H}(G, \mu)$. This is why we will abbreviate $\mathcal{H}(G) := \mathcal{H}(G, \mu)$.

The Hecke algebra $\mathcal{H}(G)$ shall be viewed as a generalization of the group ring in the discrete case. In particular, we obtain [8, Thm. 2]:

Proposition 1.1.11. *Let G be an l-group and let V be a (not necessarily finite dimensional) $\mathbb{C}$-vector space with a G-action. We call V a* smooth G-module *if the isotropy groups*

$$\{\, g \in G \mid gv = v \,\}$$

are open in G for any $v \in V$. We get an isomorphism of categories

$$\{\, \textit{smooth } G\textit{-modules} \,\} \longleftrightarrow \{\, M\ \mathcal{H}(G)\textit{-module} \mid \mathcal{H}(G)M = M \,\}\,.$$

1.2 The Orbit Category

We introduce and discuss the orbit category and the subgroup category, which will be the basic objects of our theory. It will be important whether the endomorphism sets of these categories are groups. This is discussed here, too.

In the following, let G be an arbitrary topological group.

Definition 1.2.1. A *family of subgroups* $\mathcal{F}$ is a set of closed subgroups of G which is closed under conjugation and finite intersection. We do not demand that it be closed under taking subgroups, which is often required in the literature. The family $\mathcal{F}$ is called *smooth* if the subgroups are open in G.

Example 1.2.2. Let G be a topological group. Then the trivial family $\{1\}$ and

$$\begin{aligned} \mathcal{CL} &= \{\, H \subseteq G \text{ closed} \,\} & \mathcal{COP} &= \{\, H \subseteq G \text{ compact} \,\} \\ \mathcal{COC} &= \{\, H \subseteq G \mid G/H \text{ compact} \,\} & \mathcal{O} &= \{\, H \subseteq G \text{ open} \,\} \\ \mathcal{CO} &= \{\, H \subseteq G \text{ compact and open} \,\} & \mathcal{I} &= \{\, H \subseteq G \text{ open} \mid G/H \text{ finite} \,\} \end{aligned}$$

are families of subgroups, the last three ones beeing smooth.

In this section, let $\mathcal{F}$ be a smooth family of subgroups.

Remark 1.2.3. We restrict our interest to open subgroups because we want to avoid topological issues concerning the orbit category. Furthermore, there are some issues with K-theory, which will be discussed in the next chapter.

Let $H \subseteq G$ be a subgroup of finite index. Then H is open if and only if it is closed. This is quite clear because the complement of H can be described by

$$G \setminus H = \bigcup_{\substack{[g] \in G/H \\ [g] \neq [1]}} gH.$$

This is a finite union and hence the complement is open or closed, respectively, whenever H is. On the other hand, there exist examples of subgroups of finite index which are not closed (and open). Even in the case of a profinite group G we can give an example.

Before we do so, however, we must introduce ultrafilters. Let S be a non-empty set and $\mathcal{U} \subseteq \mathcal{P}(S)$ be a subset, where $\mathcal{P}(S)$ denotes the power set of S. We call $\mathcal{U}$ a *filter* if

(i) $\emptyset \notin \mathcal{U}$,

(ii) $A, B \in \mathcal{U}$ implies $A \cap B \in \mathcal{U}$,

(iii) $A \in \mathcal{U}$ and $A \subseteq B \subseteq S$ implies $B \in \mathcal{U}$.

A filter $\mathcal{U}$ is called *ultrafilter* if for any $A \subseteq S$ either $A \in \mathcal{U}$ or $S \setminus A \in \mathcal{U}$ holds. Every filter $\mathcal{U}$ is contained in an ultrafilter $\mathcal{U} \subseteq \mathcal{U}'$ by Zorn's lemma. Now we can present the example mentioned above:

Example 1.2.4. Let K be a finite group and $G = \prod_{\mathbb{Z}} K$. Let $\mathcal{U}$ be an ultrafilter of $\mathbb{Z}$ containing the filter of all cofinite subsets of $\mathbb{Z}$. We define H by

$$H = \{ (h_n)_{n\in\mathbb{Z}} \in G \mid \{ n \in \mathbb{Z} \mid h_n = 1 \} \in \mathcal{U} \} .$$

Clearly, H is a proper normal subgroup of G. Moreover, it is dense in G because $\mathcal{U}$ contains all cofinite subsets of $\mathbb{Z}$. Hence H is not open. It remains to show $[G : H] = |K|$. To see that, it suffices to show that every $g \in G$ is congruent to the constant sequence $(k)_{n\in\mathbb{Z}}$ modulo H for some $k \in K$. Fix a $g \in G$, a $k \in K$ and define $\mathbb{Z}_k = \{ n \in \mathbb{Z} \mid g_n = k \}$. Then we get

$$\mathbb{Z} = \bigcup_{k\in K} \mathbb{Z}_k.$$

Since $\mathcal{U}$ is an ultrafilter, $\mathbb{Z}_k \in \mathcal{U}$ for some $k \in K$. Therefore $gk^{-1} \in H$, i.e., $g \in G$ is congruent to $(k)_{n\in\mathbb{Z}}$, as desired.

We call G *strongly complete* if G has the property that every subgroup of finite index is open. Nikolov and Segal [38] have recently shown that every finitely generated profinite group is strongly complete.

Definition 1.2.5. Let $H \in \mathcal{F}$. The *normalizer of* G is

$$N_G H = \left\{ g \in G \;\middle|\; gHg^{-1} = H \right\}$$

and the *Weyl group of* G is defined by

$$W_G H = N_G H / H.$$

Furthermore, the *centralizer of* G is

$$C_G H = \left\{ g \in G \;\middle|\; ghg^{-1} = h \ \forall h \in H \right\}$$

and the *center* is $C(G) = C_G(G)$.

Definition 1.2.6. The *orbit category* $\mathsf{Or}_{\mathcal{F}}(G)$ *associated to a group* G *and a family* $\mathcal{F}$ is defined by

$$\begin{aligned} \mathrm{Ob}(\mathsf{Or}_{\mathcal{F}}(G)) &= \{\, G/H \mid H \in \mathcal{F} \,\} \\ \mathrm{mor}(G/H, G/K) &= \{\, f\colon G/H \to G/K \mid f(gx) = f(x)\ \forall g \in G, x \in G/H \,\}. \end{aligned}$$

Let $H \subseteq G$ be a subgroup. We denote by (H) the conjugation class of H in G. For another subgroup $K \subseteq G$, we write $(H) \leq (K)$ if H is subconjugated to K, i.e., if there exists a $g \in G$ such that $gHg^{-1} \subseteq K$.

Proposition 1.2.7. *Let $H, K \in \mathcal{F}$. Then the following statements hold.*

(i) *There is an equivariant map $G/H \to G/K$ if and only if $(H) \leq (K)$.*

(ii) *If $g \in G$ and $g^{-1}Hg \subseteq K$, then we get a well-defined G-map*

$$R_g\colon G/H \to G/K, \quad g'H \mapsto g'gK.$$

(iii) *Every G-map $G/H \to G/K$ is of the form R_g. We have $R_g = R_{g'}$ if and only if $g^{-1}g' \in K$.*

(iv) *If and only if $g^{-1}Hg \subseteq H \Rightarrow g^{-1}Hg = H$ holds for any $g \in G$, we get an isomorphism of (discrete) groups*

$$W_GH \to \mathrm{map}(G/H, G/H)^G, \quad gH \mapsto R_{g^{-1}}.$$

Proof. A proof of the first three assertions can be found in [53, Prop. 1.1.14]. The last one is immediate. ☐

Definition 1.2.8. The *subgroup category* $\mathsf{Sub}_{\mathcal{F}}(G)$ *of a group* G *and a family* $\mathcal{F}$ is given by

$$\begin{aligned} \mathrm{Ob}(\mathsf{Sub}_{\mathcal{F}}(G)) &= \{\, H \mid H \in \mathcal{F} \,\} \\ \mathrm{mor}(H, K) &= \left\{\, f\colon H \to K \;\middle|\; \exists g \in G,\ f(h) = g^{-1}hg \text{ for all } h \in H \,\right\} / \sim, \end{aligned}$$

where $f_1 \sim f_2$ if and only if there exists a $k \in K$ such that $f_1 = k^{-1}f_2k$.

Remark 1.2.9. We have a canonical projection

$$\mathrm{pr}\colon \mathsf{Or}_{\mathcal{F}}(G) \to \mathsf{Sub}_{\mathcal{F}}(G), \quad G/H \mapsto H, \quad R_g \mapsto [c(g)],$$

where $c(g)$ denotes conjugation by g. Note that $c(g) = c(g')$ if and only if $g^{-1}g' \in C_GH$. We can deduce the following identity

$$\mathrm{mor}_{\mathsf{Or}_{\mathcal{F}}(G)}(G/H, G/K)/C_GH = \mathrm{mor}_{\mathsf{Sub}_{\mathcal{F}}(G)}(H, K),$$

where $g \in C_GH$ acts by left composition with $R_{g^{-1}}$.

Sometimes the Weyl group is defined by $W_G H = N_G H/(H \cdot C_G H)$. The advantage of this definition is that we obtain

$$N_G H/(H \cdot C_G H) = \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H)$$

if H satisfies the assumption of Proposition 1.2.7 (iv). Since we are interested in both cases, we define:

Definition 1.2.10. Let G be a group and $H \in \mathcal{F}$. Then we define the *reduced Weyl group* $\widetilde{W_G H}$ by

$$\widetilde{W_G H} = N_G H/(H \cdot C_G H).$$

Definition 1.2.11. Let Γ be a small category, i.e., $\mathrm{Ob}(\Gamma)$ is a set. Then Γ is an *EI-category* if every endomorphism is an isomorphism.

The great benefit of an EI-category is that we can define an order on it.

Definition 1.2.12. Let Γ be an EI-category and denote by $\mathrm{Is}\,\Gamma$ the set of isomorphism classes. For $x \in \mathrm{Ob}(\Gamma)$ we denote by (x) the corresponding isomorphism class in $\mathrm{Is}\,\Gamma$. Then $\mathrm{Is}\,\Gamma$ forms a partially ordered set by

$$(x) \leq (y) :\Longleftrightarrow \mathrm{mor}(x, y) \neq \emptyset, \quad x, y \in \mathrm{Ob}(\Gamma).$$

Note that the EI-property guarantees $(x) = (y)$ if $(x) \leq (y)$ and $(y) \leq (x)$.

We define the *length of* $(y) \in \mathrm{Is}(\Gamma)$ *(relative* (x)*)* and *colength (relative* (x)*)* by

$$l_x(y) = \sup\left\{ n \mid (x) < (x_1) < \cdots < (x_n) = (y) \right\},$$
$$col_x(y) = \sup\left\{ n \mid (x) > (x_1) > \cdots > (x_n) = (y) \right\}.$$

The *length* and *colength* are defined by

$$l(y) = \sup_{(x) \in \mathrm{Is}(\Gamma)} l_x(y) \quad \text{and} \quad col(y) = \sup_{(x) \in \mathrm{Is}(\Gamma)} col_x(y).$$

We call Γ *of finite length* if $l(x) < \infty$ for each $(x) \in \mathrm{Is}(\Gamma)$ and *of finite colength* if $col(x) < \infty$ for each $(x) \in \mathrm{Is}(\Gamma)$.

Lemma 1.2.13. *The category* $\mathtt{Or}_{\mathcal{F}}(G)$ *is an EI-category if and only if* $\mathtt{Sub}_{\mathcal{F}}(G)$ *is an EI-category. Furthermore, this is equivalent to the condition*

$$gHg^{-1} \subseteq H \Longrightarrow gHg^{-1} = H \quad \forall g \in G,\ H \in \mathcal{F}.$$

In this case, we have

$$\mathrm{mor}_{\mathtt{Or}_{\mathcal{F}}(G)}(G/H, G/H) = W_G H \quad \text{and} \quad \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H) = \widetilde{W_G H}.$$

Proof. Let $[c(g)] \in \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H)$. By definition, we have that $[c(g)]$ is an isomorphism if and only if $c(g) : H \to H$ is an isomorphism of groups. Now the assertion follows from Proposition 1.2.7 (iv). $\square$

Example 1.2.14. Let p be a prime and $G = \mathbb{Z}_p$. The compact open subgroups of $\mathbb{Z}_p$ are $p^k\mathbb{Z}_p$ for $k \in \mathbb{N}_0$. Since $\mathbb{Z}_p$ is abelian, we obtain

$$\mathrm{mor}_{\mathtt{Sub}_{\mathcal{CO}}(G)}(p^k\mathbb{Z}_p, p^l\mathbb{Z}_p) = \begin{cases} \{\, \mathrm{inc}\colon p^k\mathbb{Z}_p \to p^l\mathbb{Z}_p \,\}, & \text{if } k \geq l, \\ \emptyset, & \text{otherwise,} \end{cases}$$

where inc denotes the canonical inclusion. We should imagine $\mathtt{Sub}_{\mathcal{CO}}(G)$ to be the following (directed) graph:

$$\cdots \to 2 \to 1 \to 0.$$

Definition 1.2.15. Let G be locally compact. Choose a left invariant Haar measure μ. The *modular function* Δ is defined by

$$\Delta\colon G \to \mathbb{R}^+, \quad g \mapsto \frac{\mu(M)}{\mu(Mg)},$$

where $M \subseteq G$ is an arbitrary subset with finite positive measure. Note that Δ depends neither on M nor on μ. If $\Delta(g) = 1$, we call g *unimodular*. If every $g \in G$ is unimodular, we call G *unimodular*.

Immediate examples of unimodular groups are abelian groups. The next lemma will provide us with some more examples.

Lemma 1.2.16. *Let G be locally compact. Then G is unimodular if there exists a compact open normal subgroup.*

Proof. Let $H \subseteq G$ be a compact open normal subgroup and $g \in G$. Since H is compact open, we get $0 < \mu(H) < \infty$. Additionally, we have $gHg^{-1} = H$ because H is normal. Now we can deduce

$$\Delta(g^{-1}) = \frac{\mu(H)}{\mu(Hg^{-1})} = \frac{\mu(H)}{\mu(gHg^{-1})} = \frac{\mu(H)}{\mu(H)} = 1,$$

and G is unimodular, as required. □

Corollary 1.2.17. *The following groups are unimodular:*

(i) compact groups,

(ii) prodiscrete l-groups.

Proof. In the case of a compact group G, we can take the entire group G as a compact open normal subgroup. Moreover, prodiscrete l-groups have compact open normal subgroups by Proposition 1.1.6. □

Lemma 1.2.18. *Let G be locally compact. If $g \in G$ is unimodular, then we obtain for every compact open subgroup $H \subseteq G$*

$$gHg^{-1} \subseteq H \Longrightarrow gHg^{-1} = H.$$

Consequently, $\mathtt{Or}_{\mathcal{CO}}(G)$ and $\mathtt{Sub}_{\mathcal{CO}}(G)$ are EI-categories if G is unimodular.

Proof. Let μ be a Haar measure and $H \subseteq G$ be a compact open subgroup. Hence we obtain $0 < \mu(H) < \infty$. Suppose $gHg^{-1} \subseteq H$ holds. Since G is unimodular, we deduce $\mu(H) = \mu(gHg^{-1})$. Finally, we get an open subset $H \setminus gHg^{-1}$ with $\mu(H \setminus gHg^{-1}) = 0$. Thus we can conclude $H \setminus gHg^{-1} = \emptyset$. □

The converse is false, i.e., there exists a non-unimodular group G such that the corresponding orbit category is an EI-category:

Example 1.2.19. Let p be a prime and

$$G = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Q}_p, a \neq 0 \right\}.$$

The corresponding Lie algebra is

$$L(G) = \left\{ \begin{pmatrix} a & b & c \\ 0 & -a & d \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Q}_p \right\},$$

and if we take the standard basis, the adjoint representation is given by the matrix

$$M_g = \begin{pmatrix} 1 & -2ab & 2abd - c & d \\ 0 & a^2 & -a^3 d & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & b & a^{-1} \end{pmatrix} \quad \text{for } g = \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix}.$$

Bourbaki [11, Chap. III, §3.16, Cor. to Prop. 55] showed that the modular function Δ is given by

$$\Delta(g) = |\det(M_g)|_p = |a^2|_p = |a|_p^2.$$

Therefore G is not unimodular, and the non-unimodular elements are precisely

$$\left\{ \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{Q}_p, a \neq 0, |a|_p \neq 1 \right\}.$$

Let $H \subseteq G$ be a compact open subgroup. Since H is open, we can find an element $h \in H$ such that

$$h = \begin{pmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with } e \neq 0.$$

Conjugation by $\tilde{g}^n \in G$ leads to

$$\begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix}^n \begin{pmatrix} 1 & e & f \\ 0 & 1 & g \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix}^{-n} = \begin{pmatrix} 1 & a^{2n} e & \tilde{f} \\ 0 & 1 & a^{-n} g \\ 0 & 0 & 1 \end{pmatrix}.$$

Assume $\tilde{g}$ is non-unimodular and hence, without loss of generality, $|a|_p > 1$. Since H is compact and $|a^{2n}e|_p \to \infty$, we deduce, on the one hand, $\tilde{g}H\tilde{g}^{-1} \nsubseteq H$. On the other hand, since H is compact, we can pick an element

$$h_0 = \begin{pmatrix} 1 & \tilde{a} & \tilde{b} \\ 0 & 1 & \tilde{d} \\ 0 & 0 & 1 \end{pmatrix} \in H \text{ with } |\tilde{d}| \text{ maximal.}$$

We have $|a^{-1}\tilde{d}|_p < |\tilde{d}|_p$ and, consequently, $h_0 \notin \tilde{g}H\tilde{g}^{-1}$. So we have neither $\tilde{g}H\tilde{g}^{-1} \subseteq H$ nor $\tilde{g}H\tilde{g}^{-1} \supseteq H$ for a non-unimodular $\tilde{g} \in G$. If $\tilde{g}$ is unimodular, it satisfies the condition $\tilde{g}H\tilde{g}^{-1} \subseteq H \Rightarrow \tilde{g}H\tilde{g}^{-1} = H$ by the previous lemma. Therefore we have constructed a non-unimodular group which satisfies

$$\tilde{g}H\tilde{g}^{-1} \subseteq H \Longrightarrow \tilde{g}H\tilde{g}^{-1} = H.$$

Note that, by a result of Raja [43], for each unimodular element $\tilde{g} \in G$ there exists a compact open subgroup $H \subseteq G$ with $\tilde{g}H\tilde{g}^{-1} = H$.

We want to give an example of an l-group G whose orbit category $\mathsf{Or}_{\mathcal{CO}}(G)$ fails to be an EI-category.

Example 1.2.20. Let p be a prime and G be the following group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a, b \in \mathbb{Q}_p \; a \neq 0 \right\}.$$

In addition, define the compact open subgroup $H \subseteq G$ by

$$H = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \in G \,\middle|\, |c|_p = 1,\ |d|_p \leq 1 \right\}$$

and let

$$g = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in G \text{ with } |a^{-1}|_p, |b|_p < 1.$$

Then we obtain

$$g^{-1}Hg = \begin{pmatrix} a^{-1} & -b \\ 0 & a \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} c & a^{-1}(bc + a^{-1}d - bc^{-1}) \\ 0 & c^{-1} \end{pmatrix}.$$

Since

$$\begin{aligned} |a^{-1}(bc + a^{-1}d - bc^{-1})|_p &= |a^{-1}| \cdot |bc + a^{-1}d - bc^{-1}|_p \\ &\leq |a^{-1}| \cdot \max\{\, |bc|_p, |a^{-1}d|_p, |bc^{-1})|_p \,\} \\ &\leq |a^{-1}|_p < 1, \end{aligned}$$

we get $g^{-1}Hg \subseteq H$ and $g^{-1}Hg \neq H$. Therefore, $\mathsf{Or}_{\mathcal{CO}}(G)$ is not an EI-category.

Proposition 1.2.21. *Let G be a topological group.*

(i) $\mathtt{Sub}_{\mathcal{I}}(G)$ is an EI-category, and for all (closed) subgroups $H \subseteq K \subseteq G$ of finite index we obtain $l_H(K) < \infty$.

(ii) Additionally, let G be unimodular. Then $\mathtt{Sub}_{\mathcal{CO}}(G)$ is an EI-category, and for all compact open subgroups $H \subseteq K \subseteq G$ we obtain $l_H(K) < \infty$.

Proof. Let us prove the first assertion. Let $H \subseteq G$ be a closed subgroup of finite index and $g \in G$ such that $gHg^{-1} \subseteq H$. Since conjugation with g is a G-isomorphism, we obtain

$$[G : H] = [G : gHg^{-1}] = [G : H][H : gHg^{-1}]$$

and thus $[H : gHg^{-1}] = 1$. Hence $H = gHg^{-1}$ and $\mathtt{Sub}_{\mathcal{I}}(G)$ is an EI-category. Now let $H \subseteq K \subseteq G$ be closed subgroups of finite index. Then we get

$$l_H(K) < \sup \{ [K : gHg^{-1}] \mid gHg^{-1} \subseteq K,\ g \in G \} < \sup_{g \in G} [G : gHg^{-1}] = [G : H].$$

Now we prove the second assertion. Let G be unimodular and choose a Haar measure μ. Then $\mathtt{Sub}_{\mathcal{CO}}(G)$ is an EI-category by Lemma 1.2.18. Let $H \subseteq K \subseteq G$ be two compact open subgroups. We obtain

$$\begin{aligned} l_H(K) &\leq \sup \{ [K : gHg^{-1}] \mid gHg^{-1} \subseteq K,\ g \in G \} \\ &= \sup \{ \frac{\mu(K)}{\mu(gHg^{-1})} \mid gHg^{-1} \subseteq K,\ g \in G \} \\ &= \sup \{ \frac{\mu(K)}{\mu(H)} \mid gHg^{-1} \subseteq K,\ g \in G \} = \frac{\mu(K)}{\mu(H)} < \infty. \end{aligned}$$

□

The author does not know of an example, where $l_H(K) = \infty$ in the case of an EI-category $\mathtt{Sub}_{\mathcal{CO}}(G)$. However, the index $[K : gHg^{-1}]$ can vary for different $g \in G$. We want to give an example.

Example 1.2.22. Let G be the group of Example 1.2.19, namely we get

$$G = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{Q}_p, a \neq 0 \right\}.$$

Then $\mathtt{Sub}_{\mathcal{CO}}(G)$ is an EI-category. Let $H \subseteq K \subseteq G$ be the following compact open subgroups

$$K = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix} \in G \ \middle| \ |a|_p = 1,\ |b|_p, |c|_p, |d|_p \leq 1 \right\},$$

$$H = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^{-1} & d \\ 0 & 0 & 1 \end{pmatrix} \in K \ \middle| \ |b|_p, |c|_p, |d|_p \leq \frac{1}{4} \right\}.$$

Let $a', b', c', d' \in \mathbb{Q}_p$ with $|a'|_p = \frac{1}{2}$ and $|b'|_p, |c'|_p, |d'|_p \leq 1$. We define

$$g = \begin{pmatrix} a' & b' & c' \\ 0 & (a')^{-1} & d' \\ 0 & 0 & 1 \end{pmatrix}$$

and obtain $gHg^{-1} \subseteq K$ because of $|a+b|_p \leq \max\{\, |a|_p, |b|_p \,\}$ and $|ab|_p = |a|_p \cdot |b|_p$ for any $a, b \in \mathbb{Q}_p$. Note that, in our case, the absolute value of a matrix entry can increase by a factor of 4 because we multiply twice. Let μ be a Haar measure on G. We obtain $\Delta(g) = \frac{1}{4}$ by Example 1.2.19 and, consequently,

$$[K : H] = \frac{\mu(K)}{\mu(H)} < 4\frac{\mu(K)}{\mu(H)} = \frac{\mu(K)}{\Delta(g)\mu(H)} = \frac{\mu(K)}{\mu(gHg^{-1})} = [K : gHg^{-1}].$$

Therefore, the index may vary. However, in this example, it cannot tend to infinity for the following reason. Conjugation by g increases the absolute value of at least one matrix entry by a factor of $|a'|_p$ or $|a'|_p^{-1}$, respectively (cf. Example 1.2.19). But K is compact, which means that the absolute value of that matrix entry is bounded.

Let Γ be a category. The endomorphism sets of Γ have a canonical semigroup structure. If Γ happens to be an EI-category, the endomorphism sets are even groups. Since we have seen in Example 1.2.20 that the orbit category is not an EI-category in general, we must take semigroups into account. So let G be a semigroup. We call an element $g \in G$ *torsionfree* if $g^n \neq g^m$ holds for $n \neq m$ with $n, m \in \mathbb{N}$.

Lemma 1.2.23. *Let G be a group, $\mathcal{F}$ a family and $H \in \mathcal{F}$ a subgroup. If the semigroup $\operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H)$ is not a group, it contains a torsionfree element. The same holds for $\operatorname{mor}_{\mathtt{Or}_{\mathcal{F}}(G)}(G/H, G/H)$.*

Proof. By Remark 1.2.9, it suffices to prove the assertion for $\mathtt{Sub}_{\mathcal{F}}(G)$. Suppose that $\operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H)$ is not a group. Hence there exists a morphism

$$[c(g)] \in \operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H)$$

which is not an isomorphism. Since $c(g)$ is injective, it cannot be surjective. Therefore, we get

$$g^n H g^{-n} \subsetneq g^k H g^{-k} \quad \text{for } n > k$$

for the corresponding images and obtain $c(g)^n \neq c(g)^k$ for $n \neq k$. Thus $[c(g)]$ is the desired torsionfree element. $\square$

A *locally finite semigroup* is a semigroup such that every finitely generated subgroup is finite (cf. Definition A.3 for the notion of locally finite for groups).

Corollary 1.2.24. *If the semigroup $\operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H)$ is locally finite, then it is already a group. The same holds for $\operatorname{mor}_{\mathtt{Or}_{\mathcal{F}}(G)}(G/H, G/H)$. In this case, we obtain*

$$\widetilde{W_G H} = \operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H, H) \quad \text{and} \quad W_G H = \operatorname{mor}_{\mathtt{Or}_{\mathcal{F}}(G)}(G/H, G/H).$$

It will be of interest to know whether the endomorphism sets (or even the morphism sets) of $\mathsf{Sub}_{\mathcal{F}}(G)$ or $\mathsf{Or}_{\mathcal{F}}(G)$ are finite. If G is a prodiscrete l-group and $\mathcal{F} = \mathcal{CO}$, this is the case by the following proposition.

Proposition 1.2.25. *Let G be a prodiscrete l-group. Then $\mathsf{Or}_{\mathcal{CO}}(G)$ is an EI-category, and for every compact open subgroup $H \subseteq G$ the sets $\operatorname{mor}_{\mathsf{Sub}_{\mathcal{CO}}(G)}(H, K)$ are finite.*

Proof. Since G is unimodular by Corollary 1.2.17, the category $\mathsf{Or}_{\mathcal{CO}}(G)$ is an EI-category by Lemma 1.2.18. It remains to show that

$$\operatorname{mor}_{\mathsf{Sub}_{\mathcal{CO}}(G)}(H, K) = \{\, g \in G \mid gHg^{-1} \subseteq K \,\} / (H \cdot C_G H)$$

is finite for compact open subgroups $H, K \subseteq G$. We set

$$M'_G(H, K) = \{\, g \in G \mid gHg^{-1} \subseteq K \,\} / C_G H.$$

We fix the notation $U_L = U/(U \cap L)$ for a subgroup $U \subseteq G$ and a normal subgroup $L \subseteq G$. Since G is a prodiscrete l-group, it can be written as a limit (see proof of Proposition 1.1.6)

$$G = \lim_{L \in N(G)} G_L, \quad \text{where } N(G) = \{\, L \subseteq G \text{ compact open normal} \,\}.$$

In particular, G_L is a discrete group and the structure maps are surjective. Now we obtain (finite) subgroups $H_L, K_L \subseteq G_L$ such that $H = \lim_{L \in N(G)} H_L$ and $K = \lim_{L \in N(G)} K_L$ (with the corresponding restrictions as structure maps). The structure maps $\varphi_{L_1 L_2} \colon G_{L_1} \to G_{L_2}$ induce maps

$$\varphi'_{L_1 L_2} \colon M'_{G_{L_1}}(H_{L_1}, K_{L_1}) \to M'_{G_{L_2}}(H_{L_2}, K_{L_2}).$$

Here, we have to check, on the one hand, that $\varphi'_{L_1 L_2}(C_{G_{L_1}} H_{L_1}) \subseteq C_{G_{L_2}} H_{L_2}$ and, on the other hand, that the destination space is right. We only show the first assertion, the second one can be proven analogously. Let $g \in C_{G_{L_1}} H_{L_1}$ and $h \in H_{L_2}$. Since the structure maps are surjective, there exists a preimage $h' \in H_{L_1}$, and we can conclude

$$\begin{aligned} \varphi'_{L_1 L_2}(g) h \varphi'_{L_1 L_2}(g)^{-1} &= \varphi'_{L_1 L_2}(g) \varphi'_{L_1 L_2}(h') \varphi'_{L_1 L_2}(g^{-1}) \\ &= \varphi'_{L_1 L_2}(g h' g^{-1}) = \varphi'_{L_1 L_2}(h') = h. \end{aligned}$$

In the same way, the projections $\operatorname{pr}_L \colon G \to G_L$ induce a map

$$\operatorname{pr}_* \colon M'_G(H, K) \to \lim_{L \in N(G)} M'_{G_L}(H_L, K_L),$$

which is injective because G is Hausdorff. Note that the right hand side is compact because each $M'_{G_L}(H_L, K_L)$ is finite. Since $H \subseteq G$ is compact open and G admits a basis of topology consisting of compact open normal subgroups by Proposition 1.1.6, H has an open subgroup $H' \subseteq H$ which is normal in G. We can assume $H \subseteq K$ and obtain that

$$\operatorname{pr}_*(H') = \operatorname{pr}_{H'}^{-1}(i_{H'}) \cap \operatorname{im} \operatorname{pr}_*$$

and

$$\mathrm{pr}_{H'}^{-1}(i_{H'}) \subseteq \lim_{L\in N(G)} M'_{G_L}(H_L, K_L)$$

is open, where $\mathrm{pr}_{H'}$ denotes the canonical projection $\mathrm{pr}_{H'}\colon \lim_{L\in N(G)} M'_G(H_L, K_L) \to M'_G(H_{H'}, K_{H'})$ and $i_{H'}\colon H_{H'} \to K_{H'}$ is the canonical injection. Thus we get an injective map

$$\widetilde{\mathrm{pr}}_*\colon M'_G(H,K)/H' \to \big(\lim_{L\in N(G)} M'_{G_L}(H_L, K_L)\big)/\,\mathrm{pr}_{H'}^{-1}(i_{H'}),$$

where the right hand side is finite. Hence the left hand side is finite. Now we can deduce that $M'_G(H,K)/H$ is finite since it is a quotient of a finite set, as required. □

1.3 Modules over a Category

We introduce modules over a category and collect their main properties. These modules appear naturally if we want to study cellular homology in an equivariant setting (see Section 1.6).

In the following, let Γ be a small category and let R be a ring.

Definition 1.3.1. A covariant (contravariant) *$R\Gamma$-module* M is a covariant (contravariant) functor

$$M\colon \Gamma \to R\text{-}\mathsf{mod}.$$

A *morphism of $R\Gamma$-modules* is a natural transformation.

Convention 1.3.2. Let M be a set. We write RM or $R(M)$ for the freely generated R-module with basis M.

Remark 1.3.3. We can define im, ker, $\oplus$, $\prod$, lim, colim objectwise and $R\Gamma\text{-}\mathsf{mod}$ becomes an abelian category in this way. However, $R\Gamma\text{-}\mathsf{mod}$ is even more, it is a module category. The underlying (not necessarily unital) ring is

$$\tilde{R} = \bigoplus_{x,y\in \mathrm{Ob}(\Gamma)} R\,\mathrm{mor}(x,y), \qquad f\cdot g = \begin{cases} f\circ g, & \text{if } f\colon y\to z \text{ and } g\colon x\to y,\\ 0, & \text{otherwise,}\end{cases}$$

and the natural equivalence is given by

$$F\colon R\Gamma\text{-}\mathsf{mod} \to \tilde{R}\text{-}\mathsf{mod}, \quad M \mapsto \bigoplus_{x\in\mathrm{Ob}(\Gamma)} M(x).$$

This is shown in [23, Sec. 3]. Unfortunately, the ring $\tilde{R}$ has a unit if and only if $\mathrm{Ob}(\Gamma)$ is finite, which is very rare. In particular, $\mathrm{Ob}(\Gamma)$ is infinite in our main example which is $\mathsf{Or}_{\mathcal{CO}}(G)$ for a non-discrete l-group G.

Definition 1.3.4. A Γ-set I is a functor

$$I\colon \Gamma \to \mathsf{sets}$$

and a *morphism of* Γ*-sets* is a natural transformation. Let B be a Γ-set and M be an $R\Gamma$-module such that $B \subseteq M$. Then M is called *free with basis* B if for every $R\Gamma$-module N and every Γ-map $\varphi\colon B \to N$ there exists a lift

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d, "\mathrm{inc}"'] & N \\ M \arrow[ur, dotted, "\tilde{\varphi}"'] & \end{tikzcd}$$

Remark 1.3.5. Let $x \in \mathrm{Ob}(\Gamma)$ and define

$$R\Gamma(?,x)\colon \Gamma \to R\text{-}\mathsf{mod}, \quad y \mapsto R\,\mathrm{mor}(x,y).$$

Then $R\Gamma(?,x)$ is a free $R\Gamma$-module with basis

$$B_x\colon \Gamma \to \mathsf{sets}, \quad y \mapsto \begin{cases} \{\,\mathrm{id}\colon x \to x\,\}, & y = x, \\ \emptyset, & y \neq x. \end{cases}$$

Furthermore, let F be a free $R\Gamma$-module with basis B. Then there exists an isomorphism

$$f\colon F \to \bigoplus_{x\in\mathrm{Ob}(\Gamma)} \bigoplus_{|B(x)|} R\Gamma(?,x).$$

In particular, we have a free $R\Gamma$-module with basis B for any Γ-set B. Every $R\Gamma$-module M is a quotient of a free module by

$$\mathrm{pr}\colon \bigoplus_{x\in\mathrm{Ob}(\Gamma)} \bigoplus_{|M(x)|} R\Gamma(?,x) \to M, \quad (\mathrm{id}\colon x \to x, m_x) \mapsto m_x \in M(x).$$

Example 1.3.6. Let $\Gamma = \mathsf{Sub}_{\mathcal{CO}}(\mathbb{Z}_p)$ (see Example 1.2.14). Then we obtain

$$\begin{aligned} &R\mathsf{Sub}_{\mathcal{CO}}(G)(p^k\mathbb{Z}_p,?) = \cdots \to 0 \to R_k \to \cdots \to R_2 \xrightarrow{\mathrm{id}} R_1 \xrightarrow{\mathrm{id}} R_0 \quad \text{and} \\ &R\mathsf{Sub}_{\mathcal{CO}}(G)(?,p^k\mathbb{Z}_p) = 0 \to \cdots \to 0 \to R_k \xrightarrow{\mathrm{id}} R_{k+1} \to \cdots, \end{aligned}$$

where $R_k = R$ and the lower index just indicates the position.

Definition 1.3.7. Let M be a contravariant $R\Gamma$-module and N a covariant $R\Gamma$-module. Then we define the R-module $M \otimes_{R\Gamma} N$, which we call the *tensor product of* M *and* N, by

$$M \otimes_{R\Gamma} N = \Big(\bigoplus_{x\in\mathrm{Ob}(\Gamma)} M(x) \otimes_R N(x) \Big)/Q,$$

where the R-module Q is given by

$$Q = \langle \{\, mf \otimes n - m \otimes fn \mid m \in M(y), n \in N(x), f \in \mathrm{mor}_\Gamma(x,y), x,y \in \mathrm{Ob}(\Gamma)\,\} \rangle .$$

Here we set $mf \coloneqq M(f)(m)$ and $fn \coloneqq N(f)(n)$.

As in the classical case of R-modules, this tensor product fulfills a universal property which can be expressed by the following isomorphisms:

Lemma 1.3.8. *Let M be a contravariant $R\Gamma$-module and N a covariant $R\Gamma$-module. Then, for any R-module L, there are natural isomorphisms*

$$\hom_R(M \otimes_{R\Gamma} N, L) \xrightarrow{\cong} \hom_{R\Gamma}(M, \hom_R(N, L)),$$
$$\hom_R(M \otimes_{R\Gamma} N, L) \xrightarrow{\cong} \hom_{R\Gamma}(N, \hom_R(M, L)).$$

As in the case of ordinary R-modules, we also have induction, coinduction and restriction functors for $R\Gamma$-modules:

Definition 1.3.9. Let $F\colon \Gamma_1 \to \Gamma_2$ be a contravariant functor and M a contravariant $R\Gamma_2$-module. The *restriction by F* is the following $R\Gamma_1$-module:

$$\operatorname{res}_F(M) = M \circ F.$$

Let M be a contravariant $R\Gamma_1$-module. The *induction by F* and *coinduction by F* are given by

$$\begin{aligned} \operatorname{ind}_F(M)(??) &= M(?) \otimes_{R\Gamma_1} R\operatorname{mor}(??, F(?)) \quad \text{and} \\ \operatorname{coind}_F(M)(??) &= \hom_{R\Gamma_1}\big(R\operatorname{mor}\big(F(?), ??\big), M\big). \end{aligned}$$

In the same way, we can define restriction, induction and coinduction for covariant modules.

Definition 1.3.10. Let $F\colon \Gamma_1 \to \Gamma_2$ and $G\colon \Gamma_2 \to \Gamma_1$ be two functors. We say (F, G) is a *tensor adjoint pair of functors* if we can construct a natural isomorphism

$$F(M) \otimes_{\Gamma_2} N \xrightarrow{\cong} M \otimes_{\Gamma_1} G(N)$$

for every $M \in \operatorname{Ob}(\Gamma_1)$ and $N \in \operatorname{Ob}(\Gamma_2)$. We say (F, G) is an *adjoint pair of functors* if there exists a natural isomorphism

$$\operatorname{mor}_{\Gamma_2}(F(M), N) \xrightarrow{\cong} \operatorname{mor}_{\Gamma_1}(M, G(N))$$

for every $M \in \operatorname{Ob}(\Gamma_1)$ and $N \in \operatorname{Ob}(\Gamma_2)$.

Proposition 1.3.11. *Let $F\colon \Gamma_1 \to \Gamma_2$ be a functor. Then $(\operatorname{ind}_F, \operatorname{res}_F)$ is a tensor adjoint pair of functors. Furthermore, $(\operatorname{ind}_F, \operatorname{res}_F)$ and $(\operatorname{res}_F, \operatorname{coind}_F)$ are adjoint pairs of functors.*

Proof. The isomorphisms are given by

$$\begin{aligned} \hom_{\Gamma_2}(\operatorname{ind}_F(M), N) &\xrightarrow{\cong} \hom_{\Gamma_1}(M, \operatorname{res}_F N) \\ G &\mapsto (m \mapsto G(m \otimes \mathrm{id})) \\ \left((m \otimes \sum_{i=1}^k \lambda_i h_i) \mapsto \sum_{i=1}^k \lambda_i h_i g(m)\right) &\mapsfrom g \end{aligned}$$

and

$$\begin{aligned}
\hom_{\Gamma_2}(\operatorname{res}_F(M), N) &\xrightarrow{\cong} \hom_{\Gamma_1}(M, \operatorname{coind}_F N)\\
G &\mapsto \Big(m_y \mapsto \big(\sum_{i=1}^{n} \lambda_i(f_i\colon x_i \to y) \mapsto \sum_{i=1}^{n} \lambda_i\big(G \circ M(f_i)\big)(m_y)\big)\Big)\\
\big(m_x \mapsto g(m_x)(\mathrm{id}_x)\big) &\leftarrow\!\shortmid g
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{ind}_F(M) \otimes_{R\Gamma_2} N &\xrightarrow{\cong} M \otimes_{R\Gamma_2} \operatorname{res}_F N\\
\Big(m \otimes \sum_{i=1}^{k} \lambda_i G_i\Big) \otimes n &\mapsto m \otimes \big(\sum_{i=1}^{k} \lambda_i N(G_i)(m)\big),\\
(m \otimes \mathrm{id}) \otimes n &\leftarrow\!\shortmid m \otimes n.
\end{aligned}$$

□

Lemma 1.3.12. *The following statements hold:*

(i) Let (F, G) be a pair of tensor adjoint functors. Then F respects the property flat if G is exact.

(ii) Let (F, G) be a pair of adjoint functors. Then F respects the property projective if G is exact. Furthermore, G respects the property injective if F is exact.

Proof. We just prove the first assertion, the others can be proven analogously. So let

$$F\colon \Gamma_1 \to \Gamma_2 \quad \text{and} \quad G : \Gamma_2 \to \Gamma_1$$

be the functors from above. Let L be a flat module in Γ_1 and $0 \to M \to N$ an exact sequence in Γ_2. Then we get the following commutative diagram:

$$\begin{tikzcd}
 & \operatorname{ind}_F L \otimes_{\Gamma_2} M \arrow[r] \arrow[d, "\cong"] & \operatorname{ind}_F L \otimes_{\Gamma_2} N \arrow[d, "\cong"] \\
0 \arrow[r] & L \otimes_{\Gamma_1} \operatorname{res}_F M \arrow[r] & L \otimes_{\Gamma_1} \operatorname{res}_F N
\end{tikzcd}$$

Consequently, $\operatorname{ind}_F L$ is flat. □

Corollary 1.3.13. *Let F be a functor. Then the following statements hold:*

(i) The functor ind_F *respects the properties flat and projective.*

(ii) The functor coind_F *respects the property injective.*

(iii) The functor res_F *respects the properties flat and injective if* ind_F *is exact.*

(iv) The functor res_F *respects the property projective if* coind_F *is exact.*

At the end of this section we introduce Mackey functors which are modules over a category with some extra structure. They shall be viewed as a generalization of representation theory.

Let TGFI be the following category. The objects are topological groups and the morphisms are open embeddings $f\colon H \to G$ whose images have finite index, i.e., $f\colon H \to \operatorname{im}(f)$ is an isomorphism of topological groups and $\operatorname{im}(f)$ is an open subgroup of G. Let $\Gamma \subseteq$ TGFI be a small subcategory. Let $M\colon \Gamma \to R\text{-}\mathtt{mod}$ be a bifunctor to the category of R-modules, i.e., a pair (M_*, M^*) consisting of a covariant functor M_* and a contravariant functor M^* from Γ to $R\text{-}\mathtt{mod}$ which agree on objects. For $f \in \operatorname{mor}_\Gamma(H,G)$ we will often denote $M_*(f)\colon M(H) \to M(G)$ by ind_f and the map $M^*(f)\colon M(G) \to M(H)$ by res_f and write $\operatorname{ind}_H^G = \operatorname{ind}_f$ and $\operatorname{res}_G^H = \operatorname{res}_f$ if f is an inclusion of groups. We call such a bifunctor M a *Mackey functor* with values in R-modules if the following conditions are satisfied:

(i) Let $G \in \operatorname{Ob}(\Gamma)$ and $c(g)\colon G \to G$ be an inner automorphism. Then $c(g) \in \operatorname{mor}_\Gamma(G,G)$ and $M_*(c(g)) = \operatorname{id}\colon M(G) \to M(G)$.

(ii) Let $H, K \in \operatorname{Ob}(\Gamma)$ and $f\colon H \xrightarrow{\cong} K$ be an isomorphism of topological groups such that $f \in \operatorname{mor}_\Gamma(H,K)$. Then the compositions $\operatorname{res}_f \circ \operatorname{ind}_f$ and $\operatorname{ind}_f \circ \operatorname{res}_f$ are the identities.

(iii) *Double coset formula*
Let $H, K \subseteq G$ be two open subgroups of finite index such that $H, K, G \in \operatorname{Ob}(\Gamma)$. Then we obtain the identity

$$\operatorname{res}_G^K \circ \operatorname{ind}_H^G \;=\; \sum_{KgH \in K\backslash G/H} \operatorname{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \operatorname{res}_H^{H\cap g^{-1}Kg},$$

where $c(g)$ is conjugation with g, i.e., $c(g)(h) = ghg^{-1}$. In particular, each item on the right hand side is defined.

The classical example is

Example 1.3.14. Let $\Gamma \subseteq$ TGFI be the full subcategory of finite groups. This category is not small but $\operatorname{Is}\Gamma$ is small and that is what we really need. For a finite group G, we denote by $R_k(G)$ the representation ring corresponding to a field k. Then

$$R_k\colon \Gamma \to \mathbb{Z}\text{-}\mathtt{mod}, \quad H \mapsto R_k(H)$$

induces a Mackey functor, the covariant structure being given by the usual induction and the contravariant structure being given by the usual restriction.

1.4 G-CW-Complexes

In this section, we introduce G-CW-complexes for topological groups and collect some basic properties. At the end, we introduce classifying spaces, which are an important class of examples of G-CW-complexes.

In the following, let G be a topological group and $\mathcal{F}$ a family of subgroups. We remark that in the following, $\mathcal{F}$ need not be smooth.

Definition 1.4.1. A (left) $(G,\mathcal{F})$-CW-pair is a pair of (left) G-spaces (X,A) together with a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X$$

such that $\operatorname{colim}_{n\to\infty} X_n = X$ and for every $n \in \mathbb{N}$ there exists a family

$$\{ H_i \mid H_i \in \mathcal{F},\ i \in I_n \}$$

together with the following pushout (in the category of (left) G-spaces)

$$\begin{tikzcd} \coprod_{i\in I_n} G/H_i \times S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \coprod_{i\in I_n} G/H_i \times D^n \arrow[r] & X_n \end{tikzcd}$$

We call (X,A) *finite relative* A (or briefly *finite*) if there exist pushouts such that only finitely many I_n are not empty and these I_n are finite. Furthermore, we call (X,A)

(i) *proper* if $\mathcal{F} \subseteq \mathcal{COP}$ holds,

(ii) *coproper* if $\mathcal{F} \subseteq \mathcal{COC}$ holds,

(iii) *cofinite* if $\mathcal{F} \subseteq \mathcal{I}$ holds,

(iv) *smooth* if $\mathcal{F} \subseteq \mathcal{O}$ holds.

If $A = \emptyset$ we call X a $(G,\mathcal{F})$-*CW-complex*.

Remark 1.4.2. If (X,A) is smooth, then (X,A) is coproper if and only if (X,A) is cofinite.

Remark 1.4.3. Let X be a $(G,\mathcal{F})$-CW-complex. If $\mathcal{F}$ is smooth, then X is also an ordinary CW-complex. If $\mathcal{F}$ is not smooth, X might not be a CW-complex. An immediate example is $X = G$. This is a $(G,\{1\})$-CW-complex but G might not be a CW-complex (e.g., G is a non-discrete l-group).

Definition 1.4.4. A *homomorphism of topological groups* or sometimes briefly *group homomorphism* $\alpha\colon H \to G$ is a continuous map that is compatible with the group structures, $\operatorname{im}(\alpha) \subseteq G$ is a closed subgroup and α induces an identification $H \to \operatorname{im}(\alpha)$. The latter condition implies that $H \to \operatorname{im}(\alpha)$ is open. If α is injective, it is called *closed embedding*. Moreover, an embedding $\alpha\colon H \to G$ is called *compact, open, etc.* if $\alpha(H) \subseteq G$ has the corresponding property.

Definition 1.4.5. Let $\alpha\colon H \to G$ be a homomorphism of topological groups. Then we obtain an induction functor

$$\operatorname{ind}_\alpha\colon G\text{-}\mathtt{pairs} \to G\text{-}\mathtt{pairs}, \quad (X, A) \mapsto (G \times_\alpha X, G \times_\alpha A),$$

where a G-pair (X, A) is a G-space X together with a G-subspace A.

Lemma 1.4.6. *Let $\alpha\colon H \to G$ be a homomorphism of topological groups. Let X be an H-CW-complex. Suppose that for every isotropy group K of X the subgroup $\ker(\alpha) \cdot K = \{ h \cdot k \mid h \in \ker(\alpha), k \in K \} \subseteq H$ is closed. (This assumption is for instance satisfied if $\ker(\alpha)$ is compact, if $\ker(\alpha)$ is open, if $\ker(\alpha)$ is trivial, if X is smooth or if X is proper.)*

Then the G-space $\operatorname{ind}_\alpha X = G \times_\alpha X$ obtained by induction with α inherits a G-CW-complex structure. If X is free or smooth or proper, $\operatorname{ind}_\alpha X$ has the same property.

Proof. If X_n denotes the n-skeleton of X, the desired G-CW-complex structure on $\operatorname{ind}_\alpha X$ has as n-th skeleton $\operatorname{ind}_\alpha X_n$. Since we are working in the category of compactly generated spaces, $\operatorname{ind}_\alpha X$ carries the colimit topology with respect to the filtration $\{\operatorname{ind}_\alpha X_n \mid n \geq -1\}$ as the analogous statement is true for X and the filtration $\{X_n \mid n \geq -1\}$. Moreover, $\operatorname{ind}_\alpha$ sends an H-pushout to a G-pushout. It remains to check for a homogeneous space H/K for which $\ker(\alpha)$ acts freely on H/K that $\alpha(K)$ is a closed subgroup of G and that there is a G-homeomorphism $f\colon G \times_\alpha H/K \to G/\alpha(K)$. Recall that, by assumption, the map α induces an identification $H \to \operatorname{im}(\alpha)$ and $\operatorname{im}(\alpha) \subseteq G$ is closed. Since $\alpha^{-1}(\alpha(K)) = \ker(\alpha) \cdot K \subseteq H$ is closed for an isotropy group $K \subseteq H$, the subgroup $\alpha(K) \subseteq \operatorname{im}(\alpha)$ is closed and hence the subgroup $\alpha(K) \subseteq G$ is closed. The desired G-homeomorphism f sends the class of (g, hK) to $g\alpha(h)\alpha(K)$. It is a homeomorphism, because of the following commutative diagram, in which the top horizontal arrow is an identification sending (g, h) to $g \cdot \alpha(h)$ and in which the vertical arrows are identifications as well:

$$\begin{tikzcd}
G \times H \arrow[r] \arrow[d] & G \arrow[d] \\
G \times_\alpha H/K \arrow[r] & G/\alpha(K)
\end{tikzcd}$$

□

Remark 1.4.7 (Homomorphisms of topological groups). The assumptions, we impose on homomorphisms of topological groups rule out for the map $i\colon G_d \to G$ given by the identity map for topological groups G and G_d, where the latter one is G endowed with the discrete topology, unless G itself is discrete. This is because i does not in general send closed subgroups to closed subgroups.

Consider the projection $\operatorname{pr}\colon \mathbb{R} \times S^1 \to S^1$, which is a homomorphism of topological groups. Let $K \subseteq \mathbb{R} \times S^1$ be the subgroup $\{(n, \exp(2\pi i\theta n)) \mid n \in \mathbb{Z}\}$ for some irrational number $\theta \in \mathbb{R}$. Although $K \subseteq \mathbb{R} \times S^1$ is closed, its image in S^1 is the non-closed subgroup $\{\exp(2\pi i\theta n) \mid n \in \mathbb{Z}\}$. Lemma 1.4.6 does not apply to the $\mathbb{R} \times S^1$-CW-complex $\mathbb{R} \times S^1/K$ since K does not satisfy the required assumption.

At the end of this section, we want to introduce some $(G, \mathcal{F})$-CW-complexes of special interest.

A *classifying space for* $(G, \mathcal{F})$ is a $(G, \mathcal{F})$-CW-complex $E_{\mathcal{F}}G$ such that for any other $(G, \mathcal{F})$-CW-complex X there exists a G-map $f\colon X \to E_{\mathcal{F}}G$ which is unique up to G-homotopy. Equivalently, we can describe $E_{\mathcal{F}}G$ by demanding that for each $H \in \mathcal{F}$ the set of H-fixed points $(E_{\mathcal{F}}G)^H$ is weakly contractible. If $\mathcal{F}$ is smooth, $(E_{\mathcal{F}}G)^H$ is a CW-complex and the property weakly contractible is equivalent to the property contractible. Recall that the set of H-fixed points is defined by $X^H = \{ x \in X \mid hx = x,\ \forall h \in H \}$ for a G-CW-complex X and a subgroup $H \subseteq G$.

A survey on classifying spaces is given by Lück [32]. In particular, we have the following result [32, Thm. 1.9]:

Theorem 1.4.8. *There exists a classifying space $E_{\mathcal{F}}G$ for any $(G, \mathcal{F})$.*

Example 1.4.9. We have the following examples:

- Let $\mathcal{F}$ be a family of subgroups with $G \in \mathcal{F}$, e.g., $\mathcal{F} = \mathcal{CL}$, $\mathcal{O}$, $\mathcal{I}$. Then we can take $E_{\mathcal{F}}G = \{\bullet\}$.

- Let $\{1\}$ be the family which consists only of the trivial group. Then we briefly write $EG = E_{\{1\}}G$ and call it the *classifying space of* G.

  Let G be discrete. Then $EG = \widetilde{BG}$, where $\widetilde{BG}$ denotes the universal covering space of the Eilenberg-MacLane space BG of G.

- If $\mathcal{F} = \mathcal{COP}$, we define $\underline{E}G = E_{\mathcal{COP}}G$ and call it *the classifying space of proper actions*. If G happens to be a reductive p-adic group, we have the identity $\underline{E}G = E_{\mathcal{CO}}G$ by Lück [32, Thm. 4.13].

1.5 Equivariant Homology Theories

In order to study G-CW-complexes, we must introduce homology theories which take the G-action into account. This leads to G-homology theories, which are introduced in this section. Usually, constructions for G-homology theories do not yield a G-homology theory only for a fixed group G but for a large class of groups (e.g., see Chapter 2). Therefore it will be convenient to compare G-homology theories for different groups G. This leads to equivariant homology theories, which are also introduced in this section and will play an important role in the sequel.

Let R be a commutative ring. We point out that in the following, $\mathcal{F}$ need not be smooth.

Definition 1.5.1. A $(G, \mathcal{F})$*-homology theory* $\mathcal{H}_*^G$ *with values in* R*-modules* is a family of covariant functors

$$\mathcal{H}_n^G\colon (G, \mathcal{F})\text{-}\mathsf{CW\text{-}pairs} \to R\text{-}\mathsf{mod}, \quad n \in \mathbb{Z},$$

together with natural transformations

$$\delta_n(X,A)\colon \mathcal{H}_n^G(X,A) \to \mathcal{H}_{n-1}^G(A)(= \mathcal{H}_{n-1}^G(A,\emptyset))$$

such that the following hold for every $n \in \mathbb{Z}$:

(i) *G-homotopy invariance*
Let $f, g\colon (X,A) \to (Y,B)$ be two G-homotopic maps of $(G,\mathcal{F})$-CW-pairs, then $\mathcal{H}_n^G(f) = \mathcal{H}_n^G(g)$.

(ii) *Long exact sequence*
For every $(G,\mathcal{F})$-CW-pair (X,A) there exists a long exact sequence

$$\cdots \xrightarrow{\mathcal{H}_{n+1}^G(j)} \mathcal{H}_{n+1}^G(X,A) \xrightarrow{\delta_{n+1}^G} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X,A) \xrightarrow{\delta_n^G} \cdots,$$

where $i\colon A \hookrightarrow X$ and $j\colon X \hookrightarrow (X,A)$ are the canonical inclusions.

(iii) *Excision*
Let

$$\begin{tikzcd} X_0 \arrow[r, "i_1"] \arrow[d, "i_2"] & X_1 \arrow[d, "j_1"] \\ X_2 \arrow[r, "j_2"] & X \end{tikzcd}$$

be a G-pushout such that $i_1\colon X_0 \to X_1$ is an inclusion of $(G,\mathcal{F})$-CW-pairs and $i_2\colon X_0 \to X_2$ is cellular and the $(G,\mathcal{F})$-CW-structure of X is induced by the $(G,\mathcal{F})$-CW-structures of X_0, X_1, X_2 in the obvious way. Then we obtain an isomorphism

$$\mathcal{H}_n^G(j_1)\colon \mathcal{H}_n^G(X_1,X_0) \xrightarrow{\cong} \mathcal{H}_n^G(X,X_2).$$

(iv) *Disjoint union axiom*
Let $\{\, X_i \mid i \in I \,\}$ be a family of $(G,\mathcal{F})$-CW-complexes and let $j_i\colon X_i \hookrightarrow \coprod_{i\in I} X_i$ be the canonical inclusion. Then we get an isomorphism

$$\bigoplus_{i\in I} \mathcal{H}_n^G(j_i)\colon \bigoplus_{i\in I} \mathcal{H}_n^G(X_i) \xrightarrow{\cong} \mathcal{H}_n^G\Big(\coprod_{i\in I} X_i\Big).$$

As above, we call $\mathcal{H}_*^G$ *proper*, *coproper*, or *smooth* if the corresponding families are the ones as above.

Definition 1.5.2. A $(G,\mathcal{F})$-*cohomology theory $\mathcal{H}_G^*$ with values in R-modules* is a family of contravariant functors

$$\mathcal{H}_G^n\colon (G,\mathcal{F})\text{-}\mathsf{CW}\text{-}\mathsf{pairs} \to R\text{-}\mathsf{mod}, \quad n \in \mathbb{Z},$$

together with natural transformations

$$\delta^n(X,A)\colon \mathcal{H}_G^n(X,A) \to \mathcal{H}_G^{n+1}(A) \coloneqq \mathcal{H}_G^{n+1}(A,\emptyset)$$

such that the properties (except the disjoint union axiom) of the above definition hold (invert arrows where appropriate).

Remark 1.5.3. Even if $\mathcal{H}_G^*$ satisfies the disjoint union axiom, the G-cohomology theory $\mathcal{H}_G^* \otimes_{\mathbb{Z}} \mathbb{Q}$ fails to satisfy this axiom in general. This is the reason why we do not require the disjoint union axiom to hold.

Theorem 1.5.4. *(i) Let $T\colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be a natural transformation of two $(G,\mathcal{F})$-homology theories. If we have isomorphisms*

$$T(G/H)\colon \mathcal{H}_*^G(G/H) \xrightarrow{\cong} \mathcal{K}_*^G(G/H)$$

for every $H \in \mathcal{F}$, then $T(X,A)$ is an isomorphism for every $(G,\mathcal{F})$-CW-pair (X,A).

(ii) Let $T\colon \mathcal{H}_G^* \to \mathcal{K}_G^*$ be a natural transformation of two $(G,\mathcal{F})$-cohomology theories. If we have isomorphisms

$$T(G/H)\colon \mathcal{H}_G^*(G/H) \xrightarrow{\cong} \mathcal{K}_G^*(G/H)$$

for every $H \in \mathcal{F}$, then $T(X,A)$ is an isomorphism for every finite $(G,\mathcal{F})$-CW-pair (X,A). In addition, we obtain isomorphisms for every $(G,\mathcal{F})$-CW-pair (X,A) if $\mathcal{H}_G^*$ and $\mathcal{K}_G^*$ satisfy the disjoint union axiom.

Proof. Although this statement is called theorem, the proof is very simple. For simplicity let $A = \emptyset$. Then the assertion follows by induction on n, the long exact sequence for (X_n, X_{n-1}) and the 5-Lemma. Note that the disjoint union axiom forces $\mathcal{H}_*^G$ to be compatible with colimits. In the cohomological case, the non-finite case follows from [51, Prop. 7.66]. □

In the sequel, we have to compare families of subgroups for different groups. Thus we must introduce some compatibility conditions for those families. Before we can make this precise, let Γ be a subcategory of the category of topological groups which is closed under isomorphisms and taking finite intersections.

Definition 1.5.5. A Γ-*collection of families of subgroups* $\mathcal{F}_?$ is a collection of families of subgroups $(\mathcal{F}_G)_{G \in \mathrm{Ob}(\Gamma)}$ such that for every injective homomorphism of topological groups $\alpha\colon H \to G$ with $H, G \in \mathrm{Ob}(\Gamma)$ the following properties are satisfied:

(i) $\mathrm{ind}_\alpha(X,A)$ is a $(G,\mathcal{F}_G)$-CW-pair for every $(H,\mathcal{F}_H)$-CW-pair (X,A).

(ii) $\alpha(H) \in \mathcal{F}_{\alpha(H)}$ holds.

We call a Γ-collection of families of subgroups $\mathcal{F}_?$ *smooth* if for every $G \in \mathrm{Ob}(\Gamma)$ the family $\mathcal{F}_G$ is smooth.

Remark 1.5.6. The second property is required for the construction of our Chern character (cf. (4.1.0.2)). Note that an immediate consequence of the first property is that $\alpha(\mathcal{F}_H) \subseteq \mathcal{F}_G$, where we set $\alpha(\mathcal{F}_H) = \{\, \alpha(H') \mid H' \in \mathcal{F}_H \,\}$.

Example 1.5.7. Let Γ be the category of topological groups. Then the family $\{1\}$ and the families

$$\begin{aligned} \mathcal{CL} &= \{\, H \subseteq G \text{ closed} \,\} & \mathcal{COP} &= \{\, H \subseteq G \text{ compact} \,\} \\ \mathcal{COC} &= \{\, H \subseteq G \mid G/H \text{ compact} \,\} & \mathcal{O} &= \{\, H \subseteq G \text{ open} \,\} \\ \mathcal{CO} &= \{\, H \subseteq G \text{ compact and open} \,\} & \mathcal{I} &= \{\, H \subseteq G \text{ open} \mid G/H \text{ finite} \,\} \end{aligned}$$

of Example 1.2.2 induce Γ-collections of families of subgroups.

Definition 1.5.8. Let $\mathcal{F}_?$ be a Γ-collection of families of subgroups. A $(\Gamma, \mathcal{F}_?)$-*equivariant homology theory with values in R-modules* assigns to every topological group $G \in \mathrm{Ob}(\Gamma)$ a $(G, \mathcal{F}_G)$-homology theory $\mathcal{H}_*^G$ and comes with a so-called *induction structure*: For every injective homomorphism $\alpha\colon H \to G$ of topological groups, every $(H, \mathcal{F}_H)$-CW-pair (X, A) and $n \in \mathbb{Z}$ there exists an isomorphism

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(\mathrm{ind}_f(X, A))$$

such that the following axioms are satisfied:

(i) *Compatibility with the boundary homomorphisms*
$\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^H$;

(ii) *Functoriality*
Let $\beta\colon G \to K$ be another injective homomorphism of topological groups. Then we have for $n \in \mathbb{Z}$

$$\mathrm{ind}_{\beta\circ\alpha} \;=\; \mathcal{H}_n^K(f_1) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^K(\mathrm{ind}_{\beta\circ\alpha}(X, A)),$$

where $f_1\colon \mathrm{ind}_\beta \mathrm{ind}_\alpha(X, A) \xrightarrow{\cong} \mathrm{ind}_{\beta\circ\alpha}(X, A)$, $(k, g, x) \mapsto (k\beta(g), x)$ is the natural K-homeomorphism;

(iii) *Compatibility with conjugation*
For $n \in \mathbb{Z}$, $g \in G$ and a $(G, \mathcal{F}_G)$-CW-pair (X, A) we have the identity

$$\mathcal{H}_n^G(f_2) = \mathrm{ind}_{c(g)\colon G\to G}\colon \mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\mathrm{ind}_{c(g)\colon G\to G}(X, A)),$$

where $f_2\colon (X, A) \to \mathrm{ind}_{c(g)\colon G\to G}(X, A)$ is the G-homeomorphism which sends x to $(1, g^{-1}x)$ in $G \times_{c(g)} (X, A)$.

As above, we call $\mathcal{H}_*^?$ *proper*, *coproper* or *smooth* if the corresponding families are the ones as above.

We will usually omit Γ in the notation because it will usually be clear from the context which Γ to take.

Remark 1.5.9. For the construction of the equivariant Chern characters appearing in Chapter 4, we do not need the more general setting which appears in the literature [29, Sec. 1], where one requires that such an induction structure exists for (not necessarily injective) homomorphism of topological groups $\alpha\colon H \to G$.

Definition 1.5.10. Let $\mathcal{H}^?_*$ be an equivariant $\mathcal{F}_?$-homology theory and let G be a group. Then $\mathcal{H}^G_*$ admits a *Mackey functor on coefficients* if for all $q \in \mathbb{Z}$ the covariant functor

$$\mathcal{H}^?_q \colon \mathcal{F}_G \to R\text{-}\mathtt{mod}, \quad H \mapsto \mathcal{H}^H_q(\{\bullet\})$$

is a Mackey functor. Here, we consider $\mathcal{F}_G \subseteq \mathrm{TGFI}$ as a category in the obvious way. Furthermore, the covariant functor $\mathcal{H}^?_q$ is required to send a morphism $\alpha\colon H \to K$ to the composition

$$\mathcal{H}^H_q(\{\bullet\}) \xrightarrow{\mathrm{ind}_f} \mathcal{H}^K_q(\mathrm{ind}_\alpha\{\bullet\}) \xrightarrow{\mathcal{H}^K_q(\mathrm{pr})} \mathcal{H}^K_q(\{\bullet\}),$$

where $\mathrm{pr}\colon \mathrm{ind}_f\{\bullet\} \to \{\bullet\}$ is the obvious K-map.

Analogously, we define the notion of a Mackey functor in the cohomological case.

Lemma 1.5.11. *Let $\mathcal{H}^?_*$ be an equivariant $(\Gamma, \mathcal{F}_?)$-homology theory and $G \in \mathrm{Ob}(\Gamma)$. We consider subgroups $H, K \in \mathcal{F}_G$ and an element $g \in G$ with $gHg^{-1} \subseteq K$. Let $R_{g^{-1}}\colon G/H \to G/K$ be the G-map sending $g'H$ to $g'g^{-1}K$ and $c(g)\colon H \to K$ be the homomorphism of topological groups sending h to ghg^{-1}. Let* $\mathrm{pr}\colon (\mathrm{ind}_{c(g)\colon H\to K}\{\bullet\}) \to \{\bullet\}$ *be the projection. Then the following diagram commutes*

$$\begin{tikzcd}
\mathcal{H}^H_n(\{\bullet\}) \arrow[r, "\mathcal{H}^K_n(\mathrm{pr})\circ \mathrm{ind}_{c(g)}"] \arrow[d, "\cong"', "\mathrm{ind}^G_H"] & \mathcal{H}^K_n(\{\bullet\}) \arrow[d, "\cong"', "\mathrm{ind}^G_K"] \\
\mathcal{H}^G_n(G/H) \arrow[r, "\mathcal{H}^G_n(R_{g^{-1}})"] & \mathcal{H}^G_n(G/K)
\end{tikzcd}$$

An analogous statement is true in the cohomological case.

Proof. We define a bijective G-map

$$f_1\colon \mathrm{ind}_{c(g)\colon G\to G}\,\mathrm{ind}^G_H\{\bullet\} \to \mathrm{ind}^G_K\,\mathrm{ind}_{c(g)\colon H\to K}\{\bullet\}$$

by sending $(g_1, g_2, *) \in G \times_{c(g)} G \times_H \{\bullet\}$ to $(g_1 g g_2 g^{-1}, 1, *) \in G \times_K K \times_{c(g)} \{\bullet\}$. The condition that induction is compatible with composition of homomorphisms of topological groups implies that the composition

$$\mathcal{H}^H_n(\{\bullet\}) \xrightarrow{\mathrm{ind}^G_H} \mathcal{H}^G_n(\mathrm{ind}^G_H\{\bullet\}) \xrightarrow{\mathrm{ind}_{c(g)\colon G\to G}} \mathcal{H}^G_n(\mathrm{ind}_{c(g)\colon G\to G}\,\mathrm{ind}^G_H\{\bullet\})$$
$$\xrightarrow{\mathcal{H}^G_n(f_1)} \mathcal{H}^G_n(\mathrm{ind}^G_K\,\mathrm{ind}_{c(g)\colon H\to K}\{\bullet\})$$

agrees with the composition

$$\mathcal{H}^H_n(\{\bullet\}) \xrightarrow{\mathrm{ind}_{c(g)\colon H\to K}} \mathcal{H}^K_n(\mathrm{ind}_{c(g)\colon H\to K}\{\bullet\}) \xrightarrow{\mathrm{ind}^G_K} \mathcal{H}^G_n(\mathrm{ind}^G_K\,\mathrm{ind}_{c(g)\colon H\to K}\{\bullet\}).$$

Naturality of induction implies $\mathcal{H}^G_n(\mathrm{ind}^G_K \mathrm{pr}) \circ \mathrm{ind}^G_K = \mathrm{ind}^G_K \circ \mathcal{H}^K_n(\mathrm{pr})$. Hence the following diagram commutes

$$\begin{tikzcd}
\mathcal{H}^H_n(\{\bullet\}) \arrow[rr, "\mathcal{H}^K_n(\mathrm{pr})\circ \mathrm{ind}_{c(g)\colon H\to K}"] \arrow[d, "\mathrm{ind}^G_H"] & & \mathcal{H}^K_n(\{\bullet\}) \arrow[d, "\mathrm{ind}^G_K"] \\
\mathcal{H}^G_n(G/H)) \arrow[rr, "\mathcal{H}^G_n(\mathrm{ind}^G_K \mathrm{pr})\circ \mathcal{H}^G_n(f_1)\circ \mathrm{ind}_{c(g)\colon G\to G}"] & & \mathcal{H}^G_n(G/K)
\end{tikzcd}$$

By the axioms, the map $\mathrm{ind}_{c(g)\colon G\to G}\colon \mathcal{H}^G_n(G/H) \to \mathcal{H}^G_n(\mathrm{ind}_{c(g)\colon G\to G} G/H)$ agrees with $\mathcal{H}^G_n(f_2)$ for the map $f_2\colon G/H \to \mathrm{ind}_{c(g)\colon G\to G} G/H$ which sends $g'H$ to $(g'g^{-1}, 1H)$ in $G \times_{c(g)} G/H$. Since the composition $(\mathrm{ind}^G_K \mathrm{pr}) \circ f_1 \circ f_2$ is just $R_{g^{-1}}$, the assertion follows. $\square$

1.6 The Associated Bredon Homology Theory

In this section, we want to find an appropriate substitute for the ordinary cellular homology theory.

In the following, a family of subgroups $\mathcal{F}$ is always supposed to be smooth.

Definition 1.6.1. Let (X, A) be a $(G, \mathcal{F})$-CW-pair. We obtain a contravariant functor

$$\mathrm{map}(?, X)^G\colon \mathsf{Or}_{\mathcal{F}}(G) \to G\text{-}\mathsf{CW\text{-}pairs}, \quad G/H \mapsto \mathrm{map}(G/H, (X, A))^G = (X^H, A^H),$$

which is called the *associated* $\mathsf{Or}_{\mathcal{F}}(G)$*-space*. Note that (X^H, A^H) is a CW-pair because (X, A) is smooth.

Definition 1.6.2. Let (X, A) be a $(G, \mathcal{F})$-CW-pair. We can define the following functors

$$\begin{aligned} C^{\mathsf{Or}_{\mathcal{F}}(G)}_n(X, A)\colon \mathsf{Or}_{\mathcal{F}}(G) &\longrightarrow \mathsf{CW\text{-}pairs} \longrightarrow R\text{-}\mathsf{chain\ complexes} \\ G/H &\longmapsto (X^H, A^H) \longmapsto C^{\mathrm{cell}}_n((X^H, A^H); R) \end{aligned}$$

where C^{cell}_n denotes the ordinary cellular chain complex. The functor $C^{\mathsf{Or}_{\mathcal{F}}(G)}_*(X, A)$ is a contravariant $R\mathsf{Or}_{\mathcal{F}}(G)$-chain complex. Let M be a covariant $R\mathsf{Or}_{\mathcal{F}}(G)$-module. Then the tensor product $C^{\mathsf{Or}_{\mathcal{F}}(G)}_*(X, A) \otimes M$ is an R-chain complex and we can define the *equivariant Bredon homology* by

$$H^{\mathsf{Or}_{\mathcal{F}}(G)}_n(X, A; M) := H_n(C^{\mathsf{Or}_{\mathcal{F}}(G)}_*(X, A) \otimes M).$$

Let $\mathcal{H}^G_*$ be a $(G, \mathcal{F})$-homology theory. Then the coefficient system

$$\mathcal{H}^G_*(G/H), \quad H \in \mathcal{F},$$

yields a covariant $R\mathsf{Or}_{\mathcal{F}}(G)$-module. Hence we can define the *associated Bredon homology* by

$$\mathcal{BH}^G_n(X, A) := H_n(C^{\mathsf{Or}_{\mathcal{F}}(G)}_*(X, A) \otimes \mathcal{H}^G_*(G/?)).$$

The cohomological case is analogous. For a contravariant $R\mathsf{Or}_{\mathcal{F}}(G)$-module M, we define

$$H^n_{\mathsf{Or}_{\mathcal{F}}(G)}(X,A;M) := H_n\big(\hom_{\mathsf{Or}_{\mathcal{F}}(G)}(C_*^{\mathsf{Or}_{\mathcal{F}}(G)}(X,A),M)\big)$$

and

$$\mathcal{BH}^n_G(X,A) := H_n\big(\hom_{\mathsf{Or}_{\mathcal{F}}(G)}(C_*^{\mathsf{Or}_{\mathcal{F}}(G)}(X,A),\mathcal{H}^*_G(G/?))\big).$$

Remark 1.6.3. Let (X,A) be a $(G,\mathcal{F})$-CW-pair and M be an $R\mathsf{Or}_{\mathcal{CO}}(G)$-module. Then we obtain

$$C_n^{\mathsf{Or}_{\mathcal{F}}(G)}(X,A) = \bigoplus_{\sigma\ G/H\text{-}n\text{-cell}} R\mathsf{Or}_{\mathcal{F}}(G)(G/?,G/H).$$

Consequently, for a graded $\mathsf{Or}_{\mathcal{F}}(G)$-module M_*, we can compute Bredon homology from the chain complex whose n-term is given by

$$\mathcal{B}H^G_n(X_n,X_{n-1};M_*) = \bigoplus_{\substack{\sigma\ G/H\text{-}k\text{-cell}\\ l+k=n}} \mathcal{B}H^G_l(G/H,M_*).$$

One of the main advantages of cellular homology is the existence of an Atiyah-Hirzebruch spectral sequence. The next theorem states the existence of an equivariant Atiyah-Hirzeburch spectral sequence for Bredon homology. This consolidates the view of Bredon homology as an equivariant generalization of cellular homology.

Theorem 1.6.4. *(i) Let $\mathcal{H}^G_*$ be a $(G,\mathcal{F})$-homology theory. Then there exists a converging Atiyah-Hirzebruch spectral sequence of the following form:*

$$E^2_{p,q} = H_p^{\mathsf{Or}_{\mathcal{F}}(G)}(X,A;\mathcal{H}^G_q(G/?)) \Rightarrow \mathcal{H}^G_{p+q}(X,A).$$

(ii) Let $\mathcal{H}^*_G$ be a $(G,\mathcal{F}_?)$-cohomology theory, then there exists a converging Atiyah-Hirzebruch spectral sequence of the following form:

$$E_2^{p,q} = H^p_{\mathsf{Or}_{\mathcal{F}}(G)}(X,A;\mathcal{H}^q_G(G/?)) \Rightarrow \mathcal{H}^{p+q}_G(X,A).$$

However, in this case, we have to restrict to $(G,\mathcal{F})$-CW-pairs (X,A) which are finite relative A.

Proof. The non-equivariant proof (see e.g., [58, Thm XIII.3.2 and Thm XIII.3.6]) carries over directly to G-(co)homology theories. □

At the end of this section, we want to prove that Bredon homology associated to an equivariant homology theory is itself an equivariant homology theory.

Let $\mathcal{H}^?_*$ be an equivariant $\mathcal{F}_?$-homology theory and consider the map

$$i\colon \mathsf{Or}_{\mathcal{F}_H}(H) \to \mathsf{Or}_{\mathcal{F}_G}(G), \quad H/K \mapsto G/K.$$

Here we need $\mathcal{F}_H \subseteq \mathcal{F}_G$, which is guaranteed by Remark 1.5.6. We have a natural isomorphism

$$\mathrm{ind}_H^G\big(C_*^{\mathsf{Or}_{\mathcal{F}_H}(G)}(X,A)\big) \xrightarrow{\cong} C_*^{\mathsf{Or}_{\mathcal{F}_G}(G)}\big(\mathrm{ind}_H^G(X,A)\big)$$

of $R\mathsf{Or}_{\mathcal{F}_G}(G)$-chain complexes. Since $(\operatorname{ind}_H^G, \operatorname{res}_G^H)$ is a tensor adjoint pair, there exists a natural isomorphism

$$\begin{aligned}\big(\operatorname{ind}_H^G C_*^{\mathsf{Or}_{\mathcal{F}_H}(H)}(X,A)\big) \otimes_{R\mathsf{Or}_{\mathcal{F}_G}(G)} \mathcal{H}_q^G(G/?) \\ \xrightarrow{\cong} C_*^{\mathsf{Or}_{\mathcal{F}_H}(H)}(X,A) \otimes_{R\mathsf{Or}_{\mathcal{F}_H}(H)} \operatorname{res}_G^H \mathcal{H}_q^G(G/?).\end{aligned}$$

The induction structure on $\mathcal{H}_*^G$ induces an isomorphism of $R\mathsf{Or}_{\mathcal{F}_H}(H)$-modules

$$\mathcal{H}_q^G(H/?) \xrightarrow{\cong} \operatorname{res}_G^H \mathcal{H}_q^G(G/?).$$

By assembling together the last three maps, we obtain a chain isomorphism

$$C_*^{\mathsf{Or}_{\mathcal{F}_H}(H)}(X,A) \otimes_{R\mathsf{Or}_{\mathcal{F}_H}(H)} \mathcal{H}_q^H(H/?) \xrightarrow{\cong} C_*^{\mathsf{Or}_{\mathcal{F}_G}(G)}\big(\operatorname{ind}_H^G(X,A)\big) \otimes_{R\mathsf{Or}_{\mathcal{F}_G}(G)} \mathcal{H}_q^G(G/?),$$

which induces the required isomorphism

$$\operatorname{ind}_H^G \colon \mathcal{BH}_*^H(X,A) \xrightarrow{\cong} \mathcal{BH}_*^G\big(\operatorname{ind}_H^G(X,A)\big).$$

In the same way, we can conclude that

$$\operatorname{ind}_H^G \colon \mathcal{BH}_G^*\big(\operatorname{ind}_H^G(X,A)\big) \xrightarrow{\cong} \mathcal{BH}_H^*(X,A).$$

1.7 Linear Algebraic Groups

First, we recall the definition of a linear algebraic group; the reader who is interested in details is advised to consult [10] and [49]. Then we prove for a semisimple p-adic group G that $\operatorname{mor}_{\mathsf{Or}_{\mathcal{CO}}(G)}(G/H, G/K)$ is a finite set. This is one of the ingredients for the construction of our Chern character.

Let k be a field and $I \subseteq k[X_1, \ldots, X_n]$ be a finitely generated ideal. It defines a functor

$$G \colon k\text{-}\mathsf{algebras} \to \mathsf{sets}, \quad R \mapsto \{\, (x_1, \ldots x_n) \in R^n \mid P(x_1, \ldots, x_n) = 0 \ \forall P \in I \,\}.$$

Furthermore, suppose that we have algebraic maps

$$\begin{aligned} \mu(k) \colon G(k) \times G(k) &\to G(k), \quad (x,y) \mapsto xy \\ i(k) \colon G(k) &\to G(k), \qquad\quad\ \ x \mapsto x^{-1} \end{aligned}$$

such that $G(k)$ becomes a group. By algebraic, we mean that there exist polynomials

$$f_1, \ldots, f_n \in k[X_1, \ldots, X_n, X_1', \ldots, X_n']$$

such that $\mu(k)$ is given by

$$\mu(k)(p) = \big(f_1(p), \ldots, f_n(p)\big) \quad \text{for } p \in G(k) \times G(k) \subseteq k^{2n}$$

and $i(k)$ can be described analogously. This construction defines a functor

$$G\colon k\text{-}\mathsf{algebras} \to \mathsf{groups}, \quad R \mapsto \{\,(x_1, \dots x_n) \in R^n \mid P(x_1, \dots, x_n) = 0\ \forall P \in I\,\},$$

which is called an *algebraic group*. In particular, we get the group $G(\bar{k})$ for an algebraic closure $k \subseteq \bar{k}$ and obtain $G(\bar{k}) \subseteq \bar{k}^n$ for some n by construction. We say G is *connected* if the group $G(\bar{k})$ endowed with the Zariski topology is connected. The reason why we pass to the algebraic closure is that the Zariski topology for a non-algebraically closed field is a bit odd. For instance, for any field F the group $GL_n(\bar{F})$ is connected but for a finite field F the group $GL_n(F) \subseteq F^m$ is not connected. We remark that $G(\bar{k})$ is not in general a topological group because $G(\bar{k})$ is T_0 but may fail to be T_2.

An example of an algebraic group is the *general linear group* which is defined by

$$GL_n\colon k\text{-}\mathsf{algebras} \to \mathsf{groups}, \quad R \mapsto GL_n(R).$$

An algebraic subgroup G of GL_n is called *linear algebraic*. Let R be a k-algebra and $H \subseteq GL_n(R)$ be a subgroup. We call H *unipotent* if $h-1 \in GL_n(R)$ is nilpotent for all $h \in H$. Further, we call a connected linear algebraic group G a *reductive group* if $G(\bar{k})$ does not contain any non-trivial connected normal unipotent subgroup. Moreover, we call G a *semisimple group* if $G(\bar{k})$ does not contain any non-trivial connected normal solvable subgroup. We remark that a semisimple group is reductive because unipotent matrix groups are solvable (and even nilpotent) [22, 17.5 Cor.].

In the following, let k be a local field of characteristic 0, i.e., a finite extension of $\mathbb{Q}_p$ for a prime p. We denote by $|\cdot|_p$ the absolute value of k.

We call G a *linear algebraic p-adic group* if G is linear algebraic over k. Since we are mainly interested in $G(k)$, we will identify G and $G(k)$ in the sequel and endow $G = G(k)$ with the topology induced by k. In this way, G becomes a topological group. If G is even connected as a linear algebraic group, then G is completely determined by $G(k)$, which is proven in [10, Cor. 18.3]. This motivates our identification.

Example 1.7.1. The groups $GL_n(k)$ and $SL_n(k)$ are reductive p-adic. Furthermore, $SL_n(k)$ is semisimple p-adic. However, $GL_n(k)$ fails to be semisimple p-adic because the center $C(GL_n(\bar{k})) = \bar{k}^\times \cdot 1_n$ is a non-trivial connected normal solvable subgroup. Here $\bar{k}^\times \subseteq \bar{k}$ denotes the group of units.

Proposition 1.7.2. *Let G be a reductive p-adic group. Then G is unimodular.*

Proof. Let $\bar{k}$ be an algebraic closure of k. By [49, Sec. 8.1] there exists an algebraic subgroup $H \subseteq G$ with $H(\bar{k}) = [H(\bar{k}), H(\bar{k})]$ such that

$$G(\bar{k}) = H(\bar{k}) \cdot C(G(\bar{k})),$$

where $C(G(\bar{k}))$ denotes the center of $G(\bar{k})$. Consider the map

$$\varphi(\bar{k})\colon G(\bar{k}) \xrightarrow{Ad} \operatorname{aut}(L(G(\bar{k}))) \xrightarrow{\det} \bar{k}^\times,$$

where Ad denotes the adjoint representation and $L(G(\bar{k}))$ the Lie algebra of $G(\bar{k})$. Then $\varphi(\bar{k})$ is trivial by the last equation as is $\varphi(k)$, the restriction to k. We have the identity $\Delta = |\cdot|_p \circ \varphi(k)$ for the modular function Δ by Bourbaki [11, Chap. III, §3.16, Cor. to Prop. 55]. Hence Δ is trivial and G is unimodular. $\square$

Before we can state the next lemma, we need to make some definitions.

Definition 1.7.3. (i) Let G be a group and p a prime. Then G is called a *pro-p group* if there exist finite p-groups G_i such that $G = \lim_{i \in I} G_i$.

(ii) Let G be a group and $n \in \mathbb{N}$. We set $G^n = \langle \{ g^n \mid g \in G \} \rangle$.

(iii) Let $H \subseteq K \subseteq G$ be subgroups. Then K is called *H-characteristic* if $\varphi(H) \subseteq K$ holds for any given group homomorphism $\varphi \colon H \to G$. We call K *strongly characteristic* if K is H-characteristic for any subgroup $H \subseteq K$.

Lemma 1.7.4. *Let G be a finitely generated pro-p-group. We define the* lower p-series *by $P_1(G) = G$ and*

$$P_{i+1}(G) = \overline{P_i(G)^p[P_i(G), G]} \quad \text{for } i > 1.$$

Then $\{ P_i(G) \mid i \geq 1 \}$ forms a basis of topology consisting of strongly characteristic subgroups.

Proof. It is an immediate consequence of the definition that the groups $P_i(G)$ are strongly characteristic. The groups $P_i(G)$ form a basis of topology by [18, Prop. 1.16]. $\square$

Proposition 1.7.5. *Let G be a linear algebraic p-adic group and $H \subseteq G$ a closed subgroup. Then H contains an open finitely generated pro-p group K such that for any $n \in \mathbb{N}$ the group $K^{p^n} \subseteq K$ is open if p is odd, whereas $K^{4^n} \subseteq K$ is open if $p = 2$.*

Proof. Since $GL_n(k)$ is a Lie group over $\mathbb{Q}_p$, we can deduce by [18, Sec. 9.3, Thm. 8.32, Thm. 3.6] that there exists an open finitely generated pro-p group $K \subseteq H$ such that $P_{n+1}(K) = K^{p^n}$. Thus the assertion follows from the previous lemma. $\square$

Definition 1.7.6. Let G be a group and $H, K \subseteq G$ be subgroups. Then we set

$$M_G(H, K) = \left\{ g \in G \;\middle|\; gHg^{-1} \subseteq K \right\}.$$

Proposition 1.7.7. *Let G be a unimodular linear algebraic p-adic group. Let $H \subseteq K \subseteq G$ be compact open subgroups and denote by $C(G) \subseteq G$ the center of G. Then the set $M_G(H, K)/C(G)$ is compact.*

Proof. For simplicity we assume p is odd. Let $H \subseteq K \subseteq G$ be compact open subgroups and let $N \subseteq K$ be an open finitely generated pro-p group as in the previous proposition. Since $[K : N] < \infty$, there exists an open subgroup $N' \subseteq N$ which is normal in K. We obtain a short exact sequence

$$1 \to N' \to K \xrightarrow{\varphi} K/N' \to 1.$$

We consider the strongly characteristic subgroup K^n, where $[K : N'] = n$. We conclude $\varphi(K^n) = \{1\}$ and, consequently, $K^n \subseteq N' \subseteq N$. Let $n = d \cdot p^k$ with $p \nmid d$. Since N is a pro-p group, we obtain $N = N^d$ and, therefore, $N^{p^k} \subseteq K^n$. Thus $K^n \subseteq N$

is open because $N^{p^k} \subseteq N$ is open by the previous proposition. Therefore, we have constructed a strongly characteristic open pro-p subgroup K^n in K. Moreover, $K^n \subseteq N$ is finitely generated as an open subgroup of a finitely generated pro-p group by [44, Prop. 2.5.5]. From the lower p-series we obtain arbitrarily small open strongly characteristic subgroups $P_i(K^n) \subseteq K$. Since these groups are strongly characteristic, we obtain an embedding $M_G(H, K) \subseteq M_G(P_i(H^n), P_i(K^n))$. It is a closed embedding because $M_G(H, K) \subseteq G$ and $M_G(P_i(H^n), P_i(K^n)) \subseteq G$ are closed. We get $P_i(K^n) = \exp(\mathcal{U})$ for a sufficiently large i, where

$$\exp\colon L(G) \dashrightarrow G$$

is the exponential map, which is defined only on an open neighborhood of 0. Since $P_i(K^n)$ is compact, we can assume that $\mathcal{U} \subseteq L(G)$ is an o-subalgebra of the Lie algebra $L(G)$, where $o = \{ x \in k \mid \|x\| \leq 1 \}$ is the ring of integers. We have the adjoint representation

$$Ad\colon G \to \operatorname{aut}(L(G)) \subseteq \{ A \in GL_j(k) \mid |\det A|_p = 1 \} .$$

Here, we can restrict to matrices with determinant 1 because G is unimodular and

$$1 = \Delta(g) = |\det(Ad(g))|_p$$

by Bourbaki [11, Chap. III, §3.16, Cor. to Prop. 55]. Zariski closed sets can be described as intersections of preimages $p_i^{-1}(0)$ for polynomials $p_i \in k[X_1, \ldots, X_l]$. Since polynomials define continuous maps in the p-adic topology, Zariski closed sets are closed in p-adic topology. Hence the Zariski topology is a coarser topology than the p-adic topology. Since Ad is an algebraic map, it has a closed image in the Zariski topology [10, Chap. I, Cor. 1.4] and, therefore, also in p-adic topology. The kernel of Ad is $\ker(Ad) = C(G)$ by [10, Chap. I, 3.15], thus $C(G(?))$ is a linear algebraic group. Now we can divide out the kernel (in the category of linear algebraic groups (see [10, Sec. 6])) and get the quotient group $Q(?)$. We remark that $G(R)/C(G(R)) \subseteq Q(R)$ and equality does not hold in general, but it does for an algebraic closure $R = \bar{k}$. Moreover, the p-adic topology of $G(\bar{k})/C(G(\bar{k}))$ is just the quotient topology. Now Ad induces a bijective map of linear algebraic groups

$$Ad'\colon G(\bar{k})/C(G(\bar{k})) \to \operatorname{im} Ad(\bar{k}).$$

Since bijective maps of linear algebraic groups are isomorphisms [49, Sec. 5.3], the inverse map essentially is a polynomial and thus continuous in the p-adic topology. Consequently, Ad' is a homeomorphism with respect to the p-adic topology. Since $G/C(G) = G(k)/C(G(k)) \subseteq Q(k) \subseteq Q(\bar{k}) = G(\bar{k})/C(G(\bar{k}))$, the restriction to $G/C(G)$ yields a homeomorphism. Thus, the map

$$\widetilde{Ad}\colon G/C(G) \to GL_j(k)$$

is a closed embedding. Hence, preimages of compact sets are compact. We obtain

$$\widetilde{Ad}\big(M_G(P_i(H^n), P_i(K^n))/C(G)\big) \subseteq \big\{ f \in \operatorname{aut}(L(G)) \bigm| |\det(f)|_p = 1 \text{ and } f(\mathcal{U}') \subseteq \mathcal{U} \big\} ,$$

where we set $\mathcal{U}' = \exp(P_i(H^n))$. We can consider $L(G) = k^{j'}$ as a k-vector space and, consequently, as a (finite dimensional) $\mathbb{Q}_p$-vector space $L(G) = \mathbb{Q}_p^j$ because $k/\mathbb{Q}_p$ is a finite extension. Using this identification, we can assume $L(\mathcal{U}) = \mathbb{Z}_p^j$ and $L(\mathcal{U}') = \oplus_{i=1}^j p^{k_j}\mathbb{Z}_p$. We set $p_m = \max\{\, p^{k_1}, \dots, p^{k_j} \,\}$ and conclude

$$\begin{aligned}
&\{\, f \in \mathrm{aut}(L(G)) \mid |\det(f)|_p = 1 \text{ and } f(\mathcal{U}') \subseteq \mathcal{U} \,\} \\
&\qquad \subseteq \left\{ A \in GL_j(\mathbb{Q}_p) \;\middle|\; |\det(A)|_p = 1 \text{ and } A\Big(\bigoplus_{i=1}^{j} p^{k_j}\mathbb{Z}_p\Big) \subseteq \mathbb{Z}_p^j \right\} \\
&\qquad \subseteq \left\{ A = \begin{pmatrix} a_{11} & \cdots & a_{1j} \\ \vdots & & \vdots \\ a_{j1} & \cdots & a_{jj} \end{pmatrix} \;\middle|\; |\det(A)|_p = 1 \text{ and } |a_{mn}|_p \le p_m,\ 1 \le m, n \le j \right\}.
\end{aligned}$$

Since the latter term is a closed subset of the $(j \times j)$-matrices $M_j(\frac{1}{p_m}\mathbb{Z}_p)$, it is compact. Thus the image $\widetilde{Ad}(M_G(P_i(H^n), P_i(K^n)/C(G))$ is compact, which implies that the preimage $M_G(P_i(H^n), P_i(K^n)/C(G)$ is compact. This means that $M_G(H,K)/C(G)$ is also compact, as required. $\square$

Corollary 1.7.8. *Let G be a unimodular linear algebraic p-adic group. Let $H, K \subseteq G$ be compact open subgroups. If the center $C(G)$ is compact, then $M_G(H,K)$ is also compact. In particular, the morphism sets*

$$\mathrm{mor}_{\mathsf{Or}_{\mathcal{CO}}(G)}(G/H, G/K) = M(H,K)/K$$

are finite.

If G is not unimodular, the corresponding statement from above does not hold:

Example 1.7.9. We consider the group

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \;\middle|\; a, b \in \mathbb{Q}_p,\ a \neq 0 \right\}$$

of Example 1.2.20. Its center is $C(G) = \{\, \pm 1 \,\}$ and G is a linear algebraic p-adic group. However, G is not unimodular and $\mathsf{Or}_{\mathcal{CO}}(G)$ is not an EI-category. By Lemma 1.2.23, the morphism sets cannot be finite.

However, there exist non-unimodular groups with finite morphism sets:

Example 1.7.10. The group of Example 1.2.19 is not unimodular but it is a linear algebraic p-adic group, has trivial center, and $\mathsf{Or}_{\mathcal{CO}}(G)$ is an EI-category. We can deduce by the discussion at the end of Example 1.2.22 that for compact open subgroups $H \subseteq K \subseteq G$ the set

$$\{\, |\det Ad(g)|_p \mid c(g) \colon H \to K,\ g \in G \,\} \subseteq \mathbb{Q}_p$$

is closed. Therefore, the proof of Proposition 1.7.7 carries over directly and the morphism sets are finite.

Since reductive p-adic groups are unimodular, we can deduce by Lemma 1.2.18:

Corollary 1.7.11. *Let G be a reductive p-adic group. Then $\mathtt{Sub}_{\mathcal{CO}}(G)$ is an EI-category with finite morphism sets. If the center of G is compact, the orbit category $\mathtt{Or}_{\mathcal{CO}}(G)$ also has finite morphism sets.*

Corollary 1.7.12. *Let G be a semisimple p-adic group. Then $\mathtt{Or}_{\mathcal{CO}}(G)$ is an EI-category and the morphism sets $\operatorname{mor}_{\mathtt{Or}_{\mathcal{CO}}(G)}(G/H, G/K)$ are finite for compact open subgroups $H, K \subseteq G$.*

Proof. Since semisimple p-adic groups are reductive p-adic groups, it suffices to show that the center of G is finite.

The center $C(G)$ is again an algebraic group. Let $\bar{k}$ be an algebraic closure of k. Obviously, we obtain $C(G) \subseteq C(G(\bar{k}))$. Thus it suffices to show that $C(G(\bar{k}))$ is finite. The connected components of $C(G(\bar{k}))$ are trivial because G is semisimple. Therefore, $C(G(\bar{k}))$ is a 0-dimensional variety and hence finite. Now we can apply the previous corollary which implies that the morphism sets are finite. □

The great benefit of reductive p-adic groups is the following celebrated result by Bruhat and Tits [12]

Theorem 1.7.13. *Let G be a reductive p-adic group. Then there exists a simplicial complex βG which is a model for $\underline{E}G$. Furthermore, G acts simplicially on βG.*

The complex βG is called the affine Bruhat-Tits building *of G.*

Example 1.7.14. Let $G = SL_2(\mathbb{Q}_p)$. Up to conjugacy, there are the following maximal compact subgroups

$$K_0 = SL_2(\mathbb{Z}_p) \quad \text{and} \quad K_0 = \left\{ \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix} \in G \;\middle|\; a, b, c, d \in \mathbb{Z}_p \right\}.$$

We set $I = K_0 \cap K_1$. Then βG is the tree of Figure 1.1 whose vertices correspond to subgroups conjugated to K_0 or K_1 and whose edges correspond to subgroups conjugated to I. The action of G on βG is given by conjugation.

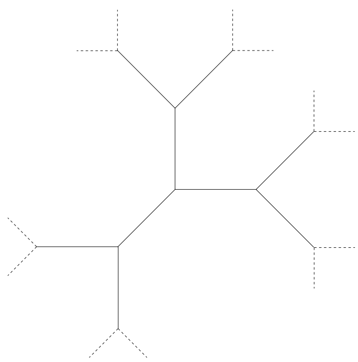

Figure 1.1: The affine Bruhat-Tits building of $SL_2(\mathbb{Q}_p)$

2 Equivariant (Co)Homology Theories

In the previous chapter, we defined equivariant (co)homology theory. However, we have not given any examples yet. This is the purpose of this chapter. In the first section, we introduce the Borel construction and prove that this is an equivariant (co)homology theory with a Mackey structure on coefficients. In the second section, we introduce equivariant bivariant K-theory, collect its basic properties and we show that it can be used to define an equivariant (co)homology theory with a Mackey structure on coefficients. At the end of the second section, we define cosheaf homology, which has the same coefficients as K-theory. We identify it with an appropriate Bredon homology. Cosheaf homology is a kind of simplicial version of Bredon homology and thus more accessible if we are dealing with simplicial complexes.

In the following, we will only consider $(G, \mathcal{F})$-CW-complexes instead of $(G, \mathcal{F})$-CW-pairs for simplicity. We can do that by the following argument. Let (X, A) be a $(G, \mathcal{F})$-CW-pair. Then the canonical inclusion $A \hookrightarrow X$ is a G-cofibration and we obtain $\mathcal{H}_*^G(X, A) = \mathcal{H}_*^G(X/A)$ for any $(G, \mathcal{F})$-homology theory $\mathcal{H}_*^G$. This is just like in the classical case.

2.1 Borel Construction

We obtain a G-homology theory with values in $\mathbb{Z}$-modules

$$\mathcal{H}_*^G(X) \;\coloneqq\; H_n(EG \times_G X),$$

where $H_n(EG \times_G X)$ is the singular homology with rational coefficients of the Borel construction $EG \times_G X$.

Note that $EG \times_G X$ does not necessarily have the homotopy type of a CW-complex. If we take $X = G$, then $EG \times_G G$ is EG. After forgetting the group action, EG does not in general have the homotopy type of a CW-complex. For instance, if G is an l-group, then EG has the homotopy type of a CW-complex if and only if G is discrete (see [33, Sec. 1]). However, if $X \to Y$ is an inclusion of G-CW-complexes and hence a G-cofibration, then $EG \times_G X \to EG \times_G Y$ is a cofibration and excision for $\mathcal{H}_*^G$ holds.

Let $\mathcal{F}_?$ be a collection of families of subgroups. Note that $\mathcal{H}_*^G$ is the evaluation of an equivariant $\mathcal{F}_?$-homology theory at G because the necessary induction structure exists by the following argument. Let $\alpha\colon H \to G$ be an injective homomorphism of topological groups. Let X be an $(H, \mathcal{F}_H)$-CW-complex. Then the G-space $G \times_\alpha X$ obtained by induction with α is a $(G, \mathcal{F}_G)$-CW-complex. There is a G-map $G \times_\alpha EH \to EG$ which is unique up to G-homotopy. Composing it with the obvious map $EH \to G \times_\alpha EH$, $e \mapsto (1, e)$ yields an α-equivariant map $f_\alpha\colon EH \to EG$. It induces

a map
$$\overline{f}_\alpha\colon EH \times_H X \to EG \times_G G \times_\alpha X, \quad (e,x) \mapsto (f_\alpha(e), 1, x).$$
We define the induction homomorphism
$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X) \to \mathcal{H}_n^G(G \times_\alpha X)$$
to be the map $H_n(\overline{f}_\alpha)$.

The map $\overline{f}_\alpha$ is a weak homotopy equivalence. It suffices to prove this in the special case $X = H/K$, in which the claim follows from the fact that α induces an isomorphism of topological groups $K \to \alpha(K)$ because α is injective. It is still an isomorphism if $\ker(\alpha)$ acts freely on H/K. This is the more general notion of induction structure we mentioned in Remark 1.5.9. Although we do not need this general setting for the upcoming Chern character, we do need this general setting in Example 4.3.2. In this case, we must generalize the definition of a collection of families of subgroups (Definition 1.5.5) and also take homomorphisms like α into account (see also Lemma 1.4.6 and Remark 1.4.7).

The previous considerations carry over directly to the cohomological version of the Borel construction, i.e., to
$$\mathcal{H}_G^*(X) \;:=\; H^*(EG \times_G X).$$
Thus, $\mathcal{H}_G^*$ is an equivariant $\mathcal{F}_?$-cohomology theory.

Proposition 2.1.1. *Let $\mathcal{F}$ be a smooth family such that $[H : K] < \infty$ for every $H, K \in \mathcal{F}$. Then the Borel construction has a Mackey structure on coefficients.*

Proof. Let $H, K \in \mathcal{F}$ and let $\alpha\colon H \to K$ be an open embedding such that $[K : \alpha(H)] < \infty$. Then $\mathrm{ind}_\alpha\colon BH \to BK$ is a finite covering space. We want to construct a transfer map $\mathrm{tr}_\alpha\colon H_*(BK) \to H_*(BH)$ as in the case of discrete groups. We remind the reader that there exists an isomorphism $H_*^{\mathcal{U}}(X) \to H_*(X)$. Here, $\mathcal{U} = (U_i)_{i\in I}$ is an open cover and $H_*^{\mathcal{U}}(X)$ denotes the homology groups of the singular chain complex in which we only consider those simplices which are contained in U_i for some $i \in I$. Since $\mathrm{ind}_\alpha\colon BH \to BK$ is a finite covering space, we can choose an open cover $\mathcal{U} = (U_1, \dots, U_n)$ such that $\mathrm{ind}_\alpha|_{U_l}$ is a homeomorphism for $i = 1, \dots, n$. Let $f\colon \Delta^k \to BK$ be a simplex such that $\mathrm{im}\, f \subseteq U_l$ for some l. Then we can lift f to a map $\tilde{f}\colon \Delta^k \to BH$. This lift is not unique but depends on the chosen base point. If we sum up the distinct lifts, this defines a map on homology
$$\mathrm{tr}_\alpha\colon H_*(BK) \to H_*(BH).$$
Obviously, we obtain $\mathrm{ind}_\alpha \mathrm{tr}_\alpha = n \cdot \mathrm{id}$ for $[K : \alpha(H)] = n$, and the proof of the double coset formula is pretty much the same as in the well known case of discrete groups. Nevertheless, I want to carry it out. Recall the double coset formula:
$$\begin{aligned}\mathrm{tr}_G^K \mathrm{ind}_H^G x &= \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{c(g)\colon H\cap g^{-1}Kg \to K} \circ \mathrm{tr}_H^{H\cap g^{-1}Kg}\, x \\ &= \sum_{KgH \in K\backslash G/H} \mathrm{ind}_{K\cap gHg^{-1}}^K \mathrm{tr}_{gHg^{-1}}^{K\cap gHg^{-1}}\, gx\end{aligned}$$

for $x \in H_*(BH)$. Let $\mathcal{U} = (U_1, \ldots, U_n)$ be an open cover of BH such that $\operatorname{ind}_H^G |_{U_k}$ and $\operatorname{ind}_{K \cap gHg^{-1}}^K |_{U_k}$ are homeomorphisms for all k. Then we can restrict to those simplices which lie entirely in some U_k and "forget" the induction maps. Consequently, we get

$$\operatorname{tr}_G^K \operatorname{ind}_H^G x = \sum_{[g] \in G/K} gx$$

for the left hand side and

$$\sum_{KgH \in K \backslash G/H} \sum_{[h] \in (gHg^{-1})/(K \cap gHg^{-1})} ghx$$

for the right hand side. We deduce that both sides coincide and get the double coset formula.

Furthermore, conjugation induces the identity on $H_*(BG)$ by the following argument. By the slice theorem [28, Thm. 1.37] there exists for every $x \in EG$ an open neighborhood $x \in U$ such that $G \times (U/G)$ and U are G-homeomorphic. Let $\mathcal{U}$ be an open cover of EG which consists of such open subsets and denote by $\mathcal{U}'$ the corresponding open cover of BG. Let $C_*^{\mathcal{U}}(EG)$ be the singular chain complex in which we only consider those simplices which are contained in some $U \in \mathcal{U}$. Analogously, we define $C_*^{\mathcal{U}'}(BG)$. Let $C_*(G)$ and $C_*(EG)$ be the singular chain complexes of G and EG. Then $C_n^{\mathcal{U}}(EG)$ and $C_n(EG)$ are free $C_n(G)$-modules, where the multiplication is given by pointwise multiplication. The non-unital ring $C(G) = \oplus_{n \in \mathbb{N}} C_n(G)$ is obviously projective in $C(G)$-$\mathtt{mod}$. Consequently, $C_n^{\mathcal{U}}(EG)$ and $C_n(EG)$ are projective $C(G)$-modules, multiplication with $f \in C_k(G)$ for $k \neq n$ being trivial. Since EG is weakly contractible, $C_*^{\mathcal{U}}(EG)$ and $C_*(EG)$ are projective $C(G)$-resolutions of $\mathbb{Z}$. An immediate consequence is that $H_*(C_*^{\mathcal{U}}(EG) \otimes_{C(G)} \mathbb{Z}) \cong H_*(C_*(EG) \otimes_{C(G)} \mathbb{Z})$. The canonical projection $\operatorname{pr}\colon EG \to BG$ induces an isomorphism $\operatorname{pr}\colon C_n^{\mathcal{U}}(EG) \otimes_{C(G)} \mathbb{Z} \to C_n^{\mathcal{U}'}(BG)$ and hence an isomorphism

$$\operatorname{pr}_*\colon H_*(C_*(EG) \otimes_{C(G)} \mathbb{Z}) \xrightarrow{\cong} H_*(BG).$$

Now we consider the map $f_{c(g)}\colon EG \to EG$ from above. Let $i\colon \operatorname{res}_{c(g)} C_*(EG) \to C_*(EG)$ be the identity. Then $(f_{c(g)})_* i^{-1}\colon C_*(EG) \to \operatorname{res}_{c(g)} C_*(EG)$ and

$$m_g\colon C_*(EG) \to \operatorname{res}_{c(g)} C_*(EG), \quad x \mapsto gx$$

are $C(G)$-homomorphisms. We obtain $m_g - (f_{c(g)})_* i^{-1} = dh + hd$ for some $C(G)$-homotopy h by the the fundamental lemma, where d denotes the boundary maps in $C_*(EG)$. Thus we get $m_g i - (f_{c(g)})_* = idh + ihd$, which induces a trivial map in $H_*(C_*(EG) \otimes_{C(G)} \mathbb{Z})$. Hence we can compute $H_*(f_{c(g)})\colon H_*(BG) \to H_*(BG)$ by $H_*(m_g i)\colon H_*(BG) \to H_*(BG)$. Obviously, we obtain $H_*(m_g i) = \mathrm{id}$, whence it follows that $H_*(f_{c(g)}) = \mathrm{id}$, as required.

The cohomological case is analogous. □

Remark 2.1.2. Since $\operatorname{ind}_\varphi \operatorname{tr}_\varphi = n \cdot \mathrm{id}$, the transfer map is always injective and induction is at least rationally surjective.

Remark 2.1.3. Let G be an l-group and denote by G_d the corresponding discrete group. Then we obtain $EG = EG_d \rtimes_{G_d} G$ by a result of Lück and Meintrup [33, Cor. 3.5]. For a G-CW-complex X we get

$$EG \rtimes_G X \simeq EG_d \rtimes_{G_d} G \rtimes_G X \simeq EG_d \rtimes_{G_d} \operatorname{res}_G^{G_d} X.$$

Thus the classifying space "cannot" distinguish between an l-group and the corresponding discrete group. As an immediate consequence, we obtain

$$H_1(BG) = G/[G,G]$$

for an l-group G.

2.2 Equivariant Bivariant K-theory

The Borel construction is a "classical" (co)homology theory, i.e., it has only one argument and G-CW-complexes as input. Equivariant bivariant K-theory KK_*^G is of a quite different nature. The origins of equivariant bivariant K-theory stem from operator theory. For instance, it has G-C^*-algebras as input. Furthermore, equivariant bivariant K-theory has two arguments. It is a functor contravariant in the first and covariant in the second argument. The complex valued functions $C_0(X)$ on a G-CW-complex X which vanish at infinity form a G-C^*-algebra. Therefore, we can study $KK_*^G(C_0(?), A)$ or $KK_*^G(A, C_0(?))$, respectively, for a G-C^*-algebra A. Unfortunately, these functors fail to be equivariant (co)homology theories. After applying some modifications, however, we obtain equivariant (co)homology theories.

First, we give a very brief survey of G-C^*-algebras and equivariant bivariant K-theory; the reader who is interested in details is advised to consult [9] and [17].

In the following, G denotes a locally compact second countable topological group.

Definition 2.2.1. A *C^*-algebra* A is a Banach algebra together with a continuous map $*\colon A \to A$ called *involution* such that for all $\lambda \in \mathbb{C}$ and $x, y \in A$ the following holds:

$$(x+y)^* = x^* + y^*, \quad (xy)^* = y^*x^*, \quad (\lambda x)^* = \bar{\lambda}x^*, \quad (x^*)^* = x \quad \text{and} \quad \|x^*x\| = \|x\|^2.$$

A *morphism of C^*-algebras*, also called *$*$-homomorphism*, $f\colon A \to B$ is a linear map such that $f(a^*) = f(a)^*$ holds for every $a \in A$. A *G-C^*-algebra* A is a C^*-algebra together with a continuous homomorphism from G to $\operatorname{aut}(A)$, where we endow $\operatorname{aut}(A)$ with the topology of pointwise norm-convergence.

Example 2.2.2. (i) Given a Hilbert space $\mathcal{H}$, the algebra of bounded operators $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, as is every closed subalgebra of $\mathcal{B}(\mathcal{H})$. Here, we endow $\mathcal{B}(\mathcal{H})$ with the topology which is induced by the operator norm. Finally, a (norm continuous) representation of G on $\mathcal{H}$ turns every G-invariant closed subalgebra $A \subseteq \mathcal{B}(\mathcal{H})$ into a G-C^*-algebra.

(ii) The multiplication map

$$\mu\colon G \times L^2(G) \to L^2(G), \quad (g, x) \mapsto g \cdot x$$

defines an embedding $\mathbb{C}G \subseteq \mathcal{B}(L^2(G))$. The closure $C_r^*(G)$ of $\mathbb{C}G$ in $\mathcal{B}(L^2(G))$ is called the *reduced group C^*-algebra of G.*

(iii) Let X be a locally compact topological space and $C_0(X)$ the algebra of (complex valued) continuous functions vanishing at infinity. Then $C_0(X)$ together with the norm

$$\|f\| = \sup_{p\in X} |f(p)|$$

is a commutative C^*-algebra. A continuous G-action on X induces a corresponding action on $C_0(X)$ which turns $C_0(X)$ into a G-C^*-algebra. Conversely, for every commutative G-C^*-algebra A there exists a locally compact G-space such that $A \cong C_0(X)$ as G-C^*-algebras.

Definition 2.2.3. A *graded C^*-algebra* is a C^*-algebra A together with a decomposition $A = A^{(0)} \oplus A^{(1)}$ into closed $*$-invariant subspaces such that $A^{(m)} \cdot A^{(n)} \subseteq A^{(m+n)}$. An element in $A^{(n)}$ is said to be *homogeneous of degree n.* A *graded G-C^*-algebra* is a C^*-algebra together with an action of G by graded $*$-automorphisms.

Definition 2.2.4. Let B be a graded C^*-algebra. A *graded pre-Hilbert B-module* is a graded right B-module $E = E^{(0)} \oplus E^{(1)}$ together with a graded B-valued inner product $\langle \cdot, \cdot \rangle : E^{(m)} \times E^{(n)} \to B^{(m+n)}$ which satisfies

(i) $\langle \cdot, \cdot \rangle$ is sesquilinear,

(ii) $\langle x, ya \rangle = \langle x, y \rangle\, a$ for all $x, y \in E$ and $a \in A$,

(iii) $\langle x, y \rangle = \langle y, x \rangle^*$ for all $x, y \in E$ and

(iv) $\langle x, x \rangle \geq 0$, with equality if and only if $x = 0$.

For $x \in E$, we set $\|x\| = \sqrt{\| \langle x, x \rangle \|}$; if E is complete with respect to this norm, it is called a *graded Hilbert B-module.*

Remark 2.2.5. A Hilbert $\mathbb{C}$-module is just a Hilbert space.

Definition 2.2.6. Let B be a (graded) G-C^*-algebra and E a (graded) Hilbert B-module. A *G-action* on E consists of a representation of G on E by bounded (graded) linear transformations such that

$$g \cdot (xb) = (g \cdot x)(g \cdot b) \quad \text{and} \quad g \cdot \langle x, y \rangle = \langle g \cdot x, g \cdot y \rangle$$

for all $g \in G$, $b \in B$ and $x, y \in E$. A (graded) Hilbert B-module with such an action will be called a *(graded) Hilbert (B, G)-module.*

Definition 2.2.7. Let E be a Hilbert B-module. We denote by $\mathcal{L}(E)$ the set of all module homomorphisms $T\colon E \to E$ for which there exists an "adjoint" module homomorphism $T^*\colon E \to E$ satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \forall x, y \in E.$$

Remark 2.2.8. (i) Actually, the assumption that T and T^* are module homomorphisms is unnecessary since homomorphisms with adjoints are automatically module homomorphisms.

(ii) As in the case of Hilbert spaces, the existence of an adjoint is enough to ensure that T is a bounded operator on the Banach space E. The converse, however, is no longer true.

Lemma 2.2.9. *The set $\mathcal{L}(E)$ has a C^*-algebra structure. Moreover, the decomposition $\mathcal{L}(E) = \mathcal{L}^{(0)}(E) \oplus \mathcal{L}^{(1)}(E)$ given by*

$$\mathcal{L}(E)^{(n)} = \left\{ T \in \mathcal{L}(E) \,\middle|\, T(E^{(m)}) \subseteq E^{(m+n)} \right\}$$

provides $\mathcal{L}(E)$ with a structure of a graded C^*-algebra. We call operators in $\mathcal{L}^0(E)$ even *and operators in $\mathcal{L}^1(E)$* odd.

Definition 2.2.10. Let $\mathcal{K}(E)$ be the closed linear span of

$$\{ \theta_{x,y} \in \mathcal{L}(E) \mid \theta_{x,y}(z) = x\, \langle y, z\rangle \} \subseteq \mathcal{L}(E).$$

Remark 2.2.11. Let E be a Hilbert $\mathbb{C}$-module. Then E is a Hilbert space by Remark 2.2.5 and $\mathcal{K}(E)$ are the *compact operators*.

Lemma 2.2.12. *The Banach space $\mathcal{K}(E)$ is a C^*-algebra.*

Definition 2.2.13. Let A and B be graded G-C^*-algebras.

(i) An *even Kasparov G-module for (A, B)* is a triple (E, ϕ, F) consisting of a (countably generated) graded Hilbert (B, G)-module E, a graded $*$-homomorphism $\phi\colon A \to \mathcal{L}(E)$ and an operator $F \in \mathcal{L}^1(E)$ such that

$$F \circ \phi(a) - \phi(a) \circ F, \quad (F^2 - 1)\phi(a), \quad (F - F^*)\phi(a) \quad \text{and} \quad (gF - F)\phi(a)$$

lie in $\mathcal{K}(E)$ for all $a \in A$ and $g \in G$. The collection of all even Kasparov G-modules will be denoted by $\mathcal{E}_0^G(A, B)$.

(ii) An *odd Kasparov G-module for (A, B)* is a triple (E, ϕ, F) which satisfies the above conditions but without grading. The collection of all odd Kasparov G-modules will be denoted by $\mathcal{E}_1^G(A, B)$

Definition 2.2.14. Let $(E_i, \phi_i, F_i)_{i=0,1} \in \mathcal{E}_*^G(A, B)$.

(i) If there exists a (graded) unitary $u\colon E_0 \to E_1$ which intertwines ϕ_i and F_i, we call $(E_i, \phi_i.F_i)_{i=0,1}$ *unitary equivalent* and write

$$(E_0, \phi_0, F_0) \sim_u (E_1, \phi_1, F_1).$$

(ii) Suppose there exists a Kasparov G-module $(E, \phi, F) \in \mathcal{E}_*^G(A, B[0, 1])$ such that

$$(ev_i)_*(E, \phi, F) \sim_u (E_i, \phi_i, F_i) \quad \text{for } i = 0, 1, \tag{2.2.14.1}$$

where ev_t denotes the canonical evaluation at $t \in [0, 1]$. Then we write

$$(E_0, \phi_0, F_0) \simeq (E_1, \phi_1, F_1)$$

and call $(E_i, \phi_i, F_i)_{i=0,1}$ *G-homotopic.*

The corresponding groups $KK_*^G(A, B) = \mathcal{E}_*^G(A, B)/\simeq$ are called *equivariant bivariant K-theory.* If G is trivial, we will often write $KK_*(A, B)$ instead of $KK_*^{\{1\}}(A, B)$.

Theorem 2.2.15. *Bivariant equivariant K-theory has the following properties:*

(i) There is a composition product

$$KK_i^G(A, B) \otimes KK_j^G(B, C) \to KK_{i+j}^G(A, C).$$

(ii) Each homomorphism $\varphi\colon A \to B$ defines an element $KK_G(\varphi)$ in the group $KK_0^G(A, B)$. If $\psi\colon B \to C$ is another homomorphism, then $KK_G(\psi \circ \varphi) = KK_G(\psi) \circ KK_G(\varphi)$. The functor $KK_*^G(A, B)$ is a contravariant functor in A and a covariant functor in B. For given homomorphisms $\alpha\colon A_1 \to A_2$ and $\beta\colon B_1 \to B_2$, the induced maps are given by left multiplication by $KK_G(\alpha)$ and right multiplication by $KK_G(\beta)$.

(iii) The functor KK_*^G is invariant under G-homotopies in both variables.

(iv) Stability
The canonical inclusion $\mathrm{inc}\colon A \to \mathcal{K} \otimes A$ *defines an invertible element* $KK_G(\mathrm{inc})$. *In particular, we get isomorphisms*

$$KK_*^G(A, B) \cong KK_*^G(\mathcal{K} \otimes A, B) \quad \text{and} \quad KK_*^G(A, B) \cong KK_*^G(A, \mathcal{K} \otimes B)$$

($\mathcal{K}$ denotes the compact operators on a separable infinite dimensional Hilbert space here).

(v) Bott periodicity
There are canonical elements in $KK_1^G(A, SA)$ and in $KK_1^G(SA, A)$ which are inverse to each other (recall that $SA = C_0(\mathbb{R}) \otimes A$, where G acts trivially on $C_0(\mathbb{R})$).

Theorem 2.2.16. *Let D be a separable G-C^*-algebra. Every extension of G-C^*-algebras admitting a completely positive linear (not necessarily equivariant) splitting*

$$0 \to I \to A \to B \to 0$$

induces exact sequences of the following form:

$$\begin{tikzcd} KK_0^G(D,I) \arrow[r] & KK_0^G(D,A) \arrow[r] & KK_0^G(D,B) \arrow[d] \\ KK_1^G(D,B) \arrow[u] & KK_1^G(D,A) \arrow[l] & KK_1^G(D,I) \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} KK_0^G(B,D) \arrow[r] & KK_0^G(A,D) \arrow[r] & KK_0^G(I,D) \arrow[d] \\ KK_1^G(I,D) \arrow[u] & KK_1^G(A,D) \arrow[l] & KK_1^G(B,D) \arrow[l] \end{tikzcd}$$

If B is commutative, a completely positive linear (non equivariant) splitting is given by [9, Thm. 15.8.2/3].

Definition 2.2.17. A functor

$$F^G\colon G\text{-}C^*\text{-}\mathsf{algebras} \times G\text{-}C^*\text{-}\mathsf{algebras} \to \mathbb{Z}\text{-}\mathsf{mod}$$

with the previous property is called *excisive* or sometimes *split exact*.

Equivariant bivariant K-theory can be viewed as an additive category KK^G with separable G-C^*-algebras as objects and $KK_0^G(A,B)$ as the set of morphisms between two objects A and B. Composition of morphisms is given by the product. There is a canonical functor $P\colon G\text{-}C^*\text{-}\mathsf{algebras} \to KK^G$ which is the identity on objects and sends equivariant $*$-homomorphisms to the corresponding KK-elements. Equivariant bivariant K-theory satisfies the following universal property [52]:

Theorem 2.2.18. *An additive functor F from G-C^*-$\mathsf{algebras}$ into an additive category $\mathcal{A}$ factorizes uniquely over KK^G if and only if it is homotopy invariant, stable and excisive. That is, given such a functor F, there exists a unique functor $\tau\colon KK^G \to \mathcal{A}$ such that $F = \tau \circ P$.*

Now we have the machinery to define some $(G,\mathcal{F})$-(co)homology theories.

Theorem 2.2.19. *Let A be a G-C^*-algebra and $\mathcal{F}$ be any family of subgroups. Then the functors*

$$K_*^{(G,A)}(?)\colon (G,\mathcal{F})\text{-}\mathsf{CW\text{-}complexes} \to \mathbb{Z}\text{-}\mathsf{mod}, \quad X \mapsto \operatorname*{colim}_{Y\subseteq X \text{ } G\text{-compact}} KK_*^G(C_0(Y),A)$$

$$K^*_{(G,A)}(?)\colon (G,\mathcal{F})\text{-}\mathsf{CW\text{-}complexes} \to \mathbb{Z}\text{-}\mathsf{mod}, \quad X \mapsto KK_*^G(A, C_0(X))$$

define a $(G,\mathcal{F})$-(co)homology theory.

Proof. By Theorem 2.2.15, we get G-homotopy invariance and the existence of a suspension isomorphism. We can deduce Mayer-Vietoris by Theorem 2.2.16. Moreover, $K_*^{(G,A)}$ satisfies the disjoint union axiom automatically by construction.

These conditions are equivalent to the Eilenberg-Steenrod axioms given in Definition 1.5.1. □

Remark 2.2.20. We must do the colimit construction in order to ensure that $K_*^{(G,A)}$ satisfies the disjoint union axiom. The naive approach does not satisfy the disjoint union axiom by [9, Thm. 19.7.1].

Definition 2.2.21. Let $\varphi\colon H \to G$ be an embedding and A be a G-C^*-algebra. The *restricted algebra* $\operatorname{res}_\varphi A$ is an H-C^*-algebra, where $\operatorname{res}_\varphi A = A$ as C^*-algebras and the H-action is given by

$$h \cdot a = \varphi(h)a, \quad h \in H, a \in A.$$

If $H \subseteq G$ is a subgroup, we write $\operatorname{res}_G^H A = \operatorname{res}_{\mathrm{inc}} A$ for the canonical inclusion $\mathrm{inc}\colon H \to G$. Let B be another G-C^*-algebra. As an immediate consequence of the previous construction, we obtain the *restriction homomorphism*

$$\operatorname{res}_\varphi\colon KK_G^*(A,B) \to KK_H^*(\operatorname{res}_\varphi A, \operatorname{res}_\varphi B),$$

which is natural and compatible with boundary maps.

Definition 2.2.22. Let $\varphi\colon H \to G$ be a closed embedding and A be an H-C^*-algebra. Then we can define the *induced algebra* $\operatorname{ind}_\varphi A$ as

$$\left\{ f \in C_b(G,A) \;\middle|\; \substack{hf(s)=f(s\varphi(h^{-1}))\ \forall s\in G,\ h\in H \\ \text{and } (s\varphi(H)\mapsto\|f(s)\|\in C_0(G/\varphi(H)))} \right\}$$

together with pointwise multiplication and the supremum norm. If $H \subseteq G$ is a closed subgroup, we write $\operatorname{ind}_H^G A = \operatorname{ind}_{\mathrm{inc}} A$ for the canonical inclusion $\mathrm{inc}\colon H \to G$.

Remark 2.2.23. Let $\varphi\colon H \to G$ be a closed embedding and A an H-C^*-algebra. If $A = C_0(X)$ holds for a locally compact H-space X, then we get $\operatorname{ind}_\varphi C_0(X) = C_0(G \times_\varphi X)$.

Kasparov (see [24, Chap. 3] and [25, Chap. 5]) constructed the following induction map:

Theorem 2.2.24. *Let $\varphi\colon H \to G$ be a closed embedding and A, B be H-C^*-algebras. There exists an* induction homomorphism

$$\operatorname{ind}_\varphi\colon KK_*^H(A,B) \to KK_*^G(\operatorname{ind}_\varphi A, \operatorname{ind}_\varphi B)$$

which is natural and compatible with the boundary map.

Proof. We only sketch the construction.

Suppose that $x \in KK_*^H(A,B)$ is represented by a Kasparov triple (E, ϕ, T). Similar to the construction of the induced algebras, we can form the induced $\operatorname{ind}_\varphi$-Hilbert module $\operatorname{ind}_\varphi E$ as the set

$$\left\{ \xi \in C_b(G,E) \;\middle|\; \substack{h(\xi(s))=\xi(s\varphi(h^{-1}))\text{ for all } s\in G,\ h\in H \\ \text{and } s\varphi H\mapsto\|\xi(s)\|\in C_0(G/\varphi(H))} \right\}$$

equipped with the pointwise actions and inner products. Pointwise action on the left provides an obvious induced representation $\operatorname{ind}_\varphi\colon \operatorname{ind}_\varphi A \to \mathcal{L}(\operatorname{ind}_\varphi E)$. Using a cut-off function $c\colon G \to [0,\infty)$ for the right translation action of $\varphi(H)$ on G, Kasparov constructs an operator $\tilde{T} \in \mathcal{L}(\operatorname{ind}_\varphi E)$ by the formula

$$\tilde{T}\xi(g) = \int_H c\big(g\varphi(h)\big)h\big(T(\xi(g\varphi(h)))\big)dh, \quad \xi \in \operatorname{ind}_\varphi E$$

to obtain a G-Kasparov module $(\operatorname{ind}_\varphi E, \operatorname{ind}_\varphi \phi, \tilde{T})$ which represents the element

$$\operatorname{ind}_\varphi x \in KK_*^G(\operatorname{ind}_\varphi A, \operatorname{ind}_\varphi B).$$

□

Definition 2.2.25. Let $\varphi\colon H \to G$ be a cocompact embedding, i.e., $G/\varphi(H)$ is compact. Then we have a canonical inclusion

$$F\colon A = A \otimes 1 \subseteq A \otimes C_0(G/\varphi(H)) \cong \operatorname{ind}_\varphi \operatorname{res}_\varphi A.$$

Now we can define the *induction by* φ by

$$\operatorname{ind}_\varphi\colon KK_*^H(\operatorname{res}_\varphi A, B) \xrightarrow{\operatorname{ind}_\varphi} KK_*^G(\operatorname{ind}_\varphi \operatorname{res}_\varphi A, \operatorname{ind}_\varphi B) \xrightarrow{F^*} KK_*^G(A; \operatorname{ind}_\varphi B).$$

Let $\varphi\colon H \to G$ be an open embedding. We have a canonical inclusion

$$G\colon A \to \operatorname{res}_\varphi \operatorname{ind}_\varphi A, \quad G(a)(g) = \begin{cases} g^{-1}(a), & \text{if } g \in \varphi(H), \\ 0, & \text{if } g \notin \varphi(H). \end{cases}$$

The *compression map with respect to* φ is defined by

$$\operatorname{comp}_\varphi\colon KK_*^G(\operatorname{ind}_\varphi A, B) \xrightarrow{\operatorname{res}_\varphi} KK_*^H(\operatorname{res}_\varphi \operatorname{ind}_\varphi A, \operatorname{res}_\varphi B) \xrightarrow{G_*} KK_*^H(A, \operatorname{res}_\varphi B).$$

Theorem 2.2.26. *Let $\varphi\colon H \to G$ be an open embedding. Then the map* $\operatorname{comp}_\varphi$ *is an isomorphism and the inverse* $(\operatorname{comp}_\varphi)^{-1}$ *induces an induction structure on $K_*^{(?,\mathbb{C})}$. Hence $K_*^{(?,\mathbb{C})}$ is an equivariant $\mathcal{O}$-homology theory.*

Let $\varphi\colon H \to G$ be cocompact. Then the map $\operatorname{ind}_\varphi$ *is an isomorphism and the inverse defines an induction structure on $K^*_{(?,\mathbb{C})}$, thus $K^*_{(?,\mathbb{C})}$ is an equivariant $\mathcal{COC}$-cohomology theory.*

Proof. The map $\operatorname{comp}_\varphi$ is an isomorphism by a result of Chabert and Echterhoff [14, Prop. 5.14]. Since H-compact subspaces of an H-space X correspond to G-compact subspaces of $\operatorname{ind}_\varphi X$, we can neglect the colimit which appears in the construction of $K_*^{(?,\mathbb{C})}$. Consequently, *comp*$_\varphi$ induces an isomorphism $K_*^{(G,\mathbb{C})}(\operatorname{ind}_\varphi ?) \to K_*^{(?,\mathbb{C})}(?)$.

The other case is well-known but lacks a proper reference. Thus we want to sketch the proof. For simplicity, we assume that $H \subseteq G$ is cocompact. Our candidate for the inverse map is

$$\varphi_G^H\colon KK_*^G(A, \operatorname{ind}_H^G B) \xrightarrow{\operatorname{res}_G^H} KK_*^H(\operatorname{res}_G^H A, \operatorname{res}_G^H \operatorname{ind}_H^G B) \xrightarrow{F_*} KK_*^H(\operatorname{res}_G^H A, B),$$

where F is the map

$$F\colon \operatorname{ind}_H^G B \to B, \quad f \mapsto f(e).$$

An immediate consequence is the identity $\varphi_G^H \circ \operatorname{ind}_H^G = \mathrm{id}$. In order to show $\operatorname{ind}_H^G \circ \varphi_G^H = \mathrm{id}$, consider a Kasparov triple $(\tilde{E}, \tilde{\varphi}, \tilde{T}) \in KK^G(A, \operatorname{ind}_H^G B)$. We just show the assertion for $\tilde{E}$. It suffices to show the identity

$$\operatorname{ind}_H^G(\tilde{E} \otimes_{\operatorname{ind}_H^G B} B) = \tilde{E},$$

where B is considered as a $(\operatorname{ind}_H^G B, G)$-module by $f \cdot b = f(e) \cdot b$. We get a map

$$\alpha\colon \tilde{E} \otimes_{\operatorname{ind}_H^G B} \operatorname{ind}_H^G B \to \operatorname{ind}_H^G\big(\tilde{E} \otimes_{\operatorname{ind}_H^G B} B\big), \quad e \otimes f \mapsto \big(g \mapsto g^{-1}e \otimes f(g)\big).$$

It is easily checked that α respects the scalar product for elementary tensors. Thus α respects the scalar product and is injective. Let $\phi \in C_0(G/H)$, then we obtain

$$\alpha\big(\phi \cdot (e \otimes f)\big) = \alpha\big(e \otimes (\phi \cdot f)\big) = \\ \Big(g \mapsto g^{-1}e \otimes \big(\phi(g)f(g)\big)\Big) = \Big(g \mapsto \phi(g)\big(g^{-1}e \otimes f(g)\big)\Big) = \phi \cdot \alpha(e \otimes f).$$

Furthermore, the evaluation

$$\alpha_g\colon \tilde{E} \otimes_{\operatorname{ind}_H^G B} \operatorname{ind}_H^G B \to \tilde{E} \otimes_{\operatorname{ind}_H^G B} B, \quad e \otimes f \mapsto g^{-1}e \otimes f(g)$$

is clearly surjective. Using a partition of unity argument, we can deduce that α is surjective.

Besides being an isomorphism, there are three other properties an induction structure must satisfy. However, these properties are clearly satisfied in both cases. $\square$

Remark 2.2.27. Let $\Gamma_G \subseteq \mathrm{TGFI}$ be the full subcategory with subgroups of G as objects. Let A be a G-C^*-algebra, then

$$K_*^{(H,A)}(X) = K_*^{(H,\operatorname{res}_G^H A)}(X)$$

assembles to an equivariant $(\Gamma_G, \mathcal{O})$-homology theory $K_*^{(?,A)}$ analogously to the above considerations.

If $\mathcal{F} = \mathcal{I}$, then $K_*^{(?,\mathbb{C})}$ and $K^*_{(?,\mathbb{C})}$ are both equivariant $\mathcal{I}$-(co)homology theories. In our second main example $\mathcal{F} = \mathcal{CO}$, we get an equivariant proper smooth homology theory $K_*^{(?,\mathbb{C})}$ but $K^*_{(?,\mathbb{C})}$ fails to be an equivariant proper smooth cohomology theory. Even in the simplest case, the corresponding groups are not isomorphic.

Example 2.2.28. Let $H = \{0\}$, $G = \mathbb{Z}$ and $X = \{\bullet\}$. Then we get

$$K^0_{(\{0\},\mathbb{C})}(\{\bullet\}) = \mathbb{Z} \quad \text{and} \quad K^0_{(\mathbb{Z},\mathbb{C})}(C_0(\mathbb{Z})) = 0.$$

Since the first identity is clear, it remains to show the second one. Let $C_0(\mathbb{R})_f$ be the $\mathbb{Z}$-C^*-algebra with the $\mathbb{Z}$-action being given by translation and $C_0(\mathbb{R})_t$ be the $\mathbb{Z}$-C^*-algebra with trivial $\mathbb{Z}$-action. Analogously to the classical case, we get Bott periodicity for $C_0(\mathbb{R})_f$ (see [25, Chap. 5]). Thus $\mathbb{C}$ and $C_0(\mathbb{R})_f \otimes C_0(\mathbb{R})_t$ are $KK^{\mathbb{Z}}$-equivalent and we obtain

$$KK_*^{\mathbb{Z}}(\mathbb{C}, C_0(\mathbb{Z})) = KK_*^{\mathbb{Z}}(C_0(\mathbb{R})_f \otimes C_0(\mathbb{R})_t, C_0(\mathbb{Z})).$$

With the aid of Bott periodicity, we can conclude

$$KK_*^{\mathbb{Z}}(C_0(\mathbb{R})_f \otimes C_0(\mathbb{R})_t, C_0(\mathbb{Z})) = KK_*^{\mathbb{Z}}(C_0(\mathbb{R})_f, C_0(\mathbb{R})_t \otimes C_0(\mathbb{Z})).$$

However, $\mathbb{R}$ with the $\mathbb{Z}$-action given by translation is just $\mathbb{R} = E\mathbb{Z}$, the classifying space of $\mathbb{Z}$. Therefore Baum-Connes and Bott periodicity leads to

$$KK_0^{\mathbb{Z}}(C_0(\mathbb{R})_f, C_0(\mathbb{R})_t \otimes C_0(\mathbb{Z})) = KK_0(\mathbb{C}, C_0(\mathbb{R})_t \otimes C_0(\mathbb{Z}) \rtimes \mathbb{Z}) = KK_1(\mathbb{C}, C_0(\mathbb{Z}) \rtimes \mathbb{Z}).$$

We obtain from the Pimsner-Voiculescu exact sequence [9, Thm. 10.2.1] an exact sequence

$$KK_1(\mathbb{C}, C_0(\mathbb{Z})) \to KK_1(\mathbb{C}, C_0(\mathbb{Z}) \rtimes \mathbb{Z}) \to KK_0(\mathbb{C}, C_0(\mathbb{Z})) \xrightarrow{\mathrm{id} - s_*} KK_0(\mathbb{C}, C_0(\mathbb{Z})),$$

where $s\colon \mathbb{Z} \to \mathbb{Z}$ sends $n \in \mathbb{Z}$ to $n+1$. We have $C_0(\mathbb{Z}) = \oplus_{n\in\mathbb{Z}}\mathbb{C}$, where $\oplus$ denotes the direct sum of C^*-algebras. Since $KK_i(\mathbb{C}, ?)$ is compatible with direct sums by [45, Prop. 7.13], we obtain $KK_i(\mathbb{C}, C_0(\mathbb{Z})) = \oplus_{n\in\mathbb{Z}} KK_i(\mathbb{C}, \mathbb{C})$ and an exact sequence

$$0 \to KK_1(\mathbb{C}, C_0(\mathbb{Z}) \rtimes \mathbb{Z}) \to \bigoplus_{n\in\mathbb{Z}} \mathbb{Z} \xrightarrow{\mathrm{id} - s_*} \bigoplus_{n\in\mathbb{Z}} \mathbb{Z},$$

where s_* is a shift of factors. Hence $\mathrm{id} - s_*$ is injective and we obtain

$$K^0_{(\mathbb{Z},\mathbb{C})}(C_0(\mathbb{Z})) = KK_1(\mathbb{C}, C_0(\mathbb{Z}) \rtimes \mathbb{Z}) = 0.$$

In the case of $K_*^{(?,\mathbb{C})}$, an induction structure is only given for open subgroups. There need not be an induction structure for non-open subgroups:

Example 2.2.29. Let $H = \{0\}$, $G = \mathbb{R}$ and $X = \{\bullet\}$. Then we obtain

$$K_0^{(\{0\},\mathbb{C})}(\{\bullet\}) = \mathbb{Z} \quad \text{and} \quad K_0^{(\mathbb{R},\mathbb{C})}(\mathbb{R}) = 0.$$

The first identity is clear. Moreover, we have

$$K_0^{(\mathbb{R},\mathbb{C})}(\mathbb{R}) = KK_0^{\mathbb{R}}(C_0(\mathbb{R}), \mathbb{C}) = KK_0(C_0(\mathbb{R}), \mathbb{C}) = 0,$$

where the middle identity is a result of Kasparov [24, Cor. 5.7].

It is very unsatisfactory that we do not have an adequate K-theory for $\mathcal{F} = \mathcal{CO}$ yet. So let us have another try.

Definition 2.2.30. Let $\underline{E}G = E_{\mathcal{COP}}G$ be the classifying space of proper actions. Then we define

$$K_G^*(X) = \operatorname*{colim}_{Y \subseteq \underline{E}G \ G\text{-compact}} KK_*^G(C_0(Y), C_0(X)).$$

Let $\varphi\colon H \to G$ be a closed embedding. Then we get an induction structure by

$$\begin{aligned}\mathrm{ind}_\varphi\colon K^*_H(X) = &\operatorname*{colim}_{Y\subseteq \underline{E}H\ H\text{-compact}} KK^H_*(C_0(Y), C_0(X))\\ &\xrightarrow{\mathrm{ind}_\varphi} \operatorname*{colim}_{Y\subseteq \underline{E}H\ H\text{-compact}} KK^H_*(C_0(G\times_\varphi Y), C_0(G\times_\varphi X))\\ &\xrightarrow{F} \operatorname*{colim}_{\tilde{Y}\subseteq \underline{E}G\ G\text{-compact}} KK^G_*(C_0(\underline{E}G), C_0(G\times_\varphi X)).\end{aligned}$$

We have to make F precise. Since $\underline{E}G$ is a classifying space we get a unique (up to G-homotopy) G-map $f\colon G\times_\varphi Y \to \underline{E}G$. However, $G\times_\varphi Y$ is G-compact and f factorizes over a G-compact subset $\tilde{Y} \subseteq \underline{E}G$. Therefore the construction induces a map on the colimits and further a map on KK^G_*.

Theorem 2.2.31. *Let $\mathcal{F} = \mathcal{CL}$ be the full family of closed subgroups. Then K^*_G defines an equivariant $\mathcal{CL}$-cohomology theory.*

Proof. The statement that K^*_G is a G-cohomology theory has already been proven above. It remains to show that the induction map is an isomorphism. This was shown by Chabert and Echterhoff in [14]. □

From the point of view of the previous discussion, we define $K^?_*$ by $K^?_* = K^{(?,\mathbb{C})}_*$.

Remark 2.2.32. Baum and Connes [5] suggested to study $K_*(C_0(?)\rtimes G)$. This is an equivariant proper smooth cohomology theory. Chabert and Echterhoff [14] showed that the Baum-Connes assembly map defines a natural transformation

$$\mu\colon K^*_G(X) \to K_*(C_0(X)\rtimes G),$$

which is compatible with the induction structures. If G is compact, μ is an isomorphism by the Green-Julg theorem. Hence μ is an equivalence of proper smooth cohomology theories.

Theorem 2.2.33. *If G is compact, K^*_G coincides with the classical definition of Atiyah and Segal [2], i.e., Grothendieck construction of G-vector bundles. If G is prodiscrete, K^*_G also coincides on finite proper smooth G-CW-complexes with the "classical" definition.*

Proof. If G is compact, this is the well known Swan isomorphism.

If G is locally compact, Phillips [40] described $K_*(C_0(X)\rtimes G)$ by (possible infinite dimensional) G-vector bundles. In the case of a prodiscrete group G, Sauer [46] showed that the "classical" definition, using finite dimensional G-vector bundles, induces an equivariant proper smooth cohomology theory $\mathcal{H}^*_?$. Thus the inclusion of finite dimensional G-vector bundles in arbitrary G-vector bundles induces a natural transformation $\tau\colon \mathcal{H}^*_?(??) \to K_*(C_0(??)\rtimes ?)$ of equivariant proper smooth cohomology theories. Since τ is an isomorphism for compact G, the natural transformation τ is even an equivalence, as required. □

Remark 2.2.34. For an l-group G finite vector bundles do not in general yield a proper smooth G-homology theory. An example is given by Sauer [46, p.435-437].

Corollary 2.2.35. *Let G be compact. The coefficients of K_*^G and K_G^* are given by*

$$K_*^G(\{\bullet\}) = K_G^*(\{\bullet\}) = \begin{cases} R(G), & \text{if } * \text{ is even,} \\ 0, & \text{if } * \text{ is odd,} \end{cases}$$

where $R(G)$ denotes the ring of (finite dimensional) complex G-representations. Let $\varphi\colon H \to G$ be a compact open embedding Then the maps

$$(\mathrm{ind}_\varphi)_h\colon R(H) = K_0^H(\{\bullet\}) \xrightarrow{\mathrm{comp}_\varphi} K_0^G(G/\varphi(H)) \xrightarrow{\mathrm{pr}} K_0^G(\{\bullet\}) = R(G) \qquad \text{and}$$
$$(\mathrm{ind}_\varphi)_c\colon R(H) = K_H^0(\{\bullet\}) \xleftarrow[\cong]{\mathrm{ind}_\varphi} K_G^0(G/\varphi(H)) \xrightarrow{\mathrm{pr}} K_G^0(\{\bullet\}) = R(G)$$

coincide with the ordinary induction map $\mathrm{ind}_\varphi\colon R(H) \to R(G)$*. Furthermore, the restriction maps*

$$\mathrm{res}_\varphi(\{\bullet\})\colon R(G) \to R(H)$$

of K_*^G and K_G^*, respectively, coincide with the ordinary restriction map $\mathrm{res}_\varphi\colon R(G) \to R(H)$*.*

Definition 2.2.36. Let G be compact. We define

$$\mathcal{R}(G) = \{\, f\colon G \to \mathbb{C} \mid f \text{ conjugation invariant and locally constant} \,\}$$

and call it the *group of class functions.*

Lemma 2.2.37. *Let G be compact. The character map induces an inclusion*

$$\chi\colon R(G) \otimes \mathbb{C} \hookrightarrow C(G, \mathbb{C})^G,$$

where $C(G,\mathbb{C})^G$ denotes the set of conjugation invariant continuous (complex valued) functions on G. In this case, the restriction maps are given by restricting the source. If $\varphi\colon H \to G$ is an open embedding, the induction map is given by

$$\mathrm{ind}_\varphi(f)(g) = \sum_{\substack{[s] \in G/\varphi(H) \\ s^{-1}gs \in \varphi(H)}} f(s^{-1}gs).$$

If G is an l-group, we get an isomorphism $R(G) \otimes \mathbb{C} \cong \mathcal{R}(G)$.

Proof. We only prove the last statement because this is the only one which might not be well known.

Let G be a compact l-group. Then G is profinite and admits a basis of topology which consists of normal open subgroups. Let

$$N(G) = \{\, H \subseteq G \mid H \text{ open normal} \,\}.$$

Then we obtain

$$G = \lim_{H \in N(G)} G/H \quad \text{and, consequently,} \quad R(G) \cong \operatorname*{colim}_{H \in N(G)} R(G/H).$$

The isomorphism is given in one direction by $\mathrm{res}_G^{G/H}$ and, in the other direction, by dividing out the kernel of the group homomorphism $G \to GL_n(\mathbb{C})$ associated to the representation. Note that the only compact totally disconnected subgroups of $GL_n(\mathbb{C})$ are the finite ones. Thus the kernel is automatically open. Since we have an isomorphism $R(K) \otimes \mathbb{C} \cong \mathcal{R}(K)$ for any finite group K, we get

$$R(G) \otimes \mathbb{C} \cong \operatorname*{colim}_{H \in N(G)} \mathcal{R}(G/H) \cong \mathcal{R}(G).$$

□

Proposition 2.2.38. *Let $H, K \subseteq G$ be subgroups such that H is open and K is closed. Let A be an H-C^*-algebra and M be an (A, G)-Hilbert module. Then we obtain an isomorphism*

$$\mathrm{res}_G^K \mathrm{ind}_H^G M \cong \bigoplus_{KgH \in K \backslash G/H} \mathrm{ind}_{c(g)\colon H \cap g^{-1}Kg \to K} \circ \mathrm{res}_H^{H \cap g^{-1}Kg} M.$$

Hence, the coefficients of $K_*^{(G,A)}$ and K_G^* satisfy the double coset formula and have a Mackey structure.

Proof. If K is open the well known proof of the discrete case carries over. Actually, in what follows, we only need this case. A proof of the general statement is given in [14, Lem. 6.3]. □

This is our license to write

Corollary 2.2.39. *On the category of locally compact second countable groups $K_*^?$ and $K_?^*$ define proper smooth equivariant (co)homology theories, which have a Mackey functor on coefficients. More generally, $K_*^{(?,A)}$ is an equivariant $(\Gamma_G, \mathcal{CO})$-homology theory, which has a Mackey functor on coefficients for any G-C^*-algebra A.*

At the end of this section, we want to prove that Bredon homology (for a certain coefficient system) and cosheaf homology coincide. Before we can do that, we must define cosheaf homology.

In the following, let G be an l-group.

Let X be a proper smooth G-simplicial complex, i.e., a simplicial complex X with a simplicial operation of G such that the isotropy groups are compact open. For simplicity, we will assume that X is oriented, which means the vertices of each simplex are linearly ordered (two orderings are regarded as the same if they differ by an even permutation). The ordering need not be done with any regard to the inclusion relations among simplices. But, for simplicity again, we will assume that G acts in an orientation-preserving manner on X. If σ is a simplex in X, then denote by G_σ its isotropy group in G. It is a compact open subgroup of G. Now we can form the vector space

$$C_n^G(X) = \bigoplus_{\sigma \in X^n} \mathcal{R}(G_\sigma),$$

where the direct sum is taken over the set X^n of n-simplices in X. We will write elements of $C_n^G(X)$ as finite formal sums

$$\sum_{\sigma \in X^n} \varphi_\sigma[\sigma],$$

where $\varphi_\sigma \in \mathcal{R}(G_\sigma)$. The differentials are given by

$$d_n(\varphi_\sigma[\sigma]) = \sum_{\substack{\eta \subseteq \sigma \\ \eta \in X^{n-1}}} (-1)^{\langle \eta : \sigma \rangle} \operatorname{ind}_{G_\sigma}^{G_\eta}(\varphi_\sigma)[\eta].$$

Here, $\langle \eta : \sigma \rangle$ denotes the incidence number, i.e., $\langle \eta : \sigma \rangle = 1$ if the induced orientation of η by σ and the given one on η coincide. Otherwise, we set $\langle \eta : \sigma \rangle = -1$. Now *cosheaf homology*, sometimes also called *chamber homology* is defined as

$$CH_n^G(X) = H_n(C_*^G(X)_G, (d_*)_G),$$

where $C_G^*(X)_G = C_G^*(X) \otimes_{\mathcal{R}(G)} \mathbb{C}$ denotes the complex of coinvariants.

Remark 2.2.40. Baum, Connes and Higson [6] developed a more general notion of cosheaf homology. Let X be a simplicial complex. A *cosheaf $\mathcal{A}$ on X* consists of the following data:

(i) For each simplex σ of X one has an abelian group A_σ.

(ii) For each inclusion of simplices $\eta \subseteq \sigma$ a homomorphism of abelian groups

$$\varphi_\eta^\sigma \colon A_\eta \to A_\sigma \text{ such that } \varphi_\tau^\sigma = \varphi_\tau^\eta \varphi_\eta^\sigma,$$

whenever $\tau \subseteq \eta \subseteq \sigma$, and $\varphi_\sigma^\sigma = \mathrm{id}$ for each σ.

An example of a cosheaf is given by $\mathcal{R}(G_?)$. This explains the name cosheaf homology.

Remark 2.2.41. Suppose $\Delta \subseteq X$ is a subcomplex which is a fundamental domain for the action of G on X in the sense that the G-orbit of any simplex in X contains precisely one simplex from Δ. Then the complex of coinvariants can be identified with the complex

$$\tilde{C}_0^G(\Delta) \xleftarrow{d_1} \tilde{C}_1^G(\Delta) \xleftarrow{d_2} \tilde{C}_3^G(\Delta) \xleftarrow{d_3} \cdots,$$

where $\tilde{C}_n^G(X)$ denotes the direct sum

$$\tilde{C}_n^G(X) = \bigoplus_{\sigma \in \Delta^n} \mathcal{R}(G_\sigma)$$

over the n-simplices in Δ (not X). The differential d_n is defined exactly as above. Note that the latter complex does not involve coinvariants. The subcomplex Δ is called a chamber, which motivates the name chamber homology.

The next proposition is taken from Voigt [55, Cor. 8.4].

Proposition 2.2.42. *We define the* $\mathbb{C}\mathrm{Or}_{\mathcal{CO}}(G)$*-module* $\mathcal{R}_G$ *by* $\mathcal{R}_G(G/H) = \mathcal{R}(H)$*. Then there exists a natural equivalence of proper smooth (simplicial) G-homology theories*

$$\tau\colon CH_*^G(X) \xrightarrow{\cong} H_*^{\mathrm{Or}_{\mathcal{CO}}(G)}(X;\mathcal{R}_G).$$

Proof. We define

$$\phi_n\colon C_n^G(X)_G \to C_n^{\mathrm{Or}_{\mathcal{CO}}(G)}(X)\otimes\mathcal{R}_G, \quad \varphi_\sigma[\sigma]\mapsto \sigma_{G_\sigma}\otimes\varphi_\sigma,$$

where

$$\sigma_{G_\sigma}\in C_n^{simp}(X^{G_\sigma}) = \bigoplus_{\eta\in(X^{G_\sigma})^p}\mathbb{Z}_\eta$$

denotes the canonical generator of $\mathbb{Z}_\sigma$. Note that σ is oriented. Further, there is a canonical identification of the simplicial chain complex C_*^{simp} with the cellular chain complex C_*^{cell}. First of all, we have to check that ϕ is well-defined. However, this is clear because the right hand side identifies conjugated elements. We obtain

$$\begin{aligned}\phi_n d_n(\varphi_\sigma[\sigma]) &= \phi\left(\sum_{\substack{\eta\subseteq\sigma\\ \eta\in X^n}}(-1)^{\langle\eta:\sigma\rangle}\operatorname{ind}_{G_\sigma}^{G_\eta}(\varphi_\sigma)[\eta]\right) = \sum_{\substack{\eta\subseteq\sigma\\ \eta\in X^n}}\eta_{G_\eta}\otimes(-1)^{\langle\eta:\sigma\rangle}\operatorname{ind}_{G_\sigma}^{G_\eta}(\varphi_\sigma)\\ &= \sum_{\substack{\eta\subseteq\sigma\\ \eta\in X^n}}(-1)^{\langle\eta:\sigma\rangle}\eta_{G_\sigma}\otimes\varphi_\sigma = d_n(\sigma_{G_\sigma}\otimes\varphi_\sigma) = d_n\phi_n(\varphi_\sigma).\end{aligned}$$

Thus φ is a map of chain complexes. We consider the map

$$\psi\colon C_n^{\mathrm{Or}_{\mathcal{CO}}(G)}(X)\otimes\mathcal{R}_G\to C_n^G(X)_G, \quad \sigma_H\otimes\phi_H\mapsto\operatorname{ind}_H^{G_\sigma}(\phi_H)[\sigma].$$

Obviously, on the one hand, ψ is well-defined and $\psi\circ\phi=\mathrm{id}$. On the other hand, we have

$$\phi\circ\psi(\sigma_H\otimes\phi_H) = \sigma_{G_\sigma}\otimes\operatorname{ind}_H^{G_\sigma}(\phi_h) = \sigma_H\otimes\phi_H$$

and, consequently, $\phi\circ\psi=\mathrm{id}$. □

We will see that cosheaf homology is a very explicit tool to compute Bredon homology (see the end of Section 4.1).

3 Flat and Injective Modules over a Category

As was pointed out in the introduction, the construction of the Chern character splits up into two parts. One of them is to prove that the coefficient module of a given equivariant (co)homology theory is flat (injective). This is done in this chapter. In the last section, we discuss, in more detail, the case in which the equivariant (co)homology theory is K-theory.

3.1 Flat Modules over a Category

In the first subsection, we introduce the basic notions to state the classification theorem for projective modules over an EI-category (Theorem 3.1.7). In the second subsection, we give criteria when a Mackey functor induces a flat $R\mathsf{Or}_{\mathcal{F}}(G)$-module.

3.1.1 Classification of Projective Modules

In the following, let R be a commutative ring, Γ be a small EI-category and M be a covariant $R\Gamma$-module.

For $x \in \mathrm{Ob}(\Gamma)$, we denote by $R[x] = R\,\mathrm{aut}(x)$ the group ring of the automorphism group.

Definition 3.1.1. Let $x \in \mathrm{Ob}(\Gamma)$. We define the following functors:

(i) *Extension:* $E_x \colon R[x]\text{-}\mathtt{mod} \to R\Gamma\text{-}\mathtt{mod}$, $M \mapsto M \otimes_{R[x]} R\,\mathrm{mor}(x, ?)$.

(ii) *Splitting:* $S_x \colon R\Gamma\text{-}\mathtt{mod} \to R[x]\text{-}\mathtt{mod}$ is defined by the following exact sequence

$$\bigoplus_{\substack{f\colon y \to x \\ f \text{ is not an isomorphism}}} M(y) \xrightarrow{\oplus f} M(x) \to S_x M \to 0.$$

Remark 3.1.2. Note that E_x is a special case of induction since we have $E_x = \mathrm{ind}_{\mathrm{inc}}$, where $\mathrm{inc}\colon \{\, x \,\} \to \Gamma$ denotes the canonical inclusion.

Definition 3.1.3. Let $x \in \mathrm{Ob}(\Gamma)$. We define the *inclusion functor* I_x by

$$I_x \colon R[x]\text{-}\mathtt{mod} \to R\Gamma\text{-}\mathtt{mod}, \quad I_x(M)(y) = \begin{cases} M, & \text{if } x \cong y, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.1.4. *For $x \in \mathrm{Ob}(\Gamma)$ the pairs (E_x, res_x) and (I_x, S_x) are adjoint pairs and tensor adjoint pairs.*

Proof. Since $E_x = \mathrm{ind}_{\mathrm{inc}}$, the assertion for (E_x, res_x) is proven in Proposition 1.3.11. The case of (I_x, S_x) is obvious. □

Proposition 3.1.5. *The functors S_x and E_x respect direct sums and the properties finitely generated, free, projective and flat.*

Proof. Since the tensor product respects direct sums, E_x does. Furthermore, S_x respects direct sums by definition. The identity $E_x(R[x]) = R\Gamma(x, ?)$ is trivial and, by an easy observation, we obtain

$$S_x(R\Gamma(y, ?)) = \begin{cases} R[x], & x \cong y, \\ 0, & x \ncong y. \end{cases}$$

We deduce that E_x, S_x respect the property free and thus they respect the properties projective and finitely generated.

By Lemma 1.3.12 and Lemma 3.1.4, the functor S_x respects the property flat and E_x respects the property flat by Corollary 1.3.13. □

Let M be an $R\Gamma$-module. Suppose that all S_xM are projective. Then we can choose splits $\sigma_x\colon S_xM \to M(x)$ and can define the following map

$$T\colon \bigoplus_{(x)\in\mathrm{Is}(\Gamma)} E_xS_xM \xrightarrow{\bigoplus_{(x)\in\mathrm{Is}(\Gamma)} E_x(\sigma_x)} \bigoplus_{(x)\in\mathrm{Is}(\Gamma)} E_xM(x) \xrightarrow{\bigoplus_{(x)\in\mathrm{Is}(\Gamma)} i_x(M)} M. \quad (3.1.5.1)$$

Here, $i_x(M)\colon E_x(M(x)) \to M$ is defined to be the adjoint map of $\mathrm{id}\colon \mathrm{res}_x M \to \mathrm{res}_x M$, recall that (E_x, res_x) is an adjoint pair.

Definition 3.1.6. An $R\Gamma$-module M is of *finite length* if $l(x) < \infty$ for each $x \in \mathrm{Ob}(\Gamma)$ with $M(x) \neq 0$. We define *finite colength* for an $R\Gamma$-module M in the same way.

Theorem 3.1.7. *Let M be a covariant $R\Gamma$-module of finite length and suppose that for all objects $x \in \mathrm{Ob}(\Gamma)$ the $R[x]$-module S_xM is projective. Then the map*

$$T\colon \bigoplus_{(x)\in\mathrm{Is}(\Gamma)} E_xS_xM \to M$$

is surjective. It is bijective if and only if M is projective.

Proof. The proof of surjectivity is very similar to the surjectivity part of Lemma 3.1.11, but an explicit proof is given in [29, Thm. 2.11].

Since E_x respects the property projective and S_xM is projective by assumption, M is projective if T is bijective. The other implication is proven in [28, Cor. 9.40]. □

Remark 3.1.8. The above theorem remains true if we replace "covariant" by "contravariant" and "length" by "colength".

Remark 3.1.9. Let F be a free module and choose a decomposition $F \cong \bigoplus_{i\in I} R\Gamma(x_i, ?)$. Then we have

$$\begin{aligned}\bigoplus_{(x)\in \mathrm{Is}(\Gamma)} E_x S_x \Big(\bigoplus_{i\in I} R\Gamma(x_i, ?)\Big) &\cong \bigoplus_{(x_i), i\in I} E_{x_i}\Big(\bigoplus_{|\{\, x_j | x_j \in (x_i)\,\}|} R[x_i]\Big)\\ &\cong \bigoplus_{(x_i), i\in I} \Big(\bigoplus_{|\{\, x_j | x_j \in (x_i)\,\}|} R\Gamma(x_i, ?)\Big)\\ &\cong \bigoplus_{i\in I} R\Gamma(x_i, ?) \cong F.\end{aligned}$$

However, this tells us nothing about T. The chosen splits might not be compatible with the chosen decomposition.

Suppose $S_x M$ is flat for an $R\Gamma$-module M and any $x \in \mathrm{Ob}(\Gamma)$. Since E_x respects the property flat, the $R\Gamma$-module $\oplus_{(x)\in \mathrm{Is}\,\Gamma} E_x S_x M$ is flat. Thus, M is flat if T is an isomorphism. However, the converse is false, hence the corresponding statement of Theorem 3.1.7 is false. We want to give an example:

Example 3.1.10. We consider the category Γ

$$\begin{tikzcd} x \arrow[r, bend left, "f_1"] \arrow[r, bend right, "f_2"'] & y \arrow[loop right, "g"] \end{tikzcd}$$

with the relations $f_2 = g \circ f_1$ and $g^2 = \mathrm{id}$. Let $\underline{\mathbb{Q}}$ be the constant $\mathbb{Q}\Gamma$-module, i.e., every object is $\mathbb{Q}$ and every morphism is id. For a $\mathbb{Q}\Gamma$-module M we obtain

$$M \otimes_{\mathbb{Q}\Gamma} \underline{\mathbb{Q}} = M(y) \otimes_{\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]} \mathbb{Q}.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Q}[\mathbb{Z}/2\mathbb{Z}]$-module, $\underline{\mathbb{Q}}$ is a flat $\mathbb{Q}\Gamma$-module. However, we have

$$E_x S_x \underline{\mathbb{Q}}(y) = E_x \mathbb{Q}(y) = \mathbb{Q}^2 \neq \mathbb{Q} = \underline{\mathbb{Q}}(y).$$

3.1.2 Mackey Functors

In the following, let G be a topological group and $\mathcal{F}$ a smooth family of subgroups with $[H : K] < \infty$ for any $H, K \in \mathcal{F}$. Let $M\colon \Gamma \to R$ be a Mackey functor such that $\mathcal{F} \subseteq \mathrm{Ob}(\Gamma)$. It defines a covariant $R\mathtt{Sub}_{\mathcal{F}}(G)$-module

$$M^G\colon \mathtt{Sub}_{\mathcal{F}}(G) \to R\text{-}\mathrm{MOD}, \quad H \to M(H)$$

by means of the covariant structure of the bifunctor M.

In the case of a discrete group G and $\mathcal{F} = \mathcal{CO}$, which simplifies to $\mathcal{CO} = \mathcal{FIN} = \{\, H \subseteq G \mid H \text{ finite}\,\}$, Lück [29, Thm. 5.2] showed that M^G is projective if $\mathbb{Q} \subseteq R$ and R is semisimple. We want to generalize this result to topological groups. Actually, we are only interested in the question whether M^G is flat or not.

Let $N \in \mathcal{F}$ and define

$$M_N^G(H) = \begin{cases} M^G(H), & \text{if } (N) \le (H), \\ 0, & \text{otherwise.} \end{cases}$$

For convenience, we abbreviate $I_{\mathcal{F}} = \operatorname{Is}\mathtt{Sub}_{\mathcal{F}}(G)$, which is a partially ordered set. We recall that the order is given by

$$(H) \le (K) \Longleftrightarrow \exists H' \in \mathcal{F}:\ (H') = (H) \text{ and } H' \subseteq K.$$

Now we obtain

$$M^G = \operatorname*{colim}_{N \in I_{\mathcal{F}}} M_N^G. \tag{3.1.10.1}$$

We want to show

Lemma 3.1.11. *Suppose* $\mathbb{Q} \subseteq R$, $\mathtt{Sub}_{\mathcal{F}}(G)$ *is an EI-category and* $l_K(H), [H:K] < \infty$ *for any* $H, K \in \mathcal{F}$ *with* $K \subseteq H$. *Then* M^G *is a flat* $R\mathtt{Sub}_{\mathcal{F}}(G)$*-module if for any* $H \in \mathcal{F}$ *the* $\widetilde{RW_GH}$*-module* $M^G(H)$ *is flat and the canonical projection* $\operatorname{pr}\colon M_N^G(H) \to S_H M_N^G$ *splits for any* $N \in \mathcal{F}$.

Proof. The functor $\operatorname{colim}_{N \in I_{\mathcal{F}}}$ is an exact functor. It commutes with $- \otimes_{R\mathtt{Sub}_{\mathcal{F}}(G)} L$ for every contravariant $R\mathtt{Sub}_{\mathcal{F}}(G)$-module L since $- \otimes_{R\mathtt{Sub}_{\mathcal{F}}(G)} L$ has a right adjoint, namely $\hom(L, -)$. Hence by (3.1.10.1), M^G is a flat covariant $R\mathtt{Sub}_{\mathcal{F}}(G)$-module if each $R\mathtt{Sub}_{\mathcal{F}}(G)$-module M_N^G is flat.

Next, we want to show for a fixed subgroup $N \in \mathcal{F}$ that M_N^G is flat. We set

$$\mathcal{F}^N = \{\, (H) \in I \mid (N) \le (H) \,\}.$$

Since we assume the existence of splits $\sigma_H\colon S_H M_N^G \to M_N^G(H)$, we can define a map of $R\mathtt{Sub}_{\mathcal{F}}(G)$-modules (cf. (3.1.5.1))

$$T_N\colon \bigoplus_{(H) \in I_{\mathcal{F}^N}} E_H \circ S_H M_N^G \longrightarrow M_N^G. \tag{3.1.11.1}$$

Namely, for an object K and $(H) \in I_{\mathcal{F}^N}$ the restriction of $T_N(K)$ to the summand $E_H S_H M_N^G = \operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H,K) \otimes_{RW_GH} S_H M_N^G$ sends $f \otimes x$ to $M(f) \circ \sigma_H(x)$ for $f\colon H \to K$ and $x \in S_H M_N$.

Next we show that T_N is surjective. Since $l_N(H) < \infty$ for the relative length by assumption, we can show by induction over $l_N(H)$ that $T_N(H)$ is surjective.

If $l_N(H) = 0$, then $M_N^G(H)$ is zero and thus $T_N(H)$ is obviously surjective. The induction step is done as follows. For any $R[W_GH]$-module L, there is an $R[W_GH]$-automorphism $L \xrightarrow{\cong} S_H \circ E_H L$ which is natural in L. It sends $x \in N$ to $\mathrm{id}_H \otimes_{R[W_GH]} x$. If K is another object in $\mathtt{Sub}_{\mathcal{F}}(G)$ which is not isomorphic to H, then $S_K E_H(L) = \{0\}$. This implies that $S_H T_N$ is an isomorphism. Hence, the surjectivity of $T_N(H)$ will follow if we can show that every element in the kernel of $p_H\colon M_N^G(H) \to S_H M_N^G$ lies in the image of $T_N(H)$. Such an element can be written as a finite sum of elements of the form $M_N^G(f_i)(x_i)$ for morphisms $f_i\colon H_i \to H$ which are not isomorphisms and elements $x_i \in M_N^G(H_i)$. Since $(H_i) < (H)$ and thus $l_N(H_i) < l_N(H)$, the induction hypothesis applies to H_i. Hence x_i is in the image of $T_N(H_i)$. Therefore $M_N^G(f_i)(x_i)$ is in the image of $T_N(H)$. This finishes the induction step.

Next, we show that the map T_N is injective. Fix an object $K \in \mathcal{F}^N$ and an element u in the kernel of $T_N(K)$. We have to show $u = 0$. Choose for any $(H) \in I_{\mathcal{F}^N}$ a representative $H \in (H)$. So in the sequel we get for these representatives that $H = K$ follows from $(H) = (K)$. Put

$$J(H) = \operatorname{mor}_{\mathsf{Sub}_{\mathcal{F}}(G)}(H, K)/\widetilde{W_G H}.$$

Then fix for any element $\overline{f} \in J(H)$ a homomorphism of topological groups $f\colon H \to K$ such that there exists a $g \in G$ with $f = c(g)$, the morphism which f represents in $\mathsf{Sub}_{\mathcal{F}}(G)$. For simplicity, we identify $\bar{f}$ and the fixed representative f in the sequel. So for two such homomorphisms $f_1, f_2\colon H \to K$ with $\overline{f_1} = \overline{f_2}$ we already have $f_1 = f_2$. We can find elements $x_{H,f} \in S_H M_N^G$ for $(H) \in I_{\mathcal{F}^N}$ and $\overline{f} \in J(H)$ such that only finitely many elements $x_{H,f}$ are different from zero and u can be written as

$$u = \sum_{(H)\in I_{\mathcal{F}^N}} \sum_{\overline{f}\in J(H)} (f\colon H \to K) \otimes_{R[W_G H]} x_{H,f}.$$

We want to show that all elements $x_{H,f}$ are zero. Suppose that this is not the case. Let (H_0) be maximal among those elements $(H) \in I_{\mathcal{F}^N}$ for which there is an $\overline{f} \in J(H)$ with $x_{H,f} \neq 0$, i.e., if for $(H) \in I$ the element $x_{H,f}$ is different from zero for some morphism $f\colon H \to K$ in $\mathsf{Sub}_{\mathcal{F}}(G)$ and there is a morphism $H_0 \to H$ in $\mathsf{Sub}_{\mathcal{F}}(G)$, then $(H_0) = (H)$. Fix $f_0\colon H_0 \to K$ with $x_{H_0,f_0} \neq 0$. We claim that the composition

$$\begin{aligned} A\colon \bigoplus_{(H)\in I_{\mathcal{F}^N}} E_H \circ S_H M_N^G(K) &\xrightarrow{T_N(K)} M_N^G(K) = M(K) \xrightarrow{\operatorname{res}_K^{\operatorname{im}(f_0)}} M(\operatorname{im}(f_0)) \\ &\xrightarrow{\operatorname{ind}_{f_0^{-1}\colon \operatorname{im}(f_0)\to H_0}} M(H_0) = M_N^G(H_0) \xrightarrow{\operatorname{pr}_{H_0}} S_{H_0} M_N^G \end{aligned}$$

maps u to $m \cdot x_{H_0,f_0}$ for some integer $m > 0$. This leads to a contradiction because $T_N(K)(u) = 0$ and $x_{H_0,f_0} \neq 0$.

In order to proof the claim, we consider $(H) \in I_{\mathcal{F}^N}$ and $\overline{f} \in J(H)$. It suffices to show

$$\begin{aligned} &A\left((f\colon H \to K) \otimes_{R[W_G H]} x_{H,f}\right) \\ &\qquad = \begin{cases} [K \cap N_G \operatorname{im}(f_0)\colon \operatorname{im}(f_0)] \cdot x_{H,f}, & \text{if } (H) = (H_0) \text{ and } \overline{f} = \overline{f_0}, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1.11.2}$$

This is obviously true for $x_{H,f} = 0$. Hence we only have to treat the case $x_{H,f} \neq 0$.

One easily checks that $A((f\colon H \to K) \otimes_{R[W_G H]} x_{H,f})$ is the image of $x_{H,f}$ under the composition

$$\begin{aligned} a(H,f)\colon S_H M_N^G &\xrightarrow{\sigma_{H,l}} M_N^G(H) = M(H) \xrightarrow{\mathrm{ind}_{f\colon H \to \mathrm{im}(f)}} M(\mathrm{im}(f)) \xrightarrow{\mathrm{ind}_{\mathrm{im}(f)}^K} M(K) \\ &\xrightarrow{\mathrm{res}_K^{\mathrm{im}(f_0)}} M(\mathrm{im}(f_0)) \xrightarrow{\mathrm{ind}_{f_0^{-1}\colon \mathrm{im}(f_0) \to H_0}} M(H_0) = M_N^G(H_0) \xrightarrow{\mathrm{pr}_{H_0}} S_{H_0} M_N^G. \end{aligned}$$

The double coset formula implies

$$\begin{aligned} &\mathrm{res}_K^{\mathrm{im}(f_0)} \circ \mathrm{ind}_{\mathrm{im}(f)}^K \\ &\quad = \sum_{k \in \mathrm{im}(f_0)\backslash K/\mathrm{im}(f)} \mathrm{ind}_{c(k)\colon \mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k \to \mathrm{im}(f_0)} \circ \mathrm{res}_{\mathrm{im}(f)}^{\mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k}. \end{aligned}$$

For $k \in \mathrm{im}(f_0)\backslash K/\mathrm{im}(f)$ we define the map $a(H,f)_k$ to be the composition

$$\begin{aligned} a(H,f)_k\colon S_H M_N^G &\xrightarrow{\sigma_{H,l}} M_N^G(H) = M(H) \xrightarrow{\mathrm{ind}_{f\colon H \to \mathrm{im}(f)}} M(\mathrm{im}(f)) \\ &\xrightarrow{\mathrm{res}_{\mathrm{im}(f)}^{\mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k}} M(\mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k) \\ &\xrightarrow{\mathrm{ind}_{c(k)\colon \mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k \to \mathrm{im}(f_0)}} M(\mathrm{im}(f_0)) \\ &\xrightarrow{\mathrm{ind}_{f_0^{-1}\colon \mathrm{im}(f_0) \to H_0}} M(H_0) = M_N^G(H_0) \xrightarrow{\mathrm{pr}_{H_0}} S_{H_0} M_N^G. \end{aligned}$$

Then we get

$$a(H,f) = \sum_{k \in \mathrm{im}(f_0)\backslash K/\mathrm{im}(f)} a(H,f)_k.$$

The composition

$$\mathrm{pr}_{H_0} \circ \mathrm{ind}_{f_0^{-1}\colon \mathrm{im}(f_0) \to H_0} \circ \mathrm{ind}_{c(k)\colon \mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k \to \mathrm{im}(f_0)}$$

is trivial if $c(k)\colon \mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k \to \mathrm{im}(f_0)$ is not an isomorphism. This implies

$$a(H,f) = \sum_{\substack{k \in \mathrm{im}(f_0)\backslash K/\mathrm{im}(f) \\ c(k)\colon \mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k \to \mathrm{im}(f_0) \text{ isomorphism}}} a(H,f)_k.$$

We consider $k \in \mathrm{im}(f_0)\backslash K/\mathrm{im}(f)$ such that $c(k)\colon \mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k \to \mathrm{im}(f_0)$ is an isomorphism. We get the following equality of integers

$$l_{\mathrm{im}(f) \cap k^{-1}\mathrm{im}(f_0)k}(K) = l_{\mathrm{im}(f_0)}(K) = l_{k^{-1}\mathrm{im}(f_0)k}(K).$$

This implies $k^{-1}\mathrm{im}(f_0)k \subseteq \mathrm{im}(f)$. Since H_0 has been chosen maximal among those H for which $x_{H,f} \neq 0$ for some morphism $f\colon H \to K$ and $x_{H,f} \neq 0$, we conclude

$(H) = (H_0)$. This implies $H = H_0$. Since we have $k^{-1}\operatorname{im}(f_0)k \cong H_0 = H \cong \operatorname{im}(f)$ in $\mathtt{Sub}_{\mathcal{F}}(G)$, there exists a $g \in G$ such that

$$g^{-1}\operatorname{im}(f)g = k^{-1}\operatorname{im}(f_0)k \subseteq \operatorname{im}(f).$$

Because $\mathtt{Sub}_{\mathcal{F}}(G)$ is an EI-category, we obtain $k^{-1}\operatorname{im}(f_0)k = \operatorname{im}(f)$. Hence $\overline{f_0} = \overline{f}$ in $J(H)$. This already implies $f = f_0$. We get $k \in N_G \operatorname{im}(f_0) \cap K$ and can deduce $a(H,f)_k = \mathrm{id}_{S_H M}$. We conclude

$$a(H,f) \ = \ \begin{cases} [K \cap N_G \operatorname{im}(f_0)\colon\ \operatorname{im}(f_0)] \cdot \mathrm{id}_{S_H M_N^G}, & \text{if } (H) = (H_0) \text{ and } \overline{f} = \overline{f_0}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the assertion (3.1.11.2) is true. Hence the map T_N is injective. This finishes the proof that the map T_N of (3.1.11.1) is an isomorphism of $R\mathtt{Sub}_{\mathcal{F}}(G)$-modules.

We conclude that M_N^G is isomorphic to $\bigoplus_{(H) \in I_{\mathcal{F}N}} E_H \circ S_H M_N^G$. The modules $S_H M_N^G$ are flat because they are direct summands of the modules $M^G(H)$, which are flat by assumption. Because E_H sends flat modules to flat modules, we conclude the flatness of M_N^G, which was claimed. □

Theorem 3.1.12. *Suppose $\mathbb{Q} \subseteq R$ is von Neumann regular, $l_K(H), [H : K] < \infty$ for any $H, K \in \mathcal{F}$ with $K \subseteq H$ and the corresponding semigroup $\operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H,H)$ is locally finite for every $H \in \mathcal{F}$. Let M be a Mackey functor. Then M^G is a flat $R\mathtt{Sub}_{\mathcal{F}}(G)$-module if for any $H, N \in \mathcal{F}$ the canonical projection $\mathrm{pr}\colon M^G(H) \to S_H M_N^G$ splits.*

Proof. Since $\operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H,H)$ is locally finite, it is a group by Corollary 1.2.24. This implies that $\widetilde{W_G H} = \operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H,H)$ is also locally finite. Thus by Theorem A.7 every $R\widetilde{W_G H}$-module L is flat and, in particular, so is $M^G(H)$. Now the assertion follows from by Lemma 3.1.11. □

Corollary 3.1.13. *Suppose $\mathbb{Q} \subseteq R$ is semisimple, $l_K(H), [H : K] < \infty$ for any $H, K \in \mathcal{F}$ with $K \subseteq H$ and the corresponding semigroup $\operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H,H)$ is finite for every $H \in \mathcal{F}$. Let M be a Mackey functor. Then M^G is a flat $R\mathtt{Sub}_{\mathcal{F}}(G)$-module.*

Proof. By Corollary 1.2.24, we obtain a finite group

$$\widetilde{W_G H} = \operatorname{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(H,H).$$

Thus $R\widetilde{W_G H}$ is semisimple and the splits required in Theorem 3.1.12 exist. Hence Theorem 3.1.12 is applicable. □

Lemma 3.1.14. *Let $\mathbb{Q} \subseteq R$ be von Neumann regular and $\mathtt{Sub}_{\mathcal{F}}(G)$ be an EI-category. Let M be a flat $R\mathtt{Sub}_{\mathcal{F}}(G)$-module and suppose that the group $C_G H/(H \cap C_G H)$ is locally finite for every $H \in \mathcal{F}$. Then $\operatorname{res}_{\mathrm{pr}} M$ is a flat $R\mathtt{Or}_{\mathcal{F}}(G)$-module, where $\mathrm{pr}\colon \mathtt{Or}_{\mathcal{F}}(G) \to \mathtt{Sub}_{\mathcal{F}}(G)$ denotes the canonical projection.*

Proof. We remind the reader that we have an identification (see Remark 1.2.9)

$$\begin{aligned}\mathrm{mor}_{\mathsf{Sub}_{\mathcal{I}}(G)}(H,K) &= \mathrm{mor}_{\mathsf{Or}_{\mathcal{I}}(G)}(G/H,G/K)/C_GH \\ &= \mathrm{mor}_{\mathsf{Or}_{\mathcal{I}}(G)}(G/H,G/K)/(C_GH/H\cap C_GH).\end{aligned}$$

Let N be an arbitrary $R\mathsf{Or}_{\mathcal{F}}(G)$-module. Then we obtain

$$\mathrm{ind}_{\mathrm{pr}} N(H) = N(H) \otimes_{R[C_GH/H\cap C_GH]} R.$$

Since $C_GH/H\cap C_GH$ is locally finite, R is a flat $R[C_GH/H\cap C_GH]$-module by Theorem A.7. Hence $\mathrm{ind}_{\mathrm{pr}}$ is exact. This implies that $\mathrm{res}_{\mathrm{pr}}$ respects the property flat by Corollary 1.3.13. □

Remark 3.1.15. The previous proof carries over directly to the projective case if we replace “von Neumann regular” by “semisimple”, “locally finite” by “finite” and consider coind instead of ind.

Finally, we get:

Theorem 3.1.16. *Suppose $\mathbb{Q}\subseteq R$ is von Neumann regular, $l_H(K),[H:K]<\infty$ for any $H,K\in\mathcal{F}$ with $K\subseteq H$ and the semigroup $\mathrm{mor}_{\mathsf{Or}_{\mathcal{F}}(G)}(G/H,G/H)$ is locally finite for every $H\in\mathcal{F}$. Let M be a Mackey functor. Then $\mathrm{res}_{\mathrm{pr}} M^G$ is a flat $R\mathsf{Or}_{\mathcal{F}}(G)$-module if for any $H,N\in\mathcal{F}$ the canonical projection $\mathrm{pr}\colon M^G(H)\to S_HM^G_N$ splits.*

Proof. Since $\mathrm{mor}_{\mathsf{Or}_{\mathcal{F}}(G)}(G/H,G/H)$ is locally finite, it is a group by Corollary 1.2.24. Further, $\widetilde{W_GH}$ is locally finite as a quotient of the locally finite group W_GH by Proposition A.5. Thus M^G is a flat $R\mathsf{Sub}_{\mathcal{F}}(G)$-module by Theorem 3.1.12. Finally, $C_GH/(H\cap C_GH)\subseteq W_GH$ is locally finite and the assertion follows now from Lemma 3.1.14. □

Remark 3.1.17. Let us consider the case $\mathcal{F}=\mathcal{CO}$. The space H/K is finite for compact open subgroups $K\subseteq H\subseteq G$. Therefore the condition $[H:K]<\infty$ is satisfied. We have seen in Example 1.2.20 that $\mathsf{Or}_{\mathcal{CO}}(G)$ might fail to be an EI-category. Furthermore, we can realize every discrete group H as an endomorphism group $H=\mathrm{mor}_{\mathsf{Or}_{\mathcal{CO}}(G)}(G/K,G/K)$, where

$$G = K\rtimes H \quad\text{and}\quad K=\prod_H \mathbb{Z}/2\mathbb{Z}$$

and H acts in the obvious way. We remark that G is unimodular by Lemma 1.2.16. Thus $\mathsf{Or}_{\mathcal{CO}}(G)$ is an EI-category with $l_{K'}(H')<\infty$ for all compact open subgroups $K'\subseteq H'\subseteq G$.

Corollary 3.1.18. *Let G be a (topological) group such that the corresponding semigroups $\mathrm{mor}_{\mathsf{Or}_{\mathcal{F}}(G)}(G/H,G/H)$ are finite and $l_K(H),[H:K]<\infty$ for every $H,K\in\mathcal{F}$. Let M be a Mackey functor and M^G be the covariant $R\mathsf{Sub}_{\mathcal{I}}(G)$-module. If $\mathbb{Q}\subseteq R$ is semisimple, then $\mathrm{res}_{\mathrm{pr}} M^G$ is a flat $R\mathsf{Or}_{\mathcal{I}}(G)$-module.*

Proof. Since $\mathrm{mor}_{\mathsf{Or}_{\mathcal{F}}(G)}(G/H, G/H)$ is finite, it is a group by Corollary 1.2.24. This implies that $\widetilde{W_G H}$ is finite as a quotient of the finite group $W_G H$. Therefore, $R\widetilde{W_G H}$ is semisimple and the projections $\mathrm{pr}\colon M^G(H) \to S_H M_N^G$ split. Now the assertion follows from the previous theorem. □

Corollary 3.1.19. *Let G be a prodiscrete l-group, M a Mackey functor and M^G be the covariant $R\mathsf{Sub}_{\mathcal{CO}}(G)$-module. If $\mathbb{Q} \subseteq R$ is semisimple, then M^G is flat as an $R\mathsf{Sub}_{\mathcal{CO}}(G)$-module. Moreover, $\mathrm{res}_{\mathrm{pr}} M^G$ is a flat $R\mathsf{Or}_{\mathcal{CO}}(G)$-module if the groups $C_G H/(H \cap C_G H)$ are locally finite for any compact open subgroup $H \subseteq G$.*

Proof. By Proposition 1.2.25 the groups $\widetilde{W_G H}$ are finite. By Corollary 1.2.17 and Lemma 1.2.21 we have $l_K(H) < \infty$ for all compact open subgroups $K \subseteq H \subseteq G$. Hence M^G is flat by Theorem 3.1.12. If the groups $C_G H/(H \cap C_G H)$ are locally finite, $\mathrm{res}_{\mathrm{pr}} M^G$ is flat by Lemma 3.1.14. □

Corollary 3.1.20. *Let G be a (topological) group. Let either be $\mathcal{F} = \mathcal{I}$ or G be a semisimple p-adic group and $\mathcal{F} = \mathcal{CO}$. Let M be a Mackey functor and M^G be the covariant $R\mathsf{Sub}_{\mathcal{F}}(G)$-module. If $\mathbb{Q} \subseteq R$ is semisimple, then $\mathrm{res}_{\mathrm{pr}} M^G$ is a flat $R\mathsf{Or}_{\mathcal{F}}(G)$-module.*

Proof. We want to apply Corollary 3.1.18. Hence we must show that the groups $\mathrm{mor}_{\mathsf{Or}_{\mathcal{CO}}(G)}(G/H, G/H)$ are finite and $l_K(H) < \infty$ for all $K \subseteq H$ with $K, H \in \mathcal{F}$. In the case of $\mathcal{F} = \mathcal{I}$, the first assertion is clear and the second follows from Proposition 1.2.21. In the case of a semisimple p-adic group G and $\mathcal{F} = \mathcal{CO}$, the groups $\mathrm{mor}_{\mathsf{Or}_{\mathcal{CO}}(G)}(G/H, G/H)$ are finite and $l_K(H) < \infty$ by Corollary 1.7.12. □

Remark 3.1.21. We have even more in the case of finite groups $\widetilde{W_G H}$. In fact, M^G is a colimit of projective modules. Since $\mathrm{res}_{\mathrm{pr}}$ has a right adjoint, namely coinduction with pr, it commutes with colim. Thus $\mathrm{res}_{\mathrm{pr}} M^G$ is a colimit of projective modules by Remark 3.1.15, too.

Note that we have *not* shown that M^G or $\mathrm{res}_{\mathrm{pr}} M$ are projective or that there is an isomorphism of $R\mathsf{Sub}_{\mathcal{F}}(G)$-modules

$$M \cong \bigoplus_{(H) \in I_{\mathcal{F}}} E_H \circ S_H M^G.$$

In order to prove such a stronger statement, we would need to choose the sections $\sigma_{H,N}\colon S_H M_N \to M_N(H)$ such that the following diagram commutes

$$\begin{tikzcd} S_H M_N \arrow[r, "\sigma_{H,N}"] \arrow[d, "S_H i_N"] & M_N(H) \arrow[d, "i_N(H)"] \\ S_H M_{N'} \arrow[r, "\sigma_{H,N'}"] & M_{N'}(H) \end{tikzcd}$$

for all $N, N' \in \mathcal{F}$, where $i_N\colon M_N \to M_{N'}$ denotes the inclusion. This might be impossible. For example, for $N = H$ we have $S_H M_N = M_N(H)$ and $\sigma_{H,N}$ must be the identity but $S_H i_N$ is not necessarily injective.

The next example shows that we cannot expect projectivity of M^G in general.

Example 3.1.22 (Borel construction). Let $G = \mathbb{Z}_p$ and $\mathcal{F} = \mathcal{CO} = \mathcal{I}$ (see Example 1.2.14). The Borel construction (see Section 2.1) with rational coefficients yields a Mackey functor

$$M^G(H) = \mathcal{H}_0^G(G/H) \otimes \mathbb{Q} = H_0(BH) \otimes \mathbb{Q} = \mathbb{Q}.$$

Thus M^G is isomorphic to the constant $\mathbb{Q}\mathtt{Sub}_{\mathcal{I}}(G)$-module $\underline{\mathbb{Q}}$ by Remark 2.1.2. However, $\underline{\mathbb{Q}}$ cannot be projective. In order to show that, it suffices to prove that every map

$$\varphi\colon \underline{\mathbb{Q}} \to \bigoplus_{i\in I} \mathbb{Q}\mathtt{Sub}_{\mathcal{I}}(G)(x_i, ?)$$

is trivial. Since the structure maps in $\underline{\mathbb{Q}}$ are isomorphisms, φ is completely determined by $\varphi(q_0)$ for some $q_0 \in \underline{\mathbb{Q}}(p^0\mathbb{Z}_p) = \mathbb{Q}$. We obtain

$$\varphi(q_0) = \sum_{j=1}^{n} y_{i_j} \quad \text{for some } y_{i_j} \in \mathbb{Q}\mathtt{Sub}_{\mathcal{I}}(G)(x_{i_j}, p^0\mathbb{Z}_p).$$

Let $k = \max\{\, col(x_{i_j}) \mid j = 1, \dots n\,\} + 1$. Thus we can deduce $\mathbb{Q}\mathtt{Sub}_{\mathcal{I}}(G)(x_{i_j}, p^k\mathbb{Z}_p) = 0$ and get the commutative diagram:

$$\begin{tikzcd}
q_0 \arrow[r, mapsto, "\varphi"] & \sum_{j=1}^{n} y_{i_j} \\
q_k \arrow[u, mapsto, "\mathrm{id}"] \arrow[r, mapsto, "\varphi"] & 0 \arrow[u, mapsto]
\end{tikzcd}$$

where $q_k \in \underline{\mathbb{Q}}(p^k\mathbb{Z}_p)$ is the obvious element. Hence $\varphi(q_0) = 0$ meaning that φ is trivial.

We conclude this section by stating a lifting property for flatness. Here, we cut off the module at the other end, which means we have to consider lim instead of colim.

Proposition 3.1.23. *Let $\mathcal{C}$ be an EI-category. For $x \in \mathcal{C}$ we denote by $\mathcal{C}_x$ the full subcategory of $\mathcal{C}$ which consists of all $y \le x$. Let M be a (covariant) $R\mathcal{C}$-module such that $\operatorname{res}_{\mathcal{C}_x} M$ is flat for all $x \in \mathcal{C}$. Then M is also flat.*

Proof. Let $x \in \mathcal{C}$ and define M_x by

$$M_x(y) = \begin{cases} M_x(y), & \text{if } y \le x, \\ 0, & \text{otherwise.} \end{cases}$$

We get $M = \lim_{x \in I_{\mathcal{C}}} M_x$, where $I_{\mathcal{C}}$ is the partially ordered set which is defined analogously to $I_{\mathcal{F}}$ (see the beginning of this section).

Furthermore, we have $N \otimes M_x = \operatorname{res}_{\mathcal{C}_x} N \otimes \operatorname{res}_{\mathcal{C}_x} M$ for an arbitrary (contravariant) $R\mathcal{C}$-module N. Therefore M_x is a flat $R\mathcal{C}$-module. Let $\varphi\colon A \to B$ be an injective map.

It suffices to prove that $\varphi \otimes \mathrm{id}\colon A \otimes M \to B \otimes M$ is injective. Since lim is left exact and M_x is flat, we deduce the injectivity of the following map

$$\lim_{x \in I_{\mathcal{C}}} (\varphi \otimes \mathrm{id})\colon \lim_{x \in I_{\mathcal{C}}} (A \otimes M_x) \to \lim_{x \in I_{\mathcal{C}}} (B \otimes M_x).$$

As the canonical map $C \otimes \lim_{x \in I_{\mathcal{C}}} M_x \to \lim_{x \in I_{\mathcal{C}}} (C \otimes M_x)$ is an isomorphism for every $R\mathcal{C}$-module C, the assertion follows. □

3.2 Injective Modules over a Category

This is the injective analogue of the previous section. In the first subsection, we state the classification result for injective modules over an EI-category. In the second subsection, we give criteria when a Mackey functor induces an injective $R\mathsf{Or}_{\mathcal{F}}(G)$-module. Unfortunately, the conditions are stricter than in the flat case because a derived limit comes into play. Therefore, we discuss derived limits, too.

3.2.1 Classification of Injective Modules

Definition 3.2.1. Let $x \in \mathrm{Ob}(\Gamma)$. We define the following functors:

(i) *Coextension:* $CE_x\colon R[x]\text{-}\mathtt{mod} \to R\Gamma\text{-}\mathtt{mod}$, $M \mapsto \hom(R\,\mathrm{mor}(x, ?), M)$.

(ii) *Cosplitting:* $CS_x\colon R\Gamma\text{-}\mathtt{mod} \to R[x]\text{-}\mathtt{mod}$ is defined by the following exact sequence

$$0 \to CS_x M \to M(x) \xrightarrow{\prod f} \prod_{\substack{f\colon x \to y \\ f \text{ is no isomorphism}}} M(y).$$

Remark 3.2.2. Note that CE_x is a special case, we have $CE_x = \mathrm{coind}_{\mathrm{inc}}$, where $\mathrm{inc}\colon \{\, x \,\} \to \Gamma$ is the canonical inclusion.

Lemma 3.2.3. *For $x \in \mathrm{Ob}(\Gamma)$, the pairs (res_x, CE_x) and (I_x, CS_x) are adjoint pairs.*

Proof. By Proposition 1.3.11, the first pair is an adjoint pair. The second assertion is clear. □

Corollary 3.2.4. *For $x \in \mathrm{Ob}(\Gamma)$ the functors CE_x and CS_x respect products and the property injective.*

Proof. The two functors clearly respect products. They respect the property injective by the previous lemma and Corollary 1.3.13. □

Let M be an $R\Gamma$-module. Suppose that $CS_x M$ is injective for every $x \in \mathrm{Ob}(\Gamma)$. Then we can choose splits $\sigma_x\colon M(x) \to CS_x M$ and define the following map

$$T_x\colon M \xrightarrow{\mathrm{id}_x} CE_x \,\mathrm{res}_x\, M \xrightarrow{\sigma_x} CE_x CS_x M,$$

where id_x denotes the map adjoint to $\mathrm{id}\colon \operatorname{res}_x M \to \operatorname{res}_x M$. Further, we can define

$$T = \prod_{x \in \mathrm{Is}(\Gamma)} T_x \colon M \to \prod_{x \in \mathrm{Is}(\Gamma)} CE_x CS_x M. \tag{3.2.4.1}$$

The next theorem is taken from [30, Thm. 2.14].

Theorem 3.2.5. *Let M be a contravariant $R\Gamma$-module of finite length and suppose that the $R[x]$-modules $CS_x M$ are injective for $x \in \mathrm{Ob}(\Gamma)$. Then the map*

$$T\colon M \to \prod_{x \in \mathrm{Is}(\Gamma)} CE_x CS_x M$$

is injective. It is bijective if and only if M is injective.

Remark 3.2.6. The previous theorem remains true if we replace “contravariant” by “covariant” and “length” by “colength”.

Let I be a partially ordered set and $(M_i)_{i\in I}$ be an inverse system of R-modules. By definition, this means that, for every $i \in I$, we have an R-module M_i and, for every pair $i, j \in I$ with $i \leq j$, we have R-homomorphisms $f_{ji}\colon M_j \to M_i$ such that

(i) $f_{ii} = \mathrm{id}_{M_i}$ for every $i \in I$,

(ii) $f_{ki} = f_{ji} \circ f_{kj}$ for every $i \leq j \leq k$.

In view of modules over a category, we can consider I as an EI-category by

$$\mathrm{Ob}(I) = I \quad \text{and} \quad |\operatorname{mor}_I(i,j)| = \begin{cases} 1, & \text{if } i \leq j, \\ 0, & \text{otherwise.} \end{cases}$$

The inverse system $(M_i)_{i\in I}$ of R-modules can be seen as a contravariant RI-module. Sometimes one defines an inverse system in a way that naturally yields a covariant RI-module. The following statements remain true in the covariant setting if we replace “length” by “colength”.

Before we can state the next lemma, we recall that a cofinal system $J \subseteq I$ of a partially ordered set is a subset such that for every $i \in I$ there exists a $j \in J$ such that $i \leq j$. An immediate consequence of this definition is

$$\lim_{i\in I} M_i = \lim_{j \in J} M_j$$

for an inverse system $(M_i)_{i\in I}$ and a cofinal system $J \subseteq I$.

Lemma 3.2.7. *Let $M_{i\in I}$ be an inverse system of R-modules and suppose I contains a cofinal system J which has finite length considered as an EI-category. Then we obtain*

$$\lim_{i\in I} M_i = \ker(\mathrm{id} - d), \quad \lim_{i\in I}{}^1 M_i = \operatorname{coker}(\mathrm{id} - d) \quad \textit{and} \quad \lim_{i\in I}{}^n M_i = 0 \quad \forall n > 1,$$

where

$$\mathrm{id} - d\colon \prod_{j\in J} M_j \to \prod_{j\in J} M_j, \quad (\mathrm{id} - d)(m_j)_{j_1,j_2} = \begin{cases} m_{j_1}, & \textit{if } j_1 = j_2, \\ -d_{j_1,j_2}(m_{j_1}), & \textit{if } j_1 < j_2, \\ 0, & \textit{otherwise} \end{cases}$$

and d_{j_1,j_2} are the structure maps of J.

Proof. Let $(F^n)_{n\in\mathbb{N}}\colon RI\text{-}\mathsf{mod} \to R\text{-}\mathsf{mod}$ be defined by $F^n = 0$ for all $n > 1$,

$$F^0 M_{j\in J} = \ker(\mathrm{id} - d) \quad \textit{and} \quad F^1 M_{j\in J} = \operatorname{coker}(\mathrm{id} - d).$$

It is clear that $F^0 M_{j\in J} = \lim_{j\in J} M_j$. We have to prove that $(F^n)_{n\in\mathbb{N}}$ is a universal δ-functor because $(\lim^n)_{n\in\mathbb{N}}$ is one and a universal δ-functor $(G^n)_{n\in\mathbb{N}}$ is uniquely determined by G^0. By the snake lemma, we get the desired long exact sequence from a short exact sequence. Thus $(F^n)_{n\in\mathbb{N}}$ is a δ-functor. It remains to show that F^1 vanishes on injective RI-modules. By Theorem 3.2.5, it suffices to check

$$F^1(CE_j M) = 0$$

for all R-modules M. The structure map d_{j_1,j_2} of $CE_j M$ is an isomorphism for $j_2 \leq j$ and $(CE_{j_2} M) = 0$ otherwise. Since I is of finite length, it follows easily that $\mathrm{id} - d$ is surjective and hence $F^1(CE_j M) = 0$. $\square$

Remark 3.2.8. If I is countable, a cofinal system of finite (co)length exists anyway by the following argument.

Since I is countable, we have a sequence $(i_0, i_1, i_2, \dots)$ with $I = \{\, i_n \mid n \in \mathbb{N} \,\}$. We define the sets J_n inductively by

$$J_0 = \{\, i_0 \,\}, \quad J_{n+1} = \begin{cases} J_n, & \text{if there exists a } j \in J_n \text{ with } i_{n+1} < j, \\ J_n \cup \{\, i_{n+1} \,\}, & \text{otherwise.} \end{cases}$$

We set $J = \bigcup_{n\in\mathbb{N}} J_n$. The subsystem $J \subseteq I$ is clearly cofinal. It remains to show that J has finite length. Let $\dots \leq j_1 \leq j_0$ be an increasing sequence in J. Then $j_0 \in J_k$ for some $k \in \mathbb{N}$ and hence $j_n \in J_k$ for every $n \in \mathbb{N}$. Since J_k is finite, the sequence $\dots \leq j_1 \leq j_0$ must become stationary and we obtain a cofinal system of finite length. Analogously, we get a cofinal system of finite colength.

Lemma 3.2.9. *Let $V_{i\in I}$ be an inverse system of k-vector spaces and suppose I has a cofinal system J of finite length. If the structure maps are injective and not surjective, we get*

$$\lim_{i\in I}{}^1 V_i \neq 0.$$

Proof. For simplicity let $I = \mathbb{N}$ be a tower. We define

$$\tilde{V}_i = \bigoplus_{n=i}^{\infty} V_n / V_{n+1}$$

and the obvious inclusions as structure maps. We choose projections $\mathrm{pr}_i\colon V_i \to V_{i+1}$ for all $i \in \mathbb{N}$ and get (non-canonical) isomorphisms $f_i\colon V_i \xrightarrow{\cong} \tilde{V}_i$. However, $(f_i)_{i\in\mathbb{N}}$ is an isomorphism of towers. Consequently, we can consider $(\tilde{V}_i)_{i\in\mathbb{N}}$ instead of $(V_i)_{i\in\mathbb{N}}$. We recall that $\lim^1_{i\in I} \tilde{V}_i \neq 0$ if the map

$$\mathrm{id} - d\colon \prod_{i\in\mathbb{N}} \tilde{V}_i \to \prod_{i\in\mathbb{N}} \tilde{V}_i, \quad (\dots, x_3, x_2, x_1) \mapsto (\dots, x_3 - d(x_4), x_2 - d(x_3), x_1 - d(x_2))$$

is not surjective. For every $i \in \mathbb{N}$ choose a non-zero element in

$$v_i \in V_i/V_{i+1} \subseteq \tilde{V}_i.$$

Then $(\dots, v_2, v_1) \notin \mathrm{im}(\mathrm{id} - d)$ because the preimage has to be $(\dots, \sum_{i=2}^{\infty} v_i, \sum_{i=1}^{\infty} v_i)$, but this does not lie in the direct sum. □

3.2.2 Mackey Functors

In the following, let G be a topological group and $\mathcal{F}$ a smooth family of subgroups with $[H : K] < \infty$ for any $H, K \in \mathcal{F}$. Let $M\colon \Gamma \to R$ be a Mackey functor such that $\mathcal{F} \subseteq \mathrm{Ob}(\Gamma)$. It defines a contravariant $R\mathtt{Sub}_{\mathcal{F}}(G)$-module

$$M^G\colon \mathtt{Sub}_{\mathcal{F}}(G) \to R\text{-}\mathrm{MOD}, \quad H \to M(H)$$

by means of the contravariant structure of the bifunctor M.

This section is the contravariant analogue of Section 3.1.2. Instead of flatness results, we want to prove injectivity results here. The case of a discrete group G and $\mathcal{F} = \mathcal{FIN}$ is fully understood, too. Lück [30, Thm. 5.2] showed that M^G is injective for $\mathbb{Q} \subseteq R$ and R semisimple.

For $N \in \mathcal{F}$ we define M^G_N by

$$M^G_N(H) = \begin{cases} M^G(H), & \text{if } (N) \leq (H), \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.9.1}$$

The canonical projections $\mathrm{pr}\colon M \to M_N$ induce an isomorphism

$$M^G \cong \lim_{N \in I_{\mathcal{F}}} M^G_N.$$

Here, we denote by $I_{\mathcal{F}}$ the partially ordered set which was introduced at the beginning of Section 3.1.2.

Lemma 3.2.10. *Suppose $\mathbb{Q} \subseteq R$ and $\mathtt{Sub}_{\mathcal{F}}(G)$ is an EI-category. Moreover, suppose that $[H : K] < \infty$ and*

$$\{\, (K) \in I_{\mathcal{F}} \mid (N) \leq (K) \leq (H) \,\}$$

is finite for any $N, H, K \in \mathcal{F}$. Then M^G_N is an injective $R\mathtt{Sub}_{\mathcal{F}}(G)$-module if for any $H \in \mathcal{F}$ the $R\widetilde{W_G H}$-module $CS_H M^G_N$ is injective.

Proof. Since $CS_H M_N^G$ is injective by assumption, we get a split $s(H)\colon M_N^G(H) \to CS_H M_N^G$ of the canonical inclusion $i(H)\colon CS_H M_N^G \to M_N^G(H)$. We obtain a map (cf. (3.2.4.1))

$$T_N\colon M_N^G \to \prod_{(H)\in I_{\mathcal{F}^N}} CE_H CS_H M_N^G,$$

where we set

$$\mathcal{F}^N = \{\, H \in \mathcal{F} \mid (N) \leq (H) \,\}.$$

The module $\prod_{(H)\in I_{\mathcal{F}^N}} CE_H CS_H M_N^G$ is injective because CE_H respects injective modules. Hence M_N^G is injective if T_N is an isomorphism.

First, let us show the injectivity of T_N. This does not differ very much from the surjectivity argument of T_N in Lemma 3.1.11. We remark that the condition

$$\{\, (K) \in I_{\mathcal{F}} \mid (N) \leq (K) \leq (H) \,\} < \infty$$

clearly implies $l_N(H) < \infty$. Now we show by induction over the relative length $l_N(H)$ that $T_N(H)$ is injective. If $l_N(H) = 0$, then $M_N^G(H)$ is zero and hence $T_N(H)$ is obviously injective. The induction step is done as follows: We have an isomorphism

$$T_N(H)|_{CS_H M_N^G}\colon CS_H M_N^G \to \bigl(CE_H CS_H M_N^G\bigr)(H).$$

If $0 \neq a \in CS_H M_N^G$, we can conclude $T_N(H)(a) \neq 0$. If $0 \neq a \notin CS_H M_N^G$, then, by definition, there exists a $K \in \mathcal{F}^N$ and $f\colon K \to H$ such that $(M_N^G)^*(f)(a) \neq 0$. In particular, we have $l_N(K) < l_N(H)$ and the induction hypothesis holds for K. We obtain

$$\Bigl(\prod_{(H)\in I_{\mathcal{F}^N}} CE_H CS_H M_N^G\Bigr)^*(f) \circ T_N(H)(a) = T_N(K) \circ \bigl((M_N^G)^*(f)(a)\bigr) \neq 0.$$

This finishes the induction step.

Next we want to show that T_N is surjective. Choose for any $(H) \in I_{\mathcal{F}^N}$ a representative $H \in (H)$. So in the sequel, we get for these representatives that $H = K$ follows from $(H) = (K)$. Put

$$J(H) = \operatorname{mor}_{\mathsf{Sub}_{\mathcal{F}}(G)}(H, K)/\widetilde{W_G H}.$$

Then we fix for any element $\overline{f} \in J(H)$ a homomorphism of topological groups $f\colon H \to K$ such that there exists a $g \in G$ with $f = c(g)$ the morphism which f represents in $\mathsf{Sub}_{\mathcal{F}}(G)$. For simplicity, we identify $\overline{f}$ and the fixed representative f in the sequel. So for two such homomorphisms $f_1, f_2\colon H \to K$ with $\overline{f_1} = \overline{f_2}$ we already have $f_1 = f_2$. Let $H_1, H_2, K \in \mathcal{F}^N$ and $(f_i\colon H_i \to K) \in \operatorname{mor}_{\mathsf{Sub}(G,\mathcal{F})}(H_i \to K)$ be arbitrary. We consider the following diagram

$$\begin{tikzcd}
M(H_1) \arrow[r, "\mathrm{ind}_{f_1}"] \arrow[rr, bend left, "\mathrm{ind}_{f_1}"] & M(\mathrm{im}\, f_1) \arrow[r, "\mathrm{ind}^K_{\mathrm{im}\, f_1}"] & M(K) \arrow[r, "\mathrm{res}^{\mathrm{im}\, f_2}_K"] \arrow[rr, bend left, "\mathrm{res}_{f_2}"] \arrow[d, "="] & M(\mathrm{im}\, f_2) \arrow[r, "\mathrm{ind}_{f_2^{-1}}"] & M(H_2) \arrow[d, "="] \\
M^G_N(H_1) \arrow[u, "="] & & M^G_N(K) \arrow[d, "T_N(K)"] & & M^G_N(H_2) \arrow[d, "s(H_2)"] \\
CS_{H_1}M^G_N \arrow[u, hook, "j"] \arrow[rr, "B(H_1,f_1)"] & & \prod\limits_{\bar{H}\in \mathrm{Is\,Sub}_{\mathcal{F}}(G)} CE_H CS_H M^G_N(K) \arrow[rr, "A(f_2)"] & & CS_{H_2}M^G_N
\end{tikzcd}$$

where j is the canonical inclusion, $s(H_2)$ the chosen retraction and $B(H_1, f_1)$ is the composition $T_N(K) \circ \mathrm{ind}_{f_1} \circ j$. Thus the left square commutes. The left arc commutes because of functoriality of ind (in the sequel, we will identify f_2 and $f_2\colon H_2 \to \mathrm{im}\, f_2$). By definition, we get, on the one hand, $\mathrm{ind}_{f_2^{-1}} = (\mathrm{res}_{f_2^{-1}})^{-1}$ and, on the other hand, with functoriality $\mathrm{res}_{f_2} = (res_{f_2^{-1}})^{-1}$ and finally $\mathrm{ind}_{f_2^{-1}} = \mathrm{res}_{f_2}$. Hence the right arc commutes, too. Let $A(f_2)$ be the evaluation against f_2, namely

$$\begin{aligned} A(f_2)\colon \prod_{(H)\in I_{\mathcal{F}N}} CE_H CS_H M^G_N(K) &\xrightarrow{\mathrm{pr}} CE_{H_2}CS_{H_2}M^G_N(K) \longrightarrow CS_{H_2}M^G_N \\ &\qquad\qquad\qquad \varphi \longmapsto \varphi(f_2). \end{aligned}$$

We define K_{H_2} as the adjoint of $\mathrm{id}\colon M^G_N(H_2) \to M^G_N(H_2)$. Explicitly we get

$$K_{H_2}(K)\colon M^G_N(K) \to CE_{H_2} \circ Res_{H_2} M^G_N(K), \quad m \mapsto (g \mapsto \mathrm{res}_g(m)).$$

We remind the reader that $T_N = \prod_{(H)\in I_{\mathcal{F}N}} (T_N)_H$ by definition (cf. (3.2.4.1)). We obtain $(T_N)_{H_2}(K) = s(H_2) \circ K_{H_2}(K)$. Consequently, the right square commutes and so does the entire diagram. Let

$$a(H_1, f_1, H_2, f_2)\colon CS_{H_1}M^G_N \to CS_{H_2}M^G_N, \quad a(H_1, f_1, H_2, f_2) = A(f_2) \circ B(H_1, f_1).$$

With the help of the double coset formula, we obtain

$$res^{\mathrm{im}\, f_2}_K \circ \mathrm{ind}^K_{\mathrm{im}\, f_1} = \sum_{k\,\in\, \mathrm{im}(f_2)\backslash K/\,\mathrm{im}(f_1)} \mathrm{ind}_{c(k)\colon\ \mathrm{im}(f_1)\cap k^{-1}\mathrm{im}(f_2)k \to \mathrm{im}(f_2)} \circ \mathrm{res}^{\mathrm{im}(f_1)\cap k^{-1}\mathrm{im}(f_2)k}_{\mathrm{im}(f_1)}.$$

By definition, the composition

$$\begin{aligned} &CS_{H_1}M^G_N \xrightarrow{j} M^G_N(H_1) \to M(H_1) \\ &\qquad \xrightarrow{\mathrm{ind}_{f_1}} M(\mathrm{im}(f_1)) \xrightarrow{\mathrm{res}^{\mathrm{im}(f_1)\cap k^{-1}\mathrm{im}(f_2)k}_{\mathrm{im}(f_1)}} M(\mathrm{im}(f_1) \cap k^{-1}\,\mathrm{im}(f_2)k) \end{aligned}$$

is trivial if $\operatorname{im}(f_1) \cap k^{-1} \operatorname{im}(f_2)k \neq \operatorname{im}(f_1)$. So let $\operatorname{im}(f_1) \cap k^{-1} \operatorname{im}(f_2)k = \operatorname{im}(f_1)$. This is only possible for $(H_1) \leq (H_2)$. Let $(H_1) = (H_2)$ and thus $H_1 = H_2$ by our choice of representatives. Furthermore, this implies $\operatorname{im}(f_1) \subseteq k^{-1} \operatorname{im}(f_2)k$. Since we have $k^{-1} \operatorname{im}(f_2)k \cong H_2 = H_1 \cong \operatorname{im}(f_1)$ in $\mathtt{Sub}_{\mathcal{F}}(G)$, there exists a $g \in G$ such that

$$g^{-1}(k^{-1} \operatorname{im}(f_2)k)g = \operatorname{im}(f_1) \subseteq k^{-1} \operatorname{im}(f_2)k.$$

As $\mathtt{Sub}_{\mathcal{F}}(G)$ is an EI-category, we obtain $k^{-1} \operatorname{im}(f_2)k = \operatorname{im}(f_1)$. Hence $\overline{f_2} = \overline{f_1}$ in $J(H)$. This already implies $f_2 = f_1$. Hence we have $k \in N_G \operatorname{im}(f_2) \cap K$ and deduce $a(H_1, f_1, H_2, f_2) = |K \cap N_G \operatorname{im}(f_2) : \operatorname{im}(f_2)| \cdot \mathrm{id}$. Recapitulating the above, we get

$$a(H_1, f_1, H_2, f_2) = \begin{cases} |K \cap N_G \operatorname{im}(f_2) : \operatorname{im}(f_2)| \cdot \mathrm{id}, & \text{if } H_1 = H_2 \text{ and } f_1 = f_2, \\ \text{do not know}, & \text{if } (H_1) < (H_2), \\ 0, & \text{otherwise.} \end{cases} \tag{3.2.10.1}$$

We have

$$\prod_{(H) \in I_{\mathcal{F}^N}} CE_H CS_H M_N^G(K) = \prod_{\substack{(H) \in I_{\mathcal{F}^N} \\ (N) \leq (H) \leq (K)}} CE_H CS_H M_N^G(K) = \bigoplus_{(H) \in I_{\mathcal{F}^N}} CE_H CS_H M_N^G(K),$$

since

$$\left|\{ (H) \in \mathcal{F}^N \}\right| = |\{ (H) \in I_{\mathcal{F}^N} \mid (N) \leq (H) \leq (K) \}| < \infty$$

by assumption. Therefore we can define the map

$$a(K)\colon \bigoplus_{\substack{(H) \in I_{\mathcal{F}^N}, H \subseteq K \\ (f\colon H \to K) \in \mathtt{Sub}_{\mathcal{F}}(G)}} CS_H M_N^G \xrightarrow{\oplus_{H,f} B(H,f)} \bigoplus_{(H) \in I_{\mathcal{F}^N}} CE_H CS_H M_N^G(K) \xrightarrow{\oplus_{H,f} A(f)} \bigoplus_{\substack{(H) \in I_{\mathcal{F}^N}, H \subseteq K \\ (f\colon H \to K) \in \mathtt{Sub}_{\mathcal{F}}(G)}} CS_H M_N^G.$$

With (3.2.10.1) and induction it is easily seen that $a(K)$ is surjective and hence $\oplus_{H,f} A(f)$. Obviously $\oplus_{H,f} A(f)$ is injective and hence $\oplus_{H,f} A(f)$ is an isomorphism. Consequently, $\oplus_{H,f} B(H,f)$ is surjective. Since $\oplus_{H,f} B(H,f)$ factorizes over $T_N(K)$, we obtain the surjectivity of $T_N(K)$. □

Lemma 3.2.11. *Let I be an $R\mathtt{Sub}_{\mathcal{F}}(G)$-module and define I_N analogously to* (3.2.9.1) *for $N \in \mathcal{F}$. Suppose that every $R\mathtt{Sub}_{\mathcal{F}}(G)$-module I_N is injective. Then I is an injective $R\mathtt{Sub}_{\mathcal{F}}(G)$-module if and only if for each $H \in \mathcal{F}$ and every $R\mathtt{Sub}_{\mathcal{F}_H}(G)$-submodule $L \subseteq R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)$, the term*

$$\lim_{N \in I_{\mathcal{F}_H}}{}^1 \hom_{R\mathtt{Sub}_{\mathcal{F}_H}(H)}(R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)/L, I_N^H) \;=\; 0$$

vanishes, where $I_N^H = \operatorname{res}_{\mathtt{Sub}_{\mathcal{F}_H}(H)} I_N$ and $\mathcal{F}_H$ denotes the subfamily of groups which are contained in H.

Proof. We show both directions simultaneously. To prove injectivity we have to prove exactness of the functor $\hom_{R\mathtt{Sub}_{\mathcal{F}}(G)}(-, I)$. By [20, Lem. 1, p. 136], it suffices to consider short exact sequences

$$0 \to L \to R\mathtt{Sub}_{\mathcal{F}}(G)(?, H) \to R\mathtt{Sub}_{\mathcal{F}}(G)(?, H)/L \to 0.$$

Since I_N is injective, we obtain a short exact sequence

$$\begin{aligned}0 \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I_N) \\ \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H), I_N) \\ \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(L, I_N) \to 0.\end{aligned}$$

By applying $\lim_{N\in I_{\mathcal{F}}}$ and using the isomorphism

$$\hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(L', I) = \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(L', \lim_{N\in I_{\mathcal{F}}} I_N) \xrightarrow{\mathrm{pr}_*} \lim_{N\in I_{\mathcal{F}}} \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(L', I_N),$$

we get an exact sequence

$$\begin{aligned}0 \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I) \\ \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H), I) \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(L, I) \\ \to \lim_{N\in I_{\mathcal{F}}}{}^1 \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I_N) \\ \to \lim_{N\in I_{\mathcal{F}}}{}^1 \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H), I_N).\end{aligned}$$

However, the inverse system of the last term is

$$\hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H), I_N) = I_N(H),$$

hence constant; and the last term vanishes. Thus we get an exact sequence

$$\begin{aligned}0 \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I) \\ \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H), I) \to \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(L, I) \\ \to \lim_{N\in I_{\mathcal{F}}}{}^1 \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I_N) \to 0\end{aligned}$$

and

$$\lim_{N\in I_{\mathcal{F}}}{}^1 \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I_N)$$

is the obstruction for exactness of $\hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(-, I)$.

In order to get the assertion, we have to simplify the derived limit a little bit. Since we have

$$\mathrm{ind}_{\mathsf{Sub}_{\mathcal{F}_H}(H)}^{\mathsf{Sub}_{\mathcal{F}}(G)} R\mathsf{Sub}_{\mathcal{F}_H}(H)(?,H) = R\mathsf{Sub}_{\mathcal{F}}(G)(?,H),$$

and $(\mathrm{ind}, \mathrm{res})$ is an adjoint pair of functors, we obtain

$$\hom_{R\mathsf{Sub}_{\mathcal{F}_H}(H)}(R\mathsf{Sub}_{\mathcal{F}_H}(H)(?,H)/L, I_N^H) = \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(H)(?,H)/L, I_N)$$

and consequently

$$\begin{aligned}\lim_{N\in I_{\mathcal{F}_H}}{}^1 \hom_{R\mathsf{Sub}_{\mathcal{F}_H}(H)}(R\mathsf{Sub}_{\mathcal{F}_H}(H)(?,H)/L, I_N^H) \\ = \lim_{N\in I_{\mathcal{F}}}{}^1 \hom_{R\mathsf{Sub}_{\mathcal{F}}(G)}(R\mathsf{Sub}_{\mathcal{F}}(G)(?,H)/L, I_N).\end{aligned}$$

Now the assertion follows. □

Remark 3.2.12. Let $\mathcal{F}_?$ be a Γ-collection of families of subgroups for some Γ. Then we have $\mathcal{F}_H \subseteq (\mathcal{F}_G)_H$ for any $G \in \mathrm{Ob}(\Gamma)$ and any $H \in \mathcal{F}_G$ by Remark 1.5.6. Equality does not hold in general but holds for the collections in Example 1.5.7.

Remark 3.2.13. Let $(I_j)_{j\in J}$ be an inverse system of injective R-modules which is of finite length. Suppose that $\lim^1_{j\in J} I_j = 0$ holds. Then we get by Corollary B.2 that $\lim_{j\in J} I_j$ is injective if and only if

$$\lim_{j\in J}{}^1 \hom(M, I_j) = 0$$

for every R-module M. This result is not valid for $R\Gamma$-modules (see Example B.5). The essence of the previous lemma is that in the case of $R\mathtt{Sub}_{\mathcal{F}}(G)$-modules the statement remains true if we restrict to special $R\mathtt{Sub}_{\mathcal{F}}(G)$-modules M.

Remark 3.2.14. The structure maps in

$$\hom_{R\mathtt{Sub}_{\mathcal{F}_H}(H)}(R\mathtt{Sub}_{\mathcal{F}}(G)(?, H)/L, I_N^H)$$

are injective. Furthermore, these groups are $\mathbb{Q}$-vector spaces, provided $\mathbb{Q} \subseteq R$. Suppose that $I_{\mathcal{F}_H}$ has finite length. This is the case for $\mathcal{F} = \mathcal{I}$ or G unimodular and $\mathcal{F} = \mathcal{CO}$. Then we can deduce by Lemma 3.2.9 the existence of a cofinal subsystem which has only isomorphisms as structure maps if the corresponding derived limit vanishes.

Theorem 3.2.15. *Let $\mathbb{Q} \subseteq R$ be semisimple. Suppose that $[H : K] < \infty$ and the corresponding morphism set $\mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(G/K, G/H)$ is finite for any $H, K \in \mathcal{F}$. For a Mackey functor M, the $R\mathtt{Sub}_{\mathcal{F}}(G)$-module M^G is injective if and only if*

$$\lim_{N\in I_{\mathcal{F}_H}}{}^1 \hom_{R\mathtt{Sub}_{\mathcal{F}_H}(G)}(R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)/L, M_N^H) \;=\; 0$$

for each $H \in \mathcal{F}$ and $L \subseteq R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)$.

Proof. The category $\mathtt{Sub}_{\mathcal{F}}(G)$ is an EI-category by Corollary 1.2.24. Thus

$$W_G H = \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(G/H, G/H)$$

is a finite group. Since $R\widetilde{W_G H}$ is semisimple, $CS_H M_N^G$ is an injective $RW_G H$-module. Moreover, we have

$$\begin{aligned}
|\{\,(K) \in I_{\mathcal{F}} \mid (N) \le (K) \le (H)\,\}| &= \left|\left\{\,(K) \;\middle|\; \exists g \in G:\; gNg^{-1} \subseteq K \subseteq H\,\right\}\right| \\
&\le |\,\mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(N, H)| \\
&\qquad \cdot \max\left\{\,|\left\{\,K \;\middle|\; gNg^{-1} \subseteq K \subseteq H\,\right\}| \;\middle|\; [c(g)] \in \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(N, H)\,\right\}
\end{aligned}$$

and

$$\begin{aligned}
&\max\left\{\,|\left\{\,K \;\middle|\; gNg^{-1} \subseteq K \subseteq H\,\right\}| \;\middle|\; [c(g)] \in \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(N, H)\,\right\} \\
&\qquad\qquad \le \max\left\{\,2^{[H:gNg^{-1}]} \;\middle|\; [c(g)] \in \mathrm{mor}_{\mathtt{Sub}_{\mathcal{F}}(G)}(N, H)\,\right\} < \infty.
\end{aligned}$$

Therefore M_N^G is injective for every $N \in \mathcal{F}$ by Lemma 3.2.10. Thus we get the assertion by Lemma 3.2.11. $\square$

Lemma 3.2.16. *Let $\mathbb{Q} \subseteq R$ be semisimple and $\mathtt{Sub}_{\mathcal{F}}(G)$ be an EI-category. Let M be an injective $R\mathtt{Sub}_{\mathcal{F}}(G)$-module and for every $H \in \mathcal{F}$ let the corresponding group $C_G H(H \cap C_G H)$ be locally finite. Then $\operatorname{res}_{\mathrm{pr}} M$ is an injective $R\mathtt{Or}_{\mathcal{F}}(G)$-module, where we denote by $\mathrm{pr}\colon \mathtt{Or}_{\mathcal{F}}(G) \to \mathtt{Sub}_{\mathcal{CO}}(G)$ the canonical projection.*

Proof. The functor $\operatorname{ind}_{\mathrm{pr}}$ is exact as we have seen in the proof of Lemma 3.1.14. Therefore $\operatorname{res}_{\mathrm{pr}}$ respects the property injective by Corollary 1.3.13. □

Assembling all together we obtain

Proposition 3.2.17. *Let $\mathbb{Q} \subseteq R$, $\mathtt{Or}_{\mathcal{F}}(G)$ be an EI-category and M be a Mackey functor. Suppose that the following conditions hold.*

(i) $|\{\, (K) \in I_{\mathcal{F}} \mid (N) \leq (K) \leq (H) \,\}| < \infty$ and $[H : N] < \infty$ for any $N, H \in \mathcal{F}$.

(ii) $C_G H/(H \cap C_G H)$ is locally finite for any $H \in \mathcal{F}$.

(iii) $CS_H M_N^G$ is an injective $R\widetilde{W_G H}$-module for any $H \in \mathcal{F}$.

Then $\operatorname{res}_{\mathrm{pr}} M^G$ is an injective $R\mathtt{Or}_{\mathcal{F}}(G)$-module if and only if

$$\lim_{N \in I_{\mathcal{F}_H}}{}^{1} \hom_{R\mathtt{Sub}_{\mathcal{F}_H}(H)}(R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)/L, M_N^H) \;=\; 0$$

for each $H \in \mathcal{F}$ and $L \subseteq R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)$.

Proof. Because of (i) and (iii), we can deduce by Lemma 3.2.10 and Lemma 1.2.13 that M_N^G is an injective $R\mathtt{Sub}_{\mathcal{F}}(G)$-module for every $N \in \mathcal{F}$. Now (ii) and Lemma 3.2.16 yield the injectivity of $\operatorname{res}_{\mathrm{pr}} M_N^G$. Thus the assertion follows from Lemma 3.2.11. □

Proposition 3.2.18. *Let $\mathbb{Q} \subseteq R$ and G be a prodiscrete l-group. Suppose $C_G H/(H \cap C_G H)$ is locally finite for any compact open subgroup $H \subseteq G$. Then $\operatorname{res}_{\mathrm{pr}} M^G$ is an injective $R\mathtt{Or}_{\mathcal{CO}}(G)$-module if and only if*

$$\lim_{N \in I_{\mathcal{CO}}}{}^{1} \hom_{R\mathtt{Sub}_{\mathcal{CO}}(H)}(R\mathtt{Sub}_{\mathcal{CO}}(H)(?, H)/L, M_N^H) \;=\; 0$$

for each compact open subgroup $H \subseteq G$ and $L \subseteq R\mathtt{Sub}_{\mathcal{CO}}(H)(?, H)$.

Proof. By Proposition 1.2.25 the subgroup category is an EI-category and the morphism sets are finite. Now the assertion follows from Theorem 3.2.15 and Lemma 3.2.16.
□

Theorem 3.2.19. *Let $\mathbb{Q} \subseteq R$ be semisimple. Suppose that $[H : K] < \infty$ and the corresponding morphism set $\operatorname{mor}_{\mathtt{Or}_{\mathcal{F}}(G)}(G/K, G/H)$ is finite for any $H, K \in \mathcal{F}$. For a Mackey functor M, the $R\mathtt{Or}_{\mathcal{F}}(G)$-module $\operatorname{res}_{\mathrm{pr}} M^G$ is injective if and only if*

$$\lim_{N \in I_{\mathcal{F}_H}}{}^{1} \hom_{R\mathtt{Sub}_{\mathcal{F}_H}(G)}(R\mathtt{Sub}_{\mathcal{F}}(H)(?, H)/L, M_N^H) \;=\; 0$$

for each $H \in \mathcal{F}$ and $L \subseteq R\mathtt{Sub}_{\mathcal{F}}(H)(?, H)$.

Proof. The category $\mathtt{Or}_{\mathcal{F}}(G)$ is an EI-category by Corollary 1.2.24. Thus

$$W_G H = \operatorname{mor}_{\mathtt{Or}_{\mathcal{F}}(G)}(G/H, G/H)$$

is finite. Hence $C_G H/(H \cap C_G H) \subseteq W_G H$ is finite. The assertion follows now from Theorem 3.2.15 and Lemma 3.2.16. $\square$

Corollary 3.2.20. *Let $\mathbb{Q} \subseteq R$ be semisimple. Suppose that either $\mathcal{F} = \mathcal{I}$ or that G is a semisimple p-adic group and $\mathcal{F} = \mathcal{CO}$. For a Mackey functor M, the $R\mathtt{Or}_{\mathcal{F}}(G)$-module* $\operatorname{res}_{\mathrm{pr}} M^G$ *is injective if and only if*

$$\lim_{N \in I_{\mathcal{F}_H}}{}^1 \hom_{R\mathtt{Sub}_{\mathcal{F}_H}(G)}(R\mathtt{Sub}_{\mathcal{F}}(H)(?, H)/L, M_N^H) \;=\; 0$$

for each $H \in \mathcal{F}$ and $L \subseteq R\mathtt{Sub}_{\mathcal{F}}(H)(?, H)$.

Proof. We want to apply the theorem. In the first case, let $\mathcal{F} = \mathcal{I}$ and let $H \subseteq K \subseteq G$ be closed subgroups of finite index. We obtain

$$\left|\operatorname{mor}_{\mathtt{Or}_{\mathcal{I}}(G)}(G/H, G/K)\right| = \left|\left\{\, g \in G \;\middle|\; gHg^{-1} \subseteq K \,\right\}/K\right| \leq [G : K] < \infty.$$

Therefore, the theorem is applicable.

Now let G be a semisimple p-adic group and $\mathcal{F} = \mathcal{CO}$. The orbit category is an EI-category with finite morphism sets by Corollary 1.7.12. Therefore we can apply the theorem. $\square$

3.3 Applications to K-Theory

In this section, we discuss the case where the Mackey functor is given by the coefficients of K-theory in more detail. We provide examples in which the coefficient modules are not flat or injective, respectively. At the very end of this section, as a happy ending, we give an example, where the coefficient module is injective.

Let G be a second countable locally compact group and $\mathcal{F} = \mathcal{CO}$. In this section, we assume that $\mathtt{Or}_{\mathcal{CO}}(G)$ is an EI-category. For a ring R, we denote by K_R^G the covariant or contravariant $\mathtt{Or}_{\mathcal{CO}}(G)$-module $K_0^G(G/?) \otimes R$ or $K_G^0(G/?) \otimes R$, respectively. Sometimes we consider K_R^G as a $\mathtt{Sub}_{\mathcal{CO}}(G)$-module but in those cases we will state it explicitly. Furthermore, for an $R\mathtt{Or}_{\mathcal{CO}}(G)$-module M and $N \in \mathcal{CO}$, we denote by M_N the truncated module which was defined in Subsection 3.1.2.

We recall that we have $K_{\mathbb{C}}^G(H) \subseteq C(H, \mathbb{C})^H$ for a compact open subgroup $H \subseteq G$ by Lemma 2.2.37. Thus we can think of an element $\varphi \in K_{\mathbb{C}}^G(H)$ as a continuous complex valued conjugation invariant function $\varphi\colon H \to \mathbb{C}$. This identification will be used throughout the entire section.

Lemma 3.3.1. *Let $H \subseteq G$ be a compact open subgroup. Suppose further that*

$$M = H \setminus \bigcup_{\substack{K \subsetneq H \\ \text{compact open}}} K$$

is non-empty and open. Then the constant $\mathbb{C}W_GH$-module $\mathbb{C}$ is a direct summand of $S_HK_{\mathbb{C}}^G$ and of $CS_HK_{\mathbb{C}}^G$.

Proof. Let $N \subseteq G$ be a compact open subgroup and consider the following maps

$$\varphi_1 \colon \mathbb{C} \xrightarrow{i} (K_{\mathbb{C}}^G)_N(H) \xrightarrow{\mathrm{pr}} S_H(K_{\mathbb{C}}^G)_N$$
$$\varphi_2 \colon CS_H(K_{\mathbb{C}}^G)_N \xrightarrow{\mathrm{inc}} (K_{\mathbb{C}}^G)_N(H) \xrightarrow{p} \mathbb{C},$$

where the maps i and p are induced by

$$i' \colon \{ f \colon H \to \mathbb{C} \mid f \text{ constant} \} \to C(H,\mathbb{C})^H, \quad f \mapsto f$$
$$p' \colon C(H,\mathbb{C})^H \to \mathbb{C}, \quad f \mapsto \frac{1}{vol(M)} \int_M f(h)dh$$

for a chosen Haar measure on H. Since M is assumed to be open and non-empty, we get $vol(M) > 0$. Thus p is defined. Consequently, the maps i and p are splits of each other. For a compact open subgroup $\tilde{N} \subseteq N$, we obtain a diagram

$$\begin{tikzcd}
& S_H(K_{\mathbb{C}}^G)_N \arrow[dd, "\mathrm{pr}"] \arrow[r, bend left, "s_N"] & CS_H(K_{\mathbb{C}}^G)_N \arrow[l] \arrow[dr, "\varphi_2"] & \\
\mathbb{C} \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_1"'] \arrow[rrr, "id"] & & & \mathbb{C} \\
& S_H(K_{\mathbb{C}}^G)_{\tilde{N}} \arrow[r, bend left, "s_{\tilde{N}}"] & CS_H(K_{\mathbb{C}}^G)_{\tilde{N}} \arrow[l] \arrow[uu, "\mathrm{inc}"] \arrow[ur, "\varphi_2"'] &
\end{tikzcd}$$

Note that under the identification $K_{\mathbb{C}}^G(H) \subseteq C(H,\mathbb{C})^H$, we have

$$CS_H(K_{\mathbb{C}}^G)_N = \left\{ (f \colon H \to \mathbb{C}) \in K_{\mathbb{C}}^G(H) \;\middle|\; \substack{f|_K = 0 \text{ for every compact open subgroup} \\ K \subsetneq H \text{ with } (N) \leq (K)} \right\}$$

and s_N sends $[f \colon H \to \mathbb{C}] \in S_H(K_{\mathbb{C}}^G)_N$ to $f' \colon H \to \mathbb{C}$, where

$$f'(h) = \begin{cases} 0, & \text{if } h \in K \text{ for some compact open subgroup } K \subsetneq H \text{ with } (N) \leq (K), \\ f(h), & \text{otherwise.} \end{cases}$$

We obtain maps

$$\mathbb{C} \to \operatorname*{colim}_{N \in I_{\mathcal{CO}}} S_H(K_{\mathbb{C}}^G)_N \to \mathbb{C} \quad \text{and} \quad \mathbb{C} \to \lim_{N \in I_{\mathcal{CO}}} CS_H(K_{\mathbb{C}}^G)_N \to \mathbb{C},$$

where the compositions are identities because the required parts of the diagram commute. Since we have

$$\operatorname*{colim}_{N \in I_{\mathcal{CO}}} S_H(K_{\mathbb{C}}^G)_N = S_HK_{\mathbb{C}}^G \quad \text{and} \quad \lim_{N \in I_{\mathcal{CO}}} CS_H(K_{\mathbb{C}}^G)_N = CS_HK_{\mathbb{C}}^G,$$

our assertion follows. $\square$

Corollary 3.3.2. *If there exists a compact open subgroup $H \subseteq G$ such that the corresponding group is locally finite (torsion) and the assumption of Lemma 3.3.1 is satisfied, then $K^G_{\mathbb{Q}}$ is not flat (injective) as a $\mathbb{Q}\Gamma$-module for $\Gamma = \mathtt{Sub}_{\mathcal{CO}}(G)$ or $\Gamma = \mathtt{Or}_{\mathcal{CO}}(G)$, respectively.*

Proof. This follows directly from the previous lemma and Theorem A.7 or Theorem A.8, respectively. $\square$

Example 3.3.3. Let $G = GL_2(\mathbb{Q}_p)$ and K be the compact open subgroup

$$K = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p) \;\middle|\; |\det A|_p = 1 \text{ and } a,b,c,d \in \mathbb{Z}_p \right\}.$$

Then K has exactly three maximal compact subgroups (cf. Figure 1.1), namely

$$\begin{aligned} H_1 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\middle|\; b \in p\mathbb{Z}_p \right\}, \\ H_2 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\middle|\; c \in p\mathbb{Z}_p \right\} \quad \text{and} \\ H_3 &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\middle|\; a,d \in p\mathbb{Z}_p \right\} \cup \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;\middle|\; b,c \in p\mathbb{Z}_p \right\} \end{aligned}$$

Obviously, $H_1 \cup H_2 \cup H_3 \neq K$ and $K \setminus (H_1 \cup H_2 \cup H_3)$ is open. Hence the assumption of Lemma 3.3.1 is satisfied. Moreover, we obtain

$$\mathbb{Z} = \mathbb{Q}_p^\times / \mathbb{Z}_p^\times = C_G(K)/(K \cap C_G(K)) \subseteq W_G K.$$

Therefore $K^G_{\mathbb{Q}}$ is neither flat nor injective as a $\mathbb{Q}\mathtt{Or}_{\mathcal{CO}}(G)$-module. We remark that $K^G_{\mathbb{Q}}$ is flat as a $\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)$-module by Corollary 1.7.11 and Corollary 3.1.13.

The Heisenberg group, which we will introduce in the next example, does not have a Chern character for the Borel construction (Example 4.3.2). Although a Chern character exists in the case of K-theory (see Section 5.6), we show in the next example that our approach does not work in this case.

Example 3.3.4. Let Hei be the three-dimensional Heisenberg group which is the subgroup of $GL_3(\mathbb{Z})$ consisting of upper triangular matrices with 1 on the diagonal. It has the presentation

$$\mathrm{Hei} = \langle u, v, z \mid [u,z] = 1, [v,z] = 1, [u,v] = z \rangle.$$

There is the central extension

$$1 \to \mathbb{Z} \xrightarrow{i} \mathrm{Hei} \xrightarrow{\mathrm{pr}} \mathbb{Z}^2 \to 1,$$

where i maps the generator in $\mathbb{Z}$ to the element z and pr sends z to $(0,0)$, u to $(1,0)$ and v to $(0,1)$. Let $k\colon \mathbb{Z} \to \mathbb{Z}_p$ be the obvious inclusion. The central extension yields a central extension of topological groups

$$1 \to \mathbb{Z}_p \xrightarrow{j} G \xrightarrow{p} \mathbb{Z}^2 \to 1 \tag{3.3.4.1}$$

described next. We equip $\mathbb{Z}^2$ and Hei with the discrete topology. The topological group G is the quotient of $\mathbb{Z}_p \times \mathrm{Hei}$ by the central closed subgroup given by the image of $j \times i\colon \mathbb{Z} \to K \times \mathrm{Hei}$. The homomorphism of topological groups j is induced by the inclusion of groups $\mathbb{Z}_p \to \mathbb{Z}_p \times \mathrm{Hei}$ and $p\colon G \to \mathbb{Z}^2$ is induced by the composition of the projection $\mathbb{Z}_p \times \mathrm{Hei} \to \mathrm{Hei}$ with $\mathrm{pr}\colon \mathrm{Hei} \to \mathbb{Z}^2$. We denote by $l\colon \mathrm{Hei} \to G$ the map induced by the inclusion $\mathrm{Hei} \to K \times \mathrm{Hei}$. Note that G is an l-group since $\mathbb{Z}_p$ is an l-group.

Since (3.3.4.1) is central, we obtain $\mathtt{Sub}_{\mathcal{CO}}(G) = \mathtt{Sub}_{\mathcal{CO}}(\mathbb{Z}_p)$ and $K_{\mathbb{Q}}^G$ is flat as a $\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)$-module. Moreover, it is injective, which we will prove in Lemma 3.3.10. Each proper compact open subgroup $K \subsetneq \mathbb{Z}_p$ is contained in $p\mathbb{Z}_p$ and $\mathbb{Z}_p \setminus p\mathbb{Z}_p$ is open. Since $W_G\mathbb{Z}_p = \mathbb{Z}^2$, the assumptions of Corollary 3.3.2 are fulfilled and $K_{\mathbb{Q}}^G$ can be neither flat nor injective as a $\mathbb{Q}\mathtt{Or}_{\mathcal{CO}}(G)$-module.

Note that G is a p-adic Lie group [18, Thm 8.1] because $\mathbb{Z}_p$ is an open subgroup.

In Example 4.3.2 we will see that the corresponding coefficient module for G of the Borel construction cannot be flat, either.

Before we give an example of an injective $K_{\mathbb{Q}}^G$ at the very end of this section, we discuss the case, where $K_{\mathbb{C}}^G$ is not injective.

Lemma 3.3.5. *Let $N = \prod_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$ and $G = N \rtimes \mathbb{Z}$, where $\mathbb{Z}$ acts on N in the obvious way. Then $K_{\mathbb{Q}}^G$ is not injective as a $\mathbb{Q}\mathtt{Or}_{\mathcal{CO}}(G)$-module.*

Proof. For $N \subseteq H \subseteq G$ compact open, the maps

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & K_{\mathbb{C}}^G(H) & \longrightarrow & \mathbb{C} \\ z & \longmapsto & (g \mapsto z) & & \\ & & f & \longmapsto & f(1) \end{array}$$

induce a split $\underline{\mathbb{C}} \to K_{\mathbb{C}}^G$, where $\underline{\mathbb{C}}$ denotes the constant contravariant $\mathtt{Or}_{\mathcal{CO}}(G)$-module. Thus it suffices to prove that $\underline{\mathbb{C}}$ is not injective. Since $\mathbb{Z}$ is a non-torsion group, the constant $\mathbb{C}\mathbb{Z}$-module $\mathbb{C}$ is not injective by Theorem A.8. Hence there exist a $\mathbb{C}\mathbb{Z}$-module M and an injective map $i\colon \mathbb{C} \to M$ which does not split. Let $H \subseteq G$ be a compact open subgroup. Then we already have $H \subseteq N$ because $\mathbb{Z}$ does not admit any non-trivial finite subgroup. Since $H \subseteq N$ is open, we can assume $\prod_I \mathbb{Z}/2\mathbb{Z} \subseteq H$ for a cofinite set $I \subseteq \mathbb{Z}$. We obtain

$$W_G H \subseteq \begin{cases} \mathbb{Z}, & \text{if } H = N, \\ (\mathbb{Z}/2\mathbb{Z})^k \text{ for some } k \in \mathbb{N}, & \text{otherwise.} \end{cases}$$

We define a $\prod_{\mathbb{N}} \mathbb{Z}/2\mathbb{Z}$-action on M by the trivial action. Then M canonically defines a $\mathbb{Q}\mathtt{Or}_{\mathcal{CO}}(G)$-module $\tilde{M}$ with $\tilde{M}(H) = M$ for any compact open subgroup $H \subseteq G$. We get an injective map $i\colon \underline{\mathbb{C}} \to \tilde{M}$, which does not split. Consequently, $\underline{\mathbb{C}}$ is not injective. $\square$

Proposition 3.3.6. *Let $\Gamma = \mathtt{Or}_{\mathcal{CO}}(G)$ or $\Gamma = \mathtt{Sub}_{\mathcal{CO}}(G)$, respectively. Suppose that the corresponding morphism sets $\operatorname{mor}_{\mathtt{Or}_{\mathcal{CO}}(G)}(G/K, G/H)$ or $\operatorname{mor}_{\mathtt{Sub}G\mathcal{CO}}(K,H)$, respectively, are finite. Let $H \subseteq G$ be a compact open subgroup. Suppose there exist infinitely many compact open subgroups $H_1, H_2, \ldots$ such that the sequence*

$$M_i = \bigcup_{k=1}^{i} \bigcup_{g \in G} gH_k g^{-1}$$

is properly increasing. Then $K_{\mathbb{Q}}^{G}$ is not injective as a $\mathbb{Q}\Gamma$-module.

Proof. By Corollary 2.2.39 and Theorem 3.2.19 or Theorem 3.2.15, respectively, it suffices to find a compact open subgroup $K \subseteq G$ and a $\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)$-submodule $L \subseteq \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?, K)$ such that

$$\lim_{N \in I_{\mathcal{CO}}}{}^{1} \hom_{\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)}(\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?, K)/L, (K_{\mathbb{Q}}^{H})_N) \neq 0.$$

Our candidates are $K = H$ and

$$L(N) = \begin{cases} 0, & l_N(H_i) < i \text{ for minimal } i \text{ with } (N) \leq (H_i), \\ \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(N, H), & l_N(H_i) \geq i \text{ for minimal } i \text{ with } (N) \leq (H_i), \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\begin{aligned} &\Big(\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?, H)/L(?)\Big)(N) \\ &\qquad = \begin{cases} \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(N, H), & l_N(H_i) < i \text{ for minimal } i \text{ with } (N) \leq (H_i), \\ 0, & l_N(H_i) \geq i \text{ for minimal } i \text{ with } (N) \leq (H_i), \\ \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(N, H), & \text{otherwise.} \end{cases} \end{aligned}$$

We define $\chi_{H_i} \colon H \to \mathbb{C}$ by

$$\chi_{H_i}(h) = \begin{cases} 0, & h \in M_{i-1}, \\ 1, & \text{otherwise.} \end{cases}$$

Let $N \subseteq G$ be a compact open subgroup. We obtain the inclusion

$$\begin{aligned} &\hom_{\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)}(\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)(?, H)/L, (K_{\mathbb{C}}^{G})_N) \\ &\qquad \subseteq \hom_{\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)}(\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)(?, H), (K_{\mathbb{C}}^{G})_N) = K_{\mathbb{C}}^{G}(H), \end{aligned}$$

and under this identification we have

$$\begin{aligned} &\hom_{\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)}(\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)(?, H)/L, (K_{\mathbb{C}}^{G})_N) \\ &\qquad = \{\, f \in K_{\mathbb{C}}^{G}(H) \mid f|_{\tilde{N}} = 0 \text{ for } (N) \leq (\tilde{N}) \leq (H_i),\ l_{\tilde{N}}(H_i) \geq i \,\}. \end{aligned}$$

For $N_1 \subseteq N_0 \subseteq H_i$ compact open with $l_{N_0}(H_i) = i-1$ and $l_{N_1}(H_i) = i$ we deduce the following:

$$\chi_{H_i} \in \hom(\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)/L, (K_{\mathbb{C}}^G)_{N_0}) \quad \text{and}$$
$$\chi_{H_i} \notin \hom(\mathbb{C}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)/L, (K_{\mathbb{C}}^G)_{N_1}).$$

The structure maps cannot be surjective; however, they are always injective. Thus the derived limit does not vanish by Lemma 3.2.9. □

Remark 3.3.7. The sequence M_i of the previous proposition may become stationary even if the corresponding subgroups are non-subconjugated. Let $G = (\mathbb{Z}/2)^3$ and let $H_1, \dots, H_n$ be the set of maximal subgroups, i.e., two-dimensional subvector spaces. Let $a \in H_n$ and suppose (a, b) is a basis of H_n. Pick a linearly independent $c \in G$. Then we obtain $H_k = \langle a, c\rangle$ for some $k < n$ and $a \in H_k$. Therefore the sequence M_i of Proposition 3.3.6 is not properly increasing in this case, although $H_1, \dots, H_n$ are not conjugated.

Example 3.3.8. Let K be compact and $G = \prod_{\mathbb{Z}} K$. Let $H_i \subseteq G$ be the subgroup, omitting the i-th factor. Then the corresponding sequence fulfills the assertion of Proposition 3.3.6. Hence $K_{\mathbb{Q}}^G$ is not injective.

Remark 3.3.9. Let G be an l-group. Suppose we have only finitely many maximal subgroups for each $H \subseteq G$ compact open up to conjugation. This is morally true in light of Proposition 3.3.6. Then we can define a decreasing sequence

$$H_0 = H, \quad H_k = \bigcap_{i=1}^{k} \bigcap_{j=1}^{n_i} H_{i,j}$$

of compact open subgroups. Here $\{ H_{i,0}, \dots, H_{i,n_i} \}$ denotes a system of representatives of conjugacy classes of maximal compact open subgroups. Since G admits, as an l-group, a basis of topology of compact open subgroups and $[N_1 : N_2] < \infty$ for $N_1 \subseteq N_2 \subseteq G$ compact open, we obtain that

$$\{ gH_ng^{-1} \mid g \in G,\ n \in \mathbb{N} \}$$

is a basis of topology and defines a cofinal subsystem in $\mathcal{CO}$ (of H). Our derived limit simplifies to

$$\begin{aligned}\lim_{N \in I_{\mathcal{CO}}}{}^1 \hom_{\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)}\big(\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)/L, (K_{\mathbb{Q}}^G)_N\big) \\ = \lim_{n\to\infty}{}^1 \hom_{\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)}\big(\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)/L, (K_{\mathbb{Q}}^G)_{H_n}\big).\end{aligned}$$

Finally, we want to give examples of l-groups which induce injective modules.

Lemma 3.3.10. *Let G be an abelian compact group such that every compact open subgroup has only finitely many maximal compact open subgroups (e.g., $G = \mathbb{Z}_p$). Then $K_{\mathbb{Q}}^G$ is injective.*

Proof. Since G is compact, $\mathtt{Or}_{\mathcal{CO}}(G) = \mathtt{Or}_{\mathcal{I}}(G)$ and it suffices to prove that the derived limit vanishes by Corollary 3.2.20.

We get $\dim_{\mathbb{Q}} \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(H,K) \in \{0,1\}$ for the free module because the morphism sets have only one element or are empty for an abelian group.

Let $L \subseteq \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)$ be a submodule for $H \subseteq G$ compact open. Further, we choose a basis of topology $H_1, H_2, \dots$ as in the previous remark. Assume we have a minimal n such that $L(H_n) = \mathbb{Q}$. Otherwise we would get the trivial case $L = \mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)$. Now we can conclude

$$\begin{aligned}&\lim_{l\to\infty}{}^1 \hom((\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)/L), (K^G_{\mathbb{Q}})_l)\\ &\qquad = \lim_{l\to\infty}{}^1 \hom((\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)(?,H)/L), (K^G_{\mathbb{Q}})_{n+1}) = 0.\end{aligned}$$

□

Note that the previous lemma does not use any special properties of $K^G_{\mathbb{Q}}$ but works for any $\mathbb{Q}\mathtt{Or}_{\mathcal{CO}}(G)$-module M which has a Mackey functor structure. However, the author wanted to place it after Proposition 3.3.6 and its remarks.

4 The Construction of the Equivariant Chern Character

The goal of the first section of this chapter is to construct a natural map

$$\widetilde{\mathrm{ch}}_*^G : H_*(X^?) \otimes_{\mathrm{Or}_{\mathcal{CO}}(G)} \mathcal{H}_*^G(G/?) \to \mathcal{H}_*(X).$$

Together with the flatness results of the previous chapter this yields the desired Chern character. As a consequence, we get Theorem 4.1.8, which provides a tool to compute $K_*(C_r^*(G)) \otimes \mathbb{C}$ for a semisimple p-adic group G. In the special case of $G = SL_2(\mathbb{Q}_p)$, we compute $K_*(C_r^*(G)) \otimes \mathbb{C}$ explicitly. In the second section, we discuss the cohomology case, where we get a Chern character, too. However, there is no nice application as it is in the homological case. In the third section, we give examples of groups, where a Chern character cannot exist.

In this chapter, R denotes a commutative ring with $\mathbb{Q} \subseteq R$.

4.1 The Homological Equivariant Chern Character

Let $\mathcal{H}_*^?$ be an equivariant $(\Gamma, \mathcal{F}_?)$-homology theory with values in R-modules such that $\mathcal{F}_H$ is a smooth family for every $H \in \mathrm{Ob}(\Gamma)$. Let $G \in \mathrm{Ob}(\Gamma)$ and (X, A) be a $(G, \mathcal{F}_G)$-CW-pair. We consider a subgroup $H \in \mathcal{F}_G$. We recall that every subgroup $H \in \mathcal{F}_G$ is open by definition. Consequently, our G-CW-complex (X, A) is smooth.

We want to construct an R-homomorphism

$$\underline{\mathrm{ch}}_{p,q}^G(X, A)(H) \colon H_p(X^H, A^H; R) \otimes_R \mathcal{H}_q^G(G/H) \longrightarrow \mathcal{H}_{p+q}^G(X, A), \tag{4.1.0.1}$$

where $H_p(X^H, A^H; R)$ is the cellular homology of the CW-pair (X^H, A^H) with R-coefficients. Note that X^H is a CW-complex since X is smooth. For (notational) simplicity we give the details only for $A = \emptyset$. The map is defined by the following composition

$$\begin{tikzcd}
H_p(X^H;R) \otimes_R \mathcal{H}_q^G(G/H) \\
\pi_p^s((X^H)_+) \otimes_{\mathbb{Z}} R \otimes_R \mathcal{H}_q^H(\{\bullet\}) \arrow[u, "\mathrm{hur}(X^H)\otimes_R \mathrm{ind}_H^G"', "\cong"] \arrow[d, "D^H_{p,q}(X^H)"'] \\
\mathcal{H}^H_{p+q}(X^H) \arrow[d, "\mathrm{ind}_H^G"', "\cong"] \\
\mathcal{H}^G_{p+q}(G \times_H X^H) \arrow[d, "\mathcal{H}^G_{p+q}(v_H)"'] \\
\mathcal{H}^G_{p+q}(X)
\end{tikzcd}$$

Some explanations are in order.

For every CW-complex Y let $\mathrm{hur}(Y)\colon \pi_p^s(Y_+)\otimes_{\mathbb{Z}} R \to H_p(Y;R)$ be the Hurewicz homomorphism. It is bijective since $\mathbb{Q}\subseteq R$ and therefore hur is a natural transformation of (non-equivariant) homology theories, which induces for the one-point space $Y=\{\bullet\}$ an isomorphism $\pi_p^s(\{\bullet\}_+)\otimes_{\mathbb{Z}} R \cong H_p(\{\bullet\};R)$ for $p\in\mathbb{Z}$ by a result of Serre [48].

Given a space Z and a topological group H, consider Z as a smooth H-space by the trivial action and define a map

$$D^H_{p,q}(Z)\colon \pi_p^s(Z_+)\otimes_{\mathbb{Z}} \mathcal{H}_q^H(\{\bullet\}) = \pi_p^s(Z_+)\otimes_{\mathbb{Z}} R\otimes_R \mathcal{H}_q^H(\{\bullet\}) \to \mathcal{H}^H_{p+q}(Z)$$

as follows. For an element $a\otimes b \in \pi_p^s(Z_+)\otimes_{\mathbb{Z}} \mathcal{H}_q^H(\{\bullet\})$ choose a representative $f\colon S^{p+k}\to S^k\wedge Z_+$ of a. We define $D^H_{p,q}(Z)(a\otimes b)$ to be the image of b under the composition

$$\begin{aligned}\mathcal{H}_q^H(\{\bullet\}) &\xrightarrow{\cong} \tilde{\mathcal{H}}_q^H(S^0) \xrightarrow{\sigma^{p+k}} \tilde{\mathcal{H}}^H_{p+q+k}(S^{p+k},\{\bullet\}) \\ &\xrightarrow{\tilde{\mathcal{H}}^H_{p+q+k}(f)} \tilde{\mathcal{H}}^H_{p+q+k}(S^k\wedge Z_+,\{\bullet\}) \xrightarrow{\sigma^{-k}} \tilde{\mathcal{H}}^H_{p+q}(Z_+) \xleftarrow{\cong} \mathcal{H}^H_{p+q}(Z),\end{aligned} \tag{4.1.0.2}$$

where σ denotes the suspension isomorphism. We remark that $\{\bullet\}$ is an $(H,\mathcal{F}_H)$-CW-complex because $H\in\mathcal{F}_H$ by definition (cf. Definition 1.5.5).

The G-map $v_h\colon G\times_H X^H\to X$ sends (g,x) to gx.

Lemma 4.1.1. *Let G be a group and X be a smooth $(G,\mathcal{F})$-CW-complex. We consider $H,K\in\mathcal{F}$ and $g\in G$ with $gHg^{-1}\subseteq K$. Let $L_{g^{-1}}\colon X^H\to X^K$ be the map induced by left multiplication with g^{-1}. Let $R_{g^{-1}}\colon G/H\to G/K$ be given by right multiplication* *with g^{-1}. Then the following diagram commutes:*

$$\begin{tikzcd}
H_p(X^K) \otimes \mathcal{H}_q^G(G/H) \arrow[r, "(L_{g^{-1}})_* \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes R_{g^{-1}}"] & H_p(X^H) \otimes \mathcal{H}_q^G(G/H) \arrow[d, "\underline{\mathrm{ch}}_{p,q}^G(X)(H)"] \\
H_p(X^K) \otimes \mathcal{H}_q^G(G/K) \arrow[r, "\underline{\mathrm{ch}}_{p,q}^G(X)(K)"] & \mathcal{H}_{p+q}^G(X)
\end{tikzcd}$$

Proof. We examine the following diagram:

$$\begin{tikzcd}
\pi_p^s(X^K) \otimes R \otimes \mathcal{H}_q^K(\{\bullet\}) \arrow[d, "D_{p,q}^K"] & \pi_p^s(X^K) \otimes R \otimes \mathcal{H}_q^H(\{\bullet\}) \arrow[l, "\mathrm{id} \otimes (\mathrm{pr}_* \circ \mathrm{ind}_{c(g)})"'] \arrow[r, "(L_{g^{-1}})_* \otimes \mathrm{id}"] \arrow[d, "D_{p,q}^H"] & \pi_p^s(X^H) \otimes R \otimes \mathcal{H}_q^H(\{\bullet\}) \arrow[d, "D_{p,q}^H"] \\
\mathcal{H}_{p+q}^K(X^K) \arrow[d, "\mathrm{ind}_K^G"] & \mathcal{H}_{p+q}^H(X^K) \arrow[l, "\mathrm{pr}_* \circ \mathrm{ind}_{c(g)}"'] \arrow[r, "(L_{g^{-1}})_*"] \arrow[d, "\mathrm{ind}_K^G"] & \mathcal{H}_{p+q}^H(X^H) \arrow[d, "\mathrm{ind}_H^G"] \\
\mathcal{H}_{p+q}^G(G \times_K X^K) \arrow[dr, "m"] & \mathcal{H}_{p+q}^G(G \times_{c(g)} X^K) \arrow[l, "(R_{g^{-1}} \times \mathrm{id})_*"'] \arrow[r, "(\mathrm{id} \times L_{g^{-1}})_*"] & \mathcal{H}_{p+q}^G(G \times_H X^H) \arrow[dl, "m"'] \\
& \mathcal{H}_{p+q}^G(X) &
\end{tikzcd}$$

We consider X^K as a trivial H-space in $\mathcal{H}_{p+q}^H(X^K)$. The upper left square commutes as an immediate consequence of the definition of $D_{p,q}^H$ and the naturality of ind. The upper right square commutes because $D_{p,q}^H$ is functorial. By Lemma 1.5.11, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}_n^H(\{\bullet\}) \arrow[r, "\mathrm{pr}_* \circ \mathrm{ind}_{c(g)}"] \arrow[d, "\mathrm{ind}_H^G"] & \mathcal{H}_n^K(\{\bullet\}) \arrow[d, "\mathrm{ind}_K^G"] \\
\mathcal{H}_n^G(G/H) \arrow[r, "(R_{g^{-1}})_*"] & \mathcal{H}_n^G(G/K)
\end{tikzcd}$$

Hence the lower left square commutes. Because of $\mathrm{ind}_{L_{g^{-1}}} = \mathrm{id} \times L_{g^{-1}}$, the lower right square commutes. Finally, it is obvious that the lower triangle commutes and so does the entire diagram commutes. From this and the definition of $\underline{\mathrm{ch}}_{p,q}^G$ the assertion easily follows. $\square$

Theorem 4.1.2. *Let $\mathcal{H}_*^?$ be an equivariant smooth $(\Gamma, \mathcal{F}_?)$-homology theory with values in R-modules. Let $G \in \mathrm{Ob}(\Gamma)$ be a (topological) group and suppose the $\mathsf{Or}_{\mathcal{F}_G}(G)$-module $\mathcal{H}_n^G(G/?)$ is flat for all $n \in \mathbb{Z}$. Then there is an isomorphism, called* equivariant Chern character*, of $(G, \mathcal{F}_G)$-homology theories*

$$\mathrm{ch}_*^G \colon \mathcal{BH}_*^G(X, A) \to \mathcal{H}_*^G(X, A)$$

which is natural in (X, A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$ is another group with a corresponding flat coefficient module $\mathcal{H}_n^H(H/?)$, then ch_*^G and ch_*^H are compatible with induction ind_H^G.

Proof. By Lemma 4.1.1, the maps $\underline{\mathrm{ch}}^G_{p,q}(X,A)$ define a map

$$\widetilde{\mathrm{ch}}^G_*\colon \bigoplus_{p+q=n} H_p\big(C_*^{\mathrm{Or}_{\mathcal{F}_G}(G)}(X,A)\big) \otimes_{R\mathrm{Or}_{\mathcal{F}_G}(G)} \mathcal{H}^G_q(G/-) \to \mathcal{H}^G_{p+q}(X,A).$$

By assumption, $H^G_*(G/?)$ is flat and we obtain an isomorphism

$$\bigoplus_{p+q=n} H_p\big(C_*^{\mathrm{Or}_{\mathcal{F}_G}(G)}(X,A)\big) \otimes_{R\mathrm{Or}_{\mathcal{F}_G}(G)} \mathcal{H}^G_q(H/-) \xrightarrow{\cong} \mathcal{B}\mathcal{H}^G_*(X,A).$$

The composition of these two maps yields a map

$$\mathrm{ch}^G_*\colon \mathcal{B}\mathcal{H}^G_*(X,A) \to \mathcal{H}^G_*(X,A).$$

It is not difficult to verify that ch^G_n is natural in (X,A), compatible with the boundary maps and the induction structure. In order to show that ch^G_n is an isomorphism, it suffices to check this only for G/H for all $H \in \mathcal{F}_G$. Since ch^G_n is compatible with induction, $\mathrm{ch}^G_n(G/H)$ is an isomorphism if $\mathrm{ch}^H_n(\{\bullet\})$ is one. However, this is obvious. □

Corollary 4.1.3. *Let R be semisimple and let $\mathcal{H}^?_*$ be an equivariant smooth $(\Gamma, \mathcal{F}_?)$-homology theory with values in R-modules which has a Mackey structure on coefficients. Let $G \in \mathrm{Ob}(\Gamma)$ be a (topological) group such that the corresponding semigroups $\mathrm{mor}_{\mathrm{Or}_{\mathcal{F}_G}(G)}(G/H, G/H)$ are finite and $l_K(H), [H : K] < \infty$ for every $H, K \in \mathcal{F}_G$. Then there is an isomorphism of $(G, \mathcal{F}_G)$-homology theories*

$$\mathrm{ch}^G_*\colon \mathcal{B}\mathcal{H}^G_*(X,A) \to \mathcal{H}^G_*(X,A)$$

which is natural in (X,A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$, the maps ch^G_* and ch^H_* are compatible with ind^G_H.

Proof. Combining the previous theorem with Corollary 3.1.18 yields the desired assertion. □

Corollary 4.1.4. *Let R be semisimple and let $\mathcal{H}^?_*$ be an equivariant proper smooth homology theory which has a Mackey structure on coefficients. Moreover, let G be a prodiscrete l-group such that $C_GH/(H \cap C_GH)$ is locally finite for every compact open subgroup $H \subseteq G$. Then there is an isomorphism of proper smooth G-homology theories*

$$\mathrm{ch}^G_*\colon \mathcal{B}\mathcal{H}^G_*(X,A) \to \mathcal{H}^G_*(X,A)$$

which is natural in (X,A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$, the maps ch^G_* and ch^H_* are compatible with ind^G_H.

Proof. By Corollary 3.1.19, the coefficient module is flat as an $R\mathrm{Or}_{\mathcal{CO}}(G)$-module. Now the assertions follow by Theorem 4.1.2. □

Corollary 4.1.5. *Let G be a unimodular group with locally finite semigroup $\mathrm{mor}_{\mathrm{Or}_{\mathcal{CO}}(G)}(G/H, G/H)$ for every $H \subseteq G$ compact open. Denote by $\mathcal{H}^?_*$ the Borel construction. Then there is an isomorphism of proper smooth G-homology theories*

$$\mathrm{ch}^G_* \colon \mathcal{BH}^G_*(X,A) \otimes \mathbb{Q} \to \mathcal{H}^G_*(X,A) \otimes \mathbb{Q}$$

which is natural in (X,A) and compatible with the boundary maps. If $H \subseteq G$ is a compact open subgroup, the maps ch^G_* and ch^H_* are compatible with ind^G_H.

Proof. We want to apply Theorem 3.1.16. By Proposition 1.2.21, the orbit category is an EI-category and $l_K(H) < \infty$ for all compact open subgroups $K \subseteq H \subseteq G$. It remains to construct splits for the projections

$$\mathrm{pr}\colon \mathcal{H}^G(G/H)_N \otimes \mathbb{Q} \to S_H(\mathcal{H}^G(G/?)_N \otimes \mathbb{Q}),$$

where $N \subsetneq H \subseteq G$ are compact open subgroups. However, the structure maps are rationally surjective by Remark 2.1.2. Hence the right term vanishes, and the projections split. Thus $\mathcal{H}^G(G/?)_H \otimes \mathbb{Q}$ is flat, and we can apply Theorem 4.1.2. □

We want to give an example of a group G which has infinite but locally finite semigroups $\mathrm{mor}_{\mathrm{Or}_{\mathcal{CO}}(G)}(G/H, G/H)$ and does induce a Chern character.

Proposition 4.1.6. *Let L be a locally finite discrete group and F be a finite group. Furthermore, let $N = \prod_F L$ and G be the group*

$$G = N \rtimes L,$$

where L acts on N in the obvious way. Then

$$\mathrm{mor}_{\mathrm{Or}_{\mathcal{CO}}(G)}(G/N, G/N) = W_G N = L,$$

and there is an isomorphism of proper smooth G-homology theories

$$\mathrm{ch}^G_* \colon \mathcal{BH}^G_*(X,A) \otimes \mathbb{Q} \to \mathcal{H}^G_*(X,A) \otimes \mathbb{Q},$$

where $\mathcal{H}^?_*$ denotes the Borel construction.

Proof. We want to apply the previous corollary. Since $N \subseteq G$ is a compact open normal subgroup, G is unimodular by Lemma 1.2.16. This implies

$$\mathrm{mor}_{\mathrm{Or}_{\mathcal{CO}}(G)}(G/N, G/N) = W_G N$$

by Lemma 1.2.18. Moreover, $W_G N = L$ is obvious. It remains to show that the Weyl group $W_G H$ is locally finite for every compact open subgroup $H \subseteq G$. So let $H \subseteq G$ be a compact open subgroup. We define

$$H_1 = \mathrm{pr}_N(H) \subseteq N \quad \text{and} \quad H_2 = \mathrm{pr}_L(H) \subseteq L,$$

where pr_L and pr_N are the canonical projections. We remark that pr_N is not a group homomorphism and H_1 is not a group. Since H is open, H_1 is open and, by definition, there exists a cofinite set $\tilde{L} \subseteq L$ such that $\prod_{\tilde{L}} F \subseteq H_1$. The Weyl group is a subquotient of

$$\Big(\prod_{L \setminus \tilde{L}} F\Big) \times W_L H_2.$$

The first factor is finite. Hence $W_G H$ is locally finite by Proposition A.5. Now a Chern character exists by the previous corollary. □

Corollary 4.1.7. *Let R be semisimple and let $\mathcal{H}_*^?$ be an equivariant smooth $(\Gamma, \mathcal{F}_?)$-homology theory which has a Mackey structure on coefficients. Moreover, let $G \in \mathrm{Ob}(\Gamma)$ and suppose that either $\mathcal{F}_? = \mathcal{I}$ or that G is a semisimple p-adic group and $\mathcal{F}_? = \mathcal{CO}$. Then there is an isomorphism of $(G, \mathcal{F}_G)$-homology theories*

$$\mathrm{ch}_*^G \colon \mathcal{BH}_*^G(X, A) \to \mathcal{H}_*^G(X, A)$$

which is natural in (X, A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$, the maps ch_*^G and ch_*^H are compatible with ind_H^G.

Proof. The assertion follows from Theorem 4.1.2 and Corollary 3.1.20. □

Now we can tackle the main theorem of this thesis.

Theorem 4.1.8. *Let G be a semisimple p-adic group. Then we get an isomorphism*

$$\bigoplus_{k \in \mathbb{Z}} CH_{2k+n}^G(\beta G) \cong K_n(C_r^* G) \otimes \mathbb{C}.$$

Proof. In Proposition 2.2.42, we have already shown

$$CH_*^G(\beta G) = H_*^{\mathrm{Or}_{\mathcal{CO}}(G)}(\beta G; \mathcal{R}_G).$$

Thus we obtain

$$\mathcal{B}K_n^G(\beta G) \otimes \mathbb{C} = \bigoplus_{k \in \mathbb{Z}} CH_{2k+n}^G(\beta G).$$

Further, we have the Baum-Connes assembly map

$$\mu \colon K_n^G(\beta G) \to K(C_r^* G),$$

which is known to be an isomorphism by a celebrated result of Lafforgue [26]. Finally, there exists an isomorphism

$$\mathrm{ch}_*^G(\beta G) \colon \mathcal{B}K_*^G(\beta G) \otimes \mathbb{C} \xrightarrow{\cong} K_*^G(\beta G) \otimes \mathbb{C}$$

by Corollary 2.2.39 and Corollary 4.1.7. Composing the three isomorphisms yields the desired isomorphism. □

The K-theory of the reduced C^*-algebra of a semisimple p-adic group is a very interesting object. For a survey of implications of the latter isomorphism see [6, Sec. 6].

At the end of this section, we want to calculate $K_*(C_r^*G)$ for $G = SL_2(\mathbb{Q}_p)$. This is taken from Higson and Plymen [7]. Note that the latter isomorphism was already known in this case [41] because $SL_2(G)$ operates on a tree, its Bruhat-Tits building (Example 1.7.14). We remark that for arbitrary p-adic groups computations were made in [39].

We recall that G up to conjugation has two maximal compact subgroups:

$$K_0 = SL_2(\mathbb{Z}_p) \quad \text{and} \quad K_0 = \left\{ \begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix} \in G \;\middle|\; a, b, c, d \in \mathbb{Z}_p \right\}.$$

The intersection $I = K_0 \cap K_1$ defines a fundamental domain for βG and the corresponding chain complex reduces to

$$0 \leftarrow \mathcal{R}(K_0) \oplus \mathcal{R}(K_1) \xleftarrow{\mathrm{ind}_I^{K_0} - \mathrm{ind}_I^{K_1}} \mathcal{R}(I) \leftarrow 0.$$

Although this does not look very difficult to compute, this filled almost a complete paper [7]. We just want to report the results. In the case of $CH_1^G(\beta G)$, we get

$$CH_1^G(\beta G) = \{\, f \colon \mathbb{Z}_p^\times \to \mathbb{C} \mid f(a^{-1}) = -f(a),\ f \text{ locally constant} \,\}.$$

Let $\tilde{T}$ be a system of representatives of conjugacy classes of maximal tori. For a maximal torus $T \subseteq G$, we set

$$T_c^{reg} = \{\, t \in T \mid t \neq \pm \mathrm{id} \ \text{ compact} \,\},$$

where an element is called compact if it lies in a compact subgroup. We obtain the following exact sequence

$$0 \to \bigoplus_{[T] \in \tilde{T}} C_c^\infty(T_c^{reg})^{W_G T} \to CH_0^G(\beta G) \to \bigoplus_w \mathbb{C} \to 0,$$

where the second direct sum is taken over conjugacy classes of elements $w \in G$ such that $(w \pm \mathrm{id})$ is nilpotent. Moreover, we denote by $C_c^\infty(T_c^{reg})^{W_G T}$ the complex valued locally constant functions with compact support which are invariant with respect to the action of the Weyl group $W_G T$.

4.2 The Cohomological Equivariant Chern Character

Now we want to dualize the previous construction in order to get a map

$$\underline{\mathrm{ch}}_G^{p,q}(X,A)(H) \colon \mathcal{H}_{p+q}^G(X,A) \to \hom_R(H_p(X^H, A^H; R), \mathcal{H}_G^q(G/H)). \tag{4.2.0.1}$$

Fortunately, in this case, the construction is completely analogous to the previous one. Namely, we obtain a map

$$\begin{tikzcd}
\mathcal{H}^{p+q}_G(X) \arrow[d, "\mathcal{H}^{p+q}_G(v_H)"'] \\
\mathcal{H}^{p+q}_G(G \times_H X^H) \arrow[d, "\mathrm{ind}_H^G"', "\cong"] \\
\mathcal{H}^{p+q}_G(X^H) \arrow[d, "D^H_{p,q}"', "\cong"] \\
\hom_R(\pi_p^s((X^H)_+) \otimes R, \mathcal{H}^q_H\{\bullet\}) \\
\hom_R(H_p((X^H)_+; R), \mathcal{H}^q_H\{\bullet\}) \arrow[u, "(\mathrm{hur}(X^H),\mathrm{id})", "\cong"'] \\
\hom_R(H_p((X^H)_+; R), \mathcal{H}^q_G(G/H)) \arrow[u, "(\mathrm{id},\mathrm{ind}_H^G)", "\cong"']
\end{tikzcd}$$

The only point which might be unclear is the dual $D_H^{p,q}$ of $D^H_{p,q}$.

Given a space Z and a topological group H, we consider Z as a smooth H-space by the trivial action and define a map

$$D^H_{p,q}(Z) \colon \mathcal{H}^{p+q}_H(Z) \to \hom_R(\pi_p^s(Z_+) \otimes_{\mathbb{Z}} R, \mathcal{H}^H_q(\{\bullet\}))$$

as follows. Given an element $a \in \mathcal{H}^{p+q}(Z)$ and an element in $\pi_p^s(Z_+, \{\bullet\})$ represented by a map $f \colon S^{p+q} \to S^k \wedge Z_+$, we define $D^{p,q}(a)([f]) \in \mathcal{H}^{p+q}(\{\bullet\})$ as the image of a under the composition

$$\mathcal{H}^{p+q}(Z) \xleftarrow{\cong} \tilde{\mathcal{H}}^{p+q}(Z_+) \xrightarrow{\sigma^k} \tilde{\mathcal{H}}^{p+q+k}(S^k \wedge Z_+)$$
$$\xrightarrow{\tilde{\mathcal{H}}^{p+q}(f)} \tilde{\mathcal{H}}^{p+q}(S^{p+k}) \xrightarrow{(\sigma^{p+k})^{-1}} \tilde{\mathcal{H}}^q(S^0) \xrightarrow{\cong} \mathcal{H}^q(\{\bullet\}),$$

where σ denotes the suspension isomorphism.

Lemma 4.2.1. *Let G be a group and X be a smooth G-CW-complex. Then consider open subgroups $H, K \subseteq G$ and $g \in G$ with $gHg^{-1} \subseteq K$. Let $L_{g^{-1}} \colon X^H \to X^K$ be the map induced by left multiplication with g^{-1}. Let $R_{g^{-1}} \colon G/H \to G/K$ be given by right multiplication with g^{-1}. Then the following diagram commutes*

$$\begin{tikzcd}
\hom_R(H_p(X^K; R), \mathcal{H}^q_G(G/H)) & \hom_R(H_p(X^H; R), \mathcal{H}_q^G(G/H)) \arrow[l, "((L_{g^{-1}})_*,\mathrm{id})"'] \\
\hom_R(H_p(X^K; R), \mathcal{H}^q_G(G/K) \arrow[u, "(\mathrm{id},R_{g^{-1}})"] & \mathcal{H}^G_{p+q}(X) \arrow[l, "\underline{\mathrm{ch}}^G_{p,q}(X)(K)"'] \arrow[u, "\underline{\mathrm{ch}}^G_{p,q}(X)(H)"']
\end{tikzcd}$$

Proof. The proof is essentially the same as in the homological case, which is proven in Lemma 4.1.1. □

Finally, we obtain

Theorem 4.2.2. *Let $\mathcal{H}^*_?$ be an equivariant smooth $(\Gamma, \mathcal{F}_?)$-cohomology theory with values in R-modules. Let $G \in \mathrm{Ob}(\Gamma)$ be a (topological) group and suppose the $\mathtt{Or}_{\mathcal{F}_G}(G)$-module $\mathcal{H}^n_G(G/?)$ is injective for all $n \in \mathbb{Z}$. Then, for (X, A) finite, there is an isomorphism, called* equivariant Chern character*, of $(G, \mathcal{F}_G)$-cohomology theories*

$$\mathrm{ch}^*_G \colon \mathcal{H}^*_G(X, A) \to \mathcal{BH}^*_G(X, A).$$

Furthermore, ch^*_G is natural in (X, A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$ is another group with a corresponding injective coefficient module $\mathcal{H}^*_H(H/?)$, then ch^*_G and ch^*_H are compatible with ind^G_H.

Proof. This can be proven analogously to Theorem 4.1.2. □

Corollary 4.2.3. *Let R be semisimple and let $\mathcal{H}^*_?$ be an equivariant smooth $(\Gamma, \mathcal{F}_?)$-cohomology theory with values in R-modules which has a Mackey structure on coefficients. Let $G \in \mathrm{Ob}(\Gamma)$ be a (topological) group such that the corresponding morphism set $\mathrm{mor}_{\mathtt{Or}_{\mathcal{CO}}(G)}(G/K, G/H)$ is finite and $[H : K] < \infty$ for every $H, K \in \mathcal{F}_G$. Furthermore, suppose*

$$\lim_{N \in I_{(\mathcal{F}_G)_H}}{}^1 \hom_{R\mathtt{Sub}_{(\mathcal{F}_G)_H}(H)}(R\mathtt{Sub}_{(\mathcal{F}_G)_H}(H)(?, H)/L, \mathcal{H}^*_H(H/?)) = 0$$

for every $H \in \mathcal{F}_G$ and $L \subseteq R\mathtt{Sub}_{(\mathcal{F}_G)_H}(H)(?, H)$.

Then, for (X, A) finite, there is an isomorphism, called equivariant Chern character*, of $(G, \mathcal{F}_G)$-cohomology theories*

$$\mathrm{ch}^*_G \colon \mathcal{H}^*_G(X, A) \to \mathcal{BH}^*_G(X, A).$$

Furthermore, ch^*_G is natural in (X, A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$ is another group, then ch^*_G and ch^*_H are compatible with ind^G_H.

Proof. This follows from Theorem 3.2.19 and the previous theorem. □

Corollary 4.2.4. *Let R be semisimple and let $\mathcal{H}^*_?$ be an equivariant smooth $(\Gamma, \mathcal{F}_?)$-cohomology theory with values in R-modules which has a Mackey structure on coefficients. Let $G \in \mathrm{Ob}(\Gamma)$ be a (topological) group and suppose either $\mathcal{F} = \mathcal{I}$ or that G is a semisimple p-adic group and $\mathcal{F} = \mathcal{CO}$. Furthermore, suppose*

$$\lim_{N \in I_{\mathcal{F}_H}}{}^1 \hom_{R\mathtt{Sub}_{\mathcal{F}_H}(H)}(R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)/L, \mathcal{H}^*_H(H/?)) = 0$$

for every $H \in \mathcal{F}_G$ and $L \subseteq R\mathtt{Sub}_{\mathcal{F}_H}(H)(?, H)$.

Then, for (X,A) finite, there is an isomorphism, called equivariant Chern character*, of $(G,\mathcal{F}_G)$-cohomology theories*

$$\mathrm{ch}_G^*\colon \mathcal{H}_G^*(X,A) \to \mathcal{BH}_G^*(X,A).$$

Furthermore, ch_G^* is natural in (X,A) and compatible with the boundary maps. If $H \in \mathcal{F}_G$ is another group, then ch_G^* and ch_H^* are compatible with ind_H^G.

Proof. This follows from Corollary 3.2.20. □

Corollary 4.2.5. *Let R be semisimple and let $\mathcal{H}_?^*$ be an equivariant proper smooth cohomology theory with values in R-modules which has a Mackey structure on coefficients. Let G be a prodiscrete l-group such that the groups $C_GH/(H \cap C_GH)$ are locally finite for every compact open subgroup $H \subseteq G$. Furthermore, suppose*

$$\lim_{N \in I_{\mathcal{CO}}}{}^1 \hom_{R\mathtt{Sub}_{\mathcal{CO}}(H)}(R\mathtt{Sub}_{\mathcal{CO}}(H)(?,H)/L, \mathcal{H}_H^*(H/?)) = 0$$

for every $H \in \mathcal{F}_G$ and $L \subseteq R\mathtt{Sub}_{\mathcal{CO}}(H)(?,H)$.

Then, for (X,A) finite, there is an isomorphism, called equivariant Chern character*, of proper smooth G-cohomology theories*

$$\mathrm{ch}_G^*\colon \mathcal{H}_G^*(X,A) \to \mathcal{BH}_G^*(X,A).$$

Furthermore, ch_G^* is natural in (X,A) and compatible with the boundary maps. If $H \subseteq G$ is a compact open subgroup, then ch_G^* and ch_H^* are compatible with ind_H^G.

Proof. This follows from Corollary 3.2.18. □

4.3 About the Non-Existence of Equivariant Chern Characters

We give examples of groups G and G-homology theories for which a Chern character cannot exist. Note that the existence of a Chern character implies that the corresponding equivariant Atiyah-Hirzebruch spectral sequence (see Theorem 1.6.4) collapses.

The following example shows that for a discrete group G and a given G-homology theory there need not be a Chern character, some further conditions such as an induction structure or a Mackey structure are needed.

Example 4.3.1 (Counterexample for discrete groups). Let Hei be the three-dimensional Heisenberg group, which we have already seen in Example 3.3.4. We recall that this is the subgroup of $GL_3(\mathbb{Z})$ consisting of upper triangular matrices with 1 on the diagonal. It has the presentation

$$\mathrm{Hei} = \langle u, v, z \mid [u,z] = 1, [v,z] = 1, [u,v] = z \rangle \,. \tag{4.3.1.1}$$

There is the central extension

$$1 \to \mathbb{Z} \xrightarrow{i} \mathrm{Hei} \xrightarrow{\mathrm{pr}} \mathbb{Z}^2 \to 1, \tag{4.3.1.2}$$

where i maps the generator in $\mathbb{Z}$ to the element z and pr sends z to $(0,0)$, u to $(1,0)$ and v to $(0,1)$. Put $G = \mathbb{Z}^2$ and $F = \mathbb{Z}\backslash E\mathrm{Hei}$, where $\mathbb{Z} \subseteq \mathrm{Hei}$ is the central infinite cyclic subgroup generated by z. Then the Hei-action on EHei induces via $\mathrm{pr}\colon \mathrm{Hei} \to \mathbb{Z}^2$ a $G = \mathbb{Z}^2$-action on $F = \mathbb{Z}\backslash E\mathrm{Hei}$. We claim that there cannot be a Chern character for the G-homology theory with values in $\mathbb{Q}$-modules given by

$$\mathcal{H}_*^G(X) := H_*(F \times_G X; \mathbb{Q}).$$

Suppose a G-equivariant Chern character would exist. For $X = EG$, it would yield an isomorphism

$$\bigoplus_{p+q=n} H_p^G(EG; H_q(F;\mathbb{Q})) \xrightarrow{\cong} H_n(F \times_G EG; \mathbb{Q}), \tag{4.3.1.3}$$

where the left hand side is computed by the chain complex $C_*^{\mathrm{cell}}(EG) \otimes_{\mathbb{Z}G} H_*(F;\mathbb{Q})$. Since G acts freely on F, the projection $F \times_G EG \to G\backslash F = B\mathrm{Hei}$ is a homotopy equivalence. Hence we get

$$H_n(F \times_G EG; \mathbb{Q}) \;\cong\; H_n(B\mathrm{Hei}; \mathbb{Q}).$$

The integral homology of BHei is computed in [31, Lem. 5.3] using the Lyndon-Serre spectral sequence associated to the central extension (4.3.1.2) from above. This spectral sequence converges to $H_{p+q}(B\mathrm{Hei})$ and has as E^2-term $H_p(B\mathbb{Z}^2; H_q(\mathbb{Z}))$. It does not collapse, not even after rationalization. The main point is that the group homomorphism $i\colon \mathbb{Z} \to \mathrm{Hei}$ induces the zero map $H_1(B\mathbb{Z}) \to H_1(B\mathrm{Hei})$ because z is the commutator $[u,v]$. Hence the second differential

$$d_{2,0}^2\colon H_2(B\mathbb{Z}^2; H_0(\mathbb{Z})) \to H_0(B\mathbb{Z}^2; H_1(\mathbb{Z})) = H_1(\mathbb{Z})$$

is surjective and in particular is rationally non-trivial. This implies that pr induces an isomorphism

$$H_1(B\mathrm{pr})\colon H_1(B\mathrm{Hei}) \xrightarrow{\cong} H_1(B\mathbb{Z}^2).$$

Hence the target of the hypothetical Chern character (4.3.1.3) would be given by $H_1(B\mathrm{Hei};\mathbb{Q}) \cong \mathbb{Q}^2$ for $n = 1$. Its source is

$$H_0^G(EG; H_1(F;\mathbb{Q})) \oplus H_1^G(EG; H_0(F;\mathbb{Q})) \;\cong\; H_1(\mathbb{Z};\mathbb{Q}) \oplus H_1(\mathbb{Z}^2;\mathbb{Q}) \;\cong\; \mathbb{Q}^3$$

since $F \simeq B\mathbb{Z}$ is homotopy equivalent to S^1 and G acts trivially on $H_*(F;\mathbb{Q})$. Hence there cannot be an isomorphism as described in (4.3.1.3). The Lyndon-Serre spectral sequence mentioned above can be identified with the equivariant Atiyah-Hirzebruch spectral sequence, which converges to $\mathcal{H}_{p+q}^G(EG)$ and whose E^2-term is

$$H_p^{\mathrm{Or}_{\{1\}}(G)}(EG; \mathcal{H}_q^G(G/?)) = H_1^G(EG; H_q(F;\mathbb{Q})).$$

Thus we see why the equivariant Atiyah-Hirzebruch spectral sequence does not collapse after rationalization as there is a non-trivial second differential.

We do not know an equivariant homology theory $\mathcal{H}^?_*$ whose evaluation at G is the G-homology theory $\mathcal{H}^G_*$ considered above. The problem is that the G-space F is not given in a universal way depending only on G as it would be the case if we chose F to be EG.

In the case of discrete groups, a Chern character exists for proper equivariant homology theories which have a Mackey functor structure (cf. Section 5.4). In the topological case, the machinery, we have developed, is much more restrictive. The next example shows that a Chern character does not in general exist for proper smooth equivariant homology theories which have a Mackey functor structure.

Example 4.3.2 (Counterexample for l-groups). Let K be any compact abelian l-group together with an injective group homomorphism $k\colon \mathbb{Z} \to K$. We do *not* require that k is a homomorphism of topological groups, we just demand that it is compatible with the group structure. An example is given by the p-adic integers $K = \mathbb{Z}_p$ and the obvious inclusion $k\colon \mathbb{Z} \to \mathbb{Z}_p$. The central extension (4.3.1.2) yields a central extension of topological groups

$$1 \to K \xrightarrow{j} G \xrightarrow{p} \mathbb{Z}^2 \to 1 \tag{4.3.2.1}$$

described next. We equip $\mathbb{Z}^2$ and Hei with the discrete topology. The topological group G is the quotient of $K \times \mathrm{Hei}$ by the central closed subgroup given by the image of $j \times k\colon \mathbb{Z} \to K \times \mathrm{Hei}$. The homomorphism of topological groups j is induced by the inclusion of groups $K \to K \times \mathrm{Hei}$ and $p\colon G \to \mathbb{Z}^2$ is induced by the composition of the projection $K \times \mathrm{Hei} \to \mathrm{Hei}$ with $\mathrm{pr}\colon \mathrm{Hei} \to \mathbb{Z}^2$. We denote by $l\colon \mathrm{Hei} \to G$ the map induced by the inclusion $\mathrm{Hei} \to K \times \mathrm{Hei}$. Note that G is an l-group since K is an l-group. We recall that in the case of $K = \mathbb{Z}_p$ the group G even is a p-adic Lie group by [18, Thm. 8.1]. Finally, note that this is a slight generalization of the setting in Example 3.3.4.

Next we consider the G-homology theory given by the Borel construction and singular homology with rational coefficients

$$\mathcal{H}^G_*(X) \;=\; H_*(EG \times_G X; \mathbb{Q}).$$

Suppose that there exists a G-equivariant Chern character for $\mathcal{H}^G_*$. We consider the proper smooth G-CW-complex $X = E\mathbb{Z}^2$ obtained from the free $\mathbb{Z}^2$-CW-complex $E\mathbb{Z}^2$ by restriction with $p\colon G \to \mathbb{Z}^2$. The hypothetical Chern character would give an isomorphism

$$\bigoplus_{p+q=n} H_p^{\mathrm{Or}_{\mathcal{CO}}(G)}(E\mathbb{Z}^2; H_q(EG \times_G G/?; \mathbb{Q})) \xrightarrow{\cong} \mathcal{H}^G_n(E\mathbb{Z}^2).$$

The right hand side simplifies to

$$\begin{aligned}\mathcal{H}^G_n(E\mathbb{Z}^2) &= H_n(E\mathbb{Z}^2 \times_G EG; \mathbb{Q}) = H_n(E\mathbb{Z}^2 \times_{\mathbb{Z}^2} (EG)/K; \mathbb{Q}) \\ &= \mathcal{H}^{\mathbb{Z}^2}_n(EG/K) \xrightarrow[\cong]{\mathrm{ind}_{\mathrm{pr}\colon \mathbb{Z}^2 \to \{1\}}} \mathcal{H}^{\{1\}}_n((EG/K)/\mathbb{Z}^2) = H_n(BG; \mathbb{Q}).\end{aligned}$$

We remark that $\mathrm{pr}\colon \mathbb{Z}^2 \to \{1\}$ is not injective but in Section 2.1 we constructed an induction structure in a more general setting, which is now applicable. Since every isotropy group of $E\mathbb{Z}^2$ is equal to K, $E\mathbb{Z}^2$ is contractible and the extension (4.3.2.1) is central, the hypothetical Chern character can be identified with an isomorphism

$$\bigoplus_{p+q=n} H_p(T^2; H_q(BK)) \xrightarrow{\cong} H_n(BG;\mathbb{Q}). \tag{4.3.2.2}$$

This would imply that the inclusion $j\colon K \to G$ induces a group homomorphism

$$H_1(Bj)\colon H_1(BK) \to H_1(BG),$$

which is rationally injective. Since G is an l-group, we get $H_1(BG) = G/[G,G]$ and analogously $H_1(BK) = K/[K,K]$ by Remark 2.1.3. Under these identifications $H_1(BK) \to H_1(BG)$ becomes the group homomorphism $\overline{j}\colon K \to G/[G,G]$ induced by $j\colon K \to G$. Since the following diagram commutes:

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\overline{i}"] \arrow[d, "k"] & \mathrm{Hei}/[\mathrm{Hei},\mathrm{Hei}] \arrow[d, "\overline{l}"] \\ K \arrow[r, "\overline{j}"] & G/[G,G] \end{tikzcd}$$

we obtain that $\overline{i}\colon \mathbb{Z} \to \mathrm{Hei}/[\mathrm{Hei},\mathrm{Hei}]$ is injective. However, this map is zero since $z \in \mathrm{Hei}$ is the commutator $[u,v]$. Hence the Chern character (4.3.2.2) cannot exist for G and the G-homology theory $\mathcal{H}^G_*$.

Note that, in this case, $\mathtt{Sub}_{\mathcal{CO}}(G) = \mathtt{Sub}_{\mathcal{CO}}(K)$ because the extension is central. The corresponding coefficient module admits a Mackey structure and is flat as a $\mathbb{Q}\mathtt{Sub}_{\mathcal{CO}}(G)$-module since K is compact. Hence it is impossible to construct a Chern character starting from $\mathtt{Sub}_{\mathcal{CO}}(G)$ for l-groups as it is done in Section 5.4 for discrete groups. Besides, the coefficient module cannot be flat as a $\mathbb{Q}\mathtt{Or}_{\mathcal{CO}}(G)$-module.

5 Comparison of Different Chern Characters

In the first section, we show that two natural equivalences from Bredon (co)homology to another equivariant (co)homology coincide if they coincide on coefficients. In the remaining sections, we introduce Chern characters which were known before and show that all these constructions coincide by applying the theorem of the first section.

5.1 The Comparison Theorem

In the following, we only consider the homological case. If we restrict to finite G-CW-complexes, the cohomological case carries over verbatim.

Theorem 5.1.1. *Let G be a group and $\mathcal{F}$ be a smooth family of subgroups. Let*

$$\tau_1, \tau_2 \colon \mathcal{B}H_*^G(-; M_*) \to \mathcal{B}H^G(-; M_*)$$

be two natural transformations of Bredon homology over $(G, \mathcal{F})$-CW-pairs for some (graded) coefficient module M_*. Suppose further $\tau_1(G/H) = \tau_2(G/H)$ for all $H \in \mathcal{F}$. Then we obtain $\tau_1 = \tau_2$.

Proof. Let X be a $(G, \mathcal{F})$-CW-complex and $\emptyset \subseteq X_0 \subseteq X_1 \subseteq \cdots$ be a corresponding filtration. By Remark 1.6.3, we obtain

$$\mathcal{B}H_n^G(X_n, X_{n-1}; M_*) = \bigoplus_{\substack{\sigma\ G/H\text{-}k\text{-cell} \\ l+k=n}} \mathcal{B}H_l^G(G/H, M_*).$$

Since this decomposition is natural, the natural transformations τ_1 and τ_2 respect it. Consequently, they coincide on $\mathcal{B}H_*^G(X_n, X_{n-1}; M_*)$ and hence on chain complex level. As an immediate consequence, they must coincide on $\mathcal{B}H_*^G(X; M_*)$. ◻

Corollary 5.1.2. *Let* $\mathrm{ch}_1, \mathrm{ch}_2 \colon \mathcal{B}\mathcal{H}_*^G \xrightarrow{\cong} \mathcal{H}_*^G$ *be two natural equivalences of $(G, \mathcal{F})$-homology theories such that $\mathrm{ch}_1^G(G/H) = \mathrm{ch}_2^G(G/H)$ for all $H \in \mathcal{F}$. Then we obtain* $\mathrm{ch}_1 = \mathrm{ch}_2$.

Proof. We apply the previous theorem to $(\mathrm{ch}_1)^{-1} \circ \mathrm{ch}_2$. ◻

Corollary 5.1.3. *Let* $\mathrm{ch}_1^?, \mathrm{ch}_2^? \colon \mathcal{B}\mathcal{H}_*^? \xrightarrow{\cong} \mathcal{H}_*^?$ *be two natural equivalences of equivariant $(\Gamma, \mathcal{F}_?)$-homology theories and $G \in \mathrm{Ob}(\Gamma)$. Suppose that* $\mathrm{ch}_1^H(\{\bullet\}) = \mathrm{ch}_2^H(\{\bullet\})$ *for any $H \in \mathcal{F}_G$ and* $\mathrm{ch}_1^G, \mathrm{ch}_2^G$ *are compatible with the induction structure. Then we obtain* $\mathrm{ch}_1^G = \mathrm{ch}_2^G$.

Remark 5.1.4. There is another direct approach involving the Atiyah-Hirzebruch spectral sequence. However, this does not lead to a more general result. In order to see this, let $\tau_1, \tau_2 \colon \mathcal{H}_*^G \to \mathcal{K}_*^G$ be two natural transformations of $(G, \mathcal{F})$-homology theories which coincide on coefficients. We get a commutative diagram

$$\begin{tikzcd} \mathcal{B}H_p^G(X; \mathcal{H}_q^G(G/?)) \arrow[r, Rightarrow] \arrow[d, "(\tau_1)_*=(\tau_2)_*"] & \mathcal{H}_n^G(X) \arrow[d, bend right, "\tau_1"'] \arrow[d, bend left, "\tau_2"] \\ \mathcal{B}H_p^G(X, \mathcal{K}_q^G(G/?)) \arrow[r, Rightarrow] & \mathcal{K}_n^G(X). \end{tikzcd}$$

If the spectral sequences do not collapse, we have only the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & F_{p-1,q+1}\mathcal{H}_{p+q}^G(X) \arrow[r] \arrow[d, "\tau_1=\tau_2"'] & F_{p,q}\mathcal{H}_{q+q}^G(X) \arrow[r] \arrow[d, bend right, "\tau_1"'] \arrow[d, bend left, "\tau_2"] & (E_{p,q}^\infty)_{\mathcal{H}} \arrow[r] \arrow[d, "\tau_1^\infty=\tau_2^\infty"] & 0 \\ 0 \arrow[r] & F_{p-1,q+1}\mathcal{K}_{p+q}^G(X) \arrow[r] & F_{p,q}\mathcal{K}_{p+q}^G(X) \arrow[r] & (E_{p,q}^\infty)_{\mathcal{K}} \arrow[r] & 0 \end{tikzcd}$$

Unfortunately, the outer maps do not determine the middle one in general. For example we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C} \arrow[r, "\mathrm{inc}_1"] \arrow[d, "\mathrm{id}"] & \mathbb{C} \oplus \mathbb{C} \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\varphi"] & \mathbb{C} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r, "\mathrm{inc}_1"] & \mathbb{C} \oplus \mathbb{C} \arrow[r, "\mathrm{pr}_2"] & \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

where φ can be

$$\varphi = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

for an arbitrary $z \in \mathbb{C}$.

Furthermore, if τ_1, τ_2 are not isomorphisms, the $E_{p,q}^\infty$-terms can be concentrated on different p, q for $\mathcal{H}_*^G$ and $\mathcal{K}_*^G$. Then we obtain $\tau_1^\infty = \tau_2^\infty = 0$ and we cannot recover any information. Thus we cannot conclude anything for τ_1, τ_2.

5.2 A Chern Character for Discrete Groups (Lück/Oliver)

Let G be a discrete group. Lück and Oliver [34] constructed a Chern character for topological K-theory and (finite) proper G-CW-complexes. We want to recall the construction briefly. We obtain a map $\tilde{\tau}_G^*(X)(H)$ by

$$\begin{aligned} K_G^*(X) &\xrightarrow{\mathrm{res}_G^{N_G H}} K_{N_G H}^*(X^H) \xrightarrow{\Psi} K_{C_G H}^*(X^H) \otimes R(H) \xrightarrow{\mathrm{pr}^*} K_{C_G H}^*(EG \times X^H) \otimes R(H) \\ &\xrightarrow[\cong]{\mathrm{ind}_{C_G H}^{\{1\}}} K^*(EG \times_{C_G H} X^H) \otimes R(H) \xrightarrow{\mathrm{ch} \otimes \mathrm{id}} H^*(EG \times_{C_G H} X^H; \mathbb{Q}) \otimes R(H) \\ &\xleftarrow[\cong]{\mathrm{pr}^*} H^*(X^H/C_G H; \mathbb{Q}) \otimes R(H) \cong \hom(H_*(X^H/C_G H), \mathbb{Q} \otimes R(H)). \end{aligned}$$

Some explanations should be made. The map ch denotes the ordinary Chern character. We want to define the maps Ψ and $\mathrm{ind}_{C_G H}^{\{1\}}$ now.

Let $E \to X$ be a G-vector bundle over a free G-CW-complex X. Then the induction map is given by

$$\mathrm{ind}_G^{\{1\}}(E \to X) = E/G \to X/G.$$

Let $N \lhd G$ be normal and finite. Denote by $\mathrm{Irr}(N)$ the set of isomorphism classes of irreducible (complex) N-representations. Let X be any finite proper G/N-CW-complex. For any $V \in \mathrm{Irr}(N)$ and any G-vector bundle $E \to X$, let $\hom_N(V, E)$ denote the vector bundle over X whose fiber over $x \in X$ is $\hom_N(V, E_x)$. If $H \subseteq G$ is a subgroup which centralizes N, we can regard $\hom_N(V, E)$ as an H-vector bundle by $(hf)(x) = h \cdot f(x)$ for any $h \in H$ and $f \in \hom_N(V, E)$. This induces a map

$$\Psi \colon K_G^*(X) \to K_H^*(X) \otimes R(N), \quad [E] \mapsto \sum_{V \in \mathrm{Irr}(N)} [\hom_N(V, E)] \otimes [V].$$

Since $R(?) \otimes \mathbb{Q}$ is an injective $\mathbb{Q}\mathtt{Sub}_{\mathcal{FIN}}(G)$-module by Lemma 3.2.10, the maps $\tilde{\tau}_G^*(X)(?)$ assemble to a natural transformation

$$\tau_?^* \colon K_?^*(X) \to \mathcal{B}(K_?^* \otimes \mathbb{Q})(X)$$

for finite proper G-CW-complexes, which is rationally an equivalence.

Let G be a finite group and $X = \{\bullet\}$. Furthermore, let V be a G-representation and

$$V = \sum_{W \in \mathrm{Irr}(G)} W^{n_W}$$

the decomposition into irreducible G-representations. For a subgroup $H \subseteq G$ and a G-representation W, the corresponding restricted H-representation is denoted by W_H. Then the Chern character $\tau_G^*(\{\bullet\}) = \tau_G^*(\{\bullet\})(G)$ for a finite group G is given by

$$\begin{aligned} [V] &\mapsto [V] \mapsto \sum_{W \in \mathrm{Irr}(G)} [W_{C_G G}^{n_W}] \otimes [W] \mapsto \sum_{W \in \mathrm{Irr}(G)} [W_{C_G G}^{n_W} \times EG] \otimes [W] \\ &\mapsto \sum_{W \in \mathrm{Irr}(G)} [W_{\{1\}}^{n_W} \times (EG/C_G G)] \otimes [W] \mapsto \sum_{W \in \mathrm{Irr}(G)} [\dim_{\mathbb{C}} [W_{\{1\}}^{n_W}]_{EG/C_G G} \otimes [W] \\ &\mapsto \sum_{W \in \mathrm{Irr}(G)} [\dim_{\mathbb{C}} W_{\{1\}}^{n_W}] \otimes [W] \mapsto \Big(\mathbb{Z} \to R(G),\ 1 \mapsto \sum_{W \in \mathrm{Irr}(G)} [\dim_{\mathbb{C}} W_{\{1\}}^{n_W} \otimes W]\Big). \end{aligned}$$

Therefore, $\tau_G^0\{\bullet\}$ is the canonical inclusion

$$\tau_G^0 \colon R(G) \hookrightarrow \mathbb{Q} \otimes R(G).$$

We remind the reader that $K_G^1(\{\bullet\}) = 0$.

Since $\tau_?^*$ is compatible with the induction structure, we obtain by Corollary 5.1.3:

Proposition 5.2.1. *Let G be a discrete group. If ch_G^* exists, we have the equality*

$$\tau_G^*(X) \otimes \mathbb{Q} = \mathrm{ch}_G^*(X)$$

for any finite proper G-CW-complex X.

5.3 A Cohomological Chern Character for Prodiscrete Groups (Sauer)

Sauer [46] generalized the construction of the previous section to prodiscrete groups. We want to recall the construction.

In the following, let G be a prodiscrete l-group. We set

$$N(G) = \{\, H \subseteq G \mid H \text{ compact open normal subgroup} \,\}$$
$$\text{and} \quad I_G(X) = N(G) \cap \left\{ L \subseteq \bigcap_{x \in X} G_x \right\}$$

for a finite proper smooth G-CW-complex X.

Lemma 5.3.1. *We get an isomorphism*

$$R(G) = \operatorname*{colim}_{L \in N(G)} R(G/L).$$

Proof. The isomorphism is given in one direction by $res_G^{G/L}$ and in the other direction by dividing out the kernel of the group homomorphism $G \to GL_n(\mathbb{C})$ associated to the representation. Note that the only totally disconnected compact subgroups of $GL_n(\mathbb{C})$ are the finite ones. Therefore the kernel is open. $\square$

Proposition 5.3.2. *Let* $\tilde{K}_G^*(X, A) = \operatorname{colim}_{L \in I_G(X)} K_{G/L}^*(X, A)$. *Then there exists a natural equivalence* $\Phi\colon \tilde{K}_G^* \to K_G^*$ *of equivariant proper smooth homology theories which is induced by* $\operatorname{res}_G^{G/L}\colon K_{G/L}^* \to K_G^*$.

Proof. Because of functoriality of res the map Φ is well-defined. It is a natural transformation of proper smooth G-cohomology theories because $\operatorname{res}_G^{G/L}$ is one. In order to prove that Φ is an equivalence, it suffices to show that $\Phi(G/H)$ is an isomorphism for any compact open normal subgroup $H \subseteq G$. For any $L \in I_G(X) \cap \{\, L \subseteq H \,\}$ we get

$$\begin{aligned} K_{G/L}^*(G/H) &= K_{G/L}^*\big((G/L)/(H/L)\big) \\ &= \begin{cases} R(H/L), & \text{if } * \text{ is even,} \\ 0, & \text{if } * \text{ is odd,} \end{cases} \end{aligned}$$

and

$$K_G^*(G/H) = \begin{cases} R(H), & \text{if } * \text{ is even,} \\ 0, & \text{if } * \text{ is odd.} \end{cases}$$

Now the assertion follows from the previous lemma. $\square$

Remark 5.3.3. Obviously we can perform the same construction for Bredon homology. There we get a natural equivalence of proper smooth cohomology theories

$$\Psi\colon \widetilde{\mathcal{B}K}_G^* \xrightarrow{\cong} \mathcal{B}K_G^*.$$

Lemma 5.3.4. *Let H, H' be discrete groups and $\pi\colon H \to H'$ surjective with finite kernel. Furthermore, let τ_H^* and $\tau_{H'}^*$ be the Chern characters of Section 5.2. Then we have the following identity*

$$\mathrm{res}_{\pi\colon H\to H'} \circ \mathrm{ch}_{H'}^* = \tau_H^* \circ \mathrm{res}_{\pi\colon H\to H'}\,.$$

Proof. This is the main step in the construction of this Chern character and is proven in [46, p.445-447] □

Theorem 5.3.5. *There exists a natural transformation of equivariant proper smooth cohomology theories*

$$\tau_G^*\colon K_G \xrightarrow{\cong} \mathcal{B}(K_G^* \otimes \mathbb{Q})$$

which is rational an equivalence.

Proof. The natural transformation is defined by the following diagram

$$\begin{tikzcd}
\operatorname*{colim}_{L\in I_G(X)} K_{G/L}^*(X,A) \arrow[r, "\operatorname{colim}\tau_{G/L}^*"] \arrow[d, "\Phi(X,A)"] & \operatorname*{colim}_{L\in I_G(X)} \mathcal{B}(K\otimes\mathbb{Q})_{G/L}^*(X,A) \arrow[d, "\Psi(X,A)"] \\
K_G^*(X,A) \arrow[r, dotted, "\tau_G^*"] & \mathcal{B}(K\otimes\mathbb{Q})_G^*(X,A)
\end{tikzcd}$$

We recall that Φ and Ψ are isomorphisms and $\operatorname{colim}\tau_{G/L}^*$ exists by the previous lemma. □

Remark 5.3.6. The only thing we really need is that the Chern character used is compatible with the given restriction structure. Hence we can generalize any Chern character of discrete groups (with source $K_?^*$) to a Chern character of prodiscrete groups. We can even change the source $\mathcal{H}_?^*$ if it satisfies

$$\operatorname*{colim}_{L\in N(G)} \mathcal{H}_{G/L}^*(G/H) \cong \mathcal{H}_G^*(G/H) \text{ for any compact open subgroup } H\subseteq G.$$

Proposition 5.3.7. *If* ch_G^* *exists, we obtain the equality* $\mathrm{ch}_G^* = \tau_G^* \otimes \mathbb{Q}$.

Proof. For a compact open subgroup $H \subseteq G$, the map

$$\tau_H^*(\{\bullet\})\colon R(H) \to R(H)\otimes\mathbb{Q}$$

is just the canonical inclusion. Hence the assertion follows from Corollary 5.1.3. □

5.4 A Chern Character for Discrete Groups (Lück)

Our Chern character is very much inspired by the one of Lück [29]. The main difference is that our Chern character is defined on the orbit category and the other one is defined on the subgroup category.

Let G be a discrete group and $\mathcal{H}^?_*$ an equivariant proper homology theory (for discrete groups) over R. We obtain a map

$$\begin{tikzcd}
H_p(C_GH\backslash X^H;R)\otimes_R \mathcal{H}^G_q(G/H) \\
H_p(EG\times_{C_GH} X^H;R)\otimes_R \mathcal{H}^G_q(G/H) \arrow[u, "H_p(\mathrm{pr}_1;R)\otimes\mathrm{id}"'] \\
\pi^s_p(EG\times_{C_GH} X^H;R)\otimes_{\mathbb{Z}} R\otimes \mathcal{H}^H_q(\{\bullet\}) \arrow[u, "\mathrm{hur}(EG\times_{C_GH}X^H)\otimes\mathrm{ind}^G_H"'] \arrow[d, "D^H_{p,q}(EG\times_{C_GH}X^H)"'] \\
\mathcal{H}^H_{p+q}(EG\times_{C_GH} X^H) \\
\mathcal{H}^{C_GH\times H}_{p+q}(EG\times X^H) \arrow[u, "\mathrm{ind}_{\mathrm{pr}\colon C_GH\times H\to H}"', "\cong"] \arrow[d, "\mathrm{ind}_{m_H}"'] \\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} EG\times X^H) \arrow[d, "\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H}\mathrm{pr}_2)"'] \\
\mathcal{H}^G_{p+q}(\mathrm{ind}_{m_H} X^H) \arrow[d, "\mathcal{H}^G_{p+q}(\nu_H)"'] \\
\mathcal{H}^G_{p+q}(X).
\end{tikzcd}$$

Some explanations should be made. The map $m_H\colon C_GH\times H\to H$ is just the multiplication map. The other maps were already defined in Section 4.1. Note that we use a more general notion of induction map here. In this setting, an induction map ind_φ for a group homomorphism $\varphi\colon H\to G$ exists if $\ker\varphi$ acts freely on X.

If $\mathcal{H}^?_*$ has a Mackey structure, the corresponding coefficient module is flat by Corollary 3.1.19 for $\mathbb{Q}\subseteq R$ semisimple. Consequently, the source is isomorphic to $\mathcal{BH}^?_*$. Actually, the tensor product is taken in the subgroup category, but by an adjointness argument this turns out to be Bredon homology. Lück showed [29] that this map is a natural transformation of proper homology theories. Since it is an isomorphism on coefficients, we get a natural equivalence of equivariant proper homology theories

$$\tau^?_*\colon \mathcal{BH}^?_*\to\mathcal{H}^?_*.$$

In the cohomological case, one gets by a similar construction [30] a natural equivalence

$$\tau_?^* \colon \mathcal{H}_?^* \to \mathcal{BH}_?^*.$$

Proposition 5.4.1. *Let R be a semisimple commutative ring with $\mathbb{Q} \subseteq R$. Let $\mathcal{H}_*^?$ be an equivariant proper homology theory (for discrete groups) over R. Suppose that $\mathcal{H}_*^?$ has a Mackey structure and the Chern character $\mathrm{ch}_G^?$ of Section 4.1 exists for a discrete group G. Then we obtain*

$$\mathrm{ch}_*^? = \tau_*^?.$$

If $\mathcal{H}_?^*$ is an equivariant cohomology theory with the same properties as above and ch_G^* exists, we get

$$\mathrm{ch}_?^* = \tau_?^*.$$

Furthermore, in the case of $\mathcal{H}_?^* = K_?^*$ the construction of τ in this section and the construction of Section 5.2 coincide.

Proof. The only thing we must check is

$$\mathrm{ch}_*^K(\{\bullet\}) = \tau_*^K(\{\bullet\})$$

for any finite group K. But this is clear.

In the cohomological case, the corresponding equation is immediate, too. □

5.5 A Bivariant Chern Character for Profinite Groups (Baum/Schneider)

We give a brief survey of [4] and compare this Chern character with the ones which were introduced.

In the following, let G be a profinite group. Moreover, let X be a locally finite G-simplicial complex.

Definition 5.5.1. The *Brylinski space* $\hat{X}$ is defined as follows

$$\hat{X} = \{\, (g,x) \mid g \in G \text{ and } gx = x \,\}.$$

It is endowed with a G-action by

$$g(h,x) = (ghg^{-1}, gx).$$

Let $\hat{X}$ be the Brylinski space and $\mathcal{H}(G)$ be the Hecke algebra. We denote by $H_c^*(\hat{X};\mathbb{C})$ the singular cohomology groups with compact support. Then $H_c^*(\hat{X};\mathbb{C})$ has a canonical $\mathcal{H}(G)$-action which is given by

$$\big(f \cdot \varphi\big)(\sigma) = \int_G f(g)\varphi(g\sigma), \quad f \in \mathcal{H}(G),\ \varphi \in C_c^*(\hat{X},\mathbb{C}).$$

Definition 5.5.2. Let G be a profinite group. Then we define the *equivariant bivariant homology* by

$$H^*_{G,c}(Y,X) = \prod_{n\in\mathbb{Z}} \hom_{\mathcal{H}(G)}\big(H^n_c(\hat{Y};\mathbb{C}), H^{n+*}_c(\hat{X};\mathbb{C})\big).$$

Remark 5.5.3. If $Y = \{\bullet\}$ and G is finite, this simplifies to

$$H^*_{G,c}(\{\bullet\},X) = H^*_c(\hat{X}/G;\mathbb{C}).$$

If $X = \{\bullet\}$, the Brylinski space is $\hat{X} = G$ and the right hand side vanishes except for $* = 0$. In this case, we obtain the group of class functions $\mathcal{R}(G)$ as an immediate consequence of the construction.

The last remark motivates:

Proposition 5.5.4. *Let CH^G_* be cosheaf homology. Then we have an isomorphism*

$$CH^G_*(Y) \cong H^*_{G,c}(Y,\{\bullet\}).$$

Proof. Baum and Schneider [4, p.318] proved this assertion in the special case of discrete groups. However, this argument also works for profinite groups. $\square$

Proposition 5.5.5. *Let X be a finite G-simplicial complex. Then we obtain an isomorphism*

$$\mathcal{B}K^*_G(X) \cong H^*_{G,c}(\{\bullet\},X).$$

Proof. Baum and Schneider [4, p.319] showed this for finite groups. Voigt [56] showed that the right hand side is compatible with limits. Therefore, the isomorphism carries over to the profinite case. $\square$

We want to indicate the construction of the Chern character. Let $E \to X$ be a G-vector bundle over X and let $\mathrm{pr}_X\colon \hat{X} \to X$ be the canonical projection. Then the pullback $\mathrm{pr}_X E$ is a G-vector bundle over $\hat{X}$. A point is a triple (e,g,x), where $e \in E_x$, $g \in G$ and $x \in X$ such that $gx = x$. We obtain an automorphism of G-vector bundles

$$\alpha\colon\ \mathrm{pr}_X E \to \mathrm{pr}_X E, \quad (e,g,x) \mapsto (ge,g,x).$$

This automorphism α has finite order. In order to see this, we may assume $E = X \times V$ by a result of Segal [47, Prop. 2.4]. Here V is a finite dimensional $\mathbb{C}$-vector space with smooth G-action and the G-action on $X \times V$ is diagonal. Since totally disconnected compact subgroups of $GL(V)$ are finite, there exists an open normal subgroup $H \subseteq G$ such that the G-action of V factorizes over G/H. Thus we obtain $\alpha^{[G:H]} = \mathrm{id}$. This allows us to view $\mathrm{pr}_X E$ as a vector bundle over the profinite completion $\hat{\mathbb{Z}}$, i.e., $\lim_{n\in\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$. Here we consider $\hat{X}$ as a trivial $\hat{\mathbb{Z}}$-space. Furthermore, $\mathrm{pr}_X E$ is G-equivariant and hence the class in $K^0_{\hat{\mathbb{Z}}}(\hat{X})$ is fixed by the natural action of G. Finally, we have constructed a map

$$K^0_G(X) \to K^0_{\hat{\mathbb{Z}}}(\hat{X})^G, \quad [E] \mapsto [\mathrm{pr}_X E].$$

Since $\hat{\mathbb{Z}}$ acts trivially on $\hat{X}$, we obtain $K^0_{\hat{\mathbb{Z}}}(\hat{X}) = K^0(\hat{X}) \otimes_{\mathbb{Z}} R(\hat{\mathbb{Z}})$, where $R(\hat{\mathbb{Z}})$ denotes the representation ring. For a $\mathbb{Z}$-module M we set $M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C}$. We get a natural transformation

$$\tau\colon K^0_G(X) \to (K^0_{\hat{\mathbb{Z}}}(\hat{X})_{\mathbb{C}})^G = (K^0(\hat{X}) \otimes_{\mathbb{Z}} R(\hat{\mathbb{Z}})_{\mathbb{C}})^G$$
$$\xrightarrow{\mathrm{id}\otimes \mathrm{res}^{\{0\}}_{\hat{\mathbb{Z}}}} (K^0(\hat{X})_{\mathbb{C}})^G \xrightarrow{\mathrm{ch}} \bigoplus_{j\in\mathbb{Z}} H^{2j}(\hat{X};\mathbb{C})^G = \bigoplus_{j\in\mathbb{Z}} H^{2j}_{G,c}(\{\bullet\}, X),$$

where ch denotes the ordinary Chern character. In the bivariant case, a generalization of the above considerations [4, Prop. 4] leads to $KK^G_*(C_0(X), C(Y))_{\mathbb{C}} = \hom_{\mathcal{H}(G)}(K^*(\hat{Y})_{\mathbb{C}}, K^*(\hat{X})_{\mathbb{C}})$. Then the ordinary Chern character induces a natural transformation

$$\tau^G_*(X,Y)\colon KK^G_*(C_0(X), C_0(Y)) \to \bigoplus_{j\in\mathbb{Z}} H^{*+2j}_{G,c}(Y,X).$$

Proposition 5.5.6. *Let G be profinite. Then we obtain the identities*

$$\tau^G_*(\{\bullet\}, ?)^{-1} = \mathrm{ch}^G_* \quad \textit{and} \quad \tau^*_G(?, \{\bullet\}) = \mathrm{ch}^*_G\,.$$

Proof. Let $H \subseteq G$ be compact open. Then $\tau^G_*(\{\bullet\}, G/H)$ is just the identification

$$\tau^G_*(\{\bullet\}, G/H)\colon R(H) \otimes \mathbb{C} \xrightarrow{\cong} \mathcal{R}(H).$$

By Corollary 5.1.2 we obtain the desired identity $\tau^G_*(\{\bullet\}, ?)^{-1} = \mathrm{ch}^G_*$. Analogously, we get the second identity. □

5.6 A Bivariant Chern Character for l-Groups (C. Voigt)

We want to give a rough survey of the work by Voigt ([54],[55],[57],[56]).

This Chern character is of a quite different nature. The previous homology theories have, as input, CW-complexes. This one has bornological algebras as input.

In the following, we will deal with bornologies (instead of topologies). An introduction can be found in [37, Chap. 2]. Roughly speaking, a bornology specifies the bounded sets (also called small sets) in a space. A map f is called *bounded* if f maps bounded sets to bounded sets. A bornological algebra is a bornological vector space A with an associative multiplication given as a bounded linear map $A\hat{\otimes}A \to A$, where $\hat{\otimes}$ denotes the completed bornological tensor product. We remark that we do not require the existence of a unit in a bornological algebra. A basic example of a bornological algebra is the Hecke algebra $\mathcal{H}(G)$ of an l-group G, where $\mathcal{H}(G)$ is equipped with the fine bornology, i.e., the smallest possible bornology. A module M over a bornological algebra A is called non-degenerate if the module action $A\hat{\otimes}M \to M$ is a bornological quotient map. We remark that the category of smooth representations of G is isomorphic to the category of non-degenerate $\mathcal{H}(G)$-modules. Here a representation ρ is called *smooth* if the isotropy group of any bounded set is open.

A G-algebra is a bornological algebra A which is at the same time a G-module such that the multiplication $A\hat{\otimes}A \to A$ is equivariant. Here the tensor product $A\hat{\otimes}A$ is equipped with the diagonal action, as usual. A particular example of a G-algebra is the algebra $\mathcal{K}_G$ which is defined as follows. As a bornological vector space we have $\mathcal{K}_G = \mathcal{H}(G)\hat{\otimes}\mathcal{H}(G) = \mathcal{H}(G \times G)$. The multiplication in $\mathcal{K}_G$ is given by

$$(k \cdot l)(s,t) = \int_G k(s,r)l(r,t)dr$$

and the G-action is defined by

$$(r \cdot k)(s,t) = k(r^{-1}s, r^{-1}t).$$

This algebra can be viewed as a dense subalgebra of the algebra of compact operators $\mathcal{K}(L^2(G))$ on the Hilbert space $L^2(G)$. Here $L^2(G)$ is equipped with the precompact bornology, i.e., a set is bounded if its closure is compact.

Next we define covariant modules. Let $\mathcal{O}_G$ be the space $\mathcal{H}(G)$ equipped with pointwise multiplication and the action of G by conjugation. A covariant module M is a smooth representation of G which is at the same time a non-degenerate $\mathcal{O}_G$-module. The G-module structure and the $\mathcal{O}_G$-module structure are required to be compatible in the sense that

$$s \cdot (f \cdot m) = (s \cdot f) \cdot (f \cdot m)$$

for all $s \in G, f \in \mathcal{O}_G$ and $m \in M$. A bounded linear map $f\colon M \to N$ of covariant modules is called covariant if it is $\mathcal{O}_G$-linear and equivariant.

Now we can introduce non-commutative equivariant differential forms. Let A be a G-algebra. The equivariant n-forms of A are defined by $\Omega^n_G(A) = \mathcal{O}_G\hat{\otimes}\Omega^n(A)$, where $\Omega^n(A) = A^+\hat{\otimes}A^{\hat{\otimes}n}$ and A^+ denotes the unitarization of A. The group G acts diagonally on $\Omega^n_G(A)$ and we have an obvious $\mathcal{O}_G$-module structure. In this way $\Omega^n_G(A)$ becomes a covariant module.
The equivariant Hochschild boundary $b\colon \Omega^n_G(A) \to \Omega^{n-1}_G(A)$ is defined by

$$\begin{aligned} b(f(t)\otimes x_0dx_1 \cdots dx_n) &= f(t) \otimes x_0x_1dx_2 \cdots dx_n \\ &+ \sum_{j=1}^{n-1}(-1)^j f(t) \otimes x_0dx_1 \cdots d(x_jx_{j+1}) \cdots dx_n \\ &+ (-1)^n f(t) \otimes (t^{-1} \cdot x_n)x_0dx_1 \cdots dx_{n-1}. \end{aligned}$$

Moreover, we have the equivariant Connes operator $B\colon \Omega^n_G(A) \to \Omega^{n+1}_G(A)$ which is given by

$$B(f(t) \otimes x_0dx_1 \cdots dx_n) = \sum_{i=0}^{n}(-1)^{ni} f(t) \otimes t^{-1} \cdot (dx_{n+1-i} \cdots dx_n)dx_0 \cdots dx_{n-i}.$$

It is straightforward to check that b and B are covariant maps. The natural symmetry operator T for covariant modules is of the form

$$T(f(t) \otimes \omega) = f(t) \otimes t^{-1} \cdot \omega$$

on $\Omega^n_G(A)$. One easily obtains the relations $b^2 = 0$, $B^2 = 0$ and $Bb + bB = \mathrm{id} - T$ for these operators. This shows that $\Omega_G(A)$ is a paramixed complex in the following sense.

Definition 5.6.1. A paramixed complex M of covariant modules is a sequence of covariant modules M_n together with differentials b of degree -1 and B of degree $+1$ satisfying $b^2 = 0$, $B^2 = 0$ and

$$[b, B] = bB + Bb = \mathrm{id} - T.$$

The most important examples of paramixed complexes are bounded below in the sense that $M_n = 0$ if $n < N$ for some fixed $N \in \mathbb{Z}$. In particular, the equivariant differential forms $\Omega_G(A)$ of a G-algebra A satisfy this condition for $N = 0$.

The Hodge filtration of a paramixed complex M of covariant modules is defined by

$$F^n M = bM_{n+1} \oplus \bigoplus_{j>n} M_j.$$

Clearly, $F^n M$ is closed under the operators b and B. We write

$$L^n M = F^{n-1} M / F^n M$$

for the n-th layer of the Hodge filtration. If M is bounded below such that $M_n = 0$ for $n < 0$, we define the n-th level $\theta^n M$ of the Hodge tower of M by

$$\theta^n M = \bigoplus_{j=0}^{n-1} M_j \oplus M_n / bM_{n+1}.$$

By definition, the Hodge tower of M is the projective system $\theta M = (\theta^n M)_{n \in \mathbb{N}}$.

The spaces $\theta^n M$ are equipped with the grading into even and odd forms and the differential $\partial = B + b$. In this way the Hodge tower becomes a projective system of paracomplexes in the following sense.

Definition 5.6.2. A paracomplex of covariant modules is a $\mathbb{Z}_2$-graded covariant module C with a boundary operator $\partial\colon C \to C$ of degree one such that $\partial^2 = \mathrm{id} - T$.

Chain maps of paracomplexes and homotopy equivalences are defined by the usual formulas.

Definition 5.6.3. Let G be an l-group and let A and B be G-algebras. The *equivariant bivariant periodic cyclic homology* of A and B is

$$HP^G_*(A, B) = H_*(\mathfrak{Hom}_G(\theta\Omega_G(A \hat{\otimes} \mathcal{K}_G), \theta\Omega_G(B \hat{\otimes} \mathcal{K}_G))).$$

The definition involves covariant maps between projective systems of covariant modules. Maps between projective systems are always understood in the sense

$$\hom((M_i)_{i \in I}, (N_j)_{j \in J}) = \operatorname*{colim}_{j \in J} \lim_{i \in I} \hom(M_i, N_j)$$

of pro-categories. Finally, we consider the usual differential for a hom-complex given by

$$\partial(\phi) = \phi\partial_A - (-1)^{|\phi|}\partial_B\phi$$

for a homogeneous element ϕ in order to define homology. This makes sense since the failure of the individual differentials in $\theta\Omega_G(A\hat{\otimes}\mathcal{K}_G)$ and $\theta\Omega_G(B\hat{\otimes}\mathcal{K}_G)$ to satisfy $\partial^2 = 0$ is cancelled out by naturality of the operator T.

Let A, B, C be G-algebras. A bounded morphism $f\colon A \to B$ defines an element $[f] \in HP_0^G(A, B)$. Further, two elements $[g], [h] \in HP_*^G(A, B)$ are represented by appropriate morphisms g, h. Composition of these morphisms defines a graded product on $HP_*^G(A, B)$. The induced maps

$$f^*\colon HP_*^G(B, C) \to HP_*^G(A, C) \quad \text{and} \quad f_*\colon HP_*^G(C, A) \to HP_*^G(C, B)$$

are given by left and right multiplication by $[f]$. Moreover, we have the following properties:

Theorem 5.6.4. *Let A, B be G-algebras. Then the following hold:*

(i) Diffeotopy Invariance
The functor HP_*^G is invariant under G-diffeotopies, i.e., smooth G-homotopies, in both variables.

(ii) Stability
There are natural isomorphisms

$$HP_*^G(A, B) \cong HP_*^G(A\hat{\otimes}\mathcal{K}_G, B) \cong HP_*^G(A, B\hat{\otimes}\mathcal{K}_G).$$

(iii) Excision
Let $0 \to C \to D \to E \to 0$ be an extension of bornological G-algebras with a (not necessarily equivariant) splitting. Then we obtain exact sequences

$$\begin{tikzcd} HP_0^G(A, C) \arrow[r] & HP_0^G(A, D) \arrow[r] & HP_0^G(A, E) \arrow[d] \\ HP_1^G(A, E) \arrow[u] & HP_1^G(A, D) \arrow[l] & HP_1^G(A, C) \arrow[l] \end{tikzcd}$$

and

$$\begin{tikzcd} HP_0^G(E, B) \arrow[r] & HP_0^G(D, B) \arrow[r] & HP_0^G(C, B) \arrow[d] \\ HP_1^G(C, B) \arrow[u] & HP_1^G(D, B) \arrow[l] & HP_1^G(E, B) \arrow[l] \end{tikzcd}$$

A G-C^*-algebra can be seen as a G-algebra by considering the precompact bornology, i.e., a subset is bounded if its closure is compact. Unfortunately, on the one hand, equivariant periodic homology is not homotopy invariant in general. On the other hand, the tensor product in the stability statement is the bornological tensor product, which does not coincide with the one of C^*-algebras. Thus we cannot apply Theorem 2.2.18.

Hence Voigt studies a variant of HP_*^G, which is called equivariant local cyclic homology HL_*^G. It is an equivariant generalization of bivariant local cyclic homology which was developed by Puschnigg [42]. It is still a bit more complicated since a smoothing functor comes into play, too. Thus I want to skip the explicit definition, but state the following theorem

Theorem 5.6.5. *(i) Equivariant bivariant local cyclic homology (together with the smoothing functor) fulfills homotopy invariance, stability and excision for C^*-algebras.*

(ii) HL_*^G provides a product structure. In particular, a $*$-homomorphism of G-C^*-algebras $f\colon A \to B$ defines an element $[f] \in HL_*^G(A,B)$ and the induced maps $f^*\colon HL_*^G(B,C) \to HL_*^G(A,C)$ and $f_*\colon HL_*^G(C,A) \to HL_*^G(C,B)$ are given by left and right multiplication by $[f]$, respectively.

(iii) Let X, Y be two finite proper smooth G-CW-complexes. Then we get natural isomorphisms

$$HP_*^G(C(X), C(Y)) \cong HL_*^G(C(X), C(Y)) \cong HL_*^G\big(S(C(X)), S(C(Y))\big),$$

where S denotes the smoothing functor.

Proof. The first statement is the main result of [57]. The second assertion can be found in [57, p.16], too. The third statement is proven in [56, Sec. 6]. □

Analogously to equivariant bivariant K-theory, we can consider equivariant bivariant local cyclic homology as a category HL^G, where the objects are separable G-C^*-algebras and the morphisms are given by $HL_*^G(A,B)$. We get by Theorem 2.2.18:

Corollary 5.6.6. *There exists a natural transformation of bivariant proper smooth G-homology theories*

$$\tau\colon KK_*^G(C(X), C(Y)) \to HP_*^G(C(X), C(Y)),$$

which is compatible with the product structures.

Proof. The even case is a direct consequence of Theorem 2.2.18. The odd case can be derived from the even case [57, Sec. 13]. □

In addition, we get for the coefficients [57, Prop. 13.5]:

Proposition 5.6.7. *Let $X = Y = \{\bullet\}$ and G be profinite. We obtain $HP^G(\mathbb{C},\mathbb{C}) = \mathcal{R}(G)$ and τ of the previous corollary simplifies to*

$$\tau\colon R(G) \to \mathcal{R}(G)$$

which is the map given by taking the characters.

Theorem 5.6.8. *There exists a natural equivalence of bivariant proper smooth G-homology theories*

$$\tau \otimes \mathbb{C}\colon KK_*^G(C(X), C(Y)) \otimes \mathbb{C} \longrightarrow HP_*^G(C(X), C(Y)),$$

which is compatible with the product structures.

Proof. It suffices to show that $HP_*^?(C(??), B)$ is an equivariant homology theory for any G-algebra B and that τ is compatible with the induction structures. This is proven by Voigt [56]. □

Theorem 5.6.9. *Let G be a profinite group. Then there is an equivalence of bivariant proper smooth G-homology theories (defined on finite G-simplicial complexes)*

$$HP_*^G(C(X), C(Y)) \cong H_{G,c}^*(X, Y),$$

where the latter homology theory is the one defined in Section 5.5. Further, the two Chern characters coincide under this identification.

Proof. The first assertion is proven in [55] and the second one in [56]. □

Theorem 5.6.10. *There exists an equivalence of proper smooth G-homology theories*

$$HP_*^G(C(X), \mathbb{C}) \cong \mathcal{B}K_*^G(X) \otimes \mathbb{C}.$$

If ch_*^G exists, we obtain $\mathrm{ch}_*^G = \tau^{-1} \otimes \mathbb{C}$. If G is discrete, $\tau^{-1} \otimes \mathbb{C}$ coincide with the Chern character of Section 5.4.

Proof. If we assume the first assertion, the second and last one directly follow from Proposition 5.6.7 and Corollary 5.1.3. The first assertion was proven in [56, Sec. 8]. I just want to sketch the proof. There is a more general notion of equivariant bivariant homology which was introduced in Section 5.5. The general one admits arbitrary l-groups and proper smooth G-simplicial complexes. Further, Theorem 5.6.9 and Proposition 5.5.4 generalize to l-groups, too. Finally, we composite these two identifications with the one of Proposition 2.2.42 and get the desired one for proper smooth G-simplicial complexes. However, this can be lifted to proper smooth G-CW-complexes. □

Remark 5.6.11. Let A be a G-C^*-algebra. Analogously to Section 2.2, we obtain an equivariant $(\Gamma_G, \mathcal{CO})$-homology theory $HP_*^?(C(??), A)$ with a Mackey functor structure on coefficients. Then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{B}(K^{(G,A)} \otimes_{\mathbb{Z}} \mathbb{C})_*(X) \arrow[r, "\cong"] \arrow[d, "\mathrm{ch}_*^G"] & \mathcal{B}HP_*^G(C(X), A) \arrow[d, "\mathrm{ch}_*^G"] \\
K^{(G,A)}(X) \otimes_{\mathbb{Z}} \mathbb{C} \arrow[r, "\tau\otimes\mathbb{C}"] & HP_*^G(C(X), A)
\end{tikzcd}$$

whenever the left and right ch_*^G are defined.

Appendices

A The Group Ring

Let G be a discrete group, $\mathbb{Q} \subseteq R$ be a commutative ring and RG be the corresponding group ring.

Definition A.1. Let M be a left R-module. We define the *left weak dimension* by

$$\operatorname{l.w.dim}_R M = \sup \{ n \mid \operatorname{Tor}_{n+1}^R(N, M) = 0 \text{ for every right } R\text{-Module } N \}.$$

Analogously, we can define the *right weak dimension* $\operatorname{r.w.dim}_R M$ for a right R-module M. The *global weak dimension* is

$$\begin{aligned} \operatorname{w.dim} R &= \sup \{ \operatorname{l.w.dim}_R M \mid M \text{ left } R\text{-module} \} \\ &= \sup \{ \operatorname{r.w.dim}_R M \mid M \text{ right } R\text{-module} \}. \end{aligned}$$

Proposition A.2. *Let R denote the RG-module which is endowed with the trivial G-action. Then the following holds:*

$$\operatorname{w.dim} RG = \operatorname{l.w.dim}_{RG} R = \operatorname{r.w.dim}_{RG} R.$$

Proof. In [13, Thm. X,6.2] this is shown for the projective dimension. However, a trivial modification may be used to prove our assertion. □

Definition A.3. A group is *locally finite* if every finitely generated subgroup is finite.

Remark A.4. A locally finite group is a torsion group. However, the converse, known as the Burnside problem, does not hold. In particular, there exist infinite finitely generated torsion groups. For an overview of the Burnside Problem see for example [1].

Proposition A.5. *The following statements hold:*

(i) If G is locally finite and $H \subseteq G$ is a subgroup. Then H is locally finite.

(ii) Let $N \subseteq G$ be a normal subgroup. Then G is locally finite if and only if N and G/N are locally finite.

(iii) Let $G = \operatorname{colim}_{i \in I} G_i$. Then G is locally finite if each G_i is locally finite.

Proof. (i): Let G be locally finite. Obviously any subgroup $H \subseteq G$ is locally finite.

(ii): Let $N \subseteq G$ be a normal subgroup of a locally finite group G. Let $H \subseteq G/N$ be finitely generated, say by representative classes $[h_1], \ldots, [h_n]$. Then the group $H' \subseteq G$ generated by $h_1, \ldots, h_n$ is finite because G is locally finite. However, this forces H to be finite, too. Therefore G/N is locally finite. Moreover, N is locally finite by (i).

Let $N \subseteq G$ be normal. Suppose further that N and G/N are locally finite. Let $H \subseteq G$ be finitely generated. Then $H/(H \cap N) \subseteq G/N$ is finitely generated and thus finite. We obtain a short exact sequence

$$1 \to N \cap H \to H \to H/(H \cap N) \to 1.$$

Since H is finitely generated and $H/(H \cap N)$ is finite, the group $H \cap N$ is finitely generated. This can be proven by a trivial modification of [44, Prop. 2.5.5]. Hence $H \cap N$ is finite and H must be finite.

(iii): By definition, G is a quotient of $\bigoplus_{i \in G_i} G_i$, however, this is locally finite. □

Definition A.6. We call R *von Neumann regular* if every R-module M is flat.

Theorem A.7. *Suppose that R is von Neumann regular. Then the following are equivalent:*

(i) RG is von Neumann regular.

(ii) Every RG-module is flat.

(iii) The RG-module R endowed with the trivial G-action is flat.

(iv) G is locally finite.

Proof. (i) ⇔ (ii): Trivial since (ii) is the definition of (i).
(ii) ⇔ (iii): This is Proposition A.2.
(ii) ⇒ (iv): This is done by Connell [16].
(iv) ⇒ (ii): This is proven by Auslander [3] and McLaughlin [36]. □

Theorem A.8. *Let R be semisimple. The following hold:*

(i) If G is finite, then every RG-module is injective and projective.

(ii) If the RG-module R endowed with the trivial G-action is injective, then G is a torsion group.

Proof. The first assertion is the well known Maschke theorem.

Suppose there exists a torsionfree element $g \in G$. Hence $(1-g)$ is not a zero divisor in RG and we can define

$$\varphi\colon (1-g)RG \to RG, \quad (1-g)r \mapsto r.$$

Let $\varepsilon\colon RG \to R$ be the augmentation. Since

$$\hom_{RG}(RG, R) = R,$$

each homomorphism $\psi\colon RG \to R$ is just a scalar multiple of the augmentation ε. Therefore we cannot lift $\varepsilon \circ \varphi$ to $RG \supseteq (1-g)RG$ because $(1-g)RG \subseteq \ker \varepsilon$ and $\varepsilon \circ \varphi \neq 0$. This implies that R is cannot be injective and we get a contradiction. □

B Limit Behavior of Flat, Projective and Injective Modules

We remind the reader:

- A partially ordered set is of finite length if every decreasing sequence becomes stationary.
- A partially ordered set is of finite colength if every increasing sequence becomes stationary.
- Every countable directed system admits a cofinal system of finite (co)length (see Remark 3.2.13).

Moreover, in Subsection 3.2.1, we have seen that we can consider an inverse system of R-modules $(M_i)_{i\in I}$ as a contravariant RI-module. Analogously, a directed system can be considered as a covariant RI-module. This identification will be used throughout this section. Therefore, we call a directed system $(M_i)_{i\in I}$ *projective* if it is projective as an RI-module and an inverse system $(M_i)_{i\in I}$ *injective* if it is injective as an RI-module. Furthermore, we call $(M_i)_{i\in I}$ a *directed system of projective modules* if each M_i is a projective R-module. In the same way we define an *inverse system of injective modules.*

Proposition B.1. *Let N be an R-module.*

(i) Let $(M_i)_{i\in I}$ be a directed system of R-modules which is of finite length. Then there is a converging spectral sequence of the following form:

$$E_2^{p,q} = \lim_{i\in I}{}^p \operatorname{Ext}_R^q(M_i, N) \Rightarrow Ext_R^{p+q}(\operatorname*{colim}_{i\in I} M_i, N).$$

(ii) Let $(M_i)_{i\in I}$ be an inverse system of R-modules which is of finite length. Then there are two spectral sequences

$$E_2^{p,q} = \lim_{i\in I}{}^p \operatorname{Ext}_R^q(N, M_i) \quad \textit{and} \quad E_2^{p,q} = \operatorname{Ext}_R^q(N, \lim_{i\in I}{}^p M_i),$$

which converge to the same limit.

Proof. We only prove the first statement, the second one is analogous and a bit easier.
Without loss of generality, we can assume that I itself is of finite length and not just $(M_i)_{i\in I}$. Otherwise, lim and colim vanish and the assertion is trivial. In Theorem 3.1.7 the projective directed systems were classified. Let $(Q_i)_{i\in I}$ be projective. We get

$Q_i = \oplus_{j\in J_i} P_j$ for some projective R-modules P_j and index sets J_i. If $i < j$ then $J_i \subseteq J_j$ and the structure map is given by the canonical inclusion. Let $J = \operatorname{colim}_{i\in I} J_i$, then we get $\operatorname{colim}_{i\in I} Q_i = \oplus_{j\in J} P_j$. Consequently, colim sends projective directed systems to projective R-modules. Thus we get the Grothendieck spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^q(\operatorname*{colim}_{i\in I}{}^p M_i, N) \Rightarrow L^{p+q}(\hom(-,N)\circ \operatorname*{colim}_{i\in I})(M_i),$$

where $L^n(F)$ denotes the n-th left derivation of the functor F. The functor colim is exact, so we get, on the one hand,

$$L^{p+q}(\hom(-,N)\circ \operatorname*{colim}_{i\in I})(M_i) \cong Ext_R^{p+q}(\operatorname*{colim}_{i\in I} M_i, N). \tag{B.1.1}$$

On the other hand, we have $\hom(\operatorname{colim}_{i\in I} M_i, N) \cong \lim_{i\in I} \hom(M_i, N)$ for arbitrary $(M_i)_{i\in I}$ and hence

$$L^{p+q}(\hom(-,N)\circ \operatorname*{colim}_{i\in I})(M_i) \cong L^{p+q}(\lim_{i\in I} \circ \hom(-,N))(M_i). \tag{B.1.2}$$

Now we will consider the functor $\lim_{i\in I}\circ \hom(-,N)$. If we want to establish the Grothendieck spectral sequence, $\hom(-,N)$ has to send projective directed systems to lim-acyclic (concerning the right derivations) directed systems. Let $(Q_i)_{i\in I}$ be projective and thus of the form which was described at the beginning of the proof. Then $\hom(Q_i, N) = \prod_{j\in J_i} \hom(P_j, N)$ and the structure maps are the canonical projections. Choose injective resolutions

$$0 \to \hom(P_j, N) \to I_j^0 \to I_j^1 \to \cdots$$

for every $j \in J$. We define the directed systems $(I_i^k)_{i\in I}$ by $I_i^k = \prod_{j\in J_i} I_j^k$ and the projections as structure maps. Because of Theorem 3.2.5, the directed systems $(I_i^k)_{i\in I}$ are injective and form an injective resolution

$$0 \to (\hom(Q_i,N))_{i\in I} \to (I_i^0)_{i\in I} \to (I_i^1)_{i\in I} \to \cdots .$$

Since

$$\lim_{i\in I} \hom(Q_i, N) = \prod_{j\in J} \hom(P_j, N) \quad \text{and} \quad \lim_{i\in I} I_i^k = \prod_{j\in J} I_j^k,$$

the sequence

$$0 \to \lim_{i\in I} \hom(Q_i, N) \to \lim_{i\in I} I_i^0 \to \lim_{i\in I} I_i^1 \to \cdots$$

is exact. Hence $(\hom(Q_i,N))_{i\in I}$ is lim-acyclic and we obtain the Grothendieck spectral sequence

$$E_2^{p,q} = \lim_{i\in I}{}^p \operatorname{Ext}_R^q(M_i, N) \Rightarrow L^{p+q}(\lim_{i\in I}\circ \hom(-,N))(M_i).$$

Combining this with (B.1.2) and (B.1.1) yields the assertion. □

Corollary B.2. *Let M be an R-module.*

(i) Let $(P_i)_{i\in I}$ be a directed system of projective R-modules which is of finite length. Then we obtain

$$\mathrm{Ext}^1_R(\operatorname*{colim}_{i\in I} P_i, M) \cong \lim_{i\in I}{}^1 \hom(P_i, M).$$

(ii) Let $(I_i)_{i\in I}$ be an inverse system of injective R-modules such that $(I_i)_{i\in I}$ is of finite length and $\lim^1_{i\in I} I_i = 0$. Then we obtain

$$\mathrm{Ext}^1_R(M, \lim_{i\in I} I_i) \cong \lim_{i\in I}{}^1 \hom(M, I_i).$$

Thus the limit is projective or injective if and only if the concerning $\lim^1$*-term vanishes for every R-module M.*

Proof. Since $\mathrm{Ext}^n_R(P, M) = 0$ for every projective R-module P and $n > 0$, the first assertion follows from the previous proposition. Suppose I is injective. Then we have $\mathrm{Ext}^n_R(M, I) = 0$ for every $n > 0$. As the higher derived limits vanish by Lemma 3.2.7, the second assertion follows from the previous proposition, too. $\square$

The next two examples show that the $\lim^1$-term does not in general vanish.

Example B.3 (Limits of projective modules). Let $F \subseteq \mathbb{Q}$ be a finitely generated subgroup. Then F is a finitely generated abelian group and hence free, in particular, a projective $\mathbb{Z}$-module. Note that $\mathbb{Q}$ is not projective and

$$\mathbb{Q} = \operatorname*{colim}_{\substack{F\subseteq\mathbb{Q} \\ F \text{ finitely generated}}} F.$$

Example B.4 (Limits of injective modules). We remind the reader that M is called a *divisible $\mathbb{Z}$-module* if

$$\forall a \in \mathbb{Z}, \forall n \in M \exists m \in M : n = am$$

and M is divisible if and only if M is injective.

Let $M = \oplus_{i=0}^{\infty}\mathbb{Q}e_i$ and $N_i = span_{\mathbb{Z}} \{ e_j - e_{j+1} \mid j \geq i \}$. Then each M/N_i is, as a quotient of a divisible module, divisible. However, the limit of the inverse system

$$\cdots \to M/N_2 \to M/N_1 \to M/N_0$$

is not divisible. In order to show that, consider the element $e = (\ldots, \bar{e}_2, \bar{e}_1, \bar{e}_0)$ in the inverse limit and an arbitrary $n \in \mathbb{N}$. Assume we have an $a = (\ldots, \bar{a_2}, \bar{a_1}, \bar{a_0})$ such that $na = e$. Let $a_i = \sum_{k=0}^{\infty} a_i^k \in \bigoplus_{k=0}^{\infty} \mathbb{Q}e_k$ be a representative of $\bar{a_i}$ in M. We have $M/N_i = \oplus_{l=0}^{i-1}\mathbb{Q} \oplus L$ for some L and hence $a_i^k = 0$ for all $k < i$. We define

$$i_0 = \min \{ k \mid a_i^k \notin \mathbb{Z} \} \tag{B.4.1}$$

and obtain $i_0 < \infty$. Otherwise we would get $\bar{a_i} = \lambda_i \bar{e_i}$ for some $\lambda_i \in \mathbb{Z}$ because $\mu\bar{e_i} = \mu\bar{e_j}$ for $\mu \in \mathbb{Z}$. This is a contradiction because of

$$0 = n\lambda_i e_i - e_i = (n\lambda_i - 1)e_i \notin N_i.$$

We conclude $i \leq i_0 < \infty$. Let $b, c \in M$, then we obtain

$$i_0^b \neq i_0^c \Longrightarrow b \neq c \text{ (in } M/N_i), \quad i \in \mathbb{N},$$

where i_0^b, i_0^c are defined by (B.4.1). Putting this all together, we have $\infty > i_0 > j$ for each $i \in \mathbb{N}$ and every $j \in \mathbb{N}$ and thus a contradiction.

Corollary B.2 is a nice application of the classification of $R\Gamma$-modules. However, Corollary B.2 is false for directed systems of $R\Gamma$-modules.

Example B.5. We consider the subgroup category $\mathtt{Sub}_{\mathcal{CO}}(\mathbb{Z}_p)$ (cf. Example 1.2.14). Here we have

$$\lim_{n\to\infty}{}^1 \hom\Big(\bigoplus_{k\in\mathbb{N}} \mathbb{Q}\mathtt{Sub}(\mathbb{Z}_p)(k,?)_n, \bigoplus_{k\in\mathbb{N}} R\mathtt{Sub}(\mathbb{Z}_p)(k,?)\Big) \neq 0$$
$$\lim_{n\to\infty}{}^1 \hom\Big(\prod_{k\in\mathbb{N}} CE_k(\mathbb{Q}), \prod_{k\in\mathbb{N}} CE_k(\mathbb{Q})_n\Big) \neq 0,$$

where we denote by $\mathbb{Q}\mathtt{Sub}(\mathbb{Z}_p)(k,-)_n = \mathbb{Q}\mathtt{Sub}(\mathbb{Z}_p)(k,-)_{p^n\mathbb{Z}_p}$ the truncated module as in Subsection 3.1.2 and define $CE_k(\mathbb{Q})_n$ analogously.

Proof. We have

$$\begin{aligned}\hom\big(&\bigoplus_{k\in\mathbb{N}} \mathbb{Q}\mathtt{Sub}(\mathbb{Z}_p)(k,?)_n, \bigoplus_{k\in\mathbb{N}} R\mathtt{Sub}(\mathbb{Z}_p)(k,?)\big)\\ &= \hom\big(\bigoplus_{k\in\mathbb{N}} \mathbb{Q}\mathtt{Sub}(\mathbb{Z}_p)(k,?)_n, \bigoplus_{k\in\mathbb{N}} R\mathtt{Sub}(\mathbb{Z}_p)(k,?)_n\big).\end{aligned}$$

Since

$$\Big(\bigoplus_{k\in\mathbb{N}} \mathbb{Q}\mathtt{Sub}(\mathbb{Z}_p)(k,l)\Big)_{l\in\mathbb{N}} = \Big(\cdots \hookrightarrow V_2 \hookrightarrow V_1 \hookrightarrow V_0\Big)$$

is a properly increasing sequence of vector spaces, the above hom-sequence is the sequence

$$\begin{aligned}\cdots \hookrightarrow \{\, &\varphi \in \mathrm{End}(V_0) \mid \varphi|V_n \subseteq V_n,\ n = 0,1,2\,\}\\ &\hookrightarrow \{\, \varphi \in \mathrm{End}(V_0) \mid \varphi|V_n \subseteq V_n,\ n = 0,1\,\} \hookrightarrow \mathrm{End}(V_0)\end{aligned}$$

with injective but non-surjective maps. Hence the corresponding $\lim^1$-term does not vanish by Lemma 3.2.9.

The second assertion can be proven analogously. □

Since products of injective modules are injective, one might think that injectivity behaves better under limits than under colimits. Analogously, projectivity ought to behave better under colimits than under limits. Oddly, this is not the case. We give a short survey here, which is taken from Chase [15] and Hernández [35]. In order to justify the title, we discuss the behavior of flat modules under limits, too.

Theorem B.6. *Let R be a ring. Then the following conditions are equivalent:*

(i) R is left Noetherian.

(ii) The colim *of injective left R-modules is injective.*

(iii) The direct sum of injective left R-modules is injective.

Proof. Let R be Noetherian and $I = \operatorname{colim}_{j \in J} I_j$ be a colimit of injective R-modules. Let M be a finitely generated R-module. Since R is Noetherian, there exists a resolution of projective finitely generated R-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to M.$$

Because I_j is injective and colim is exact, we obtain an exact sequence

$$\cdots \to \operatorname*{colim}_{j \in J} \hom_R(P_2, I_j) \to \operatorname*{colim}_{j \in J} \hom_R(P_1, I_j) \to \operatorname*{colim}_{j \in J} \hom_R(P_0, I_j).$$

Since P_n are finitely generated, the functors hom and colim commute. Finally, we get an exact sequence

$$\cdots \to \hom_R(P_2, I) \to \hom_R(P_1, I) \to \hom_R(P_0, I) \to 0$$

and $\operatorname{Ext}_R^n(M, I) = 0$. This proves the injectivity of I.

The implication (ii) $\Rightarrow$ (iii) is immediate. Suppose now (iii) holds. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of (left) ideals in R. Let Q_n be an injective R-module which contains R/I_n for every $n \in \mathbb{N}$. We set

$$I = \bigcup_{n=1}^{\infty} I_n \quad \text{and} \quad Q = \bigoplus_{n=1}^{\infty} Q_n.$$

We define a homomorphism $f\colon I \to Q$ by $f(a) = \sum_{n=1}^{\infty} f_n(a)$, where $f_n\colon I \to Q_n$ is the canonical homomorphism induced by the canonical projection $I \to I/I_n$. Note that for any $a \in I$ there exists an n with $a \in I_n$. Hence we obtain $f_k(a) = 0$ for any $k \geq n$ and f is well-defined. By hypothesis, Q is injective; thus there exists a homomorphism $g\colon R \to Q$ such that $g/I = f$. But then

$$f(I) \subseteq g(R) \subseteq Q_1 \oplus \cdots \oplus Q_n \quad \text{for some } n \in \mathbb{N},$$

from which it follows easily that $I = I_{n+1}$. Therefore R is Noetherian. $\square$

Unfortunately, the projective and flat cases are a little more complicated.

Definition B.7. A ring R is called *left coherent* if any finitely generated left ideal in R is finitely related.

Theorem B.8. *Let R be a ring. Then the following statements are equivalent:*

(i) R is left coherent.

(ii) The product of flat right R-modules is flat.

(iii) The product of any number of copies of R is flat as a right R-module.

Proof. [15, Thm. 2.1] □

In the general case of inverse limits, the following is known:

Theorem B.9. *Let R be an integral domain. Then the following are equivalent:*

(i) R is left coherent and $\mathrm{w.\,dim}(R) \leq 2$.

(ii) The inverse limit of flat modules is flat.

Proof. [35, Thm. 2.12] □

We remind the reader that $\mathrm{w.\,dim}(R)$ is the weak dimension which was defined in Definition A.1.

Theorem B.10. *Let R be a ring. Then the following are equivalent.*

(i) Every descending chain of principal left ideals in R becomes stationary.

(ii) Every flat right R-module is projective.

If R satisfies these conditions, R is called right perfect.

Proof. [15, Thm. 3.2] □

Theorem B.11. *Let R be a ring. Then the following statements are equivalent:*

(i) R is left coherent and right perfect.

(ii) The product of projective right R-modules is projective.

(iii) The product of any number of copies of R is projective as a right R-module.

Proof. [15, Thm. 3.3] □

Proposition B.12. *Let R be a commutative ring. Then the following are equivalent:*

(i) R is Artinian.

(ii) R is coherent and perfect.

Proof. [15, Thm. 3.4] □

Corollary B.13. *Let R be an integral domain. Then the following statements are equivalent:*

(i) R is Artinian and $\mathrm{w.\,dim}(R) \leq 2$.

(ii) Every inverse limit of projective modules is projective.

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